04-630
Data Structures and Algorithms for Engineers

Lecture 3: Complexity Analysis

Lecture 3

- Analysis of complexity of algorithms
  - Time complexity
  - Big-O Notation
  - Space complexity
- Introduction to complexity theory
  - P, NP, and NP-Complete classes of algorithm

Lecture 3

Analysis of complexity

- Performance of algorithms, time and space tradeoff, worst case and average case performance
- Big O notation
- Recurrence relationships
- Analysis of complexity of iterative and recursive algorithms
- Recursive vs. iterative algorithms: runtime memory implications
- Complexity theory: tractable vs intractable algorithmic complexity
- Example intractable problems: travelling salesman problem, Hamiltonian circuit, 3-colour problem, SAT, cliques
- Determinism and non-determinism
- P, NP, and NP-Complete classes of algorithm

Lecture 3

- Key goal:
  - Study and analyze algorithms in a language- and machine-independent manner.
- Strategy?
  - Experimental studies??
  - The Random Access Machine (RAM) model of computation.
  - Asymptotic analysis of computational complexity

Experimental Studies

- How?
  - Implement the algorithm, run it with various inputs, and record execution time.
- Limitations:
  - Difficult to compare in different hardware and software environments.
  - May not cover all input sizes (hence may be inconclusive).
  - All algorithms must be implemented to facilitate comparison (how practical or efficient is this approach?).

The RAM Model of Computation(1/2)

- Considers a hypothetical computer with the following characteristics:
  - Each *simple* operation, e.g., arithmetic, assignment, if, else, takes exactly one time step.
  - Loops and subroutines are not considered simple operations.
    - They contain many single step operations.
  - Each memory access takes exactly one time step.
    - Assumes enough memory is available (regardless if it is from cache or disk).

The RAM Model of Computation(2/2)

- We measure run time of an algorithm by counting the number of steps it takes.
- While simple and conceivably less accurate, in practice (examine the assumptions again), the model is robust enough to facilitate machine-independent analysis.
- Works out:
  - Worst case complexity
  - Average case(expected time) complexity
  - Best case complexity
- The complexities can be modelled with a numerical function of the *input size*.

Asymptotic Analysis(1/2)

- An analysis can be done on the high-level description of the algorithm or operation by considering primitive operations.
  - Primitive operations assumed to have constant runtime.
  - Assumes running time for every primitive operation is similar.
  - The number of primitive operations is proportional to the actual running time.
  - We can measure rate of growth of an algorithm's running time.

Asymptotic Analysis(2/2)

- Supports analysis that ignores language- or hardware-specific details about an algorithm
  - It only considers growth of running time w.r.t. the input size, n, and the number of primitive operations.
- Analysis is based on worst-case scenario.
  - ***Intuition***: If you optimize the algorithm for worst case inputs, it should work well for anything else.
- Provides a mathematical framework for algorithm analysis.
- Uses simple ***upper and lower bounds of time-complexity functions*** using the **Big Oh** notation.

Motivation

Complexity Theory

- Easy problems (sort a million items in a few seconds)
- Hard problems (schedule a thousand classes in a hundred years)
- What makes some problems hard and others easy (computationally) and how do we make hard problems easier?
- Complexity Theory addresses these questions

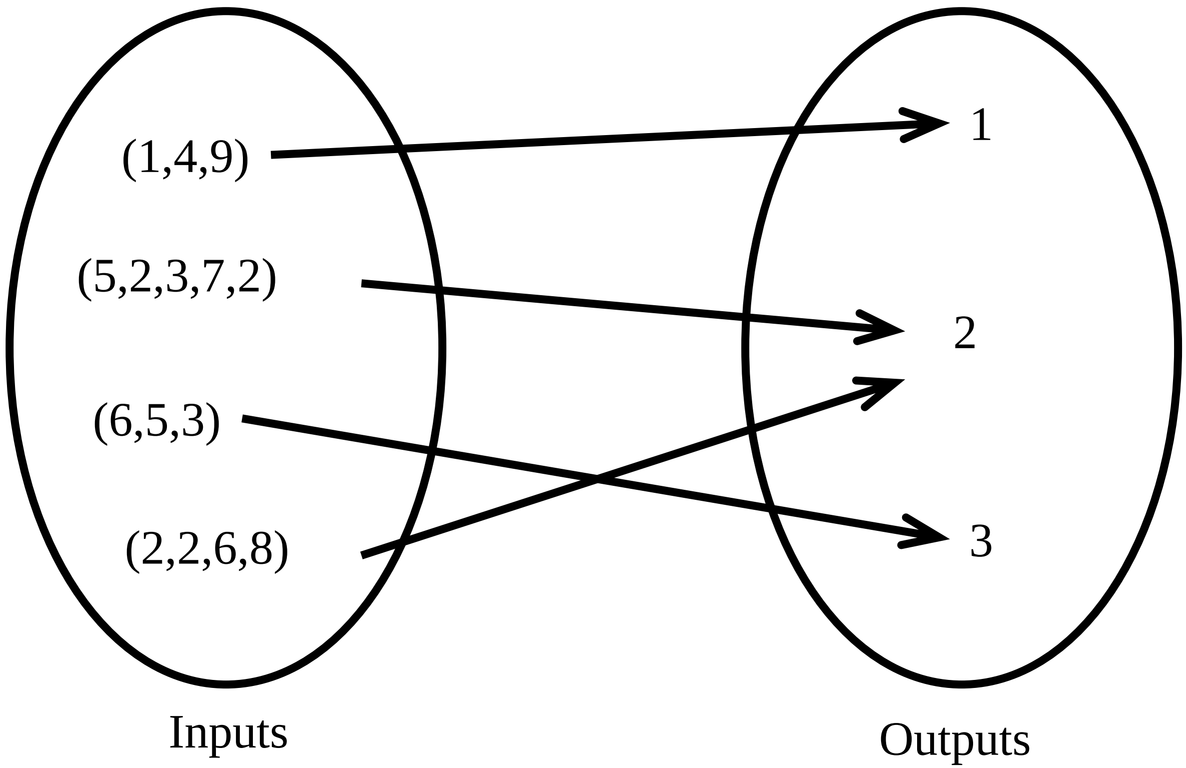
Complexity Analysis

Why do we write programs?

- to perform some specific tasks
- to solve some specific problems
- We will focus on “solving problems”
- What is a “problem”?
- We can view a problem as a mapping of “inputs” to “outputs”

Complexity Analysis

For example, Find Minimum



Complexity Analysis

How to describe a problem?

