# Inference and Estimation

## PSC7475: Week 11

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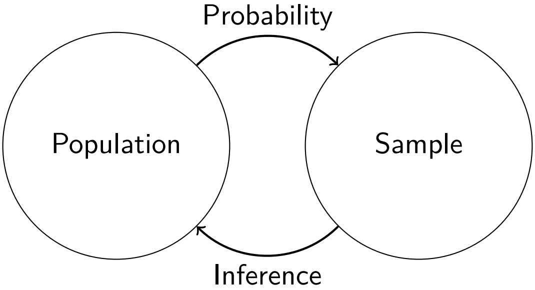
# Remember our goal




- We want to learn about the chance process that generated our data.
- Now we switch to **inference**.
  - What can I learn about the population distribution from my sample?

# What are random variables?

- What proportion of the public approves of Trump's job as president?
- Latest Gallup poll:
  - March 3-16, 2025
  - 1500 adult Americans
  - Approve (43%), Disapprove (53%)
- What can we learn about Trump approval in the population from this one sample?

**Do you approve or disapprove of the way Donald Trump is handling his job as president?**

Second-term trend

| | Approve % | Disapprove % | No opinion % |
|---|---|---|---|
| **2025** | | | |
| 2025 Mar 3-16 | 43 | 53 | 4 |
| 2025 Feb 3-16 | 45 | 51 | 5 |
| 2025 Jan 21-27 | 47 | 48 | 4 |

GALLUP

# Samples from the population

- Simple random sample of size $n$ from some population $Y_1, \ldots, Y_n$
  - $\rightsquigarrow$ i.i.d. random variables
  - e.g.: $Y_i = 1$ if $i$ approves of Trump, $Y_i = 0$ otherwise.
- **Statistical inference**: using data to guess something about the population distribution of $Y_i$.

# Point estimation

- **Quantity of interest**: some feature of the population distribution.
  - Also called: parameters.
  - These are the things we want to learn about.
- **Point estimation**: providing a single "best guess" about this q.o.i.
- Examples of quantities of interest:
  - $\mu = \mathbb{E}[Y_i]$: the population mean (turnout rate in the population).
  - $\sigma^2 = \mathbb{V}[Y_i]$: the population variance.
  - $\mu_1 - \mu_0 = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$: the population ATE.

# Estimators

- **Estimator**: function of the data that produces estimates of the q.o.i.
  - An **estimate** is one particular realization of the estimator
- Ideally we'd like to know the **estimation error**, estimator - truth
  - Problem: $\theta$ is unknown.
- Solution: figure out the properties of estimator using probability.
  - Estimator is a r.v. because it is a function of r.v.s. (the data)
  - $\rightsquigarrow$ estimator has a distribution.

# Estimating Trump's support

- Parameter $p$: **population proportion** of adults who support Trump
- There are many different possible estimators:
  - $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ the sample proportion of respondents who support Trump.
  - $\hat{p} = Y_1$ just use the first observation
  - $\hat{p} = \max(Y_1, \ldots, Y_n)$
  - $\hat{p} = 0.5$ always guess 50% support
- How good are these different estimators?

# Survey

- Assume a simple random sample of $n$ voters: $n = 1500$
- Define r.v. $Y_i$ for Trump approval:
  - $Y_i = 1 \rightsquigarrow$ respondent $i$ approves of Trump
  - $Y_i = 0 \rightsquigarrow$ respondent $i$ disapproves of Trump
- $X_i$ is **Bernoulli** with probability of success $p$
  - "success" = "selecting a Trump approver"
  - $p = \mathbb{P}(Y_i = 1)$ the population proportion of Trump approvers.
- Sample proportion is the same as the sample mean:

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{\text{number who support Trump}}{n}$$

# Sample mean properties

$$\text{sample proportion} = \text{population proportion} + \text{chance error}$$
$$\bar{Y} = p + \text{chance error}$$

- Remember: the sample mean/proportion is a random variable.
  - Different samples give different sample means.
  - Chance error "bumps" sample mean away from population mean
- $\rightsquigarrow$ $\bar{Y}$ has a distribution across repeated samples.

# Central tendency of the sample mean

- Expectation: average of the estimates across repeated samples.
  - From last week, $\mathbb{E}[\bar{Y}] = \mathbb{E}[Y_i] = p$
  - $\rightsquigarrow$ chance error is 0 on average:

$$\mathbb{E}[\bar{Y} - p] = \mathbb{E}[\bar{Y}] - p = 0$$

- **Unbiasedness**: Sample proportion is on average equal to the population proportion.

