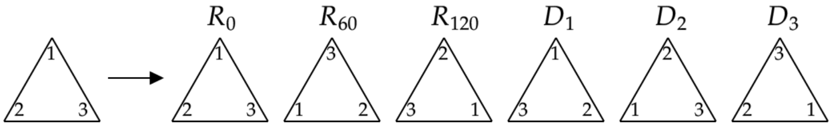
1) Compute the order of each element in the dihedral group D_6 and D_8.

→ Consider the symmetries of a triangle given in the figure below and let $D_6 = \{R_0, R_{60}, R_{120}, D_1, D_2, D_3\}$.

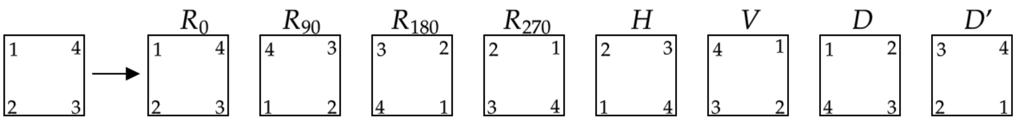


Then we have the following

element	R_0	R_{60}	R_{120}	D_1	D_2	D_3
order	1	3	3	2	2	2

For each symmetry operation, we consider the minimum number of applications that are needed to return to the original configuration. Obviously, R_0 is the identity, which always has order of 1 for any group, and the reflections across diagonals through the vertices and center must be performed twice.

Now consider the symmetries of a square shown below and let $D_8 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$.



Then it follows that

element	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D
order	1	4	2	4	2	2	2	2

2) Show that $\langle a, b | a^2 = b^2 = (ab)^2 = \mathbb{1}\rangle$ give a presentation for D_{2n} in terms of the two generators $a = s$ and $b = sr$ of order 2.

The dihedral group in terms of the rotation and reflection generators r, s is given as

$$D_{2n} = \langle r, s | r^n = s^2 = \mathbb{1},\ rs = sr^{-1}\rangle$$

We show that the representation above holds for $n = 2$.

<u>Proof</u>
First, we prove that the representation in terms of a, b implies the relations for the

representation in terms of r, s. First, notice that $a^2 = s^2 = \mathbb{1}$ and $(ab)^2 = (ssr)^2 = (s^2r)^2 = r^2 = \mathbb{1}$. Next, we see that $b^2 = (sr)^2 = \mathbb{1}$, $sr = r^{-1}s$, and $rs = sr^{-1}$.

Now, we prove that the representation in terms of r, s implies the relations for the representation in terms of a, b. Observe $s^2 = a^2 = \mathbb{1}$ and $r^2 = (ab)^2 = \mathbb{1}$. Finally, notice that $rs = sr^{-1}$ or $(sr)^2 = b^2 = \mathbb{1}$.

■

3[1.1.1]) Complete the multiplication table for D_8. Find at least one interesting pattern in the table.

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

One interesting thing to note is that there are distinct quadrants of product transformations. For products of rotations and reflections we get effectively rotations, which are shown in the upper left and lower right corners. However when the product contains one reflection and one rotation we get reflections effectively, shown in the lower left and upper right corners.

4[1.1.4])

(a) List the symmetries of a rectangle.

The symmetries are given in the second figure in problem 1 and can be put into the set D_8, which is also given in problem 1.

(b) Write the multiplication table for the symmetries of a rectangle.

The multiplication table for the symmetries of a rectangle are shown in the multiplication table in problem 3.

5[1.1.5])

(a) Find the center of D_8

The center of a group G is the set of all elements $a \in G$ such that $ab = ba$ for all $b \in G$ (i.e. a commutes with all elements of G). We can read from the table in problem 3 what elements are in the center of D_8:

$$\mathbf{Z}(D_8) = \{R_0, R_{180}\}$$

(b) Find $\mathbf{C}_{D_8}(R_{90})$ and $\mathbf{C}_{D_8}(H)$.

