Problem 1)

We have been thinking carefully about how to set up relativistic wavepackets, which will be important when we get to a formal treatment of the S-matrix and scattering theory. We used wavepacket functions $\tilde{f}(\boldsymbol{p})$ and $\tilde{g}(\boldsymbol{p})$ to describe the three-momentum parts of initial and final wavepackets. I also suggested using

$$F(t - t_0, \Delta t) = \frac{1}{\sqrt{\pi}\Delta t} e^{-(t-t_0)^2/\Delta t^2} \tag{1}$$

for the temporal part of the wavepacket. To make analyzing the propagation amplitude

$$\langle g; \mathrm{out}; \Delta | f; \mathrm{in}; \Delta \rangle \tag{2}$$

more manageable, write out an expression for it that only involves integrals over $\mathrm{d}^4 p$, $\mathrm{d}x_0$, and $\mathrm{d}y_0$. The integrand should only involve $\tilde{f}(\boldsymbol{p})$, $\tilde{g}(\boldsymbol{p})$, $1/(p^2 - m^2 + i\epsilon)$, and $F(t - t_0, \Delta t)$.

In the free, complex Klein-Gordon theory, we set up a simple scattering amplitude, for a particle or anti-particle wave-packet to evolve into some other particle or anti-particle wave-packet, respectively, at some much later time. Generically, we found that

$$\langle g; \mathrm{out}; \Delta | f; \mathrm{out}; \Delta \rangle = \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \underline{g}^*(y) \underline{f}(x) \langle 0 | T\phi(y)\phi^\dagger(x) | 0 \rangle \,. \tag{3}$$

The time-ordering handily takes care of whether we consider particles or anti-particles, and the propagator

$$\langle 0 | T\phi(y)\phi^\dagger(x) | 0 \rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i e^{-ip\cdot(y-x)}}{p^2 - m^2 + i\epsilon}. \tag{4}$$

Recall that we can write

$$\underline{f}(x) = 2i \frac{\partial f(x)}{\partial t} F(x^0 - \bar{x}^0, \Delta x^0), \tag{5}$$

where

$$f(x) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \tilde{f}(\boldsymbol{p}) e^{-ip\cdot x} \tag{6}$$

and similarly for $\underline{g}$. If we plug the propagator and the wave-packet expressions into the scattering amplitude, we find

$$\langle g | f \rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \left\{ \int \mathrm{d}^4 y \, 2i \frac{\partial g(y)}{\partial t} G(y^0 - \bar{y}^0, \Delta y^0) e^{ip\cdot y} \right\}^*$$
$$\times \left\{ \int \mathrm{d}^4 x \, 2i \frac{\partial f(x)}{\partial t} F(x^0 - \bar{x}^0, \Delta x^0) e^{ip\cdot x} \right\}. \tag{7}$$

Let us analyze one of the integrals in the curly braces:

$$
\begin{aligned}
&\int \mathrm{d}^4x\, 2i\frac{\partial f(x)}{\partial t}F(x^0-\bar{x}^0,\Delta x^0)e^{ip\cdot x} \\
&= \int \mathrm{d}^4x\, 2iF(x^0-\bar{x}^0,\Delta x^0)e^{ip\cdot x}\frac{\partial}{\partial t}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2E_{\boldsymbol{k}}}}\tilde{f}(\boldsymbol{k})e^{-ik\cdot x} \\
&= \int \mathrm{d}^4x\, F(x^0-\bar{x}^0,\Delta x^0)e^{ip\cdot x}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\sqrt{2E_{\boldsymbol{k}}}\tilde{f}(\boldsymbol{k})e^{-ik\cdot x} \\
&= \int \mathrm{d}x^0\, F(x^0-\bar{x}^0,\Delta x^0)e^{i(p^0-E_{\boldsymbol{k}})x^0}\int \mathrm{d}^3\boldsymbol{k}\,\sqrt{2E_{\boldsymbol{k}}}\tilde{f}(\boldsymbol{k})\int \frac{\mathrm{d}^3\boldsymbol{x}}{(2\pi)^3}e^{-i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{x}} \\
&= \sqrt{2E_{\boldsymbol{p}}}\tilde{f}(\boldsymbol{p})\int \mathrm{d}x^0\, F(x^0-\bar{x}^0,\Delta x^0)e^{i(p^0-E_{\boldsymbol{p}})x^0}. \qquad (8)
\end{aligned}
$$

