1) Assume x is an m-dimensional vector, prove $||x||_\infty \leq ||x||_2$ and $||x||_2 \leq \sqrt{m}||x||_\infty$.

Recall that

$$||x||_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2} \tag{1}$$

$$||x||_\infty = \max\{|x_1|, |x_2|, \ldots, |x_m|\}. \tag{2}$$

Thus, for the first inequality we have the following. Suppose that $|x_k| = ||x||_\infty$, then

$$\boxed{||x||_2 = \sqrt{||x||_\infty^2 + \sum_{i \neq k} |x_i|^2} \leq \sqrt{||x||_\infty^2} \leq ||x||_\infty}, \tag{3}$$

since $|x_i| \geq 0$ for all i.

For the second inequality, we know that $|x_i| \leq ||x||_\infty$ for all i. Hence,

$$\boxed{||x||_2 \leq \sqrt{\sum_{i=1}^{m} ||x||_\infty^2} = \sqrt{m}||x||_\infty}. \tag{4}$$

2) Show that if a matrix A is both triangular and unitary, then it is diagonal.

Suppose that $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is an upper triangular matrix we know that $a_{ij} = 0$ if $i > j$. First, we will show that A^{-1}, assuming that A is non-singular, is upper triangular. Let $A^{-1} = [b_1 b_2 \ldots b_n]$, then $A^{-1}A = \mathbb{1}$ and

$$\begin{cases} b_1 a_{11} = e_1 \\ b_2 a_{12} + b_2 a_{22} = e_2 \\ \vdots \\ b_1 a_{1n} + b_2 a_{2n} + \ldots + b_n a_{nn} = e_n, \end{cases} \tag{5}$$

where $(e_i)_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the standard basis. From lectures, we know that we can solve this system using forward substitution as follows:

$$\begin{cases} b_1 = e_1 / a_{11} \\ b_i = (e_i - \sum_{k=1}^{i-1} b_k a_{ki}) / a_{ii}. \end{cases} \tag{6}$$

Observe that

$$(b_1)_j = \frac{1}{a_{11}}\delta_{1j} \tag{7}$$

$$(b_i)_j = \frac{1}{a_{ii}}\delta_{ij} - \sum_{k=1}^{i-1}(b_k)_j\frac{a_{ki}}{a_{ii}}. \tag{8}$$

Using induction, it is easy to see that $b_{ij} = 0$ if $i > j$.

Next, we will show that A^{T} must be lower triangular, which implies that $A^\dagger = A^{*\mathrm{T}}$ is lower triangular. From the definition $(A^{\mathrm{T}})_{ij} = a_{ji}$. Since $a_{ji} = 0$ if $j > i$, then it follows that $(A^{\mathrm{T}})_{ij} = 0$ if $j > i$, which proves that A^{T} is lower triangular.

Now, let A be a unitary. Then, $A^\dagger = A^{-1}$. Hence,

$$(A^\dagger)_{ij} = a^*_{ji} = (A^{-1})_{ij}. \tag{9}$$

Recall that the inverse of A is upper triangular, so

$$a^*_{ji} = 0 \tag{10}$$

if $i > j$. That is, $a_{ji} = 0$ if $j < i$, meaning that A has no off-diagonal entries and that A is diagonal. Furthermore, it is clear now that $|a_{ii}|^2 = 1$ or $|a_{ii}| = 1$, which gives a constraint for the diagonal values of A.

The proof for a lower triangular matrix is simple. Let A be a lower triangular unitary matrix. Then $A^\dagger$ is upper triangular and unitary since $A^\dagger A = \mathbb{1}$, which we showed above must imply that $A^\dagger$ is diagonal. Hence, A must be diagonal as well.

3) Prove that matrix ∞-norm is

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|. \tag{11}$$

4) Let $||\cdot||$ denote any norm on $\mathbb{C}$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq ||A||$, where ρ is the spectral radius of A, which is the largest absolute value of an eigenvalue of A.

5) Find l_1, l_2, and l_∞ norms of the following vectors and matrices, also, please verify your results by MATLAB.

(a) $x = (3, -4, 0, \frac{3}{2})^{\mathrm{T}}$

(b) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

6) Determine SVDs of the following matrices by hand calculation and MATLAB.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$