

**1.37)** Find formulas for  $r, \theta, \phi$  in terms of  $x, y, z$ .

The equations changing  $(r, \theta, \phi)$  to  $(x, y, z)$  are

$$\begin{cases} x = r \cos \phi \sin \theta \\ y = r \sin \phi \sin \theta \\ z = r \cos \theta \end{cases}$$

It is obvious that

$$x^2 + y^2 + z^2 = r^2 \cos^2 \phi \sin^2 \theta + r^2 \sin^2 \phi \sin^2 \theta + r^2 \cos^2 \theta = r^2,$$

or

$$r = \sqrt{x^2 + y^2 + z^2}$$

Then, it is seen that

$$\theta = \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

and finally that

$$\frac{r \sin \phi \sin \theta}{r \cos \phi \sin \theta} = \tan \phi = \frac{y}{x}$$

$$\phi = \arctan \left( \frac{y}{x} \right)$$

**1.41)** Compute the gradient and Laplacian of the function  $T = r(\cos \theta + \sin \theta \cos \phi)$ . Check the Laplacian by converting  $T$  to Cartesian coordinates. Test the gradient theorem for this function, using the path shown in Fig. 1.41, from  $(0, 0, 0)$  to  $(0, 0, 2)$ .

Observe that

$$\begin{aligned} \nabla^2 T(r, \theta, \phi) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2)(\cos \theta + \sin \theta \cos \phi) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (-\sin^2 \theta \cos \theta \sin \theta \cos \phi) + \frac{1}{r^2 \sin^2 \theta} (-\cos \phi) \\ &= \frac{2(\cos \theta + \sin \theta \cos \phi)}{r} + \frac{-\sin^2 \theta \cos \phi - 2 \sin \theta \cos \theta + \cos^2 \theta \cos \phi}{r \sin \theta} - \frac{\cos \phi}{r \sin \theta} \end{aligned}$$

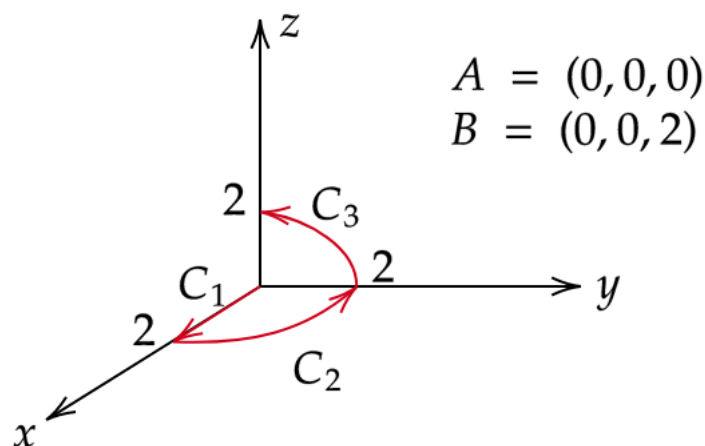
$$\boxed{\nabla^2 T = 0}$$

In Cartesian coordinates

$$T = \sqrt{x^2 + y^2 + z^2} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = z + x$$

Clearly, then

$$\boxed{\nabla^2 T = 0}$$



The gradient theorem states  $\int_C \vec{\nabla} T \cdot d\vec{l} = T(B) - T(A)$ , and the path  $C$  is shown below. Note that  $C = C_1 + C_2 + C_3$ , so  $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$ , and  $\vec{\nabla} T = \frac{\partial T}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \hat{\phi}$ . Hence, since  $d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$ , then

$$\int_C \vec{\nabla} T \cdot d\vec{l} = \int_C (\cos \theta + \sin \theta \cos \phi) dr + \int_C r(-\sin \theta + \cos \theta \cos \phi) d\theta + \int_C r \sin \theta (-\sin \phi) d\phi$$

The integrals over each path are as follows:

$$\begin{aligned} \int_{C_1} \vec{\nabla} T \cdot d\vec{l} &= \left( \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \cos(0) \right) (2) = 2 \\ \int_{C_2} \vec{\nabla} T \cdot d\vec{l} &= -2 \sin\left(\frac{\pi}{2}\right) \int_0^{\pi/2} \sin \phi d\phi = -2 \\ \int_{C_3} \vec{\nabla} T \cdot d\vec{l} &= 2 \int_{\pi/2}^0 (-\sin \phi) d\phi = 2 \end{aligned}$$

Summing up the contribution over each part, we see that

$$\boxed{\int_C \vec{\nabla} T \cdot d\vec{l} = 2,}$$

which is the same as given by the right hand side of the gradient theorem

$$\boxed{T(0, 0, 2) - T(0, 0, 0) = 2 - 0 = 0.}$$

**1.44)** Evaluate the following integrals:

(a)  $\int_2^6 (3x^2 - 2x - 1)\delta(x - 3) dx = 3(3)^2 - 2(3) - 1 = \boxed{20}$

(b)  $\int_0^5 \cos x \delta(x - \pi) dx = \cos \pi = \boxed{-1}$

(c)  $\int_0^3 x^3 \delta(x + 1) dx = \boxed{0}$

(d)  $\int_{-\infty}^{\infty} \ln(x + 3) \delta(x + 2) dx = \ln(-2 + 3) = \boxed{0}$