Problem 1)
Consider the Pauli matrix:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1}$$

(a) Obtain the eigenvalues and eigenvectors of σ_y.
(b) Find a unitary operator U which diagonalizes σ_y.

(a) We can easily see that the characteristic equation of σ_y is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1. \tag{2}$$

Thus, the components of the eigenvectors satisfy

$$-\lambda a_1 = i a_2 \Rightarrow a_2 = i\lambda a_1, \tag{3}$$

and the normalized eigenvectors are then (up to a normalizing constant)

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \leftrightarrow 1 \tag{4}$$

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} \leftrightarrow -1. \tag{5}$$

(b) We can easily diagonalize σ_y then by writing

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{6}$$

where

$$A' = U^\dagger A U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Problem 2)
Consider a rotation matrix a with complex elements a_{ij}:

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{8}$$

Find the inverse matrix a^{-1}.

Since a is a rotation matrix, it must be hermitian. We prove this as follows. Let us posit a defining property of rotation matrices: they preserve the norm of a vector under a transformation. That is, $||v|| = ||Rv||$. Using this we see that

$$v^\dagger v = v^\dagger (R^\dagger R) v \Rightarrow R^\dagger R = \mathbb{I}. \tag{9}$$

Another way to see this is by recalling from Quantum Mechanics that the angular momentum operator J is the generator of rotations. That is, we can define a generic rotation operator through an angle θ about an axis defined by a unit vector $\hat{\boldsymbol{n}}$ as $R = \exp\left(-i\theta\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{J}}\right)$, which has inverse $R^{-1} = R^\dagger$. This is a more abstract object than a matrix in general: it performs a rotation of states $|\psi\rangle \to |\psi'\rangle = R|\psi\rangle$ in the Hilbert space. We can, however, represent the operator R using a matrix, and if our Hilbert space has dimension 3 (e.g. a spin-1 system), then our rotation matrix $R \in \mathbb{C}^{3\times 3}$.

Finally, we write down explicitly

$$a^{-1} = a^\dagger = \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{pmatrix}. \tag{10}$$

Problem 3)
Using the definition of a function of a matrix given in Lect. 22 and the properties of the Pauli matrices $\sigma_i^2 = \mathbb{I}$, prove the following relation:

$$\exp(i\theta\sigma_z) = \mathbb{I}\cos\theta + i\sigma_z \sin\theta. \tag{11}$$

From the definition of the exponential of an operator, we can write

$$\exp(i\theta\sigma_z) = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta\sigma_z)^n. \tag{12}$$

Let us split the sum over even and odd n such that

$$\begin{aligned} \exp(i\theta\sigma_z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(i\theta\sigma_z)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(i\theta\sigma_z)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n}\mathbb{I} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}\sigma_z \\ &= \cos\theta\mathbb{I} + i\sin\theta\sigma_z \end{aligned}. \tag{13}$$