PHYS 604 – Electrodynamics
Study Notes

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CHAPTER 1

Coulomb's Law

In this section we deal with electrostatics, which is the idealization where the charge configurations are stationary in time, or rather, we only deal with their instantaneous properties. For electrostatics, the governing equation is Coulomb's law (we use traditional SI units):

$$\vec{\boldsymbol{F}}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|^3}, \tag{1.1}$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ $\mathrm{C^2N^{-1}m^{-2}}$ is the permittivity of free space. This is the force on charge 1 by charge 2. Notice that from Newton's 3rd law, the force $\vec{\boldsymbol{F}}_{21} = -\vec{\boldsymbol{F}}_{12}$ (equal and opposite).

Coulomb's law is an empirical observation. That is, the inverse square law is written down to match experimental observations, and indeed, it does a fantastic job explaining electrostatic phenomena. We will see that it is incomplete for a dynamic picture of electromagnetism, but for now (the next several lectures), it will be all we need.

Another empirical observation is that the force on a charge q by a configuration of charges q_i at positions $\vec{\boldsymbol{r}}_i$, respectively, is just a linear superposition of the individual forces on q by q_i:

$$\vec{\boldsymbol{F}} = q\left[\frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i|^3}\right] = q\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}). \tag{1.2}$$

The quantity $\vec{\boldsymbol{E}}$ is defined as the electrostatic field due to the charge configuration. It's rigorous definition requires a bit more care, actually, since the electric field due to q would distort the electric field due to the configuration, so

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \lim_{q\to 0} \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})}{q}. \tag{1.3}$$

In the above definitions we dealt with discrete point charges, but in general, we may deal with continuous charge distributions. Charge distributions are

idealizations of course. Realistically speaking, charge is quantized in units of $e = 1.602 \times 10^{-19}$ C, but if we have a large number of charges then the total charge of a distribution is $Q = (N_+ - N_-)e = \int \mathrm{d}q = \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\rho(\vec{\boldsymbol{r}})$ in the limit $e/Q \to 0$, where $N_\pm$ is the number of units of positive/negative charge in the solid body. The function $\rho(\vec{\boldsymbol{r}})$ is the volume charge density. From this, we can then write the electrostatic field of a continuous charge distribution with charge density $\rho(\vec{\boldsymbol{r}})$ as

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\vec{\boldsymbol{r}}'\,\rho(\vec{\boldsymbol{r}}')\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}. \tag{1.4}$$

CHAPTER 2

Gauss' Law

2.1 Derivation

Suppose that the charge distribution $\rho(\vec{\boldsymbol{r}})$ is the only source of the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$. We can define the "flux" of an electric field through some surface S as

$$\text{flux} = \int_S \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{S}}\,, \tag{2.1}$$

where $\mathrm{d}\vec{\boldsymbol{S}} = \mathrm{d}S\,\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}$ is the unit normal vector to the surface S, pointing outside the volume bounded by S.

We can derive Gauss' law as follows for the case of a point charge located at the origin by using a sphere of radius R.

$$\int \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \int \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}}{R^2} R^2\,\mathrm{d}\Omega = \frac{q}{4\pi\epsilon_0}\int \mathrm{d}\Omega = \frac{q}{\epsilon_0}. \tag{2.2}$$

This result is more general though, so let us explore this.

Notice that the divergence of the electric field produced by a point charge at the origin is given as

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{q}{4\pi\epsilon_0}\vec{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0, \tag{2.3}$$

when $r \neq 0$. If we take the volume integral over the volume bounded between a sphere of radius ϵ (which is small) and a generic surface S, we find

$$\int_V \mathrm{d}^3\vec{\boldsymbol{r}}\,\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \int_S \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{S}} - \int_{S_\epsilon} \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{r}}(\epsilon^2\,\mathrm{d}\Omega) = 0. \tag{2.4}$$

This then gives that

$$\int_S \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \frac{q}{\epsilon_0}. \tag{2.5}$$

Notice that this result holds even if the point charge is not located at the origin since we can always place the sphere of radius ϵ such that the charge is centered on it.

Let us derive this result in another way, returning to the divergence of $\vec{\boldsymbol{E}}$ at $r = 0$. Consider the vector field $\hat{\boldsymbol{r}}/r^2$, and observe that

$$\int_S \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \int_V \vec{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\, \mathrm{d}^3\vec{\boldsymbol{r}} = 4\pi. \tag{2.6}$$

We can thus define

$$\vec{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\delta^{(3)}(\vec{\boldsymbol{r}}), \tag{2.7}$$

where $\delta^{(3)}(\vec{\boldsymbol{r}})$ is the three dimensional Dirac-delta function.

