Problem 1)
Find the branch points of this function:

$$f(z) = \sqrt{z^2 + 2z - 1}. \tag{1}$$

What branch cuts can make this function single-valued?

We can write

$$f(z) = \sqrt{\left[z - \Big(-1 + \sqrt{2}\Big)\right]\left[z - \Big(-1 - \sqrt{2}\Big)\right]}. \tag{2}$$

Thus, there are branch points at $z = -1 \pm \sqrt{2}$. We can make a couple different branch cuts to make this function single valued such as those shown in Fig. 1. The first is just connecting the two points via a line on the real axis, and the second is the lines $(\infty, -1-\sqrt{2}]$ and $[-1+\sqrt{2}, \infty)$.

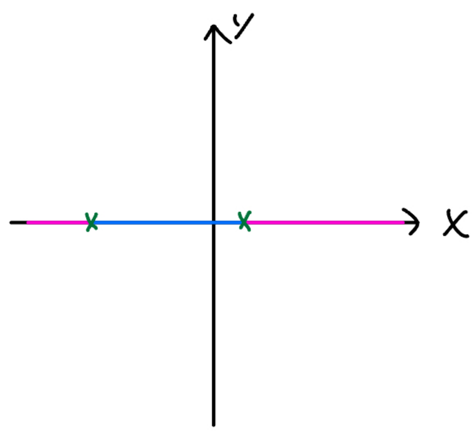

Figure 1: Sketch of branch points of the function $f(z) = \sqrt{z^2+2z-1}$ and some possible branch cuts to make the function single valued. The blue line connects the two points, and the magenta lines extend from the points either to $\pm\infty$, depending on the sign of the branch point.

Problem 2)
Find the residues and all isolated singularities of the function

$$I_2(z) = \tan z. \tag{3}$$

Observe that

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sin z}{\prod_{n=0}^{\infty}\left[1 - \frac{4z^2}{\pi^2(2n+1)^2}\right]}. \tag{4}$$

Clearly, the function becomes singular at $z = \pm(n + 1/2)\pi$ for $n = 0, 1, 2, \ldots$, and furthermore, observe that these are all isolated singularities and simple poles. Notice that in the neighborhood of these poles (let $z_n = (n + 1/2)\pi$ where $n = 0, \pm 1, \pm 2, \ldots$) we can write

$$\begin{aligned}\cos z &= \cos(w + z_n) = \cos w \cos z_n - \sin w \sin z_n = (-1)^{n+1} \sin w \\ &= (-1)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z - z_n)^{2k+1} \\ &= (z - z_n)\left[(-1)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z - z_n)^{2k}\right],\end{aligned} \tag{5}$$

where $w = z - z_n$ and we have expanded around $w = 0$. Thus, we find

$$\operatorname{Res}\{\tan z\}|_{z=z_n} = \frac{\sin z_n}{(-1)^{n+1}} = -1. \tag{6}$$

Alternatively, the "formula" for this kind of situation is that $\operatorname{Res}\{\tan z\}|_{z=z_n} = \sin z_n / [-\sin z_n] = -1$. Interestingly, this result is independent of where the singularity is located.

Problem 3)
Calculate the following real integral using the Cauchy theorem:

$$I_3 = \int_0^{2\pi} \frac{\mathrm{d}x}{2 + \cos^2 x}. \tag{7}$$

Let us write $\cos^2 x = (1 + \cos 2x)/2$ such that

$$\boxed{I_3 = 2\int_0^{2\pi} \frac{\mathrm{d}x}{5 + \cos 2x} = \frac{1}{5}\int_0^{4\pi} \frac{\mathrm{d}y}{1 + \frac{1}{5}\cos y} = \frac{2}{5}\frac{2\pi}{\sqrt{1 - (1/5)^2}} = \sqrt{\frac{2}{3}}\pi}, \tag{8}$$

where we have used the substitution $y = 2x$ and the result

$$\int_0^{2\pi} \frac{\mathrm{d}x}{1 + \epsilon \cos x} = \frac{2\pi}{\sqrt{1 - \epsilon^2}} \tag{9}$$

for $|\epsilon| < 1$. This integral was computed in lecture via the contour integral along the unit circle and the replacement $\cos(x) = (z + z^{-1})/2$, where $z = e^{ix}$. Note that the integration from $0 \to 4\pi$ is double that from $0 \to 2\pi$. This can be seen by splitting the integration into a part from $0 \to 2\pi$ and $2\pi \to 4\pi$. In the second integration, we make a substitution of the form $y' = y - 2\pi$ and observe that $\mathrm{d}y' = \mathrm{d}y$ and $\cos(y' - 2\pi) = \cos y' \cos 2\pi - \sin y' \sin 2\pi = \cos y'$.