- Input
    - Describe what an input looks like
- Output
    - Describe what an output looks like and how it relates to the input

Complexity Analysis

An instance is an assignment of values to the input variables

An instance of the Find Minimum function

$N = 10$

$(a_1, a_2, \ldots, a_N) = (5,1,7,4,3,2,3,3,0,8)$

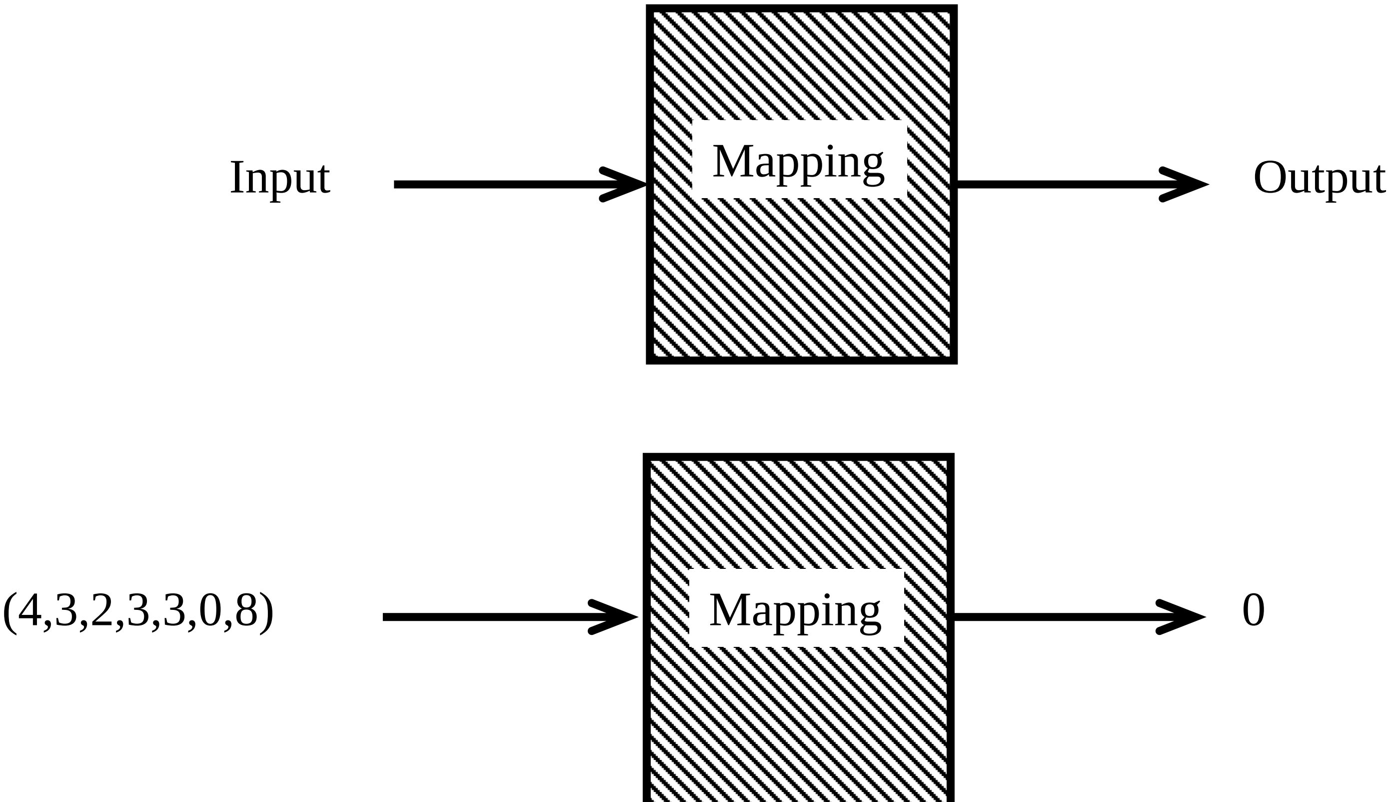
Another instance of the Find Minimum Problem

$N = 10$

$(a_1, a_2, \ldots, a_N) = (15,8,0,4,7,2,5,10,1,4)$

Complexity Analysis

A problem can be considered as a black box



Complexity Analysis

Example: Sorting

Input: A sequence of N numbers $a_1 \ldots a_n$

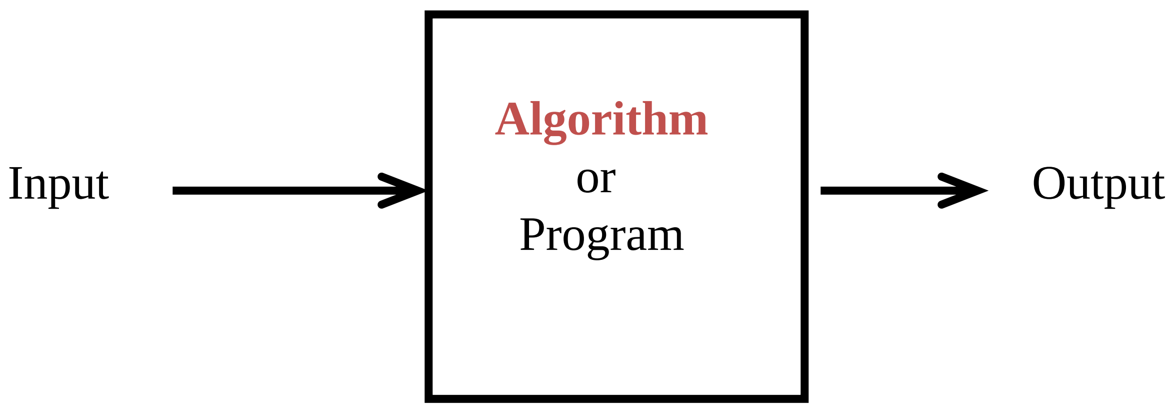
Output: the permutation (reordering) of the input sequence such that $a_1 \leq a_2 \leq \ldots \leq a_n$

Complexity Analysis

How do we solve a problem?

Write an algorithm that implements the mapping

Takes an *input* in and produces a correct *output*



Complexity Analysis

- How do we judge whether an algorithm is good or bad?
- Analyze its efficiency
  - Determined by the amount of computer resources consumed by the algorithm
- What are the important resources?
  - Amount of memory (space complexity)
  - Amount of computational time (time complexity)

Complexity Analysis

Consider the amount of resources

memory space and time

that an algorithm consumes

as a function of the size of the input to the algorithm.

Complexity Analysis

- Suppose there is an assignment statement in your program

```
x := x +1
```

- We'd like to determine:
  - The time a single execution would take
  - The number of times it is executed: **Frequency Count**

Time Complexity

- Product of execution time and frequency is *approximately* the total time taken

- But, since the execution time will be very machine dependent (and compiler dependent), we neglect it and concentrate on the frequency count

- Frequency count will vary from data set to data set (*input to the algorithm*)

Time Complexity

Program 1

```
x := x + 1
```

Frequency = 1

Program 2

```
FOR i := 1 to n
DO
   x := x + 1
END
```

Frequency = n

Program 3

```
FOR i := 1 to n
DO
   FOR j := 1 to n
   DO
      x := x + 1
   END
END
```

Frequency = n^2

Time Complexity

- Program 1
  - statement is not contained in a loop (implicitly or explicitly)
  - Frequency count is 1
- Program 2
  - statement is executed n times
- Program 3
  - statement is executed n^2 times

Big-O Notation

- 1, n, and n^2 are said to be different and increasing orders of magnitude

  (e.g. let $n = 10 \Rightarrow 1, 10, 100$)

