Problem 1)
A localized distribution of charge has a charge density

$$\rho(\vec{r}) = \frac{1}{64\pi}r^2 e^{-r}\sin^2\theta. \tag{1}$$

(a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.
(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right]. \tag{2}$$

We can make a general multipole expansion in spherical harmonics:

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\int d^3\vec{r}'\,\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\rho(\vec{r}'). \tag{3}$$

For our specific charge density

$$\Phi(\vec{r}) = \frac{1}{64\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\int_0^\infty dr'\,\frac{r_<^l}{r_>^{l+1}}r'^4 e^{-r'}\int d\Omega'\,Y_{lm}^*(\theta',\phi')\sin^2\theta'. \tag{4}$$

The angular integration can be done as follows:

$$\int d\Omega\, Y_{lm}^*(\theta,\phi)\sin^2\theta = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\underbrace{\int_0^{2\pi}d\phi\,e^{im\phi}}_{2\pi\delta_{m0}}\int_0^\pi d\theta\sin\theta P_l^m(\cos\theta)(1-\cos^2\theta)$$

$$= \sqrt{\pi(2l+1)}\,\delta_{m0}\int_{-1}^1 dx\, P_l(x)\frac{2}{3}[P_0(x)-P_2(x)] \tag{5}$$

$$= \frac{4\sqrt{\pi}}{3}\left[\delta_{l0}-\frac{1}{\sqrt{5}}\delta_{l2}\right]\delta_{m0}.$$

Putting this into our multipole expansion, we find

$$\Phi(\vec{r}) = \frac{1}{64\pi\epsilon_0}\frac{4\sqrt{\pi}}{3}\left[Y_{00}(\theta,\phi)\left(\int_0^\infty dr'\,\frac{r'^4}{r_>}e^{-r'}\right) - \frac{1}{5\sqrt{5}}Y_{20}(\theta,\phi)\left(\int_0^\infty dr'\,\frac{r_<^2}{r_>^3}r'^4 e^{-r'}\right)\right]$$

$$= \frac{1}{96\pi\epsilon_0}\left[P_0(\cos\theta)\left(\int_0^\infty dr'\,\frac{r'^4}{r_>}e^{-r'}\right) - \frac{1}{5}P_2(\cos\theta)\left(\int_0^\infty \frac{r_<^2}{r_>^3}r'^4 e^{-r'}\right)\right]. \tag{6}$$

To find the potential far away from the origin, we can take $r_< = r'$ and $r_> = r$ (valid for

r where e^{-r} is essentially zero), giving

$$
\begin{aligned}
\Phi(\vec{r}) &= \frac{1}{96\pi\epsilon_0}\left[\frac{4!}{r}P_0(\cos\theta) - \frac{6!}{5r^3}P_2(\cos\theta)\right] \\
&= \frac{1}{4\pi\epsilon_0 r}\left[1 - \frac{6P_2(\cos\theta)}{r^2}\right]
\end{aligned}
\tag{7}
$$

(b) If we are close to the origin, then we must be a bit more careful and split the integration over two portions:

$$
\int .
\tag{8}
$$

Problem 2)
Calculate the multipole moments q_{lm} of the charge distributions shown as parts (a) and (b). Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.
(c) For the charge distribution of the set (b) write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the xy-plane as a function of distance from the origin for distances greater than a.
(d) Calculate directly fro Coulomb's law the exact potential for (b) in the xy-plane. Plot it as a function of distance and compare with the result found in part (c). Divide out the asymptotic form in parts (c) and (d) to see the behavior at large distances more clearly.