1.5) Prove the BAC-CAB rule by writing out both sides in component form.

Notice that

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (B_yC_z - B_zC_y)\hat{x} + (B_zC_x - B_xC_z)\hat{y} + (B_xC_y - B_yC_x)\hat{z},$$

so

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ \left(\vec{B}\times\vec{C}\right)_x & \left(\vec{B}\times\vec{C}\right)_y & \left(\vec{B}\times\vec{C}\right)_z \end{vmatrix}$$

$$\boxed{\begin{aligned} \vec{A}\times\left(\vec{B}\times\vec{C}\right) =& [A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)]\hat{x} \\ &+ [A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_z)]\hat{y} \\ &+ [A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)]\hat{z} \end{aligned}}$$

Next, note that

$$\begin{aligned} \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right) &= (A_xC_x + A_yC_y + A_zC_z)(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \\ &\quad - (A_xB_x + A_yB_y + A_zB_z)(C_x\hat{x} + C_y\hat{y} + C_z\hat{z}) \end{aligned}$$

$$\boxed{\begin{aligned} \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right) &= [B_xA_yC_y + B_xA_zC_z - C_xA_yB_y - C_xA_zB_z]\hat{x} \\ &\quad + [B_yA_xC_x + B_yA_zC_z - C_yA_xB_x - C_yA_zB_z]\hat{y} \\ &\quad + [B_zA_xC_x + B_zA_yC_y - C_zA_xB_x - C_zA_yB_y]\hat{z}. \end{aligned}}$$

It can be easily observed that $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$ by comparing terms component-wise.

Alternatively,

$$\begin{aligned} \left(\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right)_i &= \epsilon_{ijk}A_j\left(\vec{B}\times\vec{C}\right)_k = \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m \\ &= B_i(A_jC_j) - C_i(A_iB_i) \\ \Rightarrow \vec{A}\times\left(\vec{B}\times\vec{C}\right) &= \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right). \end{aligned}$$

1.6) Prove that $\left[\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right] + \left[\vec{B}\times\left(\vec{C}\times\vec{A}\right)\right] + \left[\vec{C}\times\left(\vec{A}\times\vec{B}\right)\right] = 0$. Under what conditions does $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\times\vec{B}\right)\times\vec{C}$?

From the BAC-CAB rule, we have

$$\boxed{\begin{aligned}\left[\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right]+\left[\vec{B}\times\left(\vec{C}\times\vec{A}\right)\right]+\left[\vec{C}\times\left(\vec{A}\times\vec{B}\right)\right] &= \vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)\\ &+\vec{C}\left(\vec{B}\cdot\vec{A}\right)-\vec{A}\left(\vec{B}\cdot\vec{C}\right)\\ &+\vec{A}\left(\vec{C}\cdot\vec{B}\right)-\vec{B}\left(\vec{C}\cdot\vec{A}\right)\\ &=0,\end{aligned}}$$

since the dot product is commutative.

Notice that $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)\times\vec{C}$ whenever $\vec{B}\times\left(\vec{C}\times\vec{A}\right)$ is zero. Thus, $\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=$, meaning that $\vec{B}$ and $\vec{C}\times\vec{A}$ are scalar multiples of each other. Geometrically, $\vec{B}$ is normal to the plane spanned by $\vec{A}$ and $\vec{C}$.