- We are interested in determining the order of magnitude of the time complexity of an algorithm

Big-O Notation

- Let's look at an algorithm to print the n^{th} term of the Fibonacci sequence

0 1 1 2 3 5 8 13 21 34 …

$t_n = t_{n-1} + t_{n-2}$

$t_0 = 0$

$t_1 = 1$

Big-O Notation

```
1   procedure fibonacci {print nth term}
2      read(n)
3      if n<0
4         then print(error)
5         else if n=0
6            then print(0)
7            else if n=1
8               then print(1)
9               else
10                 fnm2 := 0;
11                 fnm1 := 1;
12                 FOR i := 2 to n DO
13                    fn := fnm1 + fnm2;
14                     fnm2 := fnm1;
15                     fnm1 := fn
16                  end
17                  print(fn);
```

step	n<0
1	1
2	1
3	1
4	1
5	0
6	0
7	0
8	0
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	0
17	0

Find out how many times each step is evaluated. For instance, steps 1-4 will evaluate once if n<0; the other steps evaluate 0 times.

Big-O Notation

```
1   procedure fibonacci {print nth term}
2      read(n)
3      if n<0
4         then print(error)
5         else if n=0
6            then print(0)
7            else if n=1
8               then print(1)
9               else
10                 fnm2 := 0;
11                 fnm1 := 1;
12                 FOR i := 2 to n DO
13                    fn := fnm1 + fnm2;
14                     fnm2 := fnm1;
15                     fnm1 := fn
16                  end
17                  print(fn);
```

step	n=0
1	1
2	1
3	1
4	0
5	1
6	1
7	0
8	0
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	0
17	0

Big-O Notation

```
1   procedure fibonacci {print nth term}
2      read(n)
3      if n<0
4         then print(error)
5         else if n=0
6            then print(0)
7            else if n=1
8               then print(1)
9               else
10                 fnm2 := 0;
11                 fnm1 := 1;
12                 FOR i := 2 to n DO
13                    fn := fnm1 + fnm2;
14                     fnm2 := fnm1;
15                     fnm1 := fn
16                  end
17                  print(fn);
```

step	n=1
1	1
2	1
3	1
4	0
5	1
6	0
7	1
8	1
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	0
17	0

Big-O Notation

```
1   procedure fibonacci {print nth term}
2       read(n)
3       if n<0
4           then print(error)
5           else if n=0
6               then print(0)
7               else if n=1
8                   then print(1)
9                   else
10                      fnm2 := 0;
11                      fnm1 := 1;
12                      FOR i := 2 to n DO
13                         fn := fnm1 + fnm2;
14                          fnm2 := fnm1;
15                          fnm1 := fn
16                       end
17                       print(fn);
```

step	n>1
1	1
2	1
3	1
4	0
5	1
6	0
7	1
8	0
9	1
10	1
11	1
12	n
13	n-1
14	n-1
15	n-1
16	n-1
17	1

Big-O Notation

step	n<0	n=0	n=1	n>1
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	0	0	0
5	0	**1**	1	1
6	0	**1**	0	0
7	0	0	**1**	1
8	0	0	**1**	0
9	0	0	0	**1**
10	0	0	0	**1**
11	0	0	0	**1**
12	0	0	0	**n**
13	0	0	0	**n-1**
14	0	0	0	**n-1**
15	0	0	0	**n-1**
16	0	0	0	**n-1**
17	0	0	0	**1**

Big-O Notation

- The cases where $n < 0$, $n = 0$, $n = 1$ are not particularly instructive or interesting

- In the case where $n > 1$, we have the total statement frequency of

  $9 + n + 4(n-1) = 5n + 5$

Big-O Notation

- $9 + n + 4(n-1) = 5n + 5$
- We write this as $O(n)$, ignoring the constants
- *This is called **Big-O notation***
- More formally, $f(n) = O(g(n))$
  where $g(n)$ is an **asymptotic upper bound** for $f(n)$
- $g(n)=n$

Big-O Notation

- The notation $f\,(n) = O\,(g(n))$ has a precise mathematical definition

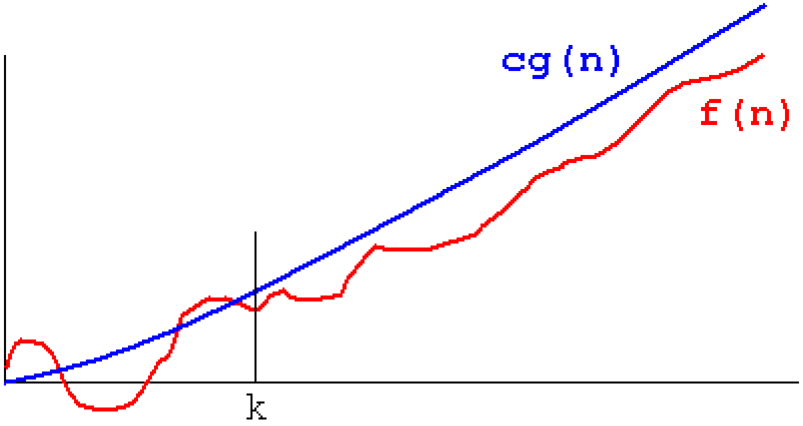
- Read $f(n) = O(g(n))$ as "f of n is big-O of g of n"

- Definition:
  Let $f,\ g\colon Z^{+} \rightarrow R^{+}$

  $f(n) = O(g(n))$ if there exist two constants c and k such that
  $f(n) \leq c\ g(n)$ for all $\mathrm{n} \geq k$

Big-O Notation

Suppose $f(n) = 2n^2+4n+10$,
and $f(n) = O(g(n))$ where $g(n)=n^2$



Proof:

$f(n) = 2n^2+4n+10$

$f(n) \leq 2n^2 + 4n^2+10n^2$ for $n \geq 1$

$f(n) \leq 16n^2$

$f(n) \leq 16g(n)$ where $c = 16$ and $k = 1$

Time & Space Complexity

- $f(n)$ will normally represent the computing time of some algorithm

  Time complexity $T(n)$

- $f(n)$ can also represent the amount of memory an algorithm will need to run

  Space complexity $S(n)$

Time Complexity

- If an algorithm has a time complexity of $O(g(n))$ it means that its execution will take no longer than a constant times $g(n)$
- More formally, $\mathrm{g(n)}$ is an **asymptotic upper bound** for $f(n)$

