1) Assume x is an m-dimensional vector, prove $||x||_\infty \leq ||x||_2$ and $||x||_2 \leq \sqrt{m}||x||_\infty$.

Recall that

$$||x||_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2} \tag{1}$$

$$||x||_\infty = \max\{|x_1|, |x_2|, \ldots, |x_m|\}. \tag{2}$$

Thus, for the first inequality we have the following. Suppose that $|x_k| = ||x||_\infty$, then

$$\boxed{||x||_2 = \sqrt{||x||_\infty^2 + \sum_{i\neq k} |x_i|^2} \leq \sqrt{||x||_\infty^2} \leq ||x||_\infty}, \tag{3}$$

since $|x_i| \geq 0$ for all i.

For the second inequality, we know that $|x_i| \leq ||x||_\infty$ for all i. Hence,

$$\boxed{||x||_2 \leq \sqrt{\sum_{i=1}^{m} ||x||_\infty^2} = \sqrt{m}||x||_\infty}. \tag{4}$$

2) Show that if a matrix A is both triangular and unitary, then it is diagonal.

Suppose that $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is an upper triangular matrix we know that $a_{ij} = 0$ if $i > j$. First, we will show that A^{-1}, assuming that A is non-singular, is upper triangular. Let $A^{-1} = [b_1 b_2 \ldots b_n]$, then $A^{-1}A = \mathbb{1}$ and

$$\begin{cases} b_1 a_{11} = e_1 \\ b_2 a_{12} + b_2 a_{22} = e_2 \\ \vdots \\ b_1 a_{1n} + b_2 a_{2n} + \ldots + b_n a_{nn} = e_n, \end{cases} \tag{5}$$

where $(e_i)_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the standard basis. From lectures, we know that we can solve this system using forward substitution as follows:

$$\begin{cases} b_1 = e_1/a_{11} \\ b_i = (e_i - \sum_{k=1}^{i-1} b_k a_{ki})/a_{ii}. \end{cases} \tag{6}$$

Observe that

$$(b_1)_j = \frac{1}{a_{11}}\delta_{1j} \tag{7}$$

$$(b_i)_j = \frac{1}{a_{ii}}\delta_{ij} - \sum_{k=1}^{i-1}(b_k)_j\frac{a_{ki}}{a_{ii}}. \tag{8}$$

Using induction, it is easy to see that $b_{ij} = 0$ if $i > j$.

Next, we will show that A^{T} must be lower triangular, which implies that $A^{\dagger} = A^{*\mathrm{T}}$ is lower triangular. From the definition $(A^{\mathrm{T}})_{ij} = a_{ji}$. Since $a_{ji} = 0$ if $j > i$, then it follows that $(A^{\mathrm{T}})_{ij} = 0$ if $j > i$, which proves that A^{T} is lower triangular.

Now, let A be a unitary. Then, $A^{\dagger} = A^{-1}$. Hence,

$$(A^{\dagger})_{ij} = a^*_{ji} = (A^{-1})_{ij}. \tag{9}$$

Recall that the inverse of A is upper triangular, so

$$a^*_{ji} = 0 \tag{10}$$

if $i > j$. That is, $a_{ji} = 0$ if $j < i$, meaning that A has no off-diagonal entries and that A is diagonal. Furthermore, it is clear now that $|a_{ii}|^2 = 1$ or $|a_{ii}| = 1$, which gives a constraint for the diagonal values of A.

The proof for a lower triangular matrix is simple. Let A be a lower triangular unitary matrix. Then $A^{\dagger}$ is upper triangular and unitary since $A^{\dagger}A = \mathbb{1}$, which we showed above must imply that $A^{\dagger}$ is diagonal. Hence, A must be diagonal as well.

3) Prove that matrix ∞-norm is

$$||A||_{\infty} = \max_i \sum_{j=1}^{n} |a_{ij}|. \tag{11}$$

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m_1} & \dots & a_{mn} \end{pmatrix}$ Let $v \in \mathbb{C}^n$, then

$$Av = \begin{pmatrix} \sum_{k=1}^{n} a_{1k}v_k \\ \vdots \\ \sum_{k=1}^{n} a_{mk}v_k \end{pmatrix}, \tag{12}$$

and

$$||Av||_{\infty} = \max_i \left| \sum_{k=1}^{n} a_{ik}v_k \right| \leq \max_i \sum_{k=1}^{n} |a_{ik}||v_k|. \tag{13}$$

Furthermore,

$$\frac{||Av||_\infty}{||v||_\infty} \leq \max_i \sum_{k=1}^{n} |a_{ik}| \frac{|v_k|}{||v||_\infty} \leq \max_i \sum_{k=1}^{n} |a_{ik}|, \tag{14}$$

Now, we will show that $||A||_\infty$ attains this maximum for some choice of v. Consider v such that $v_k = \dfrac{|a_{ik}|}{a_{ik}}$ for some $i = 1, \ldots, n$ (unless $a_{ik} = 0$, in which case we choose $v_k = 0$). Then,

$$(Av)_i = \sum_{k=1}^{n} |a_{ik}|. \tag{15}$$

Obviously, we can choose i such that it attains the maximum from Eq. (14), which proves our claim that $||A||_\infty$ is the maximum absolute row sum of A.

4) Let $||\cdot||$ denote any norm on $\mathbb{C}$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq ||A||$, where ρ is the spectral radius of A, which is the largest absolute value of an eigenvalue of A.

