

Problem 1)

Calculate the following limit:

$$I_1 = \lim_{x \rightarrow 0} \frac{x^2 \ln(1 + x^2)}{x^2 - \sin^2 x}. \quad (1)$$

Upon first glance, we have a limit of an indeterminate form $0/0$. We may use L'Hopital's rule, but the functions are quite messy to differentiate, so we expand both the numerator and denominator in Taylor series about $x = 0$. Note that $\ln(1 + x) = x - x^2/2 + \mathcal{O}(x^3)$ and $\sin^2 x = (1 - \cos 2x)/2 = [(2x)^2/2! + (2x)^4/4! + \mathcal{O}(x^6)]/2$. Thus, the limit

$$I_1 = \lim_{x \rightarrow 0} \frac{x^4}{x^2 - \frac{1}{2}[(2x)^2/2! - (2x)^4/4!]} = \frac{2(4!)}{2^4} = \frac{24}{8} = 3. \quad (2)$$

Problem 2)

Calculate the following limit of the m -th derivative at $x = 0$:

$$I_2 = \lim_{x \rightarrow 0} \frac{d^m}{dx^m} \frac{\ln(1 + x) - x}{x^2}. \quad (3)$$

Here, we can write

$$\begin{aligned} \frac{\ln(1 + x) - x}{x^2} &= \frac{1}{x^2} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - x \right] = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} x^{n-2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} x^n \\ &= \sum_{n=0}^{m-1} \frac{(-1)^{n+1}}{n+2} x^n + \frac{(-1)^{m+1}}{m+2} x^m + \sum_{n=m+1}^{\infty} \frac{(-1)^{n+1}}{n+2} x^n. \end{aligned} \quad (4)$$

The last step is mostly illustrative for this:

$$\frac{d^m}{dx^m} \frac{\ln(1 + x) - x}{x^2} = \frac{(-1)^{m+1} m!}{m+2} + \mathcal{O}(x). \quad (5)$$

Therefore, taking $x \rightarrow 0$ gives

$$I_2 = (-1)^{m+1} \frac{m!}{m+2}. \quad (6)$$

Problem 3)

Calculate the following limit at $x = y = 0$:

$$I_3 = \lim_{x \rightarrow 0, y \rightarrow 0} \nabla^2 [e^{-ax^2 - by^2} \cos ax \cos by]. \quad (7)$$

We can rewrite the operand of the laplacian as

$$e^{-ax^2-by^2} \cos ax \cos by = [e^{-ax^2} \cos ax][e^{-by^2} \cos by], \quad (8)$$

which gives

$$\nabla^2 e^{-ax^2-by^2} \cos ax \cos by = e^{-by^2} \cos by \frac{\partial^2}{\partial x^2} e^{-ax^2} \cos ax + e^{-ax^2} \cos ax \frac{\partial^2}{\partial y^2} e^{-by^2} \cos by. \quad (9)$$

Note that we can evaluate the limit of the first term and use the replacements $a \leftrightarrow b$, $x \leftrightarrow y$.

Observe that

$$\lim_{y \rightarrow 0} e^{-by^2} \cos by = 1. \quad (10)$$

Next, we evaluate the second factor to be

$$\begin{aligned} \frac{d^2}{dx^2} e^{-ax^2} \cos ax &= -2a(1 - 2ax)e^{-ax^2} \cos ax + (-2axe^{-ax^2})(-a \sin ax) \\ &\quad + e^{-ax^2}(-a^2 \cos ax). \end{aligned} \quad (11)$$

It should be clear then that

$$\lim_{x \rightarrow 0} \frac{\partial^2}{\partial x^2} e^{-ax^2} \cos ax = -2a - a^2 = -(a+1)^2 + 1. \quad (12)$$

Hence

$$\lim_{x,y \rightarrow 0} \nabla^2 e^{-ax^2-by^2} \cos ax \cos by = -(a+1)^2 - (b+1)^2 + 2. \quad (13)$$

Problem 4)

