1) Let $f : A \to B$ be a map.

(a) f is injective iff f has a left inverse.

Proof
($\Leftarrow$) Suppose $g : B \to A$ is a left inversee of f and that $f(a) = f(b)$. Then, $g \circ f(a) = g \circ f(b)$ or $g(f(a)) = g(f(b))$ or $a = b$. Hence, f is injective.

($\Rightarrow$) Suppose f is injective. Consider a function $g : B \to A$ constructed as follows. Let c be an arbitrary point in A. If $y \in f(A)$, then let $g(y) = x$ such that $f(x) = y$. We know that x is unique since f is injective. If $y \notin f(A)$ but $y \in B$, let $g(y) = c$. Thus, we observe that g is a left inverse by construction.

$\blacksquare$

(b) f is surjective iff f has a right inverse.

Proof
($\Leftarrow$) Let $h : B \to A$ be a right inverse of f. Then for each $b \in B$, we have $f \circ h(b) = b$. hence f is surjective since each $b \in B$ has a pre-imgae $h(b) \in A$.

($\Rightarrow$) Let f be surjective. Consider a function $h : B \to A$ constructed as follows. Note that for each $b \in B$, there exists an element $a \in A$ such that $f(a) = b$. let $h(b) = a$. Then by construction $f \circ h(b) = f(a) = b$. Hence h is a right inverse of f.

$\blacksquare$

(c) f is bijective iff there exists $g : B \to A$ such that $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$.

Proof
($\Leftarrow$) Let $g : B \to A$ be defined as above. Then g is a left inverse of f, meaning f is injective, and similarly g is a right inverse of f, implying f is surjective. Thus, f is bijective.

($\Rightarrow$) Let f be bijective. Then $\exists g : B \to A, g \circ f = \mathbb{1}_A$ and $\exists h : B \to A, f \circ h = id_B$. Suppose $g \neq h$. Then $\exists b \in B$ such that $g(b) \neq h(b)$. This is impossible, however, since each pre-image is unique, given that f is injective. Hence it must be the case that $g : B \to A$ is a unique inverse such that $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$.

$\blacksquare$

2) Let $f : A \to B$, $g : B \to C$ be maps.

(a) If f, g are injective, then so is $g \circ f$.

Proof
Suppose $g \circ f(a) = g \circ f(b)$. Then $g(f(a)) = g(f(b))$ or $f(a) = f(b)$ since g is injective and $a = b$ since f is injective, implying that $g \circ f$ is injective.

■

(b) If f, g are surjective, the so is $g \circ f$.

Proof
Notice that $f(A) = B$ and $g(B) = C$. Then $g \circ f(A) = g(f(A)) = g(B) = C$. Alternatively, let $z \in C$. Then $\exists y \in B, g(y) = z$. Similarly, $\exists x \in A, f(x) = y$. Hence, $g \circ f$ is surjective.

■

(c) If $f, g \in \text{Perm}(A)$, then $g \circ f \in \text{Perm}(A)$.

Proof
Observe that $f, g : A \to A$ are bijective. So $g \circ f : A \to A$ is also bijective, meaning that $g \circ f \in \text{Perm}(A)$.

■

3) Let $f : A \to B$ be a surjective map. Prove that the relation $a \sim b$ iff $f(a) = f(b)$ is an equivalence relation whose equivalence classes are the fibers of f.

Proof
Obviously $f(a) = f(a)$, so $a \sim a$, and similarly if $a \sim b$, then $b \sim a$ since $f(a) = f(b)$ is equivalent to $f(b) = f(a)$. Lastly if $a \sim b$ and $b \sim c$, then $a \sim c$ since $f(a) = f(b)$ and $f(b) = f(C)$ implies $f(a) = f(c)$.

Choose any $b \in B$. Then $f^{-1}(\{b\}) = \{a \in A | f(a) = b\}$, which is the fiber of f over b, is an equivalence class of $\sim$.

■

4[1.2.10]) Prove: Let X, Y be sets, and let $f : X \to Y$ be a 1-to-1 and onto map. Then $f^{-1} : Y \to X$ is 1-to-1 and onto also. In particular, for a set Ω, every $f \in \text{Perm}(\Omega)$ has an inverse in $\text{Perm}(\Omega)$. *

Proof
Since f is bijective, we know f^{-1} exists and that $f \circ f^{-1} = \mathbb{1}_Y$ and $f^{-1} \circ f = \mathbb{1}_X$. Suppose that $f^{-1}(x_1) = f^{-1}(x_2)$, then $f \circ f^{-1}(x_1) = f \circ f^{-1}(x_2)$ or $x_1 = x_2$, implying f^{-1} is 1-to-1. Now, let $x \in X$. Since f is surjective, then there exists $y \in Y$ such that $f(x) = y$. Hence, $f \circ f^{-1}(x) = x = f^{-1}(y)$. That is, f^{-1} is surjective.

$\blacksquare$

$\rightarrow$ If $f \in \text{Perm}(\Omega)$, then f is bijective so f^{-1} exists and is bijective. Hence $f^{-1} \in \text{Perm}(\Omega)$.

5[1.2.12]) Give an example of a map f that has a left inverse, but not an inverse.

$\rightarrow$ Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. let $f : A \rightarrow B$ such that $f(1) = 2,\ f(2) = 3,\ f(3) = 4$. Notice that $g : B \rightarrow A$ such that $g(1) = 1,\ g(2) = 1,\ g(3) = 2,\ g(4) = 3$ is a left inverse of f since $g \circ f = \mathbb{1}_A$. Hoever, since $1 \in B$ has no pre-image, there is no way to define $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1}(1) = 1$.

6[1.2.20]) Let S be a set with a finite number of elements and let $f : S \rightarrow S$ be a map.

(a) If f is onto, can f not be 1-to-1?

$\rightarrow$ Since S is finite we can write $S = \{s_1, s_2, \dots, s_n\}$. Suppose f is not 1-to-1, then $\exists s_i, s_j \in S, i \neq j$ and $f(s_i) = f(s_j)$. Then, there can be at most $n - 1$ images under f, meaning f is not onto. Thus, if f is onto, f must be 1-to-1.

(b) If f is 1-to-1, can f not be onto?

$\rightarrow$ By similar reasoning as above f must be onto. Suppose f is not onto. Then n elements in S are mapped to $n - 1$ elements in S, meaning $\exists s_i, s_j\ inS, i \neq j$ and $f(s_j) = f(s_j)$. Thus, f is not 1-to-1. This implies that if f is 1-to-1, it must be onto.

(c) Do your conclusions remain valid even if S has an infinite number of elements?

$\rightarrow$ No, they do not. Consider $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n) = n + 1$. Obviously f is 1-to-1 but not onto. Consider also $f(n) = n/2$ if n is even or $f(n) = (n + 1)/2$ if n is odd. Then, f is onto but not 1-to-1.

7[1.2.21]) As usual let $(0, 1) = \{x \in \mathbb{R} | 0 < x < 1\}$. Can you find a 1-to-1 and onto map $f : (0, 1) \rightarrow \mathbb{R}$?

$\rightarrow$ Consider $f(x) = \frac{x-1/2}{x(x-1)}$.

8[1.2.22]) As usual let $[0, 1) = (0, 1) \cup \{0\}$ and $[0, 1] = (0, 1) \cup \{0, 1\}$. Can you find a 1-to-1, onto function $f : [0, 1) \rightarrow [0, 1]$?

$\rightarrow$ Consider $f(x) = \begin{cases} 1/2^{n-1} & \text{if } x = 1/2^n \text{ for some} n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$.