1.7.2) Show that the additive group $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$ for all $z, a \in \mathbb{Z}$.

$\rightarrow$ Define the map $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ such that $z \cdot a \mapsto z + a$. We prove that this map is a group action on $\mathbb{Z}$.

Proof
(1) Let $x, y, a \in \mathbb{Z}$. Then $x \cdot (y \cdot a) = x \cdot (y + a) = x + (y + a)$ and $(x + y) \cdot a = (x + y) + a$. Obviously these are equal since the integers are associative under addition.

(2) Recall that $0 = \mathbb{1}_{\mathbb{Z}}$. Thus, $0 \cdot a = 0 + a = a$ for any integer a.

$\blacksquare$

1.7.6) Prove that a group G acts faithfully on a set A iff the kernel of the action is the set consisting only of the identity.

Proof
($\Rightarrow$) Suppose that G acts faithfully on A. That is, if g_1, g_2 are distinct elements of G, then $\sigma_{g_1} \neq \sigma_{g_2}$. By the definition of a group action, $\mathbb{1}$ is in the kernel of the group action. However, it is the only element in the kernel since if $g \neq \mathbb{1}$, then $\sigma_g \neq \sigma_{\mathbb{1}}$. Hence, there exists at least one element $b \in A$ such that $\sigma_g(b) \neq b$.

($\Leftarrow$) Suppose that the kernel of a group action $\{g \in G | \forall a \in A, ga = a\} = \{\mathbb{1}\}$ but that the group action is not faithful. Then there exist two distinct group elements g_1, g_2 such that $\sigma_{g_1} = \sigma_{g_2}$. Then $\sigma_{g_2^{-1}} \circ \sigma_{g_1} = \mathbb{1}$, which would imply that $g_2^{-1} g_1$ is in the kernel of the group action, but this is a contradiction since $g_2^{-1} g_1 \neq \mathbb{1}$. Hence, the group action must be faithful.

$\blacksquare$

1.7.14) Let G be a group and let $A = G$. Show that if G is non-abelian then the maps defined by $g \cdot a = ag$ for all $g, a \in G$ do not satisfy the axioms of a (left) group action of G on itself.

Proof
It is clear that the second requirement of the map is satisfied since $\mathbb{1} \cdot a = a\mathbb{1} = a$. However, the first is not since if $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ then $a g_2 g_1 = a(g_1 g_2)$ or $g_1 g_2 = g_2 g_1$ would contradict our assumption that the group G is non-abelian. Hence, this cannot be a (left) group action.

$\blacksquare$

1.7.15) Let G be any group and let $A = G$. Show that the maps defined by $g \cdot a = ag^{-1}$ for all $g, a \in G$ do satisfy the axioms of a (left) group action of G on itself.

Proof
(1) Notice that $g_1 \cdot (g_2 \cdot a) = g_1 \cdot (ag_2^{-1}) = ag_2^{-1}g_1^{-1} = a(g_1g_2)^{-1} = (g_1g_2) \cdot a$.

(2) Also observe that $\mathbb{1} \cdot a = a\mathbb{1}^{-1} = a\mathbb{1} = a$.

$\blacksquare$

1.7.16) Let G be any group and let $A = G$. Show that the maps defined by $g \cdot a = gag^{-1}$ for all $g, a \in G$ do satisfy the axioms of a (left) group action (this action of G on itself is called *conjugation*)

Proof
(1) We see that $g_1 \cdot (g_2 \cdot a) = g_1 \cdot g_2ag_2^{-1} = g_1g_2ag_2^{-1}g_2^{-1}$ and $(g_1g_2) \cdot a = g_1g_2a(g_1g_2)^1$, implying that this map satisfies the first property of a group action.

(2) It is trivial to see that $\mathbb{1} \cdot a = a$.

$\blacksquare$

2.2.4) For each of S_3, D_8, and Q_8 compute the centralizers of each element and find the center of each group.

$\rightarrow$ Recall $S_3 = \{\mathbb{1}_{[3]}, (2\ 3), (1\ 2), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. Thus, from previous homeworks where the multiplication tables were computed, we have the centralizers of the following elements:

$$\begin{aligned}
\mathbb{1}_{[3]} &: S_3 \\
(2\ 3) &: \{\mathbb{1}_{[3]}, (2\ 3)\} \\
(1\ 2) &: \{\mathbb{1}_{[3]}, (1\ 2)\} \\
(1\ 3) &: \{\mathbb{1}_{[3]}, (1\ 3)\} \\
(1\ 2\ 3) &: \{\mathbb{1}_{[3]}, (1\ 2\ 3), (1\ 3\ 2)\} \\
(1\ 3\ 2) &: \{\mathbb{1}_{[3]}, (1\ 2\ 3, (1\ 3\ 2))\}
\end{aligned}$$

and the center of S_3

$$Z(S_3) = \{\mathbb{1}_{[3]}\}$$

Next we have the dihedral group of order 8: $D_8 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$. Again, we have computed the multiplication tables for D_8 previously, giving the centralizers for the following elements as:

$$\begin{aligned}
R_0 &: D_8 \\
R_{90} &: \{R_0, R_{90}, R_{180}, R_{270}\} \\
R_{180} &: D_8 \\
R_{270} &: \{R_0, R_{90}, R_{180}, R_{270}\} \\
H &: \{R_0, H, R_{180}\} \\
V &: \{R_0, V, R_{180}\} \\
D &: \{R_0, D, R_{180}\} \\
D' &: \{R_0, D', R_{180}\}
\end{aligned}$$

and the center

$$Z(D_8) = \{\mathbb{1}, R_{180}\}$$

which is proven below.

