Problem 1)
Calculate the following limit:

$$I_1 = \lim_{x \to 0} \frac{x^2 \ln(1+x^2)}{x^2 - \sin^2 x}. \tag{1}$$

Upon first glance, we have a limit of an indeterminate form $0/0$. We may use L'Hopital's rule, but the functions are quite messy to differentiate, so we expand both the numerator and denominator in Taylor series about $x = 0$. Note that $\ln(1+x) = x - x^2/2 + \text{calO}(x^3)$ and $\sin^2 x = (1 - \cos 2x)/2 = [(2x)^2/2! + (2x)^4/4! + \mathcal{O}(x^6)]/2$. Thus, the limit

$$\boxed{I_1 = \lim_{x \to 0} \frac{x^4}{x^2 - \frac{1}{2}[(2x)^2/2! - (2x)^4/4!]} = \frac{2(4!)}{2^4} = \frac{24}{8} = 3}. \tag{2}$$

Problem 2)
Calculate the following limit of the $m-$th derivative at $x = 0$:

$$I_2 = \lim_{x \to 0} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{\ln(1+x) - x}{x^2}. \tag{3}$$

Here, we can write

$$\begin{aligned} \frac{\ln(1+x) - x}{x^2} &= \frac{1}{x^2}\Big[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - x\Big] = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} x^{n-2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+2} x^n \\ &= \sum_{n=0}^{m-1} \frac{(-1)^{n+1}}{n+2} x^n + \frac{(-1)^{m+1}}{m+2} x^m + \sum_{n=m+1} \frac{(-1)^{n+1}}{n+2} x^n. \end{aligned} \tag{4}$$

The last step is mostly illustrative for this:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{\ln(1+x) - x}{x^2} = \frac{(-1)^{m+1}\, m!}{m+2} + \mathcal{O}(x). \tag{5}$$

Therefore, taking $x \to 0$ gives

$$\boxed{I_2 = (-1)^{m+1} \frac{m!}{m+2}}. \tag{6}$$

Problem 3)
Calculate the following limit at $x = y = 0$:

$$I_3 = \lim_{x \to 0, y \to 0} \nabla^2 \big[e^{-ax^2 - by^2} \cos ax \cos by\big]. \tag{7}$$

We can rewrite the operand of the laplacian as

$$e^{-ax^2-by^2}\cos ax\cos by = [e^{-ax^2}\cos ax][e^{-by^2}\cos bx], \tag{8}$$

which gives

$$\nabla^2 e^{-ax^2-by^2}\cos ax\cos by = e^{-by^2}\cos by\frac{\partial^2}{\partial x^2}e^{-ax^2}\cos ax + e^{-ax^2}\cos ax\frac{\partial^2}{\partial y^2}e^{-by^2}\cos by. \tag{9}$$

Note that we can evaluate the limit of the first term and use the replacements $a \leftrightarrow b,\ x \leftrightarrow y$.

Observe that

$$\lim_{y\to 0} e^{-by^2}\cos by = 1. \tag{10}$$

Next, we evaluate the second factor to be

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}e^{-ax^2}\cos ax &= -2a(1-2ax)e^{-ax^2}\cos ax + (-2axe^{-ax^2})(-a\sin ax)\\ &\quad + e^{-ax^2}(-a^2\cos ax).\end{aligned} \tag{11}$$

It should be clear then that

$$\lim_{x\to 0}\frac{\partial^2}{\partial x^2}e^{-ax^2}\cos ax = -2a - a^2 = -(a+1)^2 + 1. \tag{12}$$

Hence

$$\boxed{\lim_{x,y\to 0}\nabla^2 e^{-ax^2-by^2}\cos ax\cos by = -(a+1)^2 - (b+1)^2 + 2}. \tag{13}$$

Problem 4)
Calculate the sum

$$I_1 = \sum_{n=1}^{N}(n^2+n+1). \tag{14}$$

Problem 5)
Calculate the sum

$$I_2 = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}. \tag{15}$$

Problem 6)
Calculate the asymptotic series for this integral at $x \gg 1$:

$$I_3 = \int_x^\infty e^{-au} \ln u \, \mathrm{d}u \,, \tag{16}$$

where $a > 0$.

Observe that we can write

$$\begin{aligned} I_3 &= \int_x^\infty e^{-au}(\ln au - \ln a)\,\mathrm{d}u = \int_x^\infty e^{-au} \ln au - \ln a \int_x^\infty e^{-au}\,\mathrm{d}u \\ &= -\frac{1}{a}\int_x^\infty e^{-v} \ln v \,\mathrm{d}v - \frac{\ln a}{a} e^{-ax} \end{aligned} \,. \tag{17}$$

We can solve this problem using integration by parts $\int f'g\,\mathrm{d}x = fg - \int fg'\,\mathrm{d}x$ with $f' = e^{-au}$ and $g = \ln u$, which gives $f = -\frac{1}{a}e^{-au}$ and $g' = 1/u$