Problem 1)
Using the approach discussed in class, calculate the evolution of $n(x,t)$ along the x-axis $-\infty < x < \infty$ described by the diffusion equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} \tag{1}$$

with the boundary conditions $n(\pm\infty, t) = 0$ and the initial condition

$$n(x,0) = n_0 e^{-x^2/a^2}. \tag{2}$$

The one-dimensional Green function for the diffusion equation (with $D = 1$) on the x-axis is

$$G(x-x',t) = \frac{1}{2\sqrt{\pi t}} e^{-(x-x')^2/4t}. \tag{3}$$

The temperature distribution at an arbitrary time t is then a spatial convolution of the Green's function with the initial shape of the temperature distribution at time $t = 0$:

$$\boxed{\begin{aligned} n(x,t) &= \int_{-\infty}^{\infty} \mathrm{d}x'\, G(x-x',t) n(x',0) = \frac{n_0}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \mathrm{d}x'\, e^{-(x-x')^2/4t} e^{-x'^2/a^2} \\ &= \frac{n_0}{2\sqrt{\pi t}} \left(2a\sqrt{\frac{\pi t}{a^2+4t}} e^{-x^2/(a^2+4t)} \right) = \frac{n_0}{\sqrt{1+4t/a^2}} e^{-x^2/(a^2+4t)}. \end{aligned}} \tag{4}$$

Problem 2)
The temperature of a planar surface at $x = 0$ changes periodically: $T(0,t) = T_0 + T_1 \cos\omega t$. Using the Fourier transform in t, calculate the temperature $T(x,t)$ away from the heated surface at $x > 0$ by solving the thermal diffusion equation

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} \tag{5}$$

with the boundary condition $T(0,t) = T_0 + T_1 \cos\omega t$ at $x = 0$ and $T(\infty, t) = T_0$ at $x = \infty$.

First, let's define $u(x,t) = T(x,t) - T_0$. It is clear that u satisfies the same diffusion equation as T but with the updated boundary conditions $u(0,t) = T_1 \cos\omega t$ and $u(\infty,t) = 0$. As suggested, let's take the Fourier transform of the diffusion equation in time:

$$\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega' t} \frac{\partial u}{\partial t} = i\omega \tilde{u}(x,\omega') = D \frac{\partial^2 \tilde{u}(x,\omega')}{\partial x^2}. \tag{6}$$

We find then that

$$\begin{aligned}\tilde{u}(x,\omega') &= Ae^{\sqrt{-i\omega'/D}\,x} + Be^{-\sqrt{-i\omega'/D}\,x} \\ &= Ae^{k'(1-i)x} + Be^{-k'(1-i)x} \\ &= Ae^{-ik'x}e^{k'x} + Be^{ik'x}e^{-k'x},\end{aligned} \tag{7}$$

where we have defined $k' = \sqrt{\omega'/2D}$. Notice that the boundary condition $u(\infty,t) = 0$ gives $\tilde{u}(\infty,t) = 0$, meaning that we must have $A = 0$.

Next, observe that the Fourier transform of $u(0,t) = T_1 \cos \omega t = \frac{T_1}{2}(e^{i\omega t} + e^{-i\omega t})$ is

$$\tilde{u}(0,\omega') = \pi T_1[\delta(\omega' - \omega) + \delta(\omega' + \omega)] = B \tag{8}$$

Thus,

$$\tilde{u}(x,\omega') = \pi T_1[\delta(\omega' - \omega) + \delta(\omega' + \omega)]e^{ik'x}e^{-k'x}. \tag{9}$$

With this, we take the inverse Fourier transform of $\tilde{u}$ and find

$$\begin{aligned}u(x,t) &= \frac{T_1}{2}\int_{-\infty}^{\infty} \mathrm{d}\omega'\, e^{i\omega' t}[\delta(\omega' - \omega) + \delta(\omega' + \omega)]e^{(-1+i)\sqrt{\omega'/2D}\,x} \\ &= \frac{T_1}{2}\left[e^{i\omega t}e^{(-1+i)\sqrt{\omega/2D}\,x} + e^{-i\omega t}e^{-(1+i)\sqrt{\omega/2D}\,x}\right] \\ &= T_1 \cos\left[\omega\left(t + \frac{x}{\sqrt{2\omega D}}\right)\right]e^{-\sqrt{\omega/2D}\,x}.\end{aligned} \tag{10}$$

Finally, we add T_0 to u and arrive at

$$\boxed{T(x,t) = T_0 + T_1 \cos\left[\omega\left(t + \frac{x}{\sqrt{2\omega D}}\right)\right] \exp\left(-\sqrt{\frac{\omega}{2D}}\,x\right)}. \tag{11}$$