In one dimension, the Dirac-delta function is defined such that $\delta(x - a) = 0$ if $x \neq a$ and

$$\int_{\mathcal{R}} f(x)\delta(x - a)\, \mathrm{d}x = \begin{cases} f(a) & \text{if } a \in \mathcal{R} \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

The three-dimensional generalization is just

$$\delta^{(3)}(\vec{\boldsymbol{r}} - \vec{\boldsymbol{a}}) = \delta(x - a_x)\delta(y - a_y)\delta(z - a_z). \tag{2.9}$$

Using these results, we have

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{q}{4\pi\epsilon_0} \vec{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = \frac{q}{\epsilon_0}\delta^{(3)}(\vec{\boldsymbol{r}}). \tag{2.10}$$

If shift our origin such that the charge q is now located at $\vec{\boldsymbol{r}}_0$, then we have

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{q}{\epsilon_0}\delta^{(3)}(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0). \tag{2.11}$$

Hence, Gauss' law for a point charge is recovered by taking the volume integral of Eq. (2.11) and using the divergence theorem to find the flux of $\vec{\boldsymbol{E}}$. Furthermore, for a charge discrete charge distribution, we replace q with the total charge enclosed by the surface S.

Now, let us consider the more general case of a charge distribution $\rho(\vec{\boldsymbol{r}})$. The divergence of the electric field is

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3\vec{\boldsymbol{r}}'\, \rho(\vec{\boldsymbol{r}}')\vec{\boldsymbol{\nabla}} \cdot \frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} = \frac{1}{\epsilon_0} \int \mathrm{d}^3\vec{\boldsymbol{r}}\, \rho(\vec{\boldsymbol{r}}')\delta^{(3)}(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') \\ &= \frac{\rho(\vec{\boldsymbol{r}})}{\epsilon_0}. \end{aligned} \tag{2.12}$$

As an useful note, we can write the charge density for a point charge at location $\vec{\boldsymbol{r}}$ as

$$\rho(\vec{\boldsymbol{r}}) = q\delta(\vec{\boldsymbol{r}}). \tag{2.13}$$

Alternatively, we can take the volume integral, which gives

$$\int \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \frac{Q}{\epsilon_0}, \tag{2.14}$$

where Q is the charge enclosed by S.

2.2 Applications

Gauss' law is only useful in some circumstancess, where there is a nice symmetry. Otherwise, the integration becomes too complicated for a direct, analytic determination of the electric field. Let us consider a few cases.

2.2.1 Spherical Symmetry

Suppose we have a distribution satisfying $\rho(\vec{\boldsymbol{r}}) = \rho(r)$, meaning that the charge distribution is rotationally invariant about some point. In this case then, we take a Gaussian surface that is a sphere of radius a:

$$\int \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{r}}(a^2 \,\mathrm{d}\Omega) = E_r(a)[4\pi a^2] = \frac{Q}{\epsilon_0} \Rightarrow E_r(a) = \frac{1}{4\pi a^2}\frac{Q}{\epsilon_0}, \tag{2.15}$$

where $Q = \int_V \rho(\vec{\boldsymbol{r}})\,\mathrm{d}^3\vec{\boldsymbol{r}} = 4\pi \int_0^a r^2\rho(r)\,\mathrm{d}r$. Thus,

$$E_r(a) = \frac{1}{a^2\epsilon_0}\int_0^a r^2\rho(r)\,\mathrm{d}r\,. \tag{2.16}$$

Notice that this integral only picks out the radial component of the field, but in fact, the angular components are easily seen to be zero since if they weren't then our problem would not be rotationally invariant. Hence, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = E_r(r)\hat{\boldsymbol{r}}$

One nice result is to consider a sphere of total charge Q. We remain agnostic about the exact distribution of charge, but we can state the electric field outside the sphere is just

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{r}}. \tag{2.17}$$

That is, outside the sphere, the field is only determined by the total charge and follows the same law as if it were a point charge at the origin.

If we specifiy a uniform charge density $\rho(\vec{\boldsymbol{r}}) = \rho_0$ inside the sphere (of radius a), we have the electric field for $r < a$ as

$$\vec{\boldsymbol{E}} = \frac{\rho_0}{\epsilon_0}\frac{1}{r^2}\left(\frac{1}{3}r^3\right)\hat{\boldsymbol{r}} = \frac{\rho_0}{3\epsilon_0}\vec{\boldsymbol{r}}. \tag{2.18}$$

2.2.2 Cylindrical Symmetry

Suppose we have a charge distribution satisfying $\rho(\vec{\boldsymbol{r}}) = \rho(s)$, where s is the distance from the $z-axis$. Then we choose a Gaussian surface that is a cylinder with radius a and length ℓ, giving

$$\int \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{s}}s\,\mathrm{d}\phi\,\mathrm{d}z = (2\pi L)sE_s = \frac{Q}{\epsilon_0} \Rightarrow \vec{\boldsymbol{E}} = \frac{Q/L}{2\pi\epsilon_0 s}\hat{\boldsymbol{s}}. \tag{2.19}$$

Again, the non-radial components of the field must be zero because of the symmetry of the problem. Typically, we denote $Q/L = \lambda$, where λ is a characteristic charge per unit length of the problem. Note that we implicitly include the integration over the caps of the cylinder, but even if E_z were nonzero, the normal vectors point in opposite directions and the problem is translationally invariant along z, meaning that there contribution cancels.

