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A note on the solutions

The following sections enumerate some problems from previous qualifying exams. Please note that the solutions are only attempts. There is no guarantee of their correctness, and it is up to the reader to critically examine them and decide on the degree of their validity. Furthermore, there may be some typos in the work left unchecked. If there are any obvious mistakes or typos, or if anything in the solutions is unclear, a message can be sent to the following email: rwhit058@odu.edu.

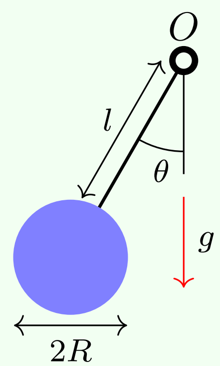
The major sections are divided by the date of the exam, from most to least recent, and the subsections divide the subject type. Note that the problem labels are maintained to correspond with their labels on the exam pdf from which they are copied.

1 May 2023

Classical Mechanics

Problem 1.1)

Write down the Lagrangian and the equation of motion for an "old clock" pendulum consisting of a weightless rod of length l connected to a disk of radius R and mass M and moving in a gravitational field with normal acceleration g. Calculate the period of small oscillations. The moment of inertia of a disk relative to its center of mass is $I = MR^2/2$.



The kinetic energy of an extended body calculated in the center of mass frame with an arbitrary origin is

$$T = \frac{Mv^2}{2} + \frac{I\omega^2}{2}. \tag{1.1}$$

If we place our origin at the center of mass, then the velocity of the body is given as

$$v^2 = \dot{x}^2 + \dot{y}^2 = (l+R)^2\dot{\theta}^2, \tag{1.2}$$

and by construction $\omega = \dot{\theta}$. Hence,

$$T = \frac{M(l+R)^2\dot{\theta}^2}{2} + \frac{MR^2\dot{\theta}^2}{4} = \frac{M[2(l+R)^2 + R^2]\dot{\theta}^2}{4}. \tag{1.3}$$

Note that we could also find this answer by placing our origin at the suspension point and using the parallel axis theorem. In this frame, our origin coincides with the axis of rotation, so all of the kinetic energy is from rotation:

$$T = \frac{I'\omega^2}{2} = \left[\frac{MR^2}{2} + M(l+R)^2\right]\frac{\dot{\theta}^2}{2} = \frac{M[2(l+R)^2 + R^2]\dot{\theta}^2}{4} \tag{1.4}$$

The Lagrangian is then given as

$$\boxed{L = \frac{M[2(l+R)^2 + R^2]\dot{\theta}^2}{4} + mg(l+R)\cos\theta}. \tag{1.5}$$

The equation of motion is given by the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = -Mg(l+R)\sin\theta - M\left[(l+R)^2 + \frac{R^2}{2}\right]\ddot{\theta} = 0 \tag{1.6}$$

$$\Rightarrow \ddot{\theta} + \frac{2(l+R)^2 + R^2}{2g(l+R)}\sin\theta = 0. \tag{1.7}$$

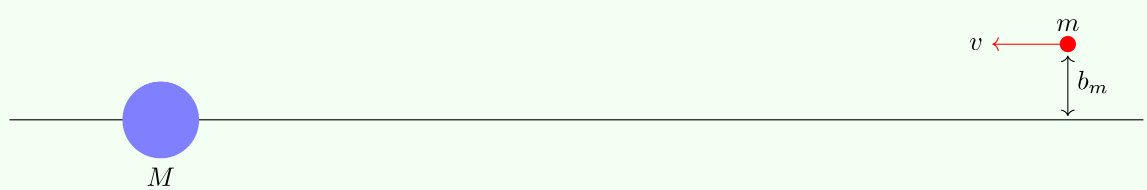
From this we can see the angular frequency of small oscillations is given by

$$\boxed{\omega = \sqrt{\frac{g(l+R)}{(l+R)^2 + R^2/2}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{(l+R)^2 + R^2/2}{g(l+R)}}}. \tag{1.8}$$

Note that in the limit where $l \ll R$, the angular frequency approaches the simple pendulum result of $\omega = \sqrt{g/l}$.

Problem 1.2)

A small asteroid of mass m is moving from infinity with a velocity v toward a planet of mass $M \gg m$ and radius R as shown below. Calculate the maximum impact parameter b_m at which the asteroid with $b < b_m$ would crush onto the planet surface, assuming that the planet does not move.



Since the planet is much more massive than the asteroid, we can treat it as being fixed. The condition for impact is that $r_{\text{min}} < R$, where $U_{\text{eff}}(r_{\text{min}}) = E = mv^2/2 > 0$. Our effective potential is given by

$$U_{\text{eff}} = \frac{M^2}{2mr^2} - \frac{\alpha}{r}, \tag{1.9}$$

where the last term represents a generic attractive $1/r$ potential. The angular momentum of the system is invariant and given by $M = mbv$. Plugging this in, and setting the effective potential equal to the energy of the system, we have

$$\frac{Eb^2}{r_{\text{min}}^2} - \frac{\alpha}{r_{\text{min}}} = E \Rightarrow r_{\text{min}} = \frac{2Eb^2}{\alpha + \sqrt{\alpha^2 + 4E^2b^2}} = \frac{mv^2b^2/\alpha}{1 + \sqrt{1 + m^2v^4b^2/\alpha^2}}. \tag{1.10}$$

The maximum impact parameter at which impact occurs is such that

$$r_{\min}(b_m) = R$$
$$R^2 + \frac{m^2v^4R^2}{\alpha^2}b_m^2 = \left(\frac{mv^2}{\alpha}b_m^2 - R\right)^2 = \left(\frac{mv^2}{\alpha}\right)^2 b_m^4 - \frac{2mv^2R}{\alpha}b_m^2 + R^2$$
$$\boxed{b_m = \sqrt{\frac{R\alpha}{mv^2}\left[2 + \frac{mv^2R}{\alpha}\right]} = R\sqrt{1 + \frac{2GM}{v^2R}}}, \tag{1.11}$$

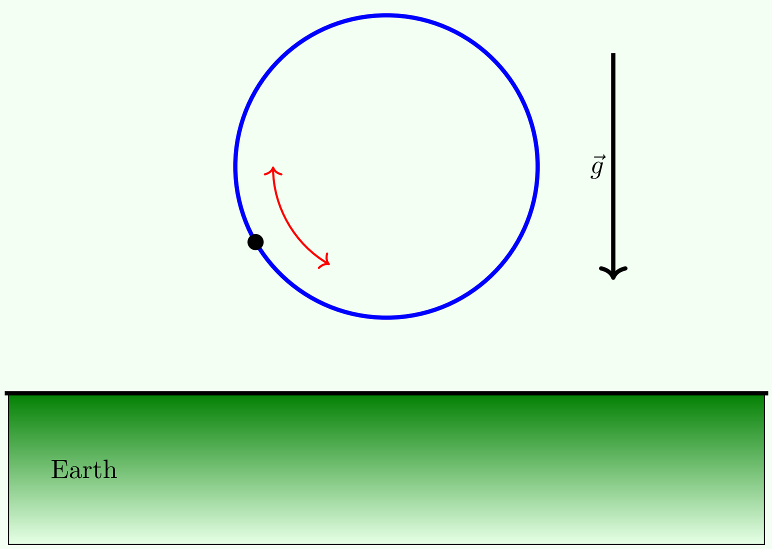
where in the last equality we have inserted $\alpha = GMm$ for gravitational potentials.

Problem 1.3)

A particle is constrained to move without friction along the surface of a sphere that is placed near the surface of the earth, where the gravitational field can be taken to be constant and uniform (see figure below).

(a) Write down the Lagrangian in spherical coordinates

(b) Derive the equations of motion.



(a) Let us call the radius of the sphere R. The position of the particle can be parameterized as

$$\boldsymbol{r} = R[\sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}], \tag{1.12}$$

where we have placed our origin at the center of the sphere and the angles θ and ϕ are defined as usual. The velocity is just the time derivative of this vector:

$$\boldsymbol{v} = R[(\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi)\hat{\boldsymbol{x}} + (\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi)\hat{\boldsymbol{y}} - \dot\theta\sin\theta\hat{\boldsymbol{z}}]. \tag{1.13}$$

Note that R is constant in time. Hence, the kinetic energy of the particle is given by

$$T = \frac{mR^2}{2}\left[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right], \tag{1.14}$$

where m is the mass of the particle. Next, we write down the potential energy of the particle, setting our reference point at the origin:

$$U = mgz = mgR\cos\theta. \tag{1.15}$$

Putting these together, the Lagrangian reads

$$\boxed{L = \frac{mR^2}{2}\left[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right] - mgR\cos\theta} \tag{1.16}$$

(b) From the Lagrangian, the equations of motion are

$$\boxed{\begin{aligned} mR^2\ddot{\theta} + mgR\sin\theta - mR^2\dot{\phi}^2\sin\theta\cos\theta = 0 \\ mR^2\ddot{\phi}\sin^2\theta = 0. \end{aligned}} \tag{1.17}$$

Notice that the second equation is really just a conservation equation. That is, ϕ is a cyclic coordinate, meaning that its conjugate momentum

$$p_\phi = mR^2\dot{\phi}\sin^2\theta \tag{1.18}$$

is a constant of motion. We can use this to rewrite our first equation of motion as

$$\ddot{\theta} + \frac{g}{R}\sin\theta - \frac{p_\phi^2}{mR^2\sin^3\theta\tan\theta} = 0. \tag{1.19}$$

From this, one recognizes the simple pendulum terms on the left and a last term from the rotating plane of oscillation.

Problem 1.4)

Physicists sometimes use the Lennard-Jones 6-12 potential

$$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

to describe the interaction between the atoms in a diatomic molecule, where A and B are constant parameters. Let the atoms in the molecule have masses m_A and m_B, respectively. For small departures from the equilibrium separation r_0, find the angular frequency of oscillations for the diatomic system in terms of A, B, and the masses. For this problem, you may assume that the methods of classical mechanics apply, and that quantum mechanical effects are negligible.

The energy of the system

$$E = \frac{\mu\dot{r}^2}{2} + \frac{M^2}{2\mu r^2} + \frac{A}{r^{12}} - \frac{B}{r^6}, \tag{1.20}$$

where $\mu = m_A m_B/(m_A + m_B)$ is the reduced mass of the system and M is the angular momentum of the system. The last three terms are the effective potential of the system, that is

$$U_{\text{eff}}(r) = \frac{M^2}{2\mu r^2} + \frac{A}{r^{12}} - \frac{B}{r^6}. \tag{1.21}$$

First, we define the equilibrium separation r_0 such that $U'_{\text{eff}}(r_0) = 0$. For small departures from this equilibrium, we can write

$$\begin{aligned} U_{\text{eff}}(r) &= U_{\text{eff}}(r_0) + U'_{\text{eff}}(r_0)(r - r_0) + \frac{U''_{\text{eff}}(r_0)}{2!}(r - r_0)^2 + ... \\ &\approx U_{\text{eff}}(r_0) + \frac{U''_{\text{eff}}(r_0)}{2!}(r - r_0)^2. \end{aligned} \tag{1.22}$$

The equation of motion for such a system is then

$$m\Delta\ddot{r} + U''_{\text{eff}}(r_0)\Delta r = 0, \tag{1.23}$$

where $\Delta r = r - r_0$ obeys the simple harmonic oscillator equation with angular frequency $\omega = \sqrt{U''_{\text{eff}}(r_0)/m}$. Notice the difficulty, though, in solving this problem, in generality, analytically:

$$U'_{\text{eff}}(r_0) = -\frac{M^2}{\mu r_0^3} - \frac{12A}{r_0^{13}} + \frac{6B}{r_0^7} = 0 \tag{1.24}$$

$$U''_{\text{eff}}(r_0) = \frac{3M^2}{\mu r_0^4} + \frac{12(13)A}{r_0^{14}} - \frac{6(7)B}{r_0^8}. \tag{1.25}$$

Solving the former of these for the equilibrium separation requires solving for the roots of a 10^{th} order polynomial, which has no known closed form solution.

For now, we will simplify our lives and assume that $M = 0$. In this case $U_{\text{eff}}(r) = V(r)$, and the equilibrium point r_0 is just

$$r_0 = \left(\frac{2A}{B}\right)^{1/6}. \tag{1.26}$$

The second derivative of the potential at this point is then

$$U''_{\text{eff}}(r_0) = 6\left(\frac{B}{2A}\right)^{1/3}\left[26A\left(\frac{B^2}{4A^2}\right) - 7B\left(\frac{B}{2A}\right)\right] = \frac{18B^2}{A}\left(\frac{B}{2A}\right)^{1/3}, \tag{1.27}$$

and the frequency of small oscillations reads

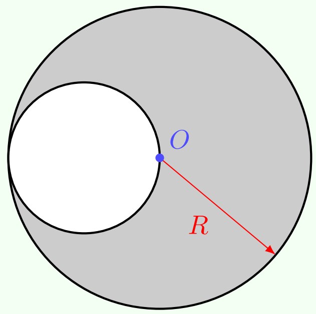
$$\boxed{\omega = 3B\sqrt{\frac{2}{mA}\left(\frac{B}{2A}\right)^{1/3}}} \tag{1.28}$$

Problem 2.1)

A round hole is cut off a homogeneous disk of radius R as shown in the Figure. The mass of the remaining part (it is solid in the figure) equals m. Find the moment of inertia of such a disk with respect to an axis perpendicular to the disk surface and going through:

(a) the point O (see Figure);

(b) its center of mass.



(a) Observe that the elements of the moment of inertia tensor are given by

$$I_{ij} = \int \mathrm{d}^3\boldsymbol{r}\,(r^2\delta_{ij} - x_i x_j)\rho(\boldsymbol{r}). \tag{1.29}$$

Since this is linear in the mass density, we can construct the hole by summing the moments of inertia from a full disk (no hole) centered at O with density $\rho = m/[\pi(R^2 - (R/2)^2)] = 4m/(3\pi R^2)$ (i.e. total mass $m_1 = \rho\pi R^2 = 4m/3$) and a smaller disk (offset by $R/2$ to the left of O) with mass density $-\rho$ (i.e. total mass $m_2 = -\rho\pi(R/2)^2 = -m/3$). That is,

$$\boxed{I = \frac{1}{2}m_1 R^2 + \left[\frac{1}{2}m_2\left(\frac{R}{2}\right)^2 + m_2\left(\frac{R}{2}\right)^2\right] = \frac{13}{24}mR^2}, \tag{1.30}$$

where we have use that for a disk with mass M and radius r, the moment of inertia about an axis perpendicular to its face is $I_{\text{disk}} = Mr^2/2$.

(b) Observe that the center of mass of any body

$$\boldsymbol{R} = \frac{1}{M}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{r}\rho(\boldsymbol{r}), \tag{1.31}$$

so for our body, we can use the same trick as above:

$$R = \frac{m_1 R_1 + m_2 R_2}{m} = -\frac{m_2}{m}\frac{R}{2} = \frac{R}{6}. \tag{1.32}$$

Hence, using the parallel axis theorem and the result above, the moment of inertia of the disk above about its center of mass is

$$\boxed{I_{\rm CM} = \frac{13}{24}mR^2 - m\left(\frac{R}{6}\right)^2 = \frac{37}{72}mR^2}. \tag{1.33}$$

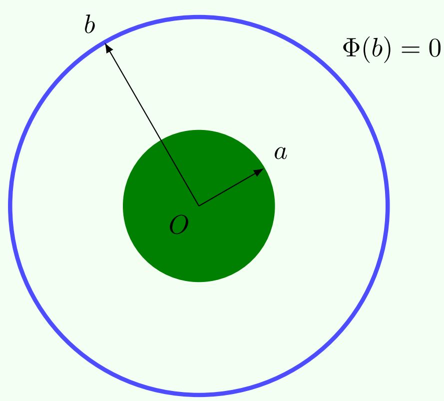
Electricity & Magnetism

Problem 2.2)

A charged sphere of radius a and centered at O has a spherically symmetric charge density $\rho(r)$ that varies radially as $\rho(r) = \alpha r^2$. The total charge of the sphere is Q.

This charged sphere is surrounded by a grounded conducting sphere of radius $b > a$ that is also centered at O (see Figure).

- **(a)** Find electric field $\mathbf{E}(\mathbf{r})$ everywhere in space.
- **(b)** Find the electrostatic potential $\Phi(\mathbf{r})$ everywhere in space.



(a) We can determine the electric field easily from Gauss' law

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{q}{\epsilon_0}, \tag{1.34}$$

which is only practically useful when we have some kine of symmetry – in this case, radial. Note that q is the charge enclosed by the surface S. For this problem, the electric field must be radial, so we choose spherical surfaces:

$$E_r(4\pi r^2) = \frac{4\pi\alpha}{\epsilon_0}\int \mathrm{d}r'\, r'^4 \Theta(r \le a) = \frac{4\pi\alpha}{5\epsilon_0} r_<^5, \tag{1.35}$$

where $r_< = \min(r, a)$. Note that this result only holds for $r < b$. The presence of the conducting sphere at $r = b$ shields the space external to this conductor from the electric field. That is, for $r \geq b$, $\boldsymbol{E} = 0$. As a last step, we should exchange α for Q by normalizing the charge density as follows:

$$Q = 4\pi\alpha \int_0^a \mathrm{d}r\, r^4 = \frac{4\pi a^5}{5}\alpha. \tag{1.36}$$

Thus, for $r < b$, we have

$$E_r = \frac{Q}{4\pi\epsilon_0 a^2}\frac{r_<^5}{a^3 r^2} = \frac{Q}{4\pi\epsilon_0 a^2}\begin{cases}(r/a)^3 & r < a \\ (a/r)^2 & a < r < b.\end{cases} \tag{1.37}$$

(b) Using the electric field, we can determine the potential by using

$$\Phi(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{r}\,. \tag{1.38}$$

Note that for $r \geq b$, our potential $\Phi = 0$ since the electric field is zero in this region and the sphere is grounded. Inside the sphere, with $r > a$, we have

$$\Phi(a < r < b) = \frac{Q}{4\pi\epsilon_0}\int_r^b \frac{\mathrm{d}r}{r^2} = \frac{Q}{4\pi\epsilon_0 b}\left(\frac{b}{r} - 1\right), \tag{1.39}$$

and for $r < a$, we have

$$\begin{aligned}\Phi(r < a) &= \frac{Q}{4\pi\epsilon_0}\left[\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{a^5}\int_r^a r^3\,\mathrm{d}r\right] \\ &= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b} + \frac{a^4 - r^4}{4a^5}\right] \\ &= \boxed{\frac{Q}{4\pi\epsilon_0 a}\left[\frac{5}{4} - \frac{a}{b} - \frac{1}{4}\left(\frac{r}{a}\right)^4\right]}\end{aligned} \tag{1.40}$$

Problem 2.3)

Two metal objects of arbitrary shape are embedded in a conducting material of uniform conductivity σ.

(a) Derive a relationship between the resistance, R, between the objects and the mutual capacitance, C.

(b) The two objects are charged to a potential difference V_0. If the battery is then disconnected, derive an expression for the potential difference as a function of time in terms of σ and ϵ_0.

(a) Ohm's law reads $\boldsymbol{J} = \sigma \boldsymbol{E}$. If we integrate over a Gaussian surface enclosing one of the spheres, which has charge Q, we find

$$I = \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{A} = \sigma \int \boldsymbol{E}\, \mathrm{d}\boldsymbol{A} = \frac{\sigma Q}{\epsilon_0}. \tag{1.41}$$

Next, we use the definition of capacitance to write

$$C = \frac{Q}{V} \Rightarrow V = I\left(\frac{\epsilon_0}{\sigma C}\right). \tag{1.42}$$

That is, the resistance

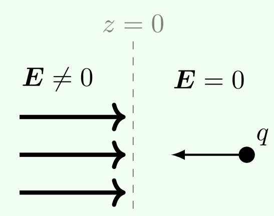
$$\boxed{R = \frac{\epsilon_0}{\sigma C}} \tag{1.43}$$

(b) The current between the spheres

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{V}{R} = \frac{Q}{RC} \Rightarrow Q(t) = Q_0 e^{-(\sigma/\epsilon_0)t} \Rightarrow \boxed{V(t) = V_0 e^{-(\sigma/\epsilon_0)t}}. \tag{1.44}$$

Problem 2.4)

A relativistic positively charged particle of charge q and mass m is traveling with velocity v_0 in the negative z-direction as shown in the figure. At $z = 0$, the particle enters a semi-infinite region $z < 0$ of homogeneous electric field directed in the positive z direction $\boldsymbol{E} = E\hat{z}$. How far does the particle penetrate into the $z < 0$ region and how much time does is spend there? Neglect the Abraham-Lorentz force of radiation reaction.



Ignoring, the Abraham-Lorentz force of radiation, we simply have

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{E}, \tag{1.45}$$

where p is the relativistic three-momentum of the particle. Since the motion is entirely constrained to the z-axis, we have

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = qE \Rightarrow p_z = p_0 + qEt, \tag{1.46}$$

where

$$p_0 = -\gamma_0 m v_0 = -\frac{m v_0}{\sqrt{1 - v_0^2/c^2}} \tag{1.47}$$

Next, we determine the velocity of the particle as a function of time through

$$\boldsymbol{p} = \gamma m \boldsymbol{v} = \frac{\mathcal{E}}{mc^2} m \boldsymbol{v} = \frac{\mathcal{E}}{c^2} \boldsymbol{v}, \tag{1.48}$$

where $\mathcal{E} = c\sqrt{(mc)^2 + \boldsymbol{p}^2}$ is the relativistic energy of the particle, so

$$v = \frac{c^2}{\mathcal{E}}[qEt + p_0] = c\frac{qEct - |p_0|c}{\sqrt{(mc^2)^2 + (qEct - |p_0|c)^2}}. \tag{1.49}$$

This is related to the position as follows:

$$z = c\int_0^t \mathrm{d}t' \, \frac{qEct' - |p_0|c}{\sqrt{(mc^2)^2 + (qEct' - |p_0|c)^2}}. \tag{1.50}$$

The above integral can be solved by utilizing the substitution

$$qEct' - |p_0|c = mc^2 \sinh u \Rightarrow qEc \, \mathrm{d}t' = mc^2 \cosh u \, \mathrm{d}u \tag{1.51}$$

such that

$$\begin{aligned} z &= c\int_{u(0)}^{u(t)} \mathrm{d}u \, \frac{mc^2}{qEc} \cosh u \frac{mc^2 \sinh u}{mc^2 \cosh u} = \frac{mc^2}{qE}\int_{u(0)}^{u(t)} \mathrm{d}u \sinh u \\ &= \frac{mc^2}{qE}[\cosh u(t) - \cosh u(0)] \\ &= \frac{1}{qE}\left[\sqrt{(mc^2)^2 + (qEct - |p_0|c)^2} - \sqrt{(mc^2)^2 + (|p_0|c)^2}\right], \end{aligned} \tag{1.52}$$

where we have used the fact that $\cosh^2 x = 1 + \sinh^2 x$.

We now have all the information needed to determine the time the particle spends in the region $z < 0$ and how far the particle penetrates within this region. The penetration distance is determined by the turning point, where

$$p = 0 \Rightarrow T = -\frac{p_0}{qE} = \frac{|p_0|}{qE} \tag{1.53}$$

so that

$$\boxed{|z(T)| = \frac{mc^2}{qE}\left|\sqrt{1 + \left(\frac{|p_0|c}{mc^2}\right)^2} - 1\right|}. \tag{1.54}$$

Next, the time the particle spends in the negative-z region is defined by the equation

$$z(t) = 0 \Rightarrow \boxed{t = \frac{|p_0|c \pm |p_0|c}{qEc} = \frac{2|p_0|}{qE} = 2T}, \tag{1.55}$$

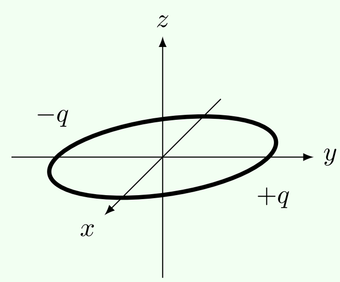
where we take the + branch since the − branch gives the entry time of the particle into the $z < 0$ region. Classically, it is obvious to us that the total time in this region is twice the time before the turning point since the position is quadratic in time, symmetric with respect to the turning point. Relativistically, though, it may not be so obvious since the dependence of the velocity and position on time is more complicated. Still, however, the total time is double the time to the turning point, and fundamentally, this is because of time-reversal symmetry. That is, the motion of the particle out of the $z > 0$ region is just the braking portion of the particle's motion in reverse.

Problem 3.1)

A thin circular ring of radius R lies in the xy-plane and is centered at the origin. It consists of two semicircles (corresponding to $y > 0$ and $y < 0$) that are uniformly charged with opposite charges $+q$ and $-q$.

Determine the electrostatic potential Φ and electric field $\boldsymbol{E}$ on the z axis (it goes through the center of the ring) and near that axis.

What is the asymptotic behavior of the field at very large distances from the ring?



