Probability: The Foundation of Uncertainty

PSC4375: Week 9

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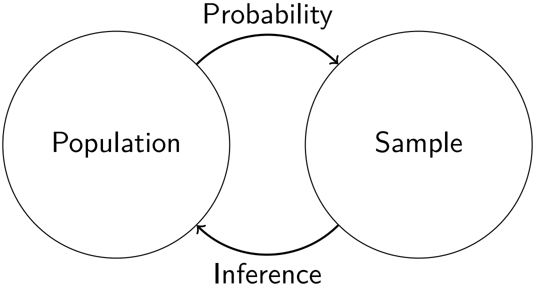
Learning about populations



Probability: formalize the uncertainty about how our data came to be

Inference: learning about the population from a sample of data

Why probability?

- Probability quantifies chance variation or uncertainty in outcomes.
  - It might rain or be sunny today, we don't know which.
- We estimated a treatment effect of 7.2, but what if we reran history?
  - Weather changes $\rightsquigarrow$ slightly different estimated effect.
- Statistical inference is a **thought experiment** about uncertainty.
  - Imagine a world where the treatment effect were 0 in the population.
  - What types of estimated effects would we see in this world by chance?
- Probability to the rescue!

Sample spaces & events

- To formalize chance, we need to define the set of possible outcomes.
- **Sample space**: Ω the set of possible outcomes.
- **Event**: any subset of outcomes in the sample space

Example: gambling

- A standard deck of playing cards has 52 cards:
  - 13 rank cards: (2,3,4,5,6,7,8,9,10,J,Q,K,A)
  - in each of 4 suits: ($\clubsuit$,$\spadesuit$,$\heartsuit$,$\diamondsuit$)
  - Hypothetical trial: pick a card, any card.
    - Uncertainty: we don't know which card we're going to get.
  - One possible outcome: picking a 4$\clubsuit$
  - Sample space:

$$
\begin{array}{ccccccccccccc}
2\clubsuit & 3\clubsuit & 4\clubsuit & 5\clubsuit & 6\clubsuit & 7\clubsuit & 8\clubsuit & 9\clubsuit & 10\clubsuit & J\clubsuit & Q\clubsuit & K\clubsuit & A\clubsuit \\
2\spadesuit & 3\spadesuit & 4\spadesuit & 5\spadesuit & 6\spadesuit & 7\spadesuit & 8\spadesuit & 9\spadesuit & 10\spadesuit & J\spadesuit & Q\spadesuit & K\spadesuit & A\spadesuit \\
2\heartsuit & 3\heartsuit & 4\heartsuit & 5\heartsuit & 6\heartsuit & 7\heartsuit & 8\heartsuit & 9\heartsuit & 10\heartsuit & J\heartsuit & Q\heartsuit & K\heartsuit & A\heartsuit \\
2\diamondsuit & 3\diamondsuit & 4\diamondsuit & 5\diamondsuit & 6\diamondsuit & 7\diamondsuit & 8\diamondsuit & 9\diamondsuit & 10\diamondsuit & J\diamondsuit & Q\diamondsuit & K\diamondsuit & A\diamondsuit
\end{array}
$$

- An event: picking a Queen, {Q$\clubsuit$,Q$\spadesuit$,Q$\heartsuit$,Q$\diamondsuit$}

What is probability?

- The probability $\mathbb{P}(A)$ represents how likely event A occurs.
- If **all outcomes equally likely**, then:

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

- Example: randomly draw 1 card:
  - probability of drawing 4♣: $\frac{1}{52}$
  - probability of drawing any ♣: $\frac{13}{52}$
- Same math, but different interpretations:
  - **Frequentist**: $\mathbb{P}()$ reflects relative frequency in a large number of trials.
  - **Bayesian**: $\mathbb{P}()$ are subjective beliefs about outcomes.
- Not our fight $\rightsquigarrow$ stick to frequentism in this class

Probability axioms

- Probability quantifies how likely or unlikely events are.
- We'll define the probability $\mathbb{P}(A)$ with three axioms:

1. (Nonnegativity) $\mathbb{P}(A) \geq 0$ for every event A
2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Addition Rule) If two events A and B are mutually exclusive

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$