04-630
Data Structures and Algorithms for Engineers

Lecture 13: Height-Balanced Trees: Red Black Trees

Height-balanced Trees

Red-Black Trees

- The goal of height-balancing is to ensure that the tree is as complete as possible and that, consequently, it has minimal height for the number of nodes in the tree

- As a result, the number of probes it takes to search the tree (and the time it takes) is minimized.

Red-Black Trees

- A perfect or a complete tree with n nodes has height $O(\log_2 n)$
  - So the time it takes to search a perfect or a complete tree with n nodes is $O(\log_2 n)$
- A skinny tree could have height $O(n)$
  - So the time it takes to search a skinny tree can be $O(n)$
- Red-Black trees are similar to AVL trees in that they allow us to construct trees which have a guaranteed search time $O(\log_2 n)$

Red-Black Trees

- A red-black tree is a binary tree whose nodes can be coloured either red or black to satisfy the following conditions:

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*

2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)

3. Each external node (leaf node) is **black.**

4. Every node must have one color –**red or black.**

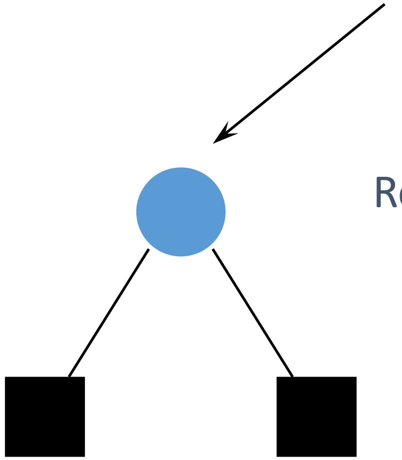
5. Root node is **black**.

Red-Black Trees

Red-black tree (condition 3)

Red-Black Trees

Undetermined colour (both red and black satisfy conditions 1 & 2)



Red-black tree (condition 3)

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
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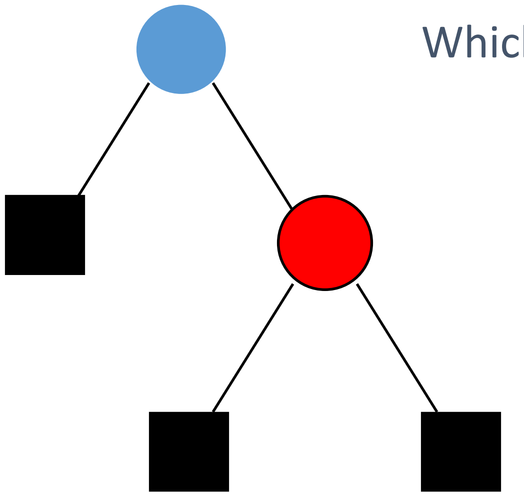
Red-Black Trees



Which color would you assign to the root to make the tree a red-black tree?

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)
3. Each external node (leaf node) is **black.**
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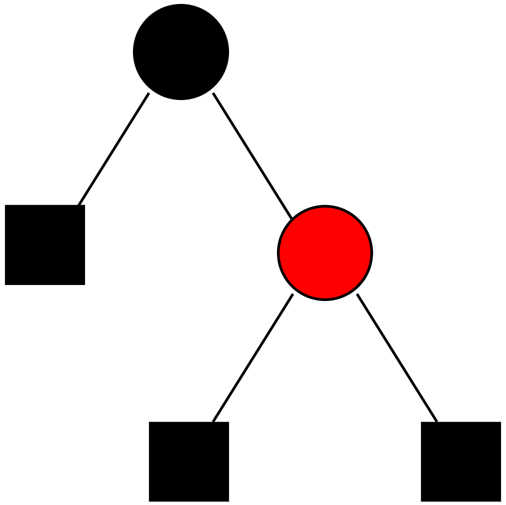
Red-Black Trees



If root was red, then right child would have to be black (because if it was red, by Condition 2 it would have to have a black parent) but then Condition 1, the black condition, would be violated ... so the root can't be red in this case.

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)
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Red-Black Trees

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
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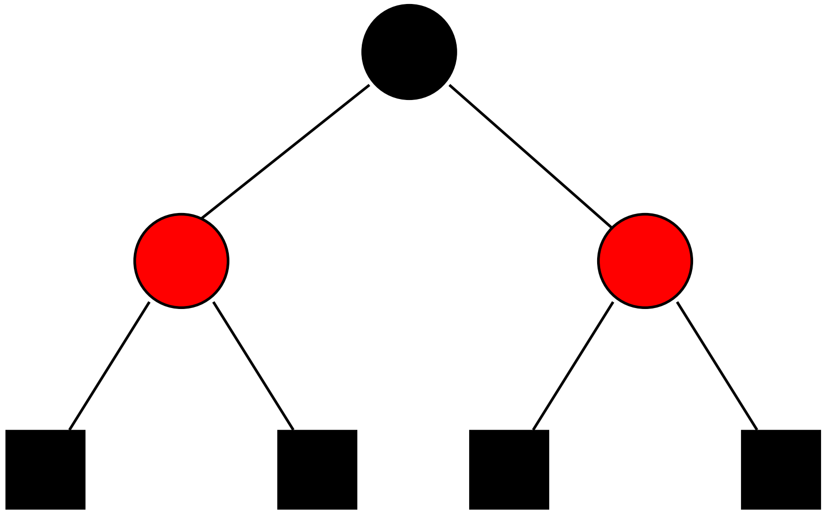
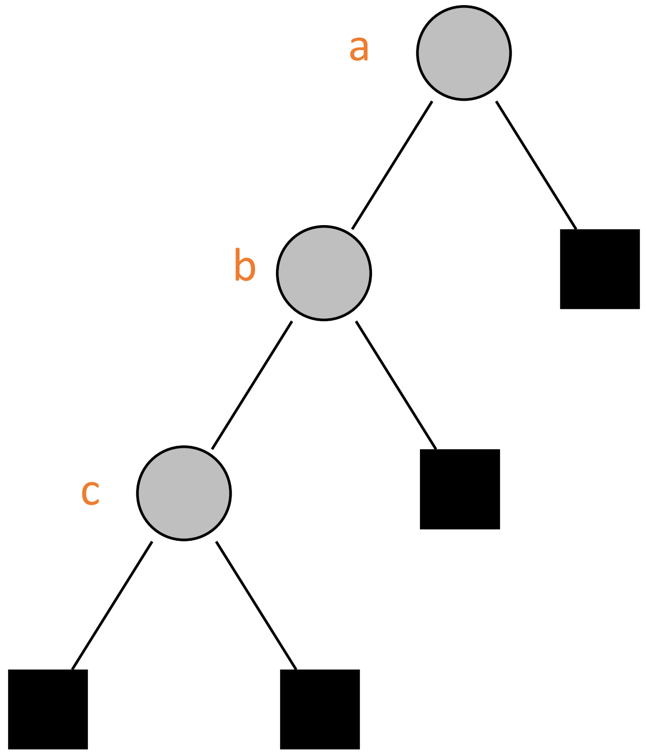
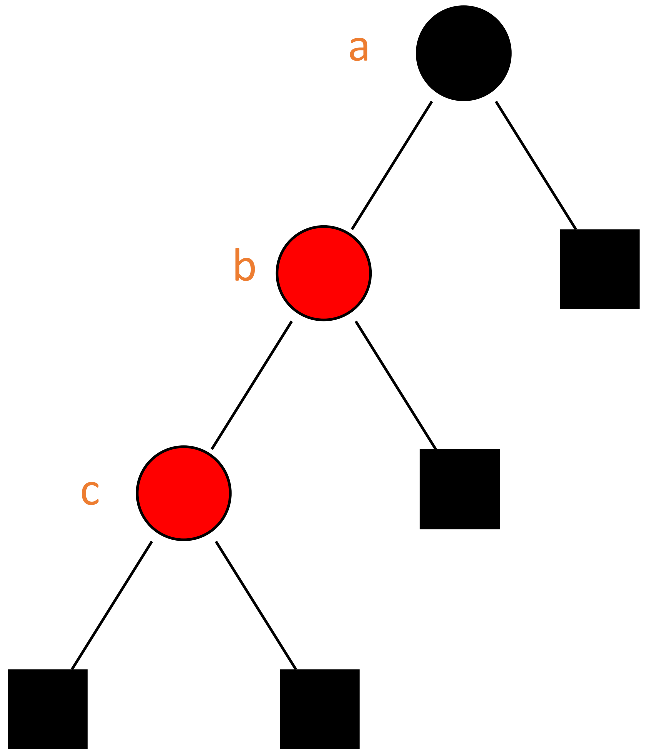


Red-Black Trees



Consider the tree shown. Can the tree be colored to make it a red black tree?

Red-Black Trees



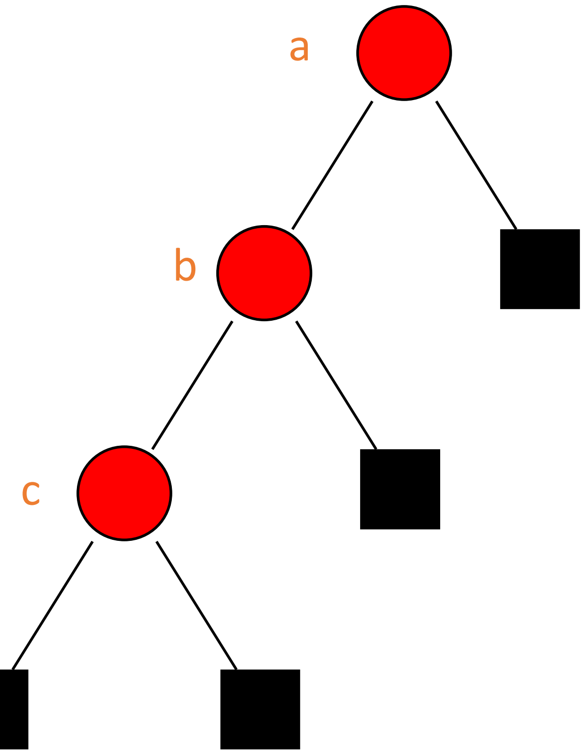
To satisfy **black condition**, either

(1) node a is black and nodes b and c are red, or

(2) nodes a, b, and c are red.