We can determine the potential via

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\boldsymbol{r}'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{1.56}$$

The charge density is simply given as

$$\rho(\boldsymbol{r}) = \frac{q}{\pi R^2}\delta(r-R)\delta(\cos\theta)\Big[\Theta(0\le\phi\le\pi) - \Theta(\pi\le\phi\le 2\pi)\Big]. \tag{1.57}$$

One can verify that this in fact gives the required total charge q for the $y > 0$ portion of the ring and $-q$ for the $y < 0$ portion of the ring. Thus, we have

$$\Phi(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\pi}\int_0^{2\pi}\frac{\Theta(0\le\phi'\le\pi) - \Theta(\pi\le\phi'\le 2\pi)}{\sqrt{r^2+R^2-2\boldsymbol{r}\cdot\boldsymbol{r}'}}\,\mathrm{d}\phi'\,. \tag{1.58}$$

Since we care first to evaluate the potential for points on the z axis, we take $\boldsymbol{r} = z\hat{\boldsymbol{z}}$, yielding

$$\Phi(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\pi}\int_0^{2\pi} \frac{\Theta(0 \leq \phi' \leq \pi) - \Theta(\pi \leq \phi' \leq 2\pi)}{\sqrt{z^2 + R^2}}\,\mathrm{d}\phi' = 0. \tag{1.59}$$

To go off the z-axis, we write

$$\begin{aligned}\boldsymbol{r} \cdot \boldsymbol{r}' &= rr'[\sin\theta\sin\theta'\cos(\phi - \phi') + \cos\theta\cos\theta'] \\ &= rR\sin\theta\cos(\phi - \phi'),\end{aligned} \tag{1.60}$$

where in the last equality, we have enforced the δ functions for the charge density. We insert this into the generic expression for the potential and find

$$\begin{aligned}\Phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{\pi}\int_0^{2\pi} \frac{\Theta(0 \leq \phi' \leq \pi) - \Theta(\pi \leq \phi' \leq 2\pi)}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos(\phi - \phi')}} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi' \left[\frac{1}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos(\phi - \phi')}} - \frac{1}{\sqrt{r^2 + R^2 + 2rR\sin\theta\cos(\phi - \phi')}}\right]\end{aligned} \tag{1.61}$$

We next want to find the potential just slightly off the z-axis, which implies that $r\sin\theta \ll R$ such that

$$\begin{aligned}\left[r^2 + R^2 \pm 2rR\sin\theta\cos(\phi - \phi')\right]^{-1/2} &= \frac{1}{\sqrt{r^2 + R^2}}\left[1 \pm \frac{2Rr\sin\theta\cos(\phi - \phi')}{r^2 + R^2}\right]^{-1/2} \\ &= \frac{1}{\sqrt{r^2 + R^2}}\left[1 \mp \frac{Rr\sin\theta}{r^2 + R^2}\cos(\phi - \phi') + \ldots\right].\end{aligned} \tag{1.62}$$

Inserting this expansion, the potential takes the form

$$\begin{aligned}\Phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{\pi\sqrt{r^2 + R^2}}\int_0^{\pi} \mathrm{d}\phi' \left[\frac{2Rr\sin\theta}{r^2 + R^2}\cos(\phi - \phi') + ...\right] \\ &\approx \frac{q}{4\pi\epsilon_0}\frac{4Rr\sin\theta\sin\phi}{\pi(r^2 + R^2)^{3/2}}.\end{aligned} \tag{1.63}$$

This also gives us the previous result on the z-axis since $\sin\theta = 0$ there. Finally, let's determine the form of the potential when $r \gg R$ such that

$$\begin{aligned}\left[r^2 + R^2 \pm 2Rr\sin\theta\cos(\phi - \phi')\right]^{-1/2} &= \frac{1}{r}\left[1 \pm \frac{2R\sin\theta\cos(\phi - \phi')}{r} + \frac{R^2}{r^2}\right]^{-1/2} \\ &= \frac{1}{r}\left[1 \mp \frac{R\sin\theta\cos(\phi - \phi')}{r} + \ldots\right].\end{aligned} \tag{1.64}$$

Thus, the potential

$$\Phi(\boldsymbol{r}) \approx \frac{q}{4\pi\epsilon_0}\frac{4R\sin\theta\sin\phi}{\pi r^2}. \tag{1.65}$$

This is just a dipole potential. Let's place our charges $+q$ at $y = d/2$ and $-q$ at $y = -d/2$. The dipole moment is then just $\boldsymbol{p} = qd\hat{\boldsymbol{y}}$, and the potential

$$\begin{aligned}\Phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_+|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_-|}\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + d^2/4 - 2\boldsymbol{r}\cdot\boldsymbol{r}_+}} - \frac{1}{\sqrt{r^2 + d^2/4 - 2\boldsymbol{r}\cdot\boldsymbol{r}_-}}\right] \\ &\approx \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{p}}{4\pi\epsilon_0\, r^2} = \frac{qd\sin\theta\sin\phi}{4\pi\epsilon_0\, r^2}. \end{aligned} \tag{1.66}$$

Next, we determine d by integrating over the position vector, weighted by the charge distribution:

$$\begin{aligned}\boldsymbol{p} &= \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{r}\rho(\boldsymbol{r}) = \frac{qR}{\pi}\int_0^{2\pi}[\cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}][\Theta(0 \le \phi \le \pi) - \Theta(\pi \le \phi \le 2\pi)] \\ &= q\frac{4R}{\pi}\hat{\boldsymbol{y}} \Rightarrow \Phi(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{4qR\sin\theta\sin\phi}{\pi r^2}. \end{aligned} \tag{1.67}$$

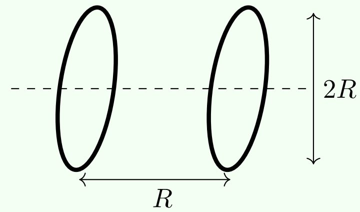
We thus see that our work is all self-consistent.

Problem 3.2)

Helmholtz coils are sometimes used by physicists to determine the charge to mass ratio of the electron. From Wikipedia:

"A Helmholtz coil is a device for producing a region of nearly uniform magnetic field, named after the German physicist Hermann von Helmholtz. It consists of two electromagnets on the same axis, carrying an equal electric current in the same direction. Besides creating magnetic fields, Helmholtz coils are also used in scientific apparatus to cancel external magnetic fields, such as the Earth's magnetic field."

Find the magnetic field between the two coils (see figure below). Let N be the number of turns in each coil, I the current, and R be the radius. Assume the coils are separated by a distance R and assume that the thickness of each coil is negligible relative to R. Express your result in terms of I, R, N, and any constants.



The magnetic field for any localized current distribution $\boldsymbol{J}$ is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3\boldsymbol{r'}\, \frac{\boldsymbol{J}(\boldsymbol{r'}) \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \tag{1.68}$$

Calculating such an integral for any point in space is likely quite difficult, if even possible for the Helmholtz coil setup, so we restrict ourselves to the axis passing through the centers of the coils. Notice that we can sum the magnetic fields produced by each coil separately, so let us calculate the field produced on this axis (the positive direction is determined by the right hand rule) by a single coil first, where the origin for now is at the center of this coil:

$$\boldsymbol{B} = \frac{\mu_0 NIR}{4\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi\, \hat{\boldsymbol{\phi}} \times (z\hat{\boldsymbol{z}} - R\hat{\boldsymbol{r}})}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 NIR^2}{2(z^2 + R^2)^{3/2}}\hat{\boldsymbol{z}}. \tag{1.69}$$

If we shift our origin to the center of these coils and sum the two contributions, we find

$$\boxed{\boldsymbol{B} = \frac{\mu_0 NIR^2}{2}\left\{\frac{1}{[(z + R/2)^2 + R^2]^{3/2}} + \frac{1}{[(z - R/2)^2 + R^2]^{3/2}}\right\}\hat{\boldsymbol{z}}}. \tag{1.70}$$

Notice that for $z \ll R$, the magnetic field

$$\boldsymbol{B} = \frac{\mu_0 NI}{2R}\left[\underbrace{\frac{16}{5\sqrt{5}}}_{1.43} - \underbrace{\frac{2304}{625\sqrt{5}}}_{1.65}\left(\frac{z}{R}\right)^4 + \ldots\right], \tag{1.71}$$

which suggests that the magnetic field varies only slightly for small deviations from the center of the coils (note: the expansion was performed with Wolfram, so it is likely not expected to be performed by hand on the exam).

Problem 3.3)

The space between the plates of a plane capacitor is filled with two layers 1 and 2 of thicknesses d_1 and d_2 and permittivities ϵ_1 and ϵ_2, respectively. Calculate:

(a) The capacitance of this capacitor

(b) Charge density at the interface between layers 1 and 2 caused by the voltage V

on the capacitor.

(a) The relevant equation here is Gauss' law for the electric displacement:

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho \longleftrightarrow \oint \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{D} = Q, \tag{1.72}$$

where ρ is the free charge density and Q is the free charged in the volume enclosed by S. Using a Gaussian pillbox, stradling both sides of one metal plate:

$$\boldsymbol{D} = \sigma \hat{\boldsymbol{n}}, \tag{1.73}$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing from the positively charged plate to the negatively charged plate. Now we can compute the electric field, assuming that both media are linear:

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon(x)}, \tag{1.74}$$

where

$$\epsilon(x) = \begin{cases} \epsilon_1 & 0 < x < d_1 \\ \epsilon_2 & d_1 < x < d_2. \end{cases} \tag{1.75}$$

From this, we can determine the potential difference between the plates via the following line integral:

$$V = \left| \int_0^{d_2} \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{E} \right| = \sigma\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right). \tag{1.76}$$

And, finally, the capacitance

$$\boxed{C = \frac{Q}{V} = A\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right)^{-1}}. \tag{1.77}$$

Notice that this reduces to the usual result $C = A\epsilon_0/d$ when $\epsilon_1 = \epsilon_2 = \epsilon_0$ and $d_1 + d_2 = d$.

(b) The bound surface charge at the interface is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = (\epsilon - \epsilon_0)\boldsymbol{E} \cdot \hat{\boldsymbol{n}}, \tag{1.78}$$

where we have used $\boldsymbol{P} = \chi_e \boldsymbol{E}$ and $\chi_e = \epsilon - \epsilon_0$. Thus,

$$\boxed{\sigma_b = (\epsilon_1 - \epsilon_0)\frac{\sigma}{\epsilon_1} - (\epsilon_2 - \epsilon_0)\frac{\sigma}{\epsilon_2} = \sigma\epsilon_0\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right)}. \tag{1.79}$$

Note that the sign depends on the relative polarizability of the dielectrics.

Quantum Mechanics

Problem 3.4)

A neutral particle with spin-1/2 and a nonzero magnetic momentum is subject to a periodic magnetic field $B(t) = B_0 \cos \omega t$ applied along the z-axis. The state vector of the spin at $t = 0$ is given by:

$$\langle \psi(0)| = \begin{pmatrix} e^{i\varphi_1} \cos\theta & e^{i\varphi_2} \sin\theta \end{pmatrix}.$$

Calculate the expectation values of the spin components $\langle S_z(t)\rangle$, $\langle S_x(t)\rangle$, and $\langle S_y(t)\rangle$ at $t > 0$.

The Hamiltonian for a neutral spin-1/2 particle immersed in a magnetic field is given by

$$H = -\frac{ge}{2mc}\boldsymbol{S}\cdot\boldsymbol{B} = -\frac{geB_0}{2mc}\cos\omega t S_z, \tag{1.80}$$

where g is the gyromagnetic factor of the particle. Observe that our eigenstates are just $|\pm\rangle$, so any arbitrary state

$$|\psi(t)\rangle = c_+(t)\,|+\rangle + c_-(t)\,|-\rangle\,. \tag{1.81}$$

We can determine how the coefficients change with time by inserting this expansion into the Schrödinger equation:

$$i\hbar\frac{\mathrm{d}\,|\psi\rangle}{\mathrm{d}t} = H\,|\psi\rangle \Rightarrow \frac{\mathrm{d}c_\pm}{\mathrm{d}t} = \pm i\alpha\cos\omega t\, c_\pm \Rightarrow c_\pm(t) = c_\pm(0)e^{\pm i(\alpha/\omega)\sin(\omega t)}, \tag{1.82}$$

where $\alpha = geB_0/(2m\hbar c)$. Thus, the state as a function of time

$$|\psi(t)\rangle = \begin{pmatrix} e^{i[\varphi_1+(\alpha/\omega)\sin\omega t]}\cos\theta \\ e^{i[\varphi_2-(\alpha/\omega)\sin\omega t]}\sin\theta \end{pmatrix} = e^{i[\varphi_2-(\alpha/\omega)\sin\omega t]}\begin{pmatrix} e^{i[(\varphi_1-\varphi_2)+2(\alpha/\omega)\sin\omega t]}\cos\theta \\ \sin\theta \end{pmatrix}. \tag{1.83}$$

The requested expectation values are then as follows:

$$\boxed{\begin{aligned} \langle S_x(t)\rangle &= \frac{\hbar}{2}\cos[(\varphi_1-\varphi_2)+2(\alpha/\omega)\sin\omega t]\sin(2\theta) \\ \langle S_y(t)\rangle &= -\frac{\hbar}{2}\sin[(\varphi_1-\varphi_2)+2(\alpha/\omega)\sin\omega t]\sin(2\theta) \\ \langle S_z(t)\rangle &= \frac{\hbar}{2}\cos(2\theta) \end{aligned}}. \tag{1.84}$$

Note that the S_z expectation value is independent of time, which is consistent with the Ehrenfest theorem given that $[H, S_z] = 0$ for all t. Also, as a sanity check, we have $\langle \boldsymbol{S}^2\rangle = 3\hbar^2/4$ as expected (again constant in time since $[\boldsymbol{S}^2, S_z] = 0$ and therefore that $[\boldsymbol{S}^2, H] = 0$.

Problem 4.1)

Assuming that the eigenfunctions for the hydrogen atom are of the form $r^\beta e^{-\gamma r} Y_{lm}(\Omega)$ with undetermined parameters β and γ, solve the Schrödinger equation. Are all eigenfunctions and eigenvalues obtained this way? Justify your answer.

The Schrödinger equation for a central potential reads

$$\left[-\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\boldsymbol{L}^2}{2\mu r^2} + V(r)\right]\psi(r) = E\psi(r). \tag{1.85}$$

If we put in the solution ansatz, we obtain

$$-\frac{\hbar^2}{2\mu}\left[\gamma^2 r^\beta - 2\gamma(\beta+1)r^{\beta-1} + \beta(\beta+1)r^{\beta-2}\right] + \frac{\hbar^2 l(l+1)}{2\mu}r^{\beta-2} - e^2 r^{\beta-1} = Er^\beta. \tag{1.86}$$

Equating the coefficients of powers of r, we obtain three equations

$$\begin{cases} E = -\frac{\hbar^2\gamma^2}{2\mu} \\ \gamma(\beta+1) = \frac{\mu e^2}{\hbar^2} \\ \beta(\beta+1) - l(l+1) = 0. \end{cases} \tag{1.87}$$

The latter equation has two solutions: $\beta = l, -l-1$, but the first is the only physically admissibly one since for $l > 1$ the second leads to non-normalizable solutions. We thus choose

$$\boxed{\beta = l} \tag{1.88}$$

The second equation then yields

$$\boxed{\gamma = \frac{\mu e^2}{\hbar^2(l+1)}}. \tag{1.89}$$

Putting this into our first equation above, the energy

$$\boxed{E = -\frac{\mu^2 e^2}{\hbar^2(l+1)^2}}. \tag{1.90}$$

This exactly reproduces the energy levels if we set $n = l+1$, but our solution ansatz does not capture the full spectrum of the Hydrogen atom, which can be observed by noting that the degeneracy for each energy level implied by the solution above is $2l+1$. The correct degeneracy, though, is n^2.

Problem 4.2)

Consider a quantum mechanical system that is described by the Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{L}}^2}{2I} + a\hat{L}_z + b\hat{L}_z^2,$$

where $\hat{\boldsymbol{L}}$ is the angular momentum operator and I, a, and b are constants.

(a) Why do the constants I, a, and b have to be real-valued parameters?

(b) What are the eigenvalues and eigenstates of $\hat{H}$

(c) Now consider $a = 4\hbar/I$, $b = 2/I$. What is the ground state and the ground state energy of the system?

(d) At time $t = 0$, the system is in the state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{3}}(|1,0\rangle + |1,1\rangle - |1,-1\rangle)$$

Here, we use the usual $|l, m\rangle$ notation to denote the angular momentum eigenstates. Determine the time evolution of the state $|\psi(t)\rangle$.

(a) Since the Hamiltonian represents a physical observable, it must be Hermitian. That is,

$$H^\dagger = \frac{\boldsymbol{L}^2}{2I^*} + a^* L_z + b^* L_z^2 = \frac{\boldsymbol{L}^2}{2I} + aL_z + bL_z^2, \tag{1.91}$$

which can only be true if I, a, and b are real.

(b) The eigenstates of the Hamiltonian are just those of $\boldsymbol{L}^2$ and L_z, where

$$\boldsymbol{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle, \quad L_z |lm\rangle = \hbar m |lm\rangle, \tag{1.92}$$

$l = 0, 1, \ldots$, and $m = -l, -l-1, \ldots, l-1, l+1$, so

$$\boxed{H |lm\rangle = \underbrace{\left(\frac{\hbar^2 l(l+1)}{2I} + a\hbar m + b\hbar^2 m^2\right)}_{E_{lm}} |lm\rangle}. \tag{1.93}$$

(c) For the constants given above, the energies read

$$E_{lm} = \frac{\hbar^2 l(l+1)}{2I} + \frac{4\hbar}{I}\hbar m + \frac{2}{I}\hbar^2 m^2 = \frac{\hbar^2}{2I}\Big(l(l+1) + 8m + 4m^2\Big). \tag{1.94}$$

From this, we see that we should minimize with respect to l and m:

$$\frac{\partial E_{lm}}{\partial l} = 2l + 1 = 0 \Rightarrow l = 1/2 \tag{1.95}$$

$$\frac{\partial E_{lm}}{\partial m} = 8 + 8m = 0 \Rightarrow m = -1. \tag{1.96}$$

The minimum for l is not physical, but since the l dependence is quadratic, the $l = 0$ and $l = 1$ states give the same contribution. Our minimum for m requires $l = 1$, so our ground state and ground state energy are

$$\boxed{|1, -1\rangle \Leftrightarrow E_{1,-1} = -\frac{\hbar^2}{I}}. \tag{1.97}$$

(d) The time evolution is given by

$$\boxed{|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle = \frac{1}{\sqrt{3}}\Big(e^{-i\hbar t/I}\,|10\rangle + e^{-7i\hbar t/I}\,|11\rangle - e^{i\hbar t/I}\,|1\ -1\rangle\Big)} \tag{1.98}$$

Problem 4.3)

Consider a hydrogen-like atom described by the Hamiltonian

$$H = \left(c\sqrt{\boldsymbol{p}^2 + (mc)^2} - mc^2\right) - \frac{Ze^2}{r},$$

where m is the mass of the electron and Ze is the nuclear charge with $Z \gg 1$. Since the typical velocity of the electron in such a system is of the order of $Z\alpha c$ (here $\alpha \approx 1/137$ is the fine structure constant and c is the speed of light), it is physically sensible to presenent the electron kinetic energy operator by its relativistic expression.

(a) Expand the kinetic energy operator in powers of $[\boldsymbol{p}/(mc)]^2$, and show that the Hamiltonian can be written as

$$H = H_0 + V \quad \text{with } H_0 = \frac{\boldsymbol{p}^2}{2m} - \frac{Ze^2}{r},$$

and V is the perturbation consisting of the leading correction to the non-relativistic kinetic energy operator. Show that the operator V can be expressed as

$$V = -\frac{1}{2mc^2}\left(H_0 + \frac{Ze^2}{r}\right)^2$$

Express your results in terms of Z, α, and mc^2.

(b) Consider the first excited level having $n = 2$, which is four-fold degenerate (spin degrees of freedom are ignored). Obtain the first-order corrections to the unperturbed energy ϵ_2, expressing your results in terms of Z, α, and mc^2. Does the perturbation lift the degeneracy of this level completely or only partially? Would you expect degeneracy to persist in high orders of perturbation theory? Justify your answers.

Hints: The unperturbed bound-state energies are

$$\epsilon_n = -\frac{(Z\alpha)^2}{2n^2}mc^2 \quad n = 1, 2, \cdots \ ,$$

where $\alpha = e^2/(\hbar c)$ is the fine structure constant. The radial wave functions with $n = 2$ are given by

$$R_{20}(x) = \frac{1}{\sqrt{2}}\Big(\frac{Z}{a_0}\Big)^{3/2}\Big(1 - \frac{x}{2}\Big)e^{-x/2}, \quad R_{21}(x) = \frac{1}{2\sqrt{6}}\Big(\frac{Z}{a_0}\Big)^{3/2} xe^{-x/2},$$

where $x = Zr/a_0$ and a_0 is the Bohr radius,

$$a_0 = \frac{\hbar^2}{me^2} = \frac{1}{\alpha}\frac{\hbar}{mc} \quad \text{and} \quad \frac{e^2}{a_0} = \alpha^2 mc^2.$$

The following integral may be useful

$$\int_0^\infty \mathrm{d}x \ x^n e^{-\gamma x} = \frac{n!}{\gamma^{n+1}}, \quad n \geq 0.$$

(a) We rewrite the Hamiltonian as requested:

$$H = mc^2\Big(\sqrt{1 + [\boldsymbol{p}/(mc)]^2} - 1\Big) - \frac{Ze^2}{r} = mc^2\Big(\frac{\boldsymbol{p}^2}{2(mc)^2} - \frac{\boldsymbol{p}^4}{8(mc)^4} + \dots\Big) - \frac{Ze^2}{r}$$
$$= \boxed{\underbrace{\frac{\boldsymbol{p}^2}{2m} - \frac{Ze^2}{r}}_{H_0} \underbrace{- \frac{\boldsymbol{p}^4}{8m^3c^2}}_{V}}. \tag{1.99}$$

Observe that we can write

$$\boldsymbol{p}^2 = 2m\Big(H_0 + \frac{Ze^2}{r}\Big), \tag{1.100}$$

allowing us to express

$$\boxed{V = -\frac{1}{8m^3c^2}\Big[2m\Big(H_0 + \frac{Ze^2}{r}\Big)\Big]^2 = -\frac{1}{2mc^2}\Big(H_0 + \frac{Ze^2}{r}\Big)^2}. \tag{1.101}$$

(b) We must diagonalize the perturbation with respect to the degenerate $n = 2$ states. Consider the generic matrix element

$$\begin{aligned}\langle\phi_{2lm}|V|\phi_{2l'm'}\rangle &= -\frac{1}{2mc^2}\langle\phi_{2lm}|\Big[H_0^2 + H_0\frac{Ze^2}{r} + \frac{Ze^2}{r}H_0 + \frac{Z^2e^4}{r^2}\Big]|\phi_{2l'm'}\rangle \\ &= -\frac{1}{2mc^2}\Big[\epsilon_2^2\delta_{ll'}\delta_{mm'} + Ze^2\epsilon_2\langle\phi_{2lm}|\frac{1}{r}|\phi_{2l'm'}\rangle + Z^2e^4\langle\phi_{2lm}|\frac{1}{r^2}|\phi_{2l'm'}\rangle\Big].\end{aligned} \tag{1.102}$$

We must determine the matrix elements

$$\begin{aligned}\langle\phi_{2lm}|\frac{1}{r^k}|\phi_{2l'm'}\rangle &= \int \mathrm{d}^3\boldsymbol{r}\,\frac{1}{r^k}R_{2l}(r)Y_{lm}(\Omega)R_{2l'}(r)Y_{l'm'}(\Omega) \\ &= \delta_{ll'}\delta_{mm'}\int_0^\infty \mathrm{d}r\,r^{2-k}R_{2l}^2(r) = \delta_{ll'}\delta_{mm'}\underbrace{\left(\frac{a_0}{Z}\right)^{3-k}\int_0^\infty \mathrm{d}x\,x^{2-k}R_{2l}^2(x)}_{I_{lk}}.\end{aligned} \tag{1.103}$$

Observe that the integral is independent of m, so we only have the four integrals as follows:

$$I_{01} = \frac{Z}{2a_0}\int_0^\infty \mathrm{d}x\,x\Big(1-\frac{x}{2}\Big)^2e^{-x} = \frac{Z}{2a_0}\int_0^\infty \mathrm{d}x\left(x - x^2 + \frac{x^3}{4}\right)e^{-x} = \frac{Z}{4a_0} \tag{1.104}$$

$$I_{02} = \frac{Z^2}{2a_0^2}\int_0^\infty \mathrm{d}x\left(1 - x + \frac{x^2}{4}\right)e^{-x} = \frac{Z^2}{4a_0^2} \tag{1.105}$$

$$I_{11} = \frac{Z}{24a_0}\int_0^\infty \mathrm{d}x\,x^3e^{-x} = \frac{Z}{4a_0} \tag{1.106}$$

$$I_{12} = \frac{Z^2}{24a_0^2}\int_0^\infty \mathrm{d}x\,x^2e^{-x} = \frac{Z^2}{12a_0^2} \tag{1.107}$$

The matrix elements are diagonal in l and m, and those elements are independent of m. Thus,

$$\boxed{\begin{aligned}\epsilon_{20}^{(1)} &= -\frac{19\epsilon_2^2}{2mc^2} \\ \epsilon_{21}^{(1)} &= -\frac{25\epsilon_2^2}{6mc^2}\end{aligned}}. \tag{1.108}$$

Problem 4.4)

Consider a particle of charge q (take q to be positive) in a magnetic field $\boldsymbol{B}(\boldsymbol{r})$. The velocity operator is given by

$$\boldsymbol{v} = \frac{1}{m}\Big[-i\hbar\boldsymbol{\nabla} - \frac{q}{c}\boldsymbol{A}(\boldsymbol{r})\Big], \quad \boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}),$$

where $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential. Show that

$$[v_i, v_j] = i\frac{q\hbar}{m^2 c}\epsilon_{ijk}B_k(\boldsymbol{r}).$$

Suppose the particle is constrained to move in the xy-plane under the influence of a uniform magnetic field directed along the $\hat{\boldsymbol{z}}$-axis. The Hamiltonian is given by

$$H = \frac{m}{2}(v_x^2 + v_y^2).$$

(a) Define the operators

$$\hat{a} = \frac{\alpha}{\sqrt{2}}(v_x + iv_y), \quad \hat{a}^\dagger = \frac{\alpha}{\sqrt{2}}(v_x - iv_y)$$

and determine α such that $[\hat{a}, \hat{a}^\dagger] = 1$. Write the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$.

(b) Obtain the eigenvalues and eigenstates of the Hamiltonian. Are the eigenvalues degenerate? Justify your answer.

Recall that $[\boldsymbol{p}, f(\boldsymbol{r})]g(\boldsymbol{r}) = g(\boldsymbol{r})[\boldsymbol{p}f(\boldsymbol{r})]$. Thus,

$$\begin{aligned}[v_i, v_j] &= \frac{1}{m^2}\left\{[p_i, p_j] - \frac{q}{c}[p_i, A_j] - \frac{q}{c}[A_i, p_j] + \frac{q^2}{c^2}[A_i, A_j]\right\} \\ &= -\frac{q}{m^2 c}\left[p_i A_j - p_j A_i\right] = i\frac{q\hbar}{m^2 c}\epsilon_{ijk}(\boldsymbol{\nabla}\times\boldsymbol{A})_k = \boxed{i\frac{q\hbar}{m^2 c}\epsilon_{ijk}B_k}. \end{aligned} \tag{1.109}$$

(a) The commutator

$$\begin{aligned}[a, a^\dagger] &= \frac{\alpha^2}{2}\left\{[v_x, v_x] - i[v_x, v_y] + i[v_y, v_x] + [v_y, v_y]\right\} = \alpha^2\frac{q\hbar B_z}{m^2 c} = 1 \\ &\Rightarrow \boxed{\alpha = \sqrt{\frac{m^2 c}{q\hbar B_z}}}. \end{aligned} \tag{1.110}$$

Next, we can rearrange the definitions of a and $a^\dagger$ to obtain

$$v_x = \frac{1}{\sqrt{2}\alpha}(a + a^\dagger), \quad v_x = \frac{1}{\sqrt{2}i\alpha}(a - a^\dagger). \tag{1.111}$$

Putting this into the Hamiltonian, we find

$$\boxed{H = \frac{m}{4\alpha^2}\left(a^\dagger a + aa^\dagger\right) = \hbar\frac{qB_z}{2m^3 c}\left(a^\dagger a + \frac{1}{2}\right)}. \tag{1.112}$$

(c) Since $a^\dagger a$ is a number operator, we know immediately how to write the spectrum of the Hamiltonian:

$$\boxed{H\left|n\right\rangle = \hbar\frac{qB_z}{2mc}(n+1/2)}, \tag{1.113}$$

where $\left|n\right\rangle$ satisfies the eigenequation $a^\dagger a\left|n\right\rangle = n\left|n\right\rangle$.

