Problem 1)

Consider again the classical complex Klein-Gordon field with the Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi\phi^*. \tag{1}$$

Repeat and write out all the steps that I showed in class for converting a real, lattice Klein-Gordon field to a quantum continuum version, but now for the complex scalar field above. Get the Heisenberg field operators $\hat{\phi}(x)$ and $\hat{\phi}^\dagger$ in terms of the creation and annihilation operators for particles and antiparticles.

In this problem, we can introduce some system with complex generalized coordinates $q_{\boldsymbol{n}}$ as in HW 4, problem 3 governed by a Lagrangian

$$L = \sum_{\boldsymbol{n}} \frac{1}{2}|\dot{q}_{\boldsymbol{n}}|^2 - \sum_{\boldsymbol{n}} \frac{m^2}{2}|q_{\boldsymbol{n}}|^2 - \sum_{\boldsymbol{n}}\sum_i \frac{\kappa}{2}|q_{\boldsymbol{n}+\hat{\boldsymbol{e}}_i} - q_{\boldsymbol{n}}|^2. \tag{2}$$

Notice that we can write this in terms of the real and imaginary parts of $q_{\boldsymbol{n}}$. For brevity, let us adopt notation where $\mathrm{Re}(q_n) = q_{R,\boldsymbol{n}}$ and $\mathrm{Im}(q_{\boldsymbol{n}}) = q_{I,\boldsymbol{n}}$

$$\begin{aligned} L = &\sum_{\boldsymbol{n}} \left\{ \frac{1}{2}\dot{q}_{R,\boldsymbol{n}}^2 - \frac{m^2}{2} q_{\boldsymbol{R,n}}^2 - \frac{\kappa}{2}[q_{R,\boldsymbol{n}+\hat{\boldsymbol{e}}_i} - q_{R,\boldsymbol{n}}]^2 \right\} \\ &+ \sum_{\boldsymbol{n}} \left\{ \frac{1}{2}\dot{q}_{I,\boldsymbol{n}}^2 - \frac{m^2}{2} q_{\boldsymbol{I,n}}^2 - \frac{\kappa}{2}[q_{I,\boldsymbol{n}+\hat{\boldsymbol{e}}_i} - q_{I,\boldsymbol{n}}]^2 \right\}. \end{aligned} \tag{3}$$

Hence, we can treat the real and imaginary parts as separate real-valued generalized coordinates and carry over the treatment from the prior homework and lectures. Summarizing, we find that the conjugate momenta to the real and imaginary parts of $q_{\boldsymbol{n}}$ are

$$p_{R,\boldsymbol{n}} = \dot{q}_{R,\boldsymbol{n}}, \quad p_{I,\boldsymbol{n}} = \dot{q}_{I,\boldsymbol{n}}, \tag{4}$$

respectively. Beware: it is tempting to conflate $p_{R,\boldsymbol{n}}$ and $p_{I,\boldsymbol{n}}$ with the real and imaginary parts of the complex generalized momentum $p_{\boldsymbol{n}}$ conjugate to $q_{\boldsymbol{n}}$, but this is not exactly the case since

$$p_{\boldsymbol{n}} = \frac{\partial L}{\partial \dot{q}_{\boldsymbol{n}}} = \frac{1}{2}\left(\frac{\partial L}{\partial \dot{q}_{R,\boldsymbol{n}}} - i\frac{\partial L}{\partial \dot{q}_{I,\boldsymbol{n}}} \right) = \frac{1}{2}(p_{R,\boldsymbol{n}} - ip_{I,\boldsymbol{n}}) = \frac{1}{2}\dot{q}_{\boldsymbol{n}}^*. \tag{5}$$