Black condition: Each root-to-frontier path contains exactly the *same number of black nodes.*

Red-Black Trees



To satisfy **black condition**, either

(1) node a is black and nodes b and c are red, or

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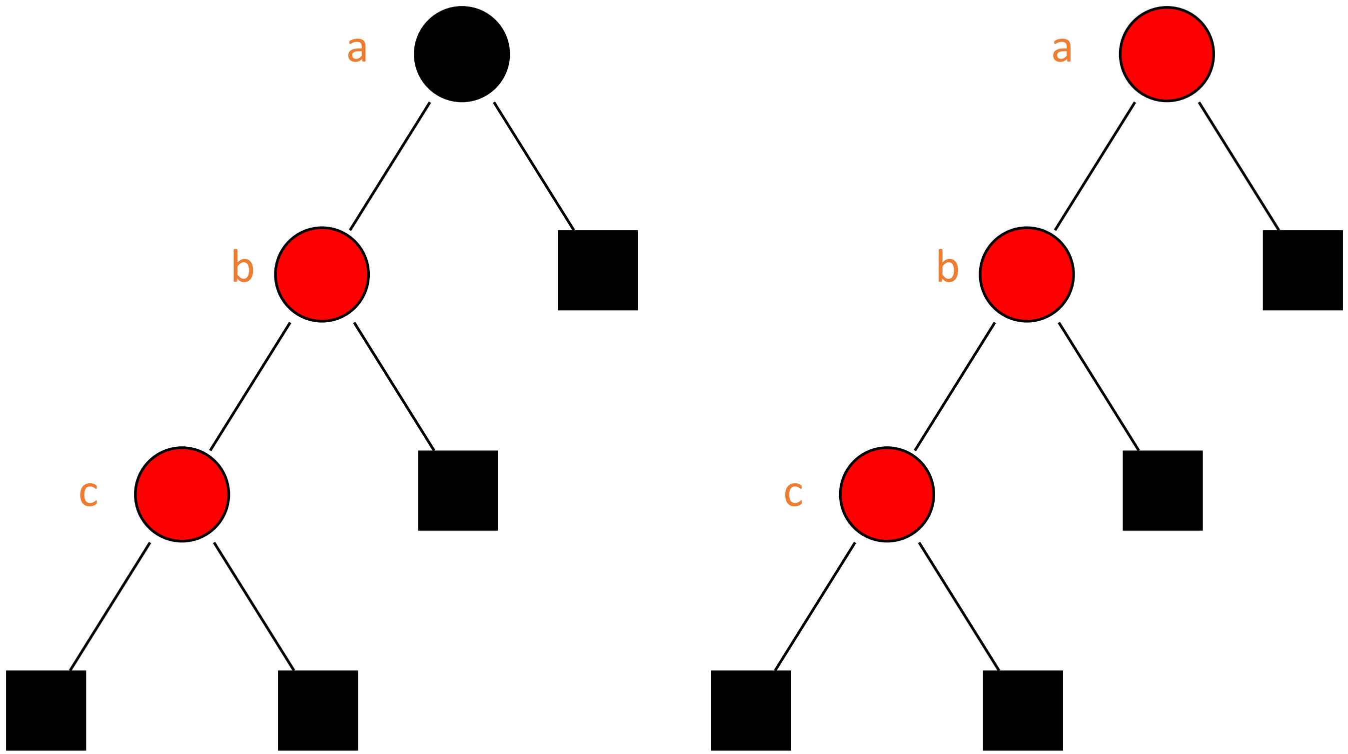
Black condition: Each root-to-frontier path contains exactly the *same number of black nodes.*

Red-Black Trees



To satisfy **black condition**, either

(1) node a is black and nodes b and c are red, or

(2) nodes a, b, and c are red.

In both cases, a **red condition is violated.**

Therefore, this is not a red-black tree (i.e., it cannot be coloured in a way that satisfies all three conditions)

Red condition: *Each red node* that is not the root *has a black parent*

Red-Black Trees

- For all n >= 1, every red-black tree of size n has height $O(\log_2 n)$

- Thus, red-black trees provide a guaranteed worst-case search time of $O(\log_2 n)$

Red-Black Trees

- Insertions and deletions can cause red and black conditions to be violated

- Trees then have to be restructured

- Restructuring is called a promotion (or rotation)
  - Single promotion
  - 2 promotion

Red-Black Trees

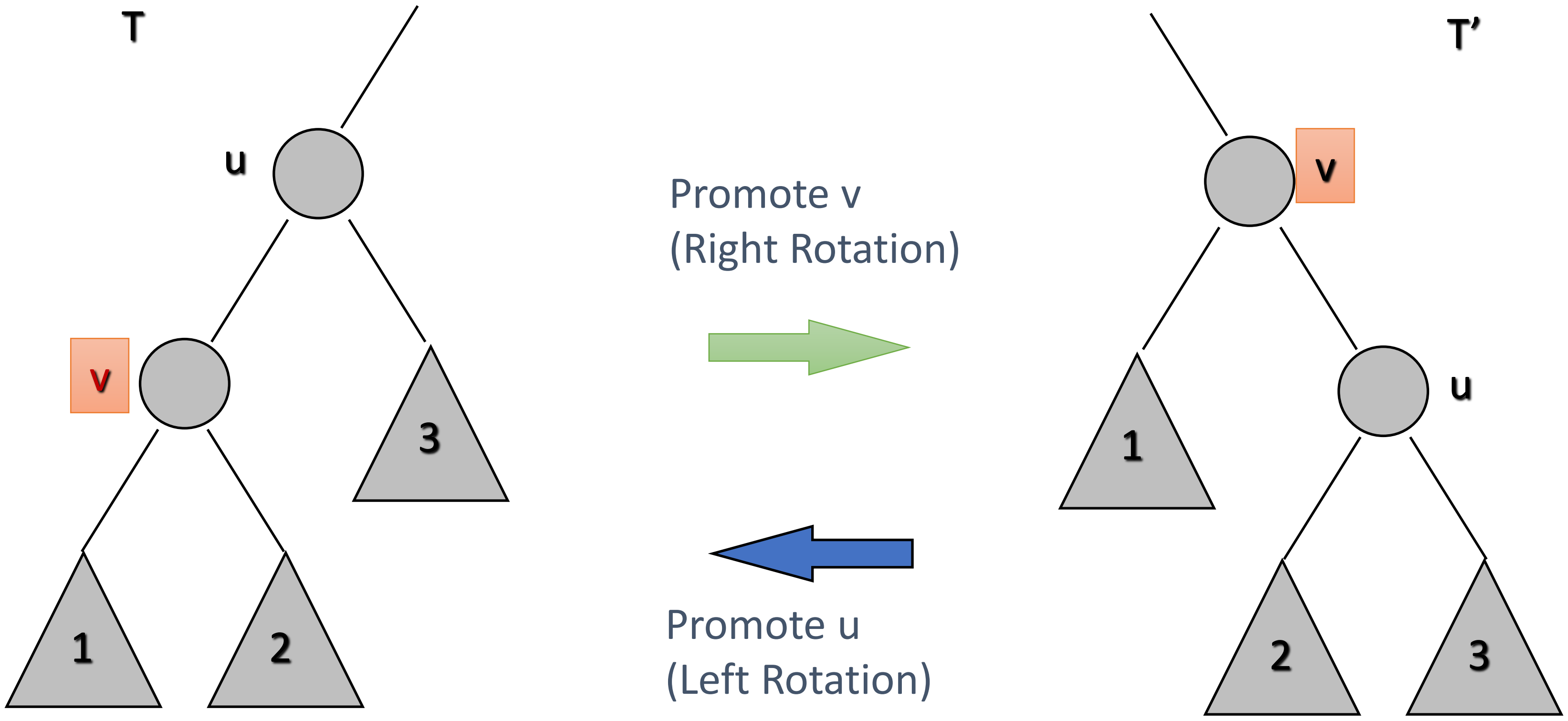
- Single promotion

- Also referred to as

  - single (left) rotation
  - single (right) rotation

- Promotes a node one level

Red-Black Trees

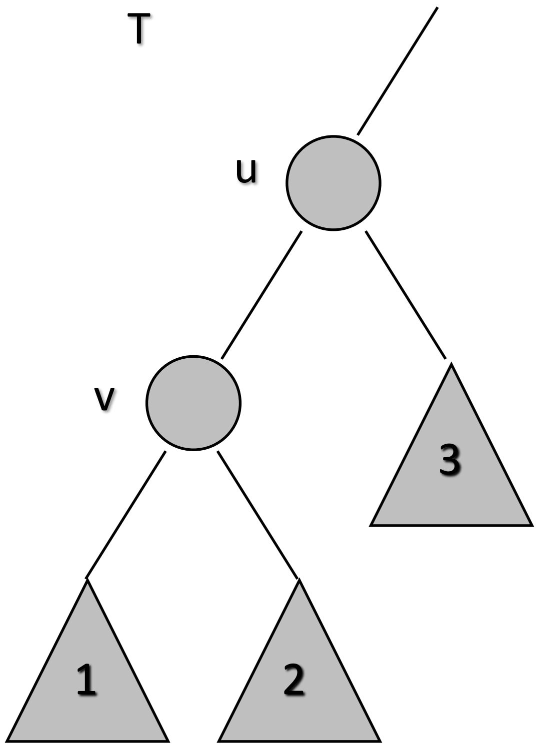


Red-Black Trees

- A single promotion (Left Rotation or Right Rotation) preserves the binary-search condition

- Same manner as an AVL rotation (*earlier*)

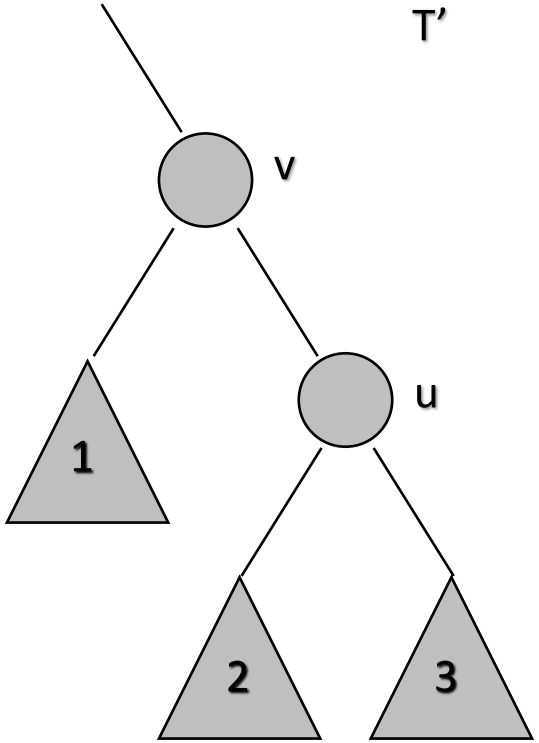
Red-Black Trees



keys(1) < key(v) < key(u)
key(v) < keys(2) < key(u)
key(u) < keys(3)

Promote v
(Right Rotation)

Promote u
(Left Rotation)



keys(1) < key(v)
key(v) < keys(2) < key(u)
key(v) < key(u) < keys(3)

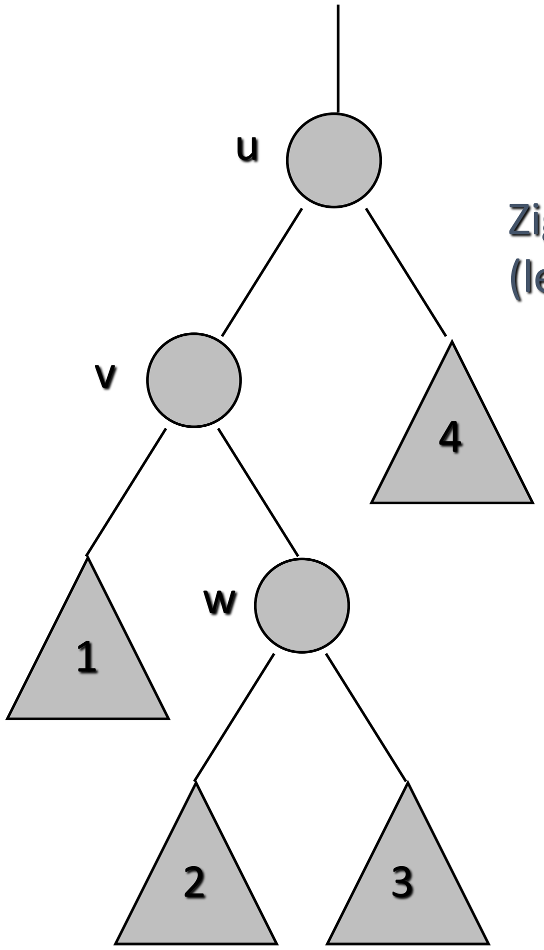
preserves the binary-search condition

Red-Black Trees

- 2-Promotion
- Zig-zag promotion
- Composed of two single promotions
- And hence preserves the binary-search condition