2 January 2023

Classical Mechanics

Problem 1.1)

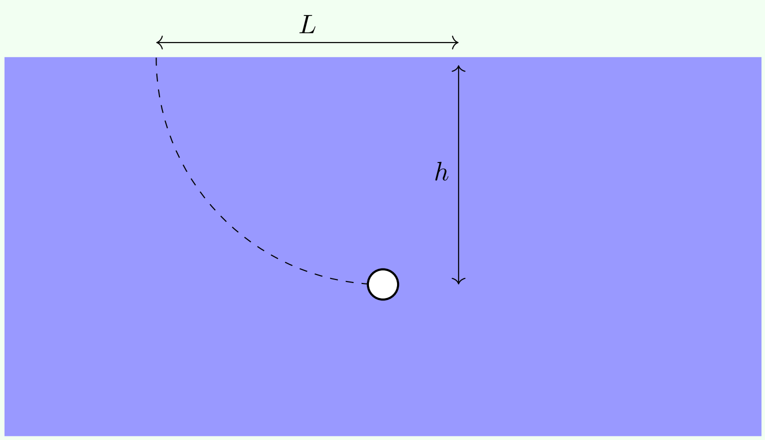
A spherical bathyscaphe of mass M and radius R is moving underwater with the velocity v_0 parallel to the surface. At $t = 0$ the engine stops and the bathyscaphe pops up. Assuming that at $t = 0$ the bathyscaphe was at the distance h from the surface, as shown in the figure below, obtain equations for:

(a) The time T for the bathyscaphe to emerge at the surface after the engine stops.

(b) The lateral distance L the bathyscaphe travels before it pops up.

Assume that: (1) the water has the mass density ρ and $M < 4\pi R^2\rho/3$, (2) The drag force acting on the bathyscaphe $\boldsymbol{F} = -\gamma\boldsymbol{v}$ is proportional to its velocity $\boldsymbol{v}$, where γ is a positive constant.

Solve the equations for T and L in the limit of $T \gg M/\gamma$.



(a) The force on the bathyscaphe is given by

$$\boldsymbol{F} = -Mg\hat{\boldsymbol{y}} - \gamma\boldsymbol{v} + \rho g\left(\frac{4}{3}\pi R^3\right)\hat{\boldsymbol{y}}, \tag{2.1}$$

which in components reads

$$M\ddot{x} = -\gamma\dot{x} \tag{2.2}$$

$$M\ddot{y} = -Mg - \gamma\dot{y} + \frac{4}{3}\pi R^3\rho g. \tag{2.3}$$

We can solve the equation for x simply:

$$\dot{x} + \frac{\gamma}{M}x = v_0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{(\gamma/M)t}x\right) = v_0 e^{(\gamma/M)t} \Rightarrow x(t) = \frac{Mv_0}{\gamma}(1 - e^{-(\gamma/M)t}). \tag{2.4}$$

We can also solve the equation for y in a similar way:

$$\dot{y} + \frac{\gamma}{M}y = -\frac{\gamma h}{M} - \Big(1 - \frac{4\pi R^3 \rho}{3M}\Big)gt$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{(\gamma/M)t}y\Big) = -\frac{\gamma h}{M}e^{(\gamma/M)t} - \Big(1 - \frac{4\pi R^3 \rho}{3M}\Big)gte^{(\gamma/M)t}$$
$$y(t) = -h - h(1 - e^{-(\gamma/M)t}) - \Big(1 - \frac{4\pi R^3 \rho}{3M}\Big)ge^{-(\gamma/M)t}\int_0^t \mathrm{d}t'\, t' e^{(\gamma/M)t'}$$
$$y(t) = -h - h(1 - e^{-(\gamma/M)t}) - \Big(\frac{M}{\gamma}\Big)^2\Big(1 - \frac{4\pi R^3 \rho}{3M}\Big)ge^{-(\gamma/M)t}\int_0^{\gamma t/M} \mathrm{d}x\, xe^x$$
$$y(t) = -h - h(1 - e^{-(\gamma/M)t}) - \Big(\frac{M}{\gamma}\Big)^2\Big(1 - \frac{4\pi R^3 \rho}{3M}\Big)g\left[\frac{\gamma}{M}t + (1 - e^{-(\gamma/M)t})\right]. \tag{2.5}$$

Assuming that the time to reach the top $T \gg \gamma/M$, we have

$$0 = -h - h\frac{\gamma T}{M} + \Big(\frac{M}{\gamma}\Big)^2\Big(\frac{4\pi R^3 \rho}{3M} - 1\Big)\frac{2\gamma T}{M}g$$
$$\boxed{T = h\left[\frac{2Mg}{\gamma}\left(\frac{4\pi R^3 \rho}{3M} - 1\right) - \frac{\gamma h}{M}\right]^{-1}}. \tag{2.6}$$

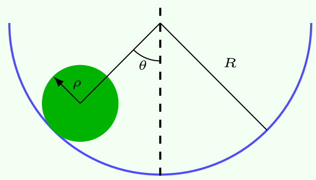
(b) Using the result above, the lateral distance the bathyscaphe travels before emerging is

$$\boxed{L = v_0 T = hv_0\left[\frac{2Mg}{\gamma}\left(\frac{4\pi R^3 \rho}{3M} - 1\right) - \frac{\gamma h}{M}\right]^{-1}}. \tag{2.7}$$

Problem 1.2)

A thin-shelled sphere of radius ρ and mass m is constrained to roll without slipping on the lower half of the inner surface of a hollow, stationary cylinder of radius R.

Take θ to be the generalized coordinate and use $I = \frac{2}{3}m\rho^2$ to find the Lagrange equation of motion that describes the motion of the shell.



Let us use cylindrical coordinates to write $x = (R-\rho)\sin\theta$ and $y = (R-\rho)\cos\theta$, where we have oriented y to point down. The velocity

$$\boldsymbol{v} = (R-\rho)\dot{\theta}(\cos\theta\hat{\boldsymbol{x}} - \cos\theta\hat{\boldsymbol{y}}) + \dot{z}\hat{\boldsymbol{z}}. \tag{2.8}$$

The kinetic energy is then

$$T = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2}{2} + \frac{1}{2}\frac{2}{5}m\rho^2\frac{v^2}{\rho^2} = \frac{7mv^2}{10} = \frac{7m}{10}\Big[(R-\rho)^2\dot{\theta}^2 + \dot{z}^2\Big], \tag{2.9}$$

so our Lagrangian reads

$$L = \frac{7m}{10}\Big[(R-\rho)^2\dot{\theta}^2 + \dot{z}^2\Big] + mg(R-\rho)\cos\theta. \tag{2.10}$$

Notice that z is a cyclic coordinate, so its motion is related to the constant of motion

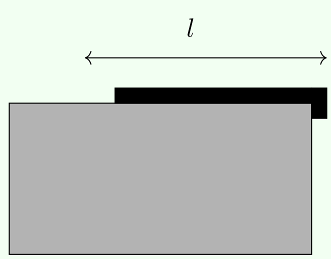
$$p_z = \frac{\partial L}{\partial \dot{z}} = \frac{7m}{5}\dot{z} \Rightarrow z = z_0 + \frac{5p_z}{7m}t. \tag{2.11}$$

On the other hand, for θ, we have

$$\boxed{\ddot{\theta} + \frac{5g}{7(R-\rho)}\sin\theta = 0}. \tag{2.12}$$

Problem 1.3)

An ideal (flexible, uniform, frictionless, etc.) rope of length l and mass M starts sliding off an ideal frictionless table as shown in the figure (the rope is initially at rest, the gravitational accleration is g, the size of the piece of the rope initially hanging off the table is y_0).



(a) Introduce some generalized coordinate and write down the Lagrangian of the system.

(b) Derive the Euler-Lagrange equations of motion.

(c) Calculate the time τ for the rope to slide half-way off the table.

(a) Let y denote the length of rope hanging off the table. The kinetic energy of the rope is then

$$T = \int \frac{\mathrm{d}m\,\dot{y}^2}{2} = \frac{M\dot{y}^2}{2}. \tag{2.13}$$

The potential energy is just

$$U = -\int \mathrm{d}m\, gy' = -\rho g \int_0^y y'\,\mathrm{d}y' = -\frac{Mgy^2}{2l}. \tag{2.14}$$

Putting these together, our Lagrangian

$$\boxed{L = \frac{M\dot{y}^2}{2} + \frac{Mgy^2}{2l}}. \tag{2.15}$$

(b) This Lagrangian yields the equation of motion

$$\boxed{\ddot{y} - \frac{g}{l}y = 0}, \tag{2.16}$$

which has solution

$$y(t) = y_0 \cosh\left(\frac{gt}{l}\right). \tag{2.17}$$

(c) The time τ for the rope to be halfway off the table is defined through

$$y(\tau) = \frac{l}{2} \Rightarrow \boxed{\tau = \frac{l}{g}\mathrm{arcosh}\left(\frac{y_0}{2l}\right)}. \tag{2.18}$$

Problem 1.4)

A smooth wire is bent into the shape of a spiral helix with a decreasing pitch. In cylindrical polar coordinates (ρ, ϕ, z) it is specified by equations $\rho = R$ and $z = \lambda\sqrt{\phi}$, where R and λ are positive constants. The z axis is vertically up (and gravity vertically down).

(a) Using z as a generalized coordinate, write down the Lagrangian for a bead of mass m threaded on the wire.

(b) Find the Lagrange equation and calculate the bead's vertical acceleration $\ddot{z}$ as a function of z and $\dot{z}$.

> **(c)** Find aceleration $\ddot{z}$ in two limits: (i) when $R \to 0$ but λ is fixed, and (ii) when $\lambda \to \infty$ but R is fixed. Discuss if the results for $\ddot{z}$ in these limits make sense.

In cylindrical coordinates

$$\boxed{L = \frac{m}{2}\Big[R^2\dot{\phi}^2 + \dot{z}^2\Big] - mgz = \frac{m}{2}\Big(1 + \frac{4R^2z^2}{\lambda^4}\Big)\dot{z}^2 - mgz}. \tag{2.19}$$

(b) The equation of motion is just

$$\boxed{m\Big(1 + \frac{4R^2z^2}{\lambda^4}\Big)\ddot{z} + \frac{8R^2z\dot{z}^2}{\lambda^4} + mg = 0}. \tag{2.20}$$

(c) In the limit $R \to 0$ with fixed λ, our acceleration is just $\ddot{z} = -g$. In the second limit $\lambda \to \infty$ with fixed R, we have $\ddot{z} = -g$.

> **Problem 2.1)**
>
> You are told that, at the known positions x_1 and x_2, an oscillating mass m has speeds v_1 and v_2. What are the amplitude and the angular frequency of the oscillations?

The position as a function of time

$$x(t) = A\cos(\omega t + \gamma), \tag{2.21}$$

and solving for t, we find

$$t = \frac{1}{\omega}\Big[\arccos\Big(\frac{x}{A}\Big) - \gamma\Big]. \tag{2.22}$$

Putting this into the expression for velocity, we have

$$|\dot{x}| = A\omega|\sin(\omega t + \gamma)| = A\omega\Big|\sin\Big(\arccos\Big(\frac{x}{A}\Big)\Big)\Big| = \omega\sqrt{A^2 - x^2}. \tag{2.23}$$

Using the boundary conditions, we have the system of equations

$$v_1 = \omega\sqrt{A^2 - x_1^2} \tag{2.24}$$

$$v_2 = \omega\sqrt{A^2 - x_2^2}. \tag{2.25}$$

Thus

$$\boxed{A = \sqrt{\frac{v_2^2x_1^2 - v_1^2x_2^2}{v_2^2 - v_1^2}}, \quad \omega = \sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}}. \tag{2.26}$$

Electricity & Magnetism

Problem 2.2)

An electron with mass m_e and momentum p_e hits a positron at rest. They annihilate, producing a pair of photons. If one of the photons emerge at angle θ to the incident electron direction, what is the second photon's angle?

From the conservation of energy and 3-momentum, we have the set of equations

$$\begin{cases} E + m_e = E_\gamma + E'_\gamma \\ p_e = E_\gamma \cos\theta + E'_\gamma \cos\theta' \\ 0 = E_\gamma \sin\theta - E'_\gamma \sin\theta'. \end{cases} \tag{2.27}$$

From the last equation, we can write

$$E_\gamma = E_{\gamma'} \frac{\sin\theta'}{\sin\theta}. \tag{2.28}$$

Putting this into the second equation,

$$p_e = E_{\gamma'} \Big(\cot\theta \sin\theta' + \cos\theta' \Big) \Rightarrow E'_\gamma = \frac{p_e}{\cot\theta\sin\theta' + \cos\theta'}. \tag{2.29}$$

Putting this into the first equation, we find

$$\begin{aligned} E + m_e &= \frac{p_e}{\cot\theta\sin\theta' + \cos\theta'}\left(1 + \frac{\sin\theta'}{\sin\theta}\right) \\ \frac{E + m_e}{p_e} &= \frac{\sin\theta + \sin\theta'}{\cos\theta\sin\theta' + \sin\theta\cos\theta'}. \end{aligned} \tag{2.30}$$

Solving for $\sin\theta'$, we find

$$\frac{\sin\theta'}{\sin\theta} = \frac{(A\cos\theta - 1) \pm A(A - \cos\theta)}{1 + A^2 - 2A\cos\theta}, \tag{2.31}$$

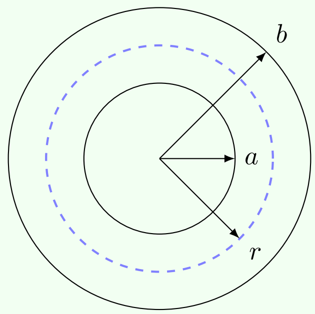
where $A = (E + m_e)/p_e$. There are two solutions here, where the $-$ yields that the two photons go off together in the same direction, leaving us with

$$\boxed{\frac{\sin\theta'}{\sin\theta} = \frac{A^2 - 1}{1 + A^2 - 2A\cos\theta} = \frac{(E + m_e)^2 - p_e^2}{p_e^2 + (E + m_e)^2 - 2p_e(E + m_e)\cos\theta} = \frac{m_e}{E - p_e\cos\theta}}. \tag{2.32}$$

Problem 2.3)

A toroid is a "donut" shaped coil, and the figure below shows an overhead cross sectional view of one. They are used in nuclear fusion reactors called tokamaks. Use Ampere's law to derive the equation for the magnitude of the magnetic field in a toroid (N turns) of inner radius a and outer radius b at a distance r midway between a and b.

Express your result in terms of a, b, current I, N and any constants.

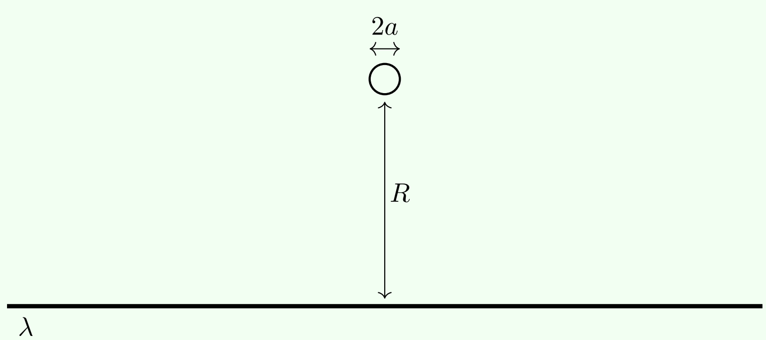


If we take our Amperian loop as shown by the dotted blue line, then

$$B(2\pi r) = \mu_0 N I \Rightarrow \boxed{\boldsymbol{B} = \frac{\mu_0 N I}{2\pi r}\hat{\boldsymbol{\phi}}}. \tag{2.33}$$

Problem 2.4)

A small neutral metallic conducting sphere with radius a is separated by a transverse distance $R \gg a$ from an infinitely long wire of negligible thicknes and charge per unit length λ.



Calculate the force between the metallic sphere and the wire.

Hint: Recall the induced electric dipole moment of a conducting sphere in a uniform electric field $\boldsymbol{E}$ is $\boldsymbol{p} = 4\pi a^3 \boldsymbol{E}$.

If we assume that $a \ll R$, then we can treat the field produced by the wire as uniform enough to induce a dipole over the sphere. Such a field is determined by Gauss' law to be

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\boldsymbol{r}}, \tag{2.34}$$

where r is the distance from the wire. The induced dipole is then just

$$\boldsymbol{p} = 4\pi\epsilon_0 a^3 \boldsymbol{E} = \frac{2\lambda a^3}{R}\hat{\boldsymbol{r}}. \tag{2.35}$$

The force on sphere by the wire is then just

$$\boxed{\boldsymbol{F} = (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E} = \frac{2\lambda a^3}{R}\frac{\partial \boldsymbol{E}}{\partial r} = -\frac{2\lambda a^3}{R}\frac{\lambda}{2\pi\epsilon_0 R^2}\hat{\boldsymbol{r}} = -\frac{\lambda^2 a^3}{\pi\epsilon_0 R^3}\hat{\boldsymbol{r}}}. \tag{2.36}$$

Problem 3.1)

The ψ' particle, which is a bound state $\bar{c}c$ of charmed quarks, has mass approximately equal to 3.7 GeV/c^2.

What is the minimal ("threshold") energy of photons necessary to produce ψ' particles in the reaction $\gamma p \to p\psi'$ from the Hall D photon source at JLab accelerator?

The threshold kinetic energy of the photon for ψ' production is defined such that

$$\sqrt{s} = \sqrt{m_p^2 + 2m_p T_{\text{th}}} = m_{\psi'} + m_p$$

$$\boxed{T_{\text{th}} = m_{\psi'} + \frac{m_{\psi'}^2}{2m_p} \approx 11.7 \text{ GeV}}. \tag{2.37}$$

Problem 3.2)

A cylinder of radius $\rho = a$ carries an azimuthal surface current $\boldsymbol{K} = f(z)\hat{\phi}$ where $f(z)$ is an arbitrary function, in cylindrical coordinate (ρ, ϕ, z).

Find expressions for $\boldsymbol{B}(0)$, the magnetic field at the origin, and $\boldsymbol{m}$, the magnetic moment of the system, as integrals involving $f(z)$.

The Biot-Savart law is given generally by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\boldsymbol{r}'\,\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{2.38}$$

Working in cylindrical coordinates and evaluating at the origin, we have

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0 a}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_{-\infty}^{\infty}\mathrm{d}z'\,\frac{f(z')\hat{\boldsymbol{\phi}}\times-(a\hat{\boldsymbol{s}}+z\hat{\boldsymbol{z}})}{(a^2+z'^2)^{3/2}} \\
&= \frac{\mu_0 a}{4\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_{-\infty}^{\infty}\mathrm{d}z'\,\frac{f(z')}{(a^2+z'^2)^{3/2}}(a\hat{\boldsymbol{z}}-z\hat{\boldsymbol{s}}) \\
&= \boxed{\frac{\mu_0 a^2}{2}\hat{\boldsymbol{z}}\int_{-\infty}^{\infty}\mathrm{d}z'\,\frac{f(z')}{(a^2+z'^2)^{3/2}}}\,.
\end{aligned} \tag{2.39}$$

The magnetic moment of the cylinder is given as

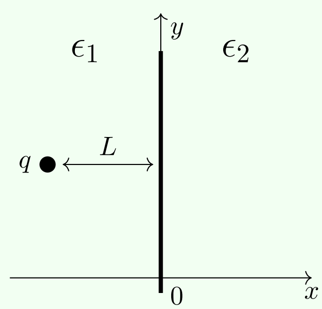
$$\boxed{\boldsymbol{m} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r}) = \frac{a}{2}\int_0^{2\pi}\mathrm{d}\phi'\int_{-\infty}^{\infty}\mathrm{d}z'\,(a\hat{\boldsymbol{s}}+z'\hat{\boldsymbol{z}})\times f(z')\hat{\boldsymbol{\phi}} = \frac{a^2}{2}\hat{\boldsymbol{z}}\int_{-\infty}^{\infty}f(z')\,\mathrm{d}z'}\,. \tag{2.40}$$

Problem 3.3)

Consider two semi-infinite dielectric media with permittivities ϵ_1 at $x<0$ and ϵ_2 at $x>0$. Let a charge q be in a medium 1 at $x_0=-L$, where L is a distance between the charge and the planar interface $x=0$ between the media. Calculate:

(a) The electric potential $\varphi(x,y,z)$ in the entire space.

(b) The force F acting on the charge.



We can construct the potential via the method of images. For the potential in region 1, let's place an image charge the same distance from the yz-plane as the real charge but of magnitude q':

$$\varphi_1 = \frac{1}{4\pi\epsilon_1}\left[\frac{q}{\sqrt{(x+L)^2+\rho^2}}+\frac{q'}{(x-L)^2+\rho^2}\right], \tag{2.41}$$

where $\rho^2 = y^2 + z^2$. In the second region, let's replace the charge q with q'' to write

$$\varphi_2 = \frac{1}{4\pi\epsilon_2}\frac{q''}{\sqrt{(x+L)^2+\rho^2}}. \tag{2.42}$$

We can solve for the image charge magnitude q' and q'' by enforcing boundary conditions:

$$(\boldsymbol{D}_2 - \boldsymbol{D}_1)\cdot\hat{\boldsymbol{x}} = 0 \Rightarrow q - q' = q'' \tag{2.43}$$

$$(\boldsymbol{E}_2 - \boldsymbol{E}_1)\cdot\hat{\boldsymbol{\rho}} = 0 \Rightarrow \epsilon_2(q+q') = \epsilon_1 q''. \tag{2.44}$$

Thus

$$q' = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}q, \quad q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}q. \tag{2.45}$$

From this, we can write

$$\boxed{\varphi(x,y,z) = \frac{q}{4\pi}\begin{cases}\frac{1}{\epsilon_1}\left\{\frac{1}{\sqrt{(x+L)^2+y^2+z^2}} + \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{\sqrt{(x-L)^2+y^2+z^2}}\right\} & x<0 \\ \frac{2}{\epsilon_1+\epsilon_2}\frac{1}{\sqrt{(x+L)^2+y^2+z^2}} & x>0\end{cases}}. \tag{2.46}$$

(b) The force on the charge in region 1 is that from the image charge q'

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{q^2}{(2L)^2}\hat{\boldsymbol{x}}. \tag{2.47}$$

Quantum Mechanics

Problem 3.4)

A particle with spin $S = 1$ is in a state described by the following bra-vector in the $\hat{S}_z$ basis:

$$\langle\psi| = \frac{1}{\sqrt{14}}(-i, 2, 3)$$

(a) Calculate the probabilities that a measurements of S_z wll give 1, 0, and -1.

(b) Calculate the expectation values of $\langle S_z\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$.

Hint: The spin-1 matrices are

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad \hat{S}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

(a) The probabilities are simply

$$\boxed{P(1) = \frac{1}{14}, \quad P(0) = \frac{2}{7}, \quad P(-1) = \frac{9}{14}}. \tag{2.48}$$

(b) The expectation values are as follows:

$$\boxed{\begin{aligned} \langle S_x\rangle &= \frac{\hbar}{\sqrt{2}}\frac{1}{14}\begin{pmatrix}-i & 2 & 3\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}i\\ 2\\ 3\end{pmatrix} = \frac{6\hbar}{7\sqrt{2}}\\ \langle S_y\rangle &= \frac{\hbar}{\sqrt{2}}\frac{1}{14}\begin{pmatrix}-i & 2 & 3\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}i\\ 2\\ 3\end{pmatrix} = 0\\ \langle S_z\rangle &= \hbar\left[\frac{1}{14} - \frac{9}{14}\right] = -\frac{4\hbar}{7}\end{aligned}}. \tag{2.49}$$

Problem 4.1)

Consider a particle of mass m subject to a δ-function potential given by

$$V(x) = -\frac{\hbar^2}{2m}v_0\delta(x).$$

Suppose the particle is initially in the bound state. Suddenly, the potential $V(x)$ is changed to $\overline{V}(x)$ by increasing the strength $v_0 \to \overline{v}_0$. Assume that this sudden change does not affect the state of the particle. Compute the probability that the particle remains in the ground state corresponding to the potential $\overline{V}(x)$. Why is this probability less than one?

Evaluate the expectation value of the Hamiltonian with the potential $\overline{V}(x)$ and obtain the energy required to change $V(x) \to \overline{V}(x)$.

The time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \frac{\hbar^2}{2m}v_0\delta(x)\psi(x) = E\psi(x). \tag{2.50}$$

If we consider $E < 0$, then the bound state solution takes the form

$$\psi(x) = \begin{cases} Ae^{\kappa x} & x < 0 \\ Be^{-\kappa x} & x > 0, \end{cases} \tag{2.51}$$

where $\kappa = \sqrt{2m|E|/\hbar^2}$. We relate κ and A, B by using the following boundary conditions:

$$\psi(0^-) = \psi(0^+) \Rightarrow A = B \tag{2.52}$$

$$\psi(0^+) - \psi(0^-) = v_0\psi(0) \Rightarrow 2A\kappa = v_0 A \Rightarrow \kappa = \frac{v_0}{2}. \tag{2.53}$$

Thus,

$$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}. \tag{2.54}$$

The probability that the after the sudden change, the particle is measured to be in the ground state of the new potential is given by

$$\boxed{P = \left|\sqrt{\kappa\bar{\kappa}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-(\kappa+\bar{\kappa})|x|}\right|^2 = \frac{4\kappa^2\bar{\kappa}^2}{(\kappa+\bar{\kappa})^2} = \frac{v_0^2\bar{v}_0^2}{(v_0+\bar{v}_0)^2}}. \tag{2.55}$$

Note that the probability is less than one because the completeness of states for the second potential includes the continuum of states, whose overlap with the initial state is not necessarily zero.

The expectation value of the Hamiltonian in the new potential is

$$\begin{aligned} \langle H \rangle &= -\frac{\hbar^2\kappa}{2m}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\kappa|x|}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \bar{v}_0\delta(x)\right]e^{-\kappa|x|} \\ &= -\frac{\hbar^2\kappa}{2m}\left[\kappa^2\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-2\kappa|x|} + \bar{v}_0\right] = -\frac{\hbar^2 v_0}{4m}\left(\frac{v_0}{2} + \bar{v}_0\right) = E - \frac{\hbar^2 v_0\bar{v}_0}{4m}. \end{aligned} \tag{2.56}$$

The energy required to change our Hamiltonian as prescribed is then

$$\boxed{\Delta E = \frac{\hbar^2 v_0\bar{v}_0}{4m} = \frac{2\bar{v}_0}{v_0}|E|}. \tag{2.57}$$

Problem 4.2)

Consider a hydrogen atom exposed to a uniform electric field $\mathcal{E}\hat{z}$ (ignore spin degrees of freedom). Calculate the corrections to the ground-state energy level up to second order in perturbation theory. You may neglect the contribution from the continuum states in the second-order calculation.

Exploiting selection rules based on parity and L_z, you will realize that you only need the following ground- and excited-state wave functions to carry out this calculation,

$$\phi_{100}(r) = R_{10}(r)\underbrace{\frac{1}{\sqrt{4\pi}}}_{Y_{00}}, \quad \phi_{n10}(r) = R_{n1}(r)\underbrace{\sqrt{\frac{3}{4\pi}}\cos\theta}_{Y_{10}}$$

Express the result in terms of the overlap integral

$$\gamma_n = \int_0^\infty \mathrm{d}r\ r^3 R_{n1}(r) R_{10}(r).$$

Note that you do not need to evaluate this integral!

