Problem 1)
For a region of volume V, calculate the following integrals taken over the surface of this region:

$$\vec{\boldsymbol{I}}_1 = \oint_S \vec{\boldsymbol{r}}(\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{n}})\,\mathrm{d}S\,, \quad \vec{\boldsymbol{I}}_2 = \oint_S (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\hat{\boldsymbol{n}}\,\mathrm{d}S\,, \tag{1}$$

where $\hat{\boldsymbol{n}}(\vec{\boldsymbol{r}})$ is a local unit vector normal to the surface and $\vec{\boldsymbol{a}}$ is a constnat vector. HINT: Try to multiply I_1 and I_2 by an auxiliary constant vector $\vec{\boldsymbol{b}}$.

For the first integral notice that

$$\begin{aligned}\vec{\boldsymbol{I}}_1 \cdot \vec{\boldsymbol{b}} &= \oint_S (\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{n}}\,\mathrm{d}S \\ &= \int_V \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{a}}\ \mathrm{d}V = \int_V \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{r}})\ \mathrm{d}V \\ &= \int_V \vec{\boldsymbol{a}} \cdot [(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{r}} + \vec{\boldsymbol{b}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{r}})]\,\mathrm{d}V = (\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{a}})V.\end{aligned} \tag{2}$$

Thus, we have

$$\vec{\boldsymbol{I}}_1 = V\vec{\boldsymbol{a}} \tag{3}$$

for any generic volume V with bounding surface S.

Similarly, for the second integration,

$$\vec{\boldsymbol{I}}_2 \quad = \oint_S (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}\,\mathrm{d}S = \int_V \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{b}} = V\vec{\boldsymbol{a}}. \tag{4}$$

Problem 2)
Calculate the following determinant:

$$\begin{vmatrix} E & V & V \\ V & E & V \\ V & V & E \end{vmatrix}. \tag{5}$$

Find E at which the determinant vanishes. Solution of this problem determines the energy levels of an electron in a triangular molecule.

The determinant of the matrix above is just

$$E(E^2 - V^2) + V(V^2 - EV) + V(EV - V^2) = E(E^2 - V^2). \tag{6}$$

If we require this be zero, then it is clear that the solutions are $E = -V, 0, V$.

Problem 3)
Calculate the inverese matrix A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}. \tag{7}$$

We will solve this by using the cofactor method, which is somewhat tedious already at 3×3 but also the most direct. First, we have $\det(A) = -18$. Next, the cofactor matrix is just

$$C = \begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix}. \tag{8}$$

The inverse matrix is then just

$$A^{-1} = \frac{1}{18} \begin{pmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{pmatrix}. \tag{9}$$

Problem 4)
Consider two Hermitian matrices:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & iv & 0 \\ -iv & 0 & 0 \\ 0 & 0 & u \end{pmatrix}, \tag{10}$$

where $a,\ b,\ c,\ u$, and v are real constants.
(a) Is the product AB a Hermitian matrix?
(b) Do A and B commute?
(c) What are the relations between a, b, c, u, v for which the answer to the questions 1 and 2 is yes?

(a) It is a general fact that if A and B are hermitian, then their product AB is also Hermitian: $(AB)^\dagger(AB) = B^\dagger A^\dagger AB = B^\dagger B = I$.

(b) It is easy to see that A and B do not commute in general:

$$AB = \begin{pmatrix} 0 & iav & 0 \\ -ibv & 0 & 0 \\ 0 & 0 & cu \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & ibv & 0 \\ -iav & 0 & 0 \\ 0 & 0 & cu \end{pmatrix}. \tag{11}$$

(c) As stated in part (a), these product of these Hermitian matrices is Hermitian for any

a, b, c, u, v. In part (b), though, we saw that the products AB and BA do not generally commute with each other. However, if $a = -b$, then $[A, B] = 0$.

Problem 5)
Find the eigenvalues and eigenvectors of this matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 1 \end{pmatrix}. \tag{12}$$

By inspection, we can see that 1 is an eigenvalue corresponding to the eigenvector $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. We therefore only have to solve for the eigenvalues and eigenvectors of the square matrix in the lower right block of A:

$$\begin{vmatrix} -\lambda & i \\ -i & 1-\lambda \end{vmatrix} = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1 = 0. \tag{13}$$

This gives the other two eigenvalues as

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}. \tag{14}$$

We then can solve for the other two eigenvectors (up to a constant) by imposing

$$-\lambda_\pm x_1 + i x_2 = 0 \Rightarrow x_2 = -i\lambda_\pm x_1. \tag{15}$$

Thus, the eigenvector

$$x_\pm = \begin{pmatrix} 1 \\ -i\lambda_\pm \end{pmatrix}. \tag{16}$$