CHAPTER 3

Maxwell Equations

We have already derived the first of Maxwell's equations, which is just Gauss' law:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{\rho(\vec{\boldsymbol{r}})}{\epsilon_0}, \tag{3.1}$$

derived from the integral form using the divergence theorem since the integral form is independent of the volume considered. Actually, it will not need any alteration when we consider non-static electric fields: the flux generally only depends on the total charge enclosed by the surface bounding the volume.

The second of Maxwell's equations concerns the flux of the electric field. Note that we can uniquely reconstruct a vector by simply knowing its curl and divergence. Observe that

$$\vec{\boldsymbol{\nabla}} \times \frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^3} = \vec{\boldsymbol{\nabla}} \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^3} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0) + \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^3} \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0) \tag{3.2}$$

$$= -\frac{3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^5} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0) + 0 = 0, \tag{3.3}$$

and therefore

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3\vec{\boldsymbol{r}}' \, \rho(\vec{\boldsymbol{r}}') \vec{\boldsymbol{\nabla}} \times \frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} = 0. \tag{3.4}$$

One needs to take some caution though. Unlike the first of Maxwell's equations, this second one will certainly change when we move past electrostatics and is a source of rich electrodynamic phenomena.

CHAPTER 4

Scalar Potential

Presumably, one can recognize that Eq. (3.4), implies that the electrostatic field is irrotational and conservative, and therefore, there exists some scalar potential Φ such that

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}}), \tag{4.1}$$

where the minus sign is a conventional factor (for a nice interpretation).

Now, we prove this fact slightly more rigorously. Generally, $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} f = 0$ by Fubini, so if we are lucky enough to find a Φ that produces $\vec{\boldsymbol{E}}$ via Eq. (4.1), then $\vec{\boldsymbol{E}}$ satisfies all the necessary properties. But, how do we know there is such a Φ for any electrostatic field, which is curlless[1]?

We state a theorem that gives necessary and sufficient conditions for the existence of a scalar potential. First, define a simply connected region $\mathcal{R}$ such that for every closed curve C in $\mathcal{R}$, then C can be shrunk continuously to a single point entirely in $\mathcal{R}$. With this, the following statements are equivalent (let $\vec{\boldsymbol{A}}$ be a continuously differentiable vector field and $\mathcal{R}$ be a simply connected region):

1. $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = 0$ for all $\vec{\boldsymbol{r}} \in \mathcal{R}$
2. $\displaystyle\oint_C \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} = 0$ for any closed curve C in $\mathcal{R}$
3. There exists $\Phi(\vec{\boldsymbol{r}})$ such that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}})$

The proofs are as follows:

(1) $\rightarrow$ (2): Let $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = 0$ for all $\vec{\boldsymbol{r}} \in \mathcal{R}$. Consider any two curves curves C_1 and C_2 with the same starting and end points $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$. If we define the closed curve $C = C_1 - C_2$ and the surface S such that $S = \partial C$, we have

$$\oint_C \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_S \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = 0. \tag{4.2}$$

[1] Is this a word?

Furthermore, this implies that

$$\int_{C_1} \vec{\boldsymbol{A}}\,\mathrm{d}\vec{\boldsymbol{r}} = \int_{C_2} \vec{\boldsymbol{A}}\cdot\mathrm{d}\vec{\boldsymbol{r}}\,, \tag{4.3}$$

meaning that the path integral of vector which has no curl is independent of the path, and we can also define a function

$$\Phi(\vec{\boldsymbol{r}}) = \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\cdot\mathrm{d}\vec{\boldsymbol{r}}'\,, \tag{4.4}$$

which is independent of the path chosen between $\vec{\boldsymbol{r}}_0$ and $\vec{\boldsymbol{r}}$.

