Probability: The Foundation of Uncertainty

PSC4375: Week 9

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Villanova University

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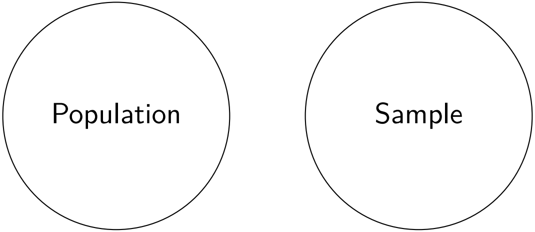
Learning about populations



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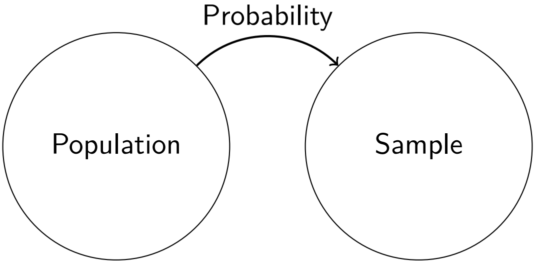

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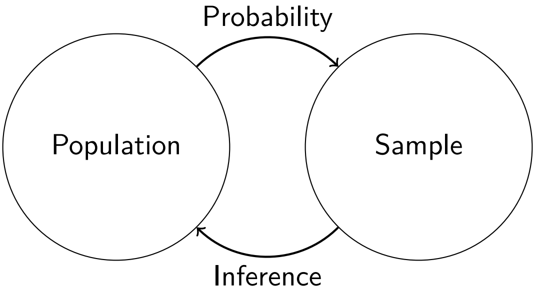

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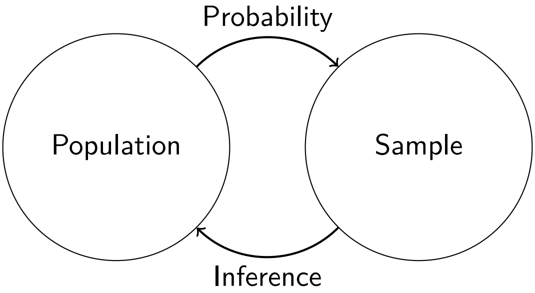

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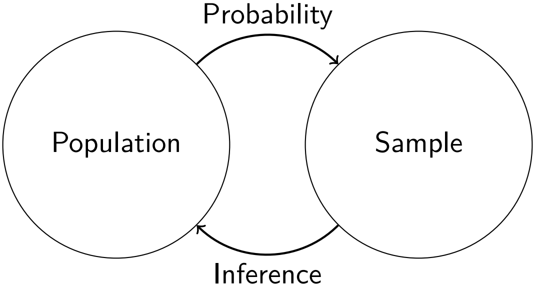

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Inference: learning about the population from a sample of data

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- Probability to the rescue!

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- **Event**: any subset of outcomes in the sample space

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- An event: picking a Queen, $\{Q\clubsuit, Q\spadesuit, Q\heartsuit, Q\diamondsuit\}$

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1. (Nonnegativity) $\mathbb{P}(A) \geq 0$ for every event A
2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Addition Rule) If two events A and B are mutually exclusive

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

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  - $\rightsquigarrow$ $\mathbb{P}(\text{Q card}) = \mathbb{P}(\text{Q}\clubsuit) + \mathbb{P}(\text{Q}\spadesuit) + \mathbb{P}(\text{Q}\heartsuit) + \mathbb{P}(\text{Q}\diamondsuit)$

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- $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

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$$\mathbb{P}(\text{teller and feminist}) = \mathbb{P}(\text{teller}) - \mathbb{P}(\text{teller and not feminist})$$

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Break time!

Conditional probability

- If we know that B has occurred, what is the probability of A?
  - Conditioning our analysis on B having occurred.
- Examples:
  - Probability of two states going to war if they are both democracies?
  - Probability of a judge issuing a pro-choice ruling if they have daughters?
  - Probability of a coup in a country if it has a presidential system?
- Conditional probability extremely useful for data analysis.

