

**Problem 1)**

Express the following delta function in terms of delta functions of the variable  $x$ :

$$\delta\left(\frac{\sin x}{x}\right). \quad (1)$$

Recall that we can write

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad (2)$$

where  $f(x)$  has simple roots  $x_i$ <sup>1</sup>. Note that  $f(x) = \sin x/x$  indeed has simple roots since we can write

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n\pi^2}\right). \quad (3)$$

From this formula, it is trivial to glean that  $\sin x/x$  has zeros when  $\sin x = 0$  or  $x = n\pi$  for  $n = \pm 1, \pm 2, \dots$ . Additionally,  $\frac{d}{dx} \sin x/x|_{x=n\pi} = [\cos x/x - \sin x/x^2]_{x=n\pi} = (-1)^n/n\pi$ .

Thus,

$$\delta\left(\frac{\sin x}{x}\right) = \sum_{n=1}^{\infty} n\pi \left[ \delta(x - n\pi) - \delta(x + n\pi) \right]. \quad (4)$$

**Problem 2)**

Calculate

$$I(z) = \Gamma(1+z)\Gamma(1-z) \quad (5)$$

at  $z = 1/4$ .

We can write (**Check this**)

$$\begin{aligned} \Gamma(1+z)\Gamma(1-z) &= z\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z} \\ \Rightarrow \Gamma(1+z)\Gamma(1-z)|_{z=1/4} &= \frac{\pi/4}{\sin \pi/4} = \frac{\pi}{2\sqrt{2}} \end{aligned} \quad (6)$$

<sup>1</sup>Otherwise the  $f'(x_i)$  term vanishes and we have to be a bit more careful with the expansion of  $f(x)$  inside the delta function.

**Problem 3)**

Using the definition of the complete elliptical integrals  $E(m)$  and  $K(m)$ , express the derivative  $\partial E(m)/\partial m$  in terms of  $K(m)$  and  $E(m)$ .

Recall that the complete elliptic integrals of the first and second kind are defined as

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \quad (7)$$

$$E(m) = \int_0^{\pi/2} d\theta \sqrt{1 - m \sin^2 \theta}, \quad (8)$$

respectively. Differentiating  $E(m)$  with respect to  $m$  we have

$$\begin{aligned} \frac{\partial E}{\partial m} &= \frac{\partial}{\partial m} \int_0^{\pi/2} d\theta \sqrt{1 - m \sin^2 \theta} = \int_0^{\pi/2} d\theta \frac{\partial}{\partial m} \sqrt{1 - m \sin^2 \theta} \\ &= -\frac{1}{2} \int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{\sqrt{1 - m \sin^2 \theta}}. \end{aligned} \quad (9)$$

Observe the following:

$$\begin{aligned} K(m) - E(m) &= \int_0^{\pi/2} d\theta \left[ \frac{1}{\sqrt{1 - m \sin^2 \theta}} - \sqrt{1 - m \sin^2 \theta} \right] \\ &= \int_0^{\pi/2} d\theta \frac{1 - (1 - m \sin^2 \theta)}{\sqrt{1 - m \sin^2 \theta}} = m \int_0^{\pi/2} d\theta \frac{\sin^2 \theta}{\sqrt{1 - m \sin^2 \theta}}. \end{aligned} \quad (10)$$

Thus,

$$\boxed{\frac{\partial E}{\partial m} = \frac{K(m) - E(m)}{m}}. \quad (11)$$

**Problem 4)**

Find the values of  $e^{\pm i\pi/2}$ ,  $e^{i\pi n}$ ,  $\ln(-1)$  where  $n = 0, \pm 1, \pm 2, \dots$

We find that

$$e^{\pm i\pi/2} = \cos(\pm\pi/2) + i \sin(\pm\pi/2) = \pm i \quad (12)$$

$$e^{i\pi n} = \cos(\pi n) + i \sin(\pi n) = (-1)^n \quad (13)$$

$$\ln(-1) = \ln(e^{i\pi}) = i\pi. \quad (14)$$

**Problem 5)**

Calculate the following series:

$$I_1 = \sum_{n=0}^{\infty} p^n \sin(qn) \quad \text{and} \quad I_2 = \sum_{n=0}^{\infty} p^n \cos(qn), \quad (15)$$

where  $p$  and  $q$  are real parameters.

HINT: Use the sum of geometric series with complex  $r$ .

Observe the following:

$$I = \sum_{n=0}^{\infty} p^n e^{iqn} = I_2 + iI_1. \quad (16)$$

That is  $I_2 = \operatorname{Re}(I)$  and  $I_1 = \operatorname{Im}(I)$ . We can use the geometric series formula with complex  $r = pe^{iq}$ , giving

$$I = \frac{1}{1 - pe^{iq}} = \frac{1}{(1 - p \cos q) - ip \sin q} = \frac{(1 - p \cos q) + ip \sin q}{(1 - p \cos q)^2 + p^2 \sin^2 q}. \quad (17)$$

Taking real and imaginary parts of  $I$ , we have

$$I_1 = \frac{p \sin q}{1 + p^2 - 2p \cos q} \quad (18)$$

$$I_2 = \frac{1 - p \cos q}{1 + p^2 - 2p \cos q}. \quad (19)$$