Thus, the scattering amplitude

$$
\begin{aligned}
\langle g|f\rangle &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\left\{\sqrt{2E_{\boldsymbol{p}}}\tilde{g}(\boldsymbol{p})\int \mathrm{d}y^0\, G(y^0-\bar{y}^0,\Delta y^0)e^{i(p^0-E_{\boldsymbol{p}})y^0}\right\}^* \\
&\quad\times\left\{\sqrt{2E_{\boldsymbol{p}}}\tilde{f}(\boldsymbol{p})\int \mathrm{d}x^0\, F(x^0-\bar{x}^0,\Delta x^0)e^{i(p^0-E_{\boldsymbol{p}})x^0}\right\} \\
&= \boxed{\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{d}x^0\,\mathrm{d}y^0\,\frac{i}{p^2-m^2+i\epsilon}2E_{\boldsymbol{p}}\tilde{g}^*(\boldsymbol{p})\tilde{f}(\boldsymbol{p})F(x^0-\bar{x}^0,\Delta x^0)G(y^0-\bar{y}^0,\Delta y^0)e^{-i(p^0-E_{\boldsymbol{p}})(y^0-x^0)}}. \qquad (9)
\end{aligned}
$$

Problem 2)

In our treatment of the charged (complex) Klein-Gordon field, we have treated ϕ and $\phi^\dagger$ as if they are independent fields, even though knowledge of ϕ determines $\phi^\dagger$. Give a more rigorous explanation for why this is reasonable than what we did in class.

The Lagrangian for the free, complex Klein-Gordon theory is

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi^*\phi. \qquad (10)$$

Let us rewrite this in terms of some scaled real and imaginary parts of ϕ, where

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad (11)$$

then

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu(\phi_1-i\phi_2)\partial^\mu(\phi_1+i\phi_2) - \frac{m^2}{2}(\phi_1-i\phi_2)(\phi_1+i\phi_2) \\
&= \left(\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2}{2}\phi_1^2\right) + \left(\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m^2}{2}\phi_2^2\right) \\
&= \mathcal{L}_1 + \mathcal{L}_2, \qquad (12)
\end{aligned}
$$

where $\mathcal{L}_{1,2}$ are real scalar KG Lagrangians. We already know how to solve the real scalar Klein-Gordon theory, and here we just have two copies:

$$\phi_{1,2}(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}(a_{\boldsymbol{p}}^{(1,2)}e^{-ip\cdot x} + a_{\boldsymbol{p}}^{(1,2)\,\dagger}e^{ip\cdot x}). \tag{13}$$

Hence

$$\begin{aligned}\phi &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\Big(\frac{a_{\boldsymbol{p}}^{(1)} + ia_{\boldsymbol{p}}^{(2)}}{\sqrt{2}}e^{-ip\cdot x} + \frac{a_{\boldsymbol{p}\dagger}^{(1)\dagger} + ia_{\boldsymbol{p}}^{(2)\dagger}}{\sqrt{2}}e^{ip\cdot x}\Big) \\ &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\Big(a_{\boldsymbol{p}}e^{-ip\cdot x} + b_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\Big) \end{aligned}\tag{14}$$

and similarly for $\phi^\dagger$. Note that we have defined $a_{\boldsymbol{p}} = (a_{\boldsymbol{p}}^{(1)} + ia_{\boldsymbol{p}}^{(2)})/\sqrt{2}$ and $b_{\boldsymbol{p}} = (a_{\boldsymbol{p}}^{(1)} - ia_{\boldsymbol{p}}^{(2)})/\sqrt{2}$.

