

1[2.3.6]) Find all the generators of the following cyclic groups:

$$(\mathbb{Z}/6\mathbb{Z}, +), ((\mathbb{Z}/5\mathbb{Z})^\times, \cdot), (2\mathbb{Z}, +), ((\mathbb{Z}/11\mathbb{Z})^\times, \cdot)$$

$$\rightarrow \mathbb{Z}/6\mathbb{Z} = \langle 1 \rangle = \langle 5 \rangle.$$

$$\rightarrow (\mathbb{Z}/5\mathbb{Z})^\times = \langle 2 \rangle = \langle 3 \rangle$$

$$\rightarrow 2\mathbb{Z} = \langle 2 \rangle = \langle -2 \rangle$$

$$\rightarrow (\mathbb{Z}/11\mathbb{Z})^\times = \langle 2 \rangle = \langle 6 \rangle = \langle 7 \rangle = \langle 8 \rangle$$

2[2.3.7]) Show that a finite group of even order has to have at least one element of order 2.

Proof

It has been proven in a previous homework problem that if a group has finite, even order then there exists $a \in G$ such that $a^2 = 1_G$ but $a \neq 1_G$. This is equivalent to saying that $o(a) = 2$, proving the above claim.

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3[2.3.11]) Let G be a group, and let $x \in G$. How are $o(x)$ and $o(x^{-1})$ related? Prove your assertion

$$\rightarrow \text{We claim that } o(x^{-1}) = o(x).$$

Proof

Observe that $o(x) = \min\{m \in \mathbb{Z}^+ | x^m = 1\}$ and that $(x^{-1})^m = (x^m)^{-1}$. Thus, if $x^m = 1_G$, then $(x^{-1})^m = (x^m)^{-1} = 1_G^{-1} = 1_G$, and $o(x^{-1}) = \min\{m \in \mathbb{Z}^+ | (x^{-1})^m = 1\} = \min\{m \in \mathbb{Z}^+ | (x^m)^{-1} = 1\} = \min\{m \in \mathbb{Z}^+ | x^m = 1\} = o(x)$.

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4[2.4.4]) Let $H = \{2^n | n \in \mathbb{Z}\}$, and let \cdot denote ordinary multiplication. Show that (H, \cdot) is isomorphic to $(\mathbb{Z}, +)$.

Proof

Consider $\varphi : \mathbb{Z} \rightarrow H$ such that $\varphi(n) = 2^n$. It is obvious that this function is bijective since if $2^n = 2^m$ then $n = m$ and $\varphi(\mathbb{Z}) = H$ by definition. We now prove that φ is homomorphic. Observe that $\varphi(n + m) = 2^{n+m}$ and $\varphi(n)\varphi(m) = 2^n 2^m$. It is clear then that $\varphi(n + m) = \varphi(n)\varphi(m)$.

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5[2.4.14]) Let G and H be groups, and let $\phi : G \rightarrow H$ be a group homomorphism. For $x \in G$, prove that $\phi(x^{-1}) = \phi(x)^{-1}$.

Proof

Recall that $\phi(1_G) = 1_H$. Notice that $x^{-1}x = 1_G$, meaning that $\phi(x^{-1}x) = \phi(x^{-1})\phi(x) = 1_H$ or $\phi(x^{-1}) = \phi(x)^{-1}$.

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6[2.6.4]) Let G be a group, and let H and K be subgroups of G . Show that $H \cap K$ is a subgroup of G .

Proof

Notice that $1 \in H \cap K$ since $1 \in H$ and $1 \in K$, meaning that $H \cap K$ is nonempty. Let $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$, implying $ab \in H$ and $ab \in K$ and $ab \in H \cap K$. Finally, suppose that $a \in H \cap K$. Then $a \in H$ and $a \in K$, which means that $a^{-1} \in H$ and $a^{-1} \in K$ and $a^{-1} \in H \cap K$.

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7[2.6.9]) Let G be a group, and assume that a and b are two elements of order 2 in G . If $ab = ba$, then what can you say about $\langle a, b \rangle$?

→ We can write $G = \langle a, b | a^2 = b^2, ab = ba \rangle = \{1_G, a, b, ab\}$.

8[2.6.12]) Let $G = \langle x, y \mid x^7 = y^3 = 1, yxy^{-1} = 1 \rangle$. What is $|G|$? Find a familiar group that is isomorphic to G .

→ Notice that the relation $yxy^{-1} = 1$ reduces to $x = y^{-1}y = 1$, so we can write $G = \{1, y, y\}$. It is clear then that this group is isomorphic to the additive group $\mathbb{Z}/3\mathbb{Z}$. This may be observed with the function from $\varphi : \mathbb{Z}/3\mathbb{Z} \rightarrow G$ such that $\varphi(\bar{n}) = y^n$.

9[2.6.19]) Let G and H be groups, and let $\theta : G \rightarrow H$ be a homomorphism. The set $\{x \in G \mid \theta(x) = 1\}$ is called the *kernel* of θ and is denoted by $\ker(\theta)$. Show that $\ker(\theta)$ is a subgroup of G .

Proof

Notice that $1_G \in \ker(\theta)$ since $\theta(1_G) = 1_H$, meaning that $\ker(\theta)$ is nonempty. Next, suppose that $a, b \in \ker(\theta)$, then $\theta(a) = \theta(b) = 1_H$. Thus, $\theta(ab) = \theta(a)\theta(b) = 1_H 1_H = 1_H$, implying that $ab \in \ker(\theta)$. Now, suppose that $a \in \ker(\theta)$. We then have that $\theta(a^{-1}a) = \theta(a^{-1})\theta(a) = \theta(a^{-1})1_H = \theta(a^{-1})$. It is also known that $\theta(a^{-1}a) = \theta(1_G) = 1_H$, implying that $\theta(a^{-1}) = 1_H$.

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10[2.6.20]) Let G and H be groups, and let $\theta : G \rightarrow H$ be a homomorphism. Let K be a subgroup of H . The set of elements of G that are mapped into K are denoted by $\theta^{-1}(K)$. In other words,

$$\theta^{-1}(K) = \{g \in G \mid \theta(g) \in K\}$$

Is $\theta^{-1}(K)$ necessarily a subgroup of G ?

Proof

Obviously $1_G \in \theta^{-1}(K)$, so it is nonempty. If $a, b \in \theta^{-1}(K)$, then $\theta(a), \theta(b) \in K$. Hence, because θ is a homomorphism $\theta(ab) = \theta(a)\theta(b) \in K$ since K is closed under multiplication. Thus, $ab \in \theta^{-1}(K)$. Similarly, if $a \in \theta^{-1}(K)$, then $\theta(a^{-1}a) = \theta(a^{-1})\theta(a) = 1_H$ or $\theta(a^{-1}) = \theta(a)^{-1}$. Notice that $\theta(a^{-1}) = \theta(a)^{-1} \in K$ since $\theta(a) \in K$.

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