Problem 1)
A hollow right circular cylinder of radius b has its axis coincident with the z-axis and its ends at $z = 0$ and $z = L$. The cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & -\pi/2 < \phi < \pi/2 \\ -V & \pi/2 < \phi < 3\pi/2. \end{cases} \tag{1}$$

(a) Find the potential inside the cylinder.
(b) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of s and ϕ and compare it with two-dimensional Problem 1 from Pset 5.

For this, we begin by looking at separable solutions of the form

$$\Phi(s, \phi, z) = R(s)T(\phi)Z(z). \tag{2}$$

For the z-dependence, we have,

$$Z_n(z) = \sin k_n z, \tag{3}$$

where $k_n = \pi n/L$ such that the end faces of the cylinder are grounded. The angular dependence is given by

$$T_m(\phi) = A_m \sin m\phi + B_m \cos m\phi, \tag{4}$$

and finally the radial solutions are from the modified Bessel equation

$$R_{nm}(s) = I_m(k_n s). \tag{5}$$

Note the second solution K_m is discarded since this leads to singular behavior at $s = 0$.

Thus, the series solution for the potential inside the cylinder is

$$\Phi(s, \phi, z) = \sum_{n=1}^{\infty} \frac{B_{0n}}{2} I_0(k_n s)\sin(k_n z) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} I_m(k_n s)\sin(k_n z)[A_{nm}\sin m\phi + B_{nm}\cos m\phi]. \tag{6}$$

We then have a Fourier series for the angular and z-dependence of Φ. Applying the BC at $s = b$, we have

$$V(\phi, z) = \sum_{n=1}^{\infty} \frac{B_{0n}}{2} I_0(k_n b)\sin(k_n z) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} I_m(k_n b)\sin(k_n z)[A_{nm}\sin m\phi + B_{nm}\cos m\phi]. \tag{7}$$

Exploiting the orthogonality of sines and cosines, we then have

$$A_{nm} = \frac{2}{\pi L I_m(k_n b)} \int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi \, V(\phi, z)\sin(m\phi)\sin(\pi n z/L) \tag{8}$$

$$B_{nm} = \frac{2}{\pi L I_m(k_n b)} \int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi \, V(\phi, z)\cos(m\phi)\sin(\pi n z/L). \tag{9}$$

Putting this into the potential we have

$$\begin{aligned}\Phi(s,\phi,z) = \frac{1}{\pi L}\sum_{n=1}\frac{I_0(n\pi s/L)}{I_0(n\pi b/L)}\sin(n\pi z/L)\left[\int_0^L \mathrm{d}z'\int_0^{2\pi}\mathrm{d}\phi'\,V(\phi',z')\sin(n\pi z'/L)\right] \\ + \frac{2}{\pi L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{I_m(n\pi s/L)}{I_m(n\pi b/L)}\sin(n\pi z/L)\left[\int_0^L \mathrm{d}z'\int_0^{2\pi}\mathrm{d}\phi'\,V(\phi',z')\sin(n\pi z'/L)\cos[m(\phi-\phi')]\right]\end{aligned} \tag{10}$$

Observe that the first term ($m = 0$) is identically zero since the integration is over V for half the ϕ' range and $-V$ for the other half of the range, which gives zero overall. Our problem then reduces to evaluating the following integral:

$$\begin{aligned}&V\int_0^L \sin(n\pi z'/L)\,\mathrm{d}z'\left[\int_{-\pi/2}^{\pi/2} - \int_{\pi/2}^{3\pi/2}\right]\mathrm{d}\phi'\cos[m(\phi-\phi')] \\ &= V\frac{L}{n\pi}[1-(-1)^n][1-(-1)^m]\int_{-\pi/2}^{\pi/2}\cos[m(\phi-\phi')]\,\mathrm{d}\phi' \\ &= V\frac{L}{n\pi}[1-(-1)^n][1-(-1)^m]\frac{1}{m}\Big\{\sin[m(\pi/2-\phi)] + \sin[m(\pi/2+\phi)]\Big\} \\ &= V\frac{2L}{nm\pi}[1-(-1)^n][1-(-1)^m]\cos m\phi\sin m\pi/2.\end{aligned} \tag{11}$$