(2) → (3): Now suppose that $\oint_C \vec{\boldsymbol{A}}\cdot\mathrm{d}\vec{\boldsymbol{r}} = 0$ for any closed curve C in $\mathcal{R}$. Because of this, we can define the function as in Eq. (4.4). Let us take the difference of Φ over a very small path increment $\mathrm{d}\vec{\boldsymbol{r}}$:

$$\mathrm{d}\Phi = \Phi(\vec{\boldsymbol{r}}+\mathrm{d}\vec{\boldsymbol{r}}) - \Phi(\vec{\boldsymbol{r}}) = -\int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}+\mathrm{d}\vec{\boldsymbol{r}}} \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\cdot\mathrm{d}\vec{\boldsymbol{r}}' + \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\cdot\mathrm{d}\vec{\boldsymbol{r}}' \tag{4.5}$$

$$= -\int_{\vec{\boldsymbol{r}}}^{\vec{\boldsymbol{r}}+\mathrm{d}\vec{\boldsymbol{r}}} \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}')\cdot\mathrm{d}\vec{\boldsymbol{r}}' = -\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})\cdot\mathrm{d}\vec{\boldsymbol{r}}\,. \tag{4.6}$$

Recall the definition of the gradient: $\mathrm{d}\Phi = \vec{\boldsymbol{\nabla}}\Phi\cdot\mathrm{d}\vec{\boldsymbol{r}}$. This is what we have, where

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}}). \tag{4.7}$$

(3) → (1): Suppose $\vec{\boldsymbol{A}} = -\vec{\boldsymbol{\nabla}}\Phi$ for some Φ. It is a general fact that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}\Phi = 0$ for any Φ, and hence that $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = 0$.

CHAPTER 5

Methods for Finding Scalar Potentials

5.1 An Example

Now that we have proved the existence of a scalar potential in a simply connected region for the electrostatic field, we move onto a discussion of its uniqueness. We can define the electrostatic potential

$$\Phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}') \cdot \mathrm{d}\vec{r}' \,. \tag{5.1}$$

We can obtain the scalar potential at any location $\vec{r}$ via any choice of path. Typically, we choose linear paths between $\vec{r}_0$ and $\vec{r}$ such that we can parameterize the path in terms of a single parameter λ between some bounds. As an examplme, consider $\vec{A} = (\vec{a} \cdot \vec{r})\vec{a}$, where $\vec{a}$ is a constant vector. Observe that this is an irrotational field

$$\begin{aligned}[\vec{\nabla} \times \vec{A}]_i &= \epsilon_{ijk} \nabla_j (\vec{a} \cdot \vec{r}) a_k = \epsilon_{ijk} \nabla_j a_\ell r_\ell a_k = \epsilon_{ijk} a_\ell a_k \nabla_j r_\ell \\ &= \epsilon_{ijk} a_\ell a_k \delta_{j\ell} = \epsilon_{ijk} a_j a_k = [\vec{a} \times \vec{a}]_i = 0. \end{aligned} \tag{5.2}$$

The scalar potential is then (choosing the origin as our reference point)

$$\Phi(\vec{r}) = -\int_0^{\vec{r}} \vec{A}(\vec{r}') \cdot \mathrm{d}\vec{r}' = -\int_0^{\vec{r}} (\vec{a} \cdot \vec{r}')\vec{a} \cdot \mathrm{d}\vec{r}' \,. \tag{5.3}$$

If we parameterize the path as $\vec{r}' = \lambda \vec{r}$, then

$$\Phi(\vec{r}) = -\int_0^1 (\vec{a} \cdot \lambda \vec{r})\vec{a} \cdot \vec{r} \, \mathrm{d}\lambda = -\frac{1}{2}(\vec{a} \cdot \vec{r})^2. \tag{5.4}$$

This is a rather contrived example, although demonstrative. In general, we want to determine Φ and $\vec{E}$ from first principles.

5.2 Singular Fields

Let us consider a point charge at the origin. Observe that the electric field is singular at $\vec{r} = 0$, so we cannot determine the potential at the origin. We can however, place our reference at ∞ and determine the scalar potential from the electric field of the point charge as

$$\begin{aligned}\Phi(\vec{r}) &= -\int_{\infty}^{\vec{r}} \frac{q}{4\pi\epsilon_0}\frac{\vec{r}'}{r'^3}\cdot \mathrm{d}\vec{r}' = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{1}\frac{\vec{r}}{\lambda^2 r^3}\cdot\vec{r}\,\mathrm{d}\lambda \\ &= -\frac{q}{4\pi\epsilon_0}\frac{1}{r}\Big[-\frac{1}{\lambda}\Big]_{\infty}^{1} = \frac{q}{4\pi\epsilon_0}\frac{1}{r}. \end{aligned} \tag{5.5}$$

We could have shifted our origin such that the charge is located at $\vec{r}_0$. This just gives the scalar potential as

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_0|}. \tag{5.6}$$

If we extend this to a continuous (finite) charge distribution, we have

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3\vec{r}'\,. \tag{5.7}$$

5.3 Doubly Connected Regions

Everything we have done so far requires simply connected regions. Consider the three-dimensional region without the z-axis. This is clearly not simply connected since we cannot shrink any closed curve containing the z-axis to $s = 0$. Consider as an example the vector field $\vec{\boldsymbol{A}} = (a/s)\hat{\boldsymbol{\phi}}$. This is an irrotational field, but if we integrate around a circle of radius ϵ, we have

$$\int_0^{2\pi}\frac{a}{\epsilon}\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}\epsilon\,\mathrm{d}\phi = 2\pi a \neq 0. \tag{5.8}$$

In this case, the potential is not well-defined. It would depend on how many times our contour wraps around the z-axis.

