

Probability: Bayes Rule

Slides from Prof. Blackwell (Harvard)

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Villanova University

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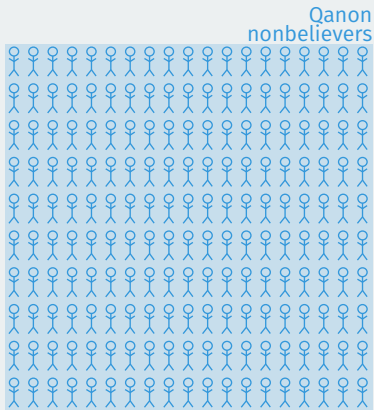
- Common response: probably believes in QAnon since believers tend to be Republicans.



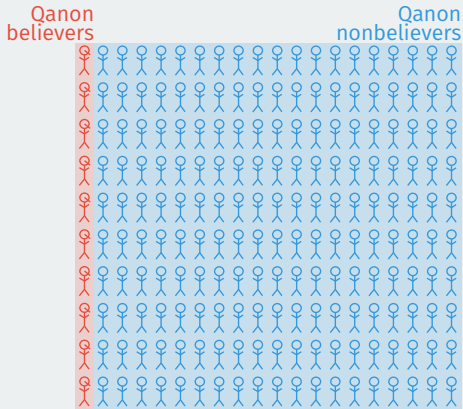
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- Common response: probably believes in QAnon since believers tend to be Republicans.
- **Base rate fallacy:** ignores how uncommon QAnon believers are!

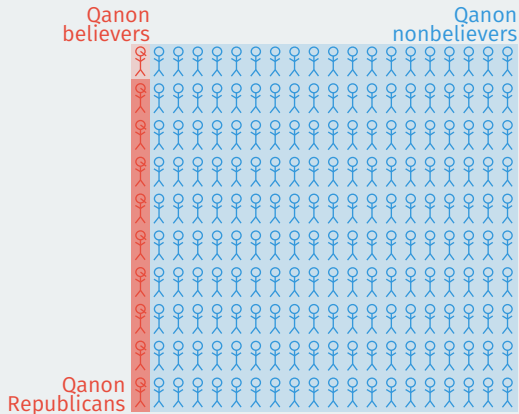
Visualizing QAnon support



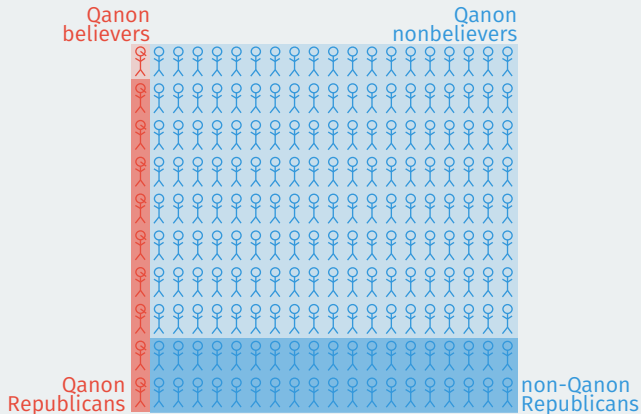
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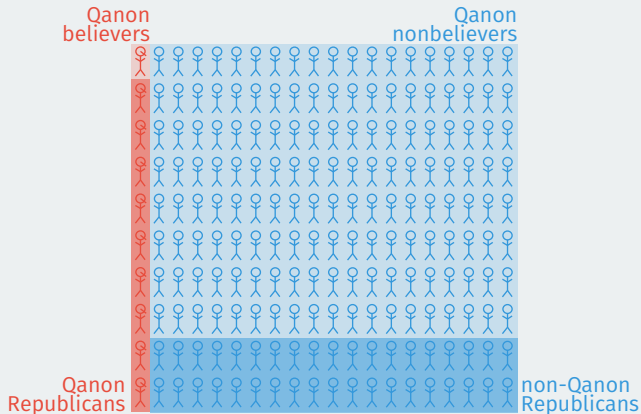
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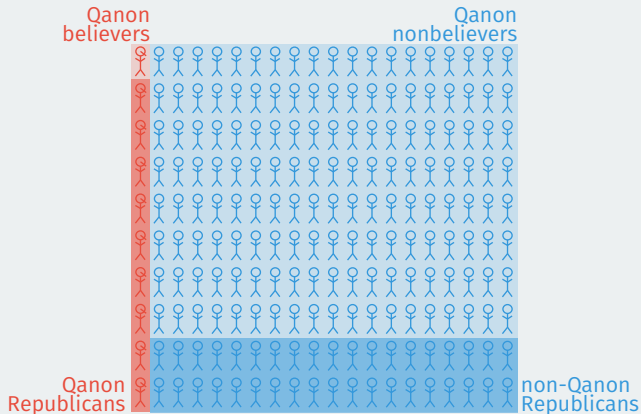


