Problem 1)

Consider the classical Lagrangian densities for the following relativistic quantum field theories,

$$\mathcal{L}_Y = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - M\bar{\psi}\psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_s^2\phi^2 - g\bar{\psi}\psi\phi - \frac{\lambda_1}{3!}\phi^3 - \frac{\lambda_2}{4!}\phi^4 \tag{1}$$

$$\mathcal{L}_V = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - M\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_V^2}{2}A^\mu A_\mu - g\bar{\psi}A\!\!\!/\psi. \tag{2}$$

We could envision each of these being proposed as moels of the interactions between spinor "nucleons" of mass M represented by the Fermi field ψ. In the first case, the interaction is then mediated by a scalar "pion" field ϕ with mass m_s, and in the second it is mediated by a vector field A^μ with mass m_V. As usual, the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. (Note that to be realistic we should really have pseudoscalar and vector interactions.) In the second theory, we would get the Maxwell field if we set $m_V = 0$.

- **(a)** Use the fast mnemonic that we developed in class for translating a classical Lagrangian density into QFT Feynman rules to write down all the Feynman rules for the two theories above. Make a comment about where each factor of "i" comes from. Use straight lines for ψ, dashed lines for ϕ, and wavy lines for A^μ.
- **(b)** Draw a two-loop diagram for the vector field case. Draw an example of a diagram that would give problems if you have not worried about the "reduction" of external leg states.
- **(c)** What happens if we then place $m_V = 0$ in the vector field? A standard way to deal with the problem is to replace $\frac{m_V^2}{2}A_\mu A^\mu \to -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$. This effectively uses the Lagrange multiplier technique to fix the Lorenz gauge condition $\partial_\mu A^\mu = 0$. What are the Feynman rules if I make this replacement? What if I further specify the gauge by fixing $\xi = 1$, where ξ is a real constant that will be chosen later?
- **(d)** Using Eq. (1), draw all the Feynman diagrams that would contribute to the $2 \to 2$ cross section, $\psi\bar{\psi} \to \psi\bar{\psi}$ through order g^2 and $g^2\lambda_1^2$.
- **(e)** Still using Eq. (1), consider the cross section for the $2 \to 2$ scattering process $\psi\psi \to \psi\psi$. Start with the general cross section expression derived in class,

$$\mathrm{d}\sigma = \frac{|M|^2}{2\sqrt{\lambda(s, m_A^2, m_B^2)}}\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\cdots\frac{\mathrm{d}^3\boldsymbol{p}_N}{(2\pi)^3 2E_N}(2\pi)^4\delta\Big(k_A + k_B - \sum_{i=1}^{N} p_i\Big), \tag{3}$$

with

$$\lambda(s, m_A^2, m_B^2) = s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 2m_A^2 m_B^2, \tag{4}$$

and derive the order g^2 expression for the unpolarized differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega\,|_{\mathrm{CM}}$ in the center-of-mass system. Since it is an unpolarized cross section, you should sum over the final and average over initial nucleon spins. Let p_A and p_B label the initial four-momenta and p_C and p_D label the final four-momenta and express your result in terms of Mandelstam variables.

(a) Let us rewrite the Yukawa and vector Lagrangians as follows:

$$\mathcal{L}_Y = \mathcal{L}_{D,\text{sym}} + \mathcal{L}_{\text{KG}} - g\bar{\psi}\psi\phi - \frac{\lambda_1}{3!}\phi^3 - \frac{\lambda_2}{4!}\phi^4 \tag{5}$$

$$\mathcal{L}_V = \mathcal{L}_{D,\text{sym}} + \mathcal{L}_{\text{P}} - g\bar{\psi}\not{A}\psi, \tag{6}$$

