1) In the toy model introduced in lecture (A, B, and C only)

a) Calculate the total cross section for $A\bar{A} \to A\bar{A}$ for the case $m_A = 0$, $m_C \gg E_A$. Remember there are two diagrams for this process.

The tree level diagrams are shown in the following image, with momenta labeled:

For diagram 1, we have

$$\mathcal{M}_1 = i(-ig)\frac{i}{s - m_C^2}(-ig), \tag{1}$$

where $s = (p_1 + p_2)^2$, and for diagram 2 we have

$$\mathcal{M}_2 = i(-ig)\frac{i}{t - m_C^2}(-ig), \tag{2}$$

where $t = (p_1 - p_3)^2$. The total amplitude is given as

$$\mathcal{M} = \frac{g^2}{s - m_C^2} + \frac{g^2}{t - m_C^2}. \tag{3}$$

We can simplify the Mandelstam variables (s and t) as

$$\begin{aligned} s &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 = \quad 2p_1 \cdot p_2 = 4E_A^2 \\ t &= p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 = -2E_A^2(1 + \cos\theta). \end{aligned} \tag{4}$$

Note that $E_A = E_{\bar{A}} = |\mathbf{p}|$ since $m_A = m_{\bar{A}} = 0$. The center of mass differential cross section is then

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2(E_A + E_{\bar{A}})^2}\frac{|\mathbf{p}'|}{|\mathbf{p}|}|\mathcal{M}|^2 && (5)\\ &= \frac{g^4}{64\pi^2(4E_A^2)}\left[\frac{1}{s - m_C^2} + \frac{1}{t - m_C^2}\right]^2 && (6)\\ &= \frac{g^4}{64\pi^2 E_A^2 m_C^4}\left[1 + \left(\frac{E_A}{m_C}\right)^2 \sin^2(\theta/2)\right], && (7) \end{aligned}$$

where E_A and $E_{\bar{A}}$ are the initial state energies of the corresponding particles and $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final state 3-momenta in the CM frame. Note that the term in square brackets was expanded in a series of powers of E_A/m_C and higher order terms were thrown away. Integrating over the solid angle Ω, we find the total cross section to be

$$\boxed{\sigma = \frac{g^4}{16\pi E_A^2 m_C^4}\left[1 + \frac{1}{2}\left(\frac{E_A}{2m_C}\right)^2\right]}. \tag{8}$$

b) Calculate the total cross section for $AA \to AA$ for the case $m_A = 0$, $m_C \gg E_A$. Remember that there are two diagrams for this process and that the integral should be taken only over half the solid angle or the S factor should be used to avoid double counting. What is the ratio of the cross sections for parts (a) and (b)?

The two possible diagrams are shown below:

As in part (a) we use our toy Feynman rules to find that

$$\mathcal{M}_1 = \frac{g^2}{(p_1 - p_3)^2 - m_C^2} \tag{9}$$

$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}, \tag{10}$$

which gives the total amplitude as

$$\mathcal{M} = g^2\left[\frac{1}{t - m_C^2} + \frac{1}{u - m_C^2}\right] \tag{11}$$

and differential cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2 E_A^2 m_C^4}\left[1 - \left(\frac{2E_A}{m_C}\right)^2\right]. \tag{12}$$

We then obtain the total cross section by integrating over the solid angle:

$$\boxed{\sigma = \frac{g^4}{32\pi E_A^2 m_C^4}\left[1 - \left(\frac{2E_A}{m_C}\right)^2\right]}. \tag{13}$$

c) Calculate the differential cross sections for $A\bar{A} \to A\bar{A}$ and $AA \to AA$ for the case $m_A = m_C = 0$.

Using our results for the amplitudes from parts (a) and (b) and inserting $m_C = 0$ we have

$$\boxed{\frac{\mathrm{d}\sigma(A\bar{A} \to A\bar{A})}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2(4E_A^2)}\left(\frac{1}{s} + \frac{1}{t}\right)^2 = \left(\frac{g^2 \tan^2(\theta/2)}{64\pi E_A^3}\right)^2} \tag{14}$$

and

$$\boxed{\frac{\mathrm{d}\sigma(AA \to AA)}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2(4E_A^2)}\left(\frac{1}{t} + \frac{1}{u}\right)^2 = \frac{1}{2}\left(\frac{g^2}{16\pi E_A^3 \sin^2\theta}\right)^2}. \tag{15}$$

d) What is the ratio of the two cross sections for part (c) at very small angles? What is the ratio for scattering at 90° in the center of mass? At 180°?