The first order correction vanishes:

$$E_1^{(1)} = \langle \phi_{100}|q\mathcal{E}z|\phi_{100}\rangle = q\mathcal{E}\,\langle \phi_{100}|z|\phi_{100}\rangle = 0, \tag{2.58}$$

where the expectation value of z vanishes because of parity selection rules. Thus, we must go to second order to see if we have a nonvanishing correction:

$$E_1^{(2)} = \sum_{nlm\neq 100} \frac{|\langle \phi_{nlm}|qEz|\phi_{100}\rangle|^2}{\epsilon_1 - \epsilon_n} = \frac{q^2E^2}{\epsilon_1}\sum_{nlm\neq 100}\frac{1}{1-1/n^2}|\langle \phi_{nlm}|z|\phi_{100}\rangle|^2. \tag{2.59}$$

All that remains is to determine the matrix elements in the sum:

$$\begin{aligned}\langle \phi_{nlm}|z|\phi_{100}\rangle &= \int \mathrm{d}^3\boldsymbol{r}\, R_{nl}(r)Y^*_{lm}(\Omega)zR_{10}(r)Y_{10}(\Omega) = \frac{1}{3}\delta_{l1}\delta_{m0}\int_0^\infty \mathrm{d}r\, r^3 R_{n_1}(r)R_{10}(r)\\ &= \frac{\gamma_n}{3}\delta_{l1}\delta_{m0}.\end{aligned} \tag{2.60}$$

Thus

$$\boxed{E_1^{(2)} = \frac{q^2E^2}{3\epsilon_1}\sum_{n=2}^{\infty}\frac{\gamma_n^2}{1-1/n^2}}. \tag{2.61}$$

Problem 4.3)

Consider a system with a three-dimensional state space. The Hamiltonian $\hat{H}$ has a non-degenerate eigenvalue E_1 with (normalized) eigenstate $|\phi_1\rangle$ and a degenerate eigenvalue E_2 with (orthonormal) eigenstates $|\phi_2\rangle$ and $|\phi_3\rangle$. Suppose at time $t = 0$, the system is in the normalized state $|\psi(0)\rangle$ given by

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{2}(|\phi_2\rangle + |\phi_3\rangle).$$

(a) At $t = 0$ the energy of the system is measured. What values can be found and with what probabilities?

(b) Suppose at $t = 0$, instead of $\hat{H}$, the observable $\hat{A}$, which in the basis $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ is represented by the following matrix

$$A = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $a > 0$ is real, is measured with the system in state $|\psi(0)\rangle$. What results can be found and with what probabilities? Do $\hat{H}$ and $\hat{A}$ commute?

(c) What is the mean value $\langle\psi(t)|\,\hat{A}\,|\psi(t)\rangle$?

(a) We have

$$P(E_1) = \frac{1}{2}, \quad P(E_2) = \frac{1}{2}. \tag{2.62}$$

(b) Notice that the state $|\phi_1\rangle$ is already an eigenstate of A with eigenvalue a, so we only have to diagonalize the block representing the subspace $\{|\phi_2\rangle, |\phi_3\rangle\}$:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1. \tag{2.63}$$

The corresponding eigenvectors and eigenvalues of A are then

$$\pm a \Leftrightarrow |\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|\phi_2\rangle \pm |\phi_3\rangle). \tag{2.64}$$

Using these states, we can write

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\chi_+\rangle. \tag{2.65}$$

From this we can see that, a measurement of the system can yield

$$P(a) = 1, \quad P(-a) = 0. \tag{2.66}$$

(c) First, we write

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}|\phi_1\rangle + \frac{1}{2}e^{-iE_2t/\hbar}(|\phi_2\rangle + |\phi_3\rangle). \tag{2.67}$$

Thus, the expectation value

$$\langle\psi(t)|A|\psi(t)\rangle = a, \tag{2.68}$$

which is time independent since $|\psi(t)\rangle$ is an eigenstate of A with eigenvalue a at all times t.

Problem 4.4)

Consider a system with a two-dimensional state space. In this space, the states $|1\rangle$ and $|2\rangle$ form an orthonormal basis. The Hamiltonian describing the system in this basis has the form

$$\hat{H} = H_{11}\left|1\right\rangle\left\langle 1\right| + H_{22}\left|2\right\rangle\left\langle 2\right| + H_{12}(\left|1\right\rangle\left\langle 2\right| + \left|2\right\rangle\left\langle 1\right|),$$

where H_{11}, H_{22}, and H_{12} are real parameters with dimension of energy.

(a) Assume $H_{12} = 0$. Write down the eigenvalues and eigenvectors of $\hat{H}$ in the basis $|1\rangle$, $|2\rangle$.

(b) Now, assume $H_{12} \neq 0$. Obtain the eigenvalues and corresponding eigenvectors of $\hat{H}$. Make sure that they reduce to the eigenvalues and eigenvectors of part (a) above in the limit $H_{12} \to 0$. It is convenient to introduce the parameter

$$\lambda = \frac{2H_{12}}{H_{11} - H_{22}} \quad \text{with } H_{11} \neq H_{22},$$

and express results in terms of λ.

(a) The Hamiltonian as a matrix in the basis $\{\left|1\right\rangle, \left|2\right\rangle\}$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}. \tag{2.69}$$

If $H_{12} = 0$, then the matrix is already diagonal with spectrum

$$\boxed{H_{11} \leftrightarrow \left|1\right\rangle, \quad H_{22} \leftrightarrow \left|2\right\rangle}. \tag{2.70}$$

(c) Assuming that the off-diagonal elements are nonzero, we diagonalize via the standard procedure:

$$\boxed{\begin{aligned} E_\pm &= \frac{(H_{11} + H_{22}) \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2}}{2} \\ &= \frac{1}{2}\left[(H_{11} + H_{22}) \pm |H_{11} - H_{22}|\left(1 + \frac{\lambda^2}{2} + \dots\right)\right], \end{aligned}}$$

where λ is as prescribed above. Note that the last equality arises from the limit that the off-diagonal elements are much smaller than the difference of the diagonal ones. This result reduces to that of part (a) if $\lambda = 0$.

3 May 2022

Classical Mechanics

Problem 1.1)

Two (1-dimensional) pendula made of massless rods of equal length L and points of masses m and M at the end are hung side-by-side. The ends of pendula are connected by a spring constant k that is in its relaxed state when both pendula hang straight down.

(a) Using the two angles ϕ_1 and ϕ_2 of the two rods with the vertical as generalized coordinates, and the small-angle approximation, write down the Lagrangian for this problem.

(b) Recast the Lagrangian into the form

$$\frac{1}{2}\dot{\boldsymbol{\phi}}\boldsymbol{T}\dot{\boldsymbol{\phi}} - \frac{1}{2}\boldsymbol{\phi}\boldsymbol{V}\boldsymbol{\phi}$$

with 2×2 matrices $\boldsymbol{T}$ and $\boldsymbol{V}$.

(c) Write down the Euler-Lagrange equations in the same matrix form, and insert the ansatz

$$\boldsymbol{\phi}(t) = \boldsymbol{a}\exp(-i\omega t)$$

to end up with an "eigenvalue" equation for $\lambda = \omega^2$.

(d) Find the possible values for $\lambda_{1,2}$ and the corresponding fundamental modes $\boldsymbol{a}_{1,2}$ (No need to normalize them).

(e) Describe and contrast the two fundamental modes: What does the motino look like in each case, and what frequency does it have?

(a) The position vectors $\boldsymbol{r}_{1,2} = L(\sin\phi_{1,2}\hat{\boldsymbol{x}} - \cos\phi_{1,2}\hat{\boldsymbol{y}})$. This allows us to write

$$\begin{aligned} L &= \frac{m_1\dot{\boldsymbol{r}}_1^2}{2} + \frac{m_2\dot{\boldsymbol{r}}_2^2}{2} - m_1 g y_1 - m_2 g y_2 - \frac{k}{2}(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 \\ &= \frac{m_1 L^2}{2}\dot{\phi}_1^2 + \frac{m_2 L^2}{2}\dot{\phi}_2^2 + m_1 g L\cos\phi_1 + m_2 g L\cos\phi_2 \qquad (3.1) \\ &\quad - \frac{kL^2}{2}\Big[(\sin\phi_1 - \sin\phi_2)^2 + (\cos\phi_1 - \cos\phi_2)^2\Big]. \qquad (3.2) \end{aligned}$$

Let us make the assumption that the angular displacements are small such that $\sin\phi_{1,2} \approx \phi_{1,2}$ and $\cos\phi_{1,2} \approx 1 - \phi_{1,2}^2/2$. The Lagrangian then becomes

$$\boxed{L = \frac{m_1 L^2}{2}\dot{\phi}_1^2 + \frac{m_2 L^2}{2}\dot{\phi}_2^2 - \frac{m_1 g L}{2}\phi_1^2 + \frac{m_2 g L}{2}\phi_2^2 - \frac{5kL^2}{4}(\phi_1 - \phi_2)^2}. \qquad (3.3)$$

(b) The matrices

$$\boxed{T = \begin{pmatrix} m_1L^2 & 0 \\ 0 & m_2L^2 \end{pmatrix}, \quad V = \begin{pmatrix} m_1gL + 5kL^2/4 & -5kL^2/2 \\ -5kL^2/2 & m_2gL + 5kL^2/4 \end{pmatrix}}. \tag{3.4}$$

(c) The Euler-Lagrange equations of motion are as follows:

$$m_1L^2\ddot{\phi}_1 + m_1gL\phi_1 + \frac{5kL^2}{2}(\phi_1 - \phi_2) = 0 \tag{3.5}$$

$$m_2L^2\ddot{\phi}_2 + m_2gL\phi_2 - \frac{5kL^2}{2}(\phi_1 - \phi_2) = 0. \tag{3.6}$$

In matrix form,

$$\begin{pmatrix} m_1L^2 & 0 \\ 0 & m_2L^2 \end{pmatrix}\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = \begin{pmatrix} m_1gL + 5kL^2/2 & -5kL^2/2 \\ -5kL^2/2 & m_2gL + 5kL^2/2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.7}$$

Let's insert the ansatz $\boldsymbol{\phi} = \boldsymbol{a}e^{-i\omega t}$, which yields

$$\begin{pmatrix} m_1gL + 5kL^2/2 - m_1L^2\omega^2 & -5kL^2/2 \\ -5kL^2/2 & m_2gL + 5kL^2/2 - m_2L^2\omega^2 \end{pmatrix}\boldsymbol{a} = 0. \tag{3.8}$$

For nontrivial solutions $\boldsymbol{a}$, we must have that the determinant of the matrix above is zero. Enforcing this conition, we obtain the quadratic equation

$$m_1m_2L^2\omega^4 + 2\Big(m_1m_2gL + \frac{5k(m_1+m_2)L^2}{8}\Big)\omega^2 + \Big(m_1m_2g^2L^2 + \frac{5k(m_1+m_2)L^2}{4}\Big) = 0 \tag{3.9}$$

(d) Solving, we find

$$\omega_\pm^2 = \Big(\frac{g}{L} + \frac{5k(m_1+m_2)}{8m_1m_2}\Big) \pm \frac{5k(m_1+m_2)}{8m_1m_2}$$

$$\Rightarrow \boxed{\omega_+^2 = \frac{g}{L}, \quad \omega_-^2 = \frac{g}{L} + \frac{5k(m_1+m_2)}{4m_1m_2}}. \tag{3.10}$$

The corresponding eigenvectors are determined as follows:

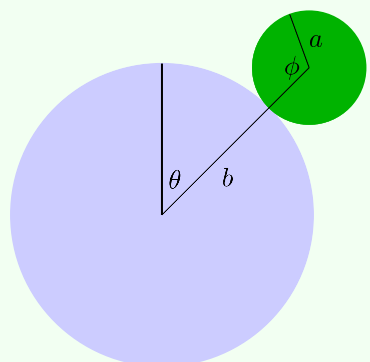
$$-\frac{5kL^2}{4}\begin{pmatrix} m_1/m_2 & 1 \\ 1 & m_2/m_1 \end{pmatrix}\begin{pmatrix} a_{1+} \\ a_{2+} \end{pmatrix} = 0 \Rightarrow \boxed{\boldsymbol{a}_+ = \begin{pmatrix} 1 \\ -m_1/m_2 \end{pmatrix}} \tag{3.11}$$

$$-\frac{5kL^2}{4}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_{1-} \\ a_{2-} \end{pmatrix} = 0 \Rightarrow \boxed{\boldsymbol{a}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}. \tag{3.12}$$

(e) The normal modes can be interpreted as follows. The "+" branch describes the oscillators moving in unison in the same direction. That is, they are essentially independent since the spring does not compress given that the displacements of the oscillators are the same. On the other hand, the "−" branch describes the oscillators moving in opposite directions, in phase.

Problem 1.2)

There is a cylinder with radius a and mass m rolling without slipping on top of another, fixed cylinder with radius b. The first cylinder starts out exactly on top of the second cylinder. (See the figure below.) Write down the equations of motion for the time period *before* the top cylinder disconnects from the bottom cylinder. Express your answer in terms of θ and its derivatives, $r = a + b$, b, and m.



Taking our origin at the center of the larger cylinder, we can write the position vector of the smaller cylinder as

$$\boldsymbol{r} = r(\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}}) \Rightarrow \dot{\boldsymbol{r}} = r\dot{\theta}(\cos\theta\hat{\boldsymbol{x}} - \sin\theta vu * y) \Rightarrow v^2 = r^2\dot{\theta}^2, \tag{3.13}$$

where $r = a + b$. The kinetic energy

$$T = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2}{2} + \frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{v^2}{a^2}\right) = \frac{3mv^2}{4}. \tag{3.14}$$

Therefore, the Lagrangian

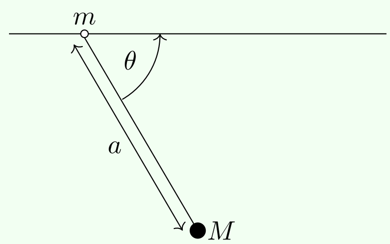
$$L = \frac{3mr^2\dot{\theta}^2}{4} - mgr\cos\theta. \tag{3.15}$$

From this, we find the equation of motion

$$\boxed{\ddot{\theta} - \frac{2g}{3r}\sin\theta = 0}. \tag{3.16}$$

Problem 1.3)

A particle of mass M is attached by a massless rod of length a to a small ring of mass m, free to slide on a fixed horizontal bar. The string moves in the vertical plane through the bar.



(a) Write down the Lagrangian and Euler-Lagrange equations for the system.

(b) Find conserved quantities.

(c) Find frequency of small oscillations, if any.

(a) Let x denote the position of the mass m and X, Y denote the coordinates of the mass M. We can express

$$X = x + a\cos\theta, \quad Y = -a\sin\theta. \tag{3.17}$$

The Lagrangian

$$\boxed{\begin{aligned} L &= \frac{m\dot{x}^2}{2} + \frac{M}{2}(\dot{X}^2 + \dot{Y}^2) - MgY \\ &= \frac{(m+M)\dot{x}^2}{2} + \frac{M}{2}(a^2\dot{\theta}^2 - 2a\dot{x}\dot{\theta}\sin\theta) + Mga\sin\theta \end{aligned}}. \tag{3.18}$$

The equations of motion are given as follows:

$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big[(m+M)\dot{x} - Ma\dot{\theta}\sin\theta\Big] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[(m+M)x + Ma\cos\theta\Big] = 0 \\ Ma^2\ddot{\theta} - Ma(g + \dot{x}\dot{\theta})\cos\theta = Ma\ddot{x}\sin\theta. \end{aligned}} \tag{3.19}$$

(b) One can see above that the momentum conjugate to x is conserved:

$$p_x = \frac{\mathrm{d}L}{\mathrm{d}\dot{x}} = (m+M)\dot{x} - Ma\dot{\theta}\sin\theta. \tag{3.20}$$

Notice that this is just the x component of the center of mass momentum. Obviously, we also have energy conservation.

(c) If we insert the conserved momentum into the equation of motion for θ above, we have

$$Ma^2\Big(1 - \frac{M}{m+M}\sin^2\theta\Big)\ddot{\theta} - \frac{Map_x}{m+M}\dot{\theta}\cos\theta - Mag\cos\theta = 0. \tag{3.21}$$

If we consider small oscillations such that $\phi = \pi/2 - \theta \ll 1$, we obtain

$$\ddot{\phi} + \Big(1 + \frac{M}{m}\Big)\frac{g}{a}\phi = 0. \tag{3.22}$$

Notice that we neglected the term $\dot{\theta}\cos\theta$ since we have a product of small quantities. From this, we find the frequency of small oscillations to be

$$\boxed{\omega^2 = \sqrt{\Big(1 + \frac{M}{m}\Big)\frac{g}{a}}}. \tag{3.23}$$

Notice that this reduces to the usual pendulum result when $m \gg M$, implying that the suspension point moves very little compared to the hanging mass.

Problem 1.4)

Two bodies move under the influence of the potential $V(r) = kr^{\alpha}$ where $\boldsymbol{r}$ is the relative coordinate and k and α are constants.

(a) If $\boldsymbol{r} = \boldsymbol{f}(t)$ is a solution of the equation of motion, show that $\boldsymbol{r} = \lambda\boldsymbol{f}(\lambda^{\sigma}t)$ is also a solution for any λ provided σ is suitably chosen.

(b) Apply the result of part (a) to the cases $\alpha = 2$ (harmonic oscillator) and $\alpha = -1$ (Kepler problem). Comment on the results and on the properties you can derive from them.

Hint: Use $m\ddot{\boldsymbol{r}} = -\nabla V(\boldsymbol{r})$.

(a) Let us denote $\boldsymbol{r}' = \lambda\boldsymbol{r}(\tau)$, where $\tau = \lambda^{\sigma}t$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = \lambda^{\sigma}\frac{\mathrm{d}}{\mathrm{d}\tau} \Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \lambda^{2\sigma}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}. \tag{3.24}$$

Additionally,

$$\frac{\partial}{\partial r'} = \frac{\partial r}{\partial r'}\frac{\partial}{\partial r} = \frac{1}{\lambda}\frac{\partial}{\partial r}. \tag{3.25}$$

Thus, for $\boldsymbol{r}'$ to satisfy the desired equation of motion

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}' = \lambda^{2\sigma}\Big(m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\boldsymbol{r}(\tau)\Big) = -\boldsymbol{\nabla}'V(r') = \lambda^{\alpha-1}\Big(-\boldsymbol{\nabla}V(r)\Big). \tag{3.26}$$

In order for $\boldsymbol{r}'(\tau)$ to satisfy the same equation of motion as $\boldsymbol{r}(t)$, we must have

$$\boxed{\sigma = \frac{\alpha}{2} - 1}. \tag{3.27}$$

(b) Observe that

$$\boxed{\alpha = 2 \Rightarrow \sigma = 0, \quad \alpha = -1 \Rightarrow \sigma = -\frac{3}{2}}. \tag{3.28}$$

The latter yields Kepler's third law since $\lambda^3 = (t/\tau)^2$.

Problem 2.1)

Find the minimal distance between two particles when one of them (having mass m) moves from infinity with velocity v and impact parameter ρ towards the second one that is initially at rest (and has mass M). The potential energy of the particle's interaction is given by $U(r) = -U_0(R/r)^2$, where r is the distance between particles, while $U_0 > 0$ and R are constants.

At the minimum separation distance, we have

$$E = \frac{\mu v^2}{2} = \frac{M^2}{2\mu r^2} - \frac{U_0 R^2}{r^2} = \frac{1}{r^2}\left[\frac{Em^2\rho^2}{\mu^2} - U_0 R^2\right]$$

$$\boxed{r_{\min} = \sqrt{\frac{(m+M)^2\rho^2}{M^2} - \frac{U_0 R^2}{E}}}. \tag{3.29}$$

Notice that if E is small enough, then r_{min} is imaginary, which implies that there is no turning point. That is, the particles are pulled together with $r = 0$ being unavoidable.

Electricity & Magnetism

Problem 2.2)

Consider an infinitely long straight wire along the z-axis. Suppose the wire gets a sudden current by $I(t) = a\delta(t)$, where a is a constant and $\delta(t)$ is the Dirac delta function. Find

(a) the electric and magnetic potentials $\Phi(\boldsymbol{r}, t)$, $\boldsymbol{A}(\boldsymbol{r}, t)$, and

(b) electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$

(a) In the Lorenz gauge, the scalar and vector potentials satisfy wave equations with the charge and current densities as sources, respectively. That is,

$$\begin{aligned}\Phi(\boldsymbol{x},t) &= \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\boldsymbol{x}' \, \frac{\rho(\boldsymbol{x}', t-|\boldsymbol{x}-\boldsymbol{x}'|/c)}{|\boldsymbol{x}-\boldsymbol{x}'|} \\ \boldsymbol{J}(\boldsymbol{x},t) &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\boldsymbol{x}' \, \frac{\boldsymbol{J}(\boldsymbol{x}', t-|\boldsymbol{x}-\boldsymbol{x}'|/c)}{|\boldsymbol{x}-\boldsymbol{x}'|}.\end{aligned} \tag{3.30}$$

The charge density is zero, while the current density reads

$$\boldsymbol{J}(\boldsymbol{x},t) = a\delta(t)\delta(x)\delta(y)\hat{\boldsymbol{z}}. \tag{3.31}$$

Putting these into the potentials, we find

$$\Phi(\boldsymbol{x},t) = 0 \tag{3.32}$$

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{x},t) &= \frac{\mu_0}{4\pi}\hat{\boldsymbol{z}}\int \mathrm{d}^3\boldsymbol{x}' \, \frac{a\delta(t-|\boldsymbol{x}-\boldsymbol{x}'|/c)\delta(x')\delta(y')}{|\boldsymbol{x}-\boldsymbol{x}'|} \\ &= \frac{\mu_0 a}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}z' \, \frac{\delta(t-|\boldsymbol{x}-z'\hat{\boldsymbol{z}}|/c)}{|\boldsymbol{x}-z'\hat{\boldsymbol{z}}|}.\end{aligned} \tag{3.33}$$

Let

$$\begin{aligned} f(z') &= t - \frac{|\boldsymbol{x}-z'\hat{\boldsymbol{z}}|}{c} = t - \frac{1}{c}\sqrt{x^2+y^2+(z-z')^2} \\ f(z') = 0 \Rightarrow z'_{\pm} &= z \pm \sqrt{c^2t^2-x^2-y^2} = z \pm \sqrt{c^2t^2-s^2} \\ \Rightarrow \left|\left.\frac{\partial f}{\partial z'}\right|_{z'=z'_{\pm}}\right| &= \frac{1}{c}\frac{z-z'}{x^2+y^2+(z-z')^2} = \pm\frac{\sqrt{c^2t^2-s^2}}{c^2t}.\end{aligned} \tag{3.34}$$

Using the property that

$$\int \mathrm{d}x \, g(x)\delta(f(x)) = \sum_i \frac{g(x_i)}{|f'(x_i)|}, \tag{3.35}$$

we find

$$\boxed{\begin{aligned}\Phi(\boldsymbol{x},t) &= 0 \\ \boldsymbol{A}(\boldsymbol{x},t) &= \frac{\mu_0 ac}{2\pi\sqrt{c^2t^2-s^2}}\hat{\boldsymbol{z}}.\end{aligned}} \tag{3.36}$$

(b) The electric and magnetic fields

$$\boxed{\begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\Phi - \frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\mu_0 ac^2}{2\pi}\frac{ct}{(c^2t^2-s^2)^{3/2}}\hat{\boldsymbol{z}} \\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} = \frac{\mu_0 acs}{2\pi(c^2t^2-s^2)^{3/2}}\hat{\boldsymbol{\phi}}.\end{aligned}} \tag{3.37}$$

Problem 2.3)

Two thin coaxial rings, each of radius a, are a distance b apart, and each uniformly charged with charges Q_1 and Q_2. The work required to bring a point charge q from infinity up to the centers of each of the two rings is W_1 and W_2, respectively. Show that the charges on the rings are

$$Q_{1,2} = \frac{4\pi\epsilon_0 a}{b^2 q}(a^2+b^2)^{1/2}\Big[(a^2+b^2)^{1/2}W_{1,2} - aW_{2,1}\Big]$$

The potential on the axis of a ring of radius a with charge Q is given by

$$V(z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{a^2+z^2}}, \tag{3.38}$$

where z is the distance from the center of the ring. The work to bring in a test charge q a distance z from the center of the ring is just $W = qV(z)$. With this, we find that

$$\begin{aligned} W_1 &= \frac{q}{4\pi\epsilon_0}\left\{\frac{Q_1}{a} + \frac{Q_2}{\sqrt{a^2+b^2}}\right\} \\ W_2 &= \frac{q}{4\pi\epsilon_0}\left\{\frac{Q_1}{\sqrt{a^2+b^2}} + \frac{Q_2}{a}\right\}. \end{aligned} \tag{3.39}$$

Solving for Q_1 in terms of W_1 and Q_2 we obtain

$$Q_1 = a\left\{\frac{4\pi\epsilon_0}{q}W_1 - \frac{Q_2}{\sqrt{a^2+b^2}}\right\}. \tag{3.40}$$

Plugging this into the second expression, we obtain

$$\begin{aligned} W_2 &= \frac{a}{\sqrt{a^2+b^2}}W_1 + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{a}{a^2+b^2}\right)Q_2 \\ &\boxed{Q_2 = \frac{4\pi\epsilon_0 a}{b^2 q}(a^2+b^2)^{1/2}\Big[(a^2+b^2)^{1/2}W_2 - aW_1\Big]}. \end{aligned} \tag{3.41}$$

For Q_1, we can interchange the labels $1 \leftrightarrow 2$ to obtain the desired expression.

Problem 2.4)

The region bounded by two concentric spherical surfaces is filled with a uniform charge density ρ_0 (constant). On the inner boundary ($r = a$) of this region, the potential is

$$\Phi(a,\theta) = V_0\cos\theta.$$

On the outer boundary ($r = b$) of this region, the potential is

$$\Phi(b, \theta) = 2V_0.$$

Find the solution of Poisson's equation in the region $a \leq r \leq b$.

We can construct the general solution in two pieces. First, we solve Laplace's equation with the prescribed boundary conditions, which yields Φ_1. Next, we will solve Poisson's equation with the prescribed charge density and grounded spheres, yielding Φ_2. The full solution will then just be $\Phi = \Phi_1 + \Phi_2$.

