Problem 1)
Calculate the following limit:

$$I_1 = \lim_{x\to 0} \frac{x^2 \ln(1+x^2)}{x^2 - \sin^2 x}. \tag{1}$$

Upon first glance, we have a limit of an indeterminate form $0/0$. We may use L'Hopital's rule, but the functions are quite messy to differentiate, so we expand both the numerator and denominator in Taylor series about $x = 0$. Note that $\ln(1+x) = x - x^2/2 + \text{calO}(x^3)$ and $\sin^2 x = (1-\cos 2x)/2 = [(2x)^2/2! + (2x)^4/4! + \mathcal{O}(x^6)]/2$. Thus, the limit

$$\boxed{I_1 = \lim_{x\to 0} \frac{x^4}{x^2 - \frac{1}{2}[(2x)^2/2! - (2x)^4/4!]} = \frac{2(4!)}{2^4} = \frac{24}{8} = 3}. \tag{2}$$

Problem 2)
Calculate the following limit of the $m-$th derivative at $x = 0$:

$$I_2 = \lim_{x\to 0} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{\ln(1+x) - x}{x^2}. \tag{3}$$

Here, we can write

$$\begin{aligned} \frac{\ln(1+x) - x}{x^2} &= \frac{1}{x^2}\Big[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} - x\Big] = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} x^{n-2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} x^n \\ &= \sum_{n=0}^{m-1} \frac{(-1)^{n+1}}{n+2} x^n + \frac{(-1)^{m+1}}{m+2} x^m + \sum_{n=m+1} \frac{(-1)^{n+1}}{n+2} x^n. \end{aligned} \tag{4}$$

The last step is mostly illustrative for this:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{\ln(1+x) - x}{x^2} = \frac{(-1)^{m+1}\, m!}{m+2} + \mathcal{O}(x). \tag{5}$$

Therefore, taking $x \to 0$ gives

$$\boxed{I_2 = (-1)^{m+1} \frac{m!}{m+2}}. \tag{6}$$

Problem 3)
Calculate the following limit at $x = y = 0$:

$$I_3 = \lim_{x\to 0, y\to 0} \nabla^2 \big[e^{-ax^2 - by^2} \cos ax \cos by\big]. \tag{7}$$

We can rewrite the operand of the laplacian as

$$e^{-ax^2-by^2}\cos ax\cos by = [e^{-ax^2}\cos ax][e^{-by^2}\cos bx], \tag{8}$$

which gives

$$\nabla^2 e^{-ax^2-by^2}\cos ax\cos by = e^{-by^2}\cos by\frac{\partial^2}{\partial x^2}e^{-ax^2}\cos ax + e^{-ax^2}\cos ax\frac{\partial^2}{\partial y^2}e^{-by^2}\cos by. \tag{9}$$

Note that we can evaluate the limit of the first term and use the replacements $a \leftrightarrow b,\ x \leftrightarrow y$.

Observe that

$$\lim_{y\to 0} e^{-by^2}\cos by = 1. \tag{10}$$

Next, we evaluate the second factor to be

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}e^{-ax^2}\cos ax = -2a(1-2ax)e^{-ax^2}\cos ax + (-2axe^{-ax^2})(-a\sin ax)\\ + e^{-ax^2}(-a^2\cos ax).\end{aligned} \tag{11}$$

It should be clear then that

$$\lim_{x\to 0}\frac{\partial^2}{\partial x^2}e^{-ax^2}\cos ax = -2a - a^2 = -(a+1)^2 + 1. \tag{12}$$

Hence

$$\boxed{\lim_{x,y\to 0}\nabla^2 e^{-ax^2-by^2}\cos ax\cos by = -(a+1)^2 - (b+1)^2 + 2}. \tag{13}$$

Problem 4)
Calculate the sum

$$I_1 = \sum_{n=1}^{N}(n^2+n+1). \tag{14}$$

We can use the Euler-Maclaurin formula

$$\sum_{n=a}^{N} f(n) = \int_a^N f(x)\,\mathrm{d}x + \frac{1}{2}[f(N)+f(a)] + \sum_{n=1}^{q}\frac{B_{2n}}{(2n)!}[f^{(2n-1)}(N) - f^{(2n-1)}(a)] + R. \tag{15}$$

to compute this series. First, we split the sum into separate terms as

$$I_1 = \sum_{n=0}^{N} n^2 + \sum_{n=0}^{N} n + N, \tag{16}$$

where we have gone ahead and evaluated the sum of the last term since it is just 1 added N times. Also note that we have adjusted out indices to start from $n = 0$ since this does not change the sum but should make the right hand side simpler to evaluate since $\frac{\mathrm{d}^n}{\mathrm{d}x^n} x^m|_{x=0} = 0$ for any $n, m \in \mathbb{N} \cup \{0\}$. The first sum is

$$\begin{aligned}\sum_{n=0}^{N} n^2 &= \int_0^N x^2 \,\mathrm{d}x + \frac{N^2}{2} + \frac{B_2}{2!}\frac{\mathrm{d}}{\mathrm{d}x}(x^2)|_{x=N} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \\ &= \frac{N}{6}(2N^2 + 3N + 1) = \frac{N(N+1)(2N+1)}{6}\end{aligned}. \tag{17}$$

