Problem 1)
Write down a general solution of the Tomas-Fermi equation

$$\nabla^2\varphi - \lambda^{-2}\varphi = f(\vec{\boldsymbol{r}}) \tag{1}$$

with the boundary condition $\varphi(x,y,0) = 0$ at a conducting plane $z = 0$ and $\varphi(x,y,z) \to 0$ at $x, y \to \infty$. The function $f(\vec{\boldsymbol{r}})$ tends to zero at infinity.

In class we found the Green function for the Tomas-Fermi problem in $\mathbb{R}^3$ (i.e. the boundary conditions are just that $G \to 0$ as $\vec{\boldsymbol{r}}' \to \infty$) as

$$G_\infty(x,y,z;x',y',z') = -\frac{e^{-|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|/\lambda}}{4\pi|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}. \tag{2}$$

We can solve the problem in the half-space $z \geq 0$ with the Green's function

$$G(x,y,z;x',y',z') = G_\infty(x,y,z;x',y',z') - G_\infty(x,y,z;x',y',-z'). \tag{3}$$

Essentially, we have used the method of images where we placed an "image source" reflected over the xy-plane from the point source described by G_∞. Notice that on the plane we then have $G(x,y,z;x',y',0) = 0$ and $G(x,y,z;x' \to \infty, y' \to \infty, z') = 0$ automatically since as $x', y' \to \infty$ we have $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| \to \infty$ and therefore $G_\infty \to 0$ under this condition by construction. The solution is then

$$\boxed{\varphi(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}y'\int_{0}^{\infty}\mathrm{d}z'\, G(x,y,z;x',y',z')f(\vec{\boldsymbol{r}}')}. \tag{4}$$

Problem 2)
Transverse displacements $u(x,t)$ of a string of length L with fixed ends are desribed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = s^2\frac{\partial^2 u}{\partial x^2} \tag{5}$$

with the boundary condition $u(0,t) = u(L,t) = 0$. At $t = 0$ a hammer of width $a < L$ hits the string which was initially at rest, $u(x,0) = 0$. The impact caused an instantaneous velocity v_0 in the central region of the string:

$$\dot{u}(x,0) = v_0 \tag{6}$$

for $|x - L/2| < a$ and $\dot{u}(x,0) = 0$ for $|x - L/2| > a$. Find the full solution $u(x,t)$ at $t > 0$.

We solved the one-dimensional wave equation on $x \in [0, L], t \geq 0$ for generic initial conditions and fixed endpoints $u(0, t) = u(L, t) = 0$ as boundary conditions as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi st}{L}\right) + B_n \sin\left(\frac{n\pi st}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right). \tag{7}$$

Notice that at $t = 0$, the position of the string is

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = 0. \tag{8}$$

Since this must be true for all x and the different modes are orthogonal, we must have $A_n = 0$ for all n. Next, we have the initial velocity of the string

$$\dot{u}(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi s}{L} \sin\left(\frac{n\pi x}{L}\right) = v_0 \theta(|x - L/2| < a). \tag{9}$$

We can use the orthogonality relations between sine modes, which gives

$$B_n = \frac{L}{n\pi s} \frac{2}{L} \int_0^L v_0 \theta(|x - L/2| < a) \, \mathrm{d}x = \frac{4v_0 a}{n\pi s}. \tag{10}$$

Putting this into the expansion for u, we have

$$\boxed{u(x,t) = \frac{4v_0 a}{\pi s} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi st}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}. \tag{11}$$

Problem 3)
Solve the Laplace equation

$$\nabla^2 u = 0 \tag{12}$$

for a circle of radius a with the boundary condition $u(a, \varphi) = u_0 \sin^3 \varphi$.

(a) Show that the solution $u(r, \varphi)$ inside the circle $r \leq a$ can be found by the separation of variables and calculate the coefficients A_n and B_n:

$$u(r,\varphi) = \sum_{n=0}^{\infty} r^n \Big[A_n \sin n\varphi + B_n \cos n\varphi \Big]. \tag{13}$$

(b) Show that the solution $u(r, \varphi)$ outside the circle $r > a$ can be found by the separation of variables and calculate the coefficients A_n and B_n:

$$u(r,\varphi) = \sum_{n=0}^{\infty} r^{-n} \Big[A_n \sin n\varphi + B_n \cos n\varphi \Big]. \tag{14}$$

The Laplacian in polar coordinates (or cylindrical coordinates with translation symmetry along the z-axis) is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}. \tag{15}$$

We can use separation of variables and write $u(r,\varphi) = R(r)T(\varphi)$, which gives

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}\varphi^2} = 0. \tag{16}$$

Both terms must be constant with respect to r, φ, and since T must be cyclic (i.e. u must be single-valued), we choose

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\phi^2} = -n^2 T, \tag{17}$$

which has solutions $\sin n\varphi$ and $\cos n\varphi$. Putting this into the separated laplace's equation, we find a differential equation for the radial part of u as

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - n^2 R = 0. \tag{18}$$

Let us propose a solution of the form $R(r) = \sum_{m=0}^{\infty} a_m r^{m+s}$. We then find that

$$\sum_{n=0}^{\infty} a_m[(m+s)^2 - n^2]r^{m+s} = 0. \tag{19}$$

We then find that either $a_m = 0$ or $m + s = \pm n$ makes each term zero. Thus, we only have two terms for R at a fixed n such that

$$R_n(r) = a_n r^n + b_n r^{-n}. \tag{20}$$

A general solution for Laplace's equation is

$$u(r,\varphi) = \sum_{n=0}^{\infty}\Big[a_n r^n + b_n r^{-n}\Big]\Big[A_n \sin n\varphi + B_n \cos n\varphi\Big]. \tag{21}$$

(a) If we wish to consider the solution of Laplace's equation with boundary conditions at $r = a$ for $r < a$, we must have $b_n = 0$ since our solution should be regular at $r = 0$. Thus,

$$u(r,\varphi) = \sum_{n=0}^{\infty} r^n \Big[A_n \sin n\varphi + B_n \cos n\varphi\Big]. \tag{22}$$

Next, we find that $\sin^3\varphi = [3\sin\varphi - \sin 3\varphi]/4$, meaning that our solution for $r < a$ must be

$$\boxed{u(r,\varphi) = \frac{u_0}{4}\left[\frac{3r}{a}\sin\varphi - \left(\frac{r}{a}\right)^3 \sin 3\varphi\right]}. \tag{23}$$

(b) Now, if we consider $r > a$, we similarly must have $a_n = 0$ so that our solution $u \to 0$ as $r \to \infty$, and thus for $r > a$

$$\boxed{u(r,\varphi) = \frac{u_0}{4}\left[\frac{3a}{r}\sin\varphi - \left(\frac{a}{r}\right)^3 \sin 3\varphi\right]}. \tag{24}$$