From this expression, we can see that only odd terms in n, m contribute, giving

$$\boxed{\Phi(s,\phi,z) = \frac{16}{\pi^2}V\sum_{n,m=0}^{\infty}\frac{(-1)^m}{(2n+1)(2m+1)}\frac{I_m\{(2n+1)\pi s/L\}}{I_m\{(2n+1)\pi b/L\}}\cos(m\phi)\sin\left[\frac{(2n+1)\pi z}{L}\right]}. \tag{12}$$

(b) If we have $z = L/2$, the potential becomes

$$\Phi(s,\phi) = \frac{16}{\pi^2}V\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\frac{I_m\{(2n+1)\pi s/L\}}{I_m\{(2n+1)\pi b/L\}}\cos(m\phi). \tag{13}$$

Taking $L \gg b$, we can use $I_\nu(x) \to (x/2)^\nu/\Gamma(\nu+1)$ when $x \ll 1$. Additionally, in this limit, $s \le b$, so

$$\begin{aligned}\Phi(s,\phi) &\approx \frac{16}{\pi^2}V\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\left(\frac{s}{b}\right)^m\cos(m\phi) \\ &= \frac{16}{\pi^2}V\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}}_{\pi/4}\,\mathrm{Re}\left\{\underbrace{\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left(\frac{s}{b}e^{i\phi}\right)^m}_{\arctan\left(\frac{s}{b}e^{i\phi}\right)}\right\} \\ &= \frac{4}{\pi}V\,\mathrm{Re}\left\{\arctan\left(\frac{s}{b}e^{i\phi}\right)\right\}.\end{aligned} \tag{14}$$

We now have to do just a touch of complex analysis. Let $w = \arctan z$. Then

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = -i\frac{e^{2iw} - 1}{e^{2iw} + 1}$$
$$iz = \frac{e^{2iw} - 1}{e^{2iw} + 1}$$
$$e^{2iw} = \frac{1 + iz}{1 - iz}$$
$$\Rightarrow w = \arctan z = \frac{1}{2i}\ln\left[\frac{1 + iz}{1 - iz}\right]. \tag{15}$$

Thus, we have

$$\mathrm{Re}(\arctan z) = \frac{1}{2}\arg\left\{\frac{1 + iz}{1 - iz}\right\} = \frac{1}{2}\arctan\left(\frac{2\,\mathrm{Re}(z)}{1 - |z|^2}\right). \tag{16}$$

Finally, we can say that

$$\boxed{\Phi(s, \phi) \approx \frac{2V}{\pi}\arctan\left(\frac{2s/b}{1 - (s/b)^2}\cos\phi\right) = \frac{2V}{\pi}\arctan\left(\frac{2sb}{b^2 - s^2}\cos\phi\right)}, \tag{17}$$

which is exactly the result of problem 1 from Pset 5 with $V_1 = V$ and $V_2 = -V$.

Problem 2)
An infinite, thin, plane sheet of conducting material has a circular hole of radius a cut in it. A thin, flat disc of the same material and slightly smaller radius lies in the plane, filling the hole, but separated from the sheet by a very narrow insulating ring. The disc is maintained at a fixed potential V, while the infinite sheet is kept at zero potential.
(a) Using appropriate cylindrical coordinates, find an integral expression involving Bessel functions for the potential at any point above the plane.
(b) Show that the potential a perpendicular distance z above the center of the disc is

$$\Phi_0(z) = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right). \tag{18}$$

(c) **Extra credit**: Show that the potential a perpendicular distance z above the edge of the disc is

$$\Phi_a(z) = \frac{V}{2}\left[1 - \frac{kz}{\pi a}K(k)\right], \tag{19}$$

where $k = 2a/(z^2 + 4a^2)^{1/2}$, and $K(k)$ is the complete elliptic integral of the first kind.

