1) Assume x is an m-dimensional vector, prove $||x||_\infty \leq ||x||_2$ and $||x||_2 \leq \sqrt{m}||x||_\infty$.

Recall that

$$||x||_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2} \tag{1}$$

$$||x||_\infty = \max\{|x_1|, |x_2|, \dots, |x_m|\}. \tag{2}$$

Thus, for the first inequality we have the following. Suppose that $|x_k| = ||x||_\infty$, then

$$\boxed{||x||_2 = \sqrt{||x||_\infty^2 + \sum_{i\neq k} |x_i|^2} \leq \sqrt{||x||_\infty^2} \leq ||x||_\infty}, \tag{3}$$

since $|x_i| \geq 0$ for all i.

For the second inequality, we know that $|x_i| \leq ||x||_\infty$ for all i. Hence,

$$\boxed{||x||_2 \leq \sqrt{\sum_{i=1}^{m} ||x||_\infty^2} = \sqrt{m}||x||_\infty}. \tag{4}$$

2) Show that if a matrix A is both triangular and unitary, then it is diagonal.

Suppose that $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is an upper triangular matrix we know that $a_{ij} = 0$ if $i > j$. Furthermore, we know that $AA^\dagger = A^\dagger A = 1$. Using the definition of matrix multiplication, we see that

$$(A^\dagger A)_{ij} = \sum_{k=1}^{n} a_{ki}^* a_{kj} = \delta_{ij}, \tag{5}$$

where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. Since A is upper triangular, it immediately follows that $A^\dagger$ is lower triangular. Hence, we find that $(A^\dagger A)_{11} = 1 = |a_{11}|^2$ or that $|a_{11}| = 1$. Next, we observe that $(A^\dagger A)_{12} = 0 = a_{11}^* a_{12}$, which implies that $a_{12} = 0$. Suppose that it is true that $a_{12}, \dots, a_{1m} = 0$ for some $m < n$, then $(A^\dagger A)_{1,m+1} = 0 = a_{11}^* a_{1,m+1}$, which implies that $a_{1,m+1} = 0$. This argument can be continued for the elements above the diagonal of A for each row. Hence, $a_{ij} = 0$ if $i > j$.

Now suppose that A is lower triangular and unitary. Then, $A^\dagger$ is upper triangular and unitary, which implies that $A^\dagger$ is diagonal. It immediately follows then that A is diagonal.

3) Prove that matrix ∞-norm is

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|. \tag{6}$$

4) Let $||\cdot||$ denote any norm on $\mathbb{C}$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq ||A||$, where ρ is the spectral radius of A, which is the largest absolute value of an eigenvalue of A.

5) Find l_1, l_2, and l_∞ norms of the following vectors and matrices, also, please verify your results by MATLAB.

(a) $x = (3, -4, 0, \frac{3}{2})^{\mathrm{T}}$

(b) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

6) Determine SVDs of the following matrices by hand calculation and MATLAB.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$