Homework 2: Monte Carlo. Due on September 17, 2024 by 23:59

In this homework assignment we are getting familiar with random number generators, and with Monte Carlo integration.

Part 1: Exploring RNGs

1. Master using a random number generator available for you. Your preference should go, if possible, with the Mersenne Twister generator.
2. Calculate 5-th moment of the random number distribution and compare with the expected value. Explore how your agreement with the expected value changes as you increase the number of generated random numbers.
3. Calculate the near-neighbor correlation for (x_i, x_{i+5}) and compare with the expected analytical value.

Part 2: Generating non-uniform distribution.

1. Write a code that can generate non-uniform distributions of random numbers based on one, or two or all three he methods, namely, the rejection method, the transformation method (when possible), and the Metropolis algorithm (importance sampling).
2. Use your code to generate the following non-uniform distributions:

$$\begin{aligned}
&(a) \qquad p(y) = \frac{1}{a}\exp\left(-\frac{y}{a}\right) \qquad \text{Poisson distribution} \\
&(b) \qquad p(y) = \frac{2}{\pi}\frac{a}{a^2+y^2} \qquad \text{Cauchy} - \text{Lorentz distribution} \\
&(c) \qquad p(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \qquad \text{Gaussian distribution}
\end{aligned}$$

   Hints: 1) The rejections method is the easiest method to implement. All other methods are optional for this assignment. 2) For the Gaussian distribution it's good to use the Box-Muller method too.
3. Analyze quality of your distributions in any way you find appropriate. (hint - plot a histogram if possible)
4. Explore if any of the above distributions are available to you with either C++, or Python, or MatLab libraries.

Part 3: Integration using the mean value and rejection.

Evaluate the following integrals using the two above methods (the mean value and rejection) for various numbers of points $N = 10, 10^2, 10^3, 10^4, 10^5$. Evaluate the errors and explain your results.

3.1 $\displaystyle\int_0^\pi \sin x dx$

3.2 $\displaystyle\int_0^1 \frac{1}{1-0.998x^2}dx$

3.3 $\displaystyle\int_0^{2\pi} x\sin(12x)\cos(24x)dx$

3.4 $\displaystyle\int_0^2 \sin^2\left[\frac{1}{x(2-x)}\right]dx$

Part 4: Multi-dimension integration by the mean method.

Compute the following integrals:

4.1 A double integral over a rectangular region:

$$\int_0^1\int_0^2 \sin(x^2+y^2)dxdy$$

4.2 A double integral over a circular region:

$$\int_{circle} e^{-(x^2+y^2)}dxdy$$

where the circle is centered at the origin with radius 1.

4.3 A double integral over a non-rectangular region

$$\int_0^1\int_0^{1-x}(x+y)dydx$$

4.4 Four-dimensional integral

$$\int_0^1\int_0^1\int_0^1\int_0^1 e^{-(x_1^2+x_2^2+x_3^2+x_4^2)}dx_1dx_2dx_3dx_4$$

4.5 Four-dimensional integral over a spherical region

$$I=\int_{sphere}\left(x^2+y^2+z^2+w^2\right)e^{-(x^2+y^2+z^2+w^2)}\,dx\,dy\,dz\,dw$$

where the domain of integration is the four-dimensional sphere (or hypersphere) defined by: $x_1^2+x_2^2+x_3^2+x_4^2\le 1$.