where $\mathcal{L}_{D,\text{sym}} = \bar{\psi}(\frac{i}{2}\overleftrightarrow{\not{\partial}} - M)\psi$ is the free Dirac Lagrangian for spin-1/2 spinor fields, $\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_s^2}{2}\phi$ is the free Klein-Gordon Lagrangian for real scalar particles, and $\mathcal{L}_P = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_V^2}{2}A^\mu A_\mu$ is the free Proca Lagrangian for a spin-1 vector particles. The vertices are simple enough to read from the Lagrangian since they are just the coefficients without any factor included for symmetries multiplied by i. We can determine the propagators easily enough from the mnemonic. For the KG scalar propagator, we list out the steps as follows:

$$D = [(ip_\mu)(-ip^\mu) - m_s^2] = p^2 - m_s^2 \to D^{-1} = \frac{1}{p^2 - m_s^2 + i\epsilon} \to \text{Prop} = \frac{i}{p^2 - m_s^2 + i\epsilon}. \tag{7}$$

Similarly, we can determine the Dirac propagator

$$D = \frac{i}{2}\gamma^\mu[(-ip_\mu) - (ip_\mu)] - M = \not{p} - M \to \text{Prop} = \frac{i}{\not{p} - M} = \frac{i(\not{p} + M)}{p^2 - M^2 + i\epsilon}. \tag{8}$$

For the Proca theory, we must massage the Lagrangian a bit to arrive at a form where we can use the mneumonic properly:

$$\begin{aligned}
\mathcal{L}_P &= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m_V^2}{2}A^\mu A_\mu \\
&= -\frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\nu A^\mu \partial_\mu A_\nu) + \frac{m_V^2}{2}A^\mu A_\mu \\
&= A^\mu\Big[-\frac{1}{2}(g_{\mu\nu}\overleftarrow{\partial^\rho}\partial_\rho - \overleftarrow{\partial_\nu}\partial_\mu) + \frac{m_V^2}{2}g_{\mu\nu}\Big]A^\nu.
\end{aligned} \tag{9}$$

Thus,

$$D_{\mu\nu} = -(p^2 - m_V^2)g_{\mu\nu} + p_\mu p_\nu. \tag{10}$$

Note that the inverse here is not simply the reciprocal, but we have $D_{\mu\nu}(D^{-1})^{\nu\rho} = \delta_\mu^{\ \rho}$. We can form the inverse as a linear combination of the tensor structures available to us as $(D^{-1})^{\mu\nu} = Ag^{\mu\nu} + Bp^\mu p^\nu$, so

$$\begin{aligned}
\delta^\mu_{\ \rho} &= [-(p^2 - m_V^2)g^{\mu\nu} + p^\mu p^\nu][Ag_{\nu\rho} + Bp_\nu p_\rho] \\
&= [-A(p^2 - m_V^2)\delta^\mu_{\ \rho} - B(p^2 - m_V^2)p^\mu p_\rho + Ap^\mu p_\rho + Bp^2 p^\mu p_\rho].
\end{aligned} \tag{11}$$

Because our tensor structures are linearly independent, we have the following equations:

$$\begin{cases} -A(p^2 - m_V^2) = 1 \\ -B(p^2 - m_V^2) + Bp^2 + A = 0. \end{cases} \tag{12}$$

Hence,

$$A = -\frac{1}{p^2 - m_V^2} \tag{13}$$

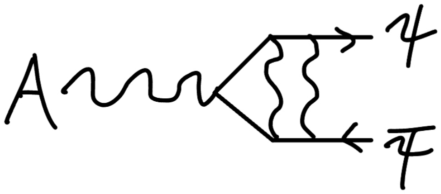
$$B = \frac{1}{m_V^2(p^2 - m_V^2)}, \tag{14}$$

and the propagator for a Proca particle is

$$\text{Prop}_{\mu\nu} = \frac{-i(g_{\mu\nu} - p_\mu p_\nu / m_V^2)}{p^2 - m_V^2 + i\epsilon}. \tag{15}$$

The Feynman rules for the Yukawa and vector theories specified by Eqs. (1) and (2) are given in Tables 1 and 2, respectively.