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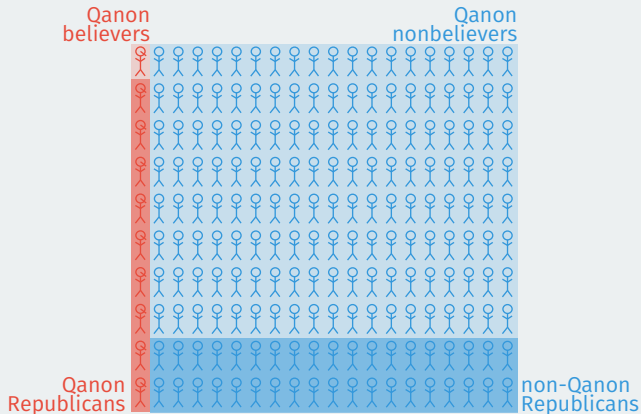
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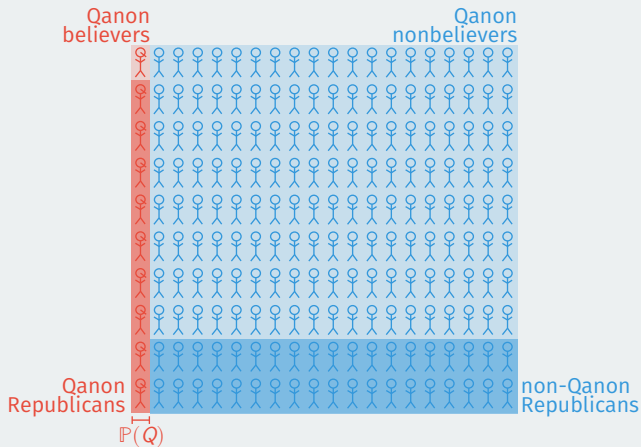
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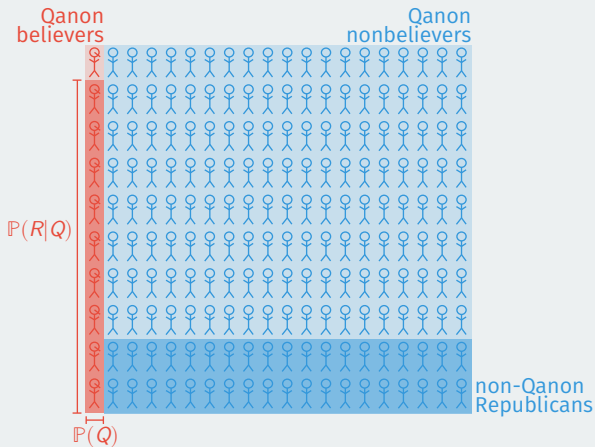
Chance a random Republican believes QAnon = $\frac{10}{20}$

