Numerical Linear Algebra — MATH 751

Midterm Test

Thursday, Oct. 13

Name: Richard Whitehill

WSU ID: h925y477

Wichita State University Honor Code

" As a Wichita State University student I will conduct myself with honor and integrity at all times. I will not lie, cheat, or steal, nor will I accept the actions of those who do."

Signature: Richard Whitehill

There are 7 questions, with point values noted next to each one. Please show all work.

Problem 1. (10 points) Prove that the inverse of a unit lower triangular matrix is unit lower triangular.

Let $L \in \mathbb{R}^{n \times n}$ be a unit (i.e. $\det(L) = 1$ and L is nonsingular) lower triangular matrix. Clearly then L^{-1} is also a unit matrix: $\det(LL^{-1}) = \det(L)\det(L^{-1}) = \det(L^{-1}) = \det(I) = 1$. Now we show that L^{-1} is lower triangular. Let $L^{-1} = [b_1 | b_2 | \cdots | b_n]$.

Then, if $L = (l_{ij})$ $[l_{ij} = 0$ if $i < j]$, we have $L^{-1}L = I$ and the equivalent system of equations

$$\begin{cases} b_1 l_{11} + b_2 l_{21} + \dots + b_n l_{n1} = e_1 \\ \qquad b_2 l_{22} + \dots + b_n l_{n2} = e_2 \\ \qquad\qquad \ddots \\ \qquad\qquad\qquad b_n l_{nn} = e_n \end{cases}$$

where $e_i \in \mathbb{R}^n$ for $i \in \{1, \dots, n\}$ and $(e_i)_j = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

This system has a unique solution, which can be determined from backward substitution:

$$\begin{cases} b_n = \dfrac{e_n}{l_{nn}} \\ b_i = \dfrac{1}{l_{ii}}\left[e_i - \displaystyle\sum_{k=1}^{i-1} b_k l_{ki} \right] \end{cases}$$

Observe that $(b_n)_j = \frac{1}{l_{nn}}(e_n)_j = \frac{1}{l_{nn}}\delta_{nj}$, meaning $(b_n)_j = 0$ unless $j = n$ and $n \geq j$ for all j. By induction, supposing $(b_k)_j = 0$ if $k < j$ for

$k = 1, \dots, i-1$, then

$$(b_i)_j = \frac{1}{l_{ii}}(e_i)_j - \sum_{k=1}^{i-1} (b_k)_j \frac{l_{ki}}{l_{ii}}$$
$$= \frac{1}{l_{ii}}\delta_{ij} - \sum_{k=1}^{i-1} (b_k)_j \frac{l_{ki}}{l_{ii}} = 0$$

if $k<j$. Thus, $(L^{-1})_{ij} = 0$ if $i<j$, implying L^{-1} is unit lower triangular if L is unit lower triangular. $\blacksquare$

Problem 2. (10 points) If A is a nonsingular symmetric matrix and has the factorization $A = LDM^T$, where L and M are unit lower triangular matrices and D is a diagonal matrix, show that $L = M$.

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular, symmetric matrix. That is, A^{-1} exists and $A^T = A$. Furthermore, suppose $A = LDM^T$, where L, M are unit lower triangular matrices and D is a diagonal matrix. Notice that this is also an LU factorization of A since M^T is upper triangular and DM^T is upper triangular as well then, where $U = DM^T$. We have proven already that L and U are unique, and hence, M is unique since $M^T = D^{-1}U$ $\left[D^{-1} = \begin{pmatrix} u_{11} & & 0 \\ & \ddots & \\ 0 & & u_{nn} \end{pmatrix}\right]$. Observe that $A^T = MDL^T = A$, and since L and M are unique $L = M$. $\blacksquare$

Problem 3. (10 points) Let $A \in \mathcal{R}^{m\times n}$, prove that AA^T and A^TA share all non-zero eigenvalues.

Let $A \in \mathbb{R}^{m\times n}$. Then $AA^T \in \mathbb{R}^{m\times m}$ and $A^TA \in \mathbb{R}^{n\times n}$.

