Problem 1)

In this problem we will continue studying the basics of classical field theory by reviewing classical electromagnetism. This exercise is based from Peskin & Schroeder's textbook, problem #2.1.

(a) Using the definition of the electromagnetic tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, show that it satisfies the Bianchi identity,

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{1}$$

(b) Using the fact that $\epsilon^{ijk} B^k = -F^{ij}$, where B^k is the k^{th} component of the magnetic field, show that

$$B^k = -\frac{\epsilon^{ijk} F^{ij}}{2}. \tag{2}$$

(c) Work through 2.1 in Peskin and Schroeder. (Tip #1: you might want to use the identities found above to find two of Maxwell's equations in part. Tip #2: you might need to use the equation of motion for the field.)

- Classical electromagnetism (with no sources) follows from the action

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{3}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu \tag{4}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}. \tag{5}$$

(a) We can easily demonstrate the validity of the Bianchi identity directly:

$$\boxed{\begin{aligned} &\partial_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \partial_\nu(\partial_\rho A_\mu - \partial_\mu A_\rho) + \partial_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &\quad = [\partial_\mu, \partial_\nu] A_\rho + [\partial_\rho, \partial_\mu] A_\nu + [\partial_\nu, \partial_\rho] A_\mu = 0 \end{aligned}},$$

where we use the fact that derivatives commute with each other.

(b) Again, the primary objective is not too difficult to establish using a well-known identity for the contraction of Levi-Civita symbols:

$$\epsilon^{ijk}\epsilon^{ijk'}B^{k'} = 2\delta^{kk'}B^{k'} = 2B^k = -\epsilon^{ijk}F^{ij} \Rightarrow \boxed{B^k = -\frac{1}{2}\epsilon^{ijk}F^{ij}}. \tag{6}$$

(c) The Lagrangian in terms of the 4-potential is given as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) \\ &= -\frac{1}{2}(g^{\sigma\alpha}g_{\nu\beta}\partial_\sigma A^\beta \partial_\alpha A^\nu - \partial_\sigma A^\nu \partial_\nu A^\sigma),\end{aligned} \tag{7}$$

and the Euler-Lagrange equation for A^μ reads

$$\begin{gathered}\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A^\mu)} = 0 \\ \frac{1}{2}\partial_\rho\left[g^{\sigma\alpha}g_{\nu\beta}\left(\delta^\rho_\sigma \delta^\beta_\mu \partial_\alpha A^\nu + \partial_\sigma A^\beta \delta^\rho_\alpha \delta^\nu_\mu\right) - \left(\delta^\rho_\sigma \delta^\nu_\mu \partial_\nu A^\sigma + \partial_\sigma A^\nu \delta^\rho_\nu \delta^\sigma_\mu\right)\right] = 0 \\ \partial_\rho(\partial^\rho A_\mu - \partial_\mu A^\rho) = 0.\end{gathered} \tag{8}$$

Note that the object in parentheses is the field-strength tensor $F^\rho_{\ \mu}$, but we can act with the metric tensor on both sides to raise the index μ and relabel $\rho \to \mu$ and $\mu \to \nu$ to obtain the typical compact presentation of the Maxwell equations:

$$\boxed{\partial_\mu F^{\mu\nu} = 0}. \tag{9}$$

Unfolding, we have

$$\frac{\partial F^{0\nu}}{\partial t} + \frac{\partial F^{i\nu}}{\partial x^i} = 0, \tag{10}$$

so that

$$\begin{gathered}\frac{\partial F^{00}}{\partial t} + \frac{\partial F^{i0}}{\partial x^i} = -\frac{\partial E^i}{\partial x^i} = -\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \\ \frac{\partial F^{0j}}{\partial \frac{\partial}{\partial t}} + \frac{\partial F^{ij}}{\partial x^i} = -\frac{\partial E^j}{\partial t} - \epsilon^{ijk}\nabla^i B^k = \left(\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t}\right)_i = 0.\end{gathered} \tag{11}$$

Note that we are lacking two equations. We can obtain these using the Bianchi identity in a slightly different form. Observe that the Bianchi identity states $\partial_{[\mu}F_{\nu\rho]} = 0$. Thus, $\epsilon^{\mu\nu\rho\sigma}\partial_{[\nu}F_{\rho\sigma]} = \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$, which is not a trivial expression. Again, unfolding, we find

$$\begin{aligned}\epsilon^{0ijk}\partial_i F_{jk} &= \epsilon^{ijk}\nabla_i \epsilon^{jkm}B^m = 2\delta_{im}\nabla_i B^m = \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \\ \epsilon^{i\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= \epsilon^{i0jk}\frac{\partial F_{jk}}{\partial t} + 2\epsilon^{ij0k}\nabla_j F_{0k} = -\epsilon^{ijk}\epsilon^{jkm}\frac{\partial B^m}{\partial t} - 2\epsilon^{ijk}\nabla_j E^k \\ &= -2\left(\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\right)_i = 0\end{aligned}.$$

Next, we can construct the energy-momentum tensor for the Maxwell theory (sans sources). From the lecture notes, we have

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - g^{\mu\nu}\mathcal{L} \\ &= -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \end{aligned} \tag{12}$$

It is not too difficult to show directly that $\partial_\mu T^{\mu\nu} = 0$, but on the other hand, $T^{\mu\nu} \neq T^{\nu\mu}$. Notice that

$$\begin{aligned} T^{\mu\nu} &= -g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} - F^{\mu\rho}\partial_\rho A^\nu + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \\ &= -g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - \partial_\rho F^{\mu\rho}A^\nu, \end{aligned} \tag{13}$$

where we have brought the derivative through the field-strength tensor since $\partial_\mu F^{\mu\nu} = 0$ in the absence of sources. Observe that this last term is exactly the $\partial_\lambda K^{\lambda\mu\nu}$ prescribed above, and therefore, we define

$$\boxed{\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} = -g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}}. \tag{14}$$

Note that this redefined energy-momentum tensor is divergenceless since

$$\partial_\mu \hat{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\rho K^{\rho\mu\nu} = 0 \tag{15}$$

and is therefore a valid energy-momentum tensor. Observe we have used that the original energy-momentum tensor is divergenceless and $\partial_\mu\partial_\rho K^{\rho\mu\nu}$ is a contraction between symmetric and antisymmetric tensors in the indices μ, ρ, which always yields zero. Recall that the field-strength tensor takes the matrix representation

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{16}$$

Hence

$$\boxed{\begin{aligned} \hat{T}^{\mu\nu} &= F^{\mu\rho}g_{\rho\sigma}F^{\sigma\nu} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \\ &= \begin{pmatrix} (\boldsymbol{E}^2 + \boldsymbol{B}^2)/2 & B_zE_y - B_yE_z & B_xE_z - B_zE_x & B_yE_x - B_xE_y \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \end{aligned}}$$

Observe that $\hat{T}^{00}$ is exactly the expected energy density of the electromagnetic field and

$$\begin{aligned}
\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} &\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ E_x & E_y & E_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= (E_y B_z - E_z B_y)\hat{\boldsymbol{x}} + (E_z B_x - E_x B_z)\hat{\boldsymbol{y}} + (E_x B_y - E_y B_x)\hat{\boldsymbol{z}} \\
&= \hat{T}^{01}\hat{\boldsymbol{x}} + \hat{T}^{02}\hat{\boldsymbol{y}} + \hat{T}^{03}\hat{\boldsymbol{z}},
\end{aligned} \tag{17}$$

again as expected.