Remember

- $f(n) \leq c\ g(n)$

  n is typically the size of the data set

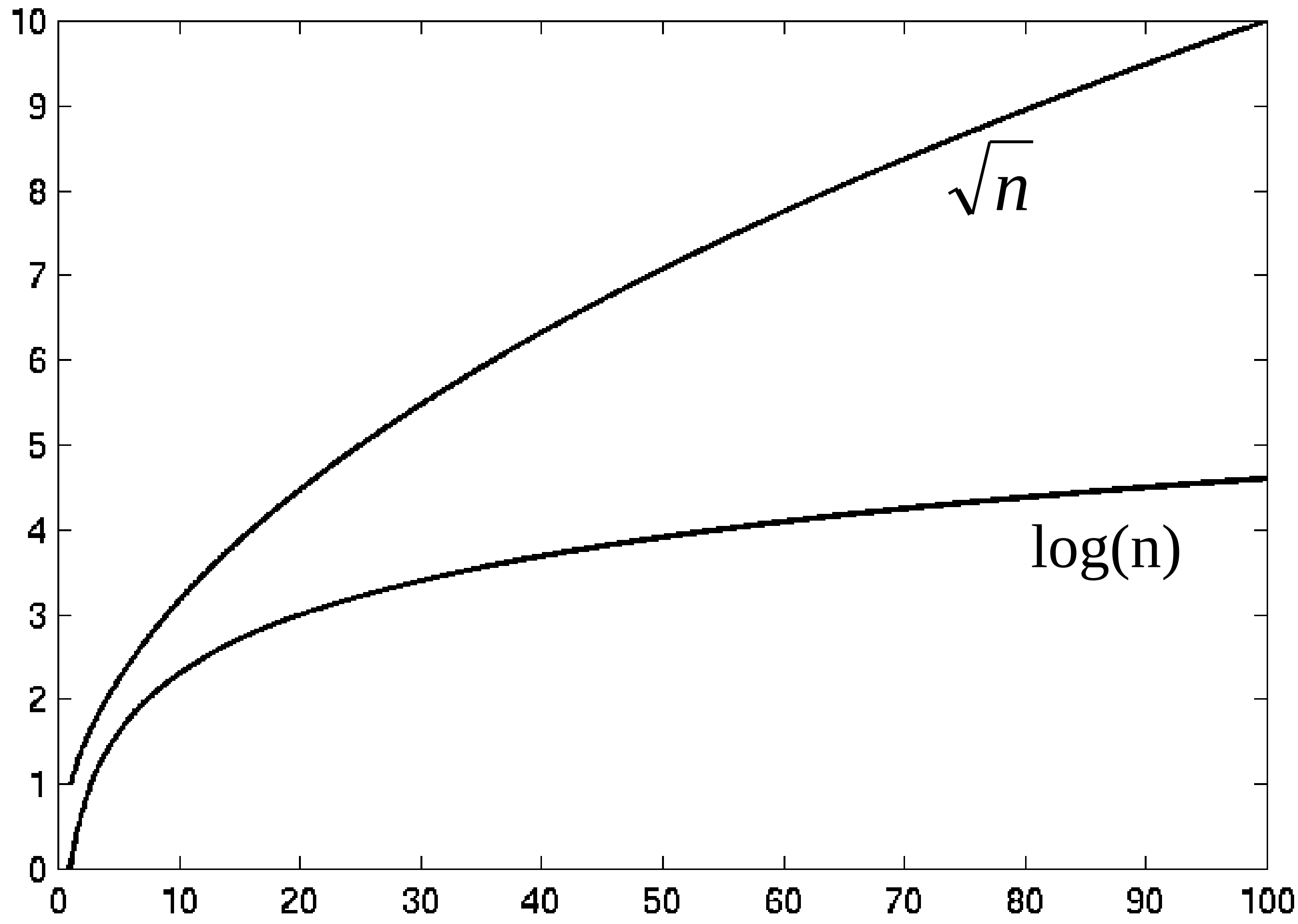
Time Complexity

$O(1)$	Constant (computing time)
$O(n)$	Linear (computing time)
$O(n^2)$	Quadratic (computing time)
$O(n^3)$	Cubic (computing time)
$O(2^n)$	Exponential (computing time)
$O(\log n)$	is faster than $O(n)$ for sufficiently large n
$O(n \log n)$	is faster than $O(n^2)$ for sufficiently large n