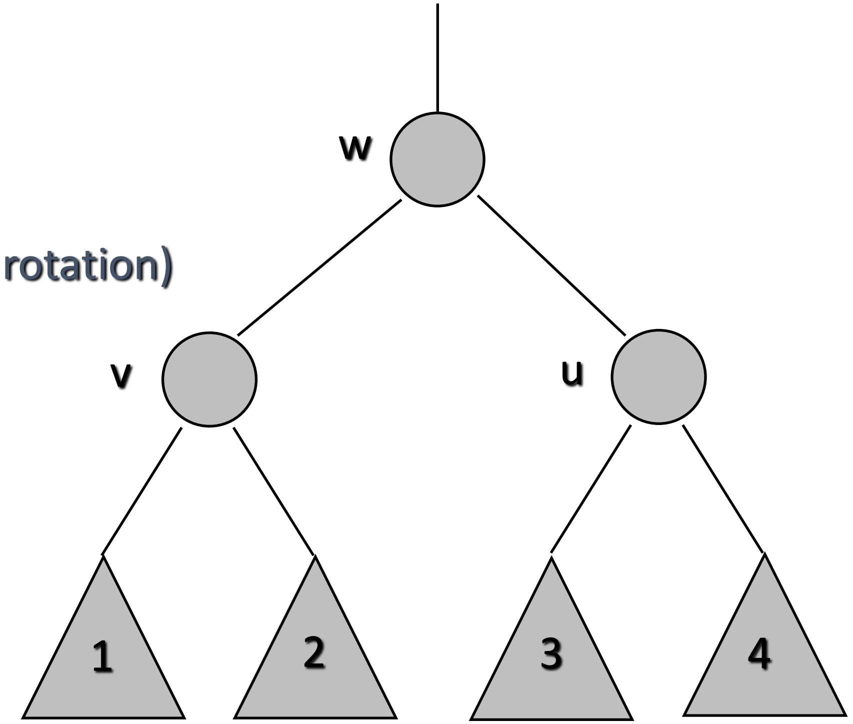
Red-Black Trees

Initial tree



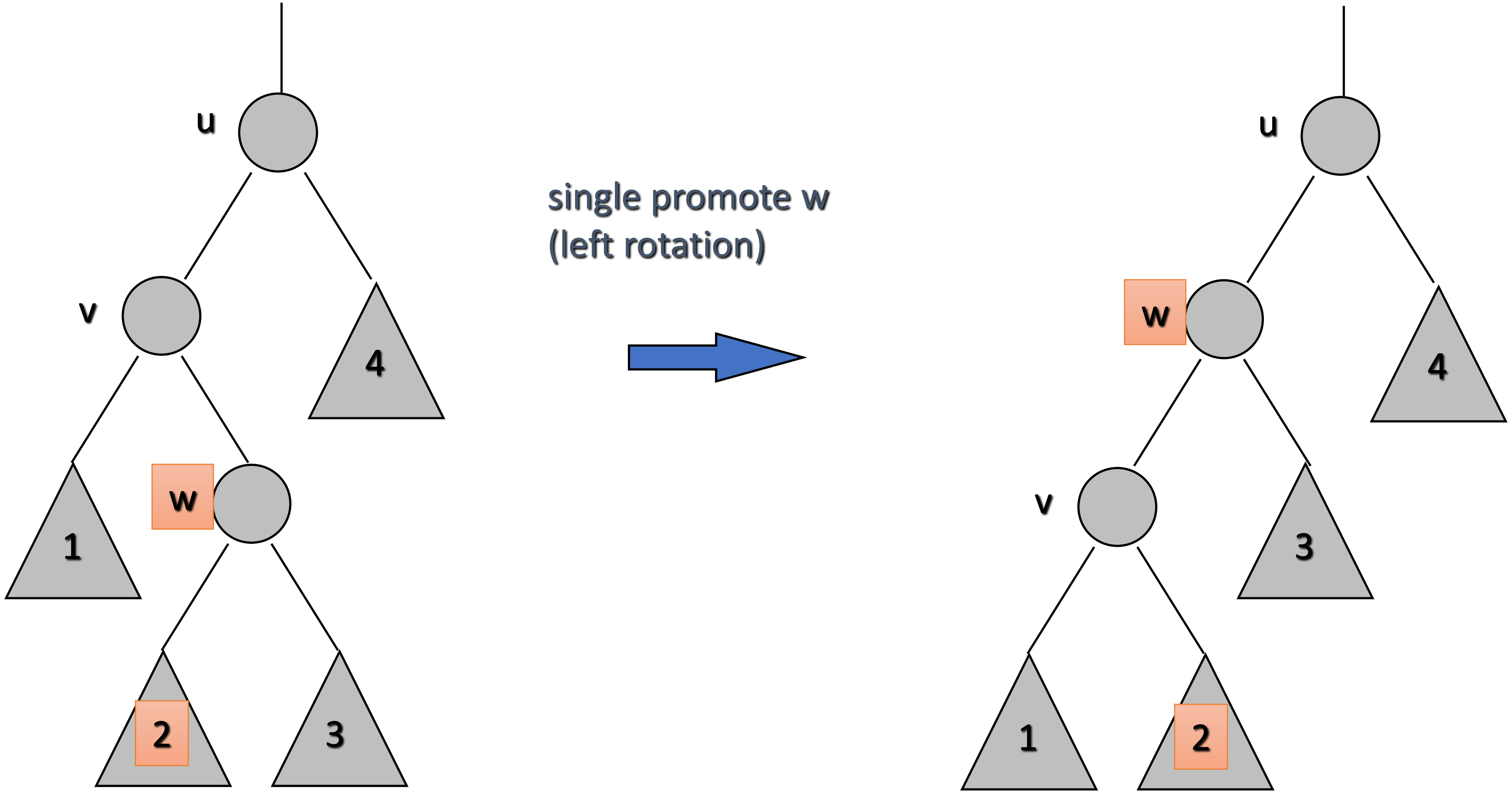
Zig-zag promote w
(left rotation, then right rotation)

Tree after zig-zag promote

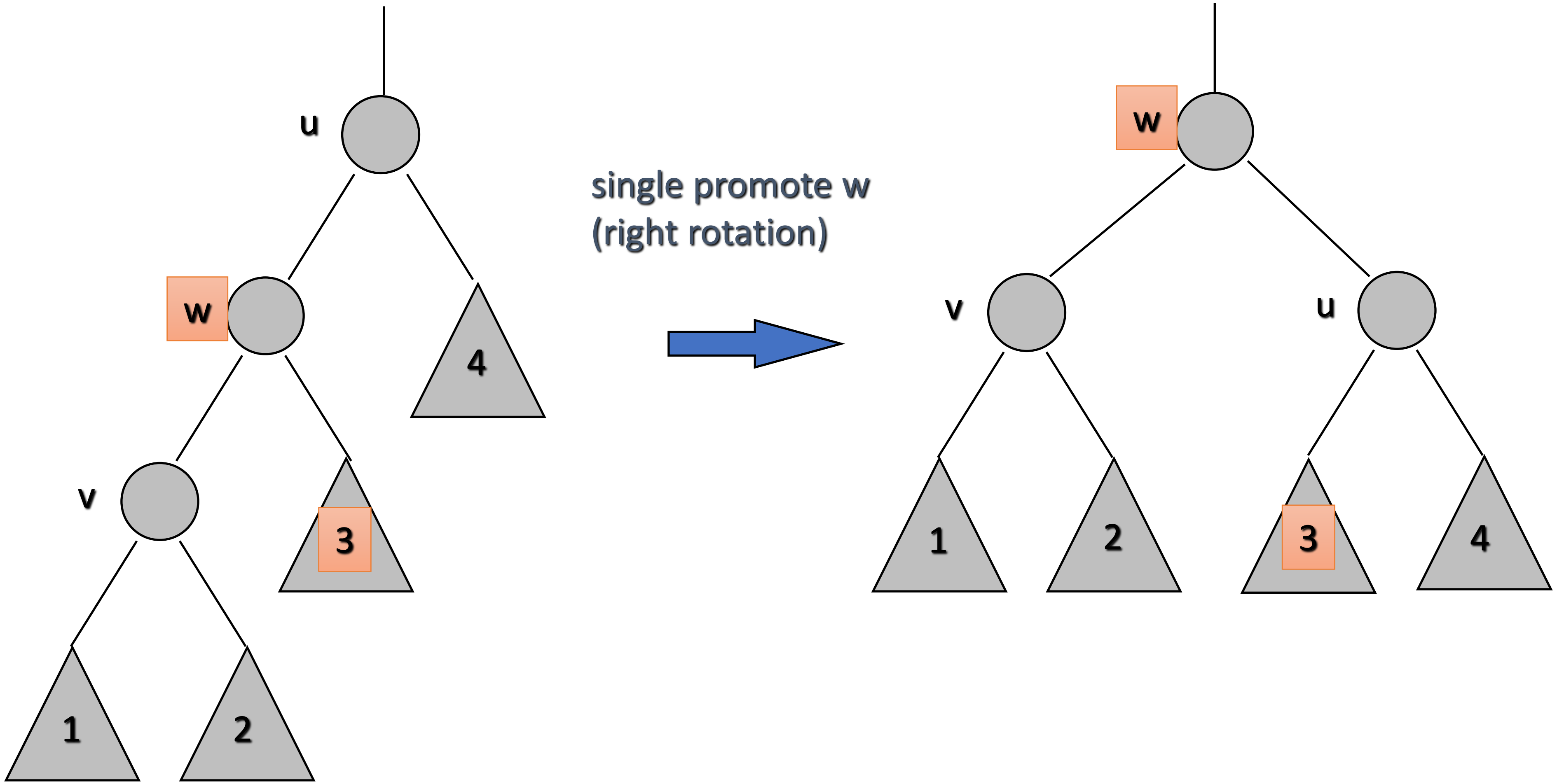


But how?

Red-Black Trees: zig-zag promote—left rotate (Step 1)

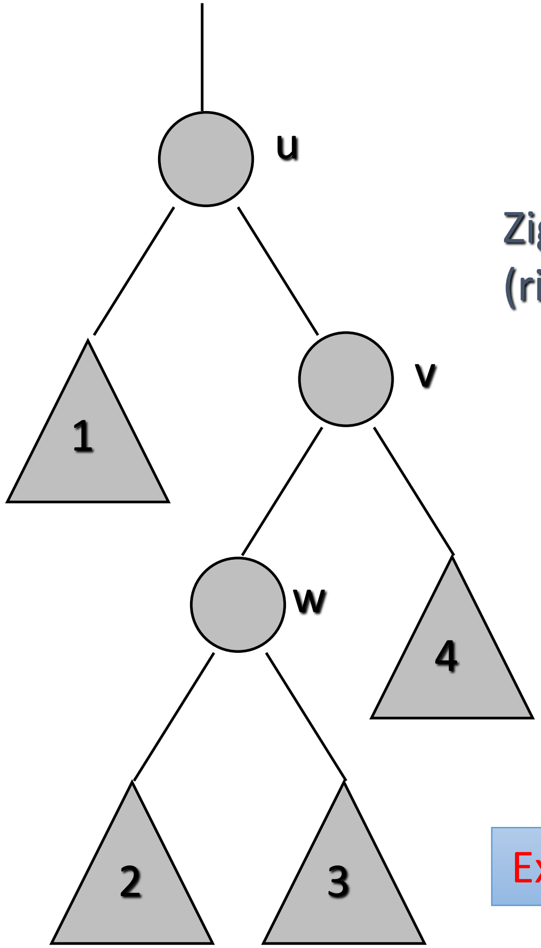


Red-Black Trees: zig-zag promote—right rotate (Step 2)