Calculate the sum

$$I_1 = \sum_{n=1}^N (n^2 + n + 1). \quad (14)$$

We can use the Euler-Maclaurin formula

$$\sum_{n=a}^N f(n) = \int_a^N f(x) dx + \frac{1}{2}[f(N) + f(a)] + \sum_{n=1}^q \frac{B_{2n}}{(2n)!} [f^{(2n-1)}(N) - f^{(2n-1)}(a)] + R. \quad (15)$$

to compute this series. First, we split the sum into separate terms as

$$I_1 = \sum_{n=0}^N n^2 + \sum_{n=0}^N n + N, \quad (16)$$

where we have gone ahead and evaluated the sum of the last term since it is just 1 added N times. Also note that we have adjusted our indices to start from $n = 0$ since this does not change the sum but should make the right hand side simpler to evaluate since $\frac{d^n}{dx^n} x^m|_{x=0} = 0$ for any $n, m \in \mathbb{N} \cup \{0\}$. The first sum is

$$\begin{aligned} \sum_{n=0}^N n^2 &= \int_0^N x^2 dx + \frac{N^2}{2} + \frac{B_2}{2!} \frac{d}{dx} (x^2)|_{x=N} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \\ &= \frac{N}{6} (2N^2 + 3N + 1) = \frac{N(N+1)(2N+1)}{6}. \end{aligned} \quad (17)$$

Moving on to the second sum we find¹

$$\sum_{n=0}^N n = \int_0^N x dx + \frac{N}{2} = \frac{N^2}{2} + \frac{N}{2} = \frac{N(N+1)}{2}. \quad (18)$$

Plugging these intermediate results into Eq. (16), we find

$$\begin{aligned} I_1 &= \frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2} + N \\ &= \frac{N}{6} \left[(N+1)((2N+1)+3) + 6 \right] = \boxed{\frac{N(N^2 + 3N + 5)}{3}}. \end{aligned} \quad (19)$$

Problem 5)

Calculate the sum

$$I_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2}. \quad (20)$$

For this problem we can use the Poisson summation formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \int_0^{\infty} f(x) dx + \frac{f(0)}{2} + 2 \sum_{n=1}^{\infty} \int_0^{\infty} f(x) \cos(2\pi n x) dx. \quad (21)$$

Note here that we can rewrite $(-1)^n = \cos n\pi$. Using this, we find

$$\int_0^{\infty} \frac{\cos kx}{x^2 + a^2} dx = \frac{\pi}{2|a|} e^{-k|a|}. \quad (22)$$

¹As Gauss did in his primary years using a much simpler and elegant rationale.

The sum is then given as

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2} &= \frac{\pi}{2|a|} e^{-\pi|a|} + \frac{1}{a^2} + 2 \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\cos[(2n+1)\pi x] + \cos[(2n-1)\pi x]}{n^2 + a^2} dx \\
 &= \frac{\pi}{2|a|} e^{-\pi|a|} + \frac{1}{2a^2} + 2 \sum_{n=1}^{\infty} \frac{\pi}{2|a|} \left[e^{-(2n+1)\pi|a|} + e^{-(2n-1)\pi|a|} \right] \\
 &= \frac{\pi}{2|a|} + \frac{1}{2a^2} + \frac{\pi}{|a|} \left[e^{-\pi|a|} \sum_{n=1}^{\infty} e^{-2n\pi|a|} + e^{\pi|a|} \sum_{n=1}^{\infty} e^{-2n\pi|a|} \right] \\
 &= \frac{\pi}{2|a|} + \frac{1}{2a^2} + \frac{\pi}{|a|} (2 \cosh \pi|a|) \frac{e^{-2\pi|a|}}{1 - e^{-2\pi|a|}} \\
 &= \frac{1}{2a^2} + \frac{\pi}{2|a|} \left[1 + 2e^{-\pi|a|} \coth \pi|a| \right].
 \end{aligned} \tag{23}$$

At this point, we can subtract the $n = 0$ term from both sides of the previous equation and obtain

$$I_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + a^2} - \frac{1}{a^2} = -\frac{1}{a^2} + \frac{\pi}{2|a|} \left[1 + 2e^{-\pi|a|} \coth \pi|a| \right]. \tag{24}$$

Problem 6)

Calculate the asymptotic series for this integral at $x \gg 1$:

$$I_3 = \int_x^{\infty} e^{-au} \ln u \, du, \tag{25}$$

where $a > 0$.

We can solve this problem using integration by parts $\int f'g \, dx = fg - \int fg' \, dx$ with $f' = e^{-au}$ and $g = \ln u$, which gives $f = -\frac{1}{a}e^{-au}$ and $g' = 1/u$