3.15) A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has three grounded sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$.

(a) Develop a general formula for the potential inside the pipe.

Notice that Laplace's equation applies inside the pipe:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We can solve this by separating the equation in terms of x and y. That is $V = X(x)Y(y)$. Since the potential at $x = 0$ and $x = b$ are not the same, we choose that $X'' - k^2 X = 0$ and $Y'' + k^2 Y = 0$, meaning that we obtain a general solution for V as a superposition of separable solutions:

$$V(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{k_n x} + B_n e^{-k_n x}\right)(C_n \sin(ky) + D_n \cos(kx))$$

Applying the first three boundary conditions we see that

$$\begin{aligned}
V(x,0) &= 0 \Rightarrow D_n = 0 \\
V(x,a) &= 0 \Rightarrow k_n = \frac{n\pi}{a} \ (n = 1, 2, \ldots) \\
V(0,0) &= 0 \Rightarrow B_n = -A_n
\end{aligned}$$

Thus,

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

Now, we may apply the final boundary condition and use the orthogonality of sine functions to obtain C_n.

$$\begin{aligned}
V_0(y) &= \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \\
\int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) \mathrm{d}y &= \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \mathrm{d}y \\
&= \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \frac{a}{2}\delta_{nm} = C_m \left[\frac{a}{2} \sinh\left(\frac{m\pi b}{a}\right)\right]
\end{aligned}$$

Thus,

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{a}\right) \frac{\sinh(n\pi x/a)}{\sinh(n\pi b/a)} \left(\frac{2}{a} \int_0^a V_0(y') \sin\left(\frac{n\pi y'}{a}\right) \mathrm{d}y'\right)}$$

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

If $V_0(y)$ is constant, then we find that

$$\frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right)\,\mathrm{d}y = \begin{cases} \frac{4V_0}{n\pi} & n = \text{ odd} \\ 0 & n = \text{ even,} \end{cases}$$

so

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi y}{a}\right)\frac{\sinh((2n+1)\pi x/a)}{\sinh((2n+1)\pi b/a)}}$$