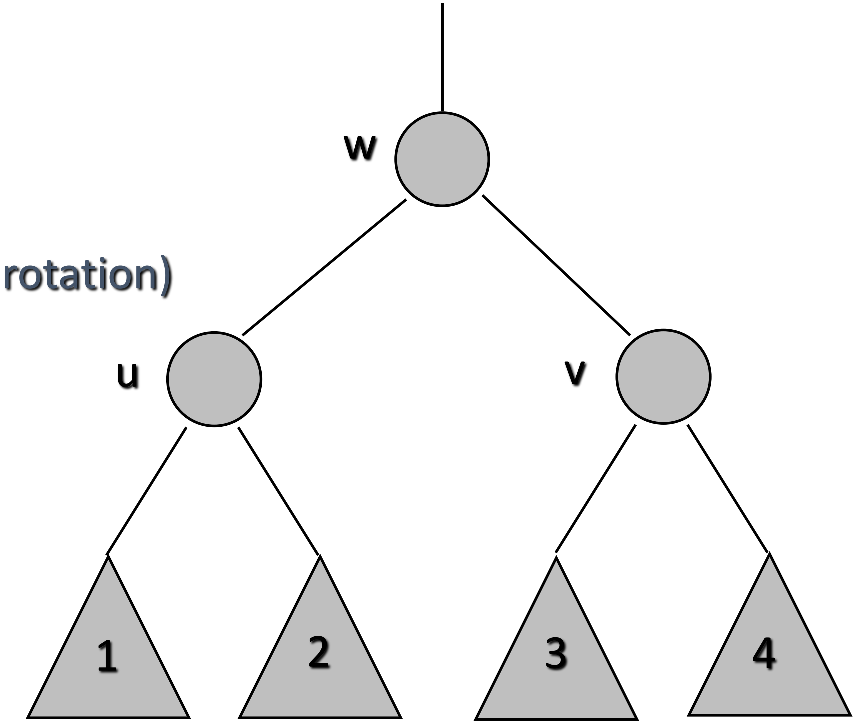
Red-Black Trees

Initial tree



Zig-zag promote w
(right rotation, then left rotation)

Tree after zig-zag promote



Exercise: Show the sequence of right rotation, left rotation steps.

Red-Black Trees

Insertions

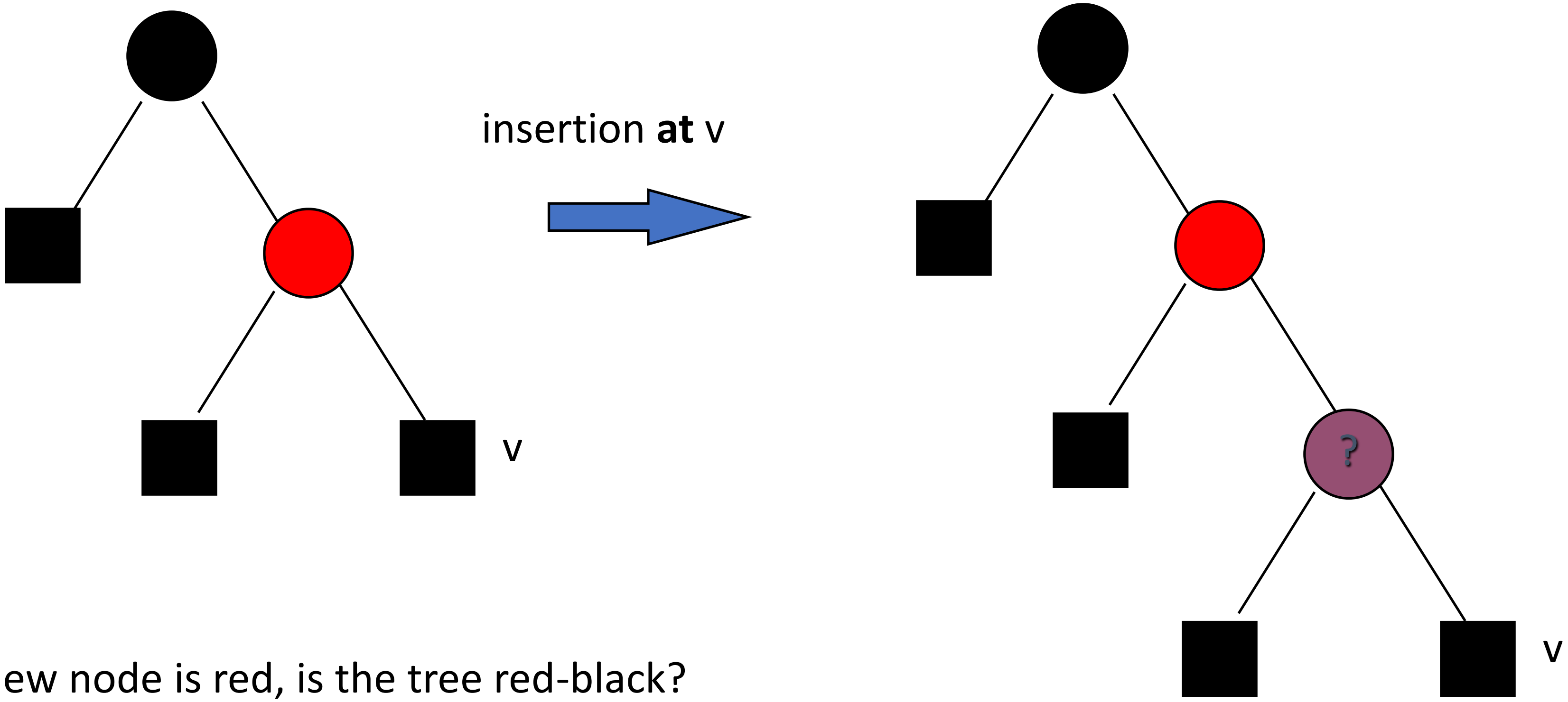
- A red-black tree can be searched in logarithmic time, i.e., O(log n), worst case
- Insertions may violate the red-black conditions necessitating restructuring
- This restructuring can also be performed in logarithmic time , i.e., O(log n).
- Thus, an insertion (or a deletion) can be performed in logarithmic time, i.e., O(log n).

Operation	Complexity
search(key)	O(log n)
insert(T, key)	O(log n)
restructure(T)	O(log n)
delete(T, key)	O(log n)

Red-Black Trees

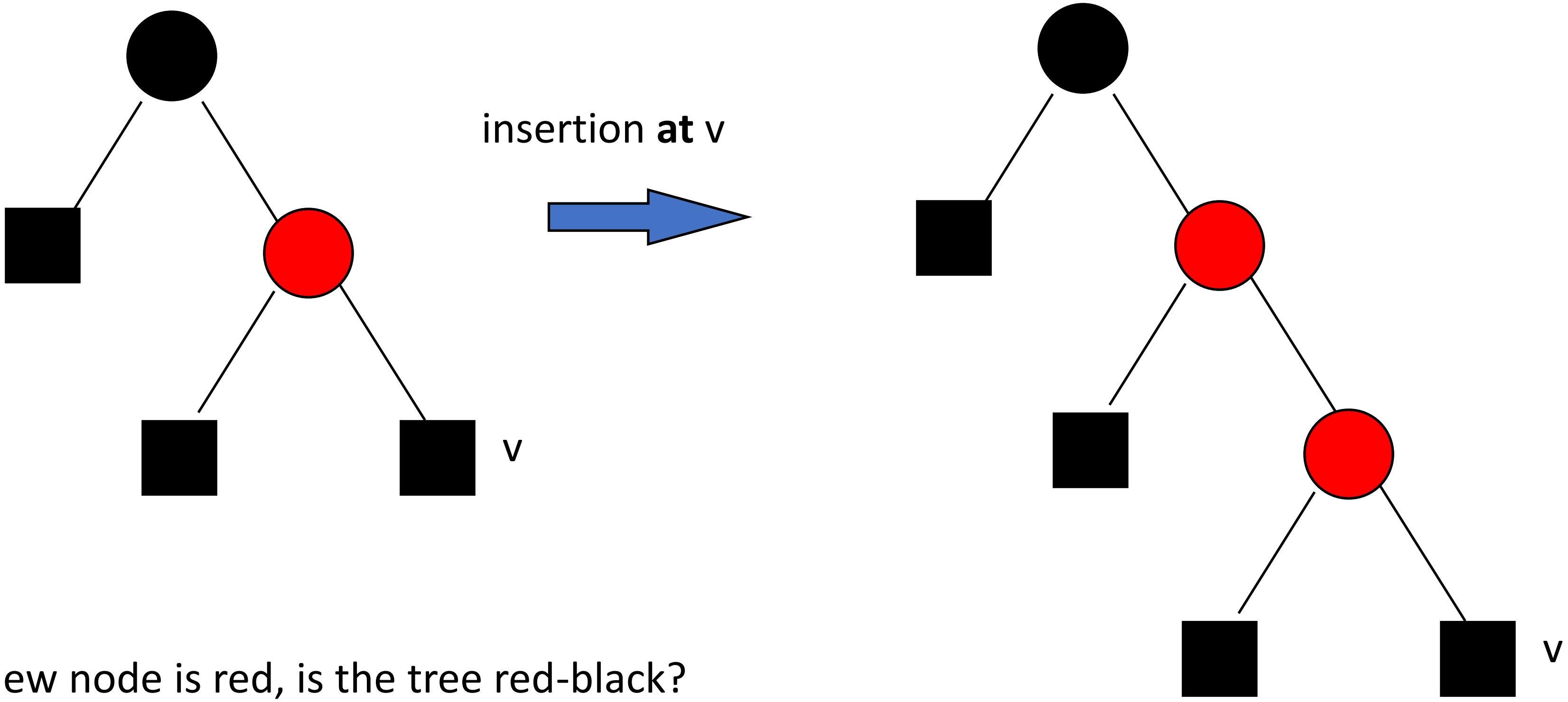
- Just as with AVL trees *(earlier),* we perform the insertion by
  - first searching the tree until an external node is reached (if the key is not already in the tree)
  - then inserting the new (internal) node
- We then have to recolour and restructure, if necessary.

Red-Black Trees



If new node is red, is the tree red-black?
If the new node is black, is the tree red-black?