In terms of these momenta and coordinates, we find

$$\begin{aligned} H = &\sum_{\boldsymbol{n}} \left\{ \frac{1}{2}p_{R,\boldsymbol{n}}^2 - \frac{m^2}{2} q_{R,\boldsymbol{n}}^2 - \frac{\kappa}{2}[q_{R,\boldsymbol{n}+\hat{\boldsymbol{e}}_i} - q_{R,\boldsymbol{n}}]^2 \right\} \\ &+ \sum_{\boldsymbol{n}} \left\{ \frac{1}{2}p_{I,\boldsymbol{n}}^2 - \frac{m^2}{2} q_{I,\boldsymbol{n}}^2 - \frac{\kappa}{2}[q_{I,\boldsymbol{n}+\hat{\boldsymbol{e}}_i} - q_{I,\boldsymbol{n}}]^2 \right\}. \end{aligned} \tag{6}$$

At this point, normal coordinates are introduced such that

$$q_{R/I,\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \bar{q}_{R/I,\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}} \tag{7}$$

$$p_{R/I,\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \bar{p}_{R/I,\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}}, \tag{8}$$

and the imposition of periodic boundary conditions in our finite space of volume L^D gives

$$\boldsymbol{k} = \frac{2\pi\bar{\boldsymbol{n}}}{L} \tag{9}$$

with the components of our reciprocal lattice vector $\bar{n}_i \in (-N/2, N/2]$. Additionally, imposing the realness of our coordinates and momenta gives that

$$\bar{q}^*_{R/I,\boldsymbol{k}} = \bar{q}_{R/I,-\boldsymbol{k}}, \quad \bar{q}^*_{R/I,\boldsymbol{k}} = \bar{p}_{R/I,-\boldsymbol{k}}. \tag{10}$$

Thus, using all the machinery built up in the previous homeworks and lectures, we can find that

$$\begin{aligned} H = &\sum_{\boldsymbol{k}} \left\{ \frac{1}{2}\bar{p}_{R,\boldsymbol{k}}\bar{p}_{R,-\boldsymbol{k}} + \frac{\omega_{\boldsymbol{k}}^2}{2}\bar{q}_{R,\boldsymbol{k}}\bar{q}_{R,-\boldsymbol{k}} \right\} \\ &+ \sum_{\boldsymbol{k}} \left\{ \frac{1}{2}\bar{p}_{I,\boldsymbol{k}}\bar{p}_{I,-\boldsymbol{k}} + \frac{\omega_{\boldsymbol{k}}^2}{2}\bar{q}_{I,\boldsymbol{k}}\bar{q}_{I,-\boldsymbol{k}} \right\}, \end{aligned} \tag{11}$$

where

$$\omega_{\boldsymbol{k}}^2 = m^2 + 2\kappa \sum_i [1 - \cos(k_i a)]. \tag{12}$$

Again, one can see using the realness condition that the real and imaginary parts of the k-modes are fully decoupled harmonic oscillators. We can then introduce the canonical commutation relations

$$[\bar{q}_{R/I,\boldsymbol{k}}, \bar{p}_{R/I,-\boldsymbol{k}'}] = i\delta_{\boldsymbol{k},\boldsymbol{k}'}, \tag{13}$$

where the rest of the combinations are identically zero. From this, we introduce the creation and annihilation operators $a^\dagger_{R/I,\boldsymbol{k}}$ and $a_{R/I,\boldsymbol{k}}$, respectively, such that

$$\bar{q}_{R/I,\boldsymbol{k}} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}(a_{R/I,\boldsymbol{k}} + a^\dagger_{R/I,-\boldsymbol{k}}) \tag{14}$$

$$\bar{p}_{R/I,\boldsymbol{k}} = -i\sqrt{\frac{\omega_{\boldsymbol{k}}}{2}}(a_{R/I,\boldsymbol{k}} - a^\dagger_{R/I,-\boldsymbol{k}}). \tag{15}$$

Observe then that the creation and annihilation satisfy the typical commutation relations:

$$[a_{R/I,\boldsymbol{k}}, a^\dagger_{R/I,\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \tag{16}$$

and thus, the Hamiltonian can be expressed as

$$H = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \Big(a^\dagger_{R,\boldsymbol{k}} a_{R,\boldsymbol{k}} + a^\dagger_{I,\boldsymbol{k}} a_{I,\boldsymbol{k}} + 1 \Big). \tag{17}$$

