Examining two methods for determining the local gravitational constant

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Abstract

One can measure the local gravitational constant $g = 9.806\,65\,\mathrm{m/s^2}$ using numerous methods with varying degrees of accuracy and precision. We present a simple method and another more sophisticated method. It is found that the results produced with the second method are more precise (i.e. less uncertain) and that it has the potential to be more accurate. Although, with better tools and setup, the results from both methods can be made better.

I. INTRODUCTION

In most undergraduate physics labs, a traditional experiment performed is one to measure the local gravitational constant. This can be done simply by dropping an object from a prescribed height and measuring the time it takes for the object to fall. With these values, we can solve the one-dimensional kinematic equation for position over time, assuming that the initial velocity is zero (or negligibly small), and isolate the local gravitational constant.

While this simple method is certainly valid, it may not be ideal. In most cases, the tools available for such an experiment are rudimentary, introducing significant human error into the calculations. An alternative method may be to record the motion of the object as it falls and use a numerical scheme to calculate the acceleration at different steps while it falls and use fitting methods to extract the local gravitational constant.

In this work, we measure the local gravitational constant using both methods and compare the results obtained.

II. METHODS

For this experiment, we need only a small set of materials. For both methods, a length measuring device and an object to drop are needed. Here, we use a meter stick to measure distances and drop a foam stress ball and baseball. A stopwatch is needed for the sim-

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pler time-based method. In this work, we use the default timer on a smartphone. And in the second method, we need a camera setup. For this work we use the default camera on a smartphone (30 fps, 1080 p HD) and a tripod to stabilize the camera.

Note that many smartphones provide the user with the ability to change the frame rate and resolution of the video taken, so to optimize the results, one may like to manipulate the settings until the best combination is obtained. Here, we have just used the default settings.

Additionally, the second method requires some software or tool for extracting the positions of the objects in each frame. For this work, the Tracker software was utilized, which has some nice features, including auto-tracking the position of the object, specifying length scales in the video, and setting up a coordinate system.

III. THEORY

The theoretical background for this work is not incredibly extensive, requiring only a knowledge of basic kinematics, finite difference methods, and parameter fitting methods which minimize χ^2.

We first review the necessary kinematics. In this work, we assume that the objects used are massive enough and dropped from small enough heights that the effects of drag are negligible. Within this regime, the general equation of motion is

$$y = v_0 t + \frac{1}{2} g t^2, \tag{1}$$

organizing our coordinate system so that the origin coincides with the start position of the object and the direction of increasing positive positions is downwards.

For our experiments, we drop our objects from rest, meaning $v_0 = 0\,\mathrm{m/s}$. In our first method we only care about the distance the objects drop (h) and the time from when the objects are dropped to when they hit the ground (T). This allows us to calculate the local gravitational constant as

$$g = \frac{2h}{T^2}. \tag{2}$$

For our second method we record the positions of our objects as a function of time with a camera and calculate accelerations at each step using finite differences. We can derive our finite difference approximations using Taylor expansions and taking weighted linear combinations to isolate the needed derivatives and achieve desired computational accuracy. The general Taylor series in question is the following:

$$f(x+ih) = \sum_{j=0}^{n} \frac{(ih)^j}{n!} f^{(n)}(x) + \mathcal{O}(h^{n+1}), \tag{3}$$

where i is not the imaginary unit but rather an integer defining how many step sizes from

x the expansion is at. The last term on the Taylor expansion is the truncation error for the expansion. Derivatives are approximated as

$$f''(x) = \sum_i \alpha_i f(x + ih). \tag{4}$$

Derivatives at interior points are calculated using a five point stencil using the central difference, and the derivatives at the endpoints of the grid are computed using a three point forward and backward difference for left and right endpoints, respectively. The relevant equations for these approximations are recorded in Appendix A.

Lastly, we discuss parameter fitting. If we have a model $f(\vec{x};\vec{\alpha})$, which describes the output values of an experiment in terms of the independent variables $\vec{x}$, the unknown parameters $\vec{\alpha}$, and data from an experiment, then the χ^2 tells us how well the model fits the data. The explicit equation is

$$\chi^2 = \sum_i \frac{[y_i - f(\vec{x}_i;\vec{\alpha})]^2}{\sigma_i^2}, \tag{5}$$

where σ_i is the uncertainty in y_i. Notice that we can fit a model to experimental data by minimizing χ^2 with respect to the parameters. That is, we enforce

$$\vec{\nabla}_{\vec{\alpha}} \chi^2 = 0. \tag{6}$$

This gives us a linear system of equations, which allows us to solve for each of the parameters in terms of the data taken and their uncertainties. In some cases we take the uncertainties to be uniform across the data set, which allows us to pull out the factor $\sigma_i^2 = \sigma^2$ as a constant and neglect it in the fitting.

