Problem 1)

Consider the classical complex Klein-Gordon field with the Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi\phi^*. \tag{1}$$

This has a global symmetry under a phase transformation $\phi \to \phi e^{i\alpha}$. Determine the Noether current corresponding to this symmetry.

First, observe that

$$\delta\mathcal{L} = \partial_\mu(e^{i\alpha}\phi)\partial^\mu(e^{-i\alpha}\phi^*) - m^2(e^{i\alpha}\phi)(e^{-i\alpha}\phi^*) - \mathcal{L} = 0. \tag{2}$$

Next, observe that under an infinitesimal phase transformation

$$\delta\phi = i\alpha\phi, \quad \delta\phi^* = -i\alpha\phi, \tag{3}$$

so the conserved current takes the form

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* \\ &= (\partial^\mu\phi^*)(i\alpha\phi) + (\partial^\mu\phi)(-i\alpha\phi^*) \\ &= i\alpha(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \\ &= -i\alpha\phi^*\overleftrightarrow{\partial}^\mu\phi, \end{aligned} \tag{4}$$

where $\overleftrightarrow{\partial}^\mu = \overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu$. Note that we can always rescale our current by a constant, so we redefine the conserved current as

$$\boxed{J^\mu = -i\phi^*\overleftrightarrow{\partial}^\mu\phi}, \tag{5}$$

which is now independent of α.

Problem 2)

Checking steps from class:

(a) Consider again the basic simple harmonic oscillator from undergraduate quantum mechanics (a.k.a. the $0+1$ scalar QFT). Show that converting the creation and annihilation operators from the Schrödinger to Heisenberg pictures gives

$$\hat{a}_H(t) = e^{-i\omega t}\hat{a}(t=0), \quad \hat{a}_H^\dagger = e^{i\omega t}\hat{a}^\dagger(t=0). \tag{6}$$

Note: I will always assume $\hbar = c = 1$. The H subscript means "Heisenberg operator".

(b) Recall that in treating the 1D lattice theory in class, I used the identity

$$\sum_j e^{ikja} = N\delta_{k0}. \tag{7}$$

Prove this expression for a general N.

(a) The Hamiltonian for the harmonic oscillator, in terms of the creation and annihilation operators, is

$$H = \omega\Big(a^\dagger a + \frac{1}{2}\Big). \tag{8}$$

These ladder operators satisfy the commutation relations

$$[a, a] = [a^\dagger, a^\dagger] = 0, \quad [a, a^\dagger] = 1. \tag{9}$$

Recall for a generic Schrödinger picture operator $\hat{A}$ that the Heisenberg picture counterpart is given as

$$\hat{A}_H(t) = e^{iHt}\hat{A}e^{-iHt}, \tag{10}$$

so

$$a_H(t) = e^{iHt}ae^{-iHt}. \tag{11}$$

We are now in a position to prove the Baker-Campbell-Hausdorf formula (really a variation of it).

Define a generic operator

$$C = e^A B e^{-A}. \tag{12}$$

Then

$$C(\lambda) = e^{\lambda A} B e^{-\lambda A}, \tag{13}$$

which obeys the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}\lambda} = AC - CA = [A, C]. \tag{14}$$

This is difficult to solve directly (perhaps impossible in all but a few nice cases), but anybody who has gone through a quantum mechanics course grows to love a good iterative solution, which is the way we proceed here:

$$C(\lambda) = C(0) + \int \mathrm{d}\lambda\, [A, C(\lambda)]. \tag{15}$$

Using this, we construct

$$C^{(n)} = B + \int \mathrm{d}\lambda\, [A, C^{(n-1)}], \tag{16}$$

where we use $C^{(0)} = C(0) = B$. We list out the first few iterative solutions

$$\begin{aligned}
C^{(1)} &= B + \int \mathrm{d}\lambda\, [A, B] = B + \lambda[A, B] \\
C^{(2)} &= B + \int \mathrm{d}\lambda\, [A, B + \lambda[A, B]] = B + \lambda[A, B] + \frac{\lambda^2}{2}[A, [A, B]] \\
C^{(3)} &= B + \lambda[A, B] + \frac{\lambda^2}{2}[A, [A, B]] + \frac{\lambda^3}{3!}[A, [A, [A, B]]].
\end{aligned} \tag{17}$$

