Problem 1)

We have been thinking carefully about how to set up relativistic wavepackets, which will be important when we get to a formal treatment of the S-matrix and scattering theory. We used wavepacket functions $\tilde{f}(\boldsymbol{p})$ and $\tilde{g}(\boldsymbol{p})$ to describe the three-momentum parts of initial and final wavepackets. I also suggested using

$$F(t-t_0,\Delta t)=\frac{1}{\sqrt{\pi}\Delta t}e^{-(t-t_0)^2/\Delta t^2} \tag{1}$$

for the temporal part of the wavepacket. To make analyzing the propagation amplitude

$$\langle g;\text{out};\Delta|f;\text{in};\Delta\rangle \tag{2}$$

more manageable, write out an expression for it that only involves integrals over d^4p, $\mathrm{d}x_0$, and $\mathrm{d}y_0$. The integrand should only involve $\tilde{f}(\boldsymbol{p})$, $\tilde{g}(\boldsymbol{p})$, $1/(p^2-m^2+i\epsilon)$, and $F(t-t_0,\Delta t)$.

In the free, complex Klein-Gordon theory, we set up a simple scattering amplitude, for a particle or anti-particle wave-packet to evolve into some other particle or anti-particle wave-packet, respectively, at some much later time. Generically, we found that

$$\langle g;\text{out};\Delta|f;\text{out};\Delta\rangle=\int \mathrm{d}^4x\,\mathrm{d}^4y\,\underline{g}^*(y)\underline{f}(x)\,\langle 0|T\phi(y)\phi^\dagger(x)|0\rangle\,. \tag{3}$$

The time-ordering handily takes care of whether we consider particles or anti-particles, and the propagator

$$\langle 0|T\phi(y)\phi^\dagger(x)|0\rangle=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(y-x)}}{p^2-m^2+i\epsilon}. \tag{4}$$

Recall that we can write

$$\underline{f}(x)=2i\frac{\partial f(x)}{\partial t}F(x^0-\bar{x}^0,\Delta x^0), \tag{5}$$

where

$$f(x)=\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\tilde{f}(\boldsymbol{p})e^{-ip\cdot x} \tag{6}$$

and similarly for $\underline{g}$. If we plug the propagator and the wave-packet expressions into the scattering amplitude, we find

$$\begin{aligned}\langle g|f\rangle=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}&\left\{\int \mathrm{d}^4y\,2i\frac{\partial g(y)}{\partial t}G(y^0-\bar{y}^0,\Delta y^0)e^{ip\cdot y}\right\}^*\\ &\times\left\{\int \mathrm{d}^4x\,2i\frac{\partial f(x)}{\partial t}F(x^0-\bar{x}^0,\Delta x^0)e^{ip\cdot x}\right\}.\end{aligned} \tag{7}$$

Problem 2)

In our treatment of the charged (complex) Klein-Gordon field, we have treated ϕ and $\phi^\dagger$ as if they are independent fields, even though knowledge of ϕ determines $\phi^\dagger$. Give a more rigorous explanation for why this is reasonable than what we did in class.

Problem 3)

Repeat the steps for quantizing the charged Klein-Gordon field, but now impose anticommutation relations on the fields rather than commutation relations. Why would one consider trying this in the first place? What happens to the Hamiltonian?

Problem 4)

Checking steps from class:

(a) In class, I went through the steps for setting up the Dirac field and showing that it gives the Dirac equation quite fast. Fill in the steps for both the symmetrized and non-symmetrized form of the Lagrangian.