1) Let $f(x) = \sqrt{x}$

a) Find the absolute and relative condition numbers of f.

The absolute and relative condition numbers for a function f are given as

$$C(x) = |f'(x)|$$
$$\kappa(x) = \left|\frac{xf'(x)}{f(x)}\right|.$$

Since $f = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$, and

$$C(x) = \frac{1}{2\sqrt{x}}$$
$$\kappa(x) = \frac{1}{2\sqrt{x}}\frac{x}{\sqrt{x}} = \frac{1}{2}.$$

b) Where is f well conditioned in an absolute sense? Where is f well conditioned in a relative sense?

In an absolute sense, we can see that the absolute condition number is inversely proportional to the root of x, so it is not well conditioned near $x = 0$, but it becomes more well conditioned at larger values of x.

In a relative sense, the function is well conditioned everywhere since $\kappa(x)$ is a half everywhere (and at zero the limit approaches a half as well).

c) Suppose $x = 10^{-17}$ is replaced by $\hat{x} = 10^{-16}$. Using the absolute condition number of f, how much of a change is expected in f due to this change in the argument?

Recall $C(x) = \frac{1}{2\sqrt{x}}$, so we expect

$$|\hat{y} - y| = C\big(10^{-17}\big)\big|10^{-17} - 10^{-16}\big| \approx 1.423 \cdot 10^{-8}.$$

2) Lagrange interpolation

a) Determine the Lagrange form of the interpolating polynomial for the data (2,1), (3,3), and (4,5). What is the degree of this interpolating polynomial?

Recall that the i^{th} Lagrange polynomial for a data set with n data points is given as

$$L_i(x) = \prod_{k \neq i}^{n} \frac{x - x_k}{x_i - x_k}.$$

Hence we have the Lagrangian basis for the given data points:

$$\begin{aligned}
L_0(x) &= \frac{(x-3)(x-4)}{(2-3)(2-4)} = \frac{(x-3)(x-4)}{2} \\
L_1(x) &= -(x-2)(x-4) \\
L_2(x) &= \frac{(x-2)(x-3)}{2}.
\end{aligned}$$

From the Lagrangian basis, we can determine an interpolating polynomial as

$$p(x) = \sum_{i=0}^{n} y_i L_i(x).$$

For this problem, we then have

$$\begin{aligned}
p(x) &= 1\left[\frac{(x-3)(x-4)}{2}\right] + 3[-(x-2)(x-4)] + 5\left[\frac{(x-2)(x-3)}{2}\right] \\
&= \frac{1}{2}(x-3)(x-4) - 3(x-2)(x-4) + \frac{5}{2}(x-2)(x-3).
\end{aligned}$$

Observe that the interpolating polynomial is of degree 2.

b) Determine the a Lagrange form of the interpolating polynomial for the data (2,2), (3,3) and (4,5). What is the degree of this interpolating polynomial?

Since the x-values are the same as in part (a), the Lagrangian basis is the same, and the interpolating polynomial is slightly modified:

$$\begin{aligned}
p(x) &= \frac{2}{2}(x-3)(x-4) - 3(x-2)(x-4) + \frac{5}{2}(x-2)(x-3) \\
&= (x-3)(x-4) - 3(x-2)(x-4) + \frac{5}{2}(x-2)(x-3).
\end{aligned}$$

Notice that the degree of the polynomial is of degree 2 as in part (a).

c) Based on your observation, is the interpolating polynomial always of degree N for $N+1$ data points? Explain why.

Yes, the interpolating polynomial is always by construction of degree n when we have $n+1$ data points.