At this point, we would like to translate our results to complex coordinates and momenta. In terms of the real and imaginary parts, we have

$$\begin{aligned} q_{\boldsymbol{n}} &= \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} (\bar{q}_{R,\boldsymbol{k}} + i\bar{q}_{I,\boldsymbol{k}}) e^{ia\boldsymbol{k}\cdot\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \bar{q}_{\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}} \\ p_{\boldsymbol{n}} &= \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{2} (\bar{p}_{R,\boldsymbol{k}} - ip_{I,\boldsymbol{k}}) e^{ia\boldsymbol{k}\cdot\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \bar{p}_{\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}}. \end{aligned} \tag{18}$$

Observe that these redefined Fourier coefficients have the desired commutation relations:

$$[\bar{q}_{\boldsymbol{k}}, \bar{p}_{-\boldsymbol{k}}] = \frac{1}{2} [q_{R,\boldsymbol{k}} + i\bar{q}_{I,\boldsymbol{k}}, \bar{p}_{R,-\boldsymbol{k}} - i\bar{p}_{I,-\boldsymbol{k}}] = i\delta_{\boldsymbol{k}',\boldsymbol{k}'} \tag{19}$$

$$[\bar{q}^*_{\boldsymbol{k}}, \bar{p}^*_{-\boldsymbol{k}}] = \frac{1}{2} [\bar{q}_{R,\boldsymbol{k}} - i\bar{q}_{I,\boldsymbol{k}}, \bar{p}_{R,-\boldsymbol{k}} + i\bar{p}_{I,-\boldsymbol{k}}] = i\delta_{\boldsymbol{k},\boldsymbol{k}'}. \tag{20}$$

Additionally, we have

$$\bar{q}_{\boldsymbol{k}} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \Big[(a_{R,\boldsymbol{k}} + ia_{I,\boldsymbol{k}}) + (a^\dagger_{R,-\boldsymbol{k}} + ia^\dagger_{I,-\boldsymbol{k}}) \Big] = \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} (a_{\boldsymbol{k}} + b^\dagger_{-\boldsymbol{k}}) \tag{21}$$

$$\bar{q}^*_{\boldsymbol{k}} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \Big[(a_{R,\boldsymbol{k}} - ia_{I,\boldsymbol{k}}) + (a^\dagger_{R,-\boldsymbol{k}} - ia^\dagger_{I,-\boldsymbol{k}}) \Big] = \frac{1}{\sqrt{\omega_k}} (b_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}), \tag{22}$$

Again, it is not too difficult to show that these redefined creation and annihilation operators satisfy the necessary commutation relations

$$[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] = \frac{1}{2} [a_{R,\boldsymbol{k}} + ia_{I,\boldsymbol{k}}, a^\dagger_{R,\boldsymbol{k}'} - ia^\dagger_{I,\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'} \tag{23}$$

$$[b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}'}] = \frac{1}{2} [a_{R,\boldsymbol{k}} - ia_{I,\boldsymbol{k}}, a^\dagger_{R,\boldsymbol{k}'} + ia^\dagger_{I,\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'} \tag{24}$$

and that our Hamiltonian

$$H = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} (a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + 1). \tag{25}$$

At this point, we can go directly and write the generalized coordinates in terms of these creation and annihilation operators:

$$q_{\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \Big(a_{\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}} + b^\dagger_{-\boldsymbol{k}} e^{ia\boldsymbol{k}\cdot\boldsymbol{n}} \Big) \tag{26}$$

$$q^\dagger_{\boldsymbol{n}} = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \Big(a^\dagger_{\boldsymbol{k}} e^{-ia\boldsymbol{k}\cdot\boldsymbol{n}} + b_{-\boldsymbol{k}} e^{-ia\boldsymbol{k}\cdot\boldsymbol{n}} \Big). \tag{27}$$

