General Math:

$$\mathrm{Re}(\arctan z) = \frac{1}{2}\arctan\left(\frac{2\,\mathrm{Re}\{z\}}{1-|z|^2}\right)$$

$$\mathrm{Im}(\arctan z) = \frac{1}{2}\arctan\left(\frac{2\,\mathrm{Im}\{z\}}{1+|z|^2}\right)$$

Gamma Function:

$$\Gamma(n+1/2) = \frac{(2n-1)!!\sqrt{\pi}}{2^n} = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Power Series:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

$$(1+x)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} (-1)^k x^k$$

Vector Calculus:

Fundamental theorem for gradients – $\displaystyle\int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} \vec{\boldsymbol{\nabla}} T \cdot \mathrm{d}\vec{\boldsymbol{l}} = T(\vec{\boldsymbol{b}}) - T(\vec{\boldsymbol{a}})$

Divergence Theorem – $\displaystyle\int_V (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}})\, \mathrm{d}^3\vec{\boldsymbol{r}} = \int_S \vec{\boldsymbol{v}} \cdot \mathrm{d}\vec{\boldsymbol{S}}$

Stoke's Theorem – $\displaystyle\int_S (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot \mathrm{d}\vec{\boldsymbol{S}} = \oint_C \vec{\boldsymbol{v}} \cdot \mathrm{d}\vec{\boldsymbol{l}}$

Green's stuff:

Green's first identity – $(\psi_1\nabla^2\psi_2 + \vec{\boldsymbol{\nabla}}\psi_1 \cdot \vec{\boldsymbol{\nabla}}\psi_2)\,\mathrm{d}^3\vec{\boldsymbol{r}} = \int_S \psi_1\vec{\boldsymbol{\nabla}}\psi_2 \cdot \hat{\boldsymbol{n}}\,\mathrm{d}S$

Green's theorem – $\int \mathrm{d}^3\vec{\boldsymbol{r}}\,(\psi_1\nabla^2\psi_2 - \psi_2\nabla^2\psi_1) = \int_S (\psi_1\vec{\boldsymbol{\nabla}}\psi_2 - \psi_2\vec{\boldsymbol{\nabla}}\psi_1)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S$

Important divergences, curls:

$$\vec{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{e}}_r}{r^2} = 4\pi\delta^{(3)}(\vec{\boldsymbol{r}})$$

$$\vec{\boldsymbol{\nabla}} \times \frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|^3} = 0$$

$$\vec{\boldsymbol{\nabla}}\frac{1}{r} = -\frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|^3}$$

$$\nabla^2\frac{1}{|\vec{\boldsymbol{r}}|} = -4\pi\delta^{(3)(\vec{\boldsymbol{r}})}$$

Coulomb's law

Force between two charges – $\vec{\boldsymbol{F}}_{12} = \frac{q_1q_2}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|^3}$

Form of the electric field – $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\vec{\boldsymbol{r}}'\,\rho(\vec{\boldsymbol{r}}')\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$

Maxwell's electrostatic equations :

Gauss' law (Integral form) – $\int_S \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{S}} = \frac{Q}{\epsilon_0}$

Gauss' law (Differential form) – $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0}$

Curl of the electric field – $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0$

Closed path integral of electric field – $\oint_C \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = 0$

Scalar potential:

Definition – $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi$

Path integral formulation – $\Phi = -\int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}}$

Potential energy of charge distribution (discrete, point charges) – $U = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{q_i q_j}{|\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_j|}$

Potential energy of charge distribution (continuous) – (1) $U = \frac{1}{2}\int \rho(\vec{\boldsymbol{r}})\Phi(\vec{\boldsymbol{r}})\,\mathrm{d}^3\vec{\boldsymbol{r}}$ (2) $U = \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\frac{\epsilon_0}{2}|\vec{\boldsymbol{E}}|^2$

Force on conducting surface – $\vec{\boldsymbol{F}} = \int_S \sigma\vec{\boldsymbol{E}}\,\mathrm{d}S$

Poission's equation:

Equation – $\nabla^2\Phi(\vec{\boldsymbol{r}}) = -\rho(\vec{\boldsymbol{r}})/\epsilon_0$

Solution for the potential (ground at ∞) – $\Phi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\vec{\boldsymbol{r}}'\,\frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$