Suppose x is an eigenvector of A with a corresponding eigenvalue λ. That is, $Ax = \lambda x$. Further, suppose that $||x|| = 1$, which is a valid choice since if x is an eigenvector of A then $x/||x||$ is also an eigenvector of A with the same eigenvalue. Then $||A|| \geq ||Ax|| = ||\lambda x|| = |\lambda|$, which is true for all eigenvalues λ of A.

5) Find l_1, l_2, and l_∞ norms of the following vectors and matrices, also, please verify your results by MATLAB.

(a) $x = (3, -4, 0, \frac{3}{2})^{\mathrm{T}}$

The hand calculations are as follows:

$$\boxed{\begin{aligned} ||x||_1 &= |3| + |-4| + |0| + \left|\frac{3}{2}\right| = \frac{17}{2} \\ ||x||_2 &= \sqrt{|3|^2 + |-4|^2 + |0|^2 + \left|\frac{3}{2}\right|^2} = \frac{\sqrt{109}}{2} \\ ||x||_\infty &= \max\{|3|, |-4|, |0|, \left|\frac{3}{2}\right|\} = 4. \end{aligned}} \tag{16}$$

The computer calculations are shown below as a screenshot of the code and the output:

```
x = np.array([3,-4,0,3/2])
print('1-norm:   %.2f'%np.linalg.norm(x,ord=1))
print('2-norm:   %.2f'%np.linalg.norm(x,ord=2))
print('inf-norm: %.2f'%np.linalg.norm(x,ord=np.inf))
```

```
1-norm:   8.50
2-norm:   5.22
inf-norm: 4.00
```

(b) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Let the matrix above be denoted by A.

The hand calculations are as follows:

$$\boxed{\begin{aligned} &||A||_1 = \max\{|2| + |-1| + |0|,\ |-1| + |2| + |-1|, |0| + |-1| + |2|\} = 4 \\ &||A||_2 = \sqrt{A^{\mathrm{T}}A} = \sqrt{4\sqrt{2}+6} \\ &||A||_\infty = 4. \end{aligned}} \tag{17}$$

Note that A is symmetric, so its 1 and ∞ norms are the same.

The work for the 2-norm of A is as follows. First, we obtain

$$A^{\mathrm{T}}A = \begin{pmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{pmatrix}. \tag{18}$$

Then we find the eigenvalues by solving the characteristic equation

$$\det(A^{\mathrm{T}}A - \lambda\mathbb{1}) = -\lambda^3 + 16\lambda^2 - 52\lambda + 16 = 0, \tag{19}$$

which has solutions

$$\lambda \in \{4, 6 \pm 4\sqrt{2}\}. \tag{20}$$

The computer calculations are shown below as a screenshot of the code and output:

```
A = np.array([[2,-1,0],[-1,2,-1],[0,-1,2]])
print('1-norm:   %.2f'%np.linalg.norm(A,ord=1))
print('2-norm:   %.2f'%np.linalg.norm(A,ord=2))
print('inf-norm: %.2f'%np.linalg.norm(A,ord=np.inf))
```

```
1-norm:   4.00
2-norm:   3.41
inf-norm: 4.00
```

6) Determine SVDs of the following matrices by hand calculation and MATLAB.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

First, we solve the eigenproblem for $A^{\mathrm{T}}A$, where we denote the matrix above as A.

$$A^{\mathrm{T}}A = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}. \tag{21}$$

Then the characteristic equation is

$$\lambda^2 - 13\lambda + 36 = 0 \Rightarrow \lambda \in \{4, 9\}. \tag{22}$$

Now, we find the eigenvectors by solving $(A^{\mathrm{T}}A - \lambda\mathbb{1})x = 0$ for each eigenvalue x. This is done using the *sympy* package in python:

$$\lambda = 9: \quad x = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}} \tag{23}$$

$$\lambda = 4: \quad x = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}. \tag{24}$$

Hence

$$\boxed{\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad V = \mathbb{1}_{2\times 2}}. \tag{25}$$

Finally, we find the columns of U by solving $u_j = Av_j/\sigma_j$ for $j = 1, 2$:

$$u_1 = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{26}$$

$$u_2 = \frac{1}{2}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & -1 \end{pmatrix} = -1, \tag{27}$$

which means

$$\boxed{U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}. \tag{28}$$

The code and output is shown in the screenshot below:

```python
A = np.array([[3,0],[0,-2]])
U,Sig,Vt = np.linalg.svd(A,full_matrices=True)
```

```python
print('U={} \n\n Sigma={} \n\n V*={}'.format(U,Sig,Vt))
```

```
U=[[1. 0.]
 [0. 1.]]

 Sigma=[3. 2.]

 V*=[[ 1.  0.]
 [-0. -1.]]
```

(b) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

We perform the SVD in the same way as above. First, we diagonalize $A^{\mathrm{T}}A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$. Incidentally, $A^{\mathrm{T}}A$ is already diagonal, so we can read off its eigenvalues and eigenvectors directly: $\lambda \in \{0, 4\}$ with corresponding eigenvectors $\begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$. Now we can write down our matrices for the SVD as follows

$$\boxed{U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}. \tag{29}$$

The code and output is shown in the screenshot below:

```python
A = np.array([[0,2],[0,0],[0,0],[0,0]])
U,Sig,Vt = np.linalg.svd(A,full_matrices=True)
```

```python
print('U={} \n\n Sigma={} \n\n V*={}'.format(U,Sig,Vt))
```

```
U=[[1. 0. 0. 0.]
 [0. 1. 0. 0.]
 [0. 0. 1. 0.]
 [0. 0. 0. 1.]]

 Sigma=[2. 0.]

 V*=[[ 0.  1.]
 [-1.  0.]]
```