

Problem 1)

Consider the classical Lagrangian densities for the following relativistic quantum field theories,

$$\mathcal{L}_Y = \frac{i}{2} \bar{\psi} \overleftrightarrow{\not{D}} \psi - M \bar{\psi} \psi + \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m_s^2 \phi^2 - g \bar{\psi} \psi \phi - \frac{\lambda_1}{3!} \phi^3 - \frac{\lambda_2}{4!} \phi^4 \quad (1)$$

$$\mathcal{L}_V = \frac{i}{2} \bar{\psi} \overleftrightarrow{\not{D}} \psi - M \bar{\psi} \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m_V^2}{2} A^\mu A_\mu - g \bar{\psi} \not{A} \psi. \quad (2)$$

We could envision each of these being proposed as models of the interactions between spinor “nucleons” of mass M represented by the Fermi field ψ . In the first case, the interaction is then mediated by a scalar “pion” field ϕ with mass m_s , and in the second it is mediated by a vector field A^μ with mass m_V . As usual, the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. (Note that to be realistic we should really have pseudoscalar and vector interactions.) In the second theory, we would get the Maxwell field if we set $m_V = 0$.

- (a) Use the fast mnemonic that we developed in class for translating a classical Lagrangian density into QFT Feynman rules to write down all the Feynman rules for the two theories above. Make a comment about where each factor of “ i ” comes from. Use straight lines for ψ , dashed lines for ϕ , and wavy lines for A^μ .
- (b) Draw a two-loop diagram for the vector field case. Draw an example of a diagram that would give problems if you have not worried about the “reduction” of external leg states.
- (c) What happens if we then place $m_V = 0$ in the vector field? A standard way to deal with the problem is to replace $\frac{m_V^2}{2} A_\mu A^\mu \rightarrow -\frac{1}{2\xi} (\partial_\mu A^\mu)^2$. This effectively uses the Lagrange multiplier technique to fix the Lorenz gauge condition $\partial_\mu A^\mu = 0$. What are the Feynman rules if I make this replacement? What if I further specify the gauge by fixing $\xi = 1$, where ξ is a real constant that will be chosen later?
- (d) Using Eq. (1), draw all the Feynman diagrams that would contribute to the $2 \rightarrow 2$ cross section, $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$ through order g^2 and $g^2\lambda_1^2$.
- (e) Still using Eq. (1), consider the cross section for the $2 \rightarrow 2$ scattering process $\psi\psi \rightarrow \psi\psi$. Start with the general cross section expression derived in class,

$$d\sigma = \frac{|M|^2}{2\sqrt{\lambda(s, m_A^2, m_B^2)}} \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3\mathbf{p}_N}{(2\pi)^3 2E_N} (2\pi)^4 \delta\left(k_A + k_B - \sum_{i=1}^N p_i\right), \quad (3)$$

with

$$\lambda(s, m_A^2, m_B^2) = s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 2m_A^2 m_B^2, \quad (4)$$

and derive the order g^2 expression for the unpolarized differential cross section $d\sigma/d\Omega|_{\text{CM}}$ in the center-of-mass system. Since it is an unpolarized cross section,

you should sum over the final and average over initial nucleon spins. Let p_A and p_B label the initial four-momenta and p_C and p_D label the final four-momenta and express your result in terms of Mandelstam variables.

(a) Let us rewrite the Yukawa and vector Lagrangians as follows:

$$\mathcal{L}_Y = \mathcal{L}_{D,\text{sym}} + \mathcal{L}_{\text{KG}} - g\bar{\psi}\psi\phi - \frac{\lambda_1}{3!}\phi^3 - \frac{\lambda_2}{4!}\phi^4 \quad (5)$$

$$\mathcal{L}_V = \mathcal{L}_{D,\text{sym}} + \mathcal{L}_P - g\bar{\psi}\not{A}\psi, \quad (6)$$

where $\mathcal{L}_{D,\text{sym}} = \bar{\psi}(\frac{i}{2}\overleftrightarrow{\not{D}} - M)\psi$ is the free Dirac Lagrangian for spin-1/2 spinor fields, $\mathcal{L}_{\text{KG}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_s^2}{2}\phi^2$ is the free Klein-Gordon Lagrangian for real scalar particles, and $\mathcal{L}_P = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_V^2}{2}A^\mu A_\mu$ is the free Proca Lagrangian for a spin-1 vector particles. The vertices are simple enough to read from the Lagrangian since they are just the coefficients without any factor included for symmetries multiplied by i . We can determine the propagators easily enough from the mnemonic. For the KG scalar propagator, we list out the steps as follows:

$$D = [(ip_\mu)(-ip^\mu) - m_s^2] = p^2 - m_s^2 \rightarrow D^{-1} = \frac{1}{p^2 - m_s^2 + i\epsilon} \rightarrow \text{Prop} = \frac{i}{p^2 - m_s^2 + i\epsilon}. \quad (7)$$