Proceeding with the first part, we write

$$\Phi(\boldsymbol{r}) = \sum_l P_l(\cos\theta) \begin{cases} A_l r^l & r < a \\ B_l r^l + C_l/r^{l+1} & a < r < b \\ D_l/r^{l+1} & r > b. \end{cases} \tag{3.42}$$

The boundary conditions are as follows:

$$\Phi(a) = V_0 \cos\theta, \quad \Phi(a^-) = \Phi(a^+), \quad \Phi(b) = 2V_0, \quad \Phi(b^-) = \Phi(b^+). \tag{3.43}$$

Enforcing them, we find

$$\begin{aligned} A_l &= \frac{V_0}{a}\delta_{l1}, \quad D_l = 2V_0 b\delta_{l0} \\ B_l &= \frac{A_l a^{2l+1} - D_l}{b^{2l+1} - a^{2l+1}} = \frac{V_0 a^2}{b^3 - a^3}\delta_{l1} - \frac{2V_0 b}{b - a}\delta_{l0} \\ C_l &= \frac{a^{2l+1}b^{2l+1}}{b^{2l+1} - a^{2l+1}} A_l + \frac{D_l a^{2l+1}}{b^{2l+1} - a^{2l+1}} = \frac{V_0 a^2 b^3}{b^3 - a^3}\delta_{l1} + \frac{2V_0 ab}{b - a}\delta_{l0}. \end{aligned} \tag{3.44}$$

From this, we find

$$\boxed{\Phi_1 = V_0\left\{\frac{2b}{b-a}\left(\frac{a}{r} - 1\right) + \frac{a^2}{b^3 - a^3}\left(r + \frac{b^3}{r^2}\right)\cos\theta\right\}}. \tag{3.45}$$

Now, we proceed to solve Poisson's equation via Gauss' law:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = E(r)(4\pi r^2) = \frac{Q_a + \rho_0(4\pi r^3/3)}{\epsilon_0} \Rightarrow E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_a}{r^2} + \frac{\rho_0}{3\pi\epsilon_0}r. \tag{3.46}$$

The potential

$$\Phi_2 = \int_r^b E(r')\,\mathrm{d}r' = \frac{Q_a}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right) + \frac{\rho_0}{6\pi\epsilon_0}(b^2 - r^2). \tag{3.47}$$

Imposing that the inner sphere is also grounded, we find

$$Q_a = -\frac{4}{3}\rho_0 ab(a+b), \tag{3.48}$$

and

$$\boxed{\Phi_2 = -\frac{\rho_0}{3\pi\epsilon_0}\left\{ab(a+b)\left(\frac{1}{r}-\frac{1}{b}\right)-\frac{1}{2}(b^2-r^2)\right\}}. \tag{3.49}$$

Problem 3.1)

The ϕ particle, which is a bound state $\bar{s}s$ of strange quarks, has mass approximately equal to 1.02 GeV/c^2.

(a) What is the minimal ("threshold") energy of electrons necessary to produce ϕ particles in the reaction $ep \to ep\phi$ at JLab electron accelerator?

(b) What is the velocity and energy (in laboratory frame) of ϕ particles produced at threshold?

(a) The threshold energy is defined such that

$$\sqrt{s} = \sqrt{m_e^2 + m_p^2 + 2m_p(m_e + T)} = m_e + m_p + m_\phi$$

$$\Rightarrow \boxed{T = \frac{(m_e+m_p+m_\phi)^2 - (m_e^2+m_p^2)}{2m_p} - m_e = m_\phi\left(1 + \frac{m_\phi + 2m_e}{2m_p}\right) \approx 1.5 \text{ GeV}}. \tag{3.50}$$

(b) The velocity and energy of the ϕ particles produced at threshold in the lab frame are determined by the Lorentz factor, which can be determined by conservation of energy in the lab frame:

$$\begin{gathered} m_e + T + m_p = \gamma(m_e + m_\phi + m_p) \\ \Rightarrow \gamma = \frac{m_e + m_p + T}{m_e + m_\phi + m_p} = 1 + \frac{m_\phi(m_\phi + 2m_e)}{2m_p(m_e + m_p + m_\phi)} \approx \frac{5}{4}. \end{gathered} \tag{3.51}$$

From this, we find the energy of the ϕ particle in the lab frame to be

$$\boxed{E_\phi = \gamma m_\phi \approx \frac{5}{4} m_\phi}. \tag{3.52}$$

The velocity of the ϕ particle in the lab frame is given by

$$\boxed{\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx \sqrt{1 - \frac{16}{25}} = \frac{3}{5}}. \tag{3.53}$$

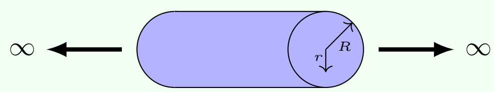
Problem 3.2)

An infinitely long, nonconducting solid cylinder of radius R has a **nonuniform volume charge density** $\rho(r)$ that is a function of the radial distance r from the axis. See the diagram below. Say that this charge density is $\rho(r) = Br^2$, where B is a constant with units of μC$/m^5$. Use Gauss' law to find the magnitude E of the resulting electric field when

(a) $0 < r < R$, and

(b) $r > R$.

Express your answers in terms of B, r, R, and any constants.



We can determine the electric field using Gauss' law and a cylindrical surface as follows:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = E(r)(2\pi r l) = \frac{1}{\epsilon_0} \int \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}) = \frac{2\pi l}{\epsilon_0} \int_0^r r' \rho(r')\, \mathrm{d}r' = \frac{2\pi l B}{\epsilon_0} \int_0^r r'^3 \Theta(r' < R)$$

$$\boxed{\boldsymbol{E} = \frac{B}{4\epsilon_0} \frac{\min^4(r, R)}{r}}. \tag{3.54}$$

Problem 3.3)

A plane electromagnetic wave with frequency ω traveling in vacuum along the z-axis (from $-\infty$) is given by

$$\boldsymbol{E}_{\mathrm{in}}(\boldsymbol{r}, t) = E_{0,\mathrm{in}} \hat{\boldsymbol{x}} \exp[i(kz - \omega t)]$$
$$\boldsymbol{B}_{\mathrm{in}}(\boldsymbol{r}, t) = E_{0,\mathrm{in}} \hat{\boldsymbol{y}} \exp[i(kz - \omega t)],$$

where $k = \omega/c$ and c is the speed of light (Gaussian units). At $z = 0$, the wave encounters an interface with a semi-infinite, linear dielectric medium filling the entire half-space $z > 0$. This medium has a dielectric constant (relative electric permittivity) $\epsilon > 1$ but unit magnetic permeability $\mu = 1$ and hence $\boldsymbol{B} = \boldsymbol{H}$. As a consequence, the medium has an index of refraction $n = \sqrt{\epsilon} > 1$ and a propagation speed $c/n < c$.

Therefore, the transmitted part of the electromagnetic wave in the medium has an electric field given by

$$\boldsymbol{E}_{\mathrm{tr}}(\boldsymbol{r},t) = E_{0,\mathrm{tr}}\hat{\boldsymbol{x}}\exp[i(nkz-\omega t)].$$

Finally, because of the boundary conditions (see below), there must also be a reflected wave going in the negative z-direction, with electric field

$$\boldsymbol{E}_{\mathrm{re}}(\boldsymbol{r},t) = E_{0,\mathrm{re}}\hat{\boldsymbol{x}}\exp[-i(kz+\omega t)]$$

(a) Determine the amplitudes $B_{0,\mathrm{tr}}$ and $B_{0,\mathrm{re}}$ of the magnetic fields of the transmitted and reflected waves,

$$\boldsymbol{B}_{\mathrm{tr}}(\boldsymbol{r},t) = B_{tr}\hat{\boldsymbol{y}}\exp[i(nkz-\omega t)]$$
$$\boldsymbol{B}_{\mathrm{re}}(\boldsymbol{r},t) = B_{0,\mathrm{re}}\hat{\boldsymbol{y}}\exp[-i(kz+\omega t)]$$

in terms of the corresponding amplitudes $E_{0,\mathrm{tr}}$ and $E_{0,\mathrm{re}}$. (It is best to use the last of Maxwell's equations, $\nabla\times\boldsymbol{E}+\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}=0$.)

(b) Using the requirement that **both** the sum of all electric fields and the sum of all magnetic fields must be continuous at $z=0$ (why?), determine the relative size of the amplitudes $E_{0,\mathrm{tr}}$ and $E_{0,\mathrm{re}}$ in terms of $E_{0,\mathrm{in}}$.

(c) Calculate the amplitude of the Poynting vector, $\boldsymbol{S}_0 = \frac{c}{4\pi}\boldsymbol{E}_0\times\boldsymbol{B}_0$, for all three waves, and show that energy is conserved (*i.e.* as much energy is carried in by the incoming wave per unit time as the reflected and transmitted waves carry out).

(a) Since we assume harmonic time-dependence, we have

$$\boldsymbol{B} = -\frac{ic}{\omega}\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{i\omega}{\omega}\frac{\partial E_x}{\partial z}\hat{\boldsymbol{y}}. \tag{3.55}$$

Thus

$$\boxed{B_{0,\mathrm{in}} = \frac{kc}{\omega}E_{0,\mathrm{in}} = E_{0,\mathrm{in}},\quad B_{0,\mathrm{tr}} = nE_{0,\mathrm{tr}},\quad B_{0,\mathrm{re}} = -E_{0,\mathrm{re}}}. \tag{3.56}$$

(b) The boundary conditions at the interface are as follows:

$$(\boldsymbol{D}_1-\boldsymbol{D}_2)\cdot\hat{\boldsymbol{z}} = \boldsymbol{E}_{1,z}-\epsilon\boldsymbol{E}_{2,z} = 0 \tag{3.57}$$
$$(\boldsymbol{E}_1-\boldsymbol{E}_2)\times\hat{\boldsymbol{z}} = -(E_{1,x}-E_{2,x})\hat{\boldsymbol{y}}+(E_{1,y}-E_{2,y})\hat{\boldsymbol{x}} = 0 \tag{3.58}$$
$$(\boldsymbol{B}_1-\boldsymbol{B}_2)\cdot\hat{\boldsymbol{z}} = B_{1,z}-B_{2,z} = 0 \tag{3.59}$$
$$(\boldsymbol{H}_1-\boldsymbol{H}_2)\times\hat{\boldsymbol{z}} = -(B_{1,x}-B_{2,x})\hat{\boldsymbol{y}}+(B_{1,y}-B_{2,y})\hat{\boldsymbol{x}} = 0. \tag{3.60}$$

The conditions on the components perpendicular to the interface are trivially satisfied, and the rest yield that the parallel components of the field are continuous, which in turn imply that the net fields in each media are the same:

$$E_{0,\text{in}} + E_{0,\text{re}} = E_{0,\text{tr}} \tag{3.61}$$
$$E_{0,\text{in}} - E_{0,\text{re}} = nE_{0,\text{tr}}. \tag{3.62}$$

Solving, we find

$$\boxed{\frac{E_{0,\text{re}}}{E_{0,\text{in}}} = \frac{n-1}{2}, \quad \frac{E_{0,\text{tr}}}{E_{0,\text{in}}} = \frac{n+1}{2}}. \tag{3.63}$$

(c) The Poynting vector for each wave

$$\begin{aligned} \boldsymbol{S}_{0,\text{in}} &= \frac{c}{4\pi}|E_{0,\text{in}}|^2\hat{\boldsymbol{z}} \\ \boldsymbol{S}_{0,\text{re}} &= -\frac{c}{4\pi}|E_{0,\text{re}}|^2\hat{\boldsymbol{z}}. \\ \boldsymbol{S}_{0,\text{tr}} &= \frac{c}{4\pi}n^2|E_{0,\text{tr}}|^2\hat{\boldsymbol{z}} \end{aligned} \tag{3.64}$$

The sum of the amplitudes of these vectors (dropping the factors of $c/4\pi$ for brevity) is

$$\boxed{|E_{0,\text{in}}|^2 + |E_{0,\text{re}}|^2 = |E_{0,\text{in}}|^2\left(1 + \frac{n-1}{2}\right) = |E_{0,\text{in}}|^2\frac{n+1}{2} = |E_{0,\text{tr}}|^2}. \tag{3.65}$$

Quantum Mechanics

Problem 3.4)

A particle of mass m is trapped by a very thin spherical shell of radius R modeled by the potential $U(r) = -V\delta(r-R)$ with $V > 0$. Consider only the s-state with zero orbital momentum and obtain:

(a) The equation for the ground state energy of the bound state.

(b) The critical radius R_c below which the bound state in the well disappears.

Schrödinger's equation in spherical coordinates takes the form

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} + \left[\frac{\boldsymbol{L}^2}{2mr^2} + V(r)\right]\psi = E\psi. \tag{3.66}$$

Writing $\psi(\boldsymbol{r}) = R(r)Y_{lm}(\Omega)$, the Schrödinger equation reduces to

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}R}{\mathrm{d}r} - \left[\frac{l(l+1)}{r^2} + v(r)\right]R = -\epsilon R, \tag{3.67}$$

where $v(r) = 2mV(r)/\hbar^2$ and $\epsilon = 2mE/\hbar^2$. Let us write $R(r) = u(r)/r$. The reduced radial equation is then

$$u''_{El}(r) + \left[\epsilon - \frac{l(l+1)}{r^2} - v(r)\right]u_{El}(r) = 0. \tag{3.68}$$

The s-wave equation in the δ-function potential reads

$$u''_{E0}(r) + \Big[\epsilon + v\delta(r-R)\Big]u_{E0}(r) = 0. \tag{3.69}$$

Separating into two regions, we have

$$u(r) = \begin{cases} A\cosh(\kappa x) + B\sinh(\kappa x) & x < R \\ Ce^{-\kappa x} & x > R, \end{cases} \tag{3.70}$$

where $\kappa = \sqrt{|\epsilon|}$. Since we consider a bound state $\epsilon < 0$. We have three boundary conditions:

$$(1):\ u(r\to 0) \to 0 \Rightarrow A = 0 \tag{3.71}$$

$$(2):\ u(R^-) = u(R^+) \Rightarrow B\sinh(\kappa R) = Ce^{-\kappa R} \tag{3.72}$$

$$(3):\ u'(R^+) - u'(R^-) = -vu(R) \Rightarrow -\kappa\Big[Ce^{-\kappa R} + B\cosh(\kappa R)\Big] = -vCe^{-\kappa R}. \tag{3.73}$$

The first condition is because our system behaves as if the potential is infinite in the region $r < 0$. From these equations, the condition on the energy of the bound state takes the form of the following transcendental equation:

$$\boxed{\frac{\sinh(\kappa R)}{\sinh(\kappa R) + \cosh(\kappa R)} = \frac{1}{2}\Big(1 - e^{-2\kappa R}\Big) = \frac{\kappa}{v}}. \tag{3.74}$$

(b) There is always a solution at $\kappa = 0$, which implies $E = 0$ but violates our original assumptions which we used to construct our wave function. Thus, we discard it and look for solutions with $\kappa > 0$. Notice, however, that the left-hand-side is a constant minus a decreasing exponential, while the right-hand side is a linear function with positive slope. Both are monotonically increasing functions, but the left-hand-side has a monotonically decreasing derivative while the right-hand-side has a constant slope of $1/v$. Thus, there will not be a second solution if

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\kappa}\frac{1}{2}\Big(1 - e^{-2\kappa R}\Big)\Big|_{\kappa=0} = R < \frac{1}{v} = R_c}. \tag{3.75}$$

Problem 4.1)

An electron is at a fixed position in an oscillating magnetic field

$$\boldsymbol{B}(t) = B_0 \cos(\omega t)\hat{\boldsymbol{z}},$$

wherer B_0 and ω are constants.

(a) Write down the Hamiltonian for this system.

(b) The electron is at time $t = 0$ in the spin state with eigenvalue $\hbar/2$ with respect to the x-axis. Determine the spin state of the electron at later times.

(c) Obtain the probability of obtaining $-\hbar/2$ if one measures S_x.

(a) The Hamiltonian for this system is

$$\boxed{H = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\frac{g_e e}{2mc} B_0 \cos(\omega t) S_z}. \tag{3.76}$$

(b) Since the Hamiltonian commutes with itself at different times, the time evolution operator

$$U(t) = e^{-\frac{i}{\hbar}\int_0^t H(t)\mathrm{d}t} = e^{i\lambda(t)\sigma_z}, \tag{3.77}$$

where $\lambda(t) = (\omega_0/\omega)\sin(\omega t)$ and $\omega_0 = g_e e B_0/(4mc)$. The state at an arbitrary time t is then

$$\boxed{|\psi(t)\rangle = e^{i\lambda(t)\sigma_z}|+\rangle = e^{i\lambda(t)}|+\rangle}. \tag{3.78}$$

(c) From the above work, we can see that the probability of obtaining $-\hbar/2$ upon measurement of S_x is just

$$\boxed{P(S_x = \hbar/2) = |\langle +_x|+\rangle|^2 = \left|\frac{1}{\sqrt{2}}e^{i\lambda(t)}\right|^2 = \frac{1}{2}}. \tag{3.79}$$

Problem 4.2)

Define a coherent state

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

where α is an arbitrary complex number and $|n\rangle$ is the eigenstate of the harmonic oscillator of energy $\hbar\omega(n+1/2)$.

(a) Show that $\langle\alpha|\alpha\rangle = 1$ and $|\alpha\rangle = \exp(-|\alpha|^2/2)\exp\left(\alpha a^\dagger\right)|0\rangle$.

(b) Show that coherent states are eigenstates of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$.

(a) We can prove that the coherent state is normalized as follows:

$$\boxed{\langle\alpha|\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}\alpha^m}{\sqrt{n!\,m!}}\langle n|m\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{(|\alpha|^2)^n}{n!} = 1}. \tag{3.80}$$

Next, we prove the form of $|\alpha\rangle$ in terms of the raising operator:

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\alpha^n(\alpha a^\dagger)^n|0\rangle = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}|0\rangle}, \tag{3.81}$$

where we have used that $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $|n\rangle = (a^\dagger)^n|0\rangle/\sqrt{n!}$.

(b) Observe that

$$\boxed{\begin{aligned} a|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}a|n\rangle = e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{(n-1)!}}|n-1\rangle \\ &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^{n+1}}{\sqrt{n!}}|n\rangle = \alpha|n\rangle\,. \end{aligned}}$$

Problem 4.3)

Consider a particle of charge q and mass m in one dimension in a harmonic oscillator potential and under the influence of a uniform electric field. The Hamiltonian reads

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2, \quad \hat{V} = -qE\hat{x}.$$

Assume that the electric field is weak, so that a perturbative calculation is permissible. The eigenenergies and eigenstates of the harmonic oscillator are well known:

$$\hat{H}_0|n\rangle = \hbar\omega(n+1/2)|n\rangle = \epsilon_n|n\rangle\,.$$

(a) Calculate the correction (up to including second order) to a generic energy level.

(b) Obtain the exact eigenenergies of $\hat{H}$ and compare them with the results obtained in part (a) above.

(c) Without doing any detailed calculation, explain why the third order correction to a generic level vanishes.

Hint: Note that, if the first-order correction $E_n^{(1)}$ vanishes, then the third-order correction to a non-degenerate energy level due to a perturbation $\hat{V}$ is simply given by

$$E_n^{(3)} = \sum_{a,b\neq n} \frac{V_{na}V_{ab}V_{bn}}{(\epsilon_a - \epsilon_n)(\epsilon_b - \epsilon_n)}.$$

(a) The first order perturbative correction to the energies

$$E_n^{(1)} = -qE \langle n|x|n\rangle = 0, \tag{3.82}$$

where we have used that

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \Rightarrow \langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\sqrt{m+1}\delta_{n,m+1} + \sqrt{m}\delta_{n,m-1}\Big). \tag{3.83}$$

Next, we consider the second order correction to the energies:

$$\begin{aligned} E_n^{(2)} = (qE)^2 \sum_{m\neq n} \frac{|\langle n|x|m\rangle|^2}{\epsilon_n - \epsilon_m} &= \frac{(qE)^2}{\hbar\omega}\frac{\hbar}{2m\omega}\sum_{m\neq n}\frac{|\sqrt{m+1}\delta_{n,m+1} + \sqrt{m}\delta_{n,m-1}|^2}{n-m} \\ &= \frac{(qE)^2}{2m\omega^2}\Big[n-(n+1)\Big] = \boxed{-\frac{(qE)^2}{2m\omega^2}}. \end{aligned} \tag{3.84}$$

(b) Observe that

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 - qEx = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left[x - \left(\frac{qE}{m\omega^2}\right)^2\right] - \frac{(qE)^2}{2m\omega^2}. \tag{3.85}$$

Thus, the exact energies of this system are

$$\boxed{E_n = \hbar\omega(n+1/2) - \frac{(qE)^2}{2m\omega^2}}. \tag{3.86}$$

(c) We can see that the second order correction calculated in part (a) matches the second term of the exact energies. Thus, all the higher order corrections must vanish.

Problem 4.4)

Two nonidentical spin-1/2 particles interact via the Hamiltonian

$$\hat{H} = A(\sigma_z^{(1)} + \sigma_z^{(2)}) + B\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)},$$

where σ_x, σ_y, and σ_z are the Pauli σ-matrices and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The "(1)" and "(2)" superscripts label particles 1 and 2 respectively. A and B are real constants.

Find the energy eigenvalues.

Hint: Notice that you can classify the states in terms of the eigenstates of the total spin operators $\hat{S}_z = \frac{\hbar}{2}(\sigma_z^{(1)} + \sigma_z^{(2)})$ and $\hat{S}^2 = \frac{\hbar^2}{4}(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})^2$ since they commute with $\hat{H}$.

We can rewrite the Hamiltonian as

$$H = A\Big(\sigma_z^{(1)} + \sigma_z^{(2)}\Big) + B\left[\frac{1}{2}\Big(\sigma_+^{(1)}\sigma_-^{(2)} + \sigma_-^{(1)}\sigma_+^{(2)}\Big) + \sigma_z^{(1)}\sigma_z^{(2)}\right], \tag{3.87}$$

where $\sigma_\pm$ represents the raising and lowering operators. Observe that the eigenstates are just the tensor product states of the spin states of particles 1 and 2:

$$\boxed{H\left|{++}\right\rangle = (2A+B)\left|{++}\right\rangle, \quad H\left|{\pm\mp}\right\rangle = -B\left|{\pm\mp}\right\rangle, \quad H\left|{--}\right\rangle = (-2A+B)\left|{--}\right\rangle}. \tag{3.88}$$

Alternatively, note that the composite angular momentum states are also eigenstates of the Hamiltonian. The $|1, \pm 1\rangle$ states are already enumerated, so all that remains are the following

$$H\left|10\right\rangle = -B\left|10\right\rangle, \quad H\left|00\right\rangle = -B\left|00\right\rangle. \tag{3.89}$$

4 January 2022

Classical Mechanics

Problem 1.1)

With the Lagrangian $\mathcal{L} = T - V$, the Euler-Lagrange equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

We can modify these equations by introducing a function $\mathcal{F}(\dot{q}_i)$ that depends on the velocities only, writing expanded equations as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0$$

Let the Lagrangian for a system with one degree of freedom, x, be given by

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$$

and

$$\mathcal{F} = \eta \dot{x}^2,$$

where $\eta > 0$.

- **(a)** Write down the expanded equation of motion defined above.
- **(b)** What system is described by this equation of motion?
- **(c)** Find an ansatz for the general solution (you may write $x(t)$ as a potentially complex-valued function which simplifies the math).
- **(d)** Show that your ansatz solves the expanded equation of motion.
- **(e)** Discuss the three difference cases for the solution, depending on k, m, and η.

(a) The expanded equation of motion is given as follows:

$$\boxed{m\ddot{x} + 2\eta\dot{x} + kx = 0}. \tag{4.1}$$

(b) The system described above is a simple, damped harmonic oscillator (a mass on a spring with non-negligible drag from the fluid it is placed in).

(c) We can write a solution ansatz as $x(t) = Ae^{i\omega t}$. Note that the physical solution is the real part of this, where A is complex and therefore encodes a phase as well.

(d) Plugging in our ansatz, we have

$$A\Big(-m\omega^2 + 2i\eta\omega - k\Big)e^{i\omega t} = 0 \Rightarrow \boxed{\omega_\pm = i\beta \pm \sqrt{\omega_0^2 - \beta^2}}, \tag{4.2}$$

where we have defined $\omega_0 = \sqrt{k/m}$ and $\beta = \eta/m$.

(e) There are three situations, governed by the sign of the radicand. First, if $\omega_0 > \beta$, then we have an underdamped situation

$$\boxed{x(t) = Ae^{-\beta t}\cos(\Omega t + \gamma)}, \tag{4.3}$$

where $\Omega = \sqrt{\omega_0^2 - \beta^2}$ and A, γ are constants determined by initial conditions. Next, if $\omega_0 = \beta$, then we have perfectly damped motion

$$\boxed{x(t) = e^{-\beta t}(A + Bt)}, \tag{4.4}$$

where A, B are constants determined by initial conditions. Note that in this case, the radicand is zero and $\omega_+ = \omega_-$. It turns out that $te^{-\beta t}$ is a linearly independent solution from the purely damped exponential, which can be checked *a posteriori*, or derived by factoring the second order differential operator as in the difference of squares. Lastly, we have the overdamped situation, where

$$\boxed{x(t) = e^{-\beta t}\Big[Ae^{-\kappa t} + Be^{\kappa t}\Big]}, \tag{4.5}$$

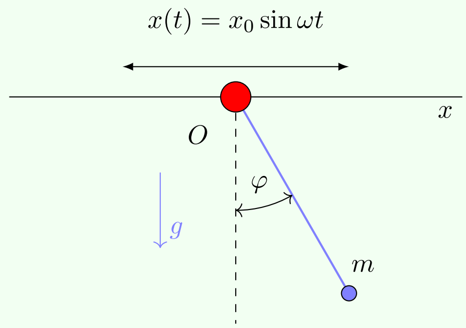
where again A, B are determined by initial conditions and $\kappa = \sqrt{\beta^2 - \omega_0^2}$.

Problem 1.2)

A mathematical pendulum of length l and mass m is in a gravitational field with normal acceleration g along the $-y$ axis. The suspension point O of the pendulum is driven by a motor along x in such a way that the suspension point oscillates as shown in the Figure.

(a) Write down the Lagrangian of the system in terms of time-dependent coordinate of the suspension point $x(t)$ and the angle $\varphi(t)$.

(b) Write down a dynamic equation for $\varphi(t)$ and solve it for small-amplitude oscillations of $\varphi(t) \ll 1$ caused by the oscillating suspension point with $x(t) = x_0 \sin\omega t$ and $x_0 \ll l$.



(a) The coordinates of the mass are given by

$$x = x_s + l \sin \varphi, \quad y = -l \cos \varphi, \tag{4.6}$$

where x_s is the horizontal position of the suspension point, so

$$\dot{x} = \dot{x}_s + l\dot{\varphi} \cos \varphi, \quad \dot{y} = l\dot{\varphi} \sin \varphi. \tag{4.7}$$

From this, we see that the Lagrangian

$$\begin{aligned} L &= \frac{m}{2}\Big[(\dot{x}_s + l\dot{\varphi}\cos\varphi)^2 + l^2\dot{\varphi}^2 \sin^2\varphi\Big] + mgl\cos\varphi \\ &= \boxed{\frac{m}{2}\Big[l^2\dot{\varphi}^2 + 2l\dot{x}_s\dot{\varphi}\cos\varphi + \dot{x}_s^2\Big] + mgl\cos\varphi}. \end{aligned} \tag{4.8}$$

(b) We can write down the equation of motion as follows:

$$ml^2\ddot{\varphi} + ml(\ddot{x}_s \cos\varphi - \dot{x}_s\dot{\varphi}\sin\varphi) + mgl\sin\varphi = 0. \tag{4.9}$$

If we introduce the prescribed motion of the suspension point and assume small oscillations

$$\boxed{\ddot{\varphi} + \Big(\frac{g}{l} - \frac{x_0\omega}{l}\cos\omega t\Big)\varphi = \frac{x_0\omega^2}{l}\sin\omega t}. \tag{4.10}$$

If we assume that $x_0\omega \ll g$, then the equation of motion looks like a driven oscillator with natural frequency $\omega_0 = \sqrt{g/l}$. Thus,

$$\boxed{\varphi(t) = A\cos(\omega_0 t + \gamma) + \frac{x_0/l}{(\omega_0/\omega)^2 - 1}\sin\omega t}, \tag{4.11}$$

where A, γ are constants to be determined by initial conditions.

