1)

(a) Write the equation for the tangent line to $y = f(x)$ at $x = p$.

→ The equation of a line passing through $p, f(p)$ with slope m is given as

$$y - f(p) = m(x - p)$$

Since the line is tangent to the curve at $x = p$, the slope $m = f'(p)$. Hence,

$$y - f(p) = f'(p)(x - p) \Rightarrow y = f'(p)x + [f(p) - pf'(p)]$$

(b) Solve for the x-intercept of the line in equation (a).

The x-intercept is defined as the value of x such that $y = 0$. We see that

$$0 = f'(p)x + [f(p) - pf'(p)] \Rightarrow x = \frac{pf'(p) - f(p)}{f'(p)} = p - \frac{f(p)}{f'(p)}$$

assuming that $f'(p) \neq 0$.

(c) Write the equation for the line that intersects the curve $y = f(x)$ at $x = p$ and $x = q$.

The slope of the line passing through $(p, f(p))$ and $(q, f(q))$ is

$$m = \frac{f(p) - f(q)}{p - q}$$

so we see

$$y - f(p) = \frac{f(p) - f(q)}{p - q}(x - p) \Rightarrow y = \frac{f(p) - f(q)}{p - q}(x - p) + f(p)$$

(d) Solve for the x-intercept of the line in equation (c).

We can solve for the x-intercept for the equation in part (c) as we did for that in part (b):

$$\begin{aligned}
0 &= \frac{f(p) - f(q)}{p - q}(x - p) + f(p) \\
\frac{f(p) - f(q)}{p - q}(x - p) &= -f(p) \\
x - p &= -f(p)\frac{p - q}{f(p) - f(q)} \\
x &= p - \frac{f(p)(p - q)}{f(p) - f(q)}
\end{aligned}$$

2) Starting with (0,1), perform two iterations of Newton iteration on the following systems of nonlinear equations

$$\begin{cases} 4x_1^2 - x_2^2 = 0 \\ 4x_1x_2^2 - x_1 = 1 \end{cases}$$

It has been shown that for the system

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$$

we can find successive approximations to the root as follows:

$$\vec{x}_{n+1} = \vec{x}_n - J^{-1} f(\vec{x}_n)$$

where

$$\vec{x}_n = \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix} \quad f(\vec{x}_n) = \begin{pmatrix} f_1(x_1^{(n)}, x_2^{(n)}) \\ f_2(x_1^{(n)}, x_2^{(n)}) \end{pmatrix}$$
$$J(x_1, x_2) = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}$$

For our problem $f_1 = 4x_1^2 - x_2^2$ and $f_2 = 4x_1 x_2^2 - x_1 - 1$, so Newton iteration gives

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{pmatrix} = \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix} - \begin{pmatrix} 8x_1^{(n)} & -2x_2^{(n)} \\ 4(x_2^{(n)})^2 - 1 & 8x_1^{(n)} x_2^{(n)} \end{pmatrix}^{-1} \begin{pmatrix} 4(x_1^{(n)})^2 - (x_2^{(n)})^2 \\ 4x_1^{(n)}(x_2^{(n)})^2 - x_1^{(n)} - 1 \end{pmatrix}$$

If we begin with $x_1^{(0)} = 0$ and $x_2^{(0)} = 1$, then

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix}$$

Repeating for a second time we see

$$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 8/3 & -1 \\ 0 & 4/3 \end{pmatrix}^{-1} \begin{pmatrix} 7/36 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix} - \begin{pmatrix} -5/24 \\ -3/4 \end{pmatrix} = \begin{pmatrix} 13/24 \\ 5/4 \end{pmatrix}$$

3) Write a program to compute the square root of m by bisection method when $m = 7,\ 13,\ 17$. The interval $[a, b]$ where the square root is located is $[s, s+1]$, where s is an integer and $s^2 < m < (s+1)^2$. Please find the approximate solution with 7 iteration steps, i.e. find $c_0, c_1, \ldots, c_7$.