Finally, we have the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The centralizers are straightforward to glean from the definition of the group:

$$\begin{aligned}
\pm 1 &: Q_8 \\
\pm i &: \{\pm 1, \pm i\} \\
\pm j &: \{\pm 1, \pm j\} \\
\pm k &: \{\pm 1, \pm k\}
\end{aligned}$$

and the center of the group is

$$Z(Q_8) = \{\pm 1\}$$

2.2.5) In each of parts (a) to (c) show that for the specified group G and subgroup A of G, $C_G(A) = A$ and $N_G(A) = G$.

(a) $G = S_3$ and $A = \{1, (123), (132)\}$.

Recall $S_3 = \{\mathbb{1}_{[3]}, (2\ 3), (1\ 2), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. Then,

$$\begin{aligned}
C_G(A) &= \{\sigma \in S_3 | \forall \tau \in A, \sigma\tau = \tau\sigma\} = C_G(\mathbb{1}_{[3]}) \cap C_G((123)) \cap C_G((132)) = \{\mathbb{1}_{[3]}, (123), (132)\} = A \\
N_G(A) &= \{\sigma \in S_3 | \sigma A \sigma^{-1} = A\} = \{\mathbb{1}_{[3]}, (12), (13)\} \cup A = G
\end{aligned}$$

Note that $C_G(A) \leq N_G(A)$, so by definition $C_G(A) \subset N_G(A)$.

(c) $G = D_{10}$ and $A = \{1, r, r^2, r^3, r^4\}$.

Using the generators and relations description of the dihedral group we have $D_{10} = \langle r, s | r^5 = s^2 = \mathbb{1}, rs = sr^4 \rangle$, meaning

$$\begin{aligned}
C_G(A) &= \{x \in D_{10} | \forall a \in A ax = xa\} = \{1, r^2, r^3, r^4, r^5\} = A \\
N_G(A) &= \{x \in D_{10} | xAx^{-1} = A\} = \{s, sr, sr^2, sr^3, sr^4\} \cup A = G
\end{aligned}$$

Note that if $a \in A$ then $sr^k a(sr^k)^{-1} = sr^k a r^{-k} s = sr^k r^{-k} as = sas = a^{-1}$. Observe that $A = \langle r \rangle$, so if $a \in A$, then $a^{-1} \in A$, which gives us the first part of the union for $N_G(A)$.

2.2.6) Let H be a subgroup of the group G.

(a) Show that $H \leq N_G(H)$. Give an example to show that this is not necessarily true if H is not a subgroup.

Proof
Recall that $N_G(H) = \{g \in G | gHg^{-1} = H\}$ and $gHg^{-1} = \{ghg^{-1} | h \in H\}$. We know that $H, N_G(H)$ are both subgroups of G so we must simply show that $H \subset N_G(H)$. That is, we must prove that $hHh^{-1} = H$. This is simple to observe since H is closed under products and if $a \in H$, then $a \in hHh^{-1}$ since $h(h^{-1}ah)h^{-1} = a$.

$\blacksquare$

$\rightarrow$ The above result hinges on the assumption that H is a subgroup of G. Consider the following example: $G = \mathbb{Z}/3\mathbb{Z}$ and $H = \{\bar{1}\}$. Obviously, H is not a subgroup of G since it is not closed under addition and $N_G(H) = \mathbb{Z}/3\mathbb{Z}$.

(b) Show that $H \leq C_G(H)$ iff H is abelian.

Proof
($\Rightarrow$) Suppose that $H \leq C_G(H)$. Then it is clear that for any two elements $h, a \in H$ that $ha = ah$, implying that H is abelian.

($\Leftarrow$) Now, suppose that H is abelian. Then, for any two $h, a \in H$ we have $ha = ah$. Thus, fixing h, it is clear that $h \in C_G(H)$, and since we have shown previously that $C_G(H) \leq G$, it immediately follows that $H \leq C_G(H)$.

$\blacksquare$

2.2.7) Let $n \in \mathbb{Z}$ with $n \geq 3$. Prove the following:

(a) $Z(D_{2n}) = \{1\}$ if n is odd.

$\rightarrow$ Recall the generators and relations representation of the dihedral group: $D_{2n} = \langle r, s | r^n = s^2 = \mathbb{1}, rs = sr^{-1} \rangle$. Obviously, $\mathbb{1}$ commutes with any element, implying that $\mathbb{1} \in Z(D_{2n})$. Next, we see s does not commute with any element of the dihedral group since $rs = sr^{-1}$. Finally, we see that $r^k s r^m = sr^{-k} r^m = sr^m r^{n-k}$. We must have $k = n - k$ for these elements to commute. That is, $n = 2k$. However, if n is odd, then n cannot be written as twice an integer. Thus, we arrive at the conclusion that if n is odd, then $Z(D_{2n}) = \{\mathbb{1}\}$.

(b) $Z(D_{2n}) = \{1, r^k\}$ if $n = 2k$.

From the reasoning above, we see that if n is even, then $n = 2k$ for some integer k, implying that $r^k \in Z(D_{2n})$ and $Z(D_{2n}) = \{\mathbb{1}, r^k\}$