Numerical Linear Algebra — MATH 751

Final Exam

Friday, December 16

Name: Richard Whitehill

WSU ID: h925y477

Wichita State University Honor Code

" As a Wichita State University student I will conduct myself with honor and integrity at all times. I will not lie, cheat, or steal, nor will I accept the actions of those who do."

Signature: Richard Whitehill

There are 7 questions, with point values noted next to each one. Please show all work.

Problem 1. (10 points) Perform LU factorization with pivoting for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 6 & 8 \end{bmatrix}$$

Please present matrix L, U, and P explicitly.

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 6 & 8 \end{pmatrix} \to \begin{pmatrix} 6 & 6 & 8 \\ 3 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 6 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 6 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Problem 2. (10 points) For the following matrix V, find its 1-norm, 2-norm and ∞−norm.

$$V = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{m \times n} \quad (m = n = 4)$$

$$\|V\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| = \max\{5, 6, 2, 2\} = 6$$

$$\|V\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}| = \max\{5, 6, 2, 2\} = 6$$

$$\|V\|_2 = \sqrt{\rho(V^T V)} = \sqrt{\rho(V^2)} = \sqrt{\rho\begin{pmatrix} 13 & 14 & 0 & 0 \\ 14 & 20 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}} = \sqrt{\tfrac{1}{2}(33 + 7\sqrt{17})} \simeq 5.562$$

* the eigenvalues of $V^T V$ were computed using python

Problem 3. (10 points) Prove that the inverse of a nonsingular block upper triangular matrix is an upper block- triangular matrix.

Let $A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ & & \ddots & \vdots \\ & & & A_{nn} \end{pmatrix}$ be a block upper triangular matrix which is nonsingular and therefore invertible. Observe that we can write $A = D + DU = D(\mathbb{1} + U)$, where $D = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{pmatrix}$ and $U = D^{-1}(A-D) = D^{-1}A - \mathbb{1}$. It is clear that D is upper triangular and U is upper triangular (with block diagonal elements equal to zero). Hence, we can write $A^{-1} = (\mathbb{1}+U)^{-1} D^{-1}$. Recall that $(\mathbb{1}+U)^{-1} = \sum_{n=0}^{\infty} (-U)^n$, which is upper triangular since U^n is just the repeated multiplization of U, which is upper triangular. Hence, $(\mathbb{1}+U)^{-1} D^{-1} = A^{-1}$ is block upper triangular since $(\mathbb{1}+U)^{-1}$ and D^{-1} are block upper triangular and diagonal, respectively.

Problem 4. (10 points) Give matrix

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix}$$

Compute the condition number of the matrix under 2-norm and investigate the convergence of Jacobi iteration and Gaussian-Seidel iteration for

$$Ax = b$$

(i.e., please determine whether Jacobi or Gaussian-Seidel iteration converges or not, or converges with some requirement.)

$$\kappa(A) = \|A^{-1}\|_2 \, \|A\|_2 = \sqrt{\rho(B^TB)}\,\sqrt{\rho(A^TA)} = \sqrt{\frac{9+\sqrt{17}}{32}}\,\sqrt{\frac{9+\sqrt{17}}{2}} = \frac{9+\sqrt{17}}{8}$$

$$\approx 1.640$$

note: $A^{-1} = \frac{1}{-4}\begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/4 \\ 0 & 1/2 \end{pmatrix} = B$

For convergence of a method defined by $Qx_{k+1} = (Q-A)x_k + b$ we must have $\|\mathbb{1} - Q^{-1}A\| < 1$

→ Jacobi: A must be diagonal dominant such that $\|\mathbb{1} - A\| < 1$, which is true here, so Jacobi iteration converges for any choice of b and x_0.

→ Gauss-Seidel: We have $Q = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ so $\mathbb{1} - Q^{-1}A = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}$ and

$$\|\mathbb{1} - Q^{-1}A\| = \begin{cases} 1/2 & \text{for 1-norm} \\ 0 & \text{for 2-norm} \\ 1/2 & \text{for } \infty\text{-norm} \end{cases} < 1.$$

Hence Gauss-Seidel iteration converges for any choice of b or x_0.

Problem 5. (20 points) Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Starting with $v_0 = [a, b]^T$, $a \geq 0$, $b \geq 0$ and a does not equal to b,

1. perform power iteration to calculate v_1, v_2, v_3 and v_4;
2. find v_{2k-1} and v_{2k};
3. does power method find any eigenvalue? If yes, what is the eigenvalue? If no, explain why power method fails.

1) $v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \to v_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \to v_3 = \begin{pmatrix} b \\ a \end{pmatrix} \to v_4 = \begin{pmatrix} a \\ b \end{pmatrix}$

2) Observe that $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$, $A^{-1} = A$

$\Rightarrow v_{2k-1} = A^{2k-1} v_0 = A^{2k} A^{-1} v_0 = A v_0 = \begin{pmatrix} b \\ a \end{pmatrix}$

$\Rightarrow v_{2k} = A^{2k} v_0 = \mathbb{1} v_0 = v_0 = \begin{pmatrix} a \\ b \end{pmatrix}$

3) Power method trivially converges since $\lambda_n = \langle A v_n, v_n \rangle = \langle v_{n+1}, v_n \rangle = \langle v_{2k}, v_{2k-1} \rangle = \langle v_{2k-1}, v_{2k} \rangle = \frac{2ab}{a^2+b^2}$. This is not necessarily an eigenvalue though. For different choices of $a, b \geq 0$, $\lambda_n = \lambda_n(a,b)$. The power method fails because A does not have a dominant eigenvalue. It has $\lambda_\pm = \pm 1$, but $|\lambda_+| = |\lambda_-|$, so it is not true that $\lim_{m \to \infty} A^m v_0 = \lambda v$, where λ is the dominant eigenvalue of A and v the corresponding eigenvector.

Problem 6. (10 points) Find a least-squares solution of the system

$$Ax = b,$$

where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

and $b = [-2, 0, 1]^T$. Use any method you want, either hand calculation or programming.

This is done by programming. We want to find x such that $\|Ax-b\|_2 = \min_v \|Av-b\|_2$, It is found that the least squares vector $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with $\|Ax-b\|_2 = \sqrt{2}$.
The code used to compute x is submitted as a separate file,

Problem 7. (30 points) For the following linear systems,

$$Ax = b$$

where $A \in \mathcal{R}^{500\times 500}$ is tridiagonal matrix.

$$\begin{bmatrix} 4 & -1 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4 \end{bmatrix}$$

and right hand side vector $b = [1, 2, 3, \dots, 500]$.

1. Use the command inv in MATLAB to find the solution (use this as exact solution later);

2. Through LU factorization, perform Gaussian-elimination to find the solution;

3. Use Jacobi method to find the solution (you need to play with the initial guess by yourself and choose ϵ by yourself such that the approximation matches at least 4 digits with exact solution), please indicate the iteration numbers;

4. Program Gauss-Seidel method to solve the linear system (Similar requirement as Jacobi method);

5. Program conjugate gradient method to solve the linear system (Similar requirement as above).

Meanwhile, please find the CPU time of each method for solving the linear system. (Note: this is a summary of the methods for solving linear system, please re-use your previous codes.)

The output is recorded in another file as well as the python script used to complete the computations.