Problem 1) Calculate the series

$$I_1 = \sum_{n=1}^{\infty} \frac{1}{4^n}, \quad I_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}. \tag{1}$$

Recall that for a geometric series $\sum_{n=0}^{\infty} ar^n = a/(1-r)$.

For I_1, we can write

$$\boxed{I_1 = \sum_{n=0}^{\infty} \frac{1}{4^n} - 1 = \frac{1}{1-1/4} - 1 = \frac{1}{3}}. \tag{2}$$

For I_2, $a = 1$ and $r = -1/3$, meaning I_2

$$\boxed{I_2 = \frac{1}{1+1/3} = \frac{3}{4}}. \tag{3}$$

Problem 2) Calculate the sum

$$I_3 = \sum_{n=1}^{N} \left[\frac{1}{n} - \frac{1}{1+n}\right]. \tag{4}$$

What is the limit of I_3 at $N \to \infty$.

We can split I_3 into two series as

$$I_3 = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1}, \tag{5}$$

and we can shift indices on the second series with the substitution $m = n+1$ such that

$$\boxed{I_3 = \sum_{n=1}^{N} \frac{1}{n} - \sum_{m=2}^{N+1} \frac{1}{m} = 1 - \frac{1}{N+1} = \frac{N}{N+1}}. \tag{6}$$

The infinite series

$$\boxed{\lim_{N\to\infty} I_3 = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = 1} \tag{7}$$

since $1/(N+1) \to 0$ as $N \to \infty$

Problem 3) Does the alternating series

$$I_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{7/6}} \tag{8}$$

converge absolutely?

The series I_1 converges absolutely:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{7/6}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{7/6}} < \int_1^{\infty} \frac{\mathrm{d}x}{x^{7/6}} = -6x^{-1/6}\Big|_1^{\infty} = 6. \tag{9}$$

Since the series is bounded above and below I_1 is absolutely convergent.

Problem 4) Does the series

$$I_2 = \sum_{n=1}^{\infty} \frac{x}{(nx+1)[(n+1)x+1]} \tag{10}$$

with $x > 0$ converge uniformly?

Notice that we can write the series as

$$I_2 = \sum_{n=1}^{\infty} \left[\frac{1}{nx+1} - \frac{1}{(n+1)x+1} \right] = \sum_{n=1}^{\infty} \frac{1}{nx+1} - \sum_{n=1}^{\infty} \frac{1}{(n+1)x+1}, \tag{11}$$

The N^{th} partial sum is given as

$$I_{2,N} = \sum_{n=1}^{N} \frac{1}{nx+1} - \sum_{n=1}^{N} \frac{1}{(n+1)x+1}, \tag{12}$$

and the second sum can be rewritten by shifting indices to $m = n+1$ such that it is clear that the partial sum is simply

$$I_{2,N} = \frac{1}{x+1} - \frac{1}{(N+1)x+1}. \tag{13}$$

Taking the limit $N \to \infty$, we find

$$\boxed{I_2 = \frac{1}{x+1}}. \tag{14}$$

which is continuous on $x > 0$, implying that I_2 is uniformly convergent on $x > 0$.

Problem 5) Calculate the infinite product

$$I_3 = \prod_{n=1}^{\infty} \left[1 + \frac{(-1)^n}{n+1}\right]. \tag{15}$$

We can heuristically solve this problem by writing the first several terms of the product:

$$\begin{aligned} I_3 &= \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{5}\right)\left(1 - \frac{1}{6}\right)\cdots \\ &= \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{5}{6} \cdots = \boxed{\frac{1}{2}}. \end{aligned} \tag{16}$$

More formally, we can write the product in terms of its even and odd terms as follows

$$\begin{aligned} I_3 &= \frac{1}{2} \prod_{k=1}^{\infty} \left[1 + \frac{(-1)^{2k}}{2k+1}\right]\left[1 + \frac{(-1)^{2k+1}}{(2k+1)+1}\right] \\ &= \frac{1}{2} \prod_{k=1}^{\infty} \frac{2k+2}{2k+1} \cdot \frac{2k+1}{2k+2} = \frac{1}{2}. \end{aligned} \tag{17}$$