

3.20) Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma_\theta = \frac{\epsilon_0}{2R} \sum_{\ell=0}^{\infty} (2\ell+1)^2 C_\ell P_\ell(\cos \theta)$$

where

$$C_\ell = \int_0^\pi V_0(\theta) P_\ell(\cos \theta) \sin \theta \, d\theta$$

In spherical coordinates, the solution to Laplace's equation is given as follows:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos \theta) \begin{cases} A_\ell r^\ell & r \leq R \\ \frac{B_\ell}{r^{\ell+1}} & r \geq R \end{cases}$$

We are given the boundary condition $V(R, \theta) = V_0(\theta)$, and we know from previous work that $V_{\text{in}} = V_{\text{out}}$. Applying these gives us

$$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos \theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos \theta)$$

From this it is clear that $B_\ell = A_\ell R^{2\ell+1}$. Using the orthogonality of the Legendre polynomials, we see that

$$\int_0^\pi V_0(\theta) P_{\ell'}(\cos \theta) \, d(\cos \theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi P_{\ell'}(\cos \theta) P_\ell(\cos \theta) \, d(\cos \theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

Thus

$$\begin{aligned} A_\ell &= \frac{2\ell+1}{2} \frac{1}{R^\ell} C_\ell \\ B_\ell &= \frac{2\ell+1}{2} R^{\ell+1} C_\ell \end{aligned}$$

where

$$C_\ell = \int_0^\pi V_0(\theta) P_\ell(\cos \theta) \, d(\cos \theta)$$

We can solve for the charge density on the surface of the sphere using another boundary condition

$$\begin{aligned} \sigma(\theta) &= -\epsilon_0 \left[\frac{\partial V_{\text{in}}}{\partial r} - \frac{\partial V_{\text{out}}}{\partial r} \right]_{r=R} \\ &= -\epsilon_0 \sum_{\ell=0}^{\infty} \left[-(\ell+1) \frac{B_\ell}{R^{\ell+2}} - \ell A_\ell R^{\ell-1} \right] P_\ell(\cos \theta) \\ &= \epsilon_0 \sum_{\ell=0}^{\infty} \left[(\ell+1) \frac{1}{R^{\ell+2}} R^{\ell+1} - \ell \frac{1}{R^\ell} R^{\ell-1} \right] \frac{2\ell+1}{2} C_\ell P_\ell(\cos \theta) \end{aligned}$$

giving

$$\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{\ell=0}^{\infty} (2\ell+1)^2 C_{\ell} P_{\ell}(\cos \theta)$$

3.25) Find the potential outside an infinitely long metal pipe, of radius R , placed at right angles to an otherwise uniform electric field \vec{E}_0 . Find the surface charge induced on the pipe.

The solution to Laplace's equation in cylindrical coordinates (assuming that the problem is invariant under translations in z) is

$$V(s, \phi) = A_0 \ln s + B_0 + \sum_{n=1}^{\infty} (A_n s^n + B_n s^{-n}) (C_n \sin(n\phi) + D_n \cos(n\phi))$$

We choose to point the “uniform” electric field along the x direction such that $\vec{E} = E_0 \hat{x}$ far away from the pipe. For this problem then, we have two relevant boundary conditions:

1. $V(R, \phi) = 0$ (we are free to choose a valid reference point)
2. $V(s \gg R, \phi) = -E_0 x = -E_0 s \cos \phi$

Note that the constant term from the potential due to the external field is zero since at $\phi = \pi/2, 3\pi/2$ the potential vanishes. Applying the first boundary condition gives

$$\begin{aligned} V(R, \phi) &= A_0 \ln R + B_0 + \sum_{n=1}^{\infty} (A_n R^n + B_n R^{-n}) (C_n \sin(n\phi) + D_n \cos(n\phi)) = 0 \\ \Rightarrow B_0 &= -A_0 \ln R \text{ and } B_n = -A_n R^{2n} \\ V(r, \phi) &= A_0 \ln\left(\frac{s}{R}\right) + \sum_{n=1}^{\infty} (s^n + R^{2n} s^{-n}) (C_n \sin(n\phi) + D_n \cos(n\phi)) \end{aligned}$$

Now, applying the second boundary condition we see that $A_0 = 0$ (since we need the \ln behavior to vanish at large s) and $s^{-n} \rightarrow 0$, leaving us with

$$V(s \gg R, \phi) = \sum_{n=1}^{\infty} (C_n s^n \sin(n\phi) + D_n s^n \cos(n\phi)) = -E_0 s \cos \phi$$

Matching terms we get that $C_n = 0$, $D_1 = -E_0$, and $D_{n \neq 1} = 0$. This gives us

$$V(s, \phi) = E_0 \cos \theta \left(\frac{R^2}{s} - s \right)$$

Finally, we can find the surface charge density as

$$\sigma(\phi) = -\epsilon_0 \frac{\partial V}{\partial s} \Big|_{s=R} = 2\epsilon_0 E_0 \cos \phi$$