1) In the toy model introduced in lecture (A, B, and C only)

a) Calculate the total cross section for $A\bar{A} \to A\bar{A}$ for the case $m_A = 0$, $m_C \gg E_A$. Remember there are two diagrams for this process.

The tree level diagrams are shown in the following image, with momenta labeled:

For diagram 1, we have

$$\mathcal{M}_1 = i(-ig)\frac{i}{s - m_C^2}(-ig), \tag{1}$$

where $s = (p_1 + p_2)^2$, and for diagram 2 we have

$$\mathcal{M}_2 = i(-ig)\frac{i}{t - m_C^2}(-ig), \tag{2}$$

where $t = (p_1 - p_3)^2$. The total amplitude is given as

$$\mathcal{M} = \frac{g^2}{s - m_C^2} + \frac{g^2}{t - m_C^2}. \tag{3}$$

We can simplify the Mandelstam variables (s and t) as

$$\begin{aligned} s &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 = \quad 2p_1 \cdot p_2 = 4E_A^2 \\ t &= p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 = -2E_A^2(1 + \cos\theta). \end{aligned} \tag{4}$$

Note that $E_A = E_{\bar{A}} = |\mathbf{p}|$ since $m_A = m_{\bar{A}} = 0$. The center of mass differential cross section is then

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2(E_A + E_{\bar{A}})^2}\frac{|\mathbf{p}'|}{|\mathbf{p}|}|\mathcal{M}|^2 &(5)\\ &= \frac{g^4}{64\pi^2(4E_A^2)}\Big[\frac{1}{s - m_C^2} + \frac{1}{t - m_C^2}\Big]^2 &(6)\\ &= \frac{g^4}{64\pi^2 E_A^2 m_C^4}\Big[1 + \Big(\frac{E_A}{m_C}\Big)^2 \sin^2(\theta/2)\Big], &(7) \end{aligned}$$

where E_A and $E_{\bar{A}}$ are the initial state energies of the corresponding particles and $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final state 3-momenta in the CM frame. Note that the term in square brackets was expanded in a series of powers of E_A/m_C and higher order terms were thrown away. Integrating over the solid angle Ω, we find the total cross section to be

$$\boxed{\sigma = \frac{g^4}{16\pi E_A^2 m_C^4}\left[1 + \frac{1}{2}\Big(\frac{E_A}{2m_C}\Big)^2\right]}. \tag{8}$$

b) Calculate the total cross section for $AA \to AA$ for the case $m_A = 0$, $m_C \gg E_A$. Remember that there are two diagrams for this process and that the integral should be taken only over half the solid angle or the S factor should be used to avoid double counting. What is the ratio of the cross sections for parts (a) and (b)?

The two possible diagrams are shown below:

As in part (a) we use our toy Feynman rules to find that

$$\mathcal{M}_1 = \frac{g^2}{(p_1 - p_3)^2 - m_C^2} \tag{9}$$

$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}, \tag{10}$$

which gives the total amplitude as

$$\mathcal{M} = g^2 \Big[\frac{1}{t - m_C^2} + \frac{1}{u - m_C^2} \Big] \tag{11}$$

and differential cross section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2 E_A^2 m_C^4} \left[1 - \left(\frac{2E_A}{m_C} \right)^2 \right]. \tag{12}$$

We then obtain the total cross section by integrating over the solid angle:

$$\boxed{\sigma = \frac{g^4}{32\pi E_A^2 m_C^4} \left[1 - \left(\frac{2E_A}{m_C} \right)^2 \right]}. \tag{13}$$

c) Calculate the differential cross sections for $A\bar{A} \to A\bar{A}$ and $AA \to AA$ for the case $m_A = m_C = 0$.

Using our results for the amplitudes from parts (a) and (b) and inserting $m_C = 0$ we have

$$\boxed{\frac{\mathrm{d}\sigma(A\bar{A} \to A\bar{A})}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2 (4E_A^2)} \left(\frac{1}{s} + \frac{1}{t} \right)^2 = \left(\frac{g^2 \tan^2(\theta/2)}{64\pi E_A^3} \right)^2} \tag{14}$$

and

$$\boxed{\frac{\mathrm{d}\sigma(AA \to AA)}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2 (4E_A^2)} \left(\frac{1}{t} + \frac{1}{u} \right)^2 = \frac{1}{2} \left(\frac{g^2}{16\pi E_A^3 \sin^2\theta} \right)^2}. \tag{15}$$

d) What is the ratio of the two cross sections for part (c) at very small angles? What is the ratio for scattering at 90° in the center of mass? At 180°?

