Probability: The Foundation of Uncertainty

PSC7475: Week 9

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Villanova University

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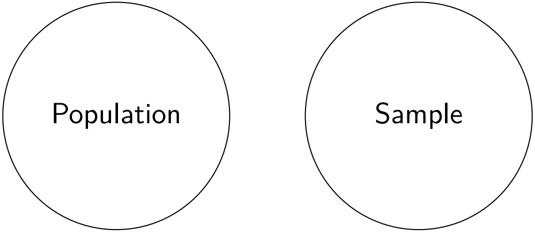
Learning about populations



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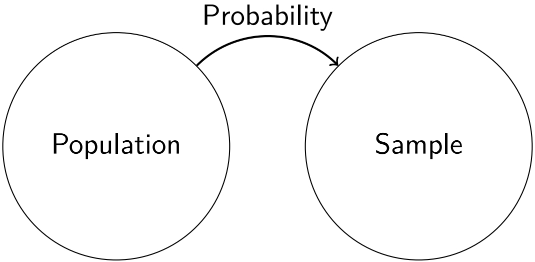

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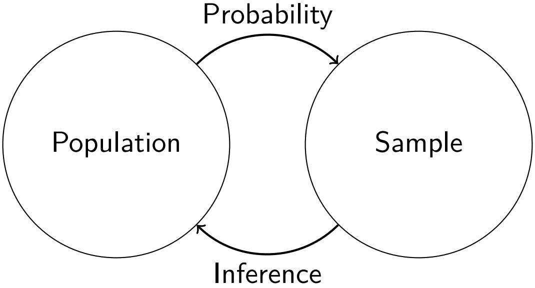

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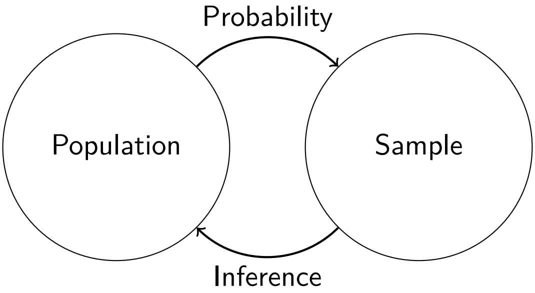

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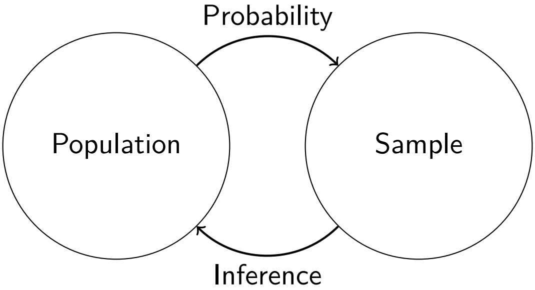

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Inference: learning about the population from a sample of data

Why probability?

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- Probability to the rescue!

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- **Event**: any subset of outcomes in the sample space

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  - Sample space:

$$\begin{array}{ccccccccccccc}
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2\diamondsuit & 3\diamondsuit & 4\diamondsuit & 5\diamondsuit & 6\diamondsuit & 7\diamondsuit & 8\diamondsuit & 9\diamondsuit & 10\diamondsuit & J\diamondsuit & Q\diamondsuit & K\diamondsuit & A\diamondsuit
\end{array}$$

- An event: picking a Queen, {Q$\clubsuit$,Q$\spadesuit$,Q$\heartsuit$,Q$\diamondsuit$}

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- Not our fight $\rightsquigarrow$ stick to frequentism in this class

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1. (Nonnegativity) $\mathbb{P}(A) \geq 0$ for every event A
2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Addition Rule) If two events A and B are mutually exclusive

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

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  - $\rightsquigarrow \mathbb{P}(\text{Q card}) = \mathbb{P}(\text{Q}\clubsuit) + \mathbb{P}(\text{Q}\spadesuit) + \mathbb{P}(\text{Q}\heartsuit) + \mathbb{P}(\text{Q}\diamondsuit)$

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- $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

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$$\mathbb{P}(\text{teller and feminist}) = \mathbb{P}(\text{teller}) - \mathbb{P}(\text{teller and not feminist})$$

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Men	29	43	2	74
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Break time!

