**1)** Prove that the following equations have at least one solution in the given intervals.

(a) $x - (\ln x)^3 = 0, \quad [5, 7]$

$\rightarrow$ Notice that $5 - (\ln 5)^3 \approx 0.831 > 0$ and $7 - (\ln 7)^3 \approx -3.68 < 0$. Hence by the Intermediate Value Theorem (IVT), we are guranteed that at least one solution to the above equation exists on the compact interval between 5 and 7 since the function $f(x) = x - (\ln x)^3$ is continuous on $\mathbb{R}$.

(b) $5x\cos(\pi x) - 2x^2 + 3 = 0, \quad [0, 2]$

$\rightarrow$ Notice that $5(0)\cos(\pi(0)) - 2(0)^2 + 3 = 3 > 0$ and $5(1)\cos(\pi) - 2(1)^2 + 3 = -5 - 2 + 3 = -4 < 0$. Thus, we a solution to the above equation exists in the interval $[0, 2]$ by the IVT, given that $f(x) = 5x\cos(\pi x) - 2x^2 + 3$ is continuous on $\mathbb{R}$. In fact, we are guranteed a solution exists on $[0, 1]$ and $[0, 2]$, observing that $5(2)\cos(2\pi) - 2(2)^2 + 3 = 10 - 8 + 3 = 5 > 0$.

**2)** Verify that the function $||\cdot||_1$ defined on $\mathbb{R}^n$ by

$$||\vec{x}||_1 = \sum_{i=1}^{n} |x_i|$$

is a norm on $\mathbb{R}^n$.

$\rightarrow$ We show that $||\cdot||_1$ is a norm by proving that it satisfies the following properties. Let $\vec{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

i) Observe that $||\vec{x}||_1 \geq 0$ since for each $i$ we have $|x_i| \geq 0$, meaning the sum $\sum_{i=1}^{n} |x_i| \geq 0$. Furthermore, notice that if $\vec{x} = 0$, then $\sum_{i=1}^{n} |0| = 0$, and if $\sum_{i=1}^{n} |x_i| = 0$, then for each $i$ we must have $|x_i| = 0$, implying that $\vec{x} = 0$.

ii) Let $\alpha \in \mathbb{R}$. Observe that $||\alpha\vec{x}||_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum_{i=1}^{n} |\alpha||x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|||\vec{x}||_1$.

iii) It can be shown that for any two real numbers $x, y$ that the following inequality holds: $|x + y| \leq |x| + |y|$. Hence we have $||\vec{x} + \vec{y}||_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n}(|x_i| + |y_i|) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = ||\vec{x}||_1 + ||\vec{y}||_1$.

Since

**3)** Find $l_1$, $l_2$, and $l_\infty$ norms of the following vectors or matrices.

(a) $x = (2, 1, -3, 4)^T$

$$\begin{aligned} ||x||_1 &= |2| + |1| + |-3| + |4| = 2 + 1 + 3 + 4 = 10 \\ ||x||_2 &= \sqrt{(2)^2 + (1)^2 + (-3)^2 + (4)^2} = \sqrt{4 + 1 + 9 + 16} = \sqrt{30} \\ ||x||_\infty &= \max\{|2|, |1|, |-3|, |4|\} = 4 \end{aligned}$$

(b) $x = (\sin k, \cos k, 2^k)^T$

$$\begin{aligned}
||x||_1 &= |\sin k| + |\cos k| + 2^k \\
||x||_2 &= \sqrt{\sin^2 k + \cos^2 k + (2^k)^2} = \sqrt{1 + 4^k} \\
||x||_\infty &= \max\{|\sin k|, |\cos k|, 2^k\} = 2^k
\end{aligned}$$

(c)

$$\begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
||A||_1 &= \max\{|10| + |0|, |15| + |1|\} = 16 \\
||A||_\infty &= \max\{|10| + |15|, |0| + |1|\} = 25 \\
||A||_2 &= \sqrt{\rho(A^T A)} = \sqrt{\max\{163 \pm 3\sqrt{2941}\}} \\
&= \sqrt{163 + 3\sqrt{2941}} \approx 18.047
\end{aligned}$$

(d)

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned}
||A||_1 &= \max\{|2| + |-1| + |0|, |-1| + |2| + |-1|, |0| + |-1| + |2|\} = 4 \\
||A||_\infty &= 4 \text{ (since the matrix is symmetric about the diagonal)} \\
||A||_2 &= \sqrt{\max\{4, 6 \pm 4\sqrt{2}\}} = \sqrt{6 + 4\sqrt{2}} \approx 11.657
\end{aligned}$$

Note: I used the *sympy* package in python to solve for the eigenvalues of $A^T A$.