(b) The diagrams for this part are shown below. On the left, we have a scattering process where the vector particle produces a particle-antiparticle pair, and on the right, we have simply the self-energy diagram for the particle in the vector theory.



(c) If we place $m_V = 0$ in the original vector Lagrangian, then our equations for A and B given in Eq. (13) leave B unconstrained. If we introduce the term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ with ξ as our Lagrange multiplier, then we have instead

$$D = -p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu, \tag{16}$$

which gives us the equations

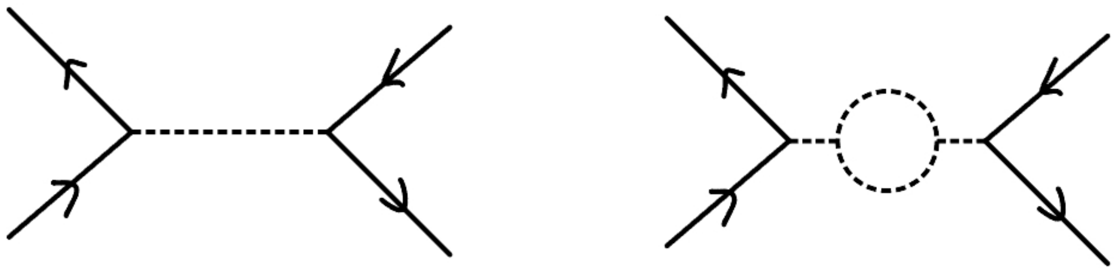
$$\begin{cases} -Ap^2 = 1 \\ -Bp^2 + A\left(1 - \frac{1}{\xi}\right) + B\left(1 - \frac{1}{\xi}\right)p^2 = 0, \end{cases} \tag{17}$$

which yields the propagator

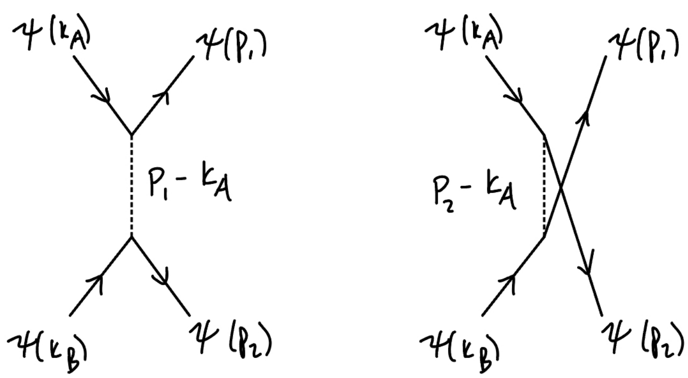
$$\text{Prop}_{\mu\nu} = \frac{-i[g_{\mu\nu} - (1 - \xi) p_\mu p_\nu / p^2]}{p^2 + i\epsilon}. \tag{18}$$

A typical choice for the undetermined Lagrange multiplier is $\xi = 1$, which corresponds to Feynman gauge.

(d) The diagrams for this part are shown below. On the left, we have the s-channel where the two annihilate, produce a virtual ϕ, which subsequently produces another $\psi\bar{\psi}$ pair. On the right, we include a scalar loop on the virtual exchange line.



(e) The leading order diagrams contributing to the $\psi\psi \to \psi\psi$ scattering amplitude are shown below.



