1.37) Find formulas for r, θ, ϕ in terms of x, y, z.

The equations changing (r, θ, ϕ) to (x, y, z) are

$$\begin{cases} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases}$$

It is obvious that

$$x^2 + y^2 + z^2 = r^2\cos^2\phi\sin^2\phi + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\theta = r^2,$$

or

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}}$$

Then, it is seen that

$$\boxed{\theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)}$$

and finally that

$$\frac{r\sin\phi\sin\theta}{r\cos\phi\sin\theta} = \tan\phi = \frac{y}{x}$$

$$\boxed{\phi = \arctan\left(\frac{y}{x}\right)}$$

1.41) Compute the gradient and Laplacian of the function $T = r(\cos\theta + \sin\theta\cos\phi)$. Check the Laplacian by converting T to Cartesian coordinates. Test the gradient theorem for this function, using the path shown in Fig. 1.41, from $(0, 0, 0)$ to $(0, 0, 2)$.

Observer that

$$\begin{aligned} \vec{\nabla}^2 T(r, \theta, \phi) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right)(\cos\theta + \sin\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\cos\theta\sin\theta\cos\phi\right) + \frac{1}{r^2\sin^2\theta}(-\cos\phi) \\ &= \frac{2(\cos\theta + \sin\theta\cos\phi)}{r} + \frac{-\sin^2\theta\cos\phi - 2\sin\theta\cos\theta + \cos^2\theta\cos\phi}{r\sin\theta} - \frac{\cos\phi}{r\sin\theta} \end{aligned}$$

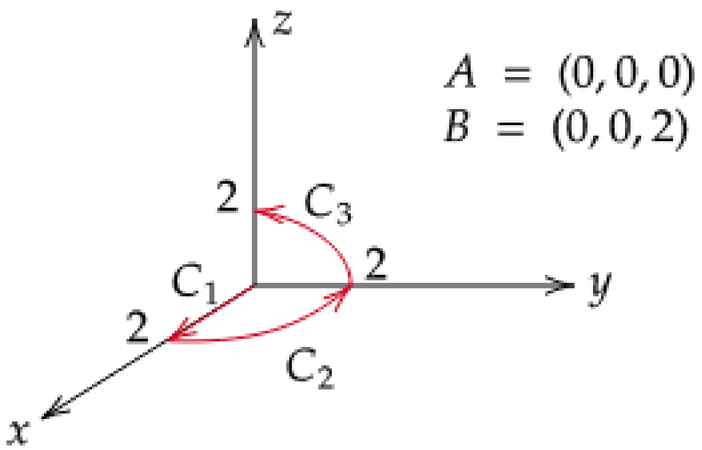
$$\boxed{\vec{\nabla}^2 T = 0}$$

In Cartesian coordinates

$$T = \sqrt{x^2 + y^2 + z^2}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) = z + x$$

Clearly, then

$$\boxed{\vec{\nabla}^2 T = 0}$$



The gradient theorem states $\int_C \vec{\nabla}T \cdot \mathrm{d}\vec{l} = T(B) - T(A)$, and the path C is shown below. Note that $C = C_1 + C_2 + C_3$, so $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$, and $\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi} = (\cos\theta + \sin\theta\cos\phi)\hat{r} + (-\sin\theta + \cos\theta\cos\phi)\hat{\theta} - \sin\phi\hat{\phi}$. Hence, since $\mathrm{d}\vec{l} = \hat{r}\,\mathrm{d}r + \hat{\theta}r\,\mathrm{d}\theta + \hat{\phi}r\sin\theta\,\mathrm{d}\phi$. Then,

$$\int_C \vec{\boldsymbol{\nabla}}T \cdot \mathrm{d}\vec{\boldsymbol{l}} = \int_C (\cos\theta + \sin\theta\cos\phi)\,\mathrm{d}r + \int_C r(-\sin\theta + \cos\theta\cos\phi)\,\mathrm{d}\theta + \int_C r\sin\theta(-\sin\phi)\,\mathrm{d}\phi$$

The integrals over each path are as follows:

$$\int_{C_1} \vec{\boldsymbol{\nabla}}T \cdot \mathrm{d}\vec{\boldsymbol{l}} = \left(\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\cos(0)\right)(2) = 2$$

$$\int_{C_2} \vec{\boldsymbol{\nabla}}T \cdot \mathrm{d}\vec{\boldsymbol{l}} = -2\sin\left(\frac{pi}{2}\right)\int_0^{\pi/2} \sin\phi\,\mathrm{d}\phi = -2$$

$$\int_{C_3} \vec{\boldsymbol{\nabla}}T \cdot \mathrm{d}\vec{\boldsymbol{l}} = 2\int_{\pi/2}^{0} (-\sin\phi)\,\mathrm{d}\phi = 2$$

Summing up the contribution over each part, we see that

$$\boxed{\int_C \vec{\boldsymbol{\nabla}}T \cdot \mathrm{d}\vec{\boldsymbol{l}} = 2,}$$

which is the same as given by the right hand side of the gradient theorem

$$\boxed{T(0,0,2) - T(0,0,0) = 2 - 0 = 0.}$$

1.44) Evaluate the following integrals:

(a) $\int_2^6 (3x^2 - 2x - 1)\delta(x-3)\,\mathrm{d}x = 3(3)^2 - 2(3) - 1 = \boxed{20}$

(b) $\int_0^5 \cos x\,\delta(x-\pi)\,\mathrm{d}x = \cos\pi = \boxed{-1}$

(c) $\int_0^3 x^3\delta(x+1)\,\mathrm{d}x = \boxed{0}$

(d) $\int_{-\infty}^{\infty} \ln(x+3)\delta(x+2)\,\mathrm{d}x = \ln(-2+3) = \boxed{0}$