The ratio of the two cross sections is as follows:

$$R = \frac{\mathrm{d}\sigma(A\bar{A} \to A\bar{A})}{\mathrm{d}\sigma(AA \to AA)} = \frac{1}{8}\sin^4\theta\tan^4(\theta/2). \tag{16}$$

At small angles, the ratio is

$$R(\theta) = \frac{\theta^8}{128}, \tag{17}$$

which is incredibly small.

Furthermore,

$$R(90°) = \frac{1}{8}, \tag{18}$$

and

$$R(180°) = \infty. \tag{19}$$

2) The Higgs boson has been discovered, cannot decay into two real W or Z pairs because its mass is too low. However, consider another HIggs boson with a sufficiently high mass and spin 0. Its coupling will be proportional to the Higgs mass divided by the W mass, i.e. the vertex factor will be proportional to $-igM_H/M_W$, where g is a dimensionless number. From dimensional analysis, how do you expect the Higgs decay width to vary with the unknown Higgs mass in this very high mass region?

The amplitude should have dimensions of momentum, and since the coupling parameter g is dimensionless, we might expect that the amplitude $\mathcal{M} \propto M_H^2$, meaning that the squared amplitude $|\mathcal{M}|^2 \propto M_H^4$. Thus, the decay width

$$\Gamma \propto |\mathcal{M}|^2 \propto M_H^4, \tag{20}$$

where the denominator carries a factor of M_W^2 such that the dimensions are correct, which makes some intuitive sense, since this would be quite a heavy particle, meaning that it would be more likely to decay quickly ($\tau 1/\Gamma$).

3) Show that $S^\dagger S \neq 1$ but $S^\dagger\gamma^0 S = \gamma^0$.

Recall that

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}, \tag{21}$$

where $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$ and $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz boost factor (in natural units). Thus,

$$\begin{aligned} S^\dagger S &= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 + a_-^2\sigma_1^2 & 2a_+a_-\sigma_1 \\ 2a_+a_-\sigma_1 & a_-^2\sigma_1^2 + a_+^2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\sqrt{\gamma^2-1}\sigma_1 \\ -\sqrt{\gamma^2-1}\sigma_1 & \gamma \end{pmatrix} = \gamma\begin{pmatrix} 1 & -v\sigma_1 \\ -v\sigma_1 & 1 \end{pmatrix} \neq 1 \end{aligned}. \tag{22}$$

Checking the second identity, we have

$$S^\dagger \gamma^0 S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 - a_-^2\sigma_1^2 & 0 \\ 0 & -a_+^2 + a_-^2\sigma_1^2 \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0. \tag{24}$$

4) Using the plane-wave solutions of the Dirac equation, verify the completeness relation

$$\sum_{s=1,2} u^{(s)}\overline{u}^{(s)} = \not{p} + m, \tag{25}$$

where $\not{p} = \gamma^\mu p_\mu$ and natural units are used.

The spinors

$$u^{(1)} = \sqrt{p^0+m}\begin{pmatrix} \chi_1 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_1 \end{pmatrix} \quad u^{(2)} = \sqrt{p^0+m}\begin{pmatrix} \chi_2 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_2 \end{pmatrix}, \tag{26}$$

where $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For a generic spinor $u^{(s)}$

$$\begin{aligned} u^{(s)}\overline{u}^{(s)} &= (p^0+m)\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s \end{pmatrix}\begin{pmatrix} \chi_s^\dagger & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s^\dagger \end{pmatrix} \\ &= (p^0+m)\begin{pmatrix} \chi_s\chi_s^\dagger & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s\chi_s^\dagger \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\chi_s\chi_s^\dagger & -\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{(p^0+m)^2}\chi_s\chi_s^\dagger \end{pmatrix}. \end{aligned} \tag{27}$$

Noting that $\sum_{s=1,2}\chi_s\chi_s^\dagger = 1$, we have

$$\begin{aligned} \sum_{s=1,2} u^{(s)}\overline{u}^{(s)} &= (p^0+m)\begin{pmatrix} 1 & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} & -\frac{\mathbf{p}^2}{(p^0+m)^2} \end{pmatrix} = (p^0+m)\begin{pmatrix} 1 & -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m} & -\frac{p^0-m}{p^0+m} \end{pmatrix} \\ &= \begin{pmatrix} p^0+m & 0 \\ 0 & -p^0+m \end{pmatrix} + \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix}. \end{aligned} \tag{28}$$