Similarly, we can determine the Dirac propagator

$$D = \frac{i}{2}\gamma^\mu[(-ip_\mu) - (ip_\mu)] - M = \not{p} - M \rightarrow \text{Prop} = \frac{i}{\not{p} - M} = \frac{i(\not{p} + M)}{p^2 - M^2 + i\epsilon}. \quad (8)$$

For the Proca theory, we must massage the Lagrangian a bit to arrive at a form where we can use the mnemonic properly:

$$\begin{aligned} \mathcal{L}_P &= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m_V^2}{2}A^\mu A_\mu \\ &= -\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\nu A^\mu \partial_\mu A_\nu) + \frac{m_V^2}{2}A^\mu A_\mu \\ &= A^\mu \left[-\frac{1}{2}(g_{\mu\nu}\overleftrightarrow{\partial}^\rho \partial_\rho - \overleftrightarrow{\partial}_\nu \partial_\mu) + \frac{m_V^2}{2}g_{\mu\nu} \right] A^\nu. \end{aligned} \quad (9)$$

Thus,

$$D_{\mu\nu} = -(p^2 - m_V^2)g_{\mu\nu} + p_\mu p_\nu. \quad (10)$$

Note that the inverse here is not simply the reciprocal, but we have $D_{\mu\nu}(D^{-1})^{\nu\rho} = \delta_\mu^\rho$. We can form the inverse as a linear combination of the tensor structures available to us as $(D^{-1})^{\mu\nu} = Ag^{\mu\nu} + Bp^\mu p^\nu$, so

$$\begin{aligned} \delta_\rho^\mu &= [-(p^2 - m_V^2)g^{\mu\nu} + p^\mu p^\nu][Ag_{\nu\rho} + Bp_\nu p_\rho] \\ &= [-A(p^2 - m_V^2)\delta_\rho^\mu - B(p^2 - m_V^2)p^\mu p_\rho + Ap^\mu p_\rho + Bp^2 p^\mu p_\rho]. \end{aligned} \quad (11)$$

Because our tensor structures are linearly independent, we have the following equations:

$$\begin{cases} -A(p^2 - m_V^2) = 1 \\ -B(p^2 - m_V^2) + Bp^2 + A = 0. \end{cases} \quad (12)$$

Hence,

$$A = -\frac{1}{p^2 - m_V^2} \quad (13)$$

$$B = \frac{1}{m_V^2(p^2 - m_V^2)}, \quad (14)$$

and the propagator for a Proca particle is

$$\text{Prop}_{\mu\nu} = \frac{-i(g_{\mu\nu} - p_\mu p_\nu / m_V^2)}{p^2 - m_V^2 + i\epsilon}. \quad (15)$$

The Feynman rules for the Yukawa and vector theories specified by Eqs. (1) and (2) are given in Tables 1 and 2, respectively.

(b) The diagrams for this part are shown below. On the left, we have a scattering process where the vector particle produces a particle-antiparticle pair, and on the right, we have simply the self-energy diagram for the particle in the vector theory.

(c) If we place $m_V = 0$ in the original vector Lagrangian, then our equations for A and B given in Eq. (13) leave B unconstrained. If we introduce the term $(\partial_\mu A^\mu)^2 / 2\xi$ with ξ as our Lagrange multiplier, then we have instead

$$D = -p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu, \quad (16)$$

which gives us the equations

$$\begin{cases} -Ap^2 = 1 \\ -Bp^2 + A\left(1 - \frac{1}{\xi}\right) + B\left(1 - \frac{1}{\xi}\right)p^2 = 0 \end{cases}, \quad (17)$$

which yields the propagator

$$\text{Prop}_{\mu\nu} = \frac{-i[g_{\mu\nu} - (1 - \xi)p_\mu p_\nu / p^2]}{p^2 + i\epsilon}. \quad (18)$$

A typical choice for the undetermined Lagrange multiplier is $\xi = 1$, which corresponds to Feynman gauge.

(d) The leading order diagrams contributing to the $\psi\psi \rightarrow \psi\psi$ scattering amplitude are shown below.