Problem 4)
Calculate the following real integral using the Cauchy theorem:

$$I_4 = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4}. \tag{10}$$

Notice that $f(z) = 1/(1+z^4) \approx \frac{1}{|z|^4} \to 0$ as $|z| \to \infty$, and therefore we can write

$$I_4 = \oint_C \frac{\mathrm{d}z}{1+z^4}, \tag{11}$$

where $C = (-\infty, \infty) + \Gamma$, where Γ is the path along semi-circle in either the upper or lower half plane.

$$1 + z^4 = (z^2+i)(z^2-i) = (z - i\sqrt{i})(z + i\sqrt{i})(z - \sqrt{i})(z + \sqrt{i}). \tag{12}$$

Note that $\sqrt{i} = e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = (1+i)/\sqrt{2}$. There are then two roots in the upper half plane ($z = \sqrt{i}, i\sqrt{i}$) and another two in the lower half plane ($z = -\sqrt{i}, -i\sqrt{i}$), so we choose Γ in the upper half plane since it is oriented in a way compatible inherently with Cauchy's theorem (although the contour in the lower half plane just comes in with a minus sign which is not prohibitively more complicated). Thus, the integral

$$\begin{aligned} I_4 &= 2\pi i\left[\frac{1}{(z^2+i)(z+\sqrt{i})}\bigg|_{z=\sqrt{i}} + \frac{1}{(z+i\sqrt{i})(z^2-i)}\bigg|_{z=i\sqrt{i}}\right] \\ &= 2\pi i\left[\frac{1}{(2i)(2\sqrt{i})} + \frac{1}{(2i\sqrt{i})(-2i)}\right] \\ &= \pi\left[\frac{1}{2\sqrt{i}} - \frac{1}{2i\sqrt{i}}\right] \\ &= \pi\left(\frac{-1+i}{2i\sqrt{i}}\right) = \boxed{\frac{\pi}{\sqrt{2}}}. \end{aligned} \tag{13}$$

Problem 5)
Calculate the following real integral using the Cauchy theorem:

$$I_5(a) = \int_0^{\infty} \frac{x \sin ax}{b^2+x^2}\,\mathrm{d}x\,, \tag{14}$$

where $a > 0$.

Let us write

$$I_5 = \frac{1}{2}\,\mathrm{Im}\left\{\int_{-\infty}^{\infty} \frac{xe^{iax}}{b^2+x^2}\,\mathrm{d}x\right\}. \tag{15}$$

Since $f(z) = z/(b^2 + z^2) \to 0$ as $|z| \to \infty$ and $a > 0$, we can write

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{b^2 + x^2}\,\mathrm{d}x = \oint_C \frac{ze^{iaz}}{b^2 + z^2}\,\mathrm{d}z\,, \tag{16}$$

where $C = (-\infty, \infty) + \Gamma$, where Γ is the semi-circle in the upper half-plane, which encloses the simple pole of the integrand $z = ib$. Hence,

$$I_5 = \frac{1}{2}\,\mathrm{Im}\left\{2\pi i \frac{ibe^{ia(ib)}}{2ib}\right\} = \frac{\pi}{2}\,\mathrm{Im}\big\{ie^{-ab}\big\} = \frac{\pi}{2}e^{-ab}. \tag{17}$$

Note that the result does not depend on the sign of b, so we append the absolute value sign to b in our expression and arrive at

$$\boxed{I_5 = \frac{\pi}{2}e^{-a|b|}}\,. \tag{18}$$

We can check our answer as follows. Note that

$$\begin{aligned} I_5 &= \frac{1}{2}\,\mathrm{Im}\left\{-i\frac{\partial}{\partial a}\int_{-\infty}^{\infty} \frac{e^{iax}}{b^2 + x^2}\,\mathrm{d}x\right\} = \frac{1}{2}\,\mathrm{Im}\left\{-i\frac{\partial}{\partial a}\frac{\pi}{|b|}e^{-a|b|}\right\} \\ &= \frac{1}{2}\,\mathrm{Im}\big\{i\pi e^{-a|b|}\big\} = \frac{\pi}{2}e^{-a|b|}, \end{aligned} \tag{19}$$

which is exactly the result quoted above.