Rearranging gives

$$\boxed{\sum_{s=1,2} u^{(s)}\overline{u}^{(s)} = \gamma_0 p^0 + \boldsymbol{\gamma}\cdot\mathbf{p} + m = \gamma_\mu p^\mu + m = \not{p} + m}. \tag{29}$$

5) In class we calculated the differential cross section for $e^+e^- \to \mu^+\mu^-$ and obtained

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta), \tag{30}$$

where s is the total energy squared in the center of mass, θ is the angle between the e^- and μ^- in the center of mass, and where we ignored the electron and muon masses. Redo the calculuation for $e^+e^- \to \tau^+\tau^-$ for an energy range in which the electron mass can be ignored, but not the τ mass . In particular, what is the factor by which the *total* cross section is suppressed by the τ mass for the case in which the colliding electron and positron each have an energy of 2 GeV.

We worked out the unpolarized squared amplitude for the process 2 leptons → 2 leptons to be

$$\Big|\sum_{\text{spin}} \mathcal{M}_i\Big|^2 = \frac{g^4}{4s} L^e_{\mu\nu} L^{\mu\nu}_\tau, \tag{31}$$

where the unpolarized leptonic tensor for an arbitrary lepton appears generically as

$$\begin{aligned} L_{\mu\nu} &= \mathrm{Tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right] \\ &= \mathrm{Tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{k}\right] + m\Big[\mathrm{Tr}\left[\gamma^\mu\not{p}\gamma^\nu\right] + \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\not{k}\right]\Big] + m^2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] \\ &= 4(p^\mu k^\nu + k^\mu p^\nu - (p\cdot k - m^2)g^{\mu\nu}), \end{aligned} \tag{32}$$

where p and k are the incoming and outgoing momenta of the lepton leg. In the kinematic regime specified in the problem statement we have that

$$\begin{aligned} \Big|\sum_{\text{spin}} \mathcal{M}_i\Big|^2 =& \frac{4g^4}{s}(p^\mu_{e^-}p^\nu_{e^+} + p^\mu_{e^-}p^\nu_{e^+} - p_{e^-}\cdot p_{e^+}g^{\mu\nu})\times \\ & (p^{\tau^-}_\mu p^{\tau^+}_\nu + p^{\tau^-}_\mu p^{\tau^+}_\nu - (p_{\tau^-}\cdot p_{\tau^+} - m_\tau^2)g^{\mu\nu}) \\ =& \frac{8g^4}{s}\left[(p_{e^-}\cdot p_{\tau^+})(p_{e^+}\cdot p_{\tau^-}) + (p_{e^-}\cdot p_{\tau^-})(p_{e^+}\cdot p_{\tau^+}) - (p_{e^-}\cdot p_{e^+})m_\tau^2\right] \\ =& \frac{8g^4}{s}\left[(EE' + pp'\cos\theta)^2 + (EE' - pp'\cos\theta)^2 - m_\tau^2(E^2+p^2)\right]. \end{aligned} \tag{33}$$

Note that from the conservation of energy, we must have $E = E'$ and $p' = \sqrt{E^2 - m_\tau^2}$, where $p = E$ since we neglect the electron mass. Expanding, cancelling, and simplifying we find

$$\Big|\sum_{\text{spin}} \mathcal{M}_i\Big|^2 = \frac{16(4\pi)^2\alpha^2E^2}{s}(E^2 - m_\tau^2)(1+\cos^2\theta), \tag{34}$$

where we used the fact that the coupling constant $g = e = \sqrt{4\pi\alpha}$, so $g^4 = (4\pi)^2\alpha^2$. Thus, in the center of mass frame, the cross section is

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{4}\frac{1}{64\pi^2(4E^2)}\frac{\sqrt{E^2-m_\tau^2}}{E}\frac{256\pi^2\alpha^2E^2}{s}(E^2-m_\tau^2)(1+\cos^2\theta) \\ &= \boxed{\frac{\alpha^2}{4s}(1+\cos^2\theta)\Big[1 - \Big(\frac{m_\tau}{E}\Big)^2\Big]^{3/2}}. \end{aligned} \tag{35}$$

The suppression factor is the last factor in the cross section.

If we have $E = 2\,\mathrm{GeV}$ then the suppression factor is

$$\left[1 - \left(\frac{m_\tau}{E}\right)^2\right]^{3/2} = 0.0967, \tag{36}$$

which means that the $e^+e^- \to \tau^+\tau^-$ cross section is only about 10% of the $e^+e^- \to \mu^+\mu^-$ cross section.