

Problem 1)

Calculate the following integral:

$$I_1 = \int_0^\infty \frac{x^{1/4} dx}{a^2 + x^2}. \quad (1)$$

Let us make use of the substitution $x = ay$, which allows us to write

$$I_1 = \frac{a^{1/4}a}{a^2} \int_0^\infty \frac{y^{1/4}}{1 + y^2} dy = a^{-3/4} \int_0^\infty \frac{y^{1/4}}{1 + y^2} dy. \quad (2)$$

Next, let us write $y = e^{t/2}$, which gives

$$I_1 = \frac{1}{2}a^{-3/4} \int_{-\infty}^\infty \frac{e^{t/8}e^{t/2}}{1 + e^t} dt = \frac{1}{2}a^{-3/4} \int_{-\infty}^\infty \frac{e^{5t/8}}{1 + e^t} dt = \frac{\pi}{2a^{3/4} \sin(5\pi/8)}. \quad (3)$$

Problem 2)

Calculate the following integral at $s \gg 1$:

$$I_2 = \int_1^\infty e^{s(x-x^2)} dx. \quad (4)$$

Notice that we can write $x^2 - x = (x - 1) + (x - 1)^2$. If s is sufficiently large, then $\exp[s(x - x^2)] \approx \exp[-s(x - 1)]$, meaning

$$I_2 \approx e^s \int_1^\infty e^{-sx} dx = \frac{1}{s}. \quad (5)$$

Problem 3)

Calculate the following integral at $s \gg 1$:

$$I_3 = \int_0^\infty \exp\left(-\frac{s^2}{x} - x\right) dx. \quad (6)$$

Let $f(x; s) = s/x + x/s$. Notice then that $f'(x; s) = -s/x^2 + 1/s$, implying that f has a minimum at $x = s$ in the integration region, allowing us to write

$$f(x; s) = f(s) + \frac{f''(s)}{2}(x - s)^2 + \dots = 2 + \frac{1}{s^2}(x - s)^2 + \dots \quad (7)$$

which makes

$$I_3 \approx e^{-2s} \int_0^\infty e^{-(x-s)^2/s} dx. \quad (8)$$

This is wrong – the expansion for $f(x; s)$ actually only converges for $|x - s| < s$ – need to think a little bit more about this

Problem 4)

Calculate the following integral at $s \gg 1$:

$$I_4 = \int_0^\infty x^\alpha e^{-sx^2} dx, \quad (9)$$

where $\alpha > 0$ is an arbitrary (not necessarily integer) number.

Notice that we can write $x^\alpha = e^{\alpha \ln x}$ such that our integral becomes

$$I_4 = \int_0^\infty e^{-s(x^2 - \alpha \ln x)} dx, \quad (10)$$

and if we denote $f(x) = x^2 - \alpha \ln x$, observe that f attains a minimum at $x = \sqrt{\alpha/2}$, which is positive since $\alpha > 0$. Hence, we can write

$$\begin{aligned} I_4 &\approx e^{-\frac{s\alpha}{2}[1-\ln(\alpha/2)]} \int_{-\infty}^\infty e^{-2s[x-\sqrt{\alpha/2}]^2} dx \\ &= e^{-\frac{s\alpha}{2}[1-\ln(\alpha/2)]} \sqrt{\frac{\pi}{2s}} = \left(\frac{\alpha}{2e}\right)^{s\alpha/2} \sqrt{\frac{\pi}{2s}}. \end{aligned} \quad (11)$$