Red-Black Trees

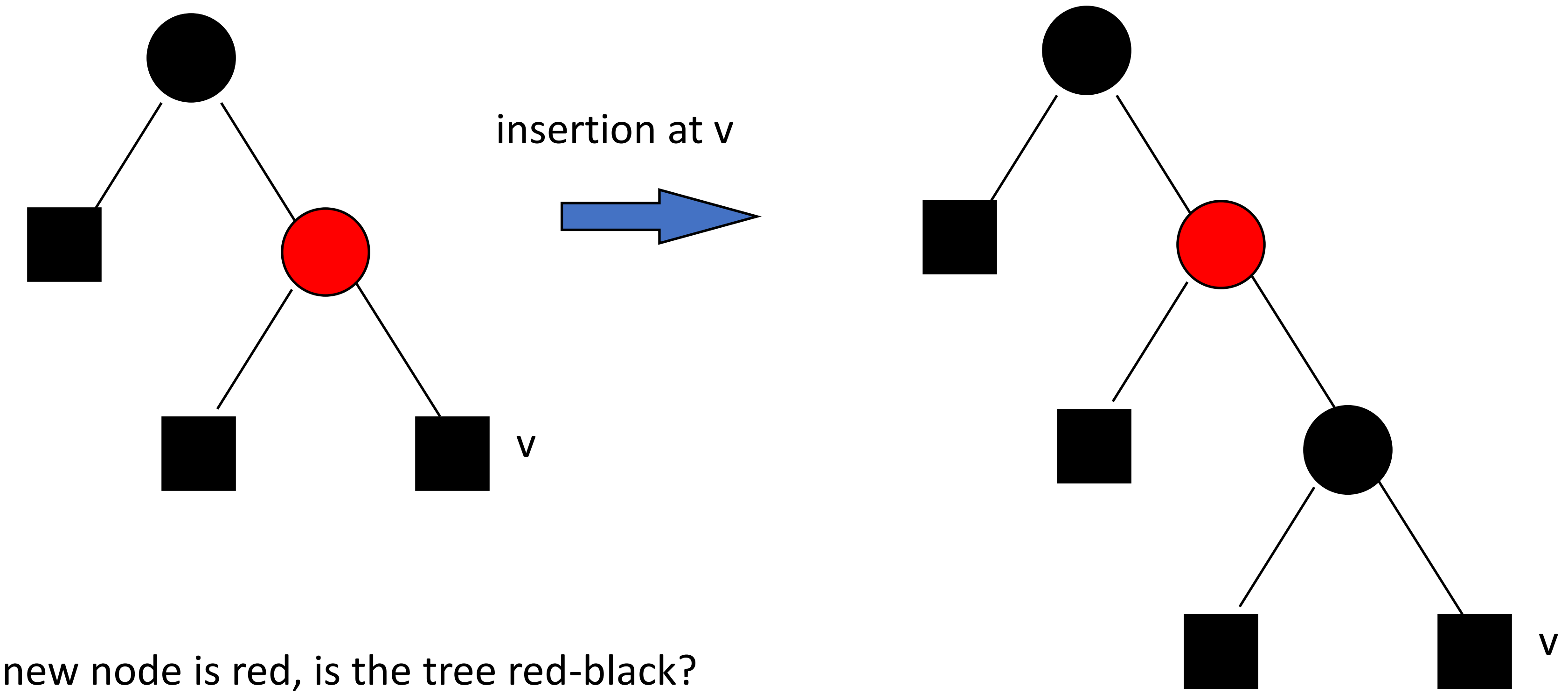


If new node is red, is the tree red-black?

No. Violates red condition---non-root red node has red parent.

If the new node is black, is the tree red-black?

Red-Black Trees



If new node is red, is the tree red-black?

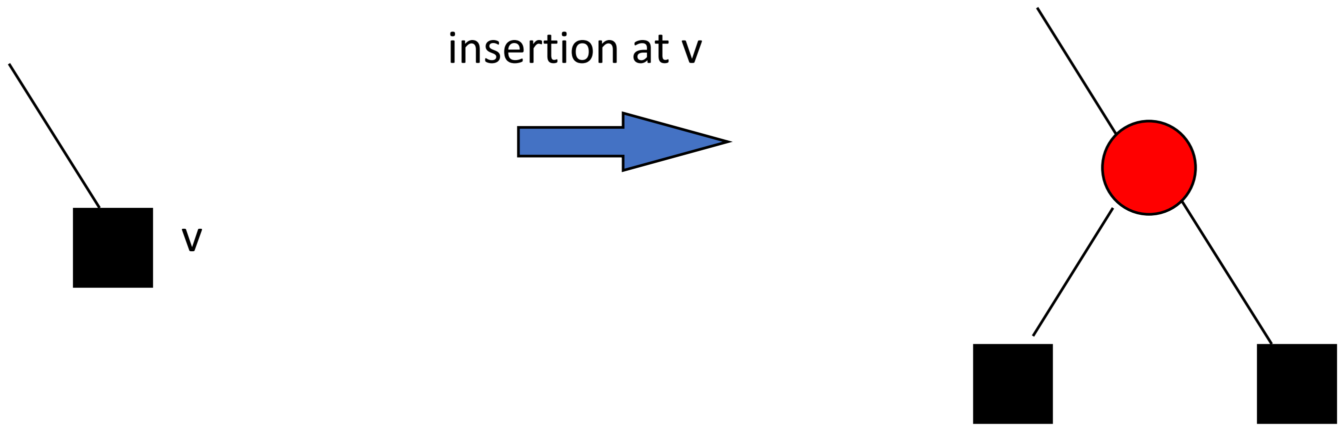
If the new node is black, is the tree red-black?

No. Violates black condition---some paths have 2 black nodes, others have 3.

Red-Black Trees

- Recolouring:
    - Colour new node red.
    - This preserves the black condition.
    - but may violate the red condition.
- Red condition can be violated only if the parent of an internal node is also red.
- Must transform this ‘almost red-black tree’ into a red-black tree.

Red-Black Trees



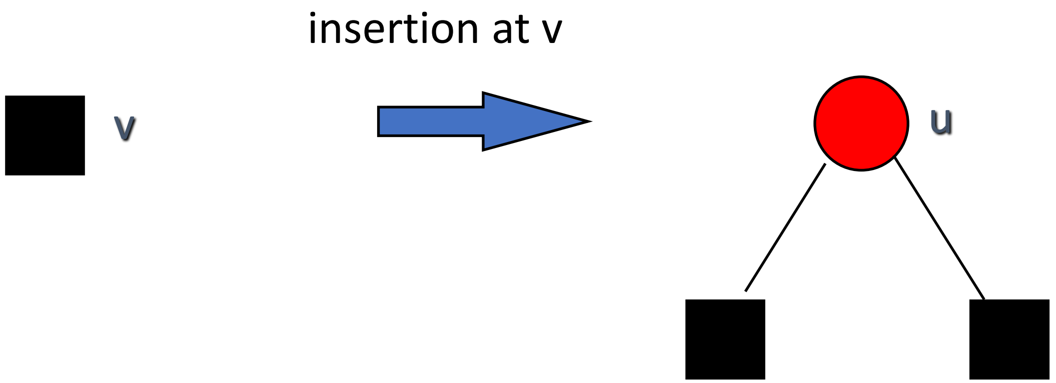
Red-Black Trees

- Recolouring and restructuring algorithm
  - The node u is a red node in a BST, T
  - u is the only candidate violating node
  - Apart from u, the tree T is red-black

Red-Black Trees

- Case 1:

  - u is the root
  - T is red-black

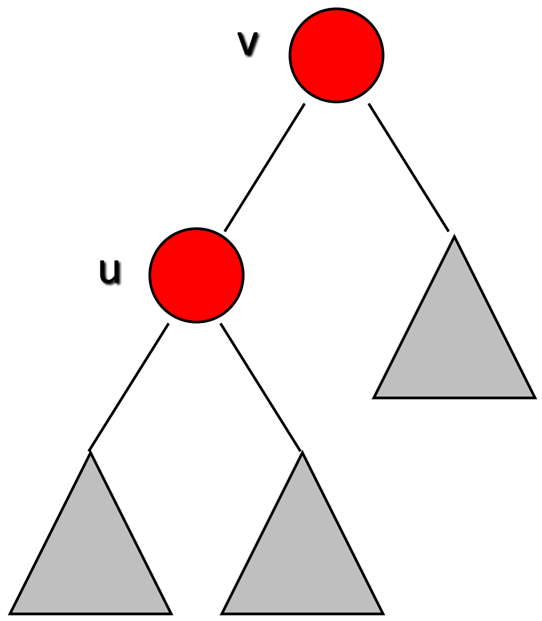


Red-Black Trees

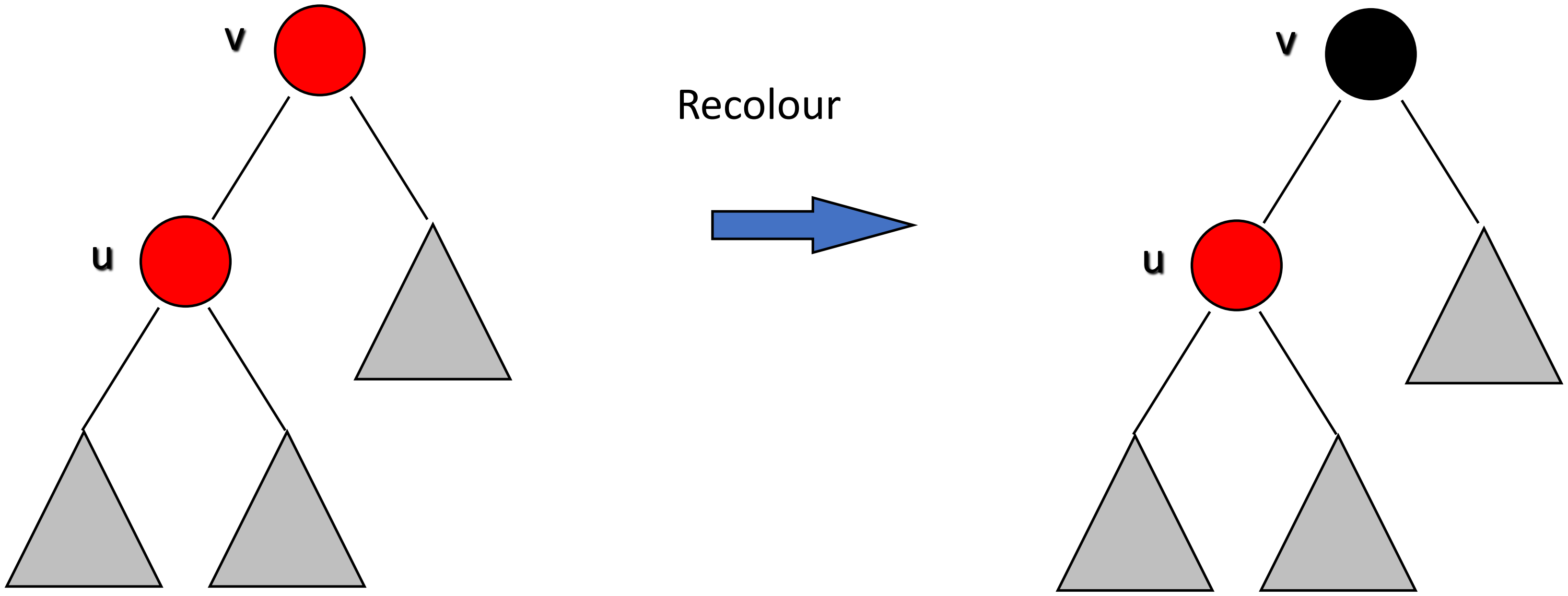
- Case 2:

  - u is not the root
  - its parent v is the root
  - **Colour v black**

    - Since v is the parent and the root,
      it is on the path to all external nodes
      and therefore, the black condition is satisfied