Problem 1.3)

The Lagrangian $L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i)$ is invariant under rotations about an axis whose direction is specified by the unit vector $\hat{\boldsymbol{n}}$. Here $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are the positions and velocities of particles $i = 1, \cdots, N$. Knowing that under an infinitesimal rotation by an angle η about $\hat{\boldsymbol{n}}$ a generic vector $\boldsymbol{v}$ changes as

$$\boldsymbol{v} \to \boldsymbol{v}' = \boldsymbol{v} + \eta\, \hat{\boldsymbol{n}} \times \boldsymbol{v},$$

show that the projection of the total angular momentum along $\hat{\boldsymbol{n}}$ is conserved.

Under the same rotation, the angular momentum

$$\begin{aligned}\boldsymbol{L}' = \boldsymbol{r}' \times \boldsymbol{p}' &= (\boldsymbol{r} + \eta\hat{\boldsymbol{n}} \times \boldsymbol{r}) \times (\boldsymbol{p} + \eta\hat{\boldsymbol{n}} \times \boldsymbol{p}) = \boldsymbol{L} + \eta[\boldsymbol{r} \times (\hat{\boldsymbol{n}} \times \boldsymbol{p}) - \boldsymbol{p} \times (\hat{\boldsymbol{n}} \times \boldsymbol{r})] \\ &= \boldsymbol{L} + \eta[\boldsymbol{r}(\hat{\boldsymbol{n}} \cdot \boldsymbol{p}) - \boldsymbol{p}(\hat{\boldsymbol{n}} \cdot \boldsymbol{r})] = \boldsymbol{L} + \eta\hat{\boldsymbol{n}} \times \boldsymbol{L}. \end{aligned} \tag{4.12}$$

Note that we have neglected the terms of $\mathcal{O}(\eta^2)$. From this form, it is obvious that $\boldsymbol{L}' \cdot \hat{\boldsymbol{n}} = \boldsymbol{L} \cdot \hat{\boldsymbol{n}}$.

Problem 1.4)

In a particle accelerator the momentum compaction factor α is a dimensionless number equal to the ratio of the relative change of the path length and the relative change of the momentum

$$\alpha = \frac{\delta L}{L} \Big/ \frac{\delta p}{p} = \frac{p}{L}\frac{\mathrm{d}L}{\mathrm{d}p}$$

Consider a satellite in a circular orbit. What is the momentum compaction factor?

You may assume that the change of momentum is small and slow enough that the orbit is always circular.

For circular paths, the path length is just the circumference of a circle with radius R: $L = 2\pi R$. Additionally, since the path is circular, the centrifugal and gravitational forces are always in equilibrium, allowing us to write

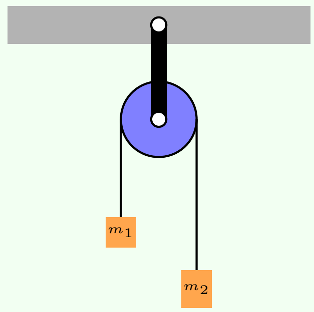
$$\frac{GMm}{R^2} = \frac{mv^2}{R} = \frac{p^2}{mR} \Rightarrow L = \frac{2\pi GMm^2}{p^2}. \tag{4.13}$$

Thus

$$\boxed{\alpha = -\frac{p}{L}\frac{4\pi GMm^2}{p^3} = -2}. \tag{4.14}$$

Problem 2.1)

In a simple Atwood machine, two masses are suspended, under constant gravity, from the ends of a flexible, massless, inextensible rope that passes over an inertialessly rotating pulley. Here, the two masses m_1 and m_2, the rope is of length l, and the constant gravitational acceleration g is downward.



Suppose that the second mass is replaced by a live monkey of equal mass that climbs up the rope at speed $v(t)$ relative to the rope. Treating the monkey's motion as a (time-dependent) constraint in the Lagrangian formalism, answer the following questions:

(a) Find the acceleration of the mass, m_1, if the monkey climbs up the rope with constant speed v.

(b) Find the acceleration of the mass m_1, if the monkey climbs up the rope with constant acceleration $\dot{v}(t) = a$.

(a) Let's solve this problem in the Lagrangian fomalism. We have two constraints. First, the length of the rope is constant $x_1 + x_2 = L$, where x_1 and x_2 are the heights of the ends of the rope from some reference. Second, we have $x_2 + \Delta x = x$, where Δx is the height of the monkey relative to the end of the rope and x is its height relative to the absolute reference point. We then have as our Lagrangian

$$\begin{aligned} L &= \frac{m_1\dot{x}_1^2}{2} + \frac{m_2\dot{x}^2}{2} - m_1 g x_1 - m_2 g x \\ &= \frac{m_1\dot{x}_1^2}{2} + \frac{m_2(v-\dot{x}_1)^2}{2} - m_1 g x_1 - m_2 g L + m_2 g x_1 - m_2 g \Delta x. \end{aligned} \tag{4.15}$$

The equation of motion for x_1 is

$$(m_1 + m_2)\ddot{x}_1 + (m_1 - m_2)g = m_2\dot{v} \Rightarrow \ddot{x}_1 = \frac{m_2\dot{v} + (m_2 - m_1)g}{m_1 + m_2}. \tag{4.16}$$

If we have constant v, then

$$\boxed{\ddot{x}_1 = \frac{m_2 - m_1}{m_1 + m_2} g}. \tag{4.17}$$

(b) If we instead have $\dot{v} = a$, where a is constant,

$$\boxed{\ddot{x}_1 = \frac{m_2 a + (m_2 - m_1)g}{m_1 + m_2}}. \tag{4.18}$$

Electricity & Magnetism

Problem 2.2)

Consider the region between two concentric spherical surfaces of radii a and b. On the inner boundary ($r = a$) of this region, the potential is constant,

$$\phi(a, \theta) = 2V_0.$$

On the outer boundary ($r = b$) of this region, the potential is given by

$$\phi(b, \theta) = V_0 \cos\theta.$$

(a) Find the potential in the region $a \le r \le b$.

(b) Now suppose that the region between two concentric spherical surfaces is filled with the inhomogeneous charge density $\rho(\boldsymbol{r}) = \lambda/r$, where λ is a constant and r is the distance to the center of the spheres. The potentials on the spherical surfaces are kept the same as in part (a).

Find the solution of the Poisson equation in the region $a \le r \le b$.

(Note: The inner and outer surfaces have surface charges that can create non-spherically symmetric potentials.)

(a) The relevant equation for this problem is Laplace's equation. Since our geometry is spherical with azimuthal symmetry, we can immediately write

$$\Phi(\boldsymbol{r}) = \sum_l P_l(\cos\theta) \begin{cases} A_l r^l & r < a \\ B_l r^l + C_l/r^{l+1} & a < r < b \\ D_l/r^{l+1} & r > b \end{cases}. \tag{4.19}$$

We have two boundary conditions are each interface:

$$(i)\ \Phi(a) = 2V_0 \tag{4.20}$$

$$(ii)\ \Phi(a^-) = \Phi(a^+) \tag{4.21}$$

$$(iii)\ \Phi(b) = V_0 \cos\theta \tag{4.22}$$

$$(iv)\ \Phi(b^-) = \Phi(b^+). \tag{4.23}$$

The first gives us that

$$\sum_l A_l a^l P_l(\cos\theta) = 2V_0$$

$$\sum_l A_l a^l \underbrace{\int_{-1}^{1} \mathrm{d}(\cos\theta)\, P_l(\cos\theta) P_m(\cos\theta)}_{2/(2m+1)\delta_{lm}} = 2V_0 \int_{-1}^{1} \mathrm{d}(\cos\theta)\, P_0(\cos\theta) P_m(\cos\theta)$$

$$\frac{2}{2m+1} A_m a^m = 4V_0 \delta_{m0} \Rightarrow A_l = 2V_0 \delta_{l0}. \tag{4.24}$$

The third condition by a similar argument and using that $P_1(\cos\theta) = \cos\theta$ yields $D_l = V_0 b^2 \delta_{l1}$. Next, we apply our continuity conditions:

$$A_l a^{2l+1} = B_l a^{2l+1} + C_l \tag{4.25}$$

$$B_l b^{2l+1} + C_l = D_l, \tag{4.26}$$

which give

$$B_l = \frac{D_l - A_l a^{2l+1}}{b^{2l+1} - a^{2l+1}} = V_0 \left[\frac{b^2}{b^3 - a^3} \delta_{l1} - \frac{2a}{b-a} \delta_{l0} \right] \tag{4.27}$$

$$C_l = -\frac{a^{2l+1}}{b^{2l+1} - a^{2l+1}} D_l + \frac{a^{2l+1} b^{2l+1}}{b^{2l+1} - a^{2l+1}} A_l = V_0 \left[\frac{2ab}{b-a} \delta_{l0} - \frac{a^3 b^2}{b^3 - a^3} \delta_{l1} \right]. \tag{4.28}$$

Thus,

$$\boxed{\Phi(r,\theta) = V_0 \begin{cases} 2 & r < a \\ \frac{2}{b/a - 1}\left(\frac{b}{r} - 1\right) + \left(\frac{1}{1-(a/b)^3} \frac{r}{b} - \frac{1}{(b/a)^3 - 1} \frac{b^2}{r^2} \right) \cos\theta & a < r < b \\ \frac{b^2}{r^2} \cos\theta & r > b \end{cases}} \tag{4.29}$$

(b) We can do as requested by superimposing the solution from above with that of two grounded conductors with the specified volume charge density between the spheres. Since the additional problem is spherically symmetric, we can use Gauss' law:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = \frac{Q}{\epsilon_0} \Rightarrow E(r) = \frac{Q_a + Q(r)}{4\pi\epsilon_0} \frac{1}{r^2}, \tag{4.30}$$

where $Q(r)$ is the charge enclosed by our Gaussian surface, which we choose to be a sphere of radius r, and Q_a is the charge on the inner sphere. We can compute the charge

$$Q(r) = 4\pi\lambda \int_a^r r\, \mathrm{d}r = 2\pi\lambda(r^2 - a^2). \tag{4.31}$$

The electric field is then

$$\boldsymbol{E} = \frac{Q_a + 2\pi\lambda(r^2 - a^2)}{4\pi\epsilon_0 r^2}, \tag{4.32}$$

and the potential is just

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\int_r^b \frac{Q_a + 2\pi\lambda(r^2 - a^2)}{r^2}\,\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\left[(Q_a - 2\pi\lambda a^2)\left(\frac{1}{r} - \frac{1}{b}\right) + 2\pi\lambda(b - r)\right]. \tag{4.33}$$

The form of the potential above gurantees that the outer sphere is grounded. We now solve for Q_a such that the inner sphere is also grounded as needed:

$$(Q_a - 2\pi\lambda a^2)\left(\frac{1}{a} - \frac{1}{b}\right) + 2\pi\lambda(b - a) = 0 \Rightarrow Q_a = -2\pi\lambda a(b - a). \tag{4.34}$$

Plugging this in, we find

$$\Phi(r) = 2\pi\lambda\left[(b - r) - a\left(\frac{b}{r} - 1\right)\right] \tag{4.35}$$

as the potential between the spheres, and using the same line integration, we see that the potential is zero for this setup when $r < a$ and $r > b$.

Now that we have our results, the composite potential with the charge density between the spheres and the potential $2V_0$ and $V_0\cos\theta$ on the inner and outer spheres is

$$\boxed{\Phi(r,\theta) = \begin{cases} 2V_0 & r < a \\ \frac{2V_0}{b/a-1}\left(\frac{b}{r} - 1\right) + V_0\left(\frac{1}{1-(a/b)^3}\frac{r}{b} - \frac{1}{(b/a)^3-1}\frac{b^2}{r^2}\right)\cos\theta + 2\pi\lambda\left[(b-r) - a\left(\frac{b}{r} - 1\right)\right] & a < r < b \\ \frac{V_0 b^2}{r^2}\cos\theta & r > b \end{cases}}. \tag{4.36}$$

Problem 2.3)

Consider a straight wave-guide of arbitrary but constant cross-section. The walls inside the wave-guide are ideal conductors (infinite conductivity) and the interior of the wave-guide is a vacuum. The lowest cutoff frequency is ω_0 and the next higher one is ω_1. The wave-guide is excited at the frequency ω such that $\omega_0 < \omega < \omega_1$.

For the waves that are propagating down the wave-guide, what are the phase velocity and the group velocity?

Recall that the relevant equation for wave-guides, regardless of the mode, is

$$(\boldsymbol{\nabla}_T^2 + \gamma^2)\psi = 0, \tag{4.37}$$

where $\psi = E_z, B_z$ and $\gamma = \mu\epsilon\omega^2 - k^2$. The allowed values of γ denoted by γ_n are determined by boundary conditions (i.e. the geometry of the wave-guide). We can define the cutoff frequency for the n^{th} mode via

$$\gamma_n^2 = \mu\epsilon\omega_n^2, \tag{4.38}$$

which allows us to express

$$k = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_n^2} = \frac{1}{c}\sqrt{\omega^2 - \omega_n^2}. \tag{4.39}$$

The phase velocity is defined by

$$\boxed{v_p = \frac{\omega}{k} = \left(\frac{k}{\omega}\right)^{-1} = \frac{c}{\sqrt{1 - (\omega_n/\omega)^2}}}, \tag{4.40}$$

and the group velocity is

$$\boxed{v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1} = c\sqrt{1 - (\omega_n/\omega)^2}}. \tag{4.41}$$

Note that these are phase and group velocities for the n^{th} mode.

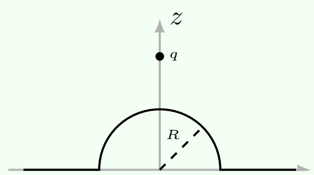
Problem 2.4)

A charge is uniformly distributed on the surface of a solid sphere, which rotates with a fixed angular velocity about an axis through its center. The sphere is in a uniform and constant magnetic field with a nonzero angle relative to the sphere's axis of rotation. Without doing any detailed calculations, describe the motion of the sphere, justifying your reasoning. What happens to the sphere's axis of rotation?

Let us orient our coordinate system so that the z-axis aligns with the angular velocity of the rotation. Similarly, let's orient the magnetic field at an angle θ relative to the z-axis fixed in the zy-plane. The rotating charges constitute a surface current $\boldsymbol{K}$, whose direction is given by $\hat{\boldsymbol{\phi}}$. The force on the sphere is proportional to the vector product $\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{K} \times \boldsymbol{B}$. Since the field is constant and we are in the realm of magnetostatics, there is no net force on the sphere. Another consideration, however is the torque on the sphere, which is proportional to $\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{r} \times (\boldsymbol{K} \times \boldsymbol{B})$, which is nonzero unless $\boldsymbol{B}$ and $\boldsymbol{K}$ are (anti-)parallel. This torque will cause the axis of rotation to precess around the magnetic field.

Problem 3.1)

A conducting surface held at zero potential consists of a plane with a hemispherical bump of radius R (see the figure below). A charge q sits a distance $r > R$ above the center of the hemispherical bump.



Calculate the force on the charge.

Hint: Use image charges. Note that you may need more than one image charge; in fact, as many as three. You should verify that your image solution satisfies the correct boundary conditions.

We know how to place two charges already. First we have q_1 as if the setup was only the plane: $q_1 = -q$ at $z = -r$. Next, we have q_2 as if the setup was only the sphere: $q_2 = -q(R/r)$ at $z = R^2/r$. Notice though that we are not done. The charges q and q_1 cancel on the plane, leaving the contribution of q_2. Meanwhile, the charges q and q_2 cancel on the hump, leaving the contribution of q_1. We should place a third charge which simultaneously cancels both of these unbalanced contributions. After some inspection, we can place a charge $q_3 = (R/r)q$ at $z = -R^2/r$, which cancels both happily on the needed surfaces. From this, the force on the charge q is

$$\boldsymbol{F} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{R/r}{(r-R^2/r)^2} + \frac{R/r}{(r+R^2/r)^2} - \frac{1}{(2r)^2}\right]\hat{\boldsymbol{z}} = -\frac{q^2}{4\pi\epsilon_0}\left[\frac{2r^3R^3}{(r^4-R^4)^4} + \frac{1}{4r^4}\right]\hat{\boldsymbol{z}} \tag{4.42}$$

Problem 3.2)

A particle with charge q and mass m moves in a parabolic potential $U(x, y, z) = k(x^2 + y^2 + z^2)/2$, and a constant magnetic field B is applied along the z-axis.

(a) Write down the equations of motion.

(b) How does the frequency of this 3D charged oscillator change owing to the presence of this constant magnetic field?

(c) Describe the trajectories of the oscillator corresponding to its different eigenfrequencies in the magnetic field.

The force acting on the charge is given as

$$m\ddot{\boldsymbol{r}} = q\dot{\boldsymbol{r}} \times B\hat{\boldsymbol{z}} - k(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}). \tag{4.43}$$

In component form, we have

$$\boxed{\begin{aligned} \ddot{x} &= \frac{qB_z}{m}\dot{y} - \frac{k}{m}x \\ \ddot{y} &= -\frac{qB_z}{m}\dot{x} - \frac{k}{m}y \\ \ddot{z} &= -\frac{k}{m}z. \end{aligned}} \tag{4.44}$$

(b) The oscillation in the z direction is unchanged as a result of the existence of the magnetic field. If we insert $x = Ae^{i\omega t}$ and $y = Be^{i\omega t}$, then

$$\begin{pmatrix} \omega^2 - \omega_0^2 & i\omega\omega_c \\ -i\omega\omega_c & \omega^2 - \omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0, \tag{4.45}$$

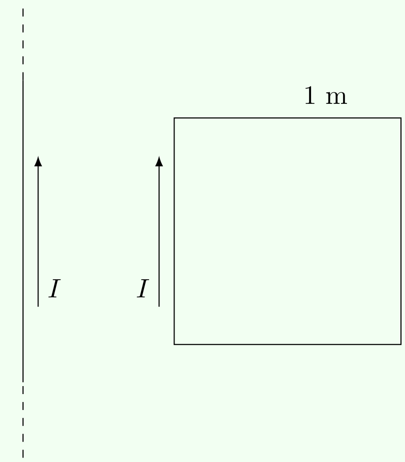
where $\omega_0 = \sqrt{k/m}$ and $\omega_c = qB_z/m$. This is solved by requiring the matrix to have zero determinant, yielding

$$\boxed{\omega_\pm^2 = \omega_0^2 + \frac{\omega_c^2}{2} \pm \omega_c\sqrt{\omega_0^2 + \frac{\omega_c^2}{4}}}. \tag{4.46}$$

The positive branch corresponds to strictly oscillatory motion. If $\omega_c\sqrt{\omega_0^2 + (\omega_c/2)^2} > \omega_0^2 + \omega_c^2/2$, then the negative branch corresponds to damped motion, which dies out over time and leaves only the first oscillation mode. If they are equal, then the

Problem 3.3)

Until 2019, the definition of the Ampere was as follows: *The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to* 2×10^{-7} *newton per meter of length.*



Obviously, this was a very impractical definition ("infinitely long, zero cross section wires"). However, one *can* approximate the implied measurement in the following way: Consider a quadratic loop of wire, with a side length of 1 m distance (and, yes, the wires are nearly infinitely thin yet totally rigid).

If we run a current I of 1 Ampere through both the square and the infinite wire, what will be the force that each conductor exerts on the other? You may use symmetry arguments as much as possible to simplify the calculation, keeping in mind that we are only interested in the **net** force.

The force on a current density is given generically by

$$\boldsymbol{F} = \int \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{J}(\boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x}), \tag{4.47}$$

which reduces to

$$\boldsymbol{F} = I \int \mathrm{d}\boldsymbol{l} \times \boldsymbol{B}(\boldsymbol{x}). \tag{4.48}$$

Before doing any integration, we should determine the magnetic field from the straight wire using Ampere's law:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B_\phi(2\pi r) = \mu_0 I \Rightarrow \boldsymbol{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}, \tag{4.49}$$

where the direction of $\hat{\boldsymbol{\phi}}$ is determined by the right-hand rule. For the purposes of calculation, let's set our origin at the lower left corner of the square with the x axis pointing to the right and y pointing up. Thus, at the square loop, the magnetic field points in the $-z$ direction. Quantitatively, we have

$$\boldsymbol{F} = \frac{\mu_0 I^2}{2\pi}\left\{ \int_0^l \frac{\mathrm{d}y\,\hat{\boldsymbol{y}} \times (-\hat{\boldsymbol{z}})}{L} + \int_0^l \frac{\mathrm{d}x\,\hat{\boldsymbol{x}} \times (-\hat{\boldsymbol{z}})}{L+x} + \int_0^l \frac{-\,\mathrm{d}y\,\hat{\boldsymbol{y}} \times (-\hat{\boldsymbol{z}})}{L+l} + \int_0^l \frac{-\,\mathrm{d}x\,\hat{\boldsymbol{x}} \times (-\hat{\boldsymbol{z}})}{L+x} \right\}$$

$$= \frac{\mu_0 I^2}{2\pi}\left\{ -\hat{\boldsymbol{x}}\frac{l}{L} + \hat{\boldsymbol{x}}\frac{l}{L+l} \right\} = \boxed{-\frac{\mu_0 I^2 l^2}{2\pi L(L+l)}\hat{\boldsymbol{x}}} \tag{4.50}$$

Quantum Mechanics

Problem 3.4)

Consider a spinless particle with mass m in the three-dimensional potential $V(r) = Cr^2$ with $C > 0$.

(a) What are the energy eigenvalues? What are the degeneracies of the three lowest energy eigenvalues?

(b) Suppose that five identical noninteracting particles with mass m move in this potential. What is the ground state energy of this system if the particles have (i) spin-1/2, (ii) spin-1?

(a) Observe that the potential is that of a harmonic oscillator with frequency $\omega = \sqrt{2C/m}$. Knowing this, we can immediately write

$$\boxed{E_{n_x,n_y,n_z} = \hbar\omega(n_x + n_y + n_z + 3/2) = \hbar\omega(n + 3/2) = E_n}. \tag{4.51}$$

The degeneracies of the three lowest eigenvalues are as follows:

$$\boxed{E_0 : \ g_0 = 1, \quad E_1 : \ g_1 = 3, \quad E_2 : \ g_2 = 6}. \tag{4.52}$$

(b) If we have five non-interacting spin-1/2 particles in this potential, the ground state energy

$$\boxed{E = 2E_0 + 3E_1 = \frac{21}{2}\hbar\omega}, \tag{4.53}$$

while if we have five non-interacting spin-1 particles the ground state energy

$$\boxed{E = 5E_0 = \frac{15}{2}\hbar\omega}. \tag{4.54}$$

The essential observation is that identical fermions cannot exist in the same state (i.e. the Pauli exclusion principle), while identical bosons can exist in the same state.

Problem 4.1)

The Pauli spin matrices in quantum mechanics are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By definition,

$$e^{\alpha\sigma_1} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\sigma_1^n.$$

(a) Calculate $e^{\alpha\sigma_1}$ as an explicit 2 by 2 matrix.

(b) Find eigenvalues and normalized eigenvectors of $e^{\alpha\sigma_1}$.

(a) Observe that $\sigma_1^2 = 1$, so

$$\boxed{\begin{aligned} e^{\alpha\sigma_1} &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\sigma_1^n = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!}\sigma_1^{2n} + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\sigma_1^{2n+1} \\ &= \cosh\alpha + \sigma_1 \sinh\alpha = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \end{aligned}} \tag{4.55}$$

(b) Recall the eigenstates of σ_1 are

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{4.56}$$

with corresponding eigenvalues ± 1. Observe that

$$\boxed{e^{\alpha\sigma_1}|\pm\rangle_x = (\cosh\alpha \pm \sinh\alpha)|\pm\rangle_x = e^{\pm\alpha}|\pm\rangle_x} \tag{4.57}$$

Problem 4.2)

At time $t = 0$ a particle in the potential $V(x) = m\omega^2 x^2/2$ is described by the wave-function

$$\psi(x,0) = A\sum_n (1/\sqrt{2})^n \psi_n(x),$$

where $\psi_n(x)$ are the orthonormal eigenfunctions of the energy with eigenvalues $E_n = (n + 1/2)\hbar\omega$.

(a) Find the normalization constant A.

(b) Write the expression for $\psi(x,t)$ for $t>0$.

(c) Show that $|\psi(x,t)|^2$ is a periodic function of time and indicate the period T.

(d) Find the expectation value of the energy.

(a) The normalization can be determined as follows:

$$\int \mathrm{d}x\,|\psi(x,0)|^2 = A^2\sum_{n,m}\left(\frac{1}{\sqrt{2}}\right)^{n+m}\underbrace{\int \mathrm{d}x\,\psi_n^*(x)\psi_m(x)}_{\delta_{nm}} = A^2\sum_n \frac{1}{2^n} = 2A^2 = 1$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{2}}}. \tag{4.58}$$

(b) We can inject time-dependence into the state using the unitary time-evolution operator:

$$\boxed{\psi(x,t) = e^{-iHt/\hbar}\psi(x,0) = \sum_n \left(\frac{1}{\sqrt{2}}\right)^{n+1} e^{-i(n+1/2)\omega t}\psi_n(x)} \tag{4.59}$$

(c) Observe that

$$|\psi(x,t)|^2 = \sum_{n,m}\left(\frac{1}{\sqrt{2}}\right)^{n+m+2} e^{-i(m-n)\omega t}\psi_n^*(x)\psi_m(x). \tag{4.60}$$

Terms with $n=m$ are constant in time, while the interference terms have period

$$T_{nm} = \frac{2\pi}{(n-m)\omega} = \frac{1}{n-m}T \Rightarrow \boxed{T = \frac{2\pi}{\omega}}. \tag{4.61}$$

Notice that T is the period of $|\psi(x,t)|^2$ since this is the largest period of the non-constant interference terms.

Problem 4.3)

A particle of mass m is moving along the x-axis (in one dimension), where its potential energy is $V(x)=0$ for all $x\le 0$ and $V(x)=V_0$ else. The particle is in an energy eigenstate with eigenvalue $0<E<V_0$.

Write down the time-independent Schrödinger equation and find a solution (determine all constants to within one overall constant). What is the probability density for the particle to be found at the classically forbidden point $x = x_0 > 0$, expressed as a fraction of the probability density for the particle to be found at $x = 0$?

