4.20) A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and the dielectric constant is ϵ_r.

We can calculate the electric displacement as follows:

$$\oint \vec{\mathbf{D}} \cdot \mathrm{d}\vec{\mathbf{a}} = Q_{\text{free}} \Rightarrow \vec{\mathbf{D}} = \begin{cases} \frac{\rho}{3} r \hat{\boldsymbol{r}} & r < R \\ \frac{\rho R^3}{3r^2} \hat{\boldsymbol{r}} & r > R \end{cases}$$

It then follows that the potential at the center of the sphere, knowing $\vec{\mathbf{E}} = \epsilon \vec{\mathbf{D}}$ is given as

$$\boxed{V(0) = -\int_{\infty}^{0} \frac{1}{\epsilon_0 \epsilon_r} D \,\mathrm{d}r = -\frac{\rho}{3\epsilon_0 \epsilon_r} \left[\int_{\infty}^{R} \frac{R^3}{r^2} \,\mathrm{d}r + \int_{R}^{0} r \,\mathrm{d}r \right] = \frac{\rho R^2}{2\epsilon_0 \epsilon_r}}$$

4.21) A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable.

We can calculate the capacitance as follows. Suppose that each copper piece has a charge Q on it (where the positive charge is on the inner tube [although this distinction does not make any difference in the capacitance calculation]). Thus, since

$$\oint \vec{\mathbf{D}} \cdot \mathrm{d}\vec{\mathbf{a}} = Q_{\text{free}} \Rightarrow \vec{\mathbf{D}} = \frac{Q}{2\pi s \ell} \hat{\boldsymbol{s}}$$

Hence, the potential difference between the plates is found to be

$$V = -\int_{b}^{a} \frac{1}{\epsilon} \frac{Q}{2\pi s \ell} \,\mathrm{d}s = \frac{Q}{2\pi \epsilon \pi \ell} \ln\left(\frac{b}{a}\right)$$

This means that the capacitance per unit length

$$\boxed{\frac{C}{\ell} = \frac{Q}{V\ell} = \frac{2\pi\epsilon_0\epsilon_r}{\ln(b/a)} = \epsilon_r \frac{C_0}{\ell}}$$

which is the expected result.