CHAPTER 6

Conservative Forces

6.1 Conservative Forces

I called the electrostatic field in a previous section "conservative". We will see why this is the case now. Recall the definition of the work done on a particle under the influence of a force $\vec{\boldsymbol{F}}$ over its motion along a curve C:

$$W = \int_C \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}}\,. \tag{6.1}$$

The work done by an electric field $\vec{\boldsymbol{E}}$ on a test charge is then

$$W = q\int_C \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}}\,, \tag{6.2}$$

and since $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0$, we know from the equivalence theorem from above that if C is a closed curve, then $W = 0$. Furthermore, since $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}$, then $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}} = 0$, and we can therefore define a scalar potential (which we will see is just the electrostatic potential energy) $U(\vec{\boldsymbol{r}})$ for the electric force such that $\vec{\boldsymbol{F}} = -\vec{\boldsymbol{\nabla}}U$, which is related to the electric potential by

$$U(\vec{\boldsymbol{r}}) = q\Phi(\vec{\boldsymbol{r}}). \tag{6.3}$$

Now, let us remind ourselves that the work done on a particle is given by the change in its kinetic energy. Starting with Newton's 2nd law $m\ddot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}U(\vec{\boldsymbol{r}})$, we can integrate this along a path, which gives

$$\begin{gathered}\int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} m\ddot{\vec{\boldsymbol{r}}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} -\vec{\boldsymbol{\nabla}}U(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} \\ W = \int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} \frac{1}{2}m\frac{\mathrm{d}\dot{\vec{\boldsymbol{r}}}^2}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2}m\Delta\dot{\vec{\boldsymbol{r}}}^2 = -\Delta U(\vec{\boldsymbol{r}}) \Rightarrow \Delta E = \Delta\Big(\frac{1}{2}m\dot{\vec{\boldsymbol{r}}}^2 + U\Big) = 0.\end{gathered} \tag{6.4}$$

We define E as the mechanical energy of a system and U as the potential energy, and clearly under a conservative force $\vec{\boldsymbol{F}}$, this is a conserved quantity.

6.2 Potential Energy of a Charge Distribution

If we consider two charges q_1 and q_2 at corresponding locations $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$, then the potential energy of this system is given by

$$U_{12} = q_1 E_2(\vec{\boldsymbol{r}}_1) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}. \tag{6.5}$$

Notice that $U_{12} = U_{21}$, which should be the case.

If we generalize to a charge distribution with N point charges, we then have the potential energy of the setup as

$$\begin{aligned} U &= 0 + (U_{12}) + (U_{13} + U_{23}) + \ldots (U_{1N} + \ldots + U_{N-1,N}) \\ &= \sum_{i=2}^{N}\sum_{j=1}^{i} U_{ij}. \end{aligned} \tag{6.6}$$

We can extend the second sum, noting that we are double counting

$$U = \frac{1}{2}\sum_i \sum_{j\neq i} U_{ij} = \frac{1}{2}\sum_i \sum_{j\neq i} q_i \Phi(\vec{\boldsymbol{r}}_j) = \frac{1}{8\pi\epsilon_0}\sum_i \sum_{j\neq i} \frac{q_i q_j}{|\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_j|}. \tag{6.7}$$

If we further generalize to a continuous charge distribution, we have

$$U = \frac{1}{2}\int \mathrm{d}^3\vec{\boldsymbol{r}}\, \rho(\vec{\boldsymbol{r}})\Phi(\vec{\boldsymbol{r}}) = \frac{\epsilon_0}{2}\int \mathrm{d}^3\vec{\boldsymbol{r}}\, [\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})]\Phi(\vec{\boldsymbol{r}}) \tag{6.8}$$

$$= -\frac{\epsilon_0}{2}\int \mathrm{d}^3\vec{\boldsymbol{r}}\, \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}}) = \frac{\epsilon_0}{2}\int |\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})|^2\, \mathrm{d}^3\vec{\boldsymbol{r}}\,. \tag{6.9}$$

From this, we can define the electric energy density as $u_E = \epsilon_0|\vec{\boldsymbol{E}}|^2/2$.

If we imagine pulling off a very small surface element $\mathrm{d}S$ a distance $\mathrm{d}x$ away from teh surface, the potential energy of the surface changes by

$$\delta U = -u_E\, \mathrm{d}S\, \mathrm{d}x \Rightarrow \delta\vec{\boldsymbol{F}} = -\frac{\mathrm{d}U}{\mathrm{d}x} = u_E\, \mathrm{d}S\,. \tag{6.10}$$

The total force on the surface is then found by integrating the energy density over the surface, which is given by

$$u_E = \frac{\sigma^2}{2\epsilon_0}, \tag{6.11}$$

where σ is the surface charge density on S.

