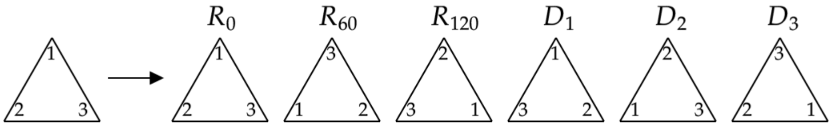
1) Compute the order of each element in the dihedral group D_6 and D_8.

$\rightarrow$ Consider the symmetries of a triangle given in the figure below and let $D_6 = \{R_0, R_{60}, R_{120}, D_1, D_2, D_3\}$.

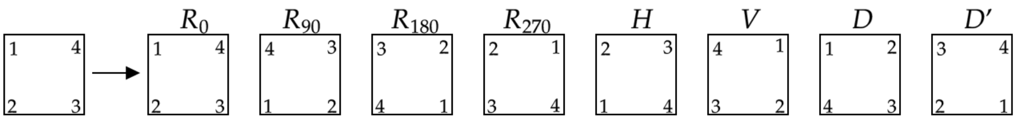


Then we have the following

element	R_0	R_{60}	R_{120}	D_1	D_2	D_3
order	1	3	3	2	2	2

For each symmetry operation, we consider the minimum number of applications that are needed to return to the original configuration. Obviously, R_0 is the identity, which always has order of 1 for any group, and the reflections across diagonals through the vertices and center must be performed twice.

Now consider the symmetries of a square shown below and let $D_8 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$.



Then it follows that

element	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D
order	1	4	2	4	2	2	2	2

2) Show that $\langle a, b | a^2 = b^2 = (ab)^2 = \mathbb{1}\rangle$ give a presentation for D_{2n} in terms of the two generators $a = s$ and $b = sr$ of order 2.

3[1.1.1]) Complete the multiplication table for D_8. Find at least one interesting pattern in the table.

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

Something interesting

4[1.1.4])

(a) List the symmetries of a rectangle.

The symmetries are given in the second figure in problem 1 can be put into the set D_8, which is also given in problem 1.

(b) Write the multiplication table for the symmetries of a rectangle.

The multiplication table for the symmetries of a rectangle are shown in the multiplication table in problem 3.

5[1.1.5])

(a) Find the center of D_8

The center of a group G is the set of all elements $a \in G$ such that $ab = ba$ for all $b \in G$ (i.e. a commutes with all elements of G). We can read from the table in problem 3 what elements are in the center of D_8:

$$\mathbf{Z}(D_8) = \{R_0, R_{180}\}$$

(b) Find $\mathbf{C}_{D_8}(R_{90})$ and $\mathbf{C}_{D_8}(H)$.

The centralizer of an element $a \in G$ is the set of all elements $b \in G$ such that a and b commute:

$$\mathbf{C}_{D_8}(R_{90}) = \{R_0, R_{90}, R_{180}, R_{270}\}$$

$$\mathbf{C}_{D_8}(H) = \{H, R_{180}, R_0, V\}$$

6[1.1.6]) Let D_6 denote the set of symmetries of an equilateral triangle. Find the multiplication table for D_6. What is the center of D_6?

	R_0	R_{60}	R_{120}	D_1	D_2	D_3
R_0	R_0	R_{60}	R_{120}	D_1	D_2	D_3
R_{60}	R_{60}	R_{120}	R_0	D_2	D_3	D_1
R_{120}	R_{120}	R_0	R_{60}	D_3	D_1	D_2
D_1	D_1	D_3	D_2	R_0	R_{120}	R_{60}
D_2	D_2	D_1	D_3	R_{60}	R_0	R_{120}
D_3	D_3	D_2	D_1	R_{120}	R_{60}	R_0

7[1.2.4]) Let $\sigma = (1\ 3\ 5)(2\ 4)$ and $\tau = (1\ 5)(2\ 3)$ be elements of S_5. Find σ^2, $\sigma\tau$, $\tau\sigma$, and $\tau\sigma^2$.

$$\begin{aligned}
\sigma^2 &= [(1\ 3\ 5)(2\ 4)][(1\ 3\ 5)(2\ 4)] = (1\ 5\ 3) \\
\sigma\tau &= [(1\ 3\ 5)(2\ 4)][(1\ 5)(2\ 3)] = (5\ 3\ 4\ 2) \\
\tau\sigma &= [(1\ 5)(2\ 3)][(1\ 3\ 5)(2\ 4)] = (1\ 2\ 4\ 3) \\
\tau\sigma^2 &= [(1\ 5)(2\ 3)](1\ 5\ 3) = (5\ 2\ 3)
\end{aligned}$$

8[1.2.5]) Construct a complete multiplication table for S_3. What is the center of S_3? If $f = (1\ 2\ 3)$, what is $\mathbf{C}_{S_3}(f)$, the centralizer of f in S_3?

$\rightarrow$ We know that $S_3 = \text{Perm}(\{1, 2, 3\}) = \{\mathbb{1}, (2\ 3), (1\ 2), (1\ 2\ 3), (1\ 3\ 2), (1\ 3)\}$. Hence,

	$\mathbb{1}$	$(2\ 3)$	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 3)$
$\mathbb{1}$	$\mathbb{1}$	$(2\ 3)$	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 3)$
$(2\ 3)$	$(2\ 3)$	$\mathbb{1}$	$(1\ 3\ 2)$	$(1\ 3)$	$(1\ 2)$	$(1\ 2\ 3)$
$(1\ 2)$	$(1\ 2)$	$(1\ 2\ 3)$	$\mathbb{1}$	$(2\ 3)$	$(1\ 3)$	$(1\ 3\ 2)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(1\ 2)$	$(1\ 3)$	$(1\ 3\ 2)$	$\mathbb{1}$	$(2\ 3)$
$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$	$\mathbb{1}$	$(1\ 2\ 3)$	$(1\ 2)$
$(1\ 3)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1\ 2\ 3)$	$(1\ 2)$	$(2\ 3)$	$\mathbb{1}$

and

$$\mathbf{C}_{S_3}(f) = \{\mathbb{1}, (1\ 2\ 3), (1\ 3\ 2)\}$$

9[1.2.6]) Let $f = (1\ 2\ 3) \in S_3$. Find the maps in the following sequence

$$\mathbb{1}_{[3]}, f, f^2, f^3, f^4, f^5, \ldots$$

Do you see a pattern?

Notice that from the table we see that

$$\begin{aligned}
&f^0 = \mathbb{1}_{[3]} \\
&f^1 = (1\ 2\ 3) \\
&f^2 = (1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2) \\
&f^3 = (1\ 2\ 3)(1\ 3\ 2) = \mathbb{1}_{[3]} \\
&\ \vdots
\end{aligned}$$

It is seen that this sequence essentially lists the elements of the cyclic group $< f >= \{f, f^2, f^3 = \mathbb{1}_{[3]}\}$.