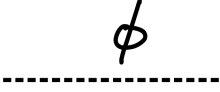

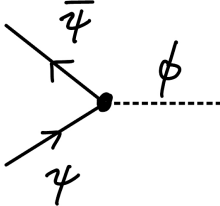
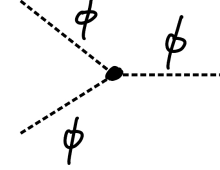
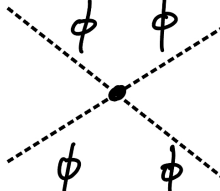
Type	Diagram	Expression
Propagator		$\frac{i}{p^2 - m^2 + i\epsilon}$
Propagator		$\frac{i(\not{p} + M)}{p^2 - M^2 + i\epsilon}$
Vertex		$-ig$
Vertex		$-i\lambda_1$
Vertex		$-i\lambda_2$

Table 1: Fundamental components for the Feynman diagrams contributing to amplitudes in the Yukawa theory and their corresponding momentum space expressions.

Translating using the Feynman rules, we have $\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u$, where

$$\begin{aligned}
 i\mathcal{M}_t &= [\bar{u}_{s_1}(p_1)(-ig)u_{s_A}(k_A)] \frac{i}{(p_1 - k_A)^2 - m_s^2 + i\epsilon} [\bar{u}_{s_2}(p_2)(-ig)u_{s_B}(k_B)] \\
 &= -\frac{ig^2}{t^2 - m_s^2} \bar{u}_{s_1}(p_1)u_{s_A}(k_A)\bar{u}_{s_2}(p_2)u_{s_B}(k_B)
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 i\mathcal{M}_u &= [\bar{u}_{s_2}(p_2)(-ig)u_{s_A}(k_A)] \frac{i}{(p_2 - k_A)^2 - m_s^2 + i\epsilon} [\bar{u}_{s_1}(p_1)(-ig)u_{s_B}(k_B)] \\
 &= -\frac{ig^2}{u^2 - m_s^2} \bar{u}_{s_2}(p_2)u_{s_A}(k_A)\bar{u}_{s_1}(p_1)u_{s_B}(k_B).
 \end{aligned} \tag{20}$$



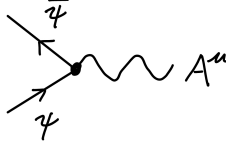
Type	Diagram	Expression
Propagator		$\frac{i(\not{p} + M)}{p^2 - M^2 + i\epsilon}$
Propagator		$\frac{-i(g_{\mu\nu} - p_\mu p_\nu / m_V^2)}{p^2 - m_V^2 + i\epsilon}$
Vertex		$-ig\gamma^\mu$

Table 2: Fundamental components for the Feynman diagrams contributing to amplitudes in the vector theory and their corresponding momentum space expressions.

The squared amplitude is then

$$|\mathcal{M}|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + 2 \operatorname{Re}\{\mathcal{M}_t^* \mathcal{M}_u\}. \quad (21)$$

Analyzing each term separately, we find

$$\begin{aligned}
|\mathcal{M}_t|^2 &= \frac{g^4}{(t^2 - m_s^2)^2} \bar{u}_{s_B}(k_B) u_{s_2}(p_2) \bar{u}_{s_A}(k_A) u_{s_1}(p_1) \bar{u}_{s_1}(p_1) u_{s_A}(k_A) \bar{u}_{s_2}(p_2) u_{s_B}(k_B) \\
&\rightarrow \frac{g^4}{(t^2 - m_s^2)^2} \operatorname{Tr}[(\not{p}_1 + M)(\not{k}_A + M)] \operatorname{Tr}[(\not{k}_B + M)(\not{p}_2 + M)] \\
&= \frac{16g^4(p_1 \cdot k_A + M^2)(p_2 \cdot k_B + M^2)}{(t^2 - m_s^2)^2} \quad (22)
\end{aligned}$$

$$|\mathcal{M}_u|^2 \rightarrow \frac{16g^4(p_2 \cdot k_A + M^2)(p_1 \cdot k_B + M^2)}{(u^2 - m_s^2)^2} \quad (23)$$

$$\begin{aligned}
\mathcal{M}_t^* \mathcal{M}_u &= \frac{g^4}{(t^2 - m_s^2)(u^2 - m_s^2)} \bar{u}_{s_B}(k_B) u_{s_1}(p_1) \bar{u}_{s_A}(k_A) u_{s_2}(p_2) \bar{u}_{s_1}(p_1) u_{s_A}(k_A) \bar{u}_{s_2}(p_2) u_{s_B}(k_B) \\
&\rightarrow \frac{g^4}{(t^2 - m_s^2)(u^2 - m_s^2)} \operatorname{Tr}[(\not{k}_B + M)(\not{p}_1 + M)(\not{k}_A + M)(\not{p}_2 + M)]. \quad (24)
\end{aligned}$$