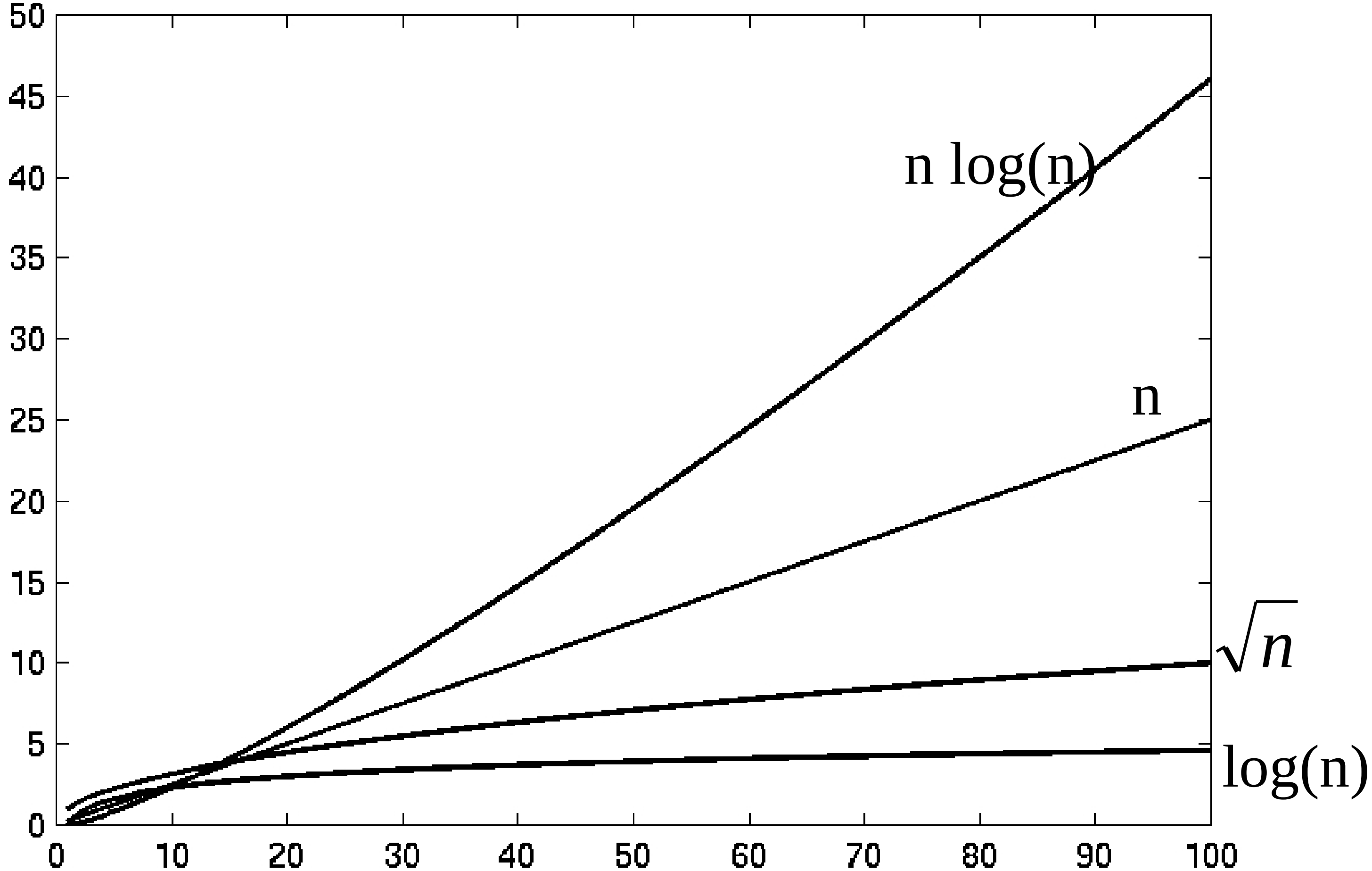
Time Complexity

n	O(1)	O(log2(n))	O(n)	O(nlog2(n))	O(n^2)	O(n^3)	O(n^4)	O(2^n)	O(n^n)
1	7	0.0	1	0.0	1	1	1	2	1
2	7	1.0	2	2.0	4	8	16	4	4
3	7	1.6	3	4.8	9	27	81	8	27
4	7	2.0	4	8.0	16	64	256	16	256
5	7	2.3	5	11.6	25	125	625	32	3125
6	7	2.6	6	15.5	36	216	1296	64	46656
7	7	2.8	7	19.7	49	343	2401	128	823543
8	7	3.0	8	24.0	64	512	4096	256	16777216
9	7	3.2	9	28.5	81	729	6561	512	3.87E+08
10	7	3.3	10	33.2	100	1000	10000	1024	1E+10
11	7	3.5	11	38.1	121	1331	14641	2048	2.85E+11
12	7	3.6	12	43.0	144	1728	20736	4096	8.92E+12
13	7	3.7	13	48.1	169	2197	28561	8192	3.03E+14
14	7	3.8	14	53.3	196	2744	38416	16384	1.11E+16
15	7	3.9	15	58.6	225	3375	50625	32768	4.38E+17
16	7	4.0	16	64.0	256	4096	65536	65536	1.84E+19
17	7	4.1	17	69.5	289	4913	83521	131072	8.27E+20
18	7	4.2	18	75.1	324	5832	104976	262144	3.93E+22
19	7	4.2	19	80.7	361	6859	130321	524288	1.98E+24
20	7	4.3	20	86.4	400	8000	160000	1048576	1.05E+26
21	7	4.4	21	92.2	441	9261	194481	2097152	5.84E+27
22	7	4.5	22	98.1	484	10648	234256	4194304	3.41E+29
23	7	4.5	23	104.0	529	12167	279841	8388608	2.09E+31
24	7	4.6	24	110.0	576	13824	331776	16777216	1.33E+33
25	7	4.6	25	116.1	625	15625	390625	33554432	8.88E+34
26	7	4.7	26	122.2	676	17576	456976	67108864	6.16E+36
27	7	4.8	27	128.4	729	19683	531441	1.34E+08	4.43E+38
28	7	4.8	28	134.6	784	21952	614656	2.68E+08	3.31E+40
29	7	4.9	29	140.9	841	24389	707281	5.37E+08	2.57E+42
30	7	4.9	30	147.2	900	27000	810000	1.07E+09	2.06E+44