# Spread of the sample mean

- **Standard error**: how big is the chance error on average?
  - This is the standard deviation of the estimator.
- Special rule for sample proportions:

$$\sqrt{\mathbb{V}(\bar{Y})} = \sqrt{\frac{p(1-p)}{n}}$$

- Problem: we don't know $p$!
- Solution: **estimate** the SE:

$$\sqrt{\mathbb{V}(\bar{Y})} = \sqrt{\frac{\bar{Y}(1-\bar{Y})}{n}} \approx 0.012$$

# Confidence intervals

- Awesome: sample proportion is correct on average.
- Awesomer: get an range of plausible values.
- **Confidence interval**: way to construct an interval that will contain the true value in some fixed proportion of repeated samples.

# CLT

$$\bar{Y} - p = \text{chance error}$$

- How can we figure out a range of plausible chance errors?
  - Find a range of plausible chance errors and add them to $\bar{Y}$
- Central limit theorem:

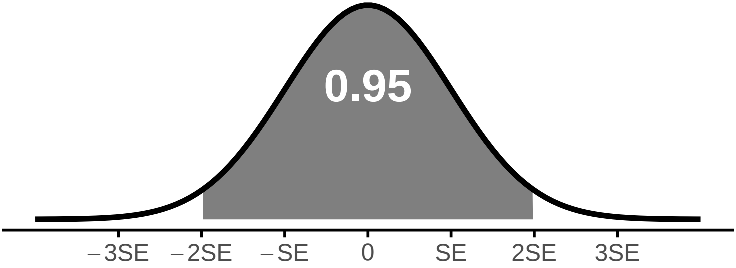
$$\bar{Y} \overset{\text{approx}}{\sim} N(\mathbb{E}(\bar{Y}), \frac{\mathbb{V}(Y_i)}{n})$$

- In this case:

$$\bar{Y} \overset{\text{approx}}{\sim} N(p, \frac{p(1-p)}{n})$$

- Chance error: $\bar{Y} - p$ is approximately normal with mean 0 and SE equal to $\sqrt{p(1-p)/n}$

# Chance errors




- We know 95% of chance errors will be within $\approx 2 \times SE$
  - (actually it's $1.96 \times SE$)
- $\rightsquigarrow$ range of plausible chance errors is $\pm 1.96 \times SE$

# Confidence interval

- First, choose a **confidence level**
  - What percent of chance errors do you want to count as "plausible"?
  - Convention is 95%
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \bar{Y} \pm z_{\alpha/2} \times SE$$

- In polling $\pm z_{\alpha/2} \times SE$ is called the **margin of error**
- $z_{\alpha/2}$ is the $N(0, 1)$ z-score that would put $\alpha/2$ of the probability density above it.
  - $\mathbb{P}(-z_{\alpha/2} < z < z_{\alpha/2}) = \alpha$
  - 90% CI $\rightsquigarrow \alpha = 0.1 \rightsquigarrow z_{\alpha/2} = 1.64$
  - 95% CI $\rightsquigarrow \alpha = 0.05 \rightsquigarrow z_{\alpha/2} = 1.96$
  - 99% CI $\rightsquigarrow \alpha = 0.01 \rightsquigarrow z_{\alpha/2} = 2.58$

# Standard normal z-scores in R

- `qnorm(x, lower.tail = FALSE)` will find the value of $z$ so that $\mathbb{P}(Z < z)$ is equal to `x`, where $Z$ is $N(0,1)$:

```r
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.644854
```

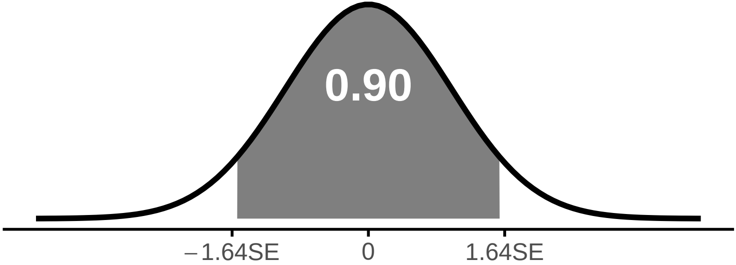
```r
qnorm(0.025, lower.tail = FALSE)
```

```
## [1] 1.959964
```

```r
qnorm(0.005, lower.tail = FALSE)
```