Moving on to the second sum we find[1]

$$\sum_{n=0}^{N} n = \int_0^N x \,\mathrm{d}x + \frac{N}{2} = \frac{N^2}{2} + \frac{N}{2} = \frac{N(N+1)}{2}. \tag{18}$$

Plugging these intermediate results into Eq. (16), we find

$$\begin{aligned}I_1 &= \frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2} + N \\ &= \frac{N}{6}\Big[(N+1)\Big((2N+1)+3\Big) + 6\Big] = \boxed{\frac{N(N^2+3N+5)}{3}}\end{aligned}. \tag{19}$$

Problem 5)
Calculate the sum

$$I_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2}. \tag{20}$$

For this problem we can use the Poisson summation formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \int_0^\infty f(x)\,\mathrm{d}x + \frac{f(0)}{2} + 2\sum_{n=1}^{\infty}\int_0^\infty f(x)\cos(2\pi n x)\,\mathrm{d}x\,. \tag{21}$$

Note here that we can rewrite $(-1)^n = \cos n\pi$. Using this, we find

$$\int_0^\infty \frac{\cos kx}{x^2 + a^2}\,\mathrm{d}x = \frac{\pi}{2|a|} e^{-k|a|}. \tag{22}$$

[1]As Gauss did in his primary years using a much simpler and elegant rationale.

The sum is then given as

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2} &= \frac{\pi}{2|a|}e^{-\pi|a|}+\frac{1}{2a^2}+\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\cos\left[(2n+1)\pi x\right]+\cos\left[(2n-1)\pi x\right]}{n^2+a^2}\,\mathrm{d}x \\
&= \frac{\pi}{2|a|}e^{-\pi|a|}+\frac{1}{2a^2}+\sum_{n=1}^{\infty}\frac{\pi}{2|a|}\left[e^{-(2n+1)\pi|a|}+e^{-(2n-1)\pi|a|}\right] \\
&= \frac{\pi}{2|a|}e^{-\pi|a|}+\frac{1}{2a^2}+\frac{\pi}{2|a|}\left[e^{-\pi|a|}\sum_{n=1}^{\infty}e^{-2n\pi|a|}+e^{\pi|a|}\sum_{n=1}^{\infty}e^{-2n\pi|a|}\right] \\
&= \frac{\pi}{2|a|}e^{-\pi|a|}+\frac{1}{2a^2}+\frac{\pi}{|a|}\cosh\pi|a|\frac{e^{-2\pi|a|}}{1-e^{-2\pi|a|}} \\
&= \frac{1}{2a^2}+\frac{\pi}{2|a|}e^{-\pi|a|}\Big[1+\coth\pi|a|\Big].
\end{aligned} \tag{23}$$

At this point, we can subtract the $n=0$ term from both sides of the previous equation and obtain

$$\boxed{I_2=\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2}-\frac{1}{a^2}=-\frac{1}{2a^2}+\frac{\pi}{2|a|}e^{-\pi|a|}\Big[1+\coth\pi|a|\Big]}. \tag{24}$$

Problem 6)
Calculate the asymptotic series for this integral at $x \gg 1$:

$$I_3=\int_x^{\infty}e^{-au}\ln u\,\mathrm{d}u\,, \tag{25}$$

where $a>0$.

We can solve this problem using integration by parts $\int f'g\,\mathrm{d}x = fg-\int fg'\,\mathrm{d}x$ with $f'=e^{-au}$ and $g=\ln u$, which gives $f=-\frac{1}{a}e^{-au}=-\frac{1}{a}f'$ and $g'=1/u$. Plugging this in, we find

$$I_3=-\frac{1}{a}\frac{e^{-au}}{u}\bigg|_x^{\infty}+\frac{1}{a}\int_x^{\infty}\frac{e^{-au}}{u}\,\mathrm{d}u=\frac{e^{-ax}}{ax}+\int_x^{\infty}\frac{e^{-au}}{au}\,\mathrm{d}u=\frac{e^{-ax}}{ax}+\frac{1}{a}E_1(ax). \tag{26}$$

Thus, we have

$$\boxed{I_3=\frac{e^{-ax}}{ax}+\frac{1}{a}e^{-ax}\sum_{n=0}^{\infty}\frac{(-1)^n}{(ax)^{n+1}}n!=\frac{e^{-ax}}{ax}\left[1+x\sum_{n=0}^{\infty}\frac{(-1)^n}{(ax)^{n+1}}n!\right]}. \tag{27}$$