Problem 1)

Consider the classical complex Klein-Gordon field with the Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*. \tag{1}$$

This has a global symmetry under a phase transformation $\phi \to \phi e^{i\alpha}$. Determine the Noether current corresponding to this symmetry.

Problem 2)

Checking steps from class:

(a) Consider again the basic simple harmonic oscillator from undergraduate quantum mechanics (a.k.a. the $0+1$ scalar QFT). Show that converting the creation and annihilation operators from the Schrödinger to Heisenberg pictures gives

$$\hat{a}_H(t) = e^{-i\omega t}\hat{a}(t=0), \quad \hat{a}^\dagger_H = e^{i\omega t}\hat{a}^\dagger(t=0). \tag{2}$$

Note: I will always assume $\hbar = c = 1$. The H subscript means "Heisenberg operator".

(b) Recall that in treating the 1D lattice theory in class, I used the identity

$$\sum_j e^{ikja} = N\delta_{k_0}. \tag{3}$$

Prove this expression for a general N.

Problem 3)

Repeat the steps from class in constructing a classical lattice field theory in D dimensions, but now include a nonlinear term as follows:

$$H = \sum_x^{N^D} \frac{\dot{q}_x^2}{2} + \sum_x^{N^D} \sum_\nu \frac{\kappa}{2}(q_{x+\nu} - q_x)^2 + \sum_x^{N^D} \frac{m^2}{2} q_x^2 + \frac{\lambda}{4!}\sum_x^{N^D} q_x^4, \tag{4}$$

where the constant λ determines the strength of the effect of the nonlinear term. For taking the continuum limit, make the same replacements I used in class, but also take $\lambda \to g/a^D$, where g is a continuum version of λ. What Hamiltonian density do you get? What is the corresponding Lagrangian density? Can you solve the quantum version of the theory again by just using a's and $a^\dagger$'s as in the linear case? If not, what prevents you from doing so? In units where $\hbar = c = 1$, what are the units of g?

Problem 4)

Show that the following Lagrangian density gives a nonrelativistic classical field that at least structurally matches the form of a single particle Schrödinger equation,

$$\mathcal{L} = \frac{i}{2}\psi^\dagger(\boldsymbol{x})\frac{\overleftrightarrow{\partial}}{\partial t}\psi(\boldsymbol{x}) - \frac{1}{2m}\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{x})\cdot\boldsymbol{\nabla}\psi(\boldsymbol{x}) - V(\boldsymbol{x})\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}). \tag{5}$$

What is the Hamiltonian density? In light of our discussion about the problems with second time derivatives when constructing relativistic wavefunction equations, what is noteworthy about this Hamiltonian?