CHAPTER 7

Laplace's and Poisson's Equations

As a general wisdom, it is easier to deal with the electric potential than the field directly since it is just one function as opposed to three different components. To that end, we derive a differential equation for Φ as follows:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \cdot -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{r}}) = -\nabla^2\Phi(\vec{\boldsymbol{r}}) = \frac{\rho(\vec{\boldsymbol{r}})}{\epsilon_0}. \tag{7.1}$$

This is Poisson's equation. If there are no sources, it reduces to Laplace's equation

$$\nabla^2\Phi(\vec{\boldsymbol{r}}) = 0. \tag{7.2}$$

As general features, these are both linear, second order partial differential equations for the scalar potential. As such, we require some boundary conditions to specify the value of Φ (or its derivative) on some surface in order to uniquely determine its form.

If we reduce the problem to one-dimension, Poisson's equation is written as

$$\frac{\partial^2\Phi}{\partial x^2} = \lambda, \tag{7.3}$$

and has solution $\Phi(x) = \lambda x^2/2 + Ax + B$, where A and B are determined by boundary conditions at some points x_1 and x_2.

(1): Dirichlet BCs simply specify the potential's value on the surface S:

$$\Phi(\vec{\boldsymbol{r}})|_{\vec{\boldsymbol{r}}\in S} = f(\vec{\boldsymbol{r}}). \tag{7.4}$$

For example, we may hold a conductor to ground, which specifies $\Phi \equiv 0$ on the surface of the conductor.

(2): Neumann BCs specify the normal derivative of the potential on the surface S as

$$\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\nabla}}\Phi = \frac{\partial\Phi}{\partial n}\Big|_{\vec{\boldsymbol{r}}\in S} = g(\vec{\boldsymbol{r}}). \tag{7.5}$$

For example, we may have a situtation where there is a charge on the surface S, which defines the electric field and therefore the gradient of the potential on S.

CHAPTER 8

Green's Theorem

Let ψ_1 and ψ_2 be continuously differentiable functions in a volume V bounded by S. If we define the vector field $\vec{\boldsymbol{A}} = \psi_1 \vec{\boldsymbol{\nabla}} \psi_2$, then from the divergence theorem we have

$$\int_S \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \int_V \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} \, \mathrm{d}^3\vec{\boldsymbol{r}}$$

$$\Rightarrow \int_S \psi_1 \frac{\partial \psi_2}{\partial n} \, \mathrm{d}S = \int_V (\psi_1 \nabla^2 \psi_2 + \vec{\boldsymbol{\nabla}} \psi_1 \cdot \vec{\boldsymbol{\nabla}} \psi_2) \, \mathrm{d}^3\vec{\boldsymbol{r}} \,. \tag{8.1}$$

This is Green's 1st identity.

If we now reverse the roles of ψ_1 and ψ_2 and subtract the equations, we find Green's theorem:

$$\int_V (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1) \, \mathrm{d}^3\vec{\boldsymbol{r}} = \int_S \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right) \mathrm{d}S \,. \tag{8.2}$$

8.1 Uniqueness of Solutions to Laplace's and Poisson's Equations

Suppose that Φ_1 and Φ_2 satisfy the equation $\nabla^2 \Phi = -\rho/\epsilon_0$. Let us define $\psi(\vec{\boldsymbol{r}}) = \Phi_1(\vec{\boldsymbol{r}}) - \Phi_2(\vec{\boldsymbol{r}})$. It should be clear that ψ solves Laplace's equation

$$\nabla^2 \psi = 0 \tag{8.3}$$

in the volume V. Using Green's 1st identity with $\psi_1 = \psi_2 = \psi$, we have

$$\int_S \psi \frac{\partial \psi}{\partial n} \, \mathrm{d}S = \int_V (\psi \nabla^2 \psi + [\vec{\boldsymbol{\nabla}} \psi]^2) \, \mathrm{d}^3\vec{\boldsymbol{r}} = \int_V (\vec{\boldsymbol{\nabla}} \psi)^2 \, \mathrm{d}^3\vec{\boldsymbol{r}} \,. \tag{8.4}$$

Dirichlet BC: Let $\Phi(\vec{\boldsymbol{r}}) = f(\vec{\boldsymbol{r}})$ on S.

Neumann BC: Let $\partial\Phi(\vec{\boldsymbol{r}})/\partial n = g(\vec{\boldsymbol{r}})$ on S.