**4)** Taylor expand the following function.

(a) $e^x$ around $x = 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O\left(x^8\right)$$

(b) $\log(x+1)$ around $x = 0$

$$\log(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} + O\left(x^8\right)$$

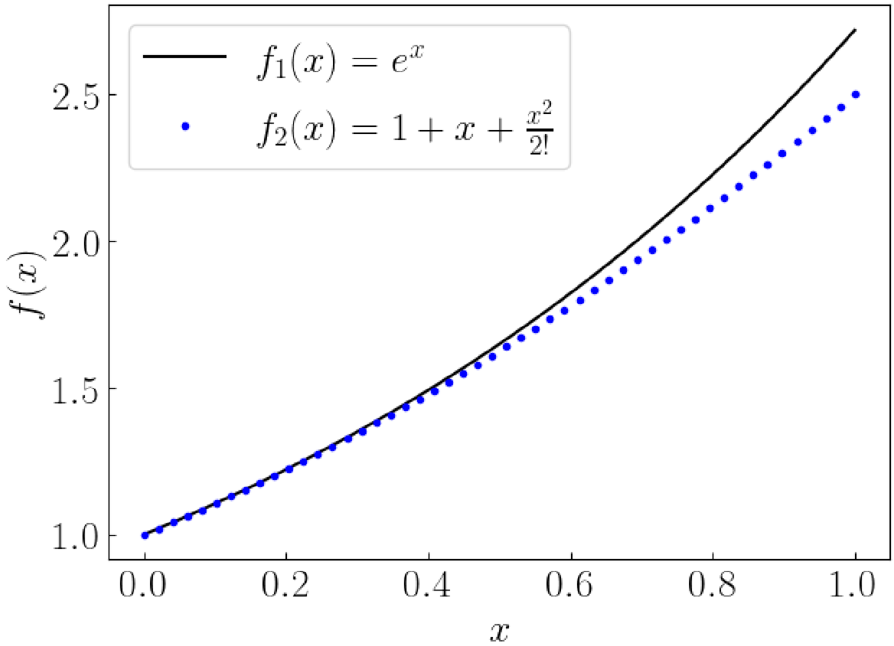
```
#!/usr/bin/env python3

import sympy as sp

x = sp.Symbol('x')
f1 = sp.exp(x)
f2 = sp.log(x + 1)

print(sp.latex(f1.series(x,0,8)))
print()
print(sp.latex(f2.series(x,0,8)))
```

**5)** Plot the function $e^x$ on $[0, 1]$ in a black solid line. On the same graph, plot the function $1 + x + \frac{x^2}{2!}$ in blue circle.




```
#!/usr/bin/env python3

import numpy as np
import matplotlib.pyplot as plt

from matplotlib import rcParams
rcParams['text.latex.preamble'] = r'\usepackage{amsmath}'
rcParams['text.usetex']         = True
rcParams['font.family']         = 'sans-serif'
rcParams['font.sans-serif']     = ['Helvetica']

x = np.linspace(0,1)
f1 = np.exp(x)
f2 = 1 + x + x**2/np.math.factorial(2)

fig, ax = plt.subplots(nrows=1,ncols=1,figsize=(7,5))
ax.plot(x,f1,'k-',label=r'$f_1(x) = e^x$')
ax.plot(x,f2,'b.',label=r'$f_2(x) = 1+x+\frac{x^2}{2!}$')
ax.legend(loc=2,fontsize=20)
```

```
ax.set_xlabel(r'$x$',size=20)
ax.set_ylabel(r'$f(x)$',size=20)
ax.tick_params(axis='both',which='major',labelsize=20,direction='in')

plt.savefig('prob5fig.png',bbox_inches='tight')
```