Up to this point, we have not dealt directly with the fields ϕ and $\phi^\dagger$ directly, but as we have seen, there are some nice interpretations that are more transparent when dealing with them as opposed to their real constituents. Observe that

$$\frac{\partial}{\partial\phi} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\phi_1} - i\frac{\partial}{\partial\phi_2}\Big) \tag{15}$$

$$\frac{\partial}{\partial\phi^*} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\phi_1} + i\frac{\partial}{\partial\phi_2}\Big). \tag{16}$$

Therefore, the two independent Euler-Lagrange equations of motion for the real scalar field ϕ_1 and ϕ_2 can be expressed and rewritten to obtain

$$\begin{cases}\dfrac{\partial\mathcal{L}}{\partial\phi_1} - \partial_\mu\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)} = 0 \\ \dfrac{\partial\mathcal{L}}{\partial\phi_2} - \partial_\mu\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)} = 0\end{cases} \Leftrightarrow \begin{cases}\dfrac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \\ \dfrac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = 0.\end{cases} \tag{17}$$

Thus, we have exchanged our two independent real scalar fields for ϕ and ϕ^*. It is this point where we may ask how ϕ and ϕ^* are independent. Of course, if we know ϕ, then we immediately know ϕ^*, and while this is true, a justification for treating them as independent degrees of freedom comes from the equivalance of the equations of motion above. The Euler-Lagrange equations for the real scalar fields and the complex scalar fields are equivalent only when we have the two equations on the right-hand-side for ϕ and ϕ^*. While one can obtain the two equations on the left from only one of those on the right, we cannot obtain both equations without both on the left or only one equation on the right from the two on the left. Hence, the sets of degrees of freedom $\{\phi_1, \phi_2\} \equiv \{\phi, \phi^*\}$.

Problem 3)

> Repeat the steps for quantizing the charged Klein-Gordon field, but now impose anticommutation relations on the fields rather than commutation relations. Why would one consider trying this in the first place? What happens to the Hamiltonian?

Writing our solutions for the field and conjugate momenta we have

$$\phi = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}(a_{\boldsymbol{p}}e^{-ip\cdot x} + b_{\boldsymbol{p}}^\dagger e^{ip\cdot x}) \tag{18}$$

$$\phi^\dagger = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}(a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} + b_{\boldsymbol{p}}e^{-ip\cdot x}) \tag{19}$$

$$\pi_\phi = i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\sqrt{\frac{E_{\boldsymbol{p}}}{2}}(a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} - b_{\boldsymbol{p}}e^{-ip\cdot x}) \tag{20}$$

$$\pi_{\phi^\dagger} = -i\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\sqrt{\frac{E_{\boldsymbol{p}}}{2}}(a_{\boldsymbol{p}}e^{-ip\cdot x} - b_{\boldsymbol{p}}^\dagger e^{ip\cdot x}). \tag{21}$$

We can take Fourier transforms and set $t = 0$ to obtain

$$\tilde{\phi}(\boldsymbol{p}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}}}(a_{\boldsymbol{p}} + b_{-\boldsymbol{p}}^\dagger) \tag{22}$$

$$\tilde{\phi}^\dagger(\boldsymbol{p}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}}}(a_{-\boldsymbol{p}}^\dagger + b_{\boldsymbol{p}}) \tag{23}$$

$$\tilde{\pi}_\phi(\boldsymbol{p}) = i\sqrt{\frac{E_{\boldsymbol{p}}}{2}}(a_{-\boldsymbol{p}}^\dagger - b_{\boldsymbol{p}}) \tag{24}$$

$$\tilde{\pi}_{\phi^\dagger}(\boldsymbol{p}) = -i\sqrt{\frac{E_{\boldsymbol{p}}}{2}}(a_{\boldsymbol{p}} - b_{-\boldsymbol{p}}^\dagger) \tag{25}$$

and

$$a_{\boldsymbol{p}} = \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\tilde{\phi}(\boldsymbol{p}) + \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\tilde{\pi}_{\phi^\dagger}(\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\phi(\boldsymbol{x}) + \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\pi_{\phi^\dagger}(\boldsymbol{x})\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$

