1) Consider the function

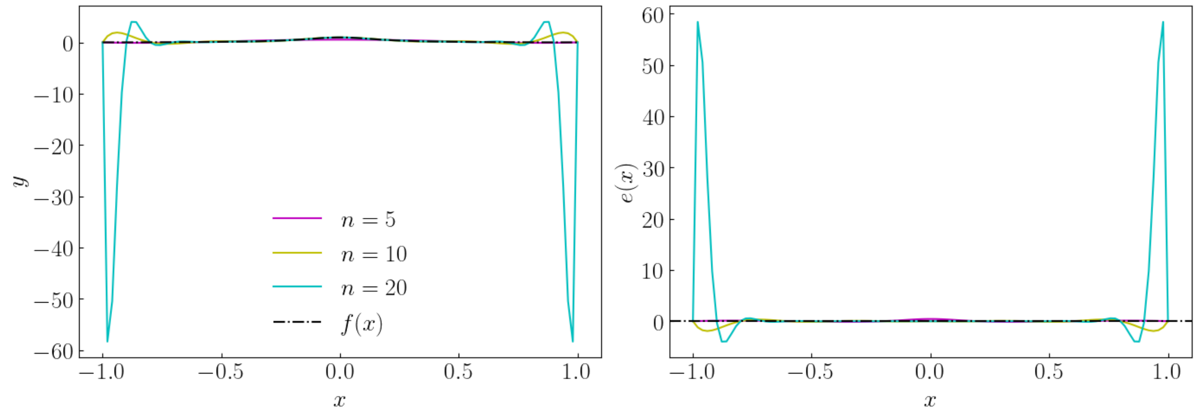
$$f(x) = \frac{1}{1+25x^2}.$$

on the interval $[-1, 1]$.

a) For $n = 5, 10, 20$ plot the error $e(x) = f(x) - p_n(x)$ where the polynomial $p_n(x)$ is computed by interpolating at the $n+1$ equally-spaced nodes over $[-1, 1]$.

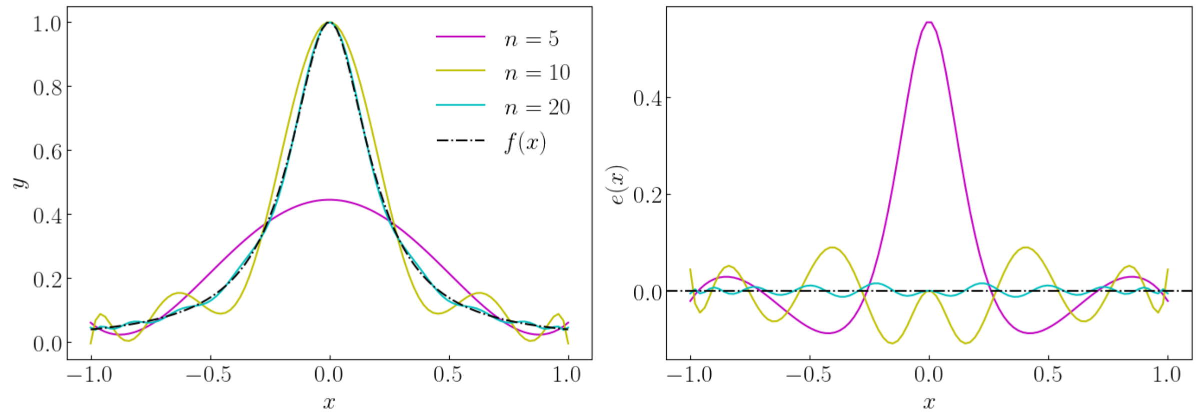
```
#!/usr/bin/env python3

import numpy as np
import matplotlib.pyplot as plt
from matplotlib import rcParams
rcParams['text.latex.preamble'] = r'\usepackage{amsmath}'
rcParams['text.usetex']         = True
rcParams['font.family']         = 'sans-serif'
rcParams['font.sans-serif']     = ['Helvetica']

def Lagrange_poly(t,x,i):
    res = 1.0
    for j in range(len(x)):
        if j == i:
            continue
        else:
            res *= (t-x[j])/(x[i]-x[j])
    return res

def Lagrange_interp(t,x,y):
    res = 0.0
    for i in range(len(x)):
        res += y[i]*Lagrange_poly(t,x,i)
    return res

def gen_plot(f,T,N,save_path,nodes='uniform'):
    F = f(T)

    ls = ['m-','y-','c-']
    fig,ax = plt.subplots(nrows=1,ncols=2,figsize=(7*2,5))
    for n in N:
        if nodes == 'uniform':
            x_data = np.linspace(-1.0,1.0,n+1)
        elif nodes == 'chebyshev':
            x_data = np.array([np.cos( (2*j+1)*np.pi / 2 / (n+1) ) for j in
    range(n+1)])
        y_data = f(x_data)

        interp = np.array([Lagrange_interp(t,x_data,y_data) for t in T])
        error  = F - interp
        ax[0].plot(T,interp,ls[N.index(n)],label=r'$n = %d$'%n)
        ax[1].plot(T,error,ls[N.index(n)])

    ax[0].plot(T,F,'k-.',label='$f(x)$')
    ax[1].axhline(y=0,color='k',linestyle='-.')
    ax[0].set_xlabel(r'$x$',size=20)
    ax[1].set_xlabel(r'$x$',size=20)
    ax[0].set_ylabel(r'$y$',size=20)
```