The ratio of the two cross sections is as follows:

$$R = \frac{\mathrm{d}\sigma(A\bar{A} \to A\bar{A})}{\mathrm{d}\sigma(AA \to AA)} = \frac{1}{8}\sin^4\theta\tan^4(\theta/2). \tag{16}$$

At small angles, the ratio is

$$R(\theta) = \frac{\theta^8}{128}, \tag{17}$$

which is incredibly small.

Furthermore,

$$R(90^\circ) = \frac{1}{8}, \tag{18}$$

and

$$R(180^\circ) = \infty. \tag{19}$$

2) The Higgs boson has been discovered, cannot decay into two real W or Z pairs because its mass is too low. However, consider another HIggs boson with a sufficiently high mass and spin 0. Its coupling will be proportional to the Higgs mass divided by the W mass, i.e. the vertex factor will be proportional to $-igM_H/M_W$, where g is a dimensionless number. From dimensional analysis, how do you expect the Higgs decay width to vary with the unknown Higgs mass in this very high mass region?

3) Show that $S^\dagger S \neq 1$ but $S^\dagger\gamma^0 S = \gamma^0$.

Recall that

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}, \tag{20}$$

where $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$ and $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz boost factor (in natural units). Thus,

$$\begin{aligned} S^\dagger S &= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 + a_-^2\sigma_1^2 & 2a_+a_-\sigma_1 \\ 2a_+a_-\sigma_1 & a_-^2\sigma_1^2 + a_+^2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\sqrt{\gamma^2-1}\sigma_1 \\ -\sqrt{\gamma^2-1}\sigma_1 & \gamma \end{pmatrix} = \gamma\begin{pmatrix} 1 & -v\sigma_1 \\ -v\sigma_1 & 1 \end{pmatrix} \neq 1 \end{aligned}. \tag{21}$$

Checking the second identity, we have

$$S^\dagger\gamma^0 S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 - a_-^2\sigma_1^2 & 0 \\ 0 & -a_+^2 + a_-^2\sigma_1^2 \end{pmatrix} = \gamma^0. \tag{22}$$

4) Using the plane-wave solutions of the Dirac equation, verify the completeness relation

$$\sum_{s=1,2} u^{(s)}\overline{u}^{(s)} = \not{p} + m, \tag{23}$$

where $\not{p} = \gamma^\mu p_\mu$ and natural units are used.

The spinors

$$u^{(1)} = \sqrt{p^0+m}\begin{pmatrix} \chi_1 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_1 \end{pmatrix} \quad u^{(2)} = \sqrt{p^0+m}\begin{pmatrix} \chi_2 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_2 \end{pmatrix}, \tag{24}$$

where $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For a generic spinor $u^{(s)}$

$$\begin{aligned} u^{(s)}\overline{u}^{(s)} &= (p^0+m)\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s \end{pmatrix}\begin{pmatrix} \chi_s^\dagger & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s^\dagger \end{pmatrix} \\ &= (p^0+m)\begin{pmatrix} \chi_s\chi_s^\dagger & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s\chi_s^\dagger \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s\chi_s^\dagger & -\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{(p^0+m)^2}\chi_s\chi_s^\dagger \end{pmatrix}. \end{aligned} \tag{25}$$

Noting that $\sum_{s=1,2}\chi_s\chi_s^\dagger = 1$, we have

$$\begin{aligned} \sum_{s=1,2} u^{(s)}\overline{u}^{(s)} &= (p^0+m)\begin{pmatrix} 1 & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} & -\frac{\mathbf{p}^2}{(p^0+m)^2} \end{pmatrix} = (p^0+m)\begin{pmatrix} 1 & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} & -\frac{p^0-m}{p^0+m} \end{pmatrix} \\ &= \begin{pmatrix} p^0+m & 0 \\ 0 & -p^0+m \end{pmatrix} + \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix}. \end{aligned} \tag{26}$$

Rearranging gives

$$\boxed{\sum_{s=1,2} u^{(s)}\overline{u}^{(s)} = \gamma_0 p^0 + \boldsymbol{\gamma}\cdot\mathbf{p} + m = \gamma_\mu p^\mu + m = \not{p} + m}. \tag{27}$$

5) In class we calculated the differential cross section for $e^+e^- \to \mu^+\mu^-$ and obtained

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta), \tag{28}$$

where s is the total energy squared in the center of mass, θ is the angle between the e^- and μ^- in the center of mass, and where we ignored the electron and muon masses. Redo the calculuation for $e^+e^- \to \tau^+\tau^-$ for an energy range in which the electron mass can be ignored, but not the τ mass . In particular, what is the factor by which the *total* cross section is suppressed by the τ mass for the case in which the colliding electron and positron each have an energy of 2 GeV.