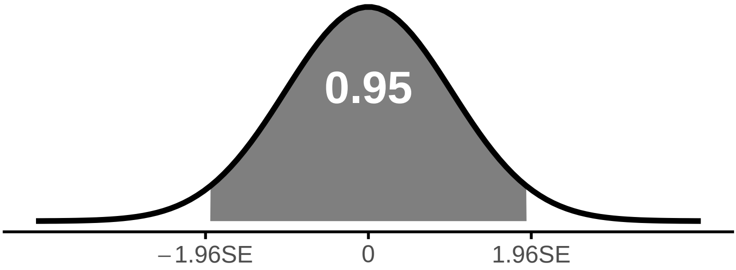
```
## [1] 2.575829
```

# Z-values



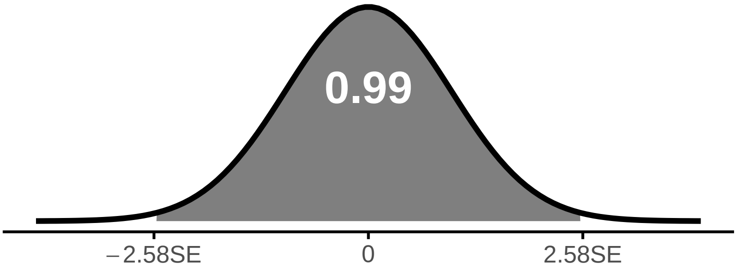

$$CI_{90} = \bar{Y} \pm 1.64 \times SE$$

# Z-values




$$CI_{95} = \bar{Y} \pm 1.96 \times SE$$

# Z-values




$$CI_{99} = \bar{Y} \pm 2.58 \times SE$$

# CIs for the Gallup Poll

- Gallup poll: $\bar{Y} = 0.43$ with an SE of 0.012.
- 90% CI:

$$[0.43 - 1.64 \times 0.012, 0.43 + 1.64 \times 0.012] = [0.410, 0.449]$$

- 95% CI:

$$[0.43 - 1.96 \times 0.012, 0.43 + 1.96 \times 0.012] = [0.406, 0.454]$$

- 99% CI:

$$[0.43 - 2.58 \times 0.012, 0.43 + 2.58 \times 0.012] = [0.399, 0.461]$$

- More confidence $\rightsquigarrow$ wider intervals

# Interpretation and simulation

- Be careful about interpretation:
  - A 95% confidence interval will contain the true value in 95% of repeated samples
  - For a particular calculated confidence interval, truth is either in it or not.
- A simulation can help our understanding:
  - Draw samples of size 1500 assuming population approval for Trump of $p = 0.4$.
  - Calculate 95% confidence intervals in each sample.
  - See how many overlap with the true population approval.

# Plotting the CIs

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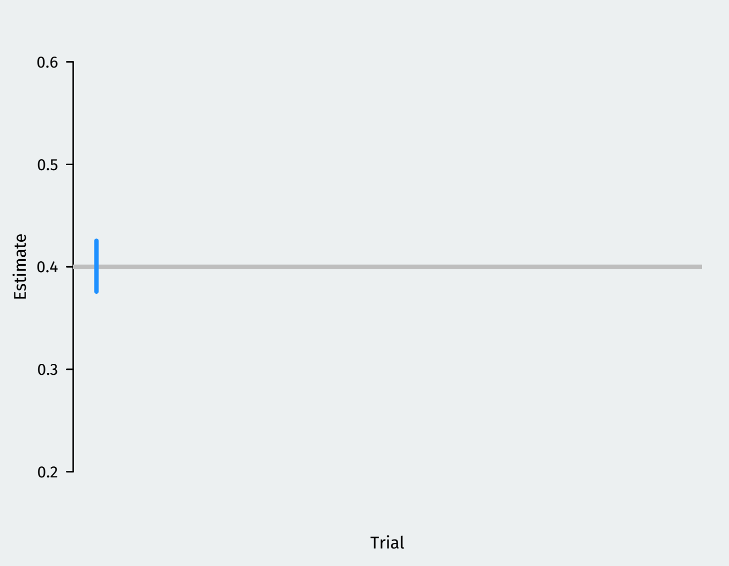

