Problem 1)
Find the Laplace transform of the following function:

$$I(t) = t^n e^{-at}, \quad a > 0 \text{ and even } n. \tag{1}$$

The Laplace transform of a function $f(t)$ is generally given as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t\,, \tag{2}$$

where the values of s are such that the integral is convergent. Thus,

$$\begin{aligned}\mathcal{L}\{I(t)\} &= \int_0^\infty t^n e^{-(a+s)t}\,\mathrm{d}t = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}(a+s)^n}\int_0^\infty e^{-(a+s)t}\,\mathrm{d}t \\ &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}(a+s)^n}\frac{1}{a+s} = \frac{n!}{(a+s)^{n+1}}.\end{aligned} \tag{3}$$

This matches the result we have in our library. That is, $\mathcal{L}\{t^n\} = n!/s^{n+1}$ and for a general function $f(t)$, $\mathcal{L}\{f(t)e^{-at}\} = F(s+a)$. Collecting these results, we end up with the same solution, which is that

$$\mathcal{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}. \tag{4}$$

Problem 2)
Solve the following equation by the Laplace transform

$$\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y = 0, \tag{5}$$

where $y(0) = 0$ and $\dot{y}(0) = v$.

If we take the Laplace transform of the equation[1], we find

$$\mathcal{L}\{\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y\} = [s^2Y(s) - sv] + 2\lambda[sY(s)] + \omega_0^2 Y(s) = 0. \tag{6}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{sv}{s^2 + 2\lambda s + \omega_0^2} = \frac{sv}{(s+\lambda)^2 + (\omega_0^2 - \lambda^2)}. \tag{7}$$

From our "library" we have

$$\mathcal{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2 + b^2} \tag{8}$$

$$\mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2 + b^2}. \tag{9}$$

[1]Strictly speaking, this means that if we have a differential equation $Df = g$, where D is a linear differential operator, then its "Laplace transform" is $\mathcal{L}\{Df\} = G(s)$.

Thus, if we write $a = \lambda$ and $b = \sqrt{\omega_0^2 - \lambda^2}$, then

$$Y(s) = v\left[\frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}\right]. \tag{10}$$

Problem 3)
A unit vector $\hat{\boldsymbol{n}}$ makes angles θ and α with the Cartesian axes z and x, respectively, and a unit vector $\hat{\boldsymbol{n}}'$ makes angles θ' and α' with z and x, respectively. Find $\cos\varphi$, where φ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

Problem 4)
Find a scalar function $\varphi(r)$ of $r = |\vec{\boldsymbol{r}}|$ which satisfies the equation

$$\vec{\boldsymbol{\nabla}} \cdot [\varphi(r)\vec{\boldsymbol{r}}] = 0. \tag{11}$$

Problem 5)
Calculate the following: (1) $\vec{\boldsymbol{\nabla}}\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{b}}]$, (2) $\vec{\boldsymbol{\nabla}}\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{b}}]$, (3) $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}$, (4) $\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})$, (5) $\vec{\boldsymbol{\nabla}}\cdot[\vec{\boldsymbol{r}}\times(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}})]$, where $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are constant vectors.