**3.1.1)** Let $\varphi : G \to H$ be a homomorphism and let $E \leq H$. Prove that $\varphi^{-1}(E) \leq G$. If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$

**Proof**

Notice that since $E$ is a subgroup of $H$, then $\mathbb{1}_H \in E$, and it is clear that $\varphi(\mathbb{1}_G) = \mathbb{1}_H$ meaning that $\mathbb{1}_G \in \varphi^{-1}(E)$.

■

**3.1.3)** Let $A$ be an abelian group and $B \leq A$. Prove that $A/B$ is abelian. Give an example of a non-abelian group $G$ containing a proper normal subgroup $N$ such that $G/N$ is abelian.

**3.1.4)** Prove that in the quotient group $G/N$, $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$

**3.1.5)** Use the preceding exercise to prove that the order of the element $gN$ in $G/N$ is $n$, where $n$ is the smallest positive integer such that $g^n \in N$. Give an example to show that the order of $gN$ in $G/N$ may be strictly smaller than the order of $g$ in $G$.

**3.1.24)** Prove that if $N \trianglelefteq G$ and $H$ is any subgroup of $G$ then $N \cap H \trianglelefteq H$.

**3.1.36)** Prove that if $G/Z(G)$ is cyclic then $G$ is abelian.[If $G/Z(G)$ is cyclic with generator $xZ(G)$, show that every element of $G$ can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.]

**3.2.1)** Which of the following are permissible orders for subgroups of a group of order 120: 1, 2, 5, 7, 9, 15, 60, 240? For each permissible order give the corresponding index.

**3.2.4)** Show that if $|G| = pq$ for some primes $p$ and $q$ (not necessarily distinct) then either $G$ is abelian or $Z(G) = 1$.

**3.2.8)** Prove that if $H$ and $K$ are finite subgroups of $G$ whose orders are relatively prime then $H \cap K = 1$.