HW 5 (due 6 Mar)

A.8 $\Psi(x) = A\cdot(\cos kx + i\sin kx)$ definite momentum case

(a) $k \longleftrightarrow p$? $p = \frac{h}{\lambda}$ and $k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$

$$\Rightarrow p = h\cdot\frac{k}{2\pi} = \hbar k$$

(b) $|\Psi(x)|^2 = A^2(\cos^2 kx + \sin^2 kx) = A^2$,

which is independent of x

(c) Why A must be infinitesimal?

$$P = \int_{x_1}^{x_2} |\Psi(x)|^2\, dx$$

$$= A^2\int_{x_1}^{x_2} dx = A^2 x\Big|_{x_1}^{x_2}$$

Total area under the curve: $A^2\cdot$ width of the range of x values.

If x-interval is infinite $\int_{-\infty}^{+\infty}$, then A^2 must be infinitesimal in order for the area to be finite.

(d) Show $\frac{d\Psi}{dx} = ik\Psi$

$$\frac{d\Psi}{dx} = A\left[-\sin(kx)\cdot k + i\cos(kx)\cdot k\right]$$

$i^2\cdot \Big|$

$$\frac{d\Psi}{dx} = ikA\left[-i\sin(kx) - \cos(kx)\right]$$

$$\frac{d\Psi}{dx} = ikA\left[\cos kx + i\sin kx\right]$$

$$\frac{d\Psi}{dx} = ik\Psi \quad \square$$

(e) Show $\Psi = A\cdot e^{ikx}$ obeys $d\Psi/dx = ik\Psi$

$$\frac{d\Psi}{dx} = A\cdot e^{ikx}\cdot ik = \Psi\cdot ik \quad \square$$

2.17 Low-temperature limit $q \ll N$ for the multiplicity of a large Einstein solid.

- Start with: $\Omega(N,q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!\,(N-1)!} \approx \frac{(q+N)!}{q!\,N!}$

$$\ln\Omega = \ln\left[\frac{(q+N)!}{q!\,N!}\right] = \ln(q+N)! - \ln q! - \ln N!$$

$\ln N! \approx N\ln N - N$

$$\approx (q+N)\cdot\ln(q+N) \cancel{- q} \cancel{- N} - q\ln q \cancel{+ q} - N\ln N \cancel{+ N}$$

$$= (q+N)\ln(q+N) - q\ln q - N\ln N$$

$$\ln(N+q) = \ln N\left(1+\frac{q}{N}\right) = \ln N + \ln\left(1+q/N\right)$$

$$\approx \ln N + q/N$$

$\ln(1+x) \approx x$

$$\ln\Omega = (q+N)\left(\ln N + \frac{q}{N}\right) - q\ln q - N\ln N$$

$$= q\ln N + \frac{q^2}{N} \cancel{+ N\ln N} + q - q\ln q \cancel{- N\ln N}$$

$$= q + q\ln\left(\frac{N}{q}\right) + \frac{q^2}{N}$$

$$\approx q + q\ln\left(\frac{N}{q}\right)$$

$q \ll N$

$$= q\left[1 + \ln\left(\frac{N}{q}\right)\right]$$

$$\Omega = e^{q}\cdot e^{q\ln N/q} = e^{q}\cdot\left(e^{\ln N/q}\right)^{q} = e^{q}\left(\frac{N}{q}\right)^{q}$$

2.24

- Single large two-state paramagnet
- Sharply peaked multiplicity function : $N_\uparrow = \frac{N}{2}$

(a) Height of the peak in Ω-function?