In either case, Eq. (8.4) gives

$$\int_V (\vec{\boldsymbol{\nabla}}\psi)^2 = 0. \tag{8.5}$$

Since this is independent of the volume V considered and the integrand is non-negative, it must be that

$$\vec{\boldsymbol{\nabla}}\psi = 0 \Rightarrow \psi(\vec{\boldsymbol{r}}) = C \text{ in } V, \tag{8.6}$$

where C is a constant. For Dirichlet boundary conditions, the solution is unique since $C = 0$ by the BCs. We cannot, however, make a similar statement about Neumann BCs: the potential is unique up to an additive constant. Since only potential differences are physically meaningful, all of our observables are uniquely determined.

8.2 Some Comments on the Proof

For a unique solution (in the sense of the previous section), we must specify either Dirichlet or Neumann BCs. We cannot however specify both over S because in general, they are inconsistent and the problem becomes overdetermined. It is possible, however, to partition S and specify different conditions on these subsets.

Another note concerns the general difficulty of solving Poisson's or Laplace's equation for a given setup. There are several methods/tricks that can be used, which may seem like magic, but because of the uniqueness of the scalar potential, *a solution is the solution* (up to some additive constant for Neumann conditions).

8.3 Uniqueness Theorem in an Infinite Region

Note that our derivation only considered finite volumes, so some more care is needed if we consider an infinite volume and hence a surface at infinity. Suppose we have two solutions Φ_1 and Φ_2 again such that $\lim_{r\to\infty} \Phi(\vec{\boldsymbol{r}}) = \Phi_\infty$. In general, $\Phi_{1,2} \sim 1/r$ or faster as $r \to \infty$. The linear combination $\psi = \Phi_1 - \Phi_2 \sim 1/r$ as $r \to \infty$ then and has a gradient $\vec{\boldsymbol{\nabla}}\psi \sim 1/r^2$ as $r \to \infty$. Hence,

$$\int_V (\vec{\boldsymbol{\nabla}}\psi)^2 \, \mathrm{d}^3\vec{\boldsymbol{r}} = \lim_{R\to\infty} \int_S \psi \frac{\partial \psi}{\partial n} R^2 \, \mathrm{d}\Omega = 0. \tag{8.7}$$

We then have the same conclusions as before about the uniqueness of Φ.

Sometimes a uniform electric field is specified at infinity. If we have $\vec{\boldsymbol{E}} \sim E_0\hat{\boldsymbol{z}}$, then $\Phi(z) = K - E_0 z$ with constant K. Uniqueness holds here because $\Phi(\vec{\boldsymbol{r}}) + E_0 z \to K + \mathcal{O}(1/r)$ as $r \to \infty$.

8.4 Boundary Conditions at a Conductor

An ideal conductor is one such that electrons are able to move freely in order to set up a charge distribution. For an isolated conductor without an external

electric field, it turns out that the most stable configuration for excess charge is the one where they are distributed over the surface (not necessarily uniformly if the surface is not a sphere as we will see). By Gauss' law then, the electric field is zero inside the conductor. On the other hand, if the conductor is immersed in an external electric field, the charges rearrange themselves, creating an induced electric field which cancels the external one in the volume of the conductor (since otherwise the charges would experience a net force and the conductor would not be in equilibrium). Since there is no net electric field inside , it is clear that the conductor is an equipotential surface, which is a defining property of a conductor. By convention, we take $\Phi \equiv 0$ for a grounded conductor[1].

Another interesting behavior is that since the conductor is in equilibrium the electric field at the surface of the conductor is normal to the surface. Otherwise, there would be a net force on the electrons, and the conductor would not be in equilibrium.

Let us now look at BCs with conductors. We can place a Gaussian pillbox straddling the surface of the conductor, which we make small enough such that the electric field is constant in the pillbox. Gauss' law then reads

$$A\vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} = \frac{\sigma A}{\epsilon_0} \Rightarrow \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} = \frac{\sigma}{\epsilon_0}. \tag{8.8}$$

This can also be written in terms of the scalar potential:

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}. \tag{8.9}$$

8.5 Capacitance and Potential Energy of Conductors

If we have a system of N conductors with total charge q_i at potential Φ_i, the total potential energy of the configuration

$$U = \frac{1}{2}\int_V \rho(\vec{\boldsymbol{r}})\Phi(\vec{\boldsymbol{r}})\,\mathrm{d}^3\vec{\boldsymbol{r}} = \frac{1}{2}\sum_{i=1}^{N} q_i\Phi_i. \tag{8.10}$$

For a given set of charges $\{q_i\}$, the potentials are determined by the field equations, which are linear, implying that

$$\Phi_i = \sum_{j=1}^{N} P_{ij}q_j, \tag{8.11}$$

or written in matrix form $\vec{\boldsymbol{\Phi}} = \mathbf{P}\vec{\boldsymbol{q}}$. Alternatively, since the matrix $\mathbf{P}$ should be nonsingular, we can invert this relationship to read

$$\vec{\boldsymbol{q}} = \mathbf{C}\vec{\boldsymbol{\Phi}}. \tag{8.12}$$

The diagonal elements of $\mathbf{C}$ give the capacitances of the conductors and the off diagonal elements give the coefficients of induction.