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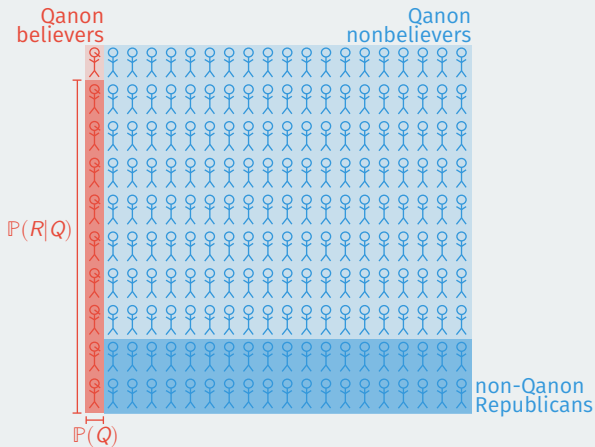
Chance a random Republican believes QAnon = $\frac{P(Q|R)}{P(R)}$

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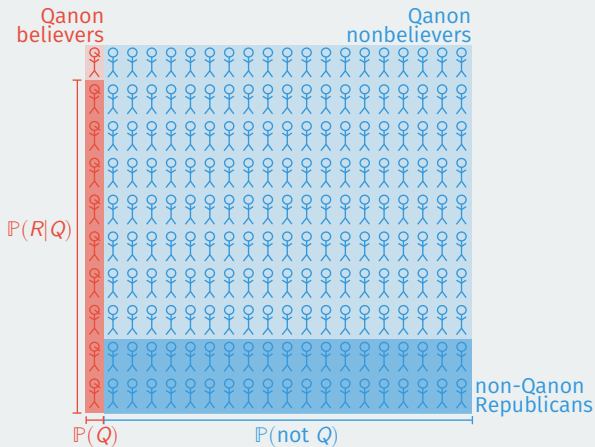
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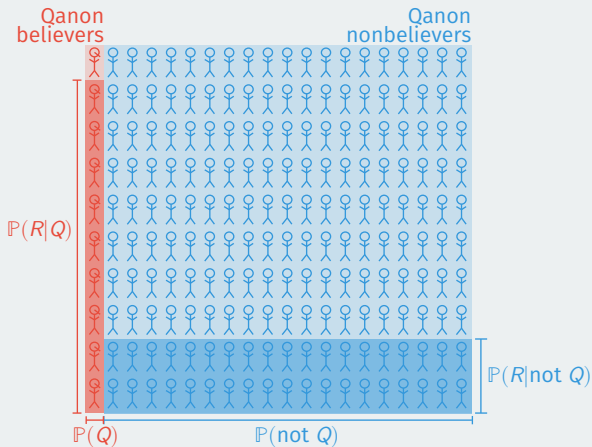
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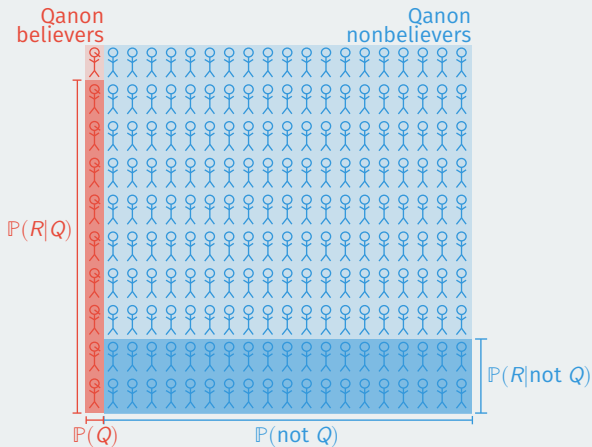
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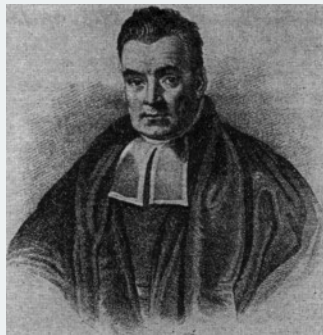
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 - How does the evidence change the chance of the hypothesis being true?

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 - $\mathbb{P}(C) = 0.001$ rough prevalence of active COVID cases.

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