Conductor boundary conditions

Potential continuity – $\Phi(\vec{\boldsymbol{r}} \in S) = \text{const} \to$ grounded conductor $\Phi \equiv 0$ on surface

Tangent electric field – $\vec{\boldsymbol{E}}_1^{\parallel} = \vec{\boldsymbol{E}}_2^{\parallel}$

Normal electric field – $(\vec{\boldsymbol{E}}_2 - \vec{\boldsymbol{E}}_1)\cdot\hat{\boldsymbol{n}}_{12} = \frac{\sigma}{\epsilon_0}$

Potential normal derivative – $\left.\frac{\partial\Phi_2}{\partial n}\right|_S - \left.\frac{\partial\Phi_1}{\partial n}\right|_S = -\frac{\sigma}{\epsilon_0}$

Capacitance:

Definition – Two conductors with charge $\pm Q$, respectively, and potential difference V between surfaces $C = Q/V$

Energy stored – $U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$

Method of images

Point charge near grounded plane – $\Phi(\vec{\boldsymbol{r}}) = \frac{q}{4\pi\epsilon_0}\Big(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|} - \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0'|}\Big)$ ($\vec{\boldsymbol{r}}_0'$ is position of charge reflected over plane – usually $z \to -z$ with choice of coordinates)

Point charge near grounded sphere – $\Phi(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon}\left(\frac{q}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{b}}|} + \frac{q'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{b}}'|}\right)$ ($q' = -(a/b)q$, $b' = (a/b)^2\vec{\boldsymbol{b}}$, where a is the sphere radius and b is the distance of the charge from the center of the sphere)

Fourier orthogonality:

$$\frac{2}{L}\int_a^{L+a} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \delta_{nm} \tag{1}$$

$$\frac{2}{L}\int_a^{L+a} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \delta_{nm} \tag{2}$$

$$\frac{2}{L}\int_a^{L+a} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = 0. \tag{3}$$

Spherical Harmonics:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

Complex conjugation – $Y_{lm}^*(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi)$

Parity tranformation – $Y_{lm}(\pi-\theta,\pi+\theta) = (-1)^l Y_{lm}(\theta,\phi)$

$$\frac{1}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) = \sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Legendre:

Legendre Normalization – $\int_{-1}^{1} \mathrm{d}x\, P_l^m(x)P_{l'}^m(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$

Recursion relations

$$(1)\ P_n(x) = \frac{1}{2n+1}[P'_{n+1}(x) - P'_{n-1}(x)] \tag{4}$$

$$(2)\ P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x). \tag{5}$$

$$\int_0^1 \mathrm{d}x\, P_l(x) = \frac{P_{l-1}(0) - P_{l+1}(0)}{2l+1}$$

First few:

$$P_0(x) = 1 \tag{6}$$
$$P_1(x) = x \tag{7}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{8}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{9}$$

$P_l(1) = 1, \quad P_l(-1) = (-1)^l$

$P_{2n+1}(0) = 0, \quad P_{2n}(0) = \dfrac{(-1)^n(2n-1)!!}{2^n n!}$

Bessel:

Bessel's equation – $s\dfrac{\mathrm{d}}{\mathrm{d}s}\Big(s\dfrac{\mathrm{d}R}{\mathrm{d}s}\Big) + (k^2s^2 - \nu^2)R = 0$ with solution $J_\nu(ks)$

2nd solution: $J_{-\nu}$ for $\nu \notin \mathbb{Z}$, N_ν (Neumann)

$J_0(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\Big(\frac{x^2}{4}\Big)^n, \quad J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\Big(\frac{x^2}{4}\Big)^n$

$J_0'(x) = -J_1(x), \quad xJ_0(x) = (xJ_1(x))'$

Asymptotic forms ($x \gg 1$):

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) \tag{10}$$
$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\sin\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) \tag{11}$$

$x \ll 1: \; J_\nu(x) = \dfrac{1}{\nu!}(x/2)^\nu, J_{-\nu\in\mathbb{N}} = \dfrac{(-1)^\alpha}{\nu!}(x/2)^{-\nu}$

Zeros: $x_{\nu n}$ such that $J_\nu(x_n) = 0$

Orthogonality: $\displaystyle\int_0^a \mathrm{d}s\, sJ_\nu(x_{\nu n}s/a)J_\nu(x_{\nu n'}s/a) = \frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{nn'}$