(a) Let us take the z-axis to be through the center of the disc. The separable solutions are then of the form

$$\Phi_{mk}(s,\phi,z) = e^{-kz}[A_m(k)\sin m\phi + B_m(k)\cos m\phi]J_m(ks). \tag{20}$$

Unlike in other problems, however, the spectrum of allowed k values is continuous ($k > 0$), not discrete. The values of m, though, are restricted to the positive integers and zero. Putting this all together, we have a generic solution for $z > 0$ of the form

$$\Phi(s,\phi,z) = \int_0^\infty \mathrm{d}k\, e^{-kz}\frac{B_0(k)}{2}J_0(ks) + \sum_{m=1}^\infty \int_0^\infty \mathrm{d}k\, e^{-kz}[A_m(k)\sin m\phi + B_m(k)\cos m\phi]J_m(ks). \tag{21}$$

The coefficients are determined via the Hankel transforms in s and Fourier series in ϕ as

$$A_m(k) = \frac{k}{\pi}\int_0^\infty \mathrm{d}s\, s\int_0^{2\pi}\mathrm{d}\phi\, V(s,\phi)\sin m\phi J_m(ks) \tag{22}$$

$$B_m(k) = \frac{k}{\pi}\int_0^\infty \mathrm{d}s\, s\int_0^{2\pi}\mathrm{d}\phi\, V(s,\phi)\cos m\phi J_m(ks). \tag{23}$$

The potential can then be written

$$\begin{aligned}\Phi(s,\phi,z) = \frac{V}{\pi}\int_0^a \mathrm{d}s'\, s'\int_0^{2\pi}\mathrm{d}\phi'\int_0^\infty \mathrm{d}k\, ke^{-kz} \\ \times\left\{\frac{1}{2}J_0(ks')J_0(ks) + \sum_{m=1}^\infty \cos[m(\phi-\phi')]J_m(ks')J_m(ks)\right\}.\end{aligned} \tag{24}$$

Since there is no ϕ dependence in the potential, we can simplify this a bit further as

$$\Phi(s,\phi,z) = V\int_0^a \mathrm{d}s'\, s'\int_0^\infty \mathrm{d}k\, ke^{-kz}J_0(ks')J_0(ks). \tag{25}$$

Additionally, recall that $xJ_0(x) = (xJ_1(x))'$ and $J_1(0) = 0$, so

$$\begin{aligned}\Phi(s,\phi,z) &= V\int_0^\infty \mathrm{d}k\, e^{-kz}J_0(ks)\int_0^a \mathrm{d}s'\, ks'J_0(ks') \\ &= \boxed{Va\int_0^\infty \mathrm{d}k\, e^{-kz}J_0(ks)J_1(ka)}.\end{aligned} \tag{26}$$

At this point – as far as I can tell – we cannot simplify this any further in full generality.

(b) If we take $s = 0$ and restrict our attention to the z-axis

$$\begin{aligned}\Phi(z) &= Va\int_0^\infty \mathrm{d}k\, e^{-kz}J_1(ka) = Va\int_0^\infty \mathrm{d}k\, e^{-kz}\frac{ka}{2}\sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1)!}\frac{(ka)^{2j}}{4^j} \\ &= \frac{V}{2}\sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1)!}\frac{a^{2(j+1)}}{4^j}\int_0^\infty \mathrm{d}k\, k^{2j+1}e^{-kz} \\ &= V\sum_{j=0}^\infty (-1)^j\frac{(2j+1)!}{j!(j+1)!2^{2j+1}}\left(\frac{a}{z}\right)^{2(j+1)}.\end{aligned} \tag{27}$$

Since we know the result already, we can compare the series of the proposal (since the power series of a function is unique within its radius of convergence – although we will not explore this question of convergence) to that obtained above:

$$\begin{aligned} 1 - \frac{z}{\sqrt{a^2+z^2}} &= 1 - \frac{1}{\sqrt{1+(a/z)^2}} = 1 - \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{j!\Gamma(1/2)}\left(\frac{a}{z}\right)^{2j} \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma[(j+1)+1/2]}{j!\sqrt{\pi}}\left(\frac{a}{z}\right)^{2(j+1)} \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{1}{(j+1)!\sqrt{\pi}}\frac{(2j+2)!\sqrt{\pi}}{2^{2(j+1)}(j+1)!}\left(\frac{a}{z}\right)^{2(j+1)} \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{(2j+1)!}{j!(j+1)!2^{2j+1}}\left(\frac{a}{z}\right)^{2(j+1)}, \end{aligned} \tag{28}$$

where we have used the fact that

$$\Gamma(n+1/2) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}. \tag{29}$$