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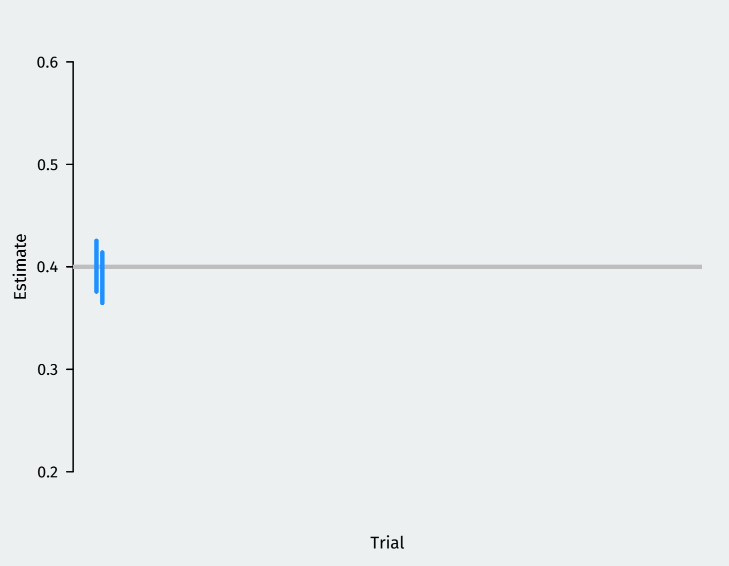

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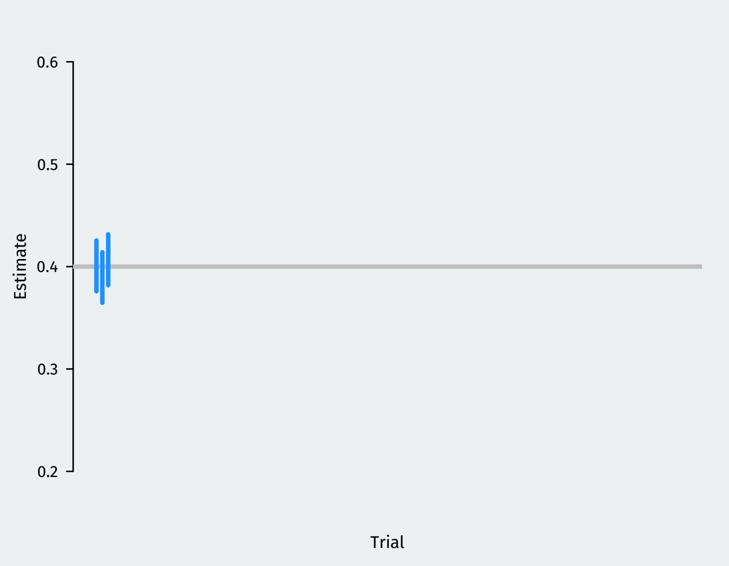

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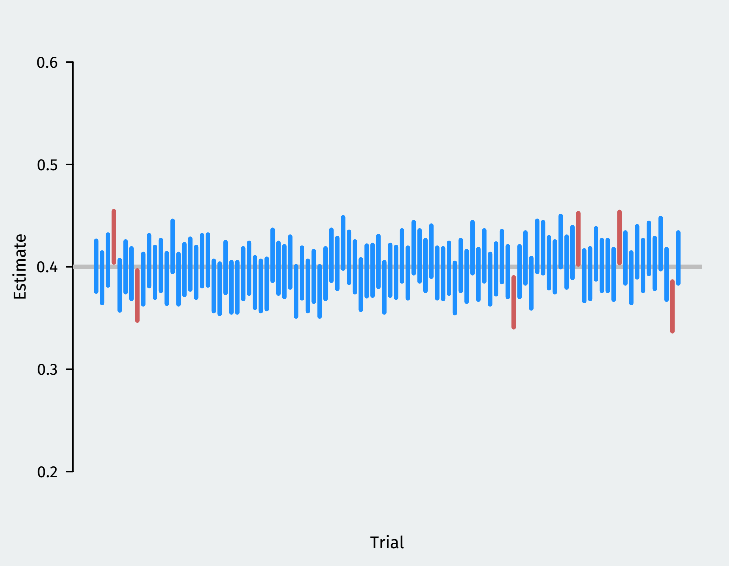


# Inference for experiments

- More interesting to compare across groups.
  - Differences in public opinion across groups
  - Difference between treatment and control groups.
- Bedrock of causal inference!