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  - There are many many smart people who are not in this class (tell your friends)
- If all outcomes equally likely:

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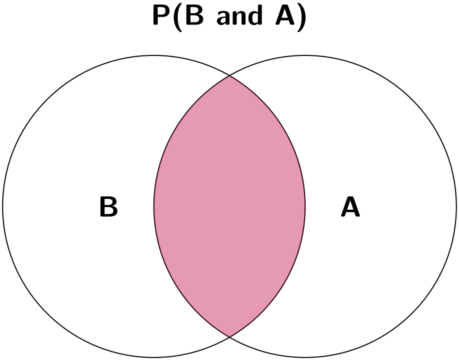

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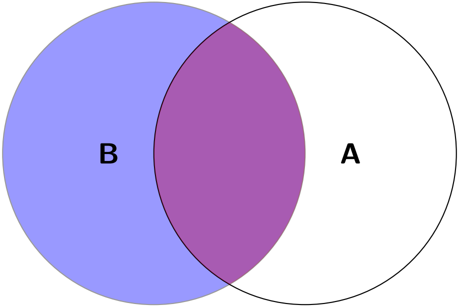

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Conditional probability rules

- Multiplication rule:

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  - Or we could just use the multiplication rule:

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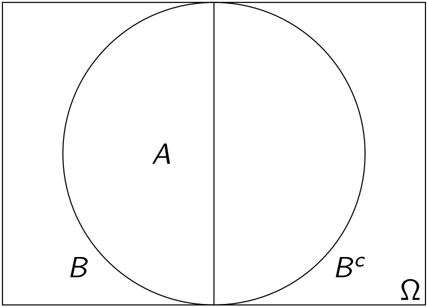
Law of total probability



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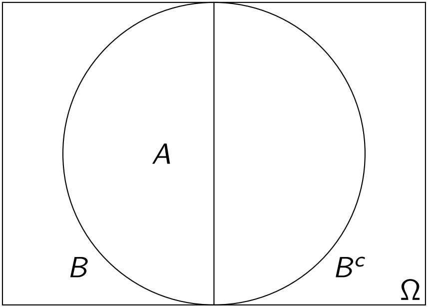

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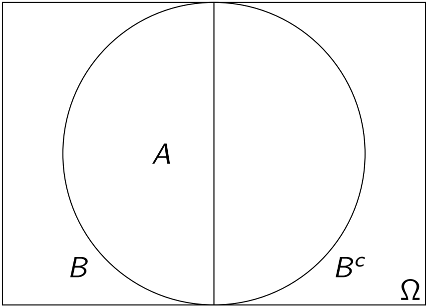

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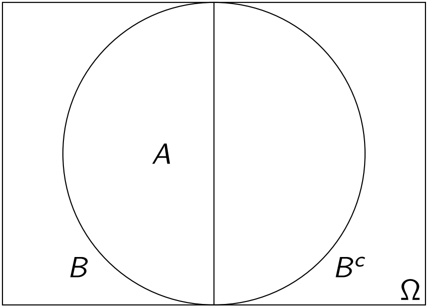

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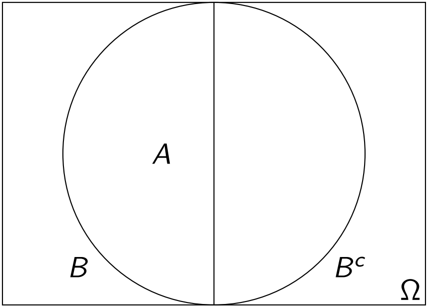

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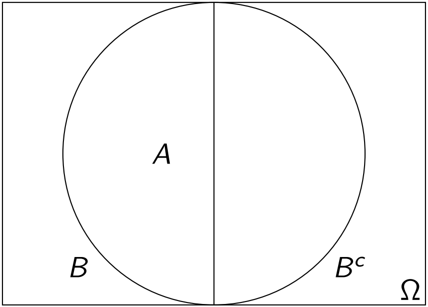

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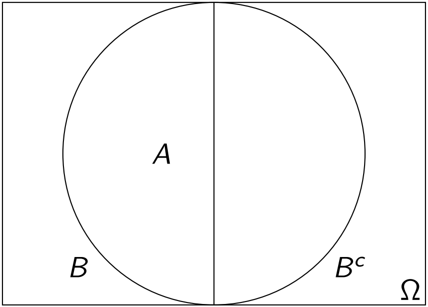

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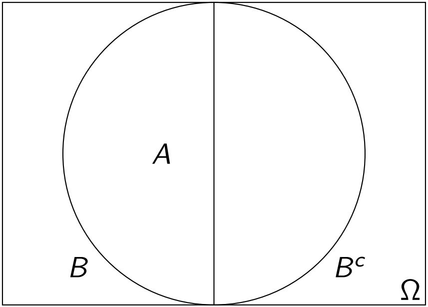

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