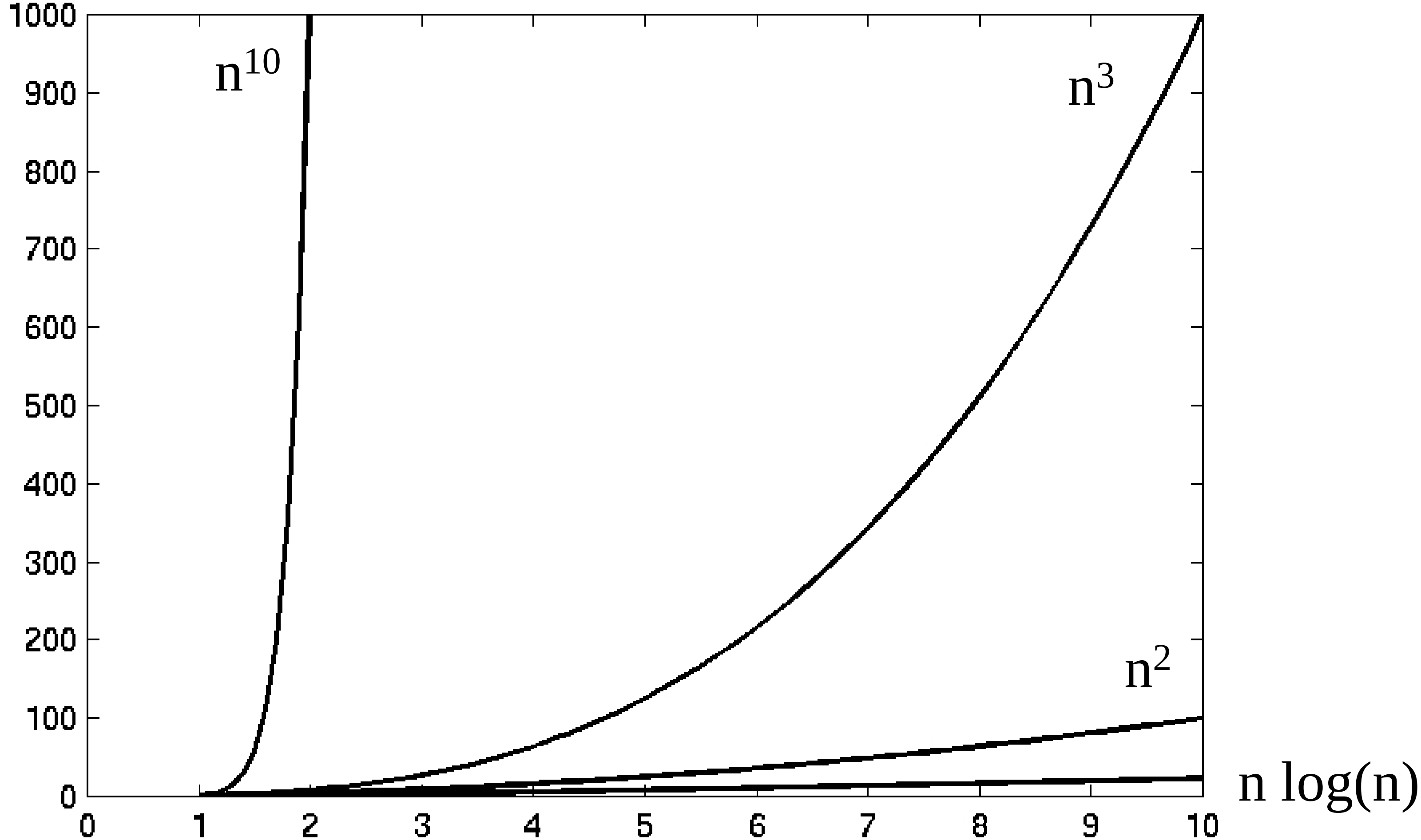
Time Complexity



Time Complexity

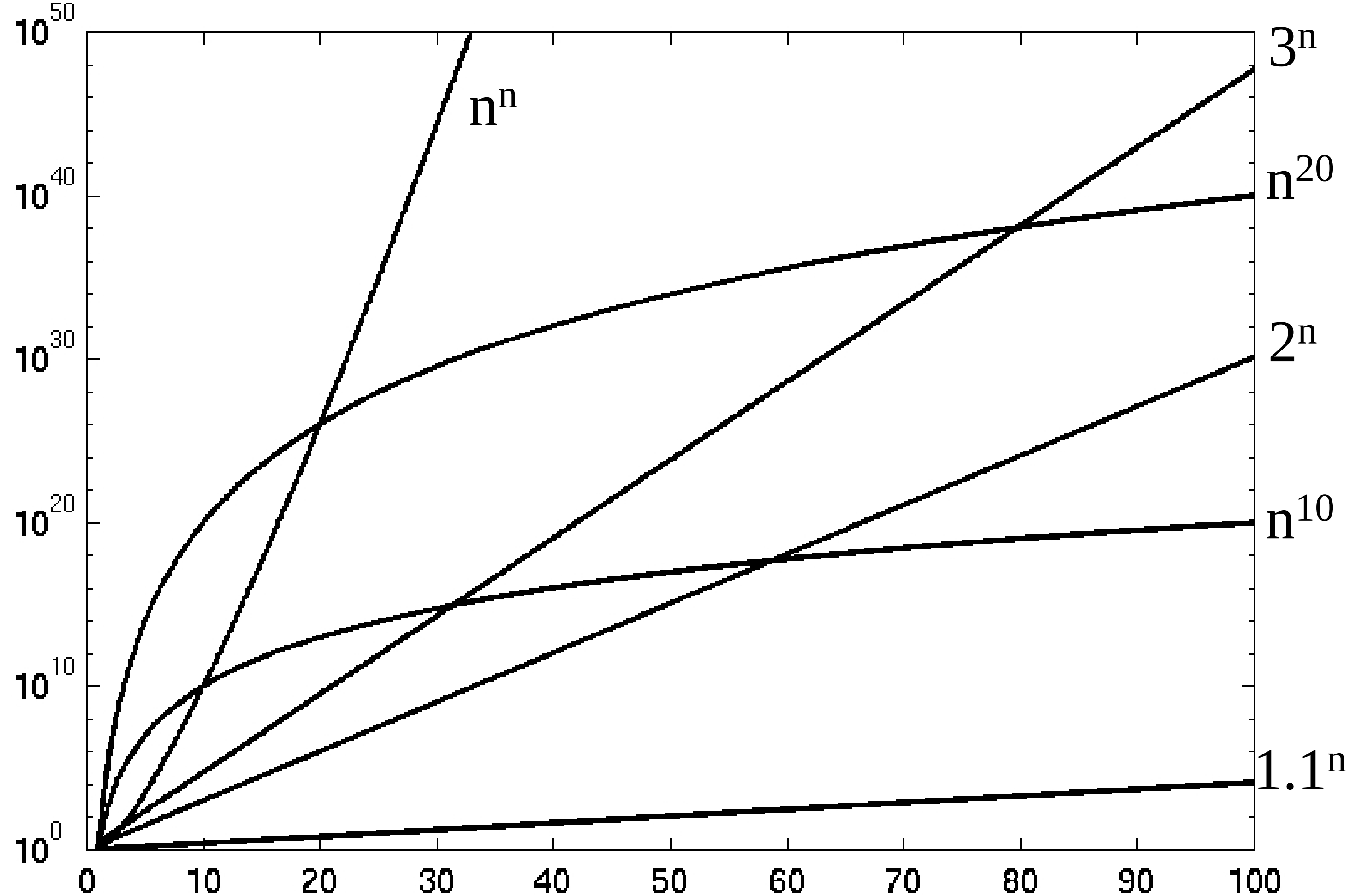


Time Complexity



Time Complexity

Log Scale



Time Complexity

$f1(n) = 10\ n + 25\ n^2$	$O(n^2)$
$f2(n) = 20\ n \log n + 5\ n$	$O(n \log n)$
$f3(n) = 12\ n \log n + 0.05\ n^2$	$O(n^2)$
$f4(n) = n^{1/2} + 3\ n \log n$	$O(n \log n)$

Time Complexity

Arithmetic of Big-O notation

if

$$T_1(n) = O(f(n)) \text{ and } T_2(n) = O(g(n))$$

then

$$T_1(n) + T_2(n) = O(max(f(n), g(n))$$

Time Complexity

Arithmetic of Big-O notation

if

$$f(n) \leq g(n)$$

then

$$O(f(n) + g(n)) = O(g(n))$$

Time Complexity

Arithmetic of Big-O notation

if

$$T_1(n) = O(f(n)) \text{ and } T_2(n) = O(g(n))$$

then

$$T_1(n)\ T_2(n) = O(f(n)\ g(n))$$

Time Complexity

Rules for computing the time complexity

- the complexity of each read, write, and assignment statement can be taken as $O(1)$
- the complexity of a sequence of statements is determined by the summation rule
- the complexity of an if statement is the complexity of the executed statements, plus the time for evaluating the condition

Time Complexity

Rules for computing the time complexity

- the complexity of an if-then-else statement is the time for evaluating the condition plus the larger of the complexities of the then and else clauses

- the complexity of a loop is the sum, over all the times around the loop, of the complexity of the body and the complexity of the termination condition

Time Complexity

- Given an algorithm, we analyze the frequency count of each statement and total the sum

- This may give a polynomial $P(n)$:

$$\mathrm{P}(n) = c_k\, n^k + c_{k\text{-}1}\, n^{k\text{-}1} + \ldots + c_1\, n + c_0$$

where the c_i are constants, c_k are non-zero, and *n* is a parameter

Time Complexity

If the big-O notation of a portion of an algorithm is given by:

$$P(n) = O(n^k)$$

and on the other hand, if any other step is executed 2^n times or more, we have:

$$c\ 2^n + P(n) = O(2^n)$$

Time Complexity

- What about computing the complexity of a recursive algorithm?
- In general, this is more difficult
- The basic technique
  - Identify a recurrence relation implicit in the recursion

    $$T(n) = f(T(k)),\ k \in \{1, 2, \ldots, n\text{-}1\}$$

  - Solve the recurrence relation by finding an expression for $T(n)$ in term which do not involve $T(k)$

Time Complexity

```
int factorial(int n) {
   int factorial_value;

   factorial_value = 0;

   /* compute factorial value recursively */

   if (n <= 1) {
      factorial_value = 1;
   }
   else {
      factorial_value = n * factorial(n-1);//recurrent
   }
   return (factorial_value);
}
```

Time Complexity

Let the time complexity of the function be $\underline{T(n)}$

… which is what we want to compute!