Translating using the Feynman rules, we have $\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u$, where

$$\begin{aligned} i\mathcal{M}_t &= \left[\bar{u}_{s_1}(p_1)(-ig)u_{s_A}(k_A)\right]\frac{i}{(p_1-k_A)^2 - m_s^2 + i\epsilon}\left[\bar{u}_{s_2}(p_2)(-ig)u_{s_B}(k_B)\right] \\ &= -\frac{ig^2}{t-m_s^2}\bar{u}_{s_1}(p_1)u_{s_A}(k_A)\bar{u}_{s_2}(p_2)u_{s_B}(k_B) \end{aligned} \tag{19}$$

$$\begin{aligned} i\mathcal{M}_u &= \left[\bar{u}_{s_2}(p_2)(-ig)u_{s_A}(k_A)\right]\frac{i}{(p_2-k_A)^2 - m_s^2 + i\epsilon}\left[\bar{u}_{s_1}(p_1)(-ig)u_{s_B}(k_B)\right] \\ &= -\frac{ig^2}{u-m_s^2}\bar{u}_{s_2}(p_2)u_{s_A}(k_A)\bar{u}_{s_1}(p_1)u_{s_B}(k_B). \end{aligned} \tag{20}$$

The squared amplitude is then

$$|\mathcal{M}|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + 2\,\mathrm{Re}\{\mathcal{M}_t^*\mathcal{M}_u\}. \tag{21}$$

Analyzing each term separately, we find

$$
\begin{aligned}
|\mathcal{M}_t|^2 &= \frac{g^4}{(t-m_s^2)^2}\bar{u}_{s_B}(k_B)u_{s_2}(p_2)\bar{u}_{s_A}(k_A)u_{s_1}(p_1)\bar{u}_{s_1}(p_1)u_{s_A}(k_A)\bar{u}_{s_2}(p_2)u_{s_B}(k_B) \\
&\to \frac{g^4}{4(t-m_s^2)^2}\operatorname{Tr}\Big[(\not{p}_1+M)(\not{k}_A+M)\Big]\operatorname{Tr}\Big[(\not{k}_B+M)(\not{p}_2+M)\Big] \\
&= \frac{16g^4}{(t-m_s^2)^2}(p_1\cdot k_A+M^2)(p_2\cdot k_B+M^2) \qquad (22)\\
|\mathcal{M}_u|^2 &\to \frac{4g^4}{(u-m_s^2)^2}(p_2\cdot k_A+M^2)(p_1\cdot k_B+M^2) \qquad (23)\\
\mathcal{M}_t^*\mathcal{M}_u &= \frac{g^4}{(t-m_s^2)(u-m_s^2)}\bar{u}_{s_B}(k_B)u_{s_1}(p_1)\bar{u}_{s_A}(k_A)u_{s_2}(p_2)\bar{u}_{s_1}(p_1)u_{s_A}(k_A)\bar{u}_{s_2}(p_2)u_{s_B}(k_B) \\
&\to \frac{g^4}{4(t-m_s^2)(u-m_s^2)}\operatorname{Tr}\Big[(\not{k}_B+M)(\not{p}_1+M)(\not{k}_A+M)(\not{p}_2+M)\Big] \\
&= \frac{g^4}{4(t-m_s^2)(u-m_s^2)} \\
&\times\Bigg\{\operatorname{Tr}\Big(\not{k}_B\not{p}_1\not{k}_A\not{p}_2\Big)+M^2\operatorname{Tr}\Big(\not{k}_B\not{p}_1+\not{k}_B\not{k}_A+\not{k}_B\not{p}_2+\not{p}_1\not{k}_A+\not{p}_1\not{p}_2+\not{k}_A\not{p}_2\Big)+4M^4\Bigg\} \\
&= \frac{g^4}{(t-m_s^2)(u-m_s^2)}\Bigg\{(k_B\cdot p_1)(k_A\cdot p_2)-(k_B\cdot k_A)(p_1\cdot p_2)+(k_B\cdot p_2)(p_1\cdot k_A) \\
&+M^2\Big(k_B\cdot p_1+k_B\cdot k_A+k_B\cdot p_2+p_1\cdot k_A+p_1\cdot p_2+k_A\cdot p_2\Big)+M^4\Bigg\}. \qquad (24)
\end{aligned}
$$