These are the Schrödinger picture operators, but we can change pictures using the typical prescription and the fact about the time dependence of the Heisenberg picture creation and annihilation operators:

$$q_{\boldsymbol{n}}(t) = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \Big(a_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t + ia\boldsymbol{k}\cdot\boldsymbol{n}} + b^\dagger_{\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t - ia\boldsymbol{k}\cdot\boldsymbol{n}} \Big) = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \Big(a_{\boldsymbol{k}} e^{-ik\cdot x} + b^\dagger_{\boldsymbol{k}} e^{ik\cdot x} \Big) \tag{28}$$

$$q^\dagger_{\boldsymbol{n}}(t) = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \Big(a^\dagger_{\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t - ia\boldsymbol{k}\cdot\boldsymbol{n}} + b_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t + ia\boldsymbol{k}\cdot\boldsymbol{n}} \Big) = \frac{1}{N^{D/2}} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \Big(a^\dagger_{\boldsymbol{k}} e^{ik\cdot x} + b_{\boldsymbol{k}} e^{-ik\cdot x} \Big). \tag{29}$$

Finally, we take the continuum limit by first taking $a \to 0$ and $N \to \infty$, holding L fixed. Defining $\phi(\boldsymbol{x}) = q_{\boldsymbol{n}}/a^{D/2}$ and similarly for the complex conjugate, we have

$$\phi(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \frac{1}{L^{D/2}\sqrt{\omega_{\boldsymbol{k}}}} \Big(a_{\boldsymbol{k}} e^{-ik\cdot x} + b^\dagger_{\boldsymbol{k}} e^{ik\cdot x} \Big) \tag{30}$$

$$\phi^\dagger(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \frac{1}{L^{D/2}\sqrt{\omega_{\boldsymbol{k}}}} \Big(a^\dagger_{\boldsymbol{k}} e^{ik\cdot x} + b_{\boldsymbol{k}} e^{-ik\cdot x} \Big). \tag{31}$$

Next, we take the infinite volume limit via $L \to \infty$ and redefining our continuum limit creation and annihilation operators via $a_{\boldsymbol{k}} \to a_{\boldsymbol{k}}/L^{D/2}$ and similarly for $b_{\boldsymbol{k}}$. Thus,

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^D \boldsymbol{k}}{(2\pi)^D \sqrt{\omega_{\boldsymbol{k}}}} \Big(a_{\boldsymbol{k}} e^{-ik\cdot x} + b^\dagger_{\boldsymbol{k}} e^{ik\cdot x} \Big) \tag{32}$$

$$\phi^\dagger(\boldsymbol{x}) = \int \frac{\mathrm{d}^D \boldsymbol{k}}{(2\pi)^D \sqrt{\omega_{\boldsymbol{k}}}} \Big(a^\dagger_{\boldsymbol{k}} e^{ik\cdot x} + b_{\boldsymbol{k}} e^{-ik\cdot x} \Big). \tag{33}$$

Problem 2)

Checking steps from class.

(a) Show that the effect of normal ordering on the Hamiltonian and Noether momentum is to eliminate any constant terms and puts $:\hat{H}:$ and $:\hat{P}_j:$ into a form that only involves number operators.

(b) Verify that the expression for the identity in the Fock space that we discussed class is

$$\hat{1} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=0}^{n} \frac{\mathrm{d}^3 \boldsymbol{p}_j}{(2\pi)^3 2E_{\boldsymbol{p}_j}} \left| p_n \right\rangle \left\langle p_n \right| \tag{34}$$

for the case of a three-excitation momentum state $|\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\rangle$.

(c) As in class, let a single excitation element of a bosonic Fock space at time t be

$$|f,1,t\rangle = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\tilde{f}(\boldsymbol{p})\,|\boldsymbol{p}\rangle = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}a^\dagger_{\boldsymbol{p}}\,|0\rangle\,\tilde{f}(\boldsymbol{p}) \tag{35}$$

with a wavepacket function $\tilde{f}(\boldsymbol{p})$. Let the coordinate space wavepacket function be defined by

$$f(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\tilde{f}(\boldsymbol{p})e^{-ip\cdot x} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\tilde{f}(\boldsymbol{p})e^{-ip^0t+i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{36}$$

Note the time dependence in the exponential despite the fact that the integral is only over spatial components. Show that

$$|f,1,t\rangle = \int \mathrm{d}^3\boldsymbol{x}\,\phi(\boldsymbol{x})\,|0\rangle\,2i\frac{\partial f(x)}{\partial t}. \tag{37}$$

(d) By using Fock states expressed like in Eq. (3) above, show directly that $a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}}$ is a density of excitations with respect to three momentum.