IV. RESULTS

In this section we briefly cover the results of the experiments, saving commentary and interpretation for later sections. For each of the methods a mean value for the local gravitational constant was extracted with an associated error. These results are tabulated in Table I.

TABLE I: Values for the local gravitational acceleration (in $\frac{\text{m}}{\text{s}}$).

	Red Ball	Baseball
Method 1	14(2)	10(1)
Method 2	9.2(2)	9.9(2)

Note that in this work we make no distinction between statistical and systematic errors. For our first method, we collect a set of time measurements, average them, and calculate an uncertainty for the average assuming a normal distribution and a uniform uncertainty for time measurements:

$$\delta\bar{t} = \frac{\delta t}{\sqrt{N}}. \tag{7}$$

Then, the error in g is found using the standard equation

$$\delta g = \sqrt{\left(\frac{\partial g}{\partial h}\delta h\right)^2 + \left(\frac{\partial g}{\partial T}\delta T\right)^2}, \qquad (8)$$

where $\delta h = 5\,\mathrm{cm}$ and $\delta T = 200\,\mathrm{ms}$. Observe that we neglect any covariance in the sample.

In our second method, we calculate the error in acceleration at each step as half the difference between the upper and lower bounds for position. The errors for position are calculated as

$$\delta y = r + 1\,\mathrm{cm} + r\sqrt{f/N}, \qquad (9)$$

where r is the radius of the ball, the centimeter comes from the uncertainty in the start position, N is the total number of frames, and f is the frame number. The last term accounts for the blurriness of the ball as it picks up speed. The final values for g are calculated as a weighted sum of the values from each drop, where the weights are those from χ^2.

V. DISCUSSION

It is seen from the previous section that the results for the local gravitational constant are somewhat mixed. For both methods, the baseball drops produced fairly good results. The second method has a small uncertainty, while the first gives a larger relative uncertainty on the order of the mean value. Both, though, agree with the accepted value as they are within one standard deviation of the accepted value.

The foam stress ball drops did not produce as good of results as the baseball drops. Both vary by more than one standard deviation from the accepted result. The uncertainties, though, are consistent with the baseball drop uncertainties. Interestingly, because of the small uncertainty for the second method, the value is only 6% smaller than the accepted value, but lies more than 3 standard deviations from it.

VI. CONCLUSION

This experiment measures the local gravitational constant with a moderate degree of success. It is seen that the tools used are not very precise. The first method introduces a very large human error from reaction time, which dominates the process. By finding more sophisticated tools or sensors, one may be able to get rid of such large time-related errors, making the overall method more accurate and precise. The second method could be improved by increasing the frame rate and resolution. Increasing the frame rate allows more data points to be taken, which decreases the step size and therefore the computational error in the numerical derivatives, which may lead to more consistent results in the calculation between trials. Furthermore, increasing the resolution of the camera could lead to better tracking of the position over longer time spans and at larger speeds, leading to less error over the course of each drop.

Regarding the computational procedures, the second method may be improved by examining the numerical techniques for differentiation in more detail and potentially improving upon them, finding methods which are more consistent between time steps. Additionally, we used the whole time domain of each video for position data. Given that the relative error in position is larger when the ball is dropped and that the absolute error is largest at the end of the drop, the calculations may be unstable in these regions. For future experiments, some analysis about the regions in which the calculation is stable may improve the results of the experiment.

Lastly, both methods would benefit from a more rigorous treatment of errors. The treatment here is quite simple, especially for propagating errors for the acceleration in the second method. With these adjustments, g may be extracted to a better degree of success.

Appendix A: Finite Differences

The forward difference used for the second derivative using a three point stencil is

$$f''(x) \approx \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}. \tag{A1}$$

The backward difference used for the second derivative using a three point stencil is

$$f''(x) \approx \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}. \tag{A2}$$

Lastly, the central difference used for the second derivative using a five point stencil is

$$f''(x) \approx \frac{-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h)}{12h^2}. \tag{A3}$$