We would like to translate these dot products into Mandelstam variables, which can be done by writing

$$s=(k_A+k_B)^2=2(k_A\cdot k_B+M^2)\Rightarrow k_A\cdot k_B=\frac{s-2M^2}{2} \qquad (25)$$

$$s=(p_1+p_2)^2=2(p_1\cdot p_2+M^2)\Rightarrow p_1\cdot p_2=\frac{s-2M^2}{2} \qquad (26)$$

$$t=(k_A-p_1)^2=2(M^2-k_A\cdot p_1)\Rightarrow k_A\cdot p_1=\frac{2M^2-t}{2} \qquad (27)$$

$$t=(k_B-p_2)^2=2(M^2-2k_B\cdot p_2)\Rightarrow k_B\cdot p_2=\frac{2M^2-t}{2} \qquad (28)$$

$$u=(k_A-p_2)^2=2(M^2-k_A\cdot p_2)\Rightarrow k_A\cdot p_2=\frac{2M^2-u}{2} \qquad (29)$$

$$u=(k_B-p_1)^2=2(M^2-k_B\cdot p_1)\Rightarrow k_B\cdot p_1=\frac{2M^2-u}{2}. \qquad (30)$$

Thus,

$$|\mathcal{M}_t|^2 = \frac{4g^4}{(t-m_s^2)^2}\frac{(4M^2-t)^2}{4} = g^4\left(\frac{t-4M^2}{t-m_s^2}\right)^2 \tag{31}$$

$$|\mathcal{M}_u|^2 = g^4\left(\frac{u-4M^2}{u-m_s^2}\right)^2 \tag{32}$$

$$2\,\mathrm{Re}\{\mathcal{M}_t\mathcal{M}_u\} = \frac{g^4}{2(t-m_s^2)(u-m_s^2)}\left\{16M^4+8M^2(s-t-u)-s^2+t^2+u^2\right\}. \tag{33}$$

Let us now look at the cross section and massage the general form of this quantity. First, the triangle function

$$\lambda(s,M^2,M^2) = s^2-4sM^2 = s(s-4M^2). \tag{34}$$

Thus,

$$\mathrm{d}^6\sigma = \frac{|\mathcal{M}|^2}{2\sqrt{s(s-4M^2)}}\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(k_A+k_B-p_1-p_2). \tag{35}$$

If we integrate over $\boldsymbol{p}_2$, then

$$\begin{aligned}\mathrm{d}^3\sigma &= \frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^2 2E_1}\int\frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_2}\frac{|\mathcal{M}|^2}{2\sqrt{s(s-4M^2)}}\delta^{(3)}(\boldsymbol{k}_A+\boldsymbol{k}_B-\boldsymbol{p}_1-\boldsymbol{p}_2)\delta(E_A+E_B-E_1-E_2)\\ &= \frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^2\sqrt{s}}\frac{|\mathcal{M}|^2}{2\sqrt{s(s-4M^2)}}\delta\Big(\sqrt{s}-2\sqrt{\boldsymbol{p}_1^2+M^2}\Big)\\ &= \frac{|\boldsymbol{p}_1|^2\,\mathrm{d}|\boldsymbol{p}_1|\,\mathrm{d}\Omega_{\mathrm{CM}}}{(2\pi)^2 s}\frac{|\mathcal{M}|^2}{2\sqrt{s(s-4M^2)}}\delta\Big(\sqrt{s}-2\sqrt{\boldsymbol{p}_1^2+M^2}\Big),\end{aligned} \tag{36}$$

where we have assumed that we are in the CM frame such that $\boldsymbol{k}_A+\boldsymbol{k}_B=0$ and $E_A+E_B=\sqrt{s}$. Finally, we can integrate over $|\boldsymbol{p}_1|$, using

$$\delta\Big(\sqrt{s}-2\sqrt{\boldsymbol{p}_1^2+M^2}\Big) = \frac{\sqrt{s}}{4|\boldsymbol{p}_1|}\delta\Big(|\boldsymbol{p}_1|-\frac{1}{2}\sqrt{s-4M^2}\Big), \tag{37}$$

to find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{CM}}} = \frac{|\mathcal{M}|^2}{64\pi^2 s}. \tag{38}$$