- Use Stirling's approx: $N! = N^N \cdot e^{-N} \cdot \sqrt{2\pi N}$

$$\Omega = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!\,(N-N_\uparrow)} = \frac{N!}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!}$$

$$= \frac{N!}{\left[\left(\frac{N}{2}\right)!\right]^2}$$

Use Stirling's approx

$$\approx \frac{N^N \cdot e^{-N} \cdot \sqrt{2\pi N}}{\left[\left(\frac{N}{2}\right)^{N/2} \cdot e^{-N/2} \cdot \sqrt{2\pi\left(\frac{N}{2}\right)}\right]^2}$$

$$\left[\left(\frac{N}{2}\right)^{N/2}\right]^2 = \left(\frac{N}{2}\right)^N$$

$$= \frac{\cancel{N^N} \cdot \cancel{e^{-N}} \cdot \sqrt{2\pi N}}{\left(\frac{\cancel{N}}{2}\right)^N \cdot \cancel{e^{-N}} \cdot \pi N}$$

$$\left(\frac{1}{2}\right)^{-N} = 2^N$$

$$= 2^N \cdot \sqrt{\frac{2}{\pi N}}$$

2.24b Formula for Ω-function in the vicinity of the peak, in terms of $x \equiv N_\uparrow - \left(\frac{N}{2}\right)$.

- Stirling's approx:

$$\Omega = \frac{N!}{N_\uparrow! \, N_\downarrow!} \approx \frac{N^N \cdot e^{-N} \cdot \sqrt{2\pi N}}{N_\uparrow^{N} \cdot e^{-N_\uparrow} \cdot \sqrt{2\pi N_\uparrow} \cdot N_\downarrow^{N} e^{-N_\downarrow} \cdot \sqrt{2\pi N_\downarrow}}$$

$$= \frac{N^N}{N_\uparrow^{N_\uparrow} \cdot N_\downarrow^{N_\downarrow}} \cdot \frac{e^{-N}}{e^{-N_\uparrow} \cdot e^{-N_\downarrow}} \cdot \sqrt{\frac{N}{2\pi \, N_\uparrow \cdot N_\downarrow}}$$

$N = N_\uparrow + N_\downarrow$

$e^{-N} = e^{-(N_\uparrow + N_\downarrow)}$

$$= \frac{N^N}{N_\uparrow^{N_\uparrow} \cdot N_\downarrow^{N_\downarrow}} \cdot \sqrt{\frac{N}{2\pi \, N_\uparrow \cdot N_\downarrow}}$$

- Now use $x \equiv N_\uparrow - \left(\frac{N}{2}\right)$

$\Rightarrow \quad N_\uparrow = x + \frac{N}{2}$ and rewrite Ω.

$N_\downarrow = N - N_\uparrow = N - \left(x + \frac{N}{2}\right) = \frac{N}{2} - x$

$$\Omega \approx \frac{N^N}{\left(\frac{N}{2} + x\right)^{x + N/2} \cdot \left(\frac{N}{2} - x\right)^{N/2 - x}} \cdot \left[\frac{N}{2\pi \cdot \left(\frac{N}{2} + x\right) \cdot \left(\frac{N}{2} - x\right)}\right]^{1/2}$$

Focus on: $\left(x + \frac{N}{2}\right)^{x + N/2} \cdot \left(\frac{N}{2} - x\right)^{N/2 - x} = \left(x + \frac{N}{2}\right)^{x} \cdot \left(x + \frac{N}{2}\right)^{N/2} \cdot \left(\frac{N}{2} - x\right)^{-x} \cdot \left(\frac{N}{2} - x\right)^{N/2}$

[note $a^n \cdot b^n = (ab)^n$]

$$= \left[\left(\frac{N}{2} + x\right) \cdot \left(\frac{N}{2} - x\right)\right]^{N/2} \cdot \left(\frac{N}{2} + x\right)^{x} \cdot \left(\frac{N}{2} - x\right)^{-x}$$