We worked out the unpolarized squared amplitude for the process 2 leptons $\to$ 2 leptons to be

$$\Big|\sum_{\text{spin}}\mathcal{M}_i\Big|^2 = \frac{g^4}{4s}L^e_{\mu\nu}L^{\mu\nu}_\tau, \tag{29}$$

where the unpolarized leptonic tensor for an arbitrary lepton appears generically as

$$\begin{aligned} L_{\mu\nu} &= \operatorname{Tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right] \\ &= \operatorname{Tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{k}\right] + m\Big[\operatorname{Tr}\left[\gamma^\mu\not{p}\gamma^\nu\right] + \operatorname{Tr}\left[\gamma^\mu\gamma^\nu\not{k}\right]\Big] + m^2\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\right] \\ &= 4(p^\mu k^\nu + k^\mu p^\nu - (p\cdot k - m^2)g^{\mu\nu}), \end{aligned} \tag{30}$$

where p and k are the incoming and outgoing momenta of the lepton leg. In the kinematic regime specified in the problem statement we have that

$$\begin{aligned} \Big|\sum_{\text{spin}}\mathcal{M}_i\Big|^2 =& \frac{4g^4}{s}(p^\mu_{e^-}p^\nu_{e^+} + p^\mu_{e^-}p^\nu_{e^+} - p_{e^-}\cdot p_{e^+}g^{\mu\nu})\times \\ & (p^{\tau^-}_\mu p^{\tau^+}_\nu + p^{\tau^-}_\mu p^{\tau^+}_\nu - (p_{\tau^-}\cdot p_{\tau^+} - m_\tau^2)g^{\mu\nu}) \\ =& \frac{8g^4}{s}\left[(p_{e^-}\cdot p_{\tau^+})(p_{e^+}\cdot p_{\tau^-}) + (p_{e^-}\cdot p_{\tau^-})(p_{e^+}\cdot p_{\tau^+}) - (p_{e^-}\cdot p_{e^+})m_\tau^2\right] \\ =& \frac{8g^4}{s}\left[(EE' + pp'\cos\theta)^2 + (EE' - pp'\cos\theta)^2 - m_\tau^2(E^2+p^2)\right]. \end{aligned} \tag{31}$$

Note that from the conservation of energy, we must have $E = E'$ and $p' = \sqrt{E^2 - m_\tau^2}$, where $p = E$ since we neglect the electron mass. Expanding, cancelling, and simplifying we find

$$\Big|\sum_{\text{spin}}\mathcal{M}_i\Big|^2 = \frac{16(4\pi)^2\alpha^2E^2}{s}(E^2 - m_\tau^2)(1+\cos^2\theta), \tag{32}$$

where we used the fact that the coupling constant $g = e = \sqrt{4\pi\alpha}$, so $g^4 = (4\pi)^2\alpha^2$. Thus, in the center of mass frame, the cross section is

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{4}\frac{1}{64\pi^2(4E^2)}\frac{\sqrt{E^2-m_\tau^2}}{E}\frac{256\pi^2\alpha^2E^2}{s}(E^2-m_\tau^2)(1+\cos^2\theta) \\ &= \boxed{\frac{\alpha^2}{4s}(1+\cos^2\theta)\Big[1 - \Big(\frac{m_\tau}{E}\Big)^2\Big]^{3/2}}. \end{aligned} \tag{33}$$

The suppression factor is the last factor in the cross section.

If we have $E = 2\,\mathrm{GeV}$ then the suppression factor is

$$\Big[1 - \Big(\frac{m_\tau}{E}\Big)^2\Big]^{3/2} = 0.0967, \tag{34}$$

which means that the $e^+e^- \to \tau^+\tau^-$ cross section is only about 10% of the $e^+e^- \to \mu^+\mu^-$ cross section.