Problem 1)
Express the following delta function in terms of delta functions of the variable x:

$$\delta\left(\frac{\sin x}{x}\right). \tag{1}$$

Recall that we can write

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \tag{2}$$

where $f(x)$ has simple roots x_i[1]. Note that $f(x) = \sin x/x$ indeed has simple roots since we can write

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{3}$$

From this formala, it is also trivial to glean that $\sin x/x$ has zeros when $\sin x = 0$ or $x = n\pi$ for $n = \pm 1, \pm 2, \ldots$. Additionally, $\frac{\mathrm{d}}{\mathrm{d}x}\sin x/x|_{x=n\pi} = [\cos x/x - \sin x/x^2]_{x=n\pi} = (-1)^n/n\pi$.

Thus,

$$\boxed{\delta\left(\frac{\sin x}{x}\right) = \sum_{n=1}^{\infty} n\pi\Big[\delta(x - n\pi) + \delta(x + n\pi)\Big]}. \tag{4}$$

Problem 2)
Calculate

$$I(z) = \Gamma(1 + z)\Gamma(1 - z) \tag{5}$$

at $z = 1/4$.

We can write

$$\boxed{\begin{aligned} \Gamma(1+z)\Gamma(1-z) &= z\Gamma(z)\Gamma(1-z) = z\frac{\pi}{\sin \pi z} \\ \Rightarrow \Gamma(1+z)\Gamma(1-z)|_{z=1/4} &= \frac{\pi/4}{\sin \pi/4} = \frac{\pi}{2\sqrt{2}} \end{aligned}}. \tag{6}$$

[1] Otherwise the $f'(x_i)$ term vanishes and we have to be a bit more careful with the expansion of $f(x)$ inside the delta function.

Problem 3)
Using the definition of the complete elliptical integrals $E(m)$ and $K(m)$, express the derivative $\partial E(m)/\partial m$ in terms of $K(m)$ and $E(m)$.

Recall that the complete elliptic integrals of the first and second kind are defined as

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} \tag{7}$$

$$E(m) = \int_0^{\pi/2} \mathrm{d}\theta\,\sqrt{1 - m\sin^2\theta}, \tag{8}$$

respectively. Differentiating $E(m)$ with respect to m we have

$$\begin{aligned} \frac{\partial E}{\partial m} &= \frac{\partial}{\partial m}\int_0^{\pi/2} \mathrm{d}\theta\,\sqrt{1 - m\sin^2\theta} = \int_0^{\pi/2} \mathrm{d}\theta\,\frac{\partial}{\partial m}\sqrt{1 - m\sin^2\theta} \\ &= -\frac{1}{2}\int_0^{\pi/2} \frac{\sin^2\theta\,\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}. \end{aligned} \tag{9}$$

Observe the following:

$$\begin{aligned} K(m) - E(m) &= \int_0^{\pi/2} \mathrm{d}\theta\left[\frac{1}{\sqrt{1 - m\sin^2\theta}} - \sqrt{1 - m\sin^2\theta}\right] \\ &= \int_0^{\pi/2} \mathrm{d}\theta\,\frac{1 - (1 - m\sin^2\theta)}{\sqrt{1 - m\sin^2\theta}} = m\int_0^{\pi/2} \mathrm{d}\theta\,\frac{\sin^2\theta}{\sqrt{1 - m\sin^2\theta}}. \end{aligned} \tag{10}$$

Thus,

$$\boxed{\frac{\partial E}{\partial m} = \frac{K(m) - E(m)}{m}}. \tag{11}$$

The final result above is not correct. It should be

$$\frac{\partial E}{\partial m} = -\frac{1}{2}\frac{K(m) - E(m)}{m} = \frac{E(m) - K(m)}{2m}. \tag{12}$$

Problem 4)
Find the values of $e^{\pm i\pi/2}$, $e^{i\pi n}$, $\ln(-1)$ where $n = 0, \pm 1, \pm 2, \ldots$.

We find that

$$\boxed{e^{\pm i\pi/2} = \cos\left(\pm\pi/2\right) + i\sin\left(\pm\pi/2\right) = \pm i} \tag{13}$$

$$\boxed{e^{i\pi n} = \cos\left(\pi n\right) + i\sin\left(\pi n\right) = (-1)^n} \tag{14}$$

$$\boxed{\ln(-1) = \ln\left(e^{i\pi}\right) = i\pi}, \tag{15}$$

restricting our definition of $\ln z$ to the principal branch (i.e. $\arg z \in [0, 2\pi)$).

Problem 5)

Calculate the following series:

$$I_1 = \sum_{n=0}^{\infty} p^n \sin\left(qn\right) \quad \text{and} \quad I_2 = \sum_{n=0}^{\infty} p^n \cos\left(qn\right), \tag{16}$$

where p and q are real parameters.

HINT: Use the sum of geometric series with complex r.

Observe the following:

$$I = \sum_{n=0}^{\infty} p^n e^{iqn} = I_2 + iI_1. \tag{17}$$

That is $I_2 = \mathrm{Re}(I)$ and $I_1 = \mathrm{Im}(I)$. We can use the geometric series formula with complex $r = pe^{iq}$, giving

$$I = \frac{1}{1 - pe^{iq}} = \frac{1}{(1 - p\cos q) - ip\sin q} = \frac{(1 - p\cos q) + ip\sin q}{(1 - p\cos q)^2 + p^2\sin^2 q}. \tag{18}$$

Taking real and imaginary parts of I, we have

$$\boxed{I_1 = \frac{p\sin q}{1 + p^2 - 2p\cos q}} \tag{19}$$

$$\boxed{I_2 = \frac{1 - p\cos q}{1 + p^2 - 2p\cos q}}. \tag{20}$$