$$b_{\boldsymbol{p}}^\dagger = \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\tilde{\phi}(-\boldsymbol{p}) - \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\tilde{\pi}_{\phi^\dagger}(-\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\phi(\boldsymbol{x}) - \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\pi_{\phi^\dagger}(\boldsymbol{x})\right)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}$$

$$a_{\boldsymbol{p}}^\dagger = \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\tilde{\phi}^\dagger(\boldsymbol{p}) - \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\tilde{\pi}_\phi(\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\phi^\dagger(\boldsymbol{x}) - \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\pi_\phi(\boldsymbol{x})\right)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}$$

$$b_{\boldsymbol{p}} = \sqrt{\frac{E_{\boldsymbol{p}}}{2}}\tilde{\phi}^\dagger(-\boldsymbol{p}) + \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\tilde{\pi}_\phi(-\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\left(\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\phi^\dagger(\boldsymbol{x}) + \frac{i}{\sqrt{2E_{\boldsymbol{p}}}}\pi_\phi(\boldsymbol{x})\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{26}$$

From here, we can see that imposing anti-commutation relations for the fields and conjugate momenta yields

$$\begin{aligned}\{a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\} &= \int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\left[\frac{\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{p}'}}}{2}\{\phi(\boldsymbol{x}),\phi^\dagger(\boldsymbol{y})\} - \frac{i}{2}\sqrt{\frac{E_{\boldsymbol{p}}}{E_{\boldsymbol{p}'}}}\{\phi(\boldsymbol{x}),\pi_\phi(\boldsymbol{y})\}\right.\\ &\quad\left. + \frac{i}{2}\sqrt{\frac{E_{\boldsymbol{p}'}}{E_{\boldsymbol{p}}}}\{\pi_{\phi^\dagger}(\boldsymbol{x}),\phi^\dagger(\boldsymbol{y})\} - \frac{1}{2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{p}'}}}\{\pi_{\phi^\dagger}(\boldsymbol{x}),\pi_\phi(\boldsymbol{y})\}\right]e^{i\boldsymbol{p}\cdot\boldsymbol{x}}e^{-i\boldsymbol{p}'\cdot\boldsymbol{y}}\\ &= 0.\end{aligned} \tag{27}$$

Problem 4)

Checking steps from class:

(a) In class, I went through the steps for setting up the Dirac field and showing that it gives the Dirac equation quite fast. Fill in the steps for both the symmetrized and non-symmetrized form of the Lagrangian.

The typical Dirac Lagrangian is

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi = \bar\psi(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = \psi^\dagger(i\partial_0 + i\gamma^0\gamma^i\partial_i - m\gamma^0)\psi. \tag{28}$$

Hence the equation of motion for ψ is

$$\frac{\partial\mathcal{L}}{\partial\psi^\dagger} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)} = (i\partial_0 + i\gamma^0\gamma^i\partial_i - m\gamma^0)\psi = 0, \tag{29}$$

and if we multiply by γ^0, we obtain

$$(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = (i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{30}$$

Next, we can define the symmetrized Lagrangian as

$$\mathcal{L}_{\text{sym}} = \frac{\mathcal{L}+\mathcal{L}^*}{2}. \tag{31}$$

Observe that

$$\begin{aligned}\mathcal{L}^* &= \psi^\dagger(-i\overleftarrow{\partial}_0 - i\gamma^0\gamma^0\gamma^i\gamma^0\overleftarrow{\partial}_i - m\gamma^0)\psi\\ &= \bar\psi(-i\gamma^0\overleftarrow{\partial}_0 + i\gamma^i\overleftarrow{\partial}_i - m)\psi\\ &= \bar\psi(-i\gamma^\mu\overleftarrow{\partial}^\mu - m)\psi + 2i\gamma^i\overleftarrow{\partial}_i.\end{aligned} \tag{32}$$

Thus

$$\mathcal{L}_{\text{sym}} = \frac{i}{2}\bar\psi(\gamma^\mu\overleftrightarrow{\partial}_\mu - m)\psi. \tag{33}$$