

**Problem 1)**

A point charge  $q$  is brought to a position a distance  $d$  away from an infinite plane conductor held at zero potential. Using the method of images, find:

- (a) the surface-charge density induced on the plane, and plot it;
- (b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;
- (c) the total force acting on the plane by integrating  $\sigma^2/2\epsilon_0$  over the whole plane;
- (d) the work necessary to remove the charge  $q$  from its position to infinity;
- (e) the potential energy between the charge  $q$  and its image [compare the answer to part(d) and discuss];
- (f) Find the answer to part (d) in electron volts for an electron originally one angstrom from the surface.

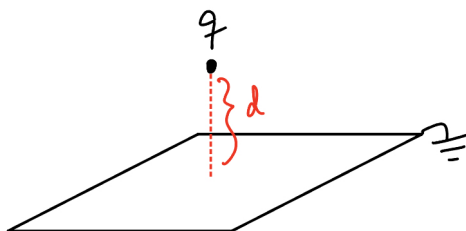


Figure 1: Sketch of setup with charge  $q$  brought to a distance  $d$  from an infinite grounded, conducting plane.

- (a) Solving this problem using the method of images, we place an image charge  $-q$  on the other side of the plane a distance  $d$  away. The potential of this setup is

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\vec{r} - d\hat{e}_z|} - \frac{1}{|\vec{r} + d\hat{e}_z|} \right). \quad (1)$$

We know that this is the correct potential since it satisfies Laplace's equation for  $z > 0$  (defining the  $z$ -axis perpendicular to the plane and the  $+$  direction pointing from the plane to the charge  $q$ ) and the boundary conditions  $\Phi(x, y, 0) = 0$ .

The relation between the potential and surface charge density is

$$\begin{aligned}
 \sigma &= \epsilon_0 \frac{\partial \Phi}{\partial z} = \frac{q}{4\pi} \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right)_{z=0} \\
 &= \frac{q}{4\pi} \left( \frac{2(z-d)}{[x^2 + y^2 + (z-d)^2]^{3/2}} - \frac{2(z+d)}{[x^2 + y^2 + (z+d)^2]^{3/2}} \right)_{z=0} \\
 &= -\frac{qd}{\pi(\rho^2 + d^2)^{3/2}},
 \end{aligned} \tag{2}$$

where we have defined  $\rho^2 = x^2 + y^2$ . Notice that the induced charge has an overall charge with sign opposite that of  $q$ .

(b) The force between the plane and point charge  $q$  can be calculated using Coulomb's law between  $q$  and  $-q$

$$\vec{F} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{2d^2}. \tag{3}$$

(c) We can also calculate the total force acting on the plane as follows:

$$\begin{aligned}
 F &= \int \frac{\sigma^2(x, y)}{2\epsilon_0} dx dy = \frac{q^2 d^2}{2\pi^2 \epsilon_0} \int_0^{2\pi} \int_0^\infty \frac{1}{(\rho^2 + d^2)^{3/2}} \rho d\rho d\phi \\
 &= \left(\frac{qd}{\pi}\right)^2 \int_0^\infty \frac{\rho}{(\rho^2 + d^2)^{3/2}} d\rho = \frac{q^2 d}{\pi^2}.
 \end{aligned} \tag{4}$$

Also, it should be clear that the force is attractive.

(d) From part(b), we can calculate the work needed to move the charge  $q$  infinitely far from the plane as follows

$$W = \int_d^\infty \frac{q^2}{4\pi\epsilon_0} \frac{1}{2z^2} dz = \frac{q^2}{4\pi\epsilon_0} \frac{1}{2d}. \tag{5}$$

Note that the sign is correct since we have to apply a force opposing the attractive force between the plane and charge.

(e) The potential energy of the system can be computed as

$$U = q\Phi_-(d\hat{e}_z) = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{2d}, \tag{6}$$

where  $\Phi_-$  is the potential due to the image charge  $-q$ . Observe that this expression says  $U = -W$ . This is essentially just a statement of the conservation of mechanical energy. The potential energy of the configuration is the negative of the work required to bring the charge  $q$  to its position.

**Problem 2)**

Using the method of images, discuss the problem of a point charge  $q$  inside a hollow, grounded, conducting sphere of inner radius  $a$ . Find

- (a) the potential inside the sphere;
- (b) the induced surface-charge density;
- (c) the magnitude and direction of the force acting on  $q$ .
- (d) Is there any change in the solution if the sphere is kept at a fixed potential  $V$ ? If the sphere has a total charge  $Q$  on its inner and outer surfaces?

**Problem 3)**

A point charge is placed a distance  $d > R$  from the center of an equally charged, isolated, conducting sphere of radius  $R$ .

- (a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?
- (b) What is the limiting value of the force of attraction when the point charge is located a distance  $a = d - R$  from the surface of the sphere, if  $a \ll R$ ?
- (c) What are the results for parts (a) and (b) if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?

**Problem 4)**

(a) Show that the work done to remove the charge  $q$  from a distance  $r > a$  to infinity against the force, Eq. (2.6) in *Jackson* textbook, of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}. \quad (7)$$

Relate this result to the electrostatic potential, Eq. (2.3) in *Jackson* textbook, and the energy discussion of Section 6.3 in Lecture 4.

(b) Repeat the calculation of the work done to remove the charge  $q$  against the force, Eq. (2.9) in *Jackson* textbook (see also Section 9.3.3 of Lecture 6), of an isolated

charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0} \left[ \frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r} \right]. \quad (8)$$

Relate the work to the electrostatic potential, Eq. (2.8) in *Jackson* textbook (see also Section 9.3.2 of Lecture 6), and the energy discussion of Section 6.3 in Lecture 4.