Problem 1)
Find the Laplace transform of the following function:

$$I(t) = t^n e^{-at}, \quad a > 0 \text{ and even } n. \tag{1}$$

The Laplace transform of a function $f(t)$ is generally given as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t\,, \tag{2}$$

where the values of s are such that the integral is convergent. Thus,

$$\begin{aligned}\mathcal{L}\{I(t)\} &= \int_0^\infty t^n e^{-(a+s)t}\,\mathrm{d}t = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}(a+s)^n}\int_0^\infty e^{-(a+s)t}\,\mathrm{d}t \\ &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}(a+s)^n}\frac{1}{a+s} = \frac{n!}{(a+s)^{n+1}}.\end{aligned} \tag{3}$$

This matches the result we have in our library. That is, $\mathcal{L}\{t^n\} = n!/s^{n+1}$ and for a general function $f(t)$, $\mathcal{L}\{f(t)e^{-at}\} = F(s+a)$. Collecting these results, we end up with the same solution, which is that

$$\boxed{\mathcal{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}}. \tag{4}$$

Problem 2)
Solve the following equation by the Laplace transform

$$\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y = 0, \tag{5}$$

where $y(0) = 0$ and $\dot{y}(0) = v$.

If we take the Laplace transform of the equation[1], we find

$$\mathcal{L}\{\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y\} = [s^2Y(s) - v] + 2\lambda[sY(s)] + \omega_0^2 Y(s) = 0. \tag{6}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{v}{s^2 + 2\lambda s + \omega_0^2} = \frac{v}{(s+\lambda)^2 + (\omega_0^2 - \lambda^2)}. \tag{7}$$

[1] Strictly speaking, this means that if we have a differential equation $Df = g$, where D is a linear differential operator, then its “Laplace transform” is $\mathcal{L}\{Df\} = G(s)$.

From our "library" we have

$$\mathcal{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2} \tag{8}$$

$$\mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}. \tag{9}$$

Thus, if we let $a = \lambda$ and $b = \sqrt{\omega_0^2 - \lambda^2}$, then

$$\boxed{y(t) = \frac{v}{\sqrt{\omega_0^2-\lambda^2}} e^{-\lambda t} \sin\left(\sqrt{\omega_0^2-\lambda^2}\, t\right)}. \tag{10}$$

This is the solution in general for any ω_0 and λ (subject to the initial conditions above), but we can specify the relative values of ω_0 and λ in order to rewrite the solution in terms of explicitly real factors . First, if $\omega_0^2 > \lambda^2$. In this case, the solution does not change since all the constants are already real.

Second, if $\omega_0^2 = \lambda^2$, we have an indeterminate form, which is defined as the limit $\sqrt{\omega_0^2-\lambda^2} \to 0$. Clearly, this is of the form $\lim_{x\to 0} \sin ax/x = a$, leaving us with

$$\boxed{y(t) = vte^{-\lambda t}}. \tag{11}$$

Finally, if $\omega_0^2 < \lambda^2$, we have $\sqrt{\omega_0^2-\lambda^2} = i\sqrt{\lambda^2-\omega_0^2}$, and using $\sin ix = i\sinh x$, we have

$$\boxed{y(t) = \frac{v}{\sqrt{\lambda^2-\omega_0^2}} e^{-\lambda t} \sinh\left(\sqrt{\lambda^2-\omega_0^2}\, t\right)}. \tag{12}$$

Now that we have enumerated all the possible combinations, we are done, and we see that we have recovered all the forms of the solution to a generic homogeneous linear, second-order ordinary differential equation with constant coefficients with initial conditions $y(0) = 0$ and $\dot{y}(0) = v$.

Problem 3)

A unit vector $\hat{\boldsymbol{n}}$ makes angles θ and α with the Cartesian axes z and x, respectively, and a unit vector $\hat{\boldsymbol{n}}'$ makes angles θ' and α' with z and x, respectively. Find $\cos\varphi$, where φ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

Note that if a vector $\vec{\boldsymbol{A}}$ "makes an angle" ϕ with some axis defined by a unit vector $\hat{\boldsymbol{e}}$, this means that $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{e}} = |\vec{\boldsymbol{A}}|\cos\phi$, and therefore the component of $\vec{\boldsymbol{A}}$ along $\hat{\boldsymbol{e}}$ is just $A_e = |\vec{\boldsymbol{A}}|\cos\phi$.

Using this fact, we can write

$$\hat{\boldsymbol{n}} = \cos\alpha\hat{\boldsymbol{x}} + \sqrt{1-\cos^2\alpha-\cos^2\theta}\,\hat{\boldsymbol{y}} - \cos\theta\hat{\boldsymbol{z}} \tag{13}$$

$$\hat{\boldsymbol{n}}' = \cos\alpha'\hat{\boldsymbol{x}} + \sqrt{1-\cos^2\alpha'-\cos^2\theta'}\,\hat{\boldsymbol{y}} + \cos\theta'\hat{\boldsymbol{z}}. \tag{14}$$

Through the dot product, we have

$$\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' = \cos\varphi = \cos\alpha\cos\alpha' + \sqrt{(1-\cos^2\alpha-\cos^2\theta)(1-\cos^2\alpha'-\cos^2\theta')} + \cos\theta\cos\theta'. \tag{15}$$

Let us simplify the term under the radical:

$$(1-\cos^2\alpha-\cos^2\theta)(1-\cos^2\alpha'-\cos^2\theta') = \quad . \tag{16}$$