The Schödinger equation reads

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \Big[v(x) - k^2\Big]\psi = E\psi, \tag{4.62}$$

where $k^2 = 2mE/\hbar^2$ and $v(x) = 2mV_0/\hbar^2\Theta(x)$. In the region $x < 0$, this reduces to a free-particle, while in the region $x > 0$, we have a particle in a classically forbidden region:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{-\kappa x} & x > 0, \end{cases} \tag{4.63}$$

where $\kappa^2 = v_0 - k^2$. We can determine two of these constants via the initial conditions

$$\psi(0^-) = \psi(0^+) \Rightarrow A + B = C \tag{4.64}$$

$$\psi'(0^-) = \psi'(0^+) \Rightarrow ik(A - B) = -\kappa C. \tag{4.65}$$

Solving this system, we have

$$\boxed{\frac{B}{A} = -\frac{\kappa + ik}{\kappa - ik}, \quad \frac{C}{A} = -\frac{2ik}{\kappa - ik}}. \tag{4.66}$$

The probability density for the particle to be found at $x_0 > 0$ relative to that at $x = 0$ is

$$\boxed{\left|\frac{\psi(x_0)}{\psi(0)}\right|^2 = \left|\frac{Ce^{-\kappa x_0}}{C}\right|^2 = e^{-2\kappa x_0}} \tag{4.67}$$

Problem 4.4)

A particle with spin-1/2 is described by a state vector:

$$|\chi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Here $\alpha = e^{i\varphi_1}\cos\theta$ and $\beta = e^{i\varphi_2}\sin\theta$ are complex amplitudes, and θ, φ_1, and φ_2 are real parameters. Calculate the probabilities to measure the spin $+1/2$ separately along each of the x, y, and z axes.

The probabilities are as follows:

$$\boxed{\begin{aligned}
P(S_z = \hbar/2) &= |\langle + | \chi \rangle|^2 = |\alpha|^2 = \cos^2\theta \\
P(S_x = \hbar/2) &= |\langle +_x | \chi \rangle|^2 = \frac{1}{2}|\alpha + \beta|^2 = \frac{1}{2}\Big(|\alpha|^2 + |\beta|^2 + \alpha^*\beta + \alpha\beta^*\Big) \\
&= \frac{1}{2}\Big[1 + 2\,\mathrm{Re}(\alpha^*\beta)\Big] = \frac{1}{2}\Big[1 + \sin(2\theta)\cos(\varphi_1 - \varphi_2)\Big] \\
P(S_y = \hbar/2) &= |\langle +_y | \chi \rangle|^2 = \frac{1}{2}|\alpha + i\beta|^2 = \frac{1}{2}\Big[1 - 2\,\mathrm{Im}(\alpha^*\beta)\Big] \\
&= \frac{1}{2}\Big[1 + \sin(2\theta)\sin(\varphi_1 - \varphi_2)\Big]
\end{aligned}}$$

5 August 2021

Classical Mechanics

Problem 1.1)

A Lagrangian for a system with infinitely many interacting particles of mass m (infinitely many degrees of freedom, $-\infty < n < +\infty$) is given as follows:

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \left[\frac{m}{2}\dot{x}_n^2 - \frac{k}{2}(x_n - x_{n-1})^2\right]$$

- **(a)** What kind of system does this Lagrangian describe?
- **(b)** Write down the Lagrangian equations of motion.
- **(c)** Show that there is a solution given by $x_n(t) = A\cos(\omega t - cn)$.
- **(d)** Find an equation for c
- **(e)** For $c \ll 1$, find the approximate magnitude of c. What kind of motion does the solution describe? What is the interpretation of the constant c?

(a) This system describes a linear chain of coupled oscillators, where the "springs" connecting each mass has the same stiffness k.

(b) We can take derivatives as follows to find the equation of motion for the m^{th} mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_m} - \frac{\partial \mathcal{L}}{\partial x_m} = 0$$

$$\sum_{n=-\infty}^{\infty} \left[m\frac{\mathrm{d}}{\mathrm{d}t}\dot{x}_n \underbrace{\frac{\partial \dot{x}_n}{\partial \dot{x}_m}}_{\delta_{nm}} + k(x_n - x_{n-1}) \underbrace{\frac{\partial}{\partial x_m}(x_n - x_{n-1})}_{\delta_{nm}-\delta_{n-1,m}} \right] = 0$$

$$m\ddot{x}_m + k[(x_m - x_{m-1}) - (x_{m+1} - x_m)] = 0$$

$$\boxed{m\ddot{x}_n - k(x_{n+1} - 2x_n + x_{n-1}) = 0}. \tag{5.1}$$

(c) We can insert the ansatz $x_n = A_n e^{i\omega t}$, which yields

$$-m\omega^2 A_n - k(A_{n+1} - 2A_n + A_{n-1}) = 0$$
$$\Rightarrow \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & -k & 2k - m\omega^2 & -k & & \\ & & -k & 2k - m\omega^2 & -k & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ A_{n-1} \\ A_n \\ A_{n+1} \\ \vdots \end{pmatrix} = 0. \tag{5.2}$$

Observe that we can use translation symmetry to solve this problem. If we translate our system to the right or left by a, where a is the separation distance between the masses, the problem is physically the same. We can encode this symmetry in a matrix S ,of which A is an eigenvector:

$$SA = \lambda A. \tag{5.3}$$

Since S moves our particles to the right by a, we have $A_{n+1} = \lambda A_n$. From this, we must impose that $|\lambda| = 1$ such that the displacements are the same, allowing us to write $\lambda = e^{-ic}$, where c is some phase. Observe that we can write

$$x_n = A_n e^{i\omega t} = A_0 e^{i(\omega t - cn)}, \tag{5.4}$$

where we have defined A_0 to be real (we have the freedom to scale our eigenvectors as desired).

(d) If we use the solution form provided Observe that

$$x_{n\pm1} = A\left[\cos(\omega t - cn)\cos(c) \pm \sin(c)\sin(\omega t - cn)\right]. \tag{5.5}$$

Plugging into the equation of motion, we find

$$2k[1 - \cos(c)] - m\omega^2 = 0 \Rightarrow \boxed{\cos c = 1 - \frac{m\omega^2}{2k}}. \tag{5.6}$$

(e) If $c \ll 1$, then we can write $\cos(c) \approx 1 - c^2/2$, which gives

$$\boxed{c = \omega\sqrt{\frac{m}{k}} = \frac{\omega}{\omega_0}}. \tag{5.7}$$

This solution describes wave motion.

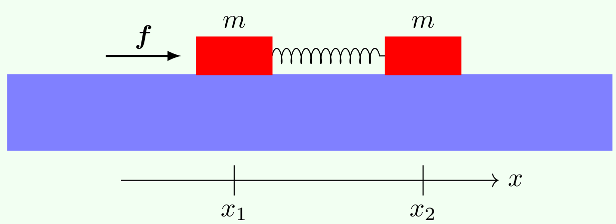
Problem 1.2)

Two bars of equal masses m connected by a weightless spring can slide with no friction along a horizontal xy-plane. The bars are initially at rest but at time $t > 0$ a constant horizontal force f is applied to the left bar, as shown in the Figure. The spring has the stiffness k and the length l in undeformed state.

(a) Write down the Lagrangian in terms of the center of mass coordinate $Q(t)$ and the distance $u(t) = x_2(t) - x_1(t)$ between the bars, assuming that they move only along the x-axis. Here $x_1(t)$ and $x_2(t)$ are the coordinates of the left and the right bars, respectively.

(b) Solve the equations of motion for $Q(t)$ and $u(t)$ with the initial conditions $x_1(0) = 0$, $x_2(0) = l$, and $\dot{x}_1(0) = \dot{x}_2(0) = 0$, where the overdot means the time derivative.

(c) Find the amplitude and the frequency of oscillations of the distance between the bars $u(t)$ at $t > 0$.



The Lagrangian in terms of x_1 and x_2 reads

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}[(x_2 - x_1) - l]^2 + f x_1. \tag{5.8}$$

We make the change of variables from $(x_1, x_2) \to (Q, u)$ as follows:

$$\begin{cases} Q = \frac{1}{2}(x_1 + x_2) \\ u = x_2 - x_1 \end{cases} \Rightarrow \begin{cases} x_1 = Q - \frac{u}{2} \\ x_2 = Q + \frac{u}{2}. \end{cases} \tag{5.9}$$

Rewriting our Lagrangian, we have

$$\boxed{L = m\dot{Q}^2 + \frac{m\dot{u}^2}{4} - \frac{k}{2}(u - l)^2 + f\left(Q - \frac{u}{2}\right)}. \tag{5.10}$$

(b) The equations of motion are then

$$2m\ddot{Q} - f = 0 \Rightarrow \boxed{Q(t) = \frac{f}{4m}t^2 + \frac{l}{2}} \tag{5.11}$$

$$\frac{m}{2}\ddot{u} + k(u-l) + \frac{f}{2} = 0 \Rightarrow \boxed{u(t) = \frac{f}{2k}\cos\left(\sqrt{\frac{2k}{m}}\,t\right) + l - \frac{f}{2k}}. \tag{5.12}$$

(c) The amplitude of the relative position of x_1 and x_2 is $f/(2k)$, and the frequency of the corresponding oscillation is $\sqrt{2k/m}$.

Problem 1.3)

A spherical pendulum consists of a particle of mass m in a gravitational field constrained to move on the inner surface of a sphere of radius R.

(a) Use the polar angle θ (measured from the downward vertical) and the azimuthal angle ϕ as generalized coordinates and obtain the Hamiltonian. Are there any conserved quantities?

(b) Obtain the equations of motion in the Hamiltonian formulation.

(c) Assume the particle performs uniform circular motion with θ fixed at θ_0. What are the values of the constants of motion (if any) in such a case?

(a) We can parameterize the position of the mass as follows:

$$\begin{aligned}
\boldsymbol{r} &= R(\sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} - \cos\theta\hat{\boldsymbol{z}}) \\
\Rightarrow \dot{\boldsymbol{r}} &= R[(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\hat{\boldsymbol{x}} + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\hat{\boldsymbol{y}} + \dot{\theta}\sin\theta\hat{\boldsymbol{z}}] \\
\Rightarrow \dot{\boldsymbol{r}}^2 &= R^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right). \qquad (5.13)
\end{aligned}$$

Hence,

$$L = \frac{mR^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgR\cos\theta. \tag{5.14}$$

From this, we can derive the conjugate momenta to θ and ϕ, respectively:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mR^2} \tag{5.15}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mR^2\sin^2\theta\dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{mR^2\sin^2\theta}. \tag{5.16}$$

In terms of the momenta, the Lagrangian reads

$$L = \frac{mR^2}{2}\Big(\frac{p_\theta^2}{(mR^2)^2} + \frac{p_\phi^2}{(mR^2\sin^2\theta)^2}\sin^2\theta\Big) + mgR\cos\theta$$
$$= \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\sin^2\theta} + mgR\cos\theta. \tag{5.17}$$

The Hamiltonian is then

$$\boxed{H = p_\theta\dot\theta + p_\phi\dot\phi - L = \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta}. \tag{5.18}$$

(b) The Hamiltonian equations of motion can be obtained in the usual way:

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}, \quad \dot p_\theta = -\frac{\partial H}{\partial\theta} = \frac{p_\phi^2}{2mR^2\sin^2\theta\tan\theta} - mgR\sin\theta \tag{5.19}$$
$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2\sin^2\theta}, \quad \dot p_\phi = -\frac{\partial H}{\partial\phi} = 0. \tag{5.20}$$

(c) If the particle undergoes circular motion, then $p_\theta = 0$, so

$$\theta(t) = \theta_0, \quad p_\phi = mR^2\sin^2\theta_0\dot\phi. \tag{5.21}$$

Problem 1.4)

- **(a)** Calculate the escape velocity v_e from the surface of a homogeneous sphere of density ρ and radius R,
- **(b)** A vertical shaft extends to the center of a homogeneous sphere of density ρ and radius R and a small mass is dropped from rest at the surface. Compare the velocity attained at the center with the escape velocity.
- **(c)** Suppose the sphere is not homogeneous but the density $\rho(r)$ is a function of the distance to the center. Find $\rho(r)$ for which the mass falls to the center with constant acceleration.

(a) The escape velocity v_e is the minimum velocity necessary for a mass m to reach a distance infinitely far from earth with no kinetic energy. That is,

$$\frac{mv_e^2}{2} - \frac{GMm}{R} = 0 \Rightarrow \boxed{v_e = \sqrt{\frac{2GM}{R}} = R\sqrt{\frac{8\pi G\rho}{3}}}. \tag{5.22}$$

(b) To compute the velocity at the center of the earth, we need to know the force on the mass at an arbitrary distance from the center. This could be obtained by integrating Newton's law of gravitation directly, but we can also be clever and repurpose Gauss' law for gravity:

$$\oint \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{a} = -4\pi G M_{\text{enc}} \Rightarrow \boldsymbol{g} = -\frac{4\pi G}{4\pi r^2}\frac{4}{3}\pi r^3 \rho \hat{\boldsymbol{r}} = -\frac{4\pi G\rho}{3}\boldsymbol{r}. \tag{5.23}$$

Using Newton's 2$^{\text{nd}}$ law, we have the equation of motion

$$m\ddot{r} = -\frac{4}{3}\pi G m \rho r \Rightarrow r(t) = R\cos(\omega t), \tag{5.24}$$

where $\omega = \sqrt{4\pi G\rho/3}$. At the center of the earth, we have

$$R\cos(\omega T) = 0 \Rightarrow \omega T = \frac{\pi}{2} \Rightarrow \boxed{|\dot{r}(T)| = R\omega = R\sqrt{\frac{4\pi G\rho}{3}} = \frac{v_e}{\sqrt{2}}}. \tag{5.25}$$

(c) Since $\rho(\boldsymbol{r}) = \rho(r)$ by assumption $\boldsymbol{g}$ will always point radially. We then have the equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = \frac{\mathrm{d}g}{\mathrm{d}r} = 0 = -4\pi G\rho(r). \tag{5.26}$$

It is clear from we can only have a uniform gravitational field if $\rho = 0$ for all r, that is, we cannot construct a planet such that the force of gravity is constant regardless of how far from the center one is.

Problem 2.1)

A harmonic oscillator with a spring constant k and mass m is damped with a force $-bv$ where v is the velocity of the mass and b is a constant. The mass is also driven by a harmonic force $F(t) = F_0 \cos\omega t$.

Given F_0, at what angular frequency ω the amplitude of the displacement is maximal?

The equation of motion for this system is

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t. \tag{5.27}$$

Let us solve for the transient motion of the system. To simplify the math, we use complex exponentials and insert an ansatz of $x = Ae^{i\omega t}$, which yields the algebraic equation

$$(k - m\omega^2 + i\omega b)A = F_0 \Rightarrow |A| = \frac{F_0}{(k - m\omega^2)^2 + \omega^2 b^2}. \tag{5.28}$$

This ampitude is maximal when the denominator is minimal:

$$\frac{\mathrm{d}}{\mathrm{d}\omega^2}\left[(k-m\omega^2)^2+\omega^2 b^2\right] = 2(m\omega^2-k)+b^2 = 0 \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}-\frac{b^2}{2m}} = \sqrt{\omega_0^2-2\beta^2}}\,. \tag{5.29}$$

Electricity & Magnetism

Problem 2.2)

The region bounded by two concentric spherical surfaces with radii R_1 and R_2 ($R_1 < R_2$) is filled with a charge density $\rho = \alpha/r$.

Find the total charge Q and spatial distribution of the electrostatic potential $\Phi(\boldsymbol{r})$ and the electric field $\boldsymbol{E}(\boldsymbol{r})$.

For a fixed total charge Q, consider the behaviors of $\Phi(\boldsymbol{r})$ and $\boldsymbol{E}(\boldsymbol{r})$ in the limit of an infinitely thin spherical shell, i.e., $R_1 \to R_2$.

The total charge

$$\boxed{Q = \int \mathrm{d}^3\boldsymbol{r}\,\rho(r) = 4\pi\int_{R_1}^{R_2}\mathrm{d}r\,\alpha r = 2\pi\alpha(R_2^2-R_1^2)}\,. \tag{5.30}$$

The electric field is determined via Gauss' law. For $r < R_1$, the electric field is zero since no charge is enclosed. For $R_1 < r < R_2$,

$$\boxed{\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{2\pi\alpha(r^2-R_1^2)}{r^2}\hat{\boldsymbol{r}} = \frac{\alpha}{2\epsilon_0}\left(1-\frac{R_1^2}{r^2}\right)\hat{\boldsymbol{r}}}\,. \tag{5.31}$$

lastly, for $r > R_2$, we effectively have a point charge:

$$\boxed{\boldsymbol{E} = \frac{\alpha}{2\epsilon_0}\frac{R_2^2-R_1^2}{r^2}}\,. \tag{5.32}$$

From this we can determine the potential as

$$\Phi(r) = \int_r^\infty E(r')\,\mathrm{d}r'\,. \tag{5.33}$$

For $r > R_2$, we have

$$\boxed{\Phi(r) = \frac{\alpha}{2\epsilon_0}\frac{R_2^2 - R_1^2}{r}}, \tag{5.34}$$

whilst for $R_1 < r < R_2$, we have

$$\begin{aligned}
\Phi(r) &= \frac{\alpha}{2\epsilon_0}\left[\frac{R_2^2}{R_1^2} - 1 + \int_r^{R_2}\left(1 - \frac{R_1^2}{r'^2}\right)\mathrm{d}r'\right] \\
&= \frac{\alpha}{2\epsilon_0}\left[\frac{R_2^2}{R_1} - R_1 + (R_2 - r) - R_1^2\left(\frac{1}{r} - \frac{1}{R_1}\right)\right] \\
&= \boxed{\frac{\alpha}{2\epsilon_0}\left[\frac{R_2^2}{R_1} - \frac{R_1^2}{r} + R_2 - r\right]}.
\end{aligned} \tag{5.35}$$

When $r < R_1$, we have

$$\boxed{\Phi(r) = \Phi(R_1) = \frac{\alpha}{2\epsilon_0}\left[\frac{R_2^2}{R_1} + R_2 - 2R_1\right]}. \tag{5.36}$$

In the limit of an infinitely thin shell, we still have the same results of the electric field inside ($\boldsymbol{E} = 0$) and outside ($\boldsymbol{E} = Q/(4\pi\epsilon_0 r^2)\hat{\boldsymbol{r}}$). For the potential, outside the shell we have $\Phi = Q/(4\pi\epsilon_0 R_2)$, and inside $\Phi = Q/(4\pi\epsilon_0 R_2)$.

Problem 2.3)

The plane $z = 0$ carries a charge such that the potential on that plane is $\Phi(x, y, 0) = V_0 \sin kx$.

Find the potential everywhere in space.

We are interested in solving Laplace's equation off the xy-plane. In Cartesian coordinates, the equation takes the form

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{5.37}$$

At this point, we can introduce a separable ansatz such that $\Phi(x, y, z) = X(x)Y(y)Z(z)$ such that

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0. \tag{5.38}$$

each of these terms is a function of a separate independent variable and their sum is a constant independent of these variables. Thus, each of the terms must be constant, and since the x-dependence is oscillatory, we set

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -q^2 \Rightarrow X = A\cos qx + B\sin qx. \tag{5.39}$$

Similarly, we can impose that

$$\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -p^2 \Rightarrow Y = C\cos py + D\sin py. \tag{5.40}$$

Putting this back into the separated equation

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - \underbrace{(q^2+p^2)}_{\gamma^2} Z = 0 \Rightarrow Z = Ee^{\gamma z} + Fe^{-\gamma z}. \tag{5.41}$$

The full solution is then

$$\Phi(x,y,z) = \int \mathrm{d}p\,\mathrm{d}q\, e^{-\sqrt{p^2+q^2}z}[A(p,q)\cos qx + B(p,q)\sin qx][C(p,q)\cos py + D(p,q)\sin py], \tag{5.42}$$

where we have discarded the increasing exponential piece of the z-dependence so that our potential is finite at large z, which is physically imposed since our potential should go to zero far from the plane. We now use our initial condition:

$$\Phi(x,y,0) = \int \mathrm{d}p\,\mathrm{d}q\,[A(p,q)\cos qx + B(p,q)\sin qx][C(p,q)\cos py + D(p,q)\sin py] = V_0\sin kx. \tag{5.43}$$

From this, we can see that $A(p,q) = 0$, $B(p,q) = \delta(p-k)$, $C(p,q) = D(p,q) = \delta(q)$. Hence, the potential reduces to

$$\boxed{\Phi(x,y,z) = V_0\sin(kx)e^{-k|z|}}. \tag{5.44}$$

Problem 2.4)

It is believed that Compton scattering by starlight quanta may be a mechanism for the energy degradation of high-energy electrons in interstellar space. An experiment has been proposed in which this phenomenon can be observed directly in the laboratory by scattering a high-energy electron beam against the intense flux of visible photons produced by a typical laser. The experimentalists have established that the laboratory energy of the scattered photon is given to an excellent approximation ($\beta \approx 1$) by the relation

$$E_f^\gamma \approx \gamma mc^2 \frac{\lambda(1-\beta\cos\theta_0)}{1+\lambda(1-\cos\theta_0)}, \quad \lambda = 2\gamma\frac{E_i^\gamma}{mc^2}$$

where E_i^γ is the laboratory energy of the incident photon, θ_0 is the photon scattering angle in the electron rest frame, m is the electron mass, c is the speed of light, and $\gamma = (1-\beta^2)^{-1/2}$. Having joined the experiment, you are asked to verify that this relation is correct. In order to do so, proceed as follows:

(a) In the rest frame of the electron, use energy and momentum conservation to express the energy $E_f^{\gamma'}$ of the scattered photon in terms of the energy of the incident photon $E_i^{\gamma'}$ and the scattering angle θ_0.

(b) Obtain the energy of the scattered photon in the laboratory frame where the eletron has the initial velocity $-\beta\hat{\boldsymbol{x}}$ with $\beta = cp_i/E$ and $E_i \gg mc^2$.

(c) The relation obtained in part (b) above still contains the incident-photon energy in the electron rest frame. Express this energy in the laboratory frame and verify the relations above.

(d) Determine how the scattering angle θ in the laboratory frame is related to the scattering angle θ_0 in the electron rest frame.

To avoid notational confusion, we adopt the following conventions. In the rest frame of the electron, energies and momenta are denoted by ϵ, p, respectively. In the lab frame, energies and momenta are denoted by E, P, respectively. Additionally, the subscripts γ and e refer to the photon and electron, and finally unprimed and primed quantities denote those before and after the scattering.

(a) We can utilize conservation of four momentum:

$$\begin{gathered} p_\gamma + p_e = p'_\gamma + p'_e \\ (p_\gamma + p_e - p'_\gamma)^2 = p_e'^2 \\ m_e^2 + 2p_\gamma \cdot p_e - 2p_\gamma \cdot p'_\gamma - 2p_e \cdot p'_\gamma = m_e^2 \\ \epsilon_\gamma m_e = \epsilon_\gamma \epsilon'_\gamma + \epsilon'_\gamma m_e \\ \boxed{\epsilon'_\gamma = \frac{\epsilon_\gamma m_e}{m_e + \epsilon_\gamma(1-\cos\theta_0)}}. \end{gathered} \tag{5.45}$$

(b) We can perform a Lorentz transformation on this expression to obtain the lab frame final state photon energy:

$$\boxed{E'_\gamma = \gamma(\epsilon'_\gamma + \boldsymbol{\beta}\cdot\boldsymbol{p}'_\gamma) = \gamma(1-\beta\cos\theta_0)\epsilon'_\gamma = \frac{\gamma(1-\beta\cos\theta_0)m_e\epsilon_\gamma}{m_e+\epsilon_\gamma(1-\cos\theta_0)}}. \tag{5.46}$$

(c) Now, we can relate the rest frame initial photon energy to the lab frame energy via another Lorentz transformation:

$$\epsilon_\gamma = \gamma(E_\gamma - \boldsymbol{\beta} \cdot \boldsymbol{P}_\gamma) = \gamma(1+\beta)E_\gamma. \tag{5.47}$$

We can put this into the expression obtained in part (b) to arrive at

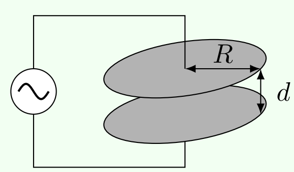
$$\boxed{E'_\gamma = \gamma^2(1+\beta)E_\gamma m_e \frac{1-\beta\cos\theta_0}{m_e + \gamma(1+\beta)E_\gamma(1-\cos\theta_0)} = \gamma m_e \frac{\lambda(1-\beta\cos\theta_0)}{1+\lambda(1-\cos\theta_0)}}. \tag{5.48}$$

(d) Finally, we can relate the scattering angle of the photon in the lab and electron rest frames as follows:

$$\boxed{\cos\theta_0 = \frac{p'_{\gamma,x}}{|\boldsymbol{p}'_\gamma|} = \frac{\gamma(P'_{\gamma,x} + \beta E'_\gamma)}{\epsilon'_\gamma} = \gamma(\cos\theta + \beta)}. \tag{5.49}$$

Problem 3.1)

A parallel plate capacitor consists of metal disks of radius R separated by an empty gap of width d. The space between the gap is a vacuum. Assume that $R \gg d$ so that the fringing fields can be neglected. The plates are connected to a generator which provides an oscillating emf $V(t) = V_0 \sin\omega t$ with the frequency $\omega \ll c/d$, where c is the speed of light.



(a) What is the magnetic field between the plates?

(b) Find the Poynting vector in the space between the capacitor plates. What is the direction of the electromagnetic energy flow?

(a) The magnetic field between the plates can be determined via the Ampere-Maxwell equation

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I + \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a}\,. \tag{5.50}$$

If we set our loop as a circle of radius r, we find

$$\boxed{\boldsymbol{B} = \hat{\boldsymbol{\phi}}\frac{1}{2\pi c r}\frac{\partial}{\partial t}E(\pi r^2) = \frac{r}{2c}\frac{\partial E}{\partial t}\hat{\boldsymbol{\phi}} = \frac{r}{2cd}\frac{\partial V}{\partial t}\hat{\boldsymbol{\phi}} = \frac{V_0\omega r}{2cd}\cos\omega t\hat{\boldsymbol{\phi}}}. \tag{5.51}$$

(b) The Poynting vector

$$\boxed{\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = -\frac{VB_\phi}{\mu_0 d}\hat{\boldsymbol{r}} = -\frac{V_0^2\omega r}{4c\mu_0 d}|\sin(2\omega t)|\hat{\boldsymbol{r}}} \,. \tag{5.52}$$

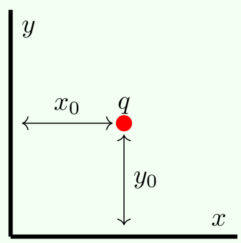
In our work of part (a), I constructed the direction of the surface normal to be in the same direction as the electric field, and the direction of the magnetic field is given by the right hand rule. For both cases, the Poynting vector points radially inward.

Problem 3.2)

A point charge q is placed between two perpendicular semi-infinite metallic plates as shown in the figure below.

(a) Calculate the electric potential $\varphi(x, y, x_0, y_0)$ produced by the charge.