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- Conditional probability extremely useful for data analysis.

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  - There are many many smart people who are not in this class (tell your friends)

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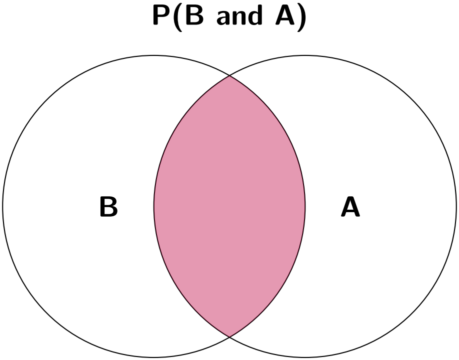
- **WARNING!** $\mathbb{P}(A|B)$ does **not**, in general, equal $\mathbb{P}(B|A)$
  - $\mathbb{P}(\text{smart}|\text{in QSS})$ is high
  - $\mathbb{P}(\text{in QSS}|\text{smart})$ is low
  - There are many many smart people who are not in this class (tell your friends)
- If all outcomes equally likely:

$$\mathbb{P}(A|B) = \frac{\text{number of outcomes in both } A \text{ and } B)}{\text{number of outcomes in just } B}$$

Conditional probability



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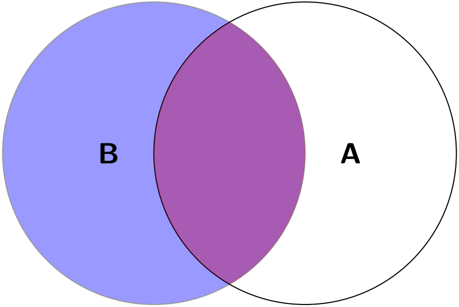

US Senate example

	Democrats	*Republicans*	*Independents*	*Total*
Men	29	43	2	74
Women	16	10	0	26
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- What is the probability of choosing a Republican who is a woman?
  - $\mathbb{P}(\text{Woman} \mid \text{Rep.}) = \frac{10/100}{53/100} \approx 0.19$

Conditional probability rules

- Multiplication rule:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A)$$

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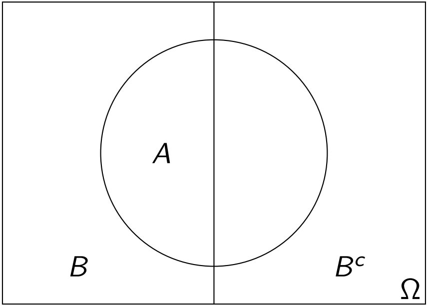
  - Let W_1 and W_2 be the events that 1st and 2nd draws are women
  - We could make a list of all possible pairs to draw and count them...
  - Or we could just use the multiplication rule:

$$\mathbb{P}(W_1 \text{ and } W_2) = \mathbb{P}(W_1)\mathbb{P}(W_2|W_1)$$

Law of total probability



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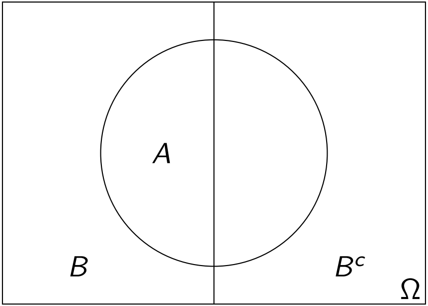

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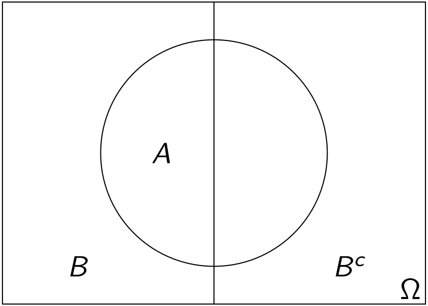

- Conditional probability lets us restate the law of total probability
- **Law of total probability**:

$$
\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(A \text{ and } B) + \mathbb{P}(A \text{ and not } B) \\
&= \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A| \text{ not } B)\mathbb{P}(\text{not } B)
\end{aligned}
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  - Knowing B occurred doesn't change the probability of A

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