5.23) Find the magnetic vector potential of a finite segment of straight wire carrying a current I. Check that your answer is consistent with Eq. 5.37.

Let us orient our coordinate sysytem such that the point at which we are evaluating the vector potential lies in the xy-plane. Thus, we have

$$\begin{aligned}\vec{\mathbf{A}} &= \frac{\mu_0 I}{4\pi}\int \frac{\mathrm{d}z}{\mathscr{r}}\hat{\boldsymbol{z}} \\ &= \frac{\mu_0 I}{4\pi}\hat{\boldsymbol{z}}\int_{z_1}^{z_2}\frac{dz}{\sqrt{r^2+z^2-2rz\cos\theta}} \\ &= \frac{\mu_0 I}{4\pi}\ln\left[\frac{\sqrt{s^2+z_2^2}+z_2}{\sqrt{s^2+z_1^2}+z_1}\right]\hat{\mathbf{z}}.\end{aligned}$$

Now, we check if this is consistent with the known magnetic field for a finite length wire. Taking the curl of the vector potential, we get

$$\begin{aligned}\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}} &= -\frac{\mu_0 I}{4\pi}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial s}\ln\left[\frac{\sqrt{s^2+z_2^2}+z_2}{\sqrt{s^2+z_1^2}+z_1}\right] \\ &= -\frac{\mu_0 I}{4\pi}\left[\frac{s}{\sqrt{s^2+s_2^2}\left(\sqrt{s^2+z_2^2}\right)+z_2}-\frac{s}{\sqrt{s^2+z_1^2}\left(\sqrt{s^2+z_1^2}+z_1\right)}\right]\hat{\boldsymbol{\phi}}.\end{aligned}$$

We simplify this by noting that

$$\begin{aligned}z &= \sqrt{s^2+z^2}\sin\theta \\ \frac{s}{\sqrt{s^2+z^2}+z} &= \frac{\sqrt{s^2+z^2}-z}{s}.\end{aligned}$$

Simplifying, we have

$$\begin{aligned}\vec{\mathbf{B}} &= -\frac{\mu_0 I}{4\pi s}\left[\frac{\sqrt{s^2+z_2^2}-z_2}{\sqrt{s^2+z_2^2}}-\frac{\sqrt{s^2+z_1^2}-z_1}{\sqrt{s^2+z_1^2}}\right]\hat{\boldsymbol{\phi}} \\ &= -\frac{\mu_0 I}{4\pi s}\left[1-\sin\theta_2-(1-\sin\theta_2)\right] \\ &= \frac{\mu_0 I}{4\pi s}\left[\sin\theta_2-\sin\theta_1\right],\end{aligned}$$

which is exactly what we expect to get.

5.35) A circular loop of wire, with radius R, lies in the xy plane (centered at the origin) and carries a current I running counterlockwise as viewed from the positive z axis.

a) What is its magnetic dipole moment?

The magnetic dipole moment of the current loop is just

$$\vec{\mathbf{m}} = I\vec{\mathbf{A}} = \pi R^2 I\hat{\mathbf{z}}$$

b) What is the (approximate) magnetic field at points far from the origin?

Far way from the origin (i.e. $r \gg R$) the dipole term in the multipole expansion dominates, so

$$\vec{\mathbf{B}} \approx \vec{\mathbf{B}}_{\text{dip}}\left(\vec{\mathbf{r}}\right) = \frac{\mu_0 I}{4}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right].$$

c) Show that, for points on the z axis, your answer is consistent with the *exact* field, when $z \gg R$.

Finally, it can be seen that above the wire on the axis passing through its center, when we are far from the wire, that

$$\vec{\mathbf{B}} \approx \frac{\mu_0 I}{2}\frac{R^2}{z^3},$$

and the dipole term gives us ($\theta = 0$):

$$\vec{\mathbf{B}}_{\text{dip}} = \frac{\mu_0 I R^2}{2z^3}.$$