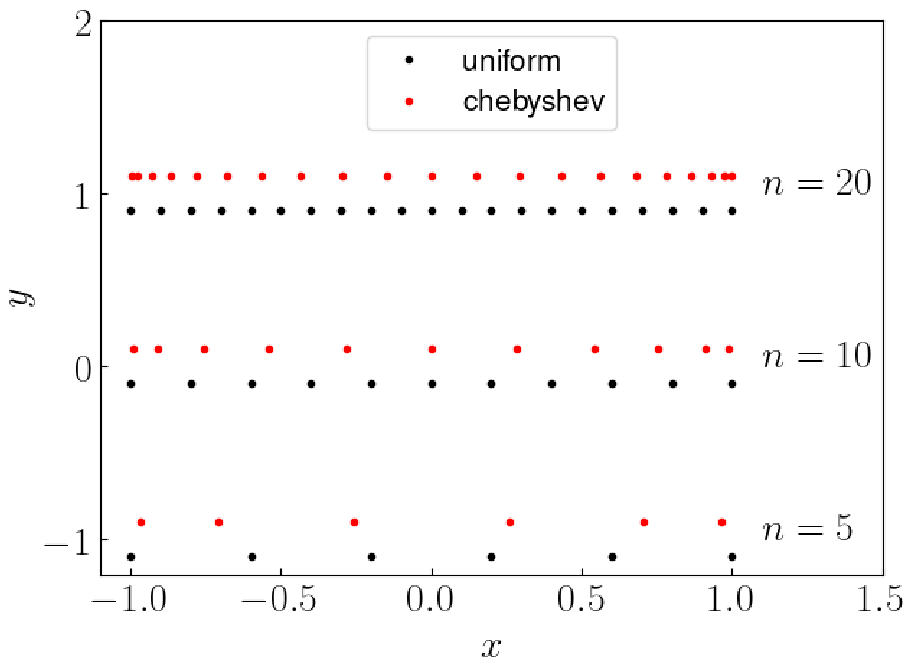
```
    ax[1].set_ylabel(r'$e(x)$',size=20)
    ax[0].tick_params(axis='both',which='major',labelsize=20,direction='in'
    )
    ax[1].tick_params(axis='both',which='major',labelsize=20,direction='in'
    )
    ax[0].legend(fontsize=20,frameon=False)
    plt.tight_layout()
    plt.savefig(save_path,bbox_inches='tight')

if __name__ == '__main__':

    f = lambda x: 1.0/(1.0 + 25.0*x**2.0)
    T = np.linspace(-1.0,1.0,100)
    N  = [5,10,20]

    gen_plot(f,T,N,'fig1.png',nodes='uniform')
    gen_plot(f,T,N,'fig2.png',nodes='chebyshev')

    uniform_nodes   = []
    chebyshev_nodes = []
    for n in N:
            uniform_nodes.append(np.linspace(-1.0,1.0,n+1))
            chebyshev_nodes.append(np.array([np.cos( (2*j+1)*np.pi / 2 / (n
    +1) ) for j in range(n+1)]))

    fix,ax = plt.subplots(nrows=1,ncols=1,figsize=(7,5))
    for i in range(len(N)):
        y = -1*np.ones(N[i] + 1) + i
        y1 = y - 0.1
        y2 = y + 0.1

        ax.plot(uniform_nodes[i],y1,'k.')
        ax.plot(chebyshev_nodes[i],y2,'r.')
    ax.plot([],[],'k.',label='uniform')
    ax.plot([],[],'r.',label='chebyshev')
    for i in range(len(N)):
        ax.text(1.1,-1+i,r'$n=%d$'%N[i],fontsize=20)
    ax.set_xlim(right=1.5)
    ax.set_ylim(top=2.0)
    ax.set_xlabel(r'$x$',size=20)
    ax.set_ylabel(r'$y$',size=20)
    ax.tick_params(axis='both',which='major',labelsize=20,direction='in')
    ax.legend(loc='upper center',fontsize=15)
    plt.savefig('fig3.png',bbox_inches='tight')
```