It is clear that these two sums match via term-by-term comparison

$$\Phi_0(z) = V\left[1 - \frac{z}{a^2+z^2}\right]. \tag{30}$$

(c) The work here is incomplete. In fact, they are mostly thoughts about how to go about solving the problem. On the edge of the disk, we can write

$$\Phi_a(z) = Va\int_0^{\infty} \mathrm{d}k\, J_0(ka)J_1(ka). \tag{31}$$

Additionally, we have the relation $J_0'(ka) = -J_1(ka)$ (where the derivative is with respect to k) such that

$$\Phi_a(z) = -Va\int_0^{\infty} \mathrm{d}k\, e^{-kz}J_0(ka)J_0'(ka). \tag{32}$$

Observe then that

$$\frac{\mathrm{d}}{\mathrm{d}k}J_0^2(ka) = \frac{\mathrm{d}(ka)}{\mathrm{d}k}\frac{\mathrm{d}}{\mathrm{d}(ka)}J_0^2(ka) = 2aJ_0(ka)J_0'(ka). \tag{33}$$

We can then write

$$
\begin{aligned}
\Phi_a(z) &= -\frac{V}{2}\int_0^\infty \mathrm{d}k\, e^{-kz}\frac{\mathrm{d}}{\mathrm{d}k}J_0^2(ka) \\
&= -\frac{V}{2}\left[e^{-kz}J_0^2(ka)\Big|_0^\infty + z\int_0^\infty \mathrm{d}k\, e^{-kz}J_0^2(ka)\right] \\
&= \frac{V}{2}\left[1 - z\int_0^\infty \mathrm{d}k\, e^{-kz}J_0^2(ka)\right].
\end{aligned} \tag{34}
$$

The second term then remains to be simplified/rewritten a bit. We could use an approach similar to that of part (b), although we would have a double sum that would have to be identified with some elliptic integral.

Problem 3)
A hollow right cylinder of radius R has its axis coincident with the z-axis and its ends at $z = 0$ and $z = L$, which are held to zero potential. The potential on the end faces is zero, while the potential on the cylindrical surface is specified by

$$\Phi(s, \phi, z)|_{s=R} = V \sin\phi \sin(\pi z/L). \tag{35}$$

Find the potential $\Phi(s, \phi, z)$ inside the cylinder.

As we found in problem 1, we can write the potential in the form

$$\Phi(s, \phi, z) = \sum_{n=1}^\infty \frac{B_{0n}}{2} I_0(n\pi s/L)\sin(n\pi z/L) \tag{36}$$

$$+ \sum_{m=1}^\infty \sum_{n=1}^\infty I_m(n\pi s/L)\sin(n\pi z/L)[A_{nm}\sin m\phi + B_{nm}\cos m\phi] \tag{37}$$

where the coefficients

$$A_{nm} = \frac{2}{\pi L I_m(n\pi R/L)}\int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi\, V(\phi, z)\sin(m\phi)\sin(n\pi z/L) \tag{38}$$

$$B_{nm} = \frac{2}{\pi L I_m(n\pi R/L)}\int_0^L \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi\, V(\phi, z)\cos(m\phi)\sin(n\pi z/L). \tag{39}$$

From these, it is clear that $B_{nm} = 0$ for all m and consequently n as well as $A_{nm} = 0$ if $m \neq 1$. We therefore, only have to look at the $m = 1$ terms in the second sum. Similarly, the terms with $n \neq 1$ also vanish by the orthogonality of the sines and cosines, leaving us with only one nonzero coefficient

$$A_{11} = \frac{2V}{\pi L I_1(\pi R/L)}(\pi)\left(\frac{L}{2}\right) = \frac{V}{I_1(\pi R/L)}. \tag{40}$$

The potential then takes the form

$$\boxed{\Phi(s,\phi,z) = V\frac{I_1(\pi s/L)}{I_1(\pi R/L)}\sin\left(\frac{\pi z}{L}\right)\sin\phi}. \tag{41}$$

Essentially then, the potential inside the hollow cylinder, is just the potential on the boundaries, modulated by the modified Bessel function I_1.