(a) For simplicity, we work in the context of a free scalar field theory (i.e. real Klein-Gordon theory). In this theory, our energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}(\partial^\rho\phi\partial_\rho\phi - m^2\phi^2). \tag{38}$$

Thus, the 4-momentum operator

$$P^\mu = \int \mathrm{d}^3\boldsymbol{x}\,T^{0\mu} = \int \mathrm{d}^3\boldsymbol{x}\left\{\dot{\phi}\partial^\mu\phi - \frac{1}{2}g^{0\mu}\Big(\partial^\rho\phi\partial_\rho\phi - m^2\phi^2\Big)\right\}. \tag{39}$$

The Hamiltonian is then

$$H = P^0 = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\left\{\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right\} \tag{40}$$

whilst the components of the momentum operator

$$P^i = \int \mathrm{d}^3\boldsymbol{x}\,\dot{\phi}\partial^i\phi. \tag{41}$$

Inserting the mode expansion of ϕ we find

$$H = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{\omega_{\boldsymbol{k}}}{2}(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}) \Rightarrow: H := \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\omega_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} \tag{42}$$

$$P^i = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{k^i}{2}(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}) \Rightarrow: P^i := \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}k^i a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}}. \tag{43}$$

It is simple to see that these normal ordered operators only contain number operators and no infinities.

(b) We verify this directly:

$$
\begin{aligned}
\hat{1}\left|\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3\right\rangle &= \sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{j=0}^{n}\frac{\mathrm{d}^3\boldsymbol{p}_j}{(2\pi)^3 2E_{\boldsymbol{p}_j}}\left|p_n\right\rangle\left\langle p_n|\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3\right\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{j=0}^{n}\frac{\mathrm{d}^3\boldsymbol{p}_j}{(2\pi)^3 2E_{\boldsymbol{p}_j}}\left|\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3\right\rangle\delta_{n3}\sum_{\sigma\in S_n}\prod_{\ell}(2\pi)^3 2E_{\boldsymbol{k}_\ell}\delta(\boldsymbol{k}_\ell-\boldsymbol{p}_{\sigma(\ell)}) \\
&= \frac{1}{3!}\sum_{\sigma\in S_n}\left|\boldsymbol{k}_{\sigma(1)},\boldsymbol{k}_{\sigma(2)},\boldsymbol{k}_{\sigma(3)}\right\rangle = \left|\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3\right\rangle.
\end{aligned} \tag{44}
$$

(c) We can invert the "Fourier transform" via the following procedure

$$
\begin{aligned}
\int\mathrm{d}^3\boldsymbol{x}\,f(x)e^{ip^0t-i\boldsymbol{p}\cdot\boldsymbol{x}} = \int\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}'}}}\tilde{f}(\boldsymbol{p}')\underbrace{\int\mathrm{d}^3\boldsymbol{x}\,e^{i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}}}_{(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}')} = \frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\tilde{f}(\boldsymbol{p}) \\
\tilde{f}(\boldsymbol{p}) = \int\mathrm{d}^3\boldsymbol{x}\,\sqrt{2E_{\boldsymbol{p}}}e^{ip\cdot x}f(x).
\end{aligned} \tag{45}
$$

Putting this into the wavepacket state

$$
\begin{aligned}
\left|f,1,t\right\rangle &= \int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}a^\dagger_{\boldsymbol{p}}\left|0\right\rangle\int\mathrm{d}^3\boldsymbol{x}\,\sqrt{2E_{\boldsymbol{p}}}e^{ip\cdot x}f(\boldsymbol{x}) \\
&= \int\mathrm{d}^3\boldsymbol{x}\,f(x)\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}2E_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}e^{ip\cdot k}\left|0\right\rangle \\
&= \int\mathrm{d}^3\boldsymbol{x}\,f(x)\Big(-2i\frac{\partial}{\partial t}\phi(x)\left|0\right\rangle\Big) \\
&= \int\mathrm{d}^3\boldsymbol{x}\,\phi(x)\left|0\right\rangle 2i\frac{\partial f(x)}{\partial t}.
\end{aligned} \tag{46}
$$

