Problem 1)

A set of three integrals for testing numerical integration:

$$\int_0^1 x^3\,\mathrm{d}x = \frac{1}{4}, \quad \int_0^1 e^x\,\mathrm{d}x = e - 1, \quad \int_0^\pi \sin x\,\mathrm{d}x\,. \tag{1}$$

(a) Write a program (that will be your code 1) that calculates an integral with a given integrand $f(x)$ in the region $[a, b]$ by one of the Newton-Cotes rules with $n \geq 1$ (the trapezoid rule, or 3-point Simpson's rule, or 5-point Boole's rule or higher) and test it on the set of three "test" integrals. Explore how the accuracy changes with the number of integrals.

(b) Upgrade/modify your code (that will be your code 2) to include the error estimate based on calculations for two step sizes h and $h/2$. You can find the error estimation in the lecture notes (see "Extrapolation and Romberg integration"). Test your code with the set of three test integrals. Now you have a tool to estimate accuracy of numerical integration.

(c) Write a program (that will be your code 3) that calculates an integral from $f(x)$ in the region $[a, b]$ by using Gauss quadratures for 10 points. Coefficients for Gauss quadratures can be found in Abramowitz and Tegun "Handbook of Mathematical Functions", or on the web. Again, test your code with the three test integrals.

(d) Find an integration routine (preferably adaptive) you can use with the language of your choice (Python, C++, or MatLab). That will be your code 4. Again, test your routine with the three test integrals.

Problem 2)

Evaluate the following integrals using your codes 2, 3, and 4:

(a) $\displaystyle\int_0^1 \frac{1}{1 - 0.998x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_0^{2\pi} x \sin(30x)\cos(50x)$

(c) $\displaystyle\int_0^1 \frac{x}{e^x - 1}\,\mathrm{d}x$

(d) $\displaystyle\int_0^1 x \sin\left(\frac{1}{x}\right)\mathrm{d}x$

Report if one of your codes fails to compute some of the integrals. Explain why.

Problem 3)

Evaluate numerically following improper integrals (you may need to modify one of your codes). There are multiple ways to evaluate improper integrals. Explain your choice and results.

(a) $\displaystyle\int_0^\infty \frac{e^{-x^2}}{x^2+1}\,\mathrm{d}x$

(b) $\displaystyle\int_0^\infty \frac{x\sin x}{x^2+1}\,\mathrm{d}x$

(c) $\displaystyle\int_0^\infty e^{-\sqrt{x}}\cos(2x)\,\mathrm{d}x$

Problem 4)

Evaluate numerically following principal value integrals using one of techniques for princpal value integrals (again, you may need to modify one of your codes).

(a) $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,\mathrm{d}x$

(b) $\displaystyle\int_{-1}^1 \left(1+\frac{1}{x}\right)\mathrm{d}x$

(c) $\displaystyle\int_{-\infty}^\infty \frac{1}{(x-1)(x^2+1)}\,\mathrm{d}x$

Problem 5)

Evaluate numerically following double integrals.

(a) $\displaystyle\int_{-1}^1 \mathrm{d}x \int_0^2 \mathrm{d}y \sin\left(x^2+y^2\right)$

(b) $\displaystyle\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{\sqrt{x+y}(1+x+y)^2}$