(b) Calculate the components of the force $F_x(x_0, y_0)$ and $F_y(x_0, y_0)$ acting on the charge as functions of its cartesian coordinates x_0 and y_0.



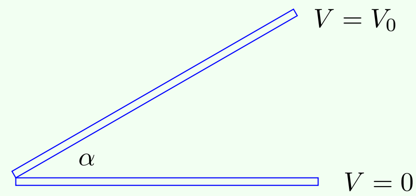
(a) Observe that we can place three image charges as follows: (1) $-q$ at $(-x_0, y_0)$, (2) $-q$ at $(x_0, -y_0)$, and (3) q at $(-x_0, -y_0)$. The resulting electric potential

$$\boxed{\begin{aligned}\varphi(\boldsymbol{r};\boldsymbol{r}_0) = \frac{q}{4\pi\epsilon_0}\Bigg\{&\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} - \frac{1}{\sqrt{(x+x_0)^2+(y-y_0)^2}} \\ &- \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2}} + \frac{1}{\sqrt{(x+x_0)^2+(y+y_0)^2}}\Bigg\}\end{aligned}} \,. \tag{5.53}$$

(b) The force acting on the charge is given by

$$\begin{aligned}\boldsymbol{F} &= \frac{q^2}{4\pi\epsilon_0}\left\{-\frac{\hat{\boldsymbol{x}}}{(2x_0)^2} - \frac{\hat{\boldsymbol{y}}}{(2y_0)^2} + \frac{(2x_0\hat{\boldsymbol{x}} + 2y_0\hat{\boldsymbol{y}})}{[(2x_0)^2+(2y_0)^2]^{3/2}}\right\} \\ &= \boxed{\frac{q^2}{16\pi\epsilon_0}\left\{\left[\frac{x_0}{(x_0^2+y_0^2)^{3/2}} - \frac{1}{x_0^2}\right]\hat{\boldsymbol{x}} + \left[\frac{y_0}{(x_0^2+y_0^2)^{3/2}} - \frac{1}{y_0^2}\right]\hat{\boldsymbol{y}}\right\}} \,.\end{aligned} \tag{5.54}$$

Problem 3.3)



Two flat conducting plates form a wedge with apex along the z-axis. The bottom plate is at ground potential and fills the half-plane $y = 0, x > 0$, while the second plate forms an angle α with the bottom one an dis at fixed potential V_0. Solve for the electric potential $V(r, \phi)$ anywhere between the 2 plates, using separation of (cylindrical variables).

Hint: The Laplacian in cylindrical coordinates can be writeen as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Pick the simplest possible solution for the Laplace equation (no Bessel functions needed!) and show it fulfills all boundary conditions.

The relevant equation is Laplace's equation, which in cylindrical coordinates takes the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{[}{2}]\Phi z = 0. \tag{5.55}$$

Notice that the setup is symmetric along z, implying that our potential cannot depend on z:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0. \tag{5.56}$$

We can introduce a separable ansatz for our potential of the form $\Phi(r, \phi) = R(r)T(\phi)$, which yields

$$\frac{r^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}\phi^2} = 0. \tag{5.57}$$

Since our potential goes from 0 to V_0 along the ϕ direction, we set

$$\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}\phi^2} = k^2 \Rightarrow T(\phi) = \begin{cases} A + B\phi & k = 0 \\ A\cosh(k\phi) + B\sinh(k\phi) & k \neq 0. \end{cases} \tag{5.58}$$

Notice that $A = 0$ for all k so that the boundary condition at $\phi = 0$ is satisfied Using this, we obtain the radial equation

$$r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + k^2 R = r\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}R}{\mathrm{d}r} + k^2 R = 0. \tag{5.59}$$

If we insert a power series $R = \sum_n a_n r^n$, we find $n = \pm k$. If $k = 0$, the second solution is $ln(r)$. The full solution for our potential is then

$$\Phi(r, \phi) = (a_0 + b_0\phi)(A_0 + B_0 \ln(r)) + \sum_{k \neq 0} \sinh(k\phi)(A_k r^k + B_k r^{-k}). \tag{5.60}$$

In order to have our potential be finite at $r = 0, \infty$, we should select

$$\boxed{\Phi(r, \phi) = V_0 \frac{\phi}{\alpha}}. \tag{5.61}$$

Quantum Mechanics

Problem 3.4)

Two spin-1/2 particles are in the state

$$|\Psi\rangle = \frac{1}{2}|\uparrow\rangle|\uparrow\rangle + \frac{\sqrt{3}}{2}|\uparrow\rangle|\downarrow\rangle$$

(a) If you measure the z-component of the total spin $S_{1z} + S_{2z}$, what values might you get and what is the probability for each?

(b) If you measure the total spin-squared $\boldsymbol{S}^2$ with $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$, what values might you get and what is the probability for each?

(a) If one measures the total spin along the z-axis, one may obtain (sans units of $\hbar$) 1 and 0 with probabilities

$$\boxed{P(1) = 1/4, \quad P(0) = 3/4}. \tag{5.62}$$

(b) Let us rewrite the state in terms of composite spin states first:

$$|\Psi\rangle = \frac{1}{2}|11\rangle + \frac{\sqrt{3}}{2\sqrt{2}}(|10\rangle + |00\rangle). \tag{5.63}$$

From this, it is clear that one may obtain 1 and 0 for s upon measurement of $\boldsymbol{S}^2$ with probabilities

$$\boxed{P(1) = 5/8, \quad P(0) = 3/8}. \tag{5.64}$$

Problem 4.1)

Consider an infinitely deep potential well of width a, i.e., a potential

$$U(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & x < 0,\ x > a. \end{cases}$$

The initial state of a particle in this well at $t = 0$ is described by the wave function

$$\Psi(x) = \begin{cases} x, & 0 < x < a/2 \\ a - x, & a/2 < x < a. \end{cases}$$

Find the probability of finding the particle on the n^{th} energy level, and estimate numerically this probability for two lowest bound states.

Find the average energy $\overline{E}$.

Hint: You may need the sum

$$\sum_{N=1,\text{odd}}^{\infty} \frac{1}{N^2} = \frac{\pi^2}{8}.$$

(a) We should first ensure that our state is normalized:

$$\int_0^{a/2} \mathrm{d}x\, x^2 + \int_{a/2}^{a} (a-x)^2 = \frac{a^3}{12}. \tag{5.65}$$

Thus, we should redefine our state as

$$\Psi(x) = \sqrt{\frac{12}{a^3}} \begin{cases} x & 0 < x < a/2 \\ a - x & a/2 < x < a \end{cases}. \tag{5.66}$$

expand the state $\Psi(x)$ in the basis of energy eigenstates:

$$\Psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x), \tag{5.67}$$

where

$$\begin{aligned} c_n &= \int \mathrm{d}x\, \psi_n^*(x)\Psi(x) = \sqrt{\frac{2}{a}} \left\{ \int_0^{a/2} \mathrm{d}x\, x \sin\left(\frac{n\pi x}{a}\right) + \int_{a/2}^{a} \mathrm{d}x\, (a-x) \sin\left(\frac{n\pi x}{2}\right) \right\} \\ &= \frac{4\sqrt{6}}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right). \end{aligned} \tag{5.68}$$

The probability to measure E_n is then

$$\boxed{P(E_n) = \begin{cases} \frac{96}{n^4\pi^4} & n = \text{odd} \\ 0 & n = \text{even} \\ . \end{cases}} \tag{5.69}$$

(b) From the above, we also can find

$$\boxed{\langle E \rangle = \sum_{n=\text{odd}} P(E_n) E_n = \frac{96}{\pi^4} E_1 \sum_{n=\text{odd}} \frac{1}{n^2} = \frac{12}{\pi^2} E_1}. \tag{5.70}$$

Problem 4.2)

The Hamiltonian for a particle of mass m is $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \lambda x$ (λ is a constant).

Find $\langle L_z \rangle$ as a function of time given that, for $t = 0$: $\langle L_x \rangle = a$, $\langle y \rangle = b$, and $\langle p_y \rangle = c$, where (a, b, c) are constant.

Recall the Ehrenfest theorem:

$$\frac{\mathrm{d}\langle A \rangle}{\mathrm{d}t} = \frac{i}{\hbar} \langle [H, A] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle. \tag{5.71}$$

If our operator does not depend explicitly on time, then the second term is zero. Thus, we can determine the time dependence of L_z as follows. First, we evalute the commutator

$$[H, L_z] = \frac{1}{2m}\Big\{[p_x^2, L_z] + [p_y^2, L_z] + [p_z^2, L_z]\Big\} + \lambda[x, L_z] = -i\hbar\lambda y. \tag{5.72}$$

This gives us that

$$\frac{\mathrm{d}\langle L_z \rangle}{\mathrm{d}t} = \lambda \langle y \rangle . \tag{5.73}$$

Next, we evaluate

$$[H, y] = \frac{1}{2m}[p_y^2, y] = -\frac{i\hbar}{m} p_y, \tag{5.74}$$

which gives

$$\frac{\mathrm{d}\langle y \rangle}{\mathrm{d}t} = \frac{\langle p_y \rangle}{m} = \frac{c}{m}. \tag{5.75}$$

where we have used that $[H, p_y] = 0$ to establish that $\langle p_y \rangle$ is independent of time. Unravelling our work, we find

$$\frac{\mathrm{d}\langle L_z \rangle}{\mathrm{d}t} = \frac{\lambda c}{m} t + \lambda b \Rightarrow \boxed{\langle L_z \rangle = \frac{\lambda c}{2m} t^2 + \lambda b t + c}. \tag{5.76}$$

Problem 4.3)

Two interacting point particles of equal mass m move along the x-axis. The most general wave function describing this system is $\psi(x_1, x_2)$. Assume the Hamiltonian for the system is given as

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + \frac{1}{4}m\omega^2(x_1 - x_2)^2$$

(a) What physical system does this Hamiltonian describe?

(b) Show that the following is a solution to the time-independent Schödinger equation:

$$\psi(x_1, x_2) = A\exp\left(i\frac{P}{\hbar}\frac{(x_1 + x_2)}{2} - \frac{(x_1 - x_2)^2}{2\sigma^2}\right).$$

(c) Express σ in terms of the other constants given and write down the eigenvalue E of the Hamiltonian for this solution.

(a) The Hamiltonian above describes a diatomic molecule bound together by a "spring" (i.e. they form a harmonic oscillator).

(b) Observe that

$$\begin{aligned}\frac{\partial\psi(x_1, x_2)}{\partial x_1} &= \left(\frac{iP}{2\hbar} - \frac{x_1 - x_2}{\sigma^2}\right)\psi(x_1, x_2)\\ \frac{\partial^2\psi(x_1, x_2)}{\partial x_1^2} &= -\frac{1}{\sigma^2}\psi(x_1, x_2) + \left(\frac{iP}{2\hbar} - \frac{x_1 - x_2}{\sigma^2}\right)^2\psi(x_1, x_2).\end{aligned} \tag{5.77}$$

Thus,

$$\begin{aligned}\frac{\partial^2\psi(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2\psi(x_1, x_2)}{\partial x_2^2} &= -\frac{1}{\sigma^2}\psi(x_1, x_2) + \left(\frac{iP}{2\hbar} - \frac{x_1 - x_2}{\sigma^2}\right)^2\psi(x_1, x_2)\\ &\quad - \frac{1}{\sigma^2}\psi(x_1, x_2) + \left(\frac{iP}{2\hbar} + \frac{x_1 - x_2}{\sigma^2}\right)^2\psi(x_1, x_2)\\ &= \left[-\frac{2}{\sigma^2} - \frac{P^2}{2\hbar^2} + \frac{2(x_1 - x_2)^2}{\sigma^2}\right]\psi(x_1, x2).\end{aligned} \tag{5.78}$$

The Hamiltonian acting on ψ then takes the form

$$\boxed{\mathcal{H}\psi(x_1, x_2) = \left[\frac{\hbar^2}{m\sigma^2} + \frac{P^2}{4m} - \frac{\hbar^2(x_1 - x_2)^2}{m\sigma^2} + \frac{m\omega^2(x_1 - x_2)^2}{4}\right]\psi(x_1, x_2)}. \tag{5.79}$$

We then see that $\psi(x_1, x_2)$ is a solution to the time-independent Schrödinger equation if σ is chosen properly.

(c) If we have $\sigma^2 = 4\hbar^2/(m^2\omega^2)$, then

$$\boxed{E = \frac{m\omega^2}{4} + \frac{P^2}{4m}}. \tag{5.80}$$

Problem 4.4)

A neutral particle with spin-1/2 and a magnetic moment proportional to is spin is in the eigenstate with the spin parallel to a constant magnetic field B_z applied along the z-axis. Calculate the probabilities $P_\uparrow(t)$ and $P_\downarrow(t)$ to observe the particle with the spin parallel and antiparallel to z after an additional constant field B_x was applied along the x-axis at $t = 0$.

Our Hamiltonian

$$H = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\gamma(\sigma_x B_x + \sigma_z B_z) = -\omega \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}, \tag{5.81}$$

where $\gamma = g_e e\hbar/(4mc)$. We now solve the eigenvalue problem:

$$\begin{vmatrix} B_z - \lambda & B_x \\ B_x & -(B_z+\lambda) \end{vmatrix} = \lambda^2 - (B_x^2 + B_z^2) = 0 \Rightarrow \lambda = \pm\sqrt{B_x^2 + B_z^2} = \pm B. \tag{5.82}$$

The corresponding eigenvectors are determined as follows:

$$\begin{pmatrix} B_z \mp B & B_x \\ B_x & -(B_z \pm \lambda) \end{pmatrix} \begin{pmatrix} a_{1,\pm} \\ a_{2,\pm} \end{pmatrix} = 0 \Rightarrow \chi_\pm = a_{1,\pm} \begin{pmatrix} 1 \\ (B_z \mp B)/B_x \end{pmatrix}. \tag{5.83}$$

Finally, we normalize our eigenvectors:

$$|a_{1,\pm}|^2 \left[1 + \frac{B_z^2 \mp 2B_z B + B^2}{B_x^2}\right] = 1 \Rightarrow \chi_\pm = \frac{1}{\sqrt{2B(B \mp B_z)}} \begin{pmatrix} B_x \\ B_z \mp B \end{pmatrix}. \tag{5.84}$$

From the problem statement, our particle has state

$$|\psi(0)\rangle = |+\rangle = \frac{B_x}{\sqrt{2B}} \left[\frac{1}{\sqrt{B - B_z}} |\chi_+\rangle + \frac{1}{\sqrt{B + B_z}} |\chi_-\rangle\right]. \tag{5.85}$$

The time evolution is given by

$$|\psi(0)\rangle = \frac{B_x}{\sqrt{2B}} \left[\frac{1}{\sqrt{B - B_z}} e^{-i\gamma Bt} |\chi_+\rangle + \frac{1}{\sqrt{B + B_z}} e^{i\gamma Bt} |\chi_-\rangle\right]. \tag{5.86}$$

We can now find the desired probabilities:

$$\begin{aligned}
P_\uparrow(t) &= |\langle +|\psi(t)\rangle|^2 \\
&= \left|\frac{B_x}{\sqrt{2B}}\left[\frac{1}{\sqrt{B-B_z}}e^{-i\gamma Bt}\frac{B_x}{\sqrt{2B(B-B_z)}} + \frac{1}{\sqrt{B+B_z}}e^{i\gamma Bt}\frac{B_x}{\sqrt{2B(B+B_z)}}\right]\right|^2 \\
&= \boxed{1 - \underbrace{\frac{B_x^2}{2B^2}\Big[1-\cos(2\gamma Bt)\Big]}_{P_\downarrow(t)}}.
\end{aligned} \tag{5.87}$$

Note that if $B_x = 0$, then the probability to be in the state $|+\rangle$ is unity for all t as expected.

6 January 2021

Classical Mechanics

Problem 1.1)

A smooth wire is bent into the shape of a spiral helix. In cylindrical polar coordinates (ρ, ϕ, z) it is specified by equations $\rho = R\phi^2$ and $z = \lambda\phi^2$, where R and λ are constants and the z-axis is vertically up (and gravity vertically down).

(a) Using z as your generalized coordinate, write down the Lagrangian for a bead of mass m threadaed on the wire.

(b) Find the Lagrange equation and find from it the expression for the bead's vertical acceleration $\ddot{z}$ as a function of z and $\dot{z}$.

(c) Find acceleration $\ddot{z}$ in two limits: (i) when $R \to 0$ but λ is fixed, and (ii) when $\lambda \to \infty$ but R is fixed. Discuss if the results for $\ddot{z}$ in these limits make sense.

We can write $x = \rho\cos\phi$ and $y = \rho\sin\phi$, so that

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2). \tag{6.1}$$

Using the conditions, we have

$$\rho = \frac{R}{\lambda}z \Rightarrow \dot{\rho} = \frac{R}{\lambda}\dot{z} \tag{6.2}$$

$$\phi = \sqrt{\frac{z}{\lambda}} \Rightarrow \dot{\phi} = \frac{\dot{z}}{2\sqrt{\lambda z}}. \tag{6.3}$$

Thus

$$\boxed{L = \frac{m}{2}\left[\frac{R^2}{\lambda^2}\left(1 + \frac{z}{4\lambda}\right) + 1\right]\dot{z}^2 - mgz}. \tag{6.4}$$

(b) Next, we take derivatives to identify the equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$

$$\boxed{\left[\frac{R^2}{\lambda^2}\left(1 + \frac{z}{4\lambda}\right) + 1\right]\ddot{z} + \frac{R^2}{8\lambda^3}\dot{z}^2 + g = 0}. \tag{6.5}$$

(c) Finally, it is simple to take the limits prescribed:

$$R \to \infty \Rightarrow \ddot{z} = -g$$
$$\lambda \to \infty \Rightarrow \ddot{z} = -g \tag{6.6}$$

Problem 1.2)

A particle with a mass m and an orbital moment L moves in an attractive potential which exerts a central force:

$$\boldsymbol{F} = -\frac{\alpha \boldsymbol{r}}{r^3} e^{-r/R},$$

where α and R are positive constants. Show that a particle does not have stable circular orbits in this potential for $L > L_c$ and calculate the threshold value of L_c.

There are two conditions for minima. First, we need to have a point where the effective potential

$$U_{\text{eff}} = \frac{L^2}{2mr^2} + V(r) \tag{6.7}$$

is an extremum, and second, at that point, we must have that the potential is concave up. For the first condition, we have

$$\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}r} = -\frac{L^2}{mr^3} + \frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{L^2}{mr^3} - F(r) = 0 \Rightarrow re^{-r/R} = \frac{L^2}{m\alpha}. \tag{6.8}$$

Note that the left-hand side is a function of r which is strictly positive and has a maximum at $r = R$ of R/e, while the right-hand-side is a positive constant. Therefore, if

$$\boxed{L > \sqrt{m\alpha R/e} = L_c}, \tag{6.9}$$

there can be no stationary points on the effective potential. We can now check that when solutions exist (i.e. $L < L_c$), one of them yields a stable orbit:

$$\begin{aligned} \frac{\mathrm{d}^2 U_{\text{eff}}}{\mathrm{d}r^2}\bigg|_{r=r_0} &= \frac{3L^2}{mR^4} - \frac{\alpha}{r_0^3}\left(2 + \frac{r_0}{R}\right)e^{-r_0/R} = \left\{\frac{3r_0}{R^4} - \frac{1}{r_0^3}\left(2 + \frac{r_0}{R}\right)\right\}\alpha e^{-r_0/R} > 0 \\ &\Rightarrow 0 < r_0 < R. \end{aligned} \tag{6.10}$$

One can easily check that $r_0 > R$ gives that $\mathrm{d}^2 U_{\text{eff}}/\mathrm{d}r^2 \mid_{r_0=R} = 0$.

Problem 1.3)

A massless inextensible string passes over a pulley located at a fixed distance above the floor. A bunch of bananas of mass m is attached at one end A of the string. A monkey of mass M is initially at the other end B. The monkey climbs the string, and his displacement $d(t)$ relative to the end B is a given function of time. The system is initially at rest, so that the initial conditions are $d(0) = \dot{d}(0) = 0$

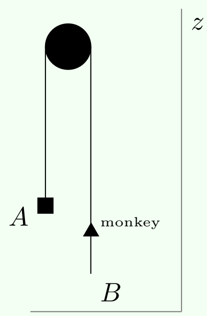
(a) Introduce suitable generalized coordinates and obtain the Lagrangian in terms of these coordinates.

(b) Show that the equation of motion for the height $Z(t)$ of the monkey above the floor is given by

$$(m + M)\ddot{Z}(t) - m\ddot{d}(t) = (m - M)g.$$

(c) Integrate the differential equation to obtain the subsequent motion.

(d) In the special case $m = M$, show that the bananas and monkey rise through equal distances, so the vertical separation between them is constant.



(a) We can write the Lagrangian simply as

$$L = \frac{m\dot{z}_A^2}{2} + \frac{M\dot{Z}^2}{2} - mgz_A - MgZ. \tag{6.11}$$

At this point, we introduce the constraints relating the heights of the monkey and bananas, respectively, from the ground:

$$Z(t) = d(t) + z_B = d(t) - z_A + L \Rightarrow z_a = d - Z + L, \tag{6.12}$$

where $L = z_A + z_B$ is a constant. From this, we write

$$\boxed{L = \frac{m(\dot{d} - \dot{Z})^2}{2} + \frac{M\dot{Z}^2}{2} - mg(d - Z + L) - MgZ}. \tag{6.13}$$

(b) We now use the Euler-Lagrange equation to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{Z}} - \frac{\partial L}{\partial Z} = -m(\ddot{d} - \ddot{Z}) + M\ddot{Z} - mg + Mg = 0$$

$$\Rightarrow \boxed{(m + M)\ddot{Z} - m\ddot{d} = (m - M)g} \tag{6.14}$$

(c) We can integrate the above equation twice as follows

$$(m+M)\dot{Z} - m\dot{d} = (m-M)gt$$

$$\boxed{Z(t) - Z_0 = \frac{md(t) + (m-M)gt^2/2}{m+M}}, \tag{6.15}$$

where we have assumed that the rope is initially stationary.

(d) If $m = M$, our equation of motion for Z is

$$\ddot{Z} = \frac{\ddot{d}}{2} \Rightarrow \ddot{z}_A = \frac{\ddot{d}}{2}. \tag{6.16}$$

Thus, integrating twice, we have

$$\boxed{Z(t) - Z_0 = z_A - z_A(0) = \frac{d(t)}{2}}. \tag{6.17}$$

That is, the monkey's and banana's displacements from their initial positions are the same at all times $t > 0$.

Problem 1.4)

A particle of mass m moves in one dimension subject to the force

$$F = -kx + \frac{a}{x^3},$$

where both k and a are positive.

- **(a)** What are the equilibrium points? Are they stable?
- **(b)** Assume that the particle undergoes small oscillations around an equilibrium point. What are the frequency and period of the oscillations?
- **(c)** Assume now that the total energy E is large so that the small oscillations approximation is not valid. The motion is not sinusoidal anymore but it is still periodic. Show that the period of the oscillations is independent of the energy E and therefore the period and frequency are still given by what was found in part (b).

Hint: You may use the following integral:

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = \pi$$

(a) The equilibrium points are defined through

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -F(x) = 0 \Rightarrow \boxed{x_\pm = \left(\frac{a}{k}\right)^{1/4}}. \tag{6.18}$$

We can determine if these are stable by considering

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right|_{x=x_\pm} = -\frac{\mathrm{d}F}{\mathrm{d}x} = k + \frac{a}{x_\pm^4} = 4k > 0. \tag{6.19}$$

Thus, these are stable equilibrium points.

(b) To find the period of small oscillations, we can Taylor expand our potential about these equilibria, yielding

$$V(x_\pm) = V(x_\pm) + \underbrace{V'(x_\pm)}_{=0}(x - x_\pm) + \frac{V''(x_\pm)}{2!}(x - x_\pm)^2. \tag{6.20}$$

The leading order restoring force is a harmonic one such that the angular frequency and period of oscillation are given by

$$\boxed{\omega = \sqrt{\frac{V''(x_\pm)}{m}} = 2\sqrt{\frac{k}{m}} \Rightarrow T = \frac{2\pi}{\omega} = \pi\sqrt{\frac{m}{k}}}, \tag{6.21}$$

respectively.

(c) In this part, we use conservation of energy to write

$$T = \sqrt{2m}\int_{x_-}^{x_+} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \tag{6.22}$$

The potential is given by $V(x) = [kx^2 + a/x^2]/2 = (kx_0^2/2)[(x/x_0)^2 + (x_0/x)^2]$, where we have used the fact that $a = kx_\pm^4$ and denoted $x_\pm = \pm x_0$. Hence

$$T = \sqrt{2m}\int_a^b \frac{\mathrm{d}x}{\sqrt{E - \frac{kx_0^2}{2}\left[(x/x_0)^2 + (x_0/x)^2\right]}}, \tag{6.23}$$

where a, b are defined such that $U(a) = U(b) = E$. Let us introduce the substitution $u = (x/x_0)^2$

$$\boxed{T = \sqrt{\frac{m}{k}}\int_{(a/x_0)^2}^{(b/x_0)^2} \frac{\mathrm{d}u}{\sqrt{\frac{2E}{kx_0^2}u - u^2 - 1}} = \sqrt{\frac{m}{k}}\int_{u_-}^{u_+} \frac{\mathrm{d}u}{\sqrt{(u-u_-)(u_+-u)}} = \pi\sqrt{\frac{m}{k}}}, \tag{6.24}$$

where

$$u_{\pm} = -\frac{E}{kx_0^2} \pm \sqrt{\left(\frac{E}{kx_0}\right)^2 - 1} \tag{6.25}$$

are the roots of the polynomial under the radical. Notice that $u_+ = (b/x_0)^2$ and $u_- = (a/x_0)^2$.

Problem 2.1)

A particle of mass is subject to an attractive central force $\boldsymbol{f}_1(\boldsymbol{r}) = \hat{\boldsymbol{r}} f(r)$ and a frictional force $\boldsymbol{f}_2(\boldsymbol{r}) = -\lambda \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the particle and $\lambda > 0$. The particle initially has an angular momentum $\boldsymbol{L}_0$ about the origin. By what time will the particle lose half of its angular momentum?

Observe that

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times (\hat{\boldsymbol{r}} f(r) - \lambda \boldsymbol{v}) = -\lambda \boldsymbol{r} \times \boldsymbol{v} = -\frac{\lambda}{m}\boldsymbol{L}, \tag{6.26}$$

which has solution

$$\boldsymbol{L}(t) = \boldsymbol{L}_0 e^{-(\lambda/m)t}. \tag{6.27}$$

Thus, the time T elapsed such that $\boldsymbol{L}(T) = \boldsymbol{L}_0/2$ satisfies

$$\frac{1}{2} = e^{-(\lambda/m)T} \Rightarrow \boxed{T = \frac{m}{\lambda}\ln(2)} \tag{6.28}$$

Electricity & Magnetism

Problem 2.2)

The largest world accelerator, LHC, is capable of accelerating protons up to the energy of 6.5 TeV, or approximately 7,000 times the rest energy of the proton.

(