Now, let's try to analyze the algorithm

Time Complexity

```
                                                        n>1

int factorial(int n)
{
   int factorial_value;                                 1

   factorial_value = 0;                                 1

   if (n <= 1) {                                        1
      factorial_value = 1;                              0
   }
   else {                                               1
      factorial_value = n * factorial(n-1);             T(n-1)
   }
   return (factorial_value);                            1
}
```

Time Complexity

$$
\begin{aligned}
T(n) &= 5 + T(n\text{-}1) \\
T(n) &= c + T(n\text{-}1) \\
T(n\text{-}1) &= c + T(n\text{-}2) \\
T(n) &= c + c + T(n\text{-}2) \\
&= 2c + T(n\text{-}2) \\
T(n\text{-}2) &= c + T(n\text{-}3) \\
T(n) &= 2c + c + T(n\text{-}3) \\
&= 3c + T(n\text{-}3)
\end{aligned}
$$

Therefore:

$$T(n) = ic + T(n\text{-}i\ )$$

Time Complexity

$T(n) = ic + T(n\text{-}i\ )$

Finally, when $i = n\text{-}1$

$$
\begin{aligned}
T(n) \quad &= (n\text{-}1)c + T(n\text{-}(n\text{-}1)\ ) \\
&= (n\text{-}1)c + T(1) \\
&= (n\text{-}1)c + d \\
&= \mathrm{cn\text{-}c} + d
\end{aligned}
$$

Hence, $T(n) = \mathrm{O}(n)$

Space Complexity

Compute the space complexity of an algorithm by analyzing the storage requirements (as a function on the input size) in the same way

Space Complexity

For example

- if you read a stream of n characters
- and only ever store a constant number of them,
- then it has space complexity $O(1)$

Space Complexity

For example

- if you read a stream of n records
- and store all of them,
- then it has space complexity $O(n)$

Space Complexity

For example

- if you read a stream of n records
- and store all of them,
- and each record causes the creation of (a constant number) of other records,
- then it still has space complexity $O(n)$

Space Complexity

For example

- if you read a stream of n records
- and store all of them,
- and each record causes the creation of a number of other records (and the number is proportional to the size of the data set n)
- then it has space complexity $O(n^2)$

Time vs Space Complexity

In general, we can often decrease the time complexity but this will involve an increase in the space complexity

and *vice versa* (decrease space, increase time)

This is the ***time-space tradeoff***

Time vs Space Complexity

For example

- the average time complexity of an iterative sort (e.g. bubble sort) is $O(n^2)$

- but we can do better:
- the average time complexity of the Quicksort is $O(n \log n)$

- But the Quicksort is recursive, and the recursion causes an increase in memory requirements (*i.e.,* an increase in space complexity)

Worst-case and average-case complexity

So far we have looked only at worst-case complexity (i.e., we have developed an upper-bound on complexity)

However, there are times when we are more interested in the average-case complexity (especially if it differs significantly)