(d) We can take the expectation value of the number operator in a generic n-"particle" wavepacket and show that its integral gives us the expected number of particles in such a state:

$$
\left|f^{(n)}\right\rangle = \int\left(\prod_{j=0}^{n}\frac{\mathrm{d}^3\boldsymbol{p}_j}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}_j}}}\right)\tilde{f}^{(n)}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)\left|\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\right\rangle, \tag{47}
$$

where the state is normalized if and only if

$$
\int\prod_{j=0}^{n}\frac{\mathrm{d}^3\boldsymbol{p}_j}{(2\pi)^3}|\tilde{f}^{(n)}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)|^2 = \frac{1}{n!}. \tag{48}
$$

Let us now take the expectation value:

$$\left\langle f^{(n)}\middle|a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\middle|f^{(n)}\right\rangle = \int \prod_{i=0}^{n} \frac{\mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}_i}}} \prod_{j=0}^{n} \frac{\mathrm{d}^3\boldsymbol{p}_j'}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}_j'}}} [\tilde{f}^{(n)}(\boldsymbol{p}_1,\dots,\boldsymbol{p}_2)]^* \tilde{f}^{(n)}(\boldsymbol{p}_1',\dots,\boldsymbol{p}_n') \\ \times \langle \boldsymbol{p}_1,\dots,\boldsymbol{p}_n|a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}|\boldsymbol{p}_1',\dots,\boldsymbol{p}_n'\rangle . \tag{49}$$

The relevant thing to determine now is the momentum matrix element of this number operator. First, observe that

$$\begin{aligned} a_{\boldsymbol{k}}\left|\boldsymbol{p}_1,\dots,\boldsymbol{p}_n\right\rangle &= \sqrt{2E_{\boldsymbol{p}_1}}\dots\sqrt{2E_{\boldsymbol{p}_n}}a_{\boldsymbol{k}}a_{\boldsymbol{p}_1}^\dagger\dots a_{\boldsymbol{p}_n}^\dagger\left|0\right\rangle \\ &= \sqrt{2E_{\boldsymbol{p}_1}}\dots\sqrt{2E_{\boldsymbol{p}_n}}\Big[(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{p}_1)+a_{\boldsymbol{p}_1}^\dagger a_{\boldsymbol{k}}\Big]a_{\boldsymbol{p}_2}^\dagger\dots a_{\boldsymbol{p}_n}^\dagger\left|0\right\rangle \\ &= (2\pi)^3\sum_{i=1}^{n}\sqrt{2E_{\boldsymbol{p}_i}}\delta(\boldsymbol{k}-\boldsymbol{p}_i)\left|\boldsymbol{p}_1,\dots,\boldsymbol{p}_{i-1},\boldsymbol{p}_{i+1},\dots,\boldsymbol{p}_n\right\rangle . \end{aligned} \tag{50}$$

Thus

$$\begin{aligned} &\langle \boldsymbol{p}_1,\dots,\boldsymbol{p}_n|a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}|\boldsymbol{p}_1',\dots,\boldsymbol{p}_n'\rangle \\ &= (2\pi)^6\sum_{i,j=1}^{n}\sqrt{2E_{\boldsymbol{p}_i}}\sqrt{2E_{\boldsymbol{p}_j}}\delta(\boldsymbol{k}-\boldsymbol{p}_i)\delta(\boldsymbol{k}-\boldsymbol{p}_j')\left\langle \boldsymbol{p}_1,\dots,\boldsymbol{p}_{i-1},\boldsymbol{p}_{i+1},\dots,\boldsymbol{p}_n\middle|\boldsymbol{p}_1',\dots,\boldsymbol{p}_{j-1}',\boldsymbol{p}_{j+1},\dots,\boldsymbol{p}_n'\right\rangle \\ &= (2\pi)^6\sum_{i,j=1}^{n}\sqrt{2E_{\boldsymbol{p}_i}}\sqrt{2E_{\boldsymbol{p}_j}}\delta(\boldsymbol{k}-\boldsymbol{p}_i)\delta(\boldsymbol{k}-\boldsymbol{p}_j')\sum_{\sigma\in S_{n-1}}\prod_m(2\pi)^3 2E_{\boldsymbol{p}_m}\delta(\boldsymbol{p}_m-\boldsymbol{p}_{\sigma(m)}') \\ &= (2\pi)^3(2\pi)^{3n}2E_{\boldsymbol{p}_1}\dots 2E_{\boldsymbol{p}_n}\sum_{i,j=1}^{n}\sum_{\sigma\in S_{n-1}}\delta(\boldsymbol{k}-\boldsymbol{p}_i)\delta(\boldsymbol{p}_i-\boldsymbol{p}_j')\prod_m\delta(\boldsymbol{p}_m-\boldsymbol{p}_{\sigma(m)}'). \end{aligned} \tag{51}$$

