Problem 1)
Calculate the following integral:

$$I_1 = \int_0^\infty \frac{x^{1/4}\,\mathrm{d}x}{a^2 + x^2}. \tag{1}$$

Let us make use of the substitution $x = ay$, which allows us to write

$$I_1 = \frac{a^{1/4}a}{a^2} \int_0^\infty \frac{y^{1/4}}{1+y^2}\,\mathrm{d}y = a^{-3/4} \int_0^\infty \frac{y^{1/4}}{1+y^2}\,\mathrm{d}y\,. \tag{2}$$

Next, let us write $y = e^{t/2}$, which gives

$$\boxed{I_1 = \frac{1}{2}a^{-3/4} \int_{-\infty}^\infty \frac{e^{t/8}e^{t/2}}{1+e^t}\,\mathrm{d}t = \frac{1}{2}a^{-3/4} \int_{-\infty}^\infty \frac{e^{5t/8}}{1+e^t}\,\mathrm{d}t = \frac{\pi}{2a^{3/4}\sin(5\pi/8)}}. \tag{3}$$

Problem 2)
Calculate the following integral at $s \gg 1$:

$$I_2 = \int_1^\infty e^{s(x-x^2)}\,\mathrm{d}x\,. \tag{4}$$

Notice that we can write $x^2 - x = (x-1) + (x-1)^2$. If s is sufficiently large, then $\exp[s(x-x^2)] \approx \exp[-s(x-1)]$, meaning

$$\boxed{I_2 \approx e^s \int_1^\infty e^{-sx}\,\mathrm{d}x = \frac{1}{s}}. \tag{5}$$

Problem 3)
Calculate the following integral at $s \gg 1$:

$$I_3 = \int_0^\infty \exp\left(-\frac{s^2}{x} - x\right)\mathrm{d}x\,. \tag{6}$$

Let $f(x;s) = s/x + x/s$. Notice then that $f'(x;s) = -s/x^2 + 1/s$, implying that f has a minimum at $x = s$ in the integration region, allowing us to write

$$f(x;s) = f(s) + \frac{f''(s)}{2}(x-s)^2 + \ldots = 2 + \frac{1}{s^2}(x-s)^2 + \ldots \tag{7}$$

which makes

$$I_3 \approx e^{-2s} \int_0^\infty e^{-(x-s)^2/s} \, \mathrm{d}x \,. \tag{8}$$

This is wrong – the expansion for $f(x;s)$ actually only converges for $|x-s| < s$ – need to think a little bit more about this

Problem 4)
Calculate the following integral at $s \gg 1$:

$$I_4 = \int_0^\infty x^\alpha e^{-sx^2} \, \mathrm{d}x \,, \tag{9}$$

where $\alpha > 0$ is an arbitrary (not necessarily integer) number.

Notice that we can write $x^\alpha = e^{\alpha \ln x}$ such that our integral becomes

$$I_4 = \int_0^\infty e^{-s(x^2 - \alpha \ln x)} \, \mathrm{d}x \,, \tag{10}$$

and if we denote $f(x) = x^2 - \alpha \ln x$, observe that f attains a minimum at $x = \sqrt{\alpha/2}$, which is positive since $\alpha > 0$. Hence, we can write

$$\boxed{\begin{aligned} I_4 &\approx e^{-\frac{s\alpha}{2}[1-\ln(\alpha/2)]} \int_{-\infty}^\infty e^{-2s\left[x-\sqrt{\alpha/2}\right]^2} \, \mathrm{d}x \\ &= e^{-\frac{s\alpha}{2}[1-\ln(\alpha/2)]} \sqrt{\frac{\pi}{2s}} = \left(\frac{\alpha}{2e}\right)^{s\alpha/2} \sqrt{\frac{\pi}{2s}}. \end{aligned}} \tag{11}$$