# Social pressure experiment

- Back to the Social Pressure Mailer GOTV example.
  - Primary election in MI 2006
- Treatment group: postcards showing their own and their neighbors' voting records.
  - Sample size of treated group, $n_T = 360$
- Control group: received nothing.
  - Sample size of the control group, $n_C = 1890$

# Outcomes

- Outcome: $X_i = 1$ if $i$ vote, 0 otherwise.
- Turnout rate (sample mean) in treated group, $\bar{X}_T = 0.37$
- Turnout rate (sample mean) in control group, $\bar{X}_C = 0.30$
- Estimated **average treatment effect**

$$\widehat{ATE} = \bar{X}_T - \bar{X}_C = 0.07$$

# Inference for the difference

- Parameter: **population ATE** $\mu_T - \mu_C$
  - $\mu_T$: Turnout rate in the population if everyone received treatment.
  - $\mu_C$: Turnout rate in the population if everyone received control.
- Estimator: $\widehat{ATE} = \bar{X}_T - \bar{X}_C$
- $\bar{X}_T$ is a r.v. with mean $\mathbb{E}[\bar{X}_T] = \mu_T$
- $\bar{X}_C$ is a r.v. with mean $\mathbb{E}[\bar{X}_C] = \mu_C$
- $\rightsquigarrow \bar{X}_T - \bar{X}_C$ is a r.v. with mean $\mu_T - \mu_C$
  - Sample difference in means is on average equal to the population difference in means.

# Simulation

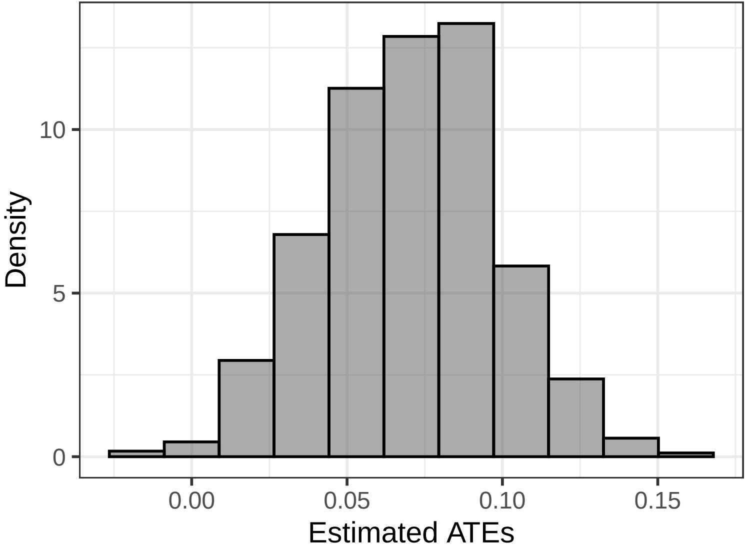
- What if these were the true population means? We would still expect some **variation** in our estimates:

```r
xt.sims <- rbinom(1000, size = 360, prob = 0.37) / 360
xc.sims <- rbinom(1000, size = 1890, prob = 0.30) / 1890

diff.sims <- as.tibble(xt.sims - xc.sims)

diff.sims %>%
ggplot(aes(x = value, y=..density..)) +
  geom_histogram(bins=11, alpha=0.5, color="black") +
  labs(x = "Estimated ATEs", y = "Density") +
  theme_bw()
```

# Simulations



## Standard error

- Is an $\widehat{ATE} = 0.07$ big?
- How much variation would we expect in the difference in means across repeated samples?
- **Variance** of our estimates:

$$\begin{aligned}\mathbb{V}(\widehat{ATE}) &= \mathbb{V}(\bar{X}_T - \bar{X}_C) = \mathbb{V}(\bar{X}_T) + \mathbb{V}(\bar{X}_C) \\ &= \frac{\mu_T(1-\mu_T)}{n_T} + \frac{\mu_C(1-\mu_C)}{n_C}\end{aligned}$$

- **Standard error** is the square root of this variance:

$$\widehat{SE}_{\widehat{ATE}} = \sqrt{\frac{\bar{X}_T(1-\bar{X}_T)}{n_T} + \frac{\bar{X}_C(1-\bar{X}_C)}{n_C}} = 0.028$$

- SE represents how far, on average, $\bar{X}_T - \bar{X}_C$ will be from $\mu_T - \mu_C$

# Confidence intervals

- We can construct confidence intervals based on the CLT like last time.

$$
\begin{aligned}
CI_{95} &= \widehat{ATE} \pm 1.96 \times \widehat{SE}_{\widehat{ATE}} \\
&= 0.07 \pm 1.96 \times 0.028 \\
&= 0.07 \pm 0.054 \\
&= [0.016, 0.124]
\end{aligned}
$$

- Range of possibilities taking into account plausible chance errors.
- 0 not included in this CI $\rightsquigarrow$ chance error as big as the estimated effect unlikely