[1] Grounding a conductor connects it to a reservoir of charge so that excess charge can either drain to "ground" or be pulled from ground.

8.6 Charged Sphere Inside a Grounded, Conducting Shell

Consider a grounded, conducting shell of radius b, and a sphere with total charge Q of radius a place inside this shell. We want to find the potential in the region $a \leq r \leq b$. This can be done by noticing that the electric field in this region is just

$$\vec{\boldsymbol{E}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}, \tag{8.13}$$

implying that the potential

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r} + \Phi_0. \tag{8.14}$$

Using the fact that the shell is grounded, we find

$$\Phi(\vec{\boldsymbol{r}}) = \frac{Q}{4\pi\epsilon_0}\Big(\frac{1}{r} - \frac{1}{b}\Big). \tag{8.15}$$

We can check that this satisfies Laplace's equation (in the region between the charged sphere and conducting shell):

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) = 0. \tag{8.16}$$

From this, we can also find the induced surface charge density on the conducting shell:

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial n} = \epsilon_0\frac{\partial\Phi}{\partial r}\Big|_{r=b} = -\frac{Q}{4\pi b^2}. \tag{8.17}$$

This result makes intuitive sense since the total induced charge on the conducting shell should be $-Q$.

CHAPTER 9

Method of Images

9.1 Charge Above an Infinite Grounded Plane

The method of images is one of the more conventional methods that can be used to solve for the potential with certain setups. Consider a charge a distance a above a conductor spanning the xy-plane. We are interested in the potential in the region $z > 0$, where Poisson's equation reads

$$\nabla^2\Phi = -4\pi q\delta^{(3)}(\vec{\boldsymbol{r}} - \vec{\boldsymbol{a}}), \tag{9.1}$$

where $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{z}}$. The potential of the point charge is simple to write down, but on first glance, it does not seem a trivial matter how to write down the potential of the plane is at any arbitary point above the plane.

We can however be clever and place a so-called "mirror charge" $-q$ a distance a below the surface of the plane. The potential of this mirror setup is

$$\Phi = \frac{q}{4\pi\epsilon_0}\Big(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{a}}|} - \frac{1}{|\vec{\boldsymbol{r}} + \vec{\boldsymbol{a}}|}\Big) \tag{9.2}$$

$$- \frac{q}{4\pi\epsilon_0}\Big(\frac{1}{\sqrt{s^2 + (z-a)^2}} - \frac{1}{\sqrt{s^2 + (z+a)^2}}\Big), \tag{9.3}$$

where $s^2 = x^2 + y^2$. From the form of Eq. (9.3), it is clear that $\Phi(x, y, 0) = 0$, meaning that the potential satisfies the BCs, and hence, this is the potential of the system since it satisfies Poisson's equation for $z > 0$:

$$\nabla^2\Phi = -4\pi q[\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{a}}) - \delta(\vec{\boldsymbol{r}} + \vec{\boldsymbol{a}})]. \tag{9.4}$$

9.2 Point Charge Near a Grounded Sphere

Consider a conducting, grounded sphere of radius a with a charge q placed a distance $b > a$ from the center of the sphere. We are interested in the potential in the region $r > a$. Again, it is not immediately clear how to write down the contribution to the potential from the conducting sphere, so we introduce an

image charge q' inside the sphere a distance b' from the center along the axis defined by the line between the center and the source charge (which we call the x-axis). The potential of the mirror setup is just

$$\Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{b}}|} + \frac{q'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{b}}'|}\right), \tag{9.5}$$

where $\vec{\boldsymbol{b}} = b\hat{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}' = b'\hat{\boldsymbol{x}}$

We can solve for q' and b' by imposing the BCs:

$$\Phi|_{x=a} = \frac{1}{4\pi\epsilon_0}\Big(\frac{q}{b-a} + \frac{q'}{a-b'}\Big) = 0 \tag{9.6}$$

$$\Phi|_{x=-a} = \frac{1}{4\pi\epsilon_0}\Big(\frac{q}{a+b} + \frac{q'}{a+b'}\Big) = 0. \tag{9.7}$$

If we make some rearrangements, we have the following for b':

$$\begin{aligned}
\frac{a+b'}{a-b'} &= \frac{a+b}{b-a} \\
\Big(1 + \frac{a+b}{b-a}\Big)b' &= a\Big(\frac{a+b}{b-a} - 1\Big) \\
\frac{2b}{b-a}b' &= \frac{2a}{b-a}a \\
b' &= \frac{a^2}{b}.
\end{aligned} \tag{9.8}$$

Plugging this into one of the BCs

$$q' = -\frac{(a - a^2/b)}{b-a}q = -\frac{a}{b}q. \tag{9.9}$$

Appendices