From here, we can insert the expressions above for the squared amplitude, imposing momentum conservation, which is equivalent to imposing the relation between Mandelstam variables $s+t+u=4M^2$. Lastly, in the CM frame, we have

$$t = 2(M^2-k_A\cdot p_1) = \frac{1}{2}\Big(4M^2+(s-4M^2)\cos\theta\Big), \tag{39}$$

where θ is the angle between $\boldsymbol{k}_A$ and $\boldsymbol{p}_1$.

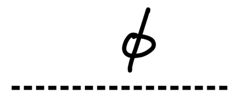
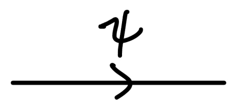
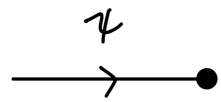
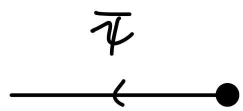
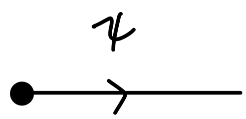
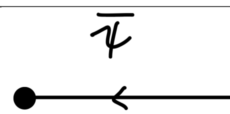
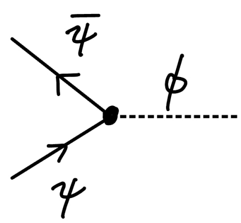
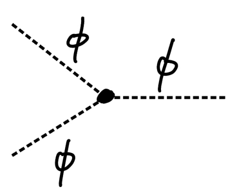
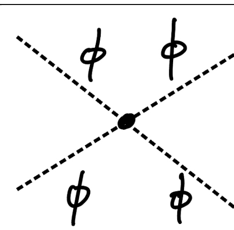
Type	Diagram	Expression
Propagator		$\frac{i}{p^2-m^2+i\epsilon}$
Propagator		$\frac{i(\not{p}+M)}{p^2-M^2+i\epsilon}$
Incoming line		$u_s(p)$
Incoming line		$v_s(p)$
Outgoing line		$\bar{u}_s(p)$
Outgoing line		$\bar{u}_s(p)$
Vertex		$-ig$
Vertex		$-i\lambda_1$
Vertex		$-i\lambda_2$

Table 1: Fundamental components for the Feynman diagrams contributing to amplitudes in the Yukawa theory and their corresponding momentum space expressions.

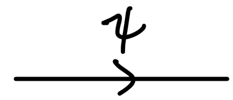
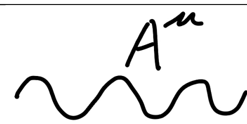
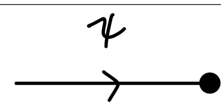
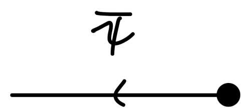
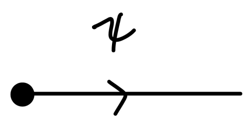
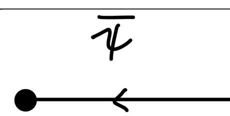
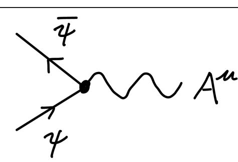
Type	Diagram	Expression
Propagator		$\frac{i(\not{p}+M)}{p^2-M^2+i\epsilon}$
Propagator		$\frac{-i(g_{\mu\nu}-p_\mu p_\nu/m_V^2)}{p^2-m_V^2+i\epsilon}$
Incoming line		$u_s(p)$
Incoming line		$v_s(p)$
Outgoing line		$\bar{u}_s(p)$
Outgoing line		$\bar{u}_s(p)$
Vertex		$-ig\gamma^\mu$

Table 2: Fundamental components for the Feynman diagrams contributing to amplitudes in the vector theory and their corresponding momentum space expressions.