We know that every matrix has an SVD decomposition such that $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $V \in \mathbb{R}^{n\times n}$. Furthermore, U and V are orthogonal matrices and $(\Sigma)_{ii} = \sigma_i$, where $\sigma_1 > \sigma_2 > \dots > \sigma_k$ $(k < n, m)$, and $(\Sigma)_{ij} = 0$ if $i \neq j$. Observe that

$$AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$$
$$A^TA = (V\Sigma^T U^T)(U\Sigma V^T) = V\Sigma^T\Sigma V^T$$

Notice that $\Sigma\Sigma^T \in \mathbb{R}^{m\times m}$ and $\Sigma^T\Sigma \in \mathbb{R}^{n\times n}$ are both diagonal, $U^{-1} = U^T$, and $V^{-1} = V^T$. Thus, these are eigendecompositions for AA^T and A^TA. Both therefore have nonzero eigenvalues $\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2$ since $\Sigma\Sigma^T = \begin{pmatrix} \sigma_1^2 & & & & 0 \\ & \ddots & & & \\ & & \sigma_k^2 & & \\ & & & 0 & \\ 0 & & & & \ddots\ 0 \end{pmatrix}$ and $\Sigma^T\Sigma = \begin{pmatrix} \sigma_1^2 & & & & 0 \\ & \ddots & & & \\ & & \sigma_k^2 & & \\ & & & 0 & \\ 0 & & & & \ddots\ 0 \end{pmatrix}$, where $\Sigma\Sigma^T$ and $\Sigma^T\Sigma$ have $m-k$ and $n-k$ zeros on their diagonals respectively. $\blacksquare$

Problem 4. (10 points) Consider the matrix

$$\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

(a) Determine an SVD for matrix A.

(b) Find A^{-1} using the SVD computed in (a).

a) $A = U\Sigma V^T \Rightarrow A^TA = V\Sigma\Sigma^T V^T$

$A^TA = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = 8,\ \lambda_2 = 2 \Rightarrow \Sigma = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

$\Rightarrow V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$u_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Big\}\ U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

b) $A^{-1} = V\Sigma^{-1}U^T = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Problem 5. (10 points) An algorithm is called **backward stable** if $\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ satisfying

$$\frac{||x - \tilde{x}||}{||x||} = O(\epsilon)$$

where ϵ is the machine error, i.e. the algorithm gives exactly the right answer to nearly the right problem. Performing $f(x) = x^2$ on a computer, is this algorithm backward stable?

***Remark:** Note $\tilde{f}(x) = fl(x) \otimes fl(x)$ in floating point arithmeic.

Observe that $\tilde{f}(x) = fl(x) \otimes fl(x) = fl(fl^2(x)) = fl^2(x)(1+\varepsilon)$

$= x^2(1+\varepsilon)^2 = \tilde{x}^2$ if $\tilde{x} = x(1+\varepsilon)$, so $f(x) = x^2$ is

backward stable since $\tilde{f}(x) = f(\tilde{x})$.

Problem 6. (25 points) For linear system

$$Ax = b$$

where A is a band matrix with lower bandwidth b_L and upper bandwidth b_U.

(1) Please write out the pseudo code (or algorithm) to solve the solution.

(2) Calculate the total flops.

1) Algorithm (solve $Ax=b$ for A being a band matrix)

→ input: $n, b_L, b_u, (a_{ij})$

→ output: x

factorization →

For $j = 1:n$

$\quad l_{jj} = 1$

$\quad$ For $i = j+1 : \min(j+1+b_L, n)$

$\qquad l_{ij} = a_{ij}/a_{jj}$ (error if $a_{jj} = 0$)

$\quad$ end

$\quad$ For $k = j+1 : \min(j+1+b_L, n)$

$\qquad$ For $m = j : \min(j+b_u)$

$\qquad\quad a_{km} = a_{km} - a_{jm} l_{kj}$

$\qquad$ end

$\quad$ end

end

forward substitution

$y_1 = b_1 / l_{11}$

For $i = 2:n$

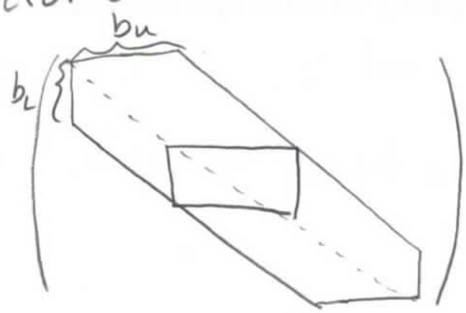
$\quad y_i = \left(b_i - \sum_{k=\max(0,i-b_L)}^{i} y_k l_{ik}\right) / l_{ii}$

end

backward substitution

$$\begin{cases} x_n = y_n / a_{nn} \\ \text{For } i = n-1 : 0 \\ \quad x_i = \left(y_i - \sum_{k=i+1}^{\min(i+1+b_u,\, n)} x_k a_{ik} \right) / a_{ii} \\ \text{end} \end{cases}$$