[note $(a+b) \cdot (a-b) = a^2 - b^2$]

$$= \left[\left(\frac{N}{2}\right)^2 - x^2\right]^{N/2} \left(\frac{N}{2} + x\right)^{x} \cdot \left(\frac{N}{2} - x\right)^{-x}$$

$$= \frac{N^N}{\left[\left(\frac{N}{2}\right)^2 - x^2\right]^{N/2} \left(\frac{N}{2}+x\right)^x \cdot \left(\frac{N}{2}-x\right)^{-x}} \cdot \left[\frac{N}{2\pi \cdot \left(\frac{N}{2}+x\right) \cdot \left(\frac{N}{2}-x\right)}\right]^{1/2}$$

$$\ln \Omega = N \ln N - \frac{N}{2} \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] - x \cdot \left[\ln\left(\frac{N}{2}+x\right) - \ln\left(\frac{N}{2}-x\right)\right]$$

$$+ \frac{1}{2}\left\{\ln N - \ln 2\pi - \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right]\right\}$$

$$= N \ln N - \frac{N}{2} \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] - x \cdot \left[\ln\left(\frac{N}{2}+x\right) - \ln\left(\frac{N}{2}-x\right)\right]$$

$$+ \left\{\ln\sqrt{\frac{N}{2\pi}} - \frac{1}{2} \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right]\right\}$$

- Size of x relative to N?
- If $x \ll N \rightarrow$ expansion of the logarithm containing 2 terms:

$$\ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] = \ln\left\{\left(\frac{N}{2}\right)^2\left[1 - \frac{x^2}{\left(\frac{N}{2}\right)^2}\right]\right\}$$

$$= \ln\left(\frac{N}{2}\right)^2 + \ln\left[1 - \left(\frac{2x}{N}\right)^2\right]$$

$\ln(1-x) \approx -x$
$\forall \; x \ll 1$

$$\approx 2\ln\left(\frac{N}{2}\right) - \left(\frac{2x}{N}\right)^2$$

Similarly,

$$\ln\left(\frac{N}{2} \pm x\right) = \ln\left[\frac{N}{2}\left(1 \pm \frac{2x}{N}\right)\right] = \ln\left(\frac{N}{2}\right) \pm \ln\left(1 \pm \frac{2x}{N}\right) \approx \ln\left(\frac{N}{2}\right) \pm \frac{2x}{N}$$

$$\Rightarrow \ln \Omega = N \ln N - \frac{N}{2}\left[2\ln\left(\frac{N}{2}\right) - \left(\frac{2x}{N}\right)^2\right]$$

$$- x\left[\left(\ln\frac{N}{2} + \frac{2x}{N}\right) - \left(\ln\frac{N}{2} - \frac{2x}{N}\right)\right] + \ln\sqrt{\frac{N}{2\pi}} - \frac{1}{2}\left[2\ln\left(\frac{N}{2}\right) - \left(\frac{2x}{N}\right)^2\right]$$

$$\approx N\ln N - N\ln\left(\frac{N}{2}\right) + \cancel{\frac{2x^2}{N}}$$

$$-\cancel{x\ln\left(\frac{N}{2}\right)} - \frac{2x^2}{N} + \cancel{x\ln\left(\frac{N}{2}\right)} - \cancel{\frac{2x^2}{N}} + \ln\sqrt{\frac{N}{2\pi}} - \ln\left(\frac{N}{2}\right) + \frac{2x^2}{N^2}$$

$$= N\ln N - N\ln\left(\frac{N}{2}\right) - \frac{2x^2}{N} + \ln\sqrt{\frac{N}{2\pi}} - \ln\left(\frac{N}{2}\right) + \frac{2x^2}{N^2}$$

$$= \cancel{N\ln N} - \cancel{N\ln N} + N\ln 2 - \frac{2x^2}{N} + \frac{1}{2}\ln N - \ln\sqrt{2\pi}$$

$$- \ln N + \ln 2 + \frac{2x^2}{N^2}$$

$$\approx N\ln 2 - \frac{2x^2}{N} - \frac{1}{2}\ln N - \ln\sqrt{2\pi} + \ln 2 + \frac{2x^2}{N}$$

$$= N\ln 2 - \frac{2x^2}{N} - \frac{1}{2}\ln N - \ln\sqrt{2} - \ln\sqrt{\pi} + \ln 2 + \frac{2x^2}{N}$$