We can now guess an explicit formula for $C^{(n)}$:

$$C^{(n)} = \sum_{k=0}^{n} \frac{\lambda^k}{k!} \underbrace{[A, [\ldots, [A}_{k \text{ times}}, B]], \tag{18}$$

which is easy enough to prove by induction. Hence, taking $n \to \infty$ and $\lambda \to 1$ to recover C, we find

$$C = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \underbrace{[A, [\ldots, [A}_{k \text{ times}}, B]]. \tag{19}$$

We have now reduced our problem to that of computing commutators between the Hamiltonian and a. Observe that

$$\begin{aligned}
[H, a] &= \omega\left(a^\dagger[a, a] + [a^\dagger, a]a\right) = -\omega a \\
[H, [H, a]] &= [H, -\omega a] = -\omega(-\omega a) = (-1)^2\omega^2 a \\
\Rightarrow \underbrace{[H, [\ldots, [H}_{n \text{ times}}, a]]] &= (-1)^n\omega^n a,
\end{aligned} \tag{20}$$

where the latter relation can be proven by induction again. Thus, putting all the pieces together

$$\boxed{a_H(t) = e^{iHt} a e^{-iHt} = \sum_{k=0}^{\infty} \frac{(-i\omega t)^k}{k!} a = e^{-i\omega t} a}. \tag{21}$$

A similar result holds for $a^\dagger$:

$$\boxed{a_H^\dagger = e^{i\omega t} a^\dagger}. \tag{22}$$

(b) Recall that this result came about in the context where $e^{ikNa} = 1$, which lead to the result

$$kNa = 2m\pi \Rightarrow k = \frac{2m\pi}{Na}, \tag{23}$$

where N is an even integer. Thus

$$\sum_{j=1}^{N} e. \tag{24}$$

$k = 2m\pi/L$, where $m \in (-N/2, N/2]$. This result is quite simple to show for $k = 0$:

$$\sum_{j=1}^{N} (e^{ija})^0 = N. \tag{25}$$

Problem 3)

Repeat the steps from class in constructing a classical lattice field theory in D dimensions, but now include a nonlinear term as follows:

$$H = \sum_{x}^{N^D} \frac{\dot{q}_x^2}{2} + \sum_{x}^{N^D} \sum_{\nu} \frac{\kappa}{2}(q_{x+\nu} - q_x)^2 + \sum_{x}^{N^D} \frac{m^2}{2} q_x^2 + \frac{\lambda}{4!} \sum_{x}^{N^D} q_x^4, \tag{26}$$

where the constant λ determines the strength of the effect of the nonlinear term. For taking the continuum limit, make the same replacements I used in class, but also take $\lambda \to g/a^D$, where g is a continuum version of λ. What Hamiltonian density do you get? What is the corresponding Lagrangian density? Can you solve the quantum version of the theory again by just using a's and $a^\dagger$'s as in the linear case? If not, what prevents you from doing so? In units where $\hbar = c = 1$, what are the units of g?

Problem 4)

Show that the following Lagrangian density gives a nonrelativistic classical field that at least structurally matches the form of a single particle Schrödinger equation,

$$\mathcal{L} = \frac{i}{2} \psi^\dagger(\boldsymbol{x}) \frac{\overleftrightarrow{\partial}}{\partial t} \psi(\boldsymbol{x}) - \frac{1}{2m} \boldsymbol{\nabla}\psi^\dagger(\boldsymbol{x}) \cdot \boldsymbol{\nabla}\psi(\boldsymbol{x}) - V(\boldsymbol{x})\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}). \tag{27}$$

What is the Hamiltonian density? In light of our discussion about the problems with second time derivatives when constructing relativistic wavefunction equations, what is noteworthy about this Hamiltonian?