Problem 1)

Consider again the classical complex Klein-Gordon field with the Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*. \tag{1}$$

Repeat and write out all the steps that I showed in class for converting a real, lattice Klein-Gordon field to a quantum continuum version, but now for the complex scalar field above. Get the Heisenberg field operators $\hat{\phi}(x)$ and $\hat{\phi}^\dagger$ in terms of the creation and annihilation operators for particles and antiparticles.

In this problem, we can introduce some system with complex generalized coordinates $q_{\boldsymbol{n}}$ as in HW 4, problem 3 governed by a Lagrangian

$$L = \sum_{\boldsymbol{n}} |\dot{q}_{\boldsymbol{n}}|^2 - \sum_{\boldsymbol{n}} m^2 |q_{\boldsymbol{n}}|^2 - \sum_{\boldsymbol{n}} \sum_i \kappa |q_{\boldsymbol{n}+\hat{\boldsymbol{e}}_i} - q_{\boldsymbol{n}}|^2. \tag{2}$$

We introduce normal coordinates such that

$$q_{\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \bar{q}_{\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}}, q_{\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \bar{q}^*_{\boldsymbol{k}} e^{-ia\boldsymbol{k}\cdot\boldsymbol{n}},, \tag{3}$$

where $p_{\boldsymbol{n}}$ is the momentum conjugate to $q_{\boldsymbol{n}}$. Placing this system in a box of finite volume $L^D = (Na)^D$ with periodic boundary conditions such that $q_{\boldsymbol{n}+N\sum_i \hat{\boldsymbol{e}}_i} = q_{\boldsymbol{n}}$, where the sum is over any subset of $\{1, \ldots, D\}$, leaving us with the condition that

$$\boldsymbol{k} = \frac{2\pi \bar{\boldsymbol{n}}}{L}, \tag{4}$$

where the components $\bar{n}_i \in (-N/2, N/2]$. Using these results, we can write the Hamiltonian in terms of normal coordinates is given by

$$H = \sum_{\boldsymbol{k}} \left\{ \frac{1}{2} \bar{p}_{\boldsymbol{k}} \bar{p}_{-\boldsymbol{k}} + \frac{\omega_{\boldsymbol{k}}^2}{2} \bar{q}_{\boldsymbol{k}} \bar{q}_{-\boldsymbol{k}} \right\}, \tag{5}$$

where $\omega_{\boldsymbol{k}}^2 = m^2 + 2\kappa \sum_i [1 - \cos(k_i a)]$ At this point we must adapt our work to the case of a complex scalar field.

Problem 2)

Checking steps from class.

(a) Show that the effect of normal ordering on the Hamiltonian and Noether momentum is to eliminate any constant terms and puts $:\hat{H}:$ and $:\hat{P}_j:$ into a form that only involves number operators.

(b) Verify that the expression for the identity in the Fock space that we discussed class is

$$\hat{1} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=0}^{n} \frac{\mathrm{d}^3 \boldsymbol{p}_j}{(2\pi)^3 2E_{\boldsymbol{p}_j}} \left| p_n \right\rangle \left\langle p_n \right| \tag{6}$$

for the case of a three-excitation momentum state $|\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\rangle$.

(c) As in class, let a single excitation element of a bosonic Fock space at time t be

$$|f, 1, t\rangle = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \tilde{f}(\boldsymbol{p}) \left| \boldsymbol{p} \right\rangle = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} a^\dagger_{\boldsymbol{p}} \left| 0 \right\rangle \tilde{f}(\boldsymbol{p}) \tag{7}$$

with a wavepacket function $\tilde{f}(\boldsymbol{p})$. Let the coordinate space wavepacket function be defined by

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \tilde{f}(\boldsymbol{p}) e^{-ip\cdot x} = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \tilde{f}(\boldsymbol{p}) e^{-ip^0 t + i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{8}$$

Note the time dependence in the exponential despite the fact that the integral is only over spatial components. Show that

$$|f, 1, t\rangle = \int \mathrm{d}^3 \boldsymbol{x}\, \phi(\boldsymbol{x}) \left| 0 \right\rangle 2i \frac{\partial f(\boldsymbol{x})}{\partial t}. \tag{9}$$

(d) By using Fock states expressed like in Eq. (3) above, show directly that $a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}$ is a density of excitations with respect to three momentum.

Problem 3)

The following is a simple undergraduate electrodynamics problem that I aim to use to motivate you to think about the interpretation of infinite energies: Let there be a continuous line of electric charge with linear density $\lambda = \mathrm{d}Q/\mathrm{d}y$ running along the y-axis from a point $-L$ to a point $+L$. Consider a position at a perpendicular distance x away from the center of the line. What is the electric potential there if I use the standard expression $\mathrm{d}V = \mathrm{d}Q\,/(4\pi\epsilon_0 r)$ for a differential element of charge? Show that the potential energy of a charge placed at that point is infinite if $L \to \infty$. Does this mean that the physics outside an infinitely long line of charge like this is pathological or ill-defined? Elaborate on the analogy with the "infinite" constant we found in the continuum limit of the lattice Klein-Gordon theory.

Problem 4)

Let $\phi_\ell(t)$ be a massless real Klein-Gordon field averaged with a function proportional to e^{-r^2/ℓ^2}, where r is the distance from the origin of spatial coordinates. That is,

$$\phi_\ell(t) = \frac{\int \mathrm{d}^3\boldsymbol{\ell}\,\phi(\boldsymbol{x})e^{-r^2/\ell^2}}{\int \mathrm{d}^3\boldsymbol{\ell}\,e^{-r^2/\ell^2}}. \tag{10}$$

Calculate the vacuum expectation value of $\phi_\ell(t)^2$,

$$\langle 0|\phi_\ell(t)^2|0\rangle\,. \tag{11}$$

The square root of this expectation value is an estimate of the size of fluctuations in thefield when probed with some kind of detector with resolution ℓ. Convert this quantity to volts. This estimate should also be roughly good for the electromagnetic field, to within a modest factor. Compute numerical values for a few distance scales of physical interest. In what situations might these 'zero point fluctuations' be of significance?