

**General Math:**

$$\operatorname{Re}(\arctan z) = \frac{1}{2} \arctan\left(\frac{2 \operatorname{Re}\{z\}}{1 - |z|^2}\right)$$

$$\operatorname{Im}(\arctan z) = \frac{1}{2} \arctan\left(\frac{2 \operatorname{Im}\{z\}}{1 + |z|^2}\right)$$

**Gamma Function:**

$$\Gamma(n + 1/2) = \frac{(2n - 1)!!\sqrt{\pi}}{2^n} = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

**Power Series:**

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1}$$

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\ln\left(\frac{1 + x}{1 - x}\right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n + 1}$$

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

$$(1 + x)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} (-1)^k x^k$$

**Fourier orthogonality:**

$$\frac{2}{L} \int_a^{L+a} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \delta_{nm} \quad (1)$$

$$\frac{2}{L} \int_a^{L+a} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) = \delta_{nm} \quad (2)$$

$$\frac{2}{L} \int_a^{L+a} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = 0. \quad (3)$$

**Spherical Harmonics:**

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}$$

Complex conjugation –  $Y_{lm}^*(\theta, \phi) = (-1)^m Y_{l,-m}(\theta, \phi)$

Parity transformation –  $Y_{lm}(\pi - \theta, \pi + \phi) = (-1)^l Y_{lm}(\theta, \phi)$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \gamma) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

**Legendre:**

Legendre Normalization –  $\int_{-1}^1 dx P_l^m(x) P_{l'}^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$

Recursion relations

$$(1) P_n(x) = \frac{1}{2n+1} [P'_{n+1}(x) - P'_{n-1}(x)] \quad (4)$$

$$(2) P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x). \quad (5)$$

$$\int_0^1 dx P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}$$

First few:

$$P_0(x) = 1 \quad (6)$$

$$P_1(x) = x \quad (7)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad (8)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad (9)$$

$$P_l(1) = 1, \quad P_l(-1) = (-1)^l$$

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = \frac{(-1)^n (2n-1)!!}{2^n n!}$$

**Bessel:**

Bessel's equation –  $s \frac{d}{ds} \left( s \frac{dR}{ds} \right) + (k^2 s^2 - \nu^2) R = 0$  with solution  $J_\nu(ks)$

2nd solution:  $J_{-\nu}$  for  $\nu \notin \mathbb{Z}$ ,  $N_\nu$  (Neumann)

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x^2}{4}\right)^n, \quad J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x^2}{4}\right)^n$$

$$J'_0(x) = -J_1(x), \quad xJ_0(x) = (xJ_1(x))'$$

Asymptotic forms ( $x \gg 1$ ):

$$J_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad (10)$$

$$N_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad (11)$$

$$x \ll 1: J_\nu(x) = \frac{1}{\nu!} (x/2)^\nu, J_{-\nu \in \mathbb{N}} = \frac{(-1)^\alpha}{\nu!} (x/2)^{-\nu}$$

Zeros:  $x_{\nu n}$  such that  $J_\nu(x_n) = 0$

$$\text{Orthogonality: } \int_0^a ds s J_\nu(x_{\nu n} s/a) J_\nu(x_{\nu n'} s/a) = \frac{a^2}{2} [J_{\nu+1}(x_{\nu n})]^2 \delta_{nn'}$$

$$\text{Modified Bessel equation: } s \frac{\partial}{\partial s} \left( s \frac{\partial R}{\partial s} \right) - (k^2 s^2 + \nu^2) R = 0$$

Modified Bessel function:

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \rightarrow \begin{cases} \frac{1}{\Gamma(\nu+1)} (x/2)^\nu & x \ll 1 \\ \frac{1}{\sqrt{2\pi x}} e^x [1 + \mathcal{O}(1/x)] & x \gg 1 \end{cases} \quad (12)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \rightarrow \begin{cases} -[\ln(x/2) + \gamma_E + \dots] & x \ll 1, \nu = 0 \\ \frac{\Gamma(\nu)}{2} (2/x)^\nu & x \ll 1, \nu \neq 0 \\ \sqrt{\frac{\pi}{2x}} e^{-x} [1 + \mathcal{O}(1/x)] & x \gg 1 \end{cases} \quad (13)$$

$$\text{Hankel Transform} - \mathcal{H}_m(s) = \int_0^\infty dk A(k) J_m(ks) \Rightarrow A(k) = k \int_0^\infty ds s J_m(ks) \mathcal{H}_m(s)$$

**Laplace's equation:**

$$\nabla^2 \Phi = 0. \quad (14)$$

Cartesian separation –  $\Phi = X(x)Y(y)Z(z)$

Cylindrical (no  $z$ -dependence) –  $\Phi = (a_0 + b_0 \ln s)(A_0 + B_0 \phi) + \sum_{\nu > 0} (a_\nu s^\nu + b_\nu s^{-\nu})(A_\nu \sin \nu \phi + B_\nu \cos \nu \phi)$

Spherical (no  $\phi$ -dependence) –  $\Phi = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$

Note: solution along  $z$ -axis  $\Rightarrow z^l \rightarrow r^l P_l(\cos \theta)$ ,  $1/z^{l+1} \rightarrow \frac{1}{r^{l+1}} P_l(\cos \theta)$

Spherical –  $\Phi = \sum_{l,m} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$

### Green's function

Spherical –  $G(\vec{x}, \vec{x}') = \sum_{l,m} g_l(r, r') Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3\vec{x}' G(\vec{x}, \vec{x}') \rho(\vec{x}') + \frac{1}{4\pi} \int_S dS' \left[ G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right]$$

Diriclet –  $G_D(\vec{x}, \vec{x}') = 0$  for  $\vec{x}' \in S$

Neumann –  $\frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = -\frac{4\pi}{S}$  for  $\vec{x}' \in S$

Dirichlet concentric spheres –  $g_l(r, r') = \frac{4\pi}{2l+1} \left[ 1 - \frac{a^{2l+1}}{b^{2l+1}} \right]^{-1} \left( r_{<}^l - \frac{a^{2l+1}}{r_{<}^{l+1}} \right) \left( \frac{1}{r_{>}^{l+1}} - \frac{r_{>}^l}{b^{2l+1}} \right)$

Neumann concentric spheres –

$$l \neq 0 : g_l(r, r') = \frac{r_{<}^l}{r_{>}^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left[ \frac{l+1}{l} (rr')^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1} \left( \frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}} \right) \right] \quad (15)$$

$$l = 0 : g_0(r, r') = \frac{1}{r_{>}} - \frac{a^2}{a^2 + b^2} \frac{1}{r'} + f(r) \quad (16)$$

### Multipole expansion

$$\text{Cartesian} - \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left( \frac{Q}{r} + \frac{\vec{P} \cdot \vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} \frac{Q_{ij} x_i x_j}{r^5} + \dots \right)$$

$$Q = \int_V d^3\vec{x} \rho(\vec{x}) \quad (17)$$

$$\vec{P} = \int_V d^3\vec{x} \rho(\vec{x}) \vec{x} \quad (18)$$

$$Q_{ij} = \int_V d^3\vec{x} \rho(\vec{x}) (3x_i x_j - r^2 \delta_{ij}) \quad (19)$$

$$\text{Spherical Harmonics} - \Phi(\vec{r}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \text{ where } q_{lm} = \int d^3\vec{r}' Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{r}')$$

### Exam 1 Carry-over:

$$\text{Surface charge density} - \sigma = \epsilon_0 \vec{E} \cdot \hat{n} = -\epsilon_0 \frac{\partial \Phi}{\partial n}$$