Note that the notation for the permutations is not technically correct since we are omitting i from the domain set and j from the codomain set, but it should be understood implicitly nevertheless that we are just forming all possible pairings of momentum delta-functions. If we put this back into the expectation value, we see that

$$\left\langle f^{(n)}\middle|a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\middle|f^{(n)}\right\rangle = n!\sum_{i=1}^{n}\int\prod_{j\neq i}\frac{\mathrm{d}^3\boldsymbol{p}_j}{(2\pi)^3}|\tilde{f}^{(n)}(\boldsymbol{p}_1,\dots,\boldsymbol{p}_{i-1},\boldsymbol{k},\boldsymbol{p}_{i+1},\dots,\boldsymbol{p}_n)|^2. \tag{52}$$

At this point, it is difficult to see how we have a number density. Recall that densities live to be integrated, so we integrate over all $\boldsymbol{k}$ as follows:

$$\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left\langle f^{(n)}\middle|a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\middle|f^{(n)}\right\rangle = \left\langle f^{(n)}\middle|\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\middle|f^{(n)}\right\rangle = n. \tag{53}$$

Since the right-hand-side is independent of the shape of our wavepacket, we see that we have a true number density operator:

$$\mathrm{d}n = a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}. \tag{54}$$

Problem 3)

The following is a simple undergraduate electrodynamics problem that I aim to use to motivate you to think about the interpretation of infinite energies: Let there be a continuous line of electric charge with linear density $\lambda = \mathrm{d}Q/\mathrm{d}y$ running along the y-axis from a point $-L$ to a point $+L$. Consider a position at a perpendicular distance x away from the center of the line. What is the electric potential there if I use the standard expression $\mathrm{d}V = \mathrm{d}Q\,/(4\pi\epsilon_0 r)$ for a differential element of charge? Show that the potential energy of a charge placed at that point is infinite if $L \to \infty$. Does this mean that the physics outside an infinitely long line of charge like this is pathological or ill-defined? Elaborate on the analogy with the "infinite" constant we found in the continuum limit of the lattice Klein-Gordon theory.

Problem 4)

Let $\phi_\ell(t)$ be a massless real Klein-Gordon field averaged with a function proportional to e^{-r^2/ℓ^2}, where r is the distance from the origin of spatial coordinates. That is,

$$\phi_\ell(t) = \frac{\int \mathrm{d}^3\boldsymbol{\ell}\, \phi(\boldsymbol{x}) e^{-r^2/\ell^2}}{\int \mathrm{d}^3\boldsymbol{\ell}\, e^{-r^2/\ell^2}}. \tag{55}$$

Calculate the vacuum expectation value of $\phi_\ell(t)^2$,

$$\langle 0|\phi_\ell(t)^2|0\rangle\,. \tag{56}$$

The square root of this expectation value is an estimate of the size of fluctuations in thefield when probed with some kind of detector with resolution ℓ. Convert this quantity to volts. This estimate should also be roughly good for the electromagnetic field, to within a modest factor. Compute numerical values for a few distance scales of physical interest. In what situations might these 'zero point fluctuations' be of significance?