2) factorization:

$A =$ 

at each row: b_u numbers on top row

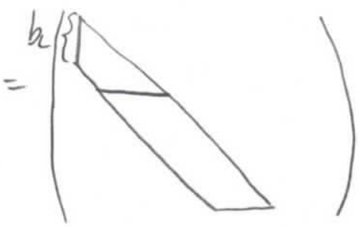
→ operations on b_L rows below

→ each row: flops $= 2 b_u b_L$

* one operation for multiplication
* one operation for subtraction

⇒ n rows: flops $= 2 n b_u b_L$

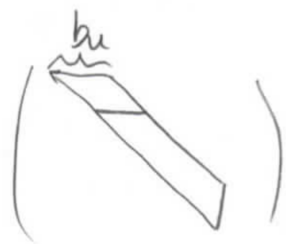
forward substitution:

$L =$ 

at each row: flops $= 2 b_L$

⇒ n rows: flops $= 2 n b_L$

backward substitution:

$U =$ 

at each row: flops $= 2 b_u$

⇒ n rows: flops $= 2 n b_u$

$$\boxed{\text{total flops: } 2 n b_u b_L + 2 n b_u + 2 n b_L}$$

→ note: it was assumed b_u, b_L much smaller than n such that the difference from the top and bottom flops compared to the middle rows is not significant.

Problem 7. (25 points) Write a MATLAB program for solving the solution of the linear system

$$Ax = b$$

where $A \in \mathcal{R}^{500\times 500}$ is tridiagonal matrix.

$$\begin{bmatrix} 4 & -1 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4 \end{bmatrix}$$

and right hand side vector b can be taken as $b_1 = [1, 1, \dots, 1]$ or $b_2 = [1, 2, 3, \dots, 500]$.

(1) Without pivoting, use the regular Gaussian-elimination to solve the linear system, i.e. you need to implement LU factorization, and forward/backward substitution.

(2) Without pivoting, use Gaussian-elimination taking into account the special structure to solve the linear system, i.e. you need to implement the code including LU factorization, and forward/backward substitution, but with smaller loop index.

(3) Directly use MATLAB command inv(A)*b to yield the solution of the linear system.

(4) For the above calculation, list the CPU time.

***Remark**: In MATLAB, you can calculate your CPU time by command *tic* as follows

```
tic
your MATLAB Code
timeElapsed = toc
```

1) $x_1 = [0.366, 0.464, 0.490, 0.497, 0.499, \dots, 0.499, 0.497,$
$0.490, 0.464, 0.366]$

$x_2 = [0.5, 1.0, 1.5, 2.0, 2.5, \dots, 247.654,$
$247.209, 244.181, 231.515, 182.879]$

2) $x_1 = [0.366, 0.464, 0.490, 0.497, 0.499, \dots, 0.499, 0.497,$
$0.490, 0.464, 0.366]$

$x_2 = [0.5, 1.0, 1.5, 2.0, 2.5, \dots, 247.654,$
$247.209, 244.181, 231.515, 182.879]$

3) LU:

$x_1 = [0.366, 0.464, 0.490, 0.497, 0.499, \dots, 0.499, 0.497,$
$0.490, 0.464, 0.366]$

$x_2 = [0.5, 1.0, 1.5, 2.0, 2.5, \dots, 247.654,$
$247.209, 244.181, 231.515, 182.879]$

Band:

$x_1 = [0.366, 0.464, 0.490, 0.497, 0.499, \dots, 0.499, 0.497,$
$0.490, 0.464, 0.366]$

$x_2 = [0.5, 1.0, 1.5, 2.0, 2.5, \dots, 247.654,$
$247.209, 244.181, 231.515, 182.879]$

4) LU:

(1) 20.388 s (2) 20.559 s

Band:

(1) 0.009 s (2) 0.010 s