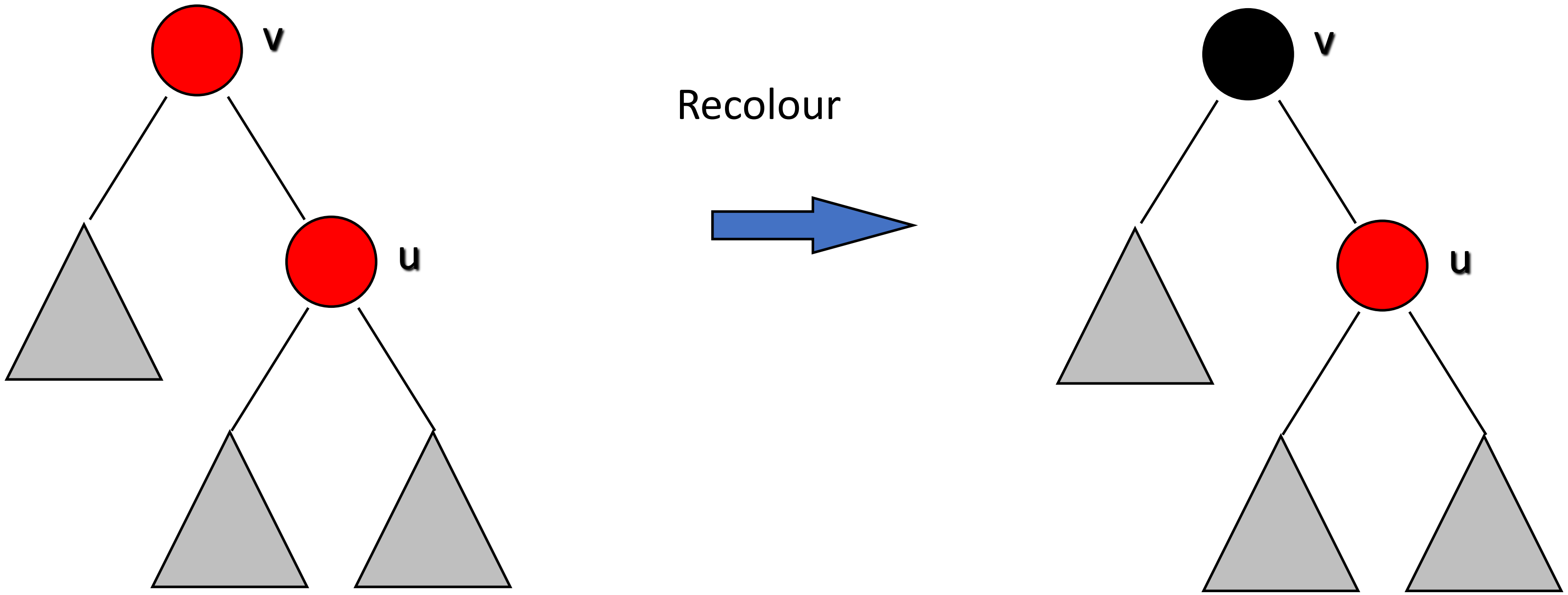


Red-Black Trees



Is there anything unexpected about this figure?

Red-Black Trees

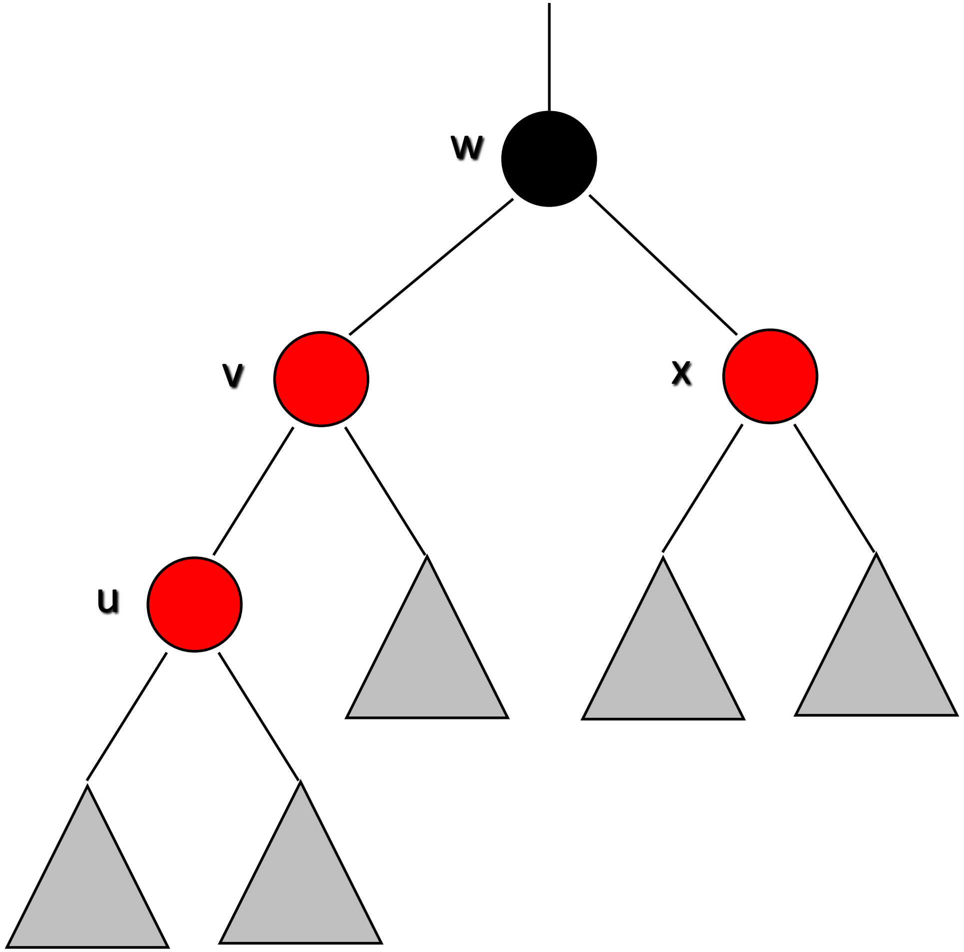


Is there anything unexpected about this figure?

Red-Black Trees

- Case 3:

  - u is not the root,
  - its parent v is not the root,
  - v is the left child of its parent w
  - (x is the right child of w, i.e., x is v's sibling)



Red-Black Trees

- Case 3.1:

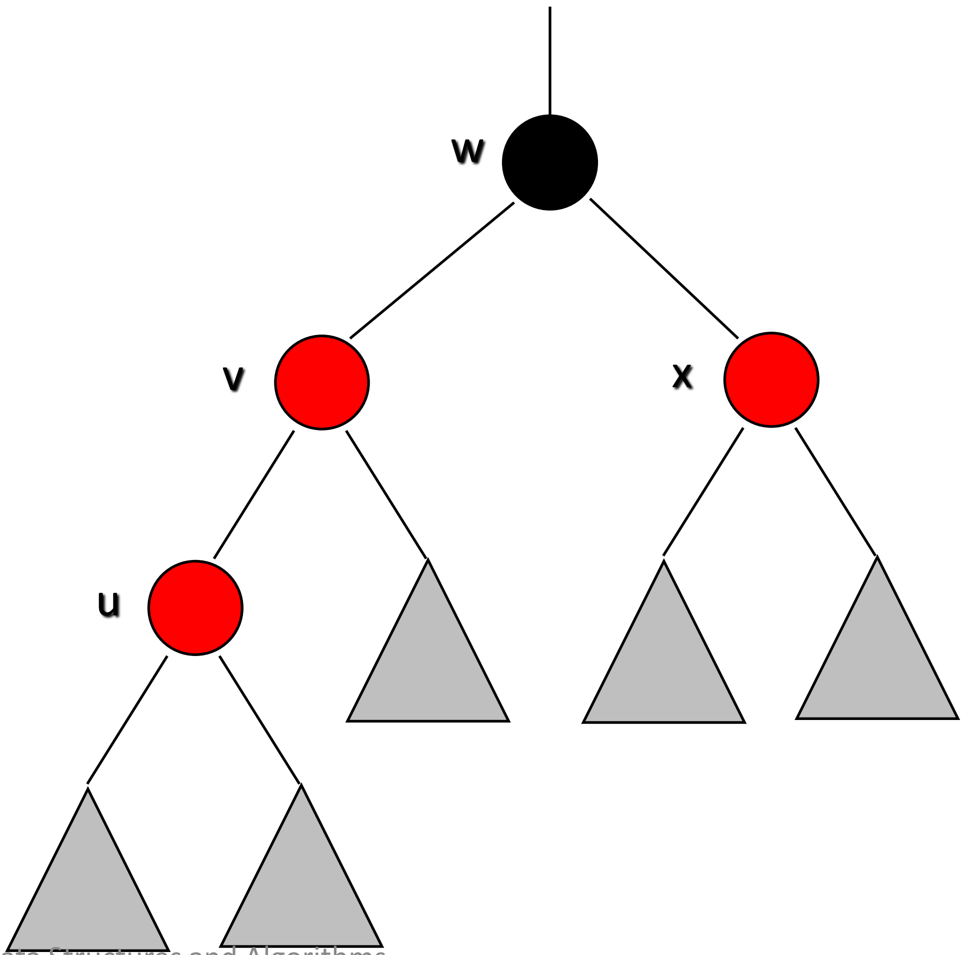
  - x is red

  - Colour v and x black and w red

  - Now repeat the restructuring with u := w

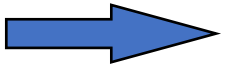
    (since the recolouring of w to red may cause a **red violation**)

Red-Black Trees



Note:
w must be black,
v must be red,
u must be red.
Why?

Recolour



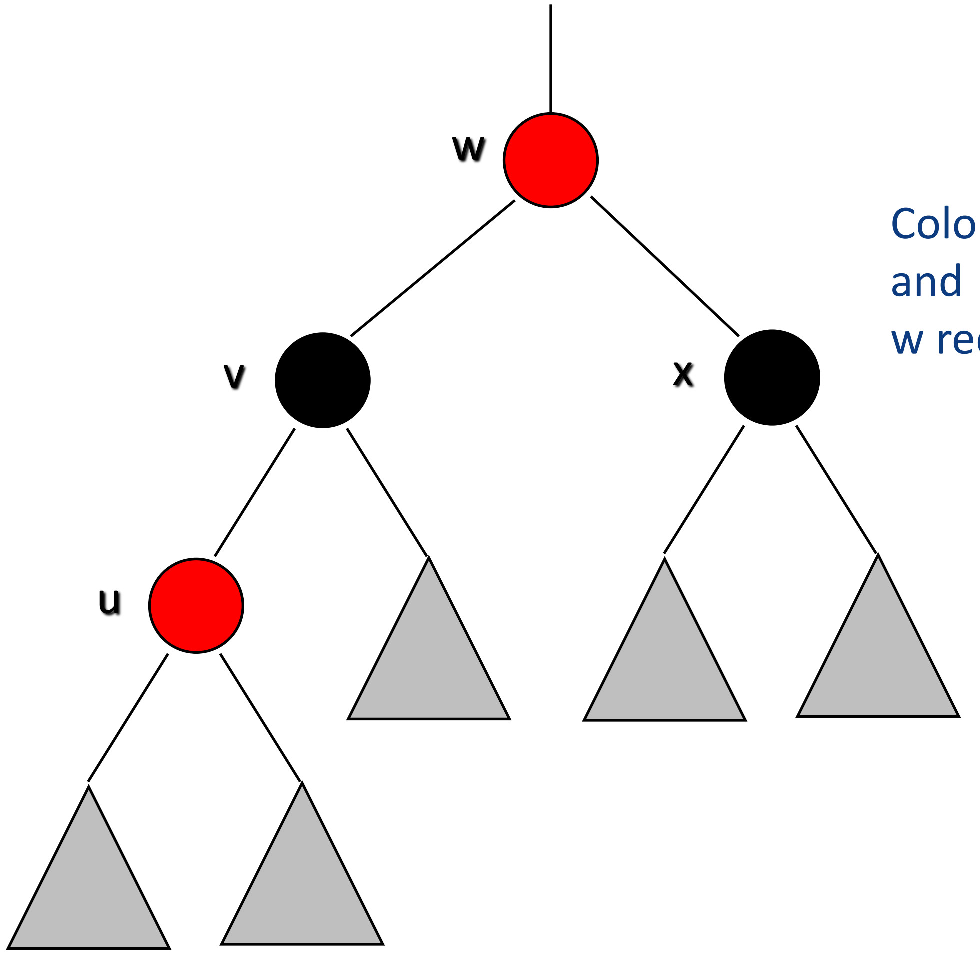
Red-Black Trees

- u must be red because we colour new nodes that way by convention (to preserve the **black condition**).

- v must be red because otherwise it would be black and then we wouldn’t have violated the red condition and we wouldn’t be restructuring anything!

