

**Problem 1)**

Consider the Pauli matrix:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (1)$$

- (a) Obtain the eigenvalues and eigenvectors of  $\sigma_y$ .
- (b) Find a unitary operator  $U$  which diagonalizes  $\sigma_y$ .

(a) We can easily see that the characteristic equation of  $\sigma_y$  is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1. \quad (2)$$

Thus, the components of the eigenvectors satisfy

$$-\lambda a_1 = i a_2 \Rightarrow a_2 = i \lambda a_1, \quad (3)$$

and the eigenvectors are then (up to some constant)

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \leftrightarrow 1, \quad \begin{pmatrix} 1 \\ -i \end{pmatrix} \leftrightarrow -1. \quad (4)$$

(b) We can easily diagonalize  $\sigma_y$  by writing

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad (5)$$

where

$$A' = U^\dagger A U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6)$$

**Problem 2)**

Consider a rotation matrix  $a$  with complex elements  $a_{ij}$ :

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \quad (7)$$

Find the inverse matrix  $a^{-1}$ .

Since  $a$  is a rotation matrix, it must be unitary. We prove this as follows. Let us posit a defining property of rotation matrices: they preserve the norm of a vector under a transformation. That is,  $\|v\| = \|Rv\|$ . Using this we see that

$$v^\dagger v = v^\dagger (R^\dagger R) v \Rightarrow R^\dagger R = \mathbb{I}. \quad (8)$$

Another way to see this is by recalling from Quantum Mechanics that the angular momentum operator  $J$  is the generator of rotations. That is, we can define a generic rotation operator through an angle  $\theta$  about an axis defined by a unit vector  $\hat{n}$  as  $R = \exp(-i\theta\hat{n} \cdot \vec{J})$ , which has inverse  $R^{-1} = R^\dagger$ . This is a more abstract object than a matrix in general: it performs a rotation of states  $|\psi\rangle \rightarrow |\psi'\rangle = R|\psi\rangle$  in the Hilbert space. We can, however, typically represent the operator  $R$  using a matrix, and if our Hilbert space has dimension 3 (e.g. a spin-1 system), then our rotation matrix  $R \in \mathbb{C}^{3 \times 3}$ .

Finally, we write down explicitly

$$a^{-1} = a^\dagger = \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{pmatrix}. \quad (9)$$

### Problem 3)

Using the definition of a function of a matrix given in Lect. 22 and the properties of the Pauli matrices  $\sigma_i^2 = \mathbb{I}$ , prove the following relation:

$$\exp(i\theta\sigma_z) = \mathbb{I} \cos \theta + i\sigma_z \sin \theta. \quad (10)$$

From the definition of the exponential of an operator, we can write

$$\exp(i\theta\sigma_z) = \sum_{n=0}^{\infty} \frac{1}{n!} (i\theta\sigma_z)^n. \quad (11)$$

Let us split the sum over even and odd  $n$  such that

$$\begin{aligned} \exp(i\theta\sigma_z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (i\theta\sigma_z)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (i\theta\sigma_z)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n} \mathbb{I} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} \sigma_z \\ &= \cos \theta \mathbb{I} + i \sin \theta \sigma_z \end{aligned} \quad (12)$$