3.20) *

In spherical coordinates, the solution to Laplace's equation is given as follows:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \begin{cases} A_\ell r^\ell & r \le R \\ \frac{B_\ell}{r^{\ell+1}} & r \ge R \end{cases}$$

We are given the boundary condition $V(R,\theta) = V_0(\theta)$, and we know from previous work that $V_{\text{in}} = V_{\text{out}}$. Applying these gives us

$$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

From this it is clear that $B_\ell = A_\ell R^{2\ell+1}$. Using the orthogonality of the Legendre polynomials, we see that

$$\int_0^\pi V_0(\theta) P_{\ell'}(\cos\theta)\,\mathrm{d}(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\,\mathrm{d}(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

Thus

$$\begin{aligned} A_\ell &= \frac{2\ell+1}{2}\frac{1}{R^\ell} C_\ell \\ B_\ell &= \frac{2\ell+1}{2} R^{\ell+1} C_\ell \end{aligned}$$

where

$$C_\ell = \int_0^\pi V_0(\theta) P_\ell(\cos\theta)\,\mathrm{d}(\cos\theta)$$

We can solve for the charge density on the surface of the sphere using another boundary condition

$$\begin{aligned} \sigma(\theta) &= -\epsilon_0 \left[\frac{\partial V_{\text{in}}}{\partial r} - \frac{\partial V_{\text{out}}}{\partial r}\right]_{r=R} \\ &= -\epsilon_0 \sum_{\ell=0}^{\infty} \left[-(\ell+1)\frac{B_\ell}{R^{\ell+2}} - \ell A_\ell R^{\ell-1}\right] P_\ell(\cos\theta) \\ &= \epsilon_0 \sum_{\ell=0}^{\infty} \left[(\ell+1)\frac{1}{R^{\ell+2}} R^{\ell+1} - \ell \frac{1}{R^\ell} R^{\ell-1}\right] \frac{2\ell+1}{2} C_\ell P_\ell(\cos\theta) \end{aligned}$$

giving

$$\boxed{\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{\ell=0}^{\infty} (2\ell+1)^2 C_\ell P_\ell(\cos\theta)}$$

3.25) *

The solution to Laplace's equation in cylindrical coordinates (assuming that the problem is invariant under translations in z) is

$$V(s,\phi) = A_0 \ln s + B_0 + \sum_{n=1}^{\infty} \left(A_n s^n + B_n s^{-n}\right)\left(C_n \sin(n\phi) + D_n \cos(n\phi)\right)$$

We choose to point the "uniform" electric field along the x direction such that $\vec{E} = E_0\hat{\boldsymbol{x}}$ far away from the pipe. For this problem then, we have two relevant boundary conditions:

1. $V(R,\phi) = 0$ (we are free to choose a valid reference point)
2. $V(s \gg R, \phi) = -E_0 x = -E_0 s\cos\phi$

Note that the constant term from the potential due to the external field is zero since at $\phi = \pi/2, 3\pi/2$ the potential vanishes. Applying the first boundary condition gives

$$\begin{aligned}
V(R,\phi) &= A_0 \ln R + B_0 + \sum_{n=1}^{\infty} \left(A_n R^n + B_n R^{-n}\right)\left(C_n \sin(n\phi) + D_n \cos(n\phi)\right) = 0 \\
&\Rightarrow B_0 = -A_0 \ln R \text{ and } B_n = -A_n R^{2n} \\
V(r,\phi) &= A_0 \ln\left(\frac{s}{R}\right) + \sum_{n=1}^{\infty} \left(s^n + R^{2n} s^{-n}\right)\left(C_n \sin(n\phi) + D_n \cos(n\phi)\right)
\end{aligned}$$

Now, applying the second boundary condition we see that $A_0 = 0$ (since we need the ln behavior to vanish at large s) and $s^{-n} \to 0$, leaving us with

$$V(s \gg R, \phi) = \sum_{n=1}^{\infty} \left(C_n s^n \sin(n\phi) + D_n s^n \cos(n\phi)\right) = -E_0 s\cos\phi$$

Matching terms we get that $C_n = 0$, $D_1 = -E_0$, and $D_{n\neq 1} = 0$. This gives us

$$\boxed{V(s,\phi) = E_0 \cos\theta\left(\frac{R^2}{s} - s\right)}$$

Finally, we can find the surface charge density as

$$\boxed{\sigma(\phi) = -\epsilon_0 \frac{\partial V}{\partial s}\Big|_{s=R} = 2\epsilon_0 E_0 \cos\phi}$$