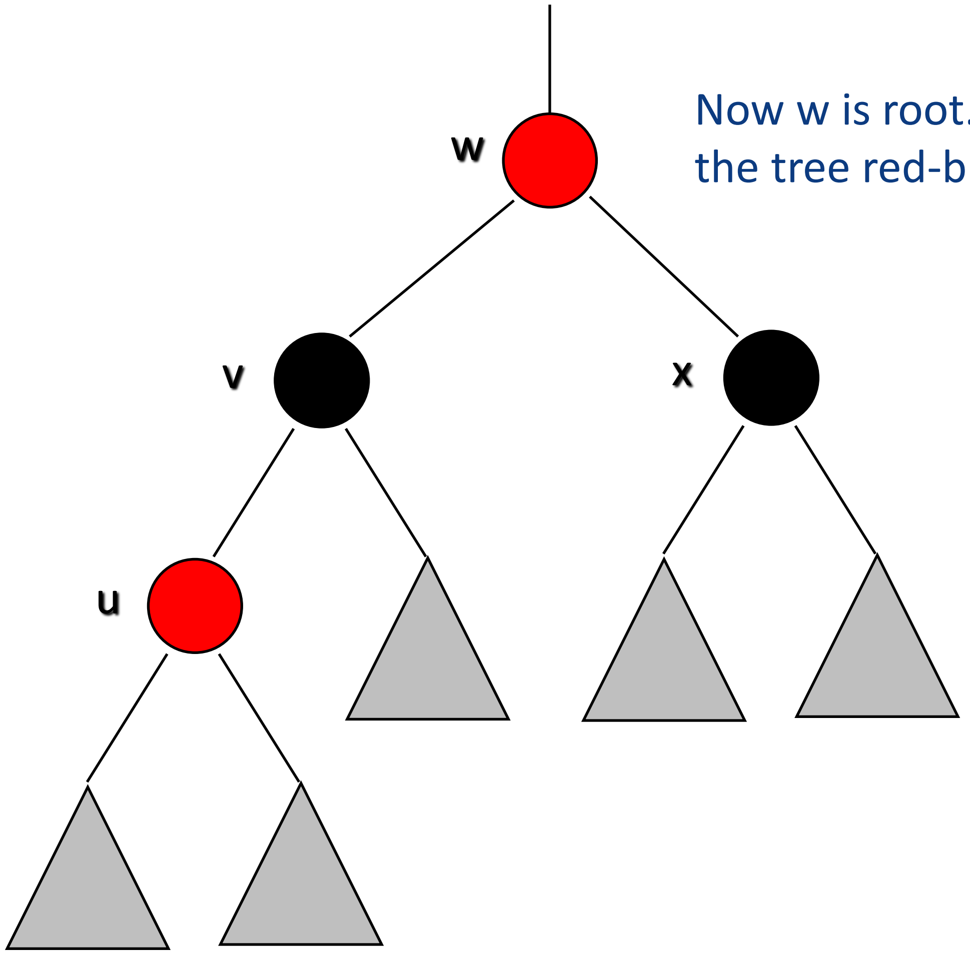
- w must be black because every red node (that isn’t the root) has a black parent (and x is red so w must be black).

Red-Black Trees



Colour v and x black
and
w red

Red-Black Trees



Now w is root. Is the tree red-black?

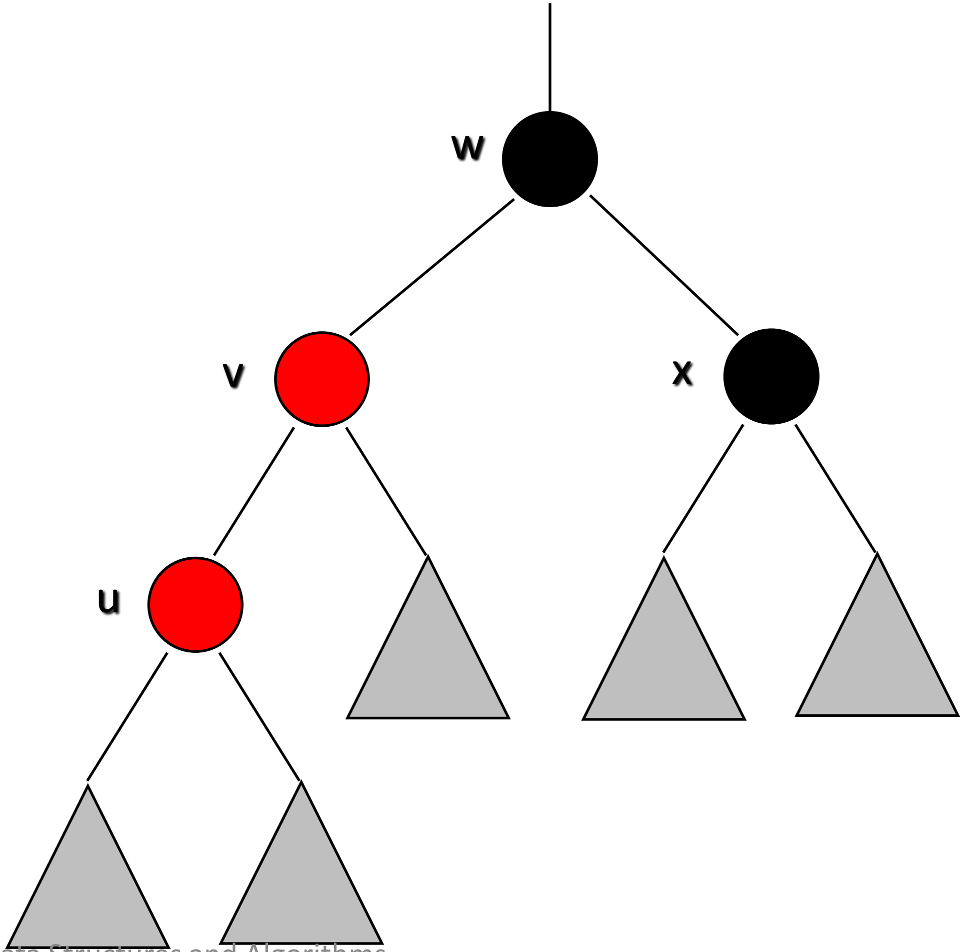
1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)
3. Each external node (leaf node) is **black.**
4. Every node must have one color –**red or black.**
5. Root node is **black**.

Red-Black Trees

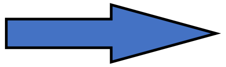
- Case 3.2:

  - x is black
  - u is the left child of v
  - Promote v
  - Colour v black
  - Colour w red

Red-Black Trees

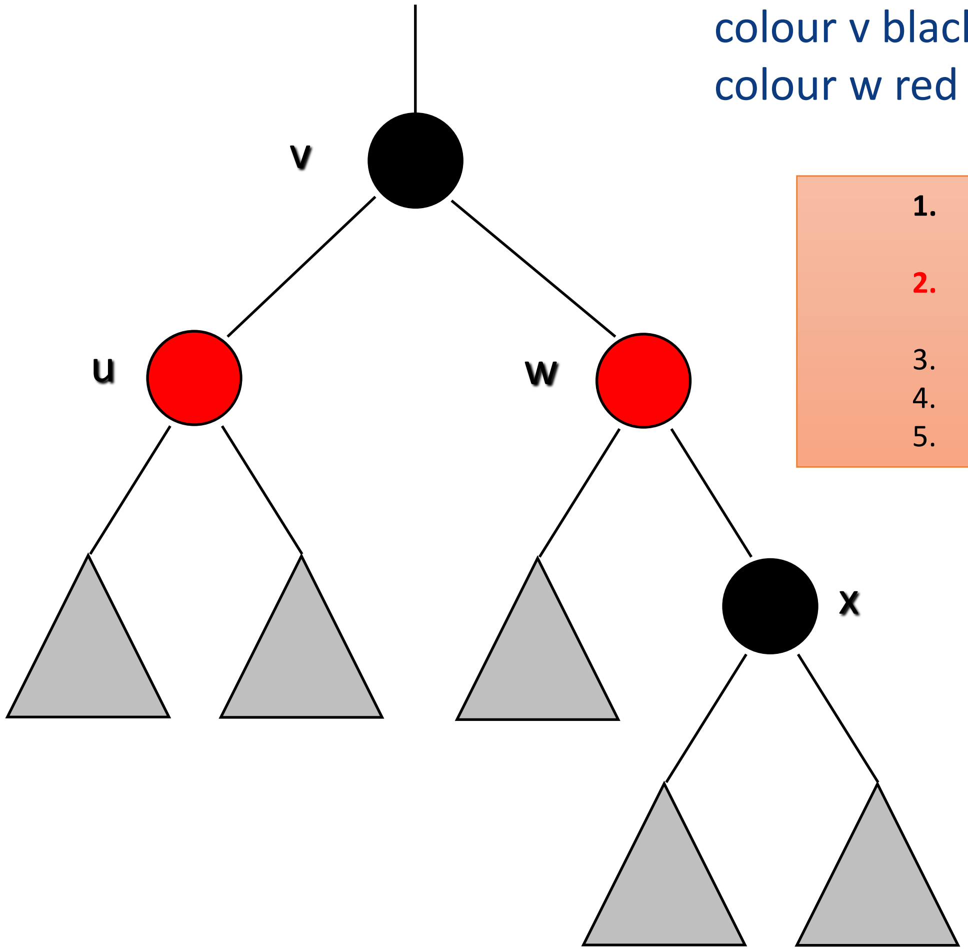


Restructure and recolour



Promote v (right rotation);
colour v black;
colour w red

Red-Black Trees



colour v black;
colour w red

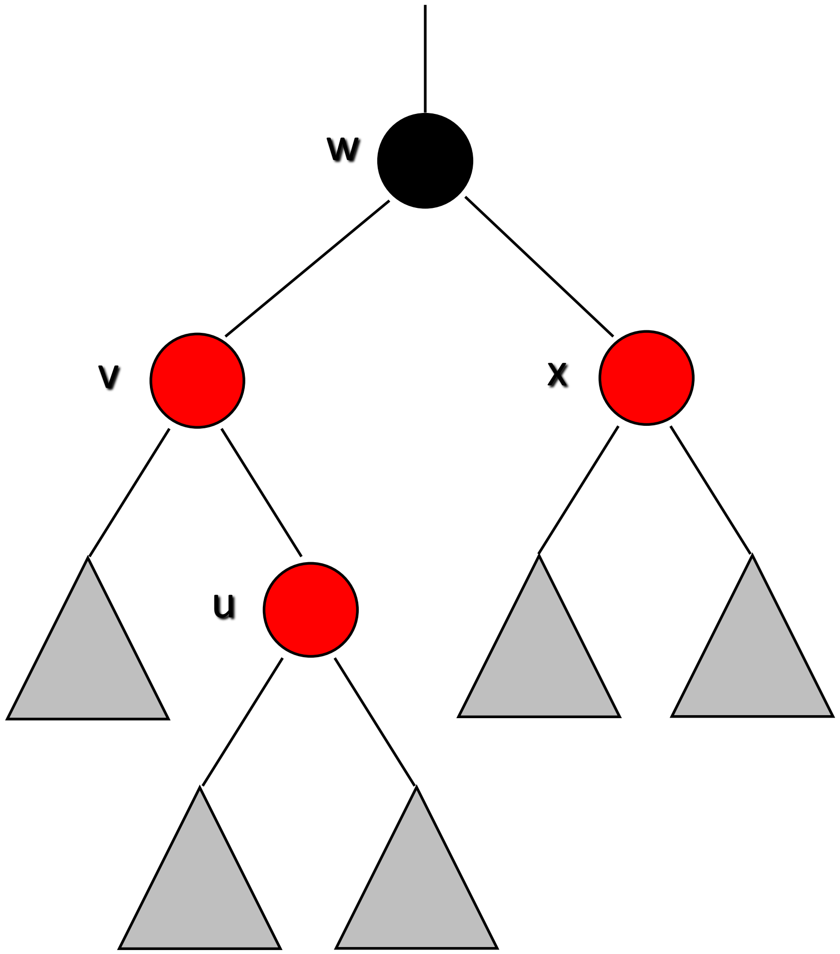
1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)
3. Each external node (leaf node) is **black.**
4. Every node must have one color –**red or black.**
5. Root node is **black**.