Worst-case and average-case complexity

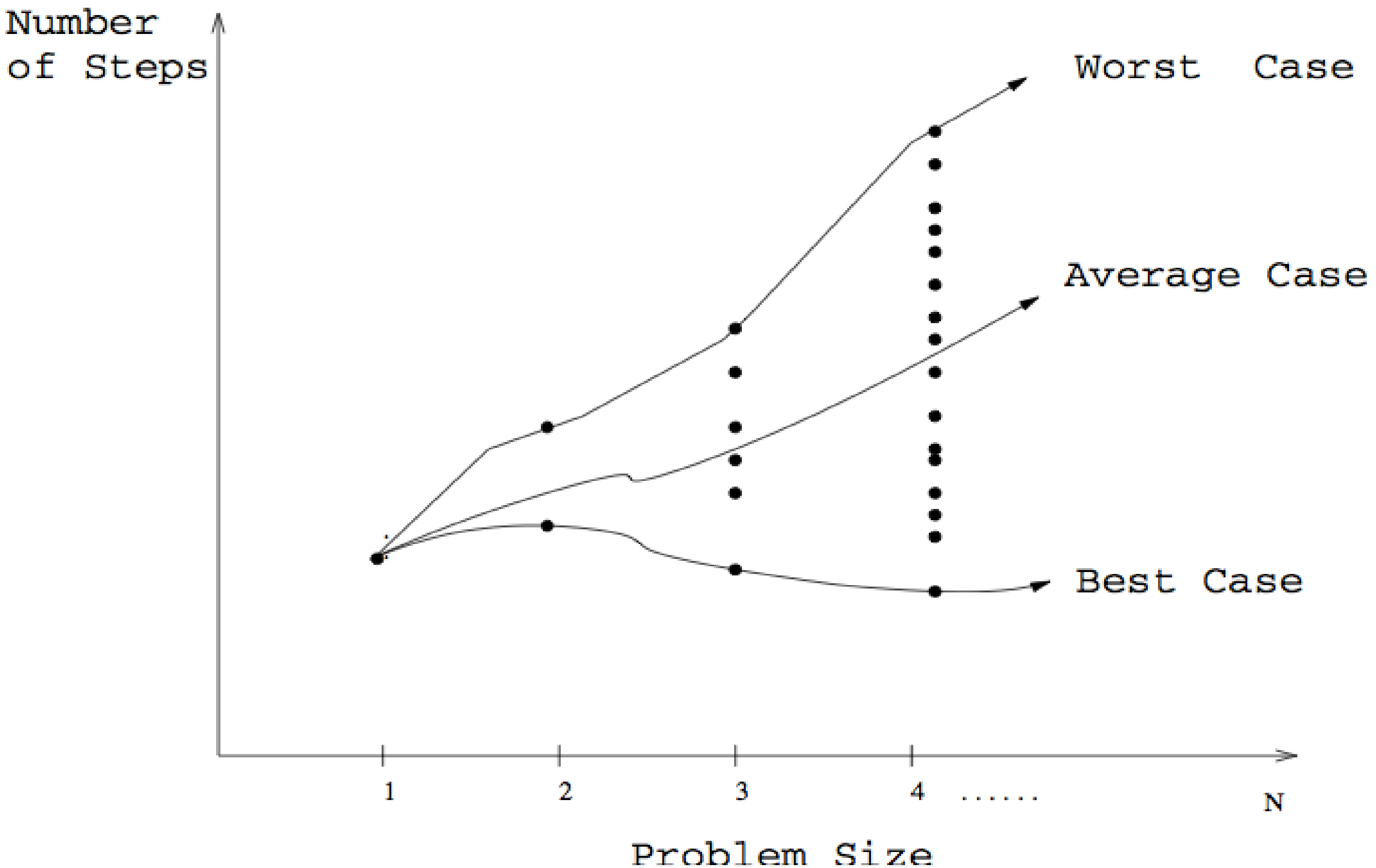
For example

the Quicksort algorithm has

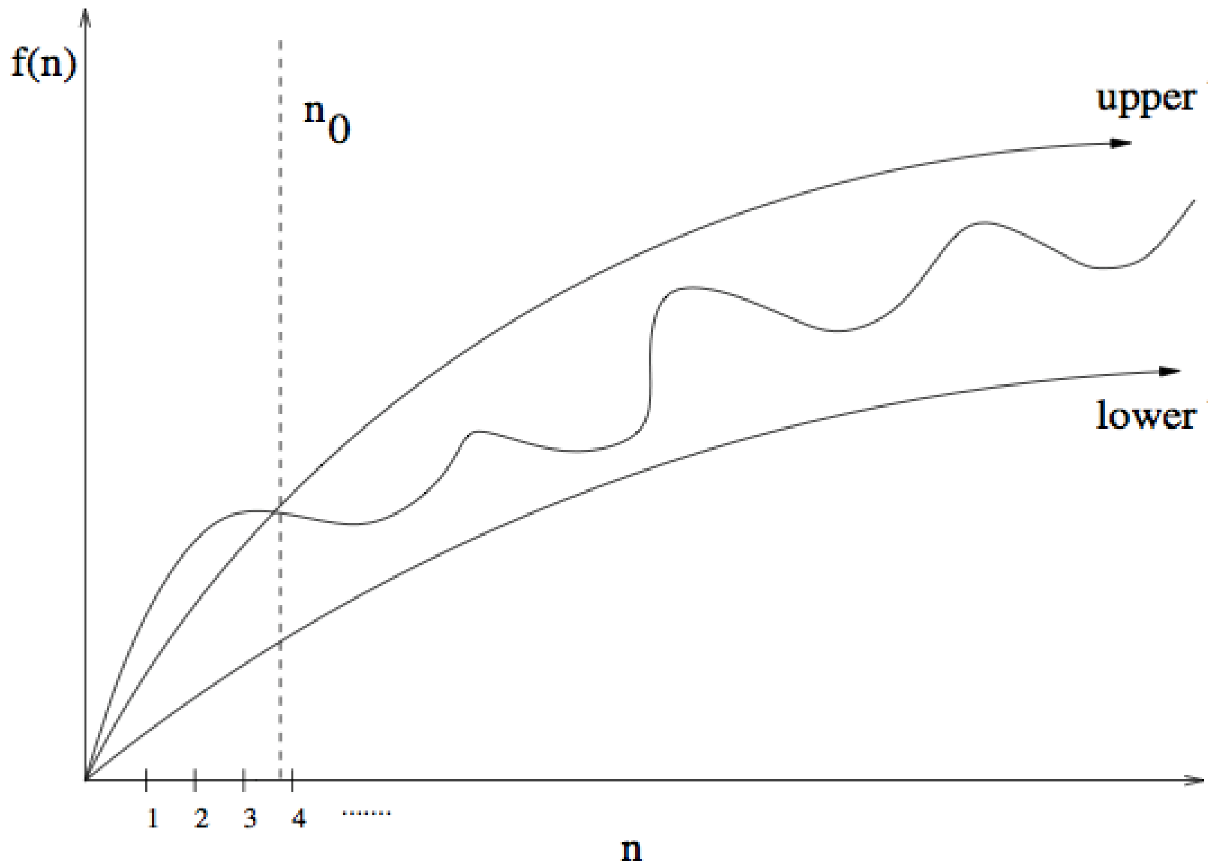
$T(n) = O(n^2)$, worst case (for inversely sorted data)

$T(n) = O(n \log_2 n)$, average case (for randomly ordered data)

Worst-case and average-case complexity



Worst-case and average-case complexity



Worst-case and average-case complexity

asymptotic upper bound aka worst case

$f(n) = \boxed{O(g(n))}$ means $c \cdot g(n)$ is an *upper bound* on $f(n)$. Thus there exists some constant c such that $f(n)$ is always $\leq c \cdot g(n)$, for large enough n (i.e. $n \geq n_0$ for some constant n_0).

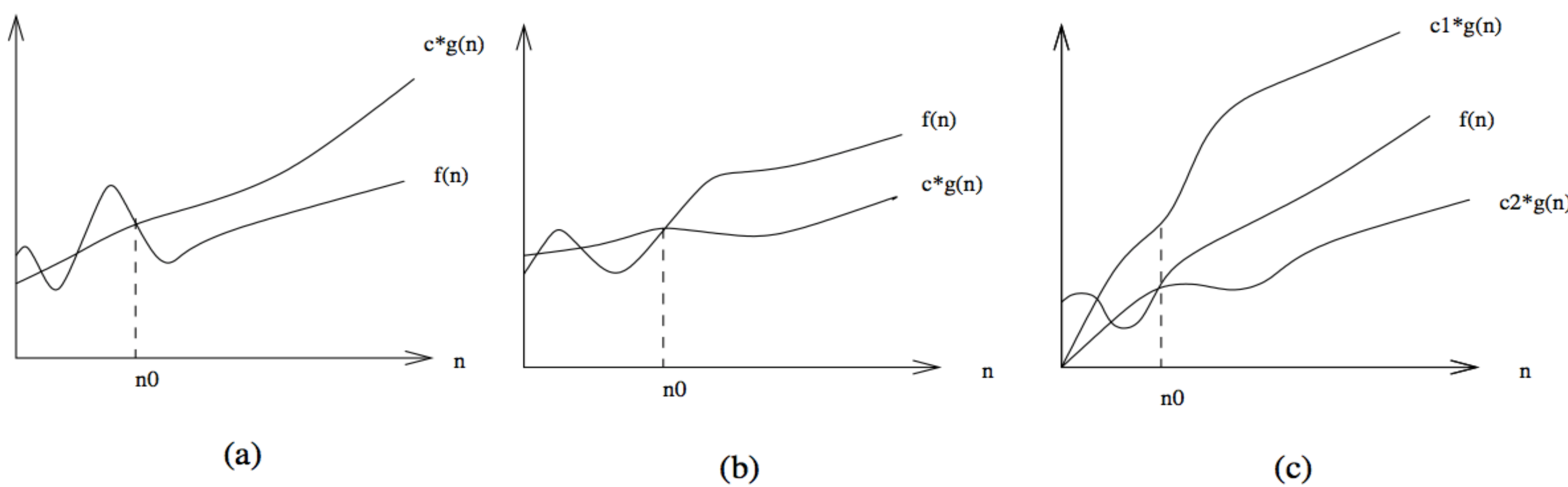
asymptotic lower bound aka best case

$f(n) = \boxed{\Omega(g(n))}$ means $c \cdot g(n)$ is a *lower bound* on $f(n)$. Thus there exists some constant c such that $f(n)$ is always $\geq c \cdot g(n)$, for all $n \geq n_0$.

asymptotic tightest bound aka best of all worst cases

$f(n) = \boxed{\Theta(g(n))}$ means $c_1 \cdot g(n)$ is an upper bound on $f(n)$ and $c_2 \cdot g(n)$ is a lower bound on $f(n)$, for all $n \geq n_0$. Thus there exist constants c_1 and c_2 such that $f(n) \leq c_1 \cdot g(n)$ and $f(n) \geq c_2 \cdot g(n)$. This means that $g(n)$ provides a nice, tight bound on $f(n)$.

Worst-case and average-case complexity



Illustrating the big (a) O, (b) Ω, and (c) Θ notations

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- Adopted and Adapted from Material by:
- David Vernon: vernon@cmu.edu ; www.vernon.eu