Problem 1)

Laplace's equation in 2D rectangular coordinates

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \tag{1}$$

The Laplace equation applies to problems in steady state heat conduction, ideal fluid flow, electrostatics, etc. Write a code for solving 2D Laplace's equation using finite-difference 2nd-order method and solve 2D Laplace's equation on the square $[0, h] \times [0, h]$ with $h = 10$ and the boundary conditions $f(x, 0) = f(x, h) = f(0, y) = 0$ and $f(h, y) = 10$.

(a) Compute electric potential as a function of x and y.

(b) Create a surface plot of the calculated potential.

(c) Run 10, 100, and 1000 iterations, and note when convergence occurs.

(d) Investigate the effect of varying the step size Δh. Draw conclusions regarding the stability and accuracy of the solution for various Δh.

(e) *Optional*: Investigate the effect of using Gauss-Seidel relaxation versus Jacobi iterations. Which converges faster?

We can discretize our space and convert this continuous partial differential equation into one relating the value of f at neighboring lattice points using the finite difference method. Letting $f(x_i, y_j) = f_{i,j}$, we have

$$\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{\Delta x^2} + \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{\Delta y^2} = 0. \tag{2}$$

Rearranging, we have

$$f_{i,j} = \frac{1}{2}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}\left[\frac{f_{i-1,j} + f_{i+1,j}}{\Delta x^2} + \frac{f_{i,j-1} + f_{i,j+1}}{\Delta y^2}\right]. \tag{3}$$

We can devise a method of relaxation based on this equation to construct a sequence for function values at each lattice point such that

$$f_{i,j}^{(n+1)} = \frac{1}{2}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}\left[\frac{f_{i-1,j}^{(n)} + f_{i+1,j}^{(n)}}{\Delta x^2} + \frac{f_{i,j-1}^{(n)} + f_{i,j+1}^{(n)}}{\Delta y^2}\right], \tag{4}$$

where we choose some initial configuration for $f^{(0)}$.

(a)

Problem 2)

Poisson's equation in 2D rectangular coordinates has a nonhomogeneous term

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4\pi\rho(x, y). \tag{5}$$

Now consider the following boundary conditions for the square region $[0, h] \times [0, h]$ with $h = \pi$,

$$f(x, 0) = \sin x, \quad f(x, h) = |\sin 2x|$$
$$f(0, y) = \frac{1}{2}\sin y, \quad f(h, y) = 1.2\sin y, \tag{6}$$

and

$$-4\pi\rho(x, y) = \begin{cases} 360 & x = y = h/3 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

(a) Compute electric potential as a function of x and y.

(b) Create a surface plot of the calculated potential.

(c) Investigate the effect of varying the step size Δh. Draw conclusions regarding the stability and accuracy of the solution for various Δh.

In this problem, we can utilize the same method of relaxation as in problem 1. We need only add the source term such that

$$f_{i,j}^{(n+1)} = \frac{1}{2}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}\left[\frac{f_{i-1,j}^{(n)} + f_{i+1,j}^{(n)}}{\Delta x^2} + \frac{f_{i,j-1}^{(n)} + f_{i,j+1}^{(n)}}{\Delta y^2} - 4\pi\rho_{i,j}\right]. \tag{8}$$

Problem 3)

The heat equation for 1D case is

$$\frac{\partial T(x, t)}{\partial t} = a\frac{\partial^2 T(x, t)}{\partial x^2}. \tag{9}$$

You are given a metal bar of length $L = 1$ m and aligned along the x-axis. It is insulated along its length though not at its ends. Initially the entire bar is at a uniform temperature of $T_0 = 100°$C and then both ends are placed in contact with ice water at $0°$C. Heat flows through the non-insulated ends only.

(a) Write a code for solving $1D$ diffusion equation.

(b) Calculate how the temperature varies along the length of the bar as a function of time.

(c) Create a 3D plot of $T(x, t)$.

(d) Vary the time and space steps in your calculation so that you obtain solutions that are stable in time and vary smoothly in both space and time.

(e) Test what happens when the von Neumann stability condition $a\Delta t/\Delta x^2 \leq 0.5$ is not satisfied.

(f) *Optional*: The diffusion equation above can be solved analytically as series. Compare the analytic and numeric solutions.

Following the same logic as the previous two problems, we discretize our space and use the finite difference method to write

$$T_{i,j+1} = T_{i,j} + \frac{a\Delta t}{\Delta x^2}\left(T_{i-1,j} - 2T_{i,j} + T_{i+1,j}\right), \tag{10}$$

where $T_{i,j} \equiv T(x_i, t_j)$. With this, we already have a natural time-sequence from this equation given that $T(x, 0)$ is supplied as well as the boundary conditions.