$$= N\ln 2 - \frac{2x^2}{N} + \ln\sqrt{\frac{2}{\pi N}} + \frac{2x^2}{N}$$

$\ln 2 = 2\cdot\ln\sqrt{2}$

- $\frac{2x^2}{N}$ is much smaller than others

$$\Omega = \exp\left(\ln 2^N - \frac{2x^2}{N} + \ln\sqrt{\frac{2}{\pi N}}\right)$$

$$\Omega = 2^N \cdot e^{-2x^2/N} \cdot \sqrt{\frac{2}{\pi N}} \qquad \forall\ x \ll N.$$

- This is a Gaussian function.

For $x = 0$: $\Omega = 2^N \cdot \sqrt{\frac{2}{\pi N}}$.

It agrees with (a).

2.24(c) Width of the peak in Ω-function?

$$\Omega = 2^N \cdot \sqrt{\frac{2}{\pi N}}\, e^{-2x^2/N}$$

- When $2x^2/N = 1$, then the Gaussian function falls off to $\frac{1}{e}$ of its peak value.

$$x^2 = N/2 \quad \Rightarrow \quad x = \sqrt{N/2}$$

- Full width of the peak would be twice of this:

$$\Delta x = 2\sqrt{N/2} = \sqrt{2N}$$

2.24(d): Flip 10^6 coins: $N = 10^6$

- Half-width of the peak in Ω-function: $\sqrt{\frac{N}{2}} = \sqrt{500\,000} \approx 700$

So, an excess of 1000 heads puts us only a little beyond the point where the Gaussian has fallen off to $1/e$ of its peak value.

- I would not be surprized to obtain ~ 1000 heads, though it might be surprizing to get an excess of exactly 1000.
- On the other hand, an excess of 10000 heads lies far outside the peak region. At this point the Gaussian has fallen off to $e^{-200} \approx 10^{-87}$ of its max. I would be surprized to get such a result and probably the coins were not fair.

$$\int_0^{\infty} e^{-ax}\,dx = ?$$

- Integrate from $x=0$ to $x=\lambda$ $(\lambda>0)$

$$I(\lambda) = \int_0^{\lambda} e^{-ax}\,dx$$

$$= -\frac{1}{a} e^{-ax}\Big|_0^{\lambda}$$

$$= -\frac{1}{a}\left(e^{-a\lambda} - e^{0}\right)$$

$$= -\frac{1}{a} e^{-a\lambda} + \frac{1}{a}$$

Limit : $\lambda \to \infty$:

$$\lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty} \int_0^{\lambda} e^{-ax}\,dx$$

$$= \lim_{\lambda\to\infty}\left(\frac{1}{a} - \frac{1}{a} e^{-a\lambda}\right)$$

$$= \frac{1}{a}$$

Prove $\quad n! = \int_0^\infty x^n \cdot e^{-x}\, dx$

Start with $\quad \int_0^\infty e^{-ax}\, dx = a^{-1} \quad$ and differentiate repeatedly:

$$\frac{d}{da}\int_0^\infty e^{-ax}\, dx = \int_0^\infty \frac{d}{da}\left(e^{-ax}\right) dx = \int_0^\infty -x \cdot e^{-ax}\, dx = -a^{-2}$$

$$\int_0^\infty \frac{d}{da}\left(-x \cdot e^{-ax}\right) dx = \int_0^\infty x^2 \cdot e^{-ax}\, dx = 1 \cdot 2\, a^{-3}$$

$$\int_0^\infty \frac{d}{da}\left(x^2 \cdot e^{-ax}\right) dx = \int_0^\infty -x^3 \cdot e^{-ax}\, dx = 1 \cdot 2 (-3) \cdot a^{-4}$$

$$\Rightarrow \int_0^\infty x^n \cdot e^{-ax}\, dx = n! \cdot a^{-(n+1)}$$

$a = 1$:

$$\int_0^\infty x^n \cdot e^{-x}\, dx = n!$$