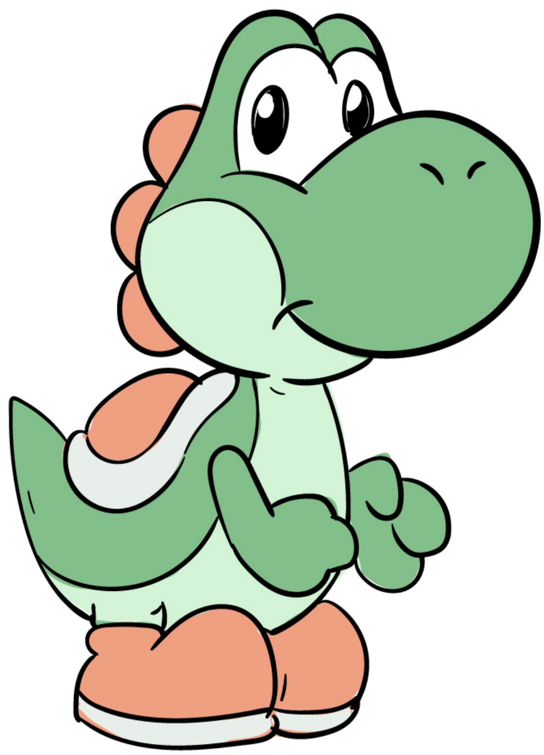
b) Describe what you observe numerically in (a) and Google **Runge's phenomenon** to study the reason.

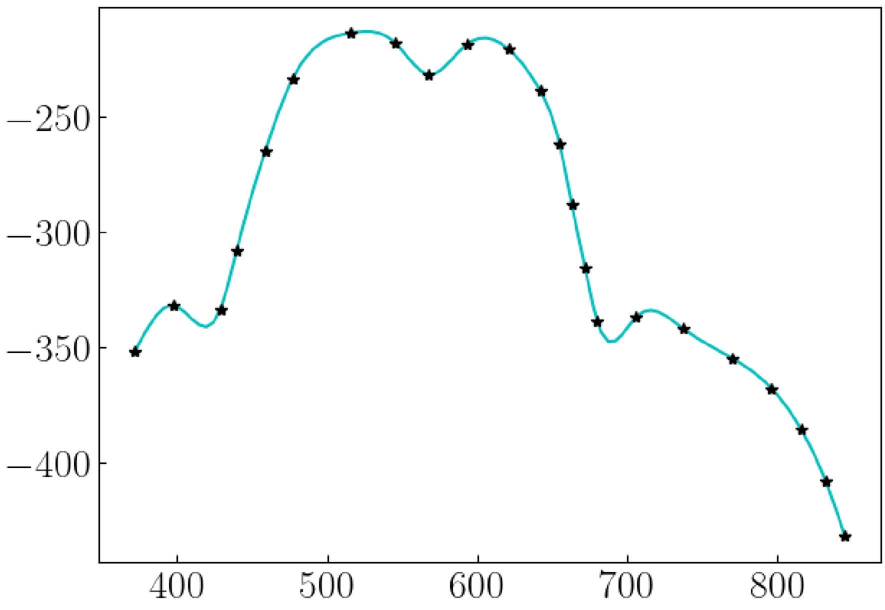
c) Repeat Part (a) but interpolate at the $n + 1$ Chebyshev nodes over $[-1, 1]$. Describe what you observe numerically.





