Problem 1)

In this problem we will continue studying the basics of classical field theory by reviewing classical electromagnetism. This exercise is based from Peskin & Schroeder's textbook, problem #2.1.

(a) Using the definition of the electromagnetic tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, show that it satisfies the Bianchi identity,

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{1}$$

(b) Using the fact that $\epsilon^{ijk} B^k = -F^{ij}$, where B^k is the k^{th} component of the magnetic field, show that

$$B^k = -\frac{\epsilon^{ijk} F^{ij}}{2}. \tag{2}$$

(c) Work through 2.1 in Peskin and Schroeder. (Tip #1: you might want to use the identities found above to find two of Maxwell's equations in part. Tip #2: you might need to use the equation of motion for the field.)

- Classical electromagnetism (with no sources) follows from the action

$$S = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad \text{where} F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A_\mu. \tag{3}$$

  Derive Maxwell's equations as the Euler Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

- Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so;

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{4}$$

  is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu \tag{5}$$

  leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}. \tag{6}$$

(a) We can easily show the Bianchi identity directly:

$$\boxed{\begin{aligned}&\partial_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \partial_\nu(\partial_\rho A_\mu - \partial_\mu A_\rho) + \partial_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu)\\ &\quad = [\partial_\mu, \partial_\nu]A_\rho + [\partial_\rho, \partial_\mu]A_\nu + [\partial_\nu, \partial_\rho]A_\mu = 0\end{aligned}},$$

where we use the fact that derivatives commute with each other.

(b) Again, the primary objective is not too difficult to establish using a well-known identity for the contraction of Levi-Civita symbols:

$$\epsilon^{ijk}\epsilon^{ijk'}B^{k'} = 2\delta^{kk'}B^{k'} = 2B^k = -\epsilon^{ijk}F^{ij} \Rightarrow \boxed{B^k = -\frac{1}{2}\epsilon^{ijk}F^{ij}}. \tag{7}$$

(c) The Lagrangian in terms of the 4-potential is given as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)\\ &= -\frac{1}{2}(g^{\sigma\alpha}g_{\nu\beta}\partial_\sigma A^\beta \partial_\alpha A^\nu - \partial_\sigma A^\nu \partial_\nu A^\sigma),\end{aligned} \tag{8}$$

and the Euler-Lagrange equation for A^μ reads

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A^\mu)} &= 0\\ \frac{1}{2}\partial_\rho\left[g^{\sigma\alpha}g_{\nu\beta}\left(\delta^\rho_\sigma\delta^\beta_\mu\partial_\alpha A^\nu + \partial_\sigma A^\beta \delta^\rho_\alpha\delta^\nu_\mu\right) - \left(\delta^\rho_\sigma\delta^\nu_\mu\partial_\nu A^\sigma + \partial_\sigma A^\nu \delta^\rho_\nu\delta^\sigma_\mu\right)\right] &= 0\\ \partial_\rho(\partial^\rho A_\mu - \partial_\mu A^\rho) &= 0.\end{aligned} \tag{9}$$

Note that the object in parentheses is the field-strength tensor $F^\rho_{\ \mu}$, but we can act with the metric tensor on both sides to raise the index μ and relabel $\rho \to \mu$ and $\mu \to \nu$ to obtain the typical compact presentation of the Maxwell equations:

$$\boxed{\partial_\mu F^{\mu\nu} = 0}. \tag{10}$$

Unfolding, we have

$$\frac{\partial F^{0\nu}}{\partial t} + \frac{\partial F^{i\nu}}{\partial x^i} = 0, \tag{11}$$

so that

$$\begin{aligned}\frac{\partial F^{00}}{\partial t} + \frac{\partial F^{i0}}{\partial x^i} &= -\frac{\partial E^i}{\partial x^i} = -\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0\\ \frac{\partial F^{0j}}{\partial \frac{\partial}{\partial t}} + \frac{\partial F^{ij}}{\partial x^i} &= -\frac{\partial E^j}{\partial t} - \epsilon^{ijk}\nabla^i B^k = \left(\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t}\right)_i = 0.\end{aligned} \tag{12}$$