Problem 4)
Find a scalar function $\varphi(r)$ of $r = |\vec{\boldsymbol{r}}|$ which satisfies the equation

$$\vec{\boldsymbol{\nabla}}\cdot[\varphi(r)\vec{\boldsymbol{r}}] = 0. \tag{17}$$

We can write the equation above as

$$\vec{\boldsymbol{\nabla}}\cdot[\varphi(r)\vec{\boldsymbol{r}}] = \varphi(r)\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{r}} + \vec{\boldsymbol{\nabla}}\varphi(r)\cdot\vec{\boldsymbol{r}} = 3\varphi(r) + x\frac{\partial\varphi(r)}{\partial x} + y\frac{\partial\varphi(r)}{\partial y} + z\frac{\partial\varphi(r)}{\partial z} = 0. \tag{18}$$

Recall the transformation between the Cartesian (x, y, z) and spherical (r, ϕ, θ) bases in $\mathbb{R}^3$ is just

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \arctan(y/x) \\ \theta = \arctan\left(z/\sqrt{x^2+y^2}\right). \end{cases} \tag{19}$$

We can use this to transform the last three terms of Eq. (18) to an explicit representation in terms of spherical components. It suffices to do this for the x-component and generalize to the other two:

$$\frac{\partial\varphi(r)}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial\varphi}{\partial r} = \frac{x}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r}, \tag{20}$$

Plugging this into Eq. (18) then gives us

$$\vec{\boldsymbol{\nabla}}\cdot[\varphi(r)\vec{\boldsymbol{r}}] = 3\varphi(r) + \frac{x^2+y^2+z^2}{r}\frac{\mathrm{d}\varphi(r)}{\mathrm{d}r} = 3\varphi(r) + r\frac{\mathrm{d}\varphi(r)}{\mathrm{d}r} = 0. \tag{21}$$

This is just a separable differential equation, which gives

$$\boxed{\varphi(r) = \frac{A}{r^3}}, \tag{22}$$

where A is a constant determined by some boundary condition.

Problem 5)
Calculate the following: (1) $\vec{\boldsymbol{\nabla}} \cdot [(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{b}}]$, (2) $\vec{\boldsymbol{\nabla}} \times [(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{b}}]$, (3) $\vec{\boldsymbol{\nabla}} \cdot [\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}]$, (4) $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})$, (5) $\vec{\boldsymbol{\nabla}} \cdot [\vec{\boldsymbol{r}} \times (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}})]$, where $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are constant vectors.

There are a few ways of approaching these kinds of problems. One way is to do the calculus and algebra via brute-force in its full glory. Another utilizes a nicer and more compact notation through tensors and the Einstein summation notation. The summation notation is that we can write sums like $\vec{\boldsymbol{A}} = \sum_{i=1}^{3} A_i \hat{\boldsymbol{e}}_i = A_i \hat{\boldsymbol{e}}_i$, dropping the explicit sum and implicitly understanding that repeated indices are summed over. Note that $i = 1, 2, 3$ corresponds to the x, y, z components of $\vec{\boldsymbol{A}}$. Additionally, we can define the Kronecker-delta symbol (effectively the three-dimensional analogue to the metric tensor) as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{23}$$

This is useful for defining the dot product between vectors as

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_i B_i = A_i B_j \delta_{ij}. \tag{24}$$

We also introduce the Levi-civita symbol as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \in \{(123), (231), (312)\} \\ 1 & \text{if } ijk \in \{(213), (132), (321)\} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

The Levi-civita symbol is totally anti-symmetric in the sense that permuting indices as $\epsilon_{ijk} = -\epsilon_{jik}$. This is useful in writing the cross product as

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = \hat{\boldsymbol{e}}_i \epsilon_{ijk} A_j B_k. \tag{26}$$

Note the following useful identity:

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{27}$$

This can be proven in a number of ways. The most direct (although at the same time the least satisfying and enlightening) is to just check that the two sides are the same for each pair (ij) and (lm) for $k = 1, 2, 3$.

(1) Using the framework established above, we have

$$\boxed{\vec{\boldsymbol{\nabla}} \cdot [(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}})\vec{\boldsymbol{b}}] = \nabla_i (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}}) b_i = \nabla_i a_j r_j b_i = a_j (\nabla_i r_j) b_i = a_j \delta_{ij} b_i = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}} \tag{28}$$

since a_j and b_i are constants by assumption.

(2)

$$\boxed{\vec{\boldsymbol{\nabla}} \times [\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{r}}]\vec{\boldsymbol{b}} = \hat{\boldsymbol{e}}_i \epsilon_{ijk} \nabla_j a_l r_l b_k = \hat{\boldsymbol{e}}_i \epsilon_{ijk} a_l \delta_{jl} b_k = \hat{\boldsymbol{e}}_i \epsilon_{ilk} a_l b_k = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}}. \tag{29}$$

(3)

$$\boxed{\vec{\boldsymbol{\nabla}} \cdot [\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}] = \nabla_i \epsilon_{ijk} a_j r_k = \epsilon_{ijk} a_j \delta_{ik} = \epsilon_{kjk} a_j = 0}. \tag{30}$$

(4)

$$\boxed{\begin{aligned} \vec{\boldsymbol{\nabla}} \times [\vec{\boldsymbol{a}} \times \vec{\boldsymbol{r}}] &= \hat{\boldsymbol{e}}_i \epsilon_{ijk} \nabla_j \epsilon_{klm} a_l r_m = \hat{\boldsymbol{e}}_i \epsilon_{imk} \epsilon_{lmk} a_l \\ &= \hat{\boldsymbol{e}}_i [\delta_{il} \delta_{mm} - \delta_{im} \delta_{lm}] a_l = 3\vec{\boldsymbol{a}} - \vec{\boldsymbol{a}} = 2\vec{\boldsymbol{a}} \end{aligned}}. \tag{31}$$