1) In the toy model introduced in lecture (A, B, and C only)

a) Calculate the total cross section for $A\bar{A} \to A\bar{A}$ for the case $m_A = 0$, $m_C \gg E_A$. Remember there are two diagrams for this process.

The tree level diagrams are shown in the following image, with momenta labeled:

For diagram 1, we have

$$\mathcal{M}_1 = i(-ig)\frac{i}{s - m_C^2}(-ig), \tag{1}$$

where $s = (p_1 + p_2)^2$, and for diagram 2 we have

$$\mathcal{M}_2 = i(-ig)\frac{i}{t - m_C^2}(-ig), \tag{2}$$

where $t = (p_1 - p_3)^2$. The total amplitude is given as

$$\mathcal{M} = \frac{g^2}{s - m_C^2} + \frac{g^2}{t - m_C^2}. \tag{3}$$

We can simplify the Mandelstam variables (s and t) as

$$\begin{aligned} s &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 = \quad 2p_1 \cdot p_2 = 4E_A^2 \\ t &= p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 = -2E_A^2(1 + \cos\theta). \end{aligned} \tag{4}$$

Note that $E_A = E_{\bar{A}} = |\mathbf{p}|$ since $m_A = m_{\bar{A}} = 0$. The center of mass differential cross section is then

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2(E_A + E_{\bar{A}})^2}\frac{|\mathbf{p}'|}{|\mathbf{p}|}|\mathcal{M}|^2 & (5)\\ &= \frac{g^4}{64\pi^2(4E_A^2)}\left[\frac{1}{s - m_C^2} + \frac{1}{t - m_C^2}\right]^2 & (6)\\ &= \frac{g^4}{64\pi^2 E_A^2 m_C^4}\left[1 + \left(\frac{E_A}{m_C}\right)^2 \sin^2(\theta/2)\right], & (7) \end{aligned}$$

where E_A and $E_{\bar{A}}$ are the initial state energies of the corresponding particles and $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final state 3-momenta in the CM frame. Note that the term in square brackets was expanded in a series of powers of E_A/m_C and higher order terms were thrown away. Integrating over the solid angle Ω, we find the total cross section to be

$$\boxed{\sigma = \frac{g^4}{16\pi E_A^2 m_C^4}\left[1 + \frac{1}{2}\left(\frac{E_A}{2m_C}\right)^2\right]}. \tag{8}$$

b) Calculate the total cross section for $AA \to AA$ for the case $m_A = 0$, $m_C \gg E_A$. Remember that there are two diagrams for this process and that the integral should be taken only over half the solid angle or the S factor should be used to avoid double counting. What is the ratio of the cross sections for parts (a) and (b)?

The two possible diagrams are shown below:

As in part (a) we use our toy Feynman rules to find that

$$\mathcal{M}_1 = \frac{g^2}{(p_1 - p_3)^2 - m_C^2} \tag{9}$$

$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}, \tag{10}$$

which gives the total amplitude as

$$\mathcal{M} = g^2 \left[\frac{1}{t - m_C^2} + \frac{1}{u - m_C^2} \right] \tag{11}$$

and differential cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2 E_A^2 m_C^4} \left[1 - \left(\frac{2E_A}{m_C} \right)^2 \right]. \tag{12}$$

We then obtain the total cross section by integrating over the solid angle:

$$\boxed{\sigma = \frac{g^4}{32\pi E_A^2 m_C^4} \left[1 - \left(\frac{2E_A}{m_C} \right)^2 \right]}. \tag{13}$$

c) Calculate the differential cross sections for $A\bar{A} \to A\bar{A}$ and $AA \to AA$ for the case $m_A = m_C = 0$.

Using our results for the amplitudes from parts (a) and (b) and inserting $m_C = 0$ we have

$$\boxed{\frac{\mathrm{d}\sigma(A\bar{A} \to A\bar{A})}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2(4E_A^2)} \left(\frac{1}{s} + \frac{1}{t} \right)^2 = \left(\frac{g^2 \tan^2(\theta/2)}{64\pi E_A^3} \right)^2} \tag{14}$$

and

$$\boxed{\frac{\mathrm{d}\sigma(AA \to AA)}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2(4E_A^2)} \left(\frac{1}{t} + \frac{1}{u} \right)^2 = \frac{1}{2} \left(\frac{g^2}{16\pi E_A^3 \sin^2\theta} \right)^2}. \tag{15}$$

d) What is the ratio of the two cross sections for part (c) at very small angles? What is the ratio for scattering at 90° in the center of mass? At 180°?

The ratio of the two cross sections is as follows:

$$R = \frac{\mathrm{d}\sigma(A\bar{A} \to A\bar{A})}{\mathrm{d}\sigma(AA \to AA)} = \frac{1}{8}\sin^4\theta\tan^4(\theta/2). \tag{16}$$

At small angles, the ratio is

$$R(\theta) = \frac{\theta^8}{128}, \tag{17}$$

which is incredibly small.

Furthermore,

$$R(90^\circ) = \frac{1}{8}, \tag{18}$$

and

$$R(180^\circ) = \infty. \tag{19}$$

2) The Higgs boson has been discovered, cannot decay into two real W or Z pairs because its mass is too low. However, consider another HIggs boson with a sufficiently high mass and spin 0. Its coupling will be proportional to the Higgs mass divided by the W mass, i.e. the vertex factor will be proportional to $-igM_H/M_W$, where g is a dimensionless number. From dimensional analysis, how do you expect the Higgs decay width to vary with the unknown Higgs mass in this very high mass region?

3) Show that $S^\dagger S \neq 1$ but $S^\dagger\gamma^0 S = \gamma^0$.

Recall that

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}, \tag{20}$$

where $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$ and $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz boost factor (in natural units). Thus,

$$\begin{aligned} S^\dagger S &= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 + a_-^2\sigma_1^2 & 2a_+a_-\sigma_1 \\ 2a_+a_-\sigma_1 & a_-^2\sigma_1^2 + a_+^2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\sqrt{\gamma^2-1}\sigma_1 \\ -\sqrt{\gamma^2-1}\sigma_1 & \gamma \end{pmatrix} = \gamma\begin{pmatrix} 1 & -v\sigma_1 \\ -v\sigma_1 & 1 \end{pmatrix} \neq 1 \end{aligned}. \tag{21}$$

Checking the second identity, we have

$$S^\dagger\gamma^0 S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 - a_-^2\sigma_1^2 & 0 \\ 0 & -a_+^2 + a_-^2\sigma_1^2 \end{pmatrix} = \gamma^0. \tag{22}$$

4) Using the plane-wave solutions of the Dirac equation, verify the completeness relation

$$\sum_{s=1,2} u^{(s)}\overline{u}^{(s)} = \not{p} + m, \tag{23}$$

where $\not{p} = \gamma^\mu p_\mu$ and natural units are used.

The spinors

$$u^{(1)} = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad u^{(2)} = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix}. \tag{24}$$

5) In class we calculated the differential cross section for $e^+e^- \to \mu^+\mu^-$ and obtained

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta), \tag{25}$$

where s is the total energy squared in the center of mass, θ is the angle between the e^- and μ^- in the center of mass, and where we ignored the electron and muon masses. Redo the calculuation for $e^+e^- \to \tau^+\tau^-$ for an energy range in which the electron mass can be ignored, but no the τ mass . In particular, what is the factor by which the *total* cross section is suppressed by the τ mass for the case in which the colliding electron and positron each have an energy of 2 GeV.