The centralizer of an element $a \in G$ is the set of all elements $b \in G$ such that a and b commute:

$$\mathbf{C}_{D_8}(R_{90}) = \{R_0, R_{90}, R_{180}, R_{270}\}$$

$$\mathbf{C}_{D_8}(H) = \{H, R_{180}, R_0, V\}$$

6[1.1.6]) Let D_6 denote the set of symmetries of an equilateral triangle. Find the multiplication table for D_6. What is the center of D_6?

	R_0	R_{60}	R_{120}	D_1	D_2	D_3
R_0	R_0	R_{60}	R_{120}	D_1	D_2	D_3
R_{60}	R_{60}	R_{120}	R_0	D_2	D_3	D_1
R_{120}	R_{120}	R_0	R_{60}	D_3	D_1	D_2
D_1	D_1	D_3	D_2	R_0	R_{120}	R_{60}
D_2	D_2	D_1	D_3	R_{60}	R_0	R_{120}
D_3	D_3	D_2	D_1	R_{120}	R_{60}	R_0

7[1.2.4]) Let $\sigma = (1\ 3\ 5)(2\ 4)$ and $\tau = (1\ 5)(2\ 3)$ be elements of S_5. Find σ^2, $\sigma\tau$, $\tau\sigma$, and $\tau\sigma^2$.

$$\begin{aligned}
\sigma^2 &= [(1\ 3\ 5)(2\ 4)][(1\ 3\ 5)(2\ 4)] = (1\ 5\ 3)\\
\sigma\tau &= [(1\ 3\ 5)(2\ 4)][(1\ 5)(2\ 3)] \quad = (5\ 3\ 4\ 2)\\
\tau\sigma &= [(1\ 5)(2\ 3)][(1\ 3\ 5)(2\ 4)] \quad = (1\ 2\ 4\ 3)\\
\tau\sigma^2 &= [(1\ 5)(2\ 3)](1\ 5\ 3) \qquad\quad = (5\ 2\ 3)
\end{aligned}$$

8[1.2.5]) Construct a complete multiplication table for S_3. What is the center of S_3? If $f = (1\ 2\ 3)$, what is $\mathbf{C}_{S_3}(f)$, the centralizer of f in S_3?

$\rightarrow$ We know that $S_3 = \text{Perm}(\{1,2,3\}) = \{\mathbb{1}, (2\ 3), (1\ 2), (1\ 2\ 3), (1\ 3\ 2), (1\ 3)\}$. Hence,

	$\mathbb{1}$	(2 3)	(1 2)	(1 2 3)	(1 3 2)	(1 3)
$\mathbb{1}$	$\mathbb{1}$	(2 3)	(1 2)	(1 2 3)	(1 3 2)	(1 3)
(2 3)	(2 3)	$\mathbb{1}$	(1 3 2)	(1 3)	(1 2)	(1 2 3)
(1 2)	(1 2)	(1 2 3)	$\mathbb{1}$	(2 3)	(1 3)	(1 3 2)
(1 2 3)	(1 2 3)	(1 2)	(1 3)	(1 3 2)	$\mathbb{1}$	(2 3)
(1 3 2)	(1 3 2)	(1 3)	(2 3)	$\mathbb{1}$	(1 2 3)	(1 2)
(1 3)	(1 3)	(1 3 2)	(1 2 3)	(1 2)	(2 3)	$\mathbb{1}$

and

$$\mathbf{C}_{S_3}(f) = \{\mathbb{1}, (1\ 2\ 3), (1\ 3\ 2)\}$$

9[1.2.6]) Let $f = (1\ 2\ 3) \in S_3$. Find the maps in the following sequence

$$\mathbb{1}_{[3]}, f, f^2, f^3, f^4, f^5, \ldots$$

Do you see a pattern?

Notice that from the table we see that

$$\begin{aligned}
f^0 &= \mathbb{1}_{[3]} \\
f^1 &= (1\ 2\ 3) \\
f^2 &= (1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2) \\
f^3 &= (1\ 2\ 3)(1\ 3\ 2) = \mathbb{1}_{[3]} \\
&\vdots
\end{aligned}$$

It is seen that this sequence essentially lists the elements of the cyclic group $\langle f \rangle = \{f, f^2, f^3 = \mathbb{1}_{[3]}\}$.