Red-Black Trees

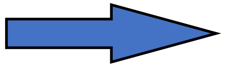
- Case 3.3:
  - x is red
  - u is the right child of v
  - Colour v and x black
  - Colour w red
  - Repeat the restructuring with u := w

    (since the recolouring of w to red may cause a red violation)

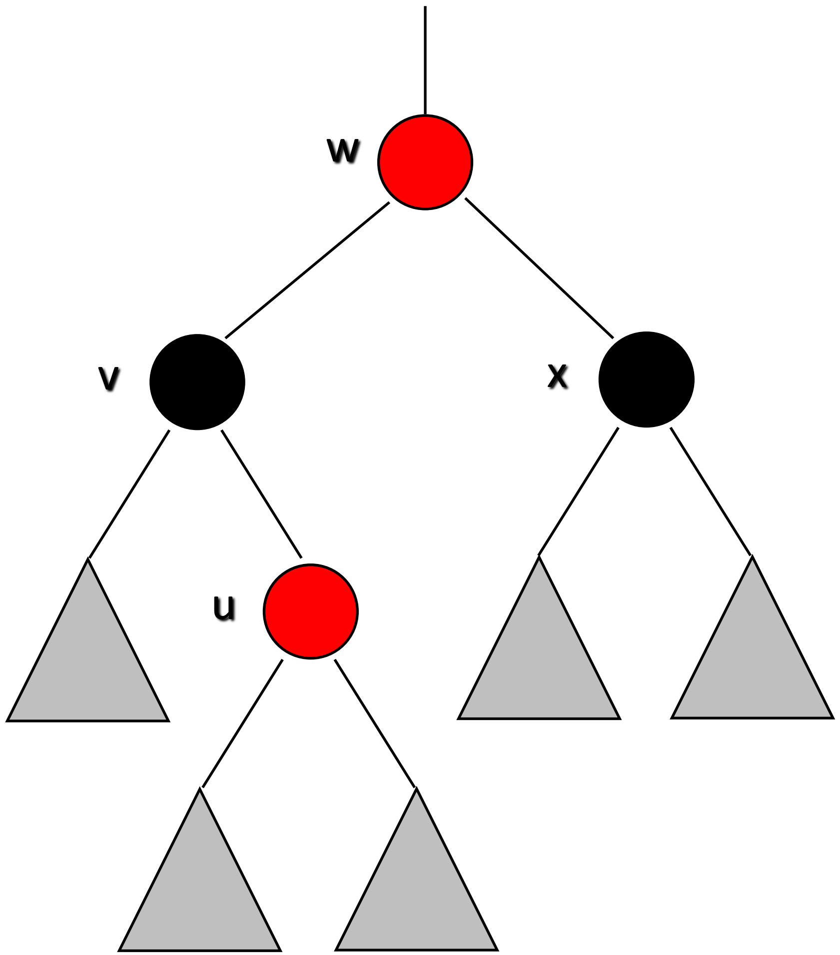
Red-Black Trees



Recolour



Red-Black Trees



Colour v and x black
Colour w red

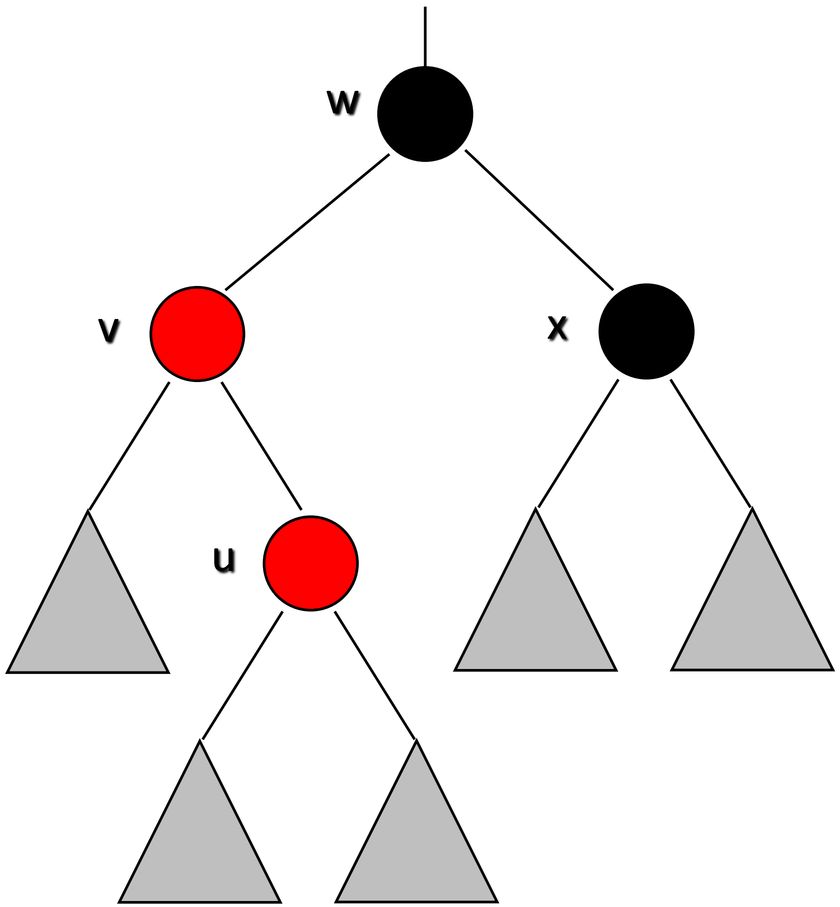
Repeat the restructuring with u := w

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)
3. Each external node (leaf node) is **black.**
4. Every node must have one color –**red or black.**
5. Root node is **black**.

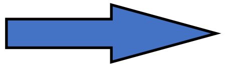
Red-Black Trees

- Case 3.4:
  - x is black
  - u is the right child of v
  - Zig-zag promote u
  - Colour u black
  - Colour w red

Red-Black Trees

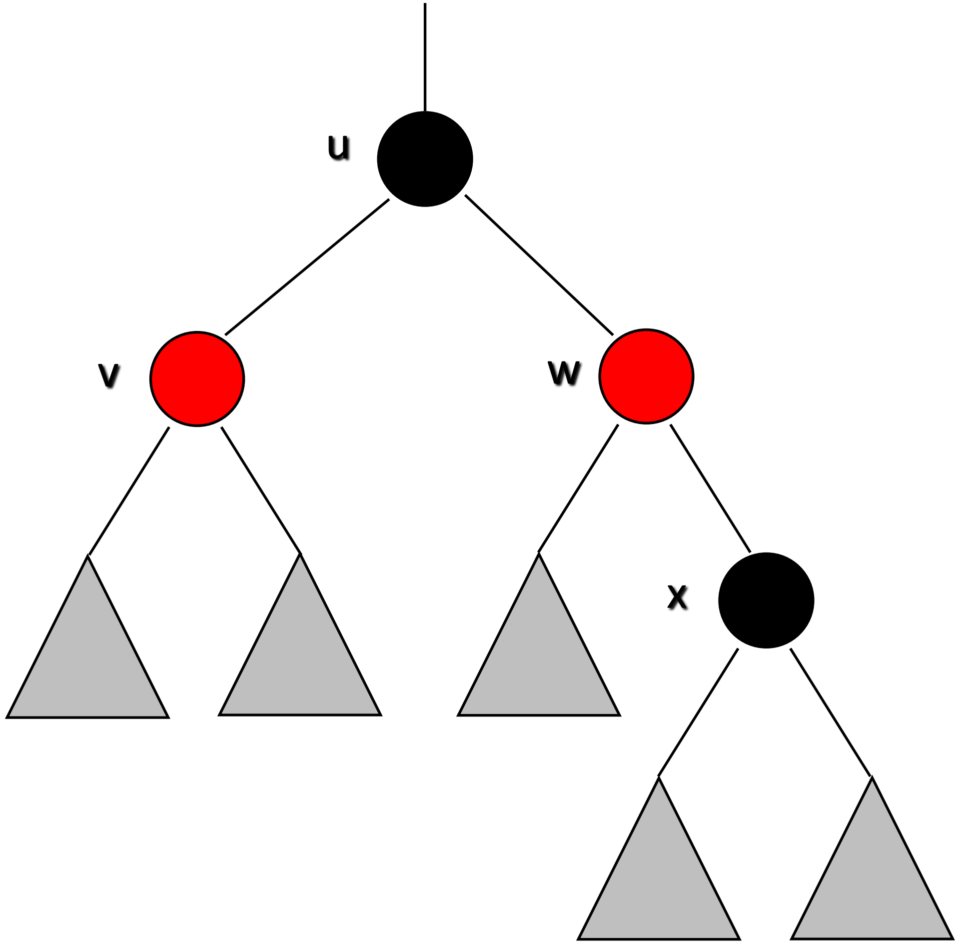


Restructure and recolour



Zig-zag promote u;
colour u black;
colour w red

Red-Black Trees



colour u black;
colour w red

1. **Black condition**: Each root-to-frontier path contains exactly the *same number of black nodes*
2. **Red condition**: *Each red node* that is not the root *has a black parent* (If a node is red, both children are black)
3. Each external node (leaf node) is **black.**
4. Every node must have one color –**red or black.**
5. Root node is **black**.

Red-Black Trees

- Case 4:

  - u is not the root,
  - its parent v is not the root,
  - v is the **right** child of its parent w
  - (x is the **left** child of w, i.e. x is v's sibling)

- This case is symmetric to case 3.

Deletion

- Deletion of node from tree
  - Deleting a node from a Red-Black tree may also need node adjustment
  - Any violation of Red-Black tree properties can also be corrected using a combination of rotation and color change operations.

- *Reading Exercise*: See the 4 cases in Section 13.4 of Introduction to Algorithms 3rd Edition by Cormen et al (2009)

Applications of Red Black Trees(1/2)

- In general, where sorting, searching , or hashing algorithms are needed.
- E.g.:
  - Implementation of sets and maps in C++ Standard Templates Library.
  - TreeMap (TreeMap (Java Platform SE 7) (oracle.com)) implemented using Red Black trees, TreeSet—based on TreeMap(TreeSet (Java Platform SE 7) (oracle.com)), and Hashmap in Java collections.
  - Computational geometry e.g., geometric range searches (Geometric range searching | ACM Computing Surveys)
  - k-means clustering

Applications of Red Black Trees(2/2)

- In general, where sorting, searching , or hashing algorithms are needed.
- E.g.:
  - Text-mining.
  - mmap and munmap operations for file/memory mapping in Linux.
  - Searching on the web.
  - Completely Fair Scheduler in Linux.
  - Searching in a dictionary.
  - Databases.

Summary

- Red black trees are height balanced.
- Search, insertion, and deletion have a worst-case complexity of O(log n).
- Insertion and deletions may lead to restructuring (promotions) and recoloring (as per the case). Insertions can lead to ten (10) distinct cases.
- Red black trees can be used in applications that require searching, sorting, and hashing algorithms.