2) Find one picture with some object and reconstruct the shape with cubic spline reconstruction.






```python
#!/usr/bin/env python3

import numpy as np
import matplotlib.pyplot as plt
from matplotlib import rcParams
rcParams['text.latex.preamble'] = r'\usepackage{amsmath}'
rcParams['text.usetex']         = True
rcParams['font.family']         = 'sans-serif'
rcParams['font.sans-serif']     = ['Helvetica']

import csv
from scipy.interpolate import interp1d

def spline(dataX,dataY,t):
    a,b,c,d = spline_coeff(dataX,dataY)
    for i in range(len(dataX)-1):
        if dataX[i] <= t and dataX[i+1] >= t:
            k = i
            break
    xk = dataX[k]
    y = a[k] + b[k]*(t-xk) + c[k]*(t-xk)**2.0 + d[k]*(t-xk)**3.0
    return y

def spline_coeff(dataX,dataY,printA=False):
    n = len(dataX) - 1
    h = np.zeros(n)
    for i in range(n):
        h[i] = dataX[i+1] - dataX[i]
    A = np.zeros([n+1,n+1])
    A[0,0]   = 1.0
    A[-1,-1] = 1.0
    for i in range(1,n):
        A[i,i-1] = h[i-1]
        A[i,i+1] = h[i]
        A[i,i]   = 2.0*(h[i]+h[i-1])
    bb = np.zeros(n+1)
```

```
    for i in range(1,n):
        bb[i] = 3.0/h[i]*(dataY[i+1]-dataY[i]) - 3.0/h[i-1]*(dataY[i]-dataY
   [i-1])
    c = np.linalg.solve(A,bb)
    a = np.copy(dataY)
    b = np.zeros(n)
    d = np.zeros(n)
    for i in range(n):
        b[i] = 1.0/h[i]*(a[i+1]-a[i]) - h[i]/3.0*(2.0*c[i]+c[i+1])
        d[i] = (c[i+1]-c[i])/3.0/h[i]
    a = a[:-1]
    c = c[:-1]
    return a,b,c,d

if __name__ == '__main__':

    file = open('prob5.txt')

    csvreader = csv.reader(file)
    header = []
    header = next(csvreader)

    x_data = []
    y_data = []
    for row in csvreader:
        x_data.append(float(row[0]))
        y_data.append(float(row[1]))
    file.close()

    x_data = np.array(x_data)
    y_data = -1.0 * np.array(y_data)

    T = np.linspace(min(x_data),max(x_data),100)
    reconstruct = np.array([spline(x_data,y_data,t) for t in T])
    f = interp1d(x_data,y_data,kind='cubic')

    a,b,c,d = spline_coeff(x_data,y_data,True)
    #for i in range(len(a)):
    #     print('%.5f  %.5f  %.5f  %.5f'%(a[i],b[i],c[i],d[i]))

    fig,ax = plt.subplots(nrows=1,ncols=1,figsize=(7,5))
    ax.plot(T,reconstruct,'c')
    ax.plot(x_data,y_data,'k*')
    ax.tick_params(axis='both',which='major',labelsize=20,direction='in')
    plt.savefig('./fig4.png',bbox_inches='tight')
```