MATH 551, Spring 2022, Midterm Exam

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Wichita State University Honor Code

" As a Wichita State University student I will conduct myself with honor and integrity at all times. I will not lie, cheat, or steal, nor will I accept the actions of those who do."

Signature: Richard Whitehill

There are 5 questions, with point values noted next to each one. Please show all work.

Problem 1. (10 Points)
(a) Convert the decimal number 19.75 into a binary number;

$$19.75 = 16 + 3.75 = 16 + 2 + 1 + 0.75 = 16 + 2 + 1 + 0.5 + 0.25$$
$$= 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^{-1} + 1 \cdot 2^{-2}$$

$$\boxed{19.75_{10} = 10011.11_2}$$

(b) Convert the following IEEE number into a decimal number.
1 10000000011 0111000000000000000000000000000000000000000000000000

$$-(1 + 2^{-2} + 2^{-3} + 2^{-4}) \cdot 2^{(2^0 + 2^1 + 2^{10}) - 1023}$$

$$-\left(1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) \cdot 2^{(1024+2+1)-1023}$$

$$-\left(\frac{23}{16}\right) \cdot 2^4 = \boxed{-23}$$

Problem 2. (20 Points) For $f(x) = x^3 - x - 2$.
(a) Prove that f has root in the interval $[1, 2]$;
(b) Find the first three approximations of bisection method on interval $[1, 2]$;
(c) How many steps of the bisection method are required to approximate the root within 10^{-13}.

a) Observe that $f(1) = 1^3 - 1 - 2 = -2$ and $f(2) = 2^3 - 2 - 2 = 4$. Thus, since $f(1) < 0$, $f(2) > 0$, and f is continuous on the closed interval $[1,2]$ the intermediate value theorem guarantees that there exists $c \in [1,2]$ such that $f(c) = 0$.

b) From above $f(1) = -2 \neq 0$ and $f(2) = 4 \neq 0$

step 1: $c = \frac{1+2}{2} = \frac{3}{2}$, $f(c) = -\frac{1}{8} \neq 0$

$\rightarrow$ compare: $f(1) f(c) = -2(-\frac{1}{8}) > 0$

$\Rightarrow 1 \rightarrow \frac{3}{2}$ and new interval $r \in [\frac{3}{2}, 2]$

step 2: $c = \frac{3/2 + 2}{2} = \frac{7}{4}$, $f(c) = \frac{103}{64}$

$\rightarrow$ compare $f(\frac{3}{2}) f(c) = -\frac{1}{8}\left(\frac{103}{64}\right) < 0$

$\Rightarrow 2 \rightarrow \frac{7}{4}$ and new interval $r \in [\frac{3}{2}, \frac{7}{4}]$

step 3: $c = \frac{3/2 + 7/4}{2} = \frac{13}{8}$

*first 3 approximations:

step	1	2	3
approx	1.5	1.75	1.625

c) The error after n steps $e_n = \frac{|2-1|}{2^{n+1}} = \frac{1}{2^{n+1}}$. Enforcing $e_n \leq 10^{-13}$, we have $2^{n+1} \geq 10^{13}$ or $n \geq 13 \log_2(10) - 1 \approx 42.2$ or $\boxed{n \geq 43}$.

Problem 3. (30 points) For $x = 1 + \frac{1}{4}\sin(2x)$.

(a) Prove that the fix point iteration will converge on the interval $[0, 2]$.

(b) Estimate the number of iterations needed to obtain approximations accurate to within 10^{-4}.

(c) Starting with $x_0 = 0$, find the first two approximations by fix point iteration.

a) Let $g(x) = 1 + \frac{1}{4}\sin(2x)$. Then $|g'(x)| = |\frac{1}{2}\cos(2x)| = \frac{1}{2}|\cos(2x)|$. Since, $|\cos(\theta)| \le 1$ for all $\theta \in \mathbb{R}$, we have $|g'(x)| \le \frac{1}{2} < 1$ for all $x \in \mathbb{R}$. Now, observe that for all $x \in \mathbb{R}$, $-1 \le \sin(2x) \le 1$ or $\frac{3}{4} \le 1 + \frac{1}{4}\sin(2x) \le \frac{5}{4}$. That is, $g(x) \in [0,2]$ if $x \in [0,2]$. Hence, since $g \in C[0,2]$, if we perform fixed point iteration with $p_0 \in [0,2]$, then the iteration will converge to $p \in [0,2]$ such that $p = g(p)$.

b) The error $|p_n - p| = e_n \le \frac{(1/2)^n}{1-1/2}|g(p_0) - p_0| \le \frac{1}{2^{n-1}}|2-0| = \frac{1}{2^{n-2}}$. Thus, if we desire $e_n \le 10^{-4}$, then $n \geq 4\log_2(10) + 2 \approx 15.3$ or $\boxed{n \geq 16}$.

c) Let $x_0 = 0 \Rightarrow x_1 = g(x_0) = 1 + \frac{1}{4}\sin(2(0)) = 1$

$\Rightarrow x_2 = g(x_1) = 1 + \frac{1}{4}\sin(2) \approx 1.227$

Problem 4. (20 Points) You can use MATLAB to perform Taylor expansion by symbolic computation, for example,

$$\begin{gathered}\textit{syms } x\\ f = 1/(5 + 4{*}\cos(x));\\ T = \textit{taylor}(f, \textit{'Order'}, 8)\end{gathered}$$

can be used to calculate 8-th order Taylor polynomial $T_8(x)$ at $x_0 = 0$ for $f(x) = \frac{1}{5+4\cos(x)}$.
Now, for

$$f(x) = e^{1-x^2},$$

use MATLAB to calculate $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, and $T_5(x)$ at $x_0 = 0$. Plot the function and its Taylor polynomials (up to 5th order) in the same figure. Please use different colors for each curve and add the legend.

The figure and source code are located in the addendum/appendix at the end of this document.

→ some plots overlap; $e^x = 1 + x + \frac{x^2}{2!} + \ldots$

$$e^{1-x^2} = e\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - + \ldots\right)$$

* cutting off at $O(x^3)$ is equivalent to cutting off at $O(x^4)$ for example so $T_2 = T_3$

Problem 5. (20 Points) For $f(x) = \arctan(x)$, find its zeros by implementing Secant method in MATLAB. In particular, please use Newton's method to calculate x_1 for Secant method.

The code and sample output are shown in the addendum/appendix.

→ it is found that

① the code is "antisymmetric": starting at x and -x give results of the same magnitude but opposite sign

② if the initial guess is not close to the root $x=0$, the iteration will not converge since f' is closer to zero from from $x=0$.