Modified Bessel equation: $s\dfrac{\partial}{\partial s}\Big(s\dfrac{\partial R}{\partial s}\Big) - (k^2s^2 + \nu^2)R = 0$

Modified Bessel function:

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \to \begin{cases} \frac{1}{\Gamma(\nu+1)}(x/2)^\nu & x \ll 1 \\ \frac{1}{\sqrt{2\pi x}} e^x [1 + \mathcal{O}(1/x)] & x \gg 1 \end{cases} \tag{12}$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \to \begin{cases} -[\ln(x/2) + \gamma_E + \ldots] & x \ll 1,\ \nu = 0 \\ \frac{\Gamma(\nu)}{2}(2/x)^\nu & x \ll 1,\ \nu \neq 0 \\ \sqrt{\frac{\pi}{2x}} e^{-x}[1 + \mathcal{O}(1/x)] & x \gg 1 \end{cases} \tag{13}$$

Hankel Transform – $\mathcal{H}_m(s) = \int_0^\infty \mathrm{d}k\, A(k) J_m(ks) \Rightarrow A(k) = k \int_0^\infty \mathrm{d}s\, s J_m(ks) \mathcal{H}_m(s)$

Laplace's equation:

$$\nabla^2 \Phi = 0. \tag{14}$$

Cartesian separation – $\Phi = X(x)Y(y)Z(z)$

Cylindrical (no z-dependence) – $\Phi = (a_0 + b_0 \ln s)(A_0 + B_0\phi) + \sum_{\nu>0} (a_\nu s^\nu + b_\nu s^{-\nu})(A_\nu \sin\nu\phi + B_\nu \cos\nu\phi)$

Spherical (no ϕ-dependence) – $\Phi = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

Note: solution along z-axis $\Rightarrow z^l \to r^l P_l(\cos\theta)$, $1/z^{l+1} \to \frac{1}{r^{l+1}} P_l(\cos\theta)$

Spherical – $\Phi = \sum_{l,m} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) Y_{lm}(\theta, \phi)$

Green's function

Spherical – $G(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') = \sum_{l,m} g_l(r, r') Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$

$$\Phi(\vec{\boldsymbol{x}}) = \frac{1}{4\pi\epsilon_0} \int_V \mathrm{d}^3\vec{\boldsymbol{x}}'\, G(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')\rho(\vec{\boldsymbol{x}}') + \frac{1}{4\pi} \int_S \mathrm{d}S' \left[G(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') \frac{\partial \Phi(\vec{\boldsymbol{x}}')}{\partial n'} - \Phi(\vec{\boldsymbol{x}}') \frac{\partial G(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')}{\partial n'} \right]$$

Diriclet – $G_D(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}') = 0$ for $\vec{\boldsymbol{x}}' \in S$

Neumann – $\frac{\partial G_N(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{\boldsymbol{x}}' \in S$

Dirichlet concentric spheres – $g_l(r,r') = \frac{4\pi}{2l+1}\Big[1 - \frac{a^{2l+1}}{b^{2l+1}}\Big]^{-1}\Big(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\Big)\Big(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\Big)$

Neumann concentric spheres –

$$l \neq 0: \;\; g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}}\Big[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\Big(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\Big)\Big] \tag{15}$$

$$l = 0: \;\; g_0(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'} + f(r) \tag{16}$$

Multipole expansion

Cartesian – $\Phi(\vec{\boldsymbol{x}}) = \frac{1}{4\pi\epsilon_0}\Big(\frac{Q}{r} + \frac{\vec{\boldsymbol{P}}\cdot\vec{\boldsymbol{x}}}{r^3} + \frac{1}{2}\sum_{i,j}\frac{Q_{ij}x_ix_j}{r^5} + \dots\Big)$

$$Q = \int_V \mathrm{d}^3\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}}) \tag{17}$$

$$\vec{\boldsymbol{P}} = \int_V \mathrm{d}^3\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\vec{\boldsymbol{x}} \tag{18}$$

$$Q_{ij} = \int_V \mathrm{d}^3\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})(3x_ix_j - r^2\delta_{ij}) \tag{19}$$

Spherical Harmonics – $\Phi(\vec{\boldsymbol{r}}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ where $q_{lm} = \int \mathrm{d}^3\vec{\boldsymbol{r}}'\,Y_{lm}^*(\theta',\phi')r'^l\rho(\vec{\boldsymbol{r}}')$