### Problem 1)

Express the following delta function in terms of delta functions of the variable x:

$$\delta\left(\frac{\sin x}{x}\right). \tag{1}$$

Recall that we can write

$$\delta(f(x)) = \sum_{i} \frac{\delta(x - x_i)}{|f'(x_i)|},\tag{2}$$

where f(x) has simple roots  $x_i^1$ . Note that  $f(x) = \sin x/x$  indeed has simple roots since we can write

$$\sin x = x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n\pi^2} \right). \tag{3}$$

From this formala, it is also trivial to glean that  $\sin x/x$  has zeros when  $\sin x = 0$  or  $x = n\pi$  for  $n = \pm 1, \pm 2, \dots^2$ . Additionally,  $\frac{\mathrm{d}}{\mathrm{d}x} \sin x/x|_{x=n\pi} = [\cos x/x - \sin x/x^2]_{x=n\pi} = (-1)^n/n\pi$ .

Thus,

$$\delta\left(\frac{\sin x}{x}\right) = \sum_{n=1}^{\infty} (-1)^n n\pi \left[\delta(x - n\pi) - \delta(x + n\pi)\right]$$
 (4)

#### Problem 2)

Calculate

$$I(z) = \Gamma(1+z)\Gamma(1-z) \tag{5}$$

at z = 1/4

We can write

$$\Gamma(1+z)\Gamma(1-z) = z\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}$$

$$\Rightarrow \Gamma(1+z)\Gamma(1-z)|_{z=1/4} = \frac{\pi/4}{\sin \pi/4} = \frac{\pi}{2\sqrt{2}}$$
(6)

<sup>&</sup>lt;sup>1</sup>Otherwise the  $f'(x_i)$  term vanishes and we have to be a bit more careful with the expansion of f(x) inside the delta function.

<sup>&</sup>lt;sup>2</sup>Note that there is not a zero when n=0 (i.e. x=0) since the 1/x factor makes the function indeterminate there and  $\sin x/x \to 1$  as  $x \to 0$ . Alternatively, from the product expansion, the factor x out front cancels with 1/x, and we take the limit of this as  $x \to 0$ , which is just an infinite product of ones.

# Problem 3)

Using the definition of the complete elliptical integrals E(m) and K(m), express the derivative  $\partial E(m)/\partial m$  in terms of K(m) and E(m).

Recall that the complete elliptic integrals of the first and second kind are defined as

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} \tag{7}$$

$$E(m) = \int_0^{\pi/2} d\theta \sqrt{1 - m\sin^2\theta}, \tag{8}$$

respectively. Differentiating E(m) with respect to m we have

$$\frac{\partial E}{\partial m} = \frac{\partial}{\partial m} \int_0^{\pi/2} d\theta \sqrt{1 - m \sin^2 \theta} = \int_0^{\pi/2} d\theta \frac{\partial}{\partial m} \sqrt{1 - m \sin^2 \theta}$$

$$= -\frac{1}{2} \int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{\sqrt{1 - m \sin^2 \theta}}.$$
(9)

Observe the following:

$$K(m) - E(m) = \int_0^{\pi/2} d\theta \left[ \frac{1}{\sqrt{1 - m \sin^2 \theta}} - \sqrt{1 - m \sin^2 \theta} \right]$$

$$= \int_0^{\pi/2} d\theta \frac{1 - (1 - m \sin^2 \theta)}{\sqrt{1 - m \sin^2 \theta}} = m \int_0^{\pi/2} d\theta \frac{\sin^2 \theta}{\sqrt{1 - m \sin^2 \theta}}.$$
(10)

Thus,

$$\frac{\partial E}{\partial m} = \frac{K(m) - E(m)}{m} \quad . \tag{11}$$

### Problem 4)

Find the values of  $e^{\pm i\pi/2}$ ,  $e^{i\pi n}$ ,  $\ln(-1)$  where  $n=0,\pm 1,\pm 2,\ldots$ 

We find that

$$e^{\pm i\pi/2} = \cos(\pm \pi/2) + i\sin(\pm \pi/2) = \pm i$$
 (12)

$$e^{i\pi n} = \cos(\pi n) + i\sin(\pi n) = (-1)^n$$
 (13)

$$\ln(-1) = \ln(e^{i\pi}) = i\pi. \tag{14}$$

## Problem 5)

Calculate the following series:

$$I_1 = \sum_{n=0}^{\infty} p^n \sin(qn)$$
 and  $I_2 = \sum_{n=0}^{\infty} p^n \cos(qn)$ , (15)

where p and q are real parameters.

HINT: Use the sum of geometric series with complex r.

Observe the following:

$$I = \sum_{n=0}^{\infty} p^n e^{iqn} = I_2 + iI_1.$$
 (16)

That is  $I_2 = \text{Re}(I)$  and  $I_1 = \text{Im}(I)$ . We can use the geometric series formula with complex  $r = pe^{iq}$ , giving

$$I = \frac{1}{1 - pe^{iq}} = \frac{1}{(1 - p\cos q) - ip\sin q} = \frac{(1 - p\cos q) + ip\sin q}{(1 - p\cos q)^2 + p^2\sin^2 q}.$$
 (17)

Taking real and imaginary parts of I, we have

$$I_{1} = \frac{p \sin q}{1 + p^{2} - 2p \cos q}$$

$$I_{2} = \frac{1 - p \cos q}{1 + p^{2} - 2p \cos q}$$

$$(18)$$

$$I_2 = \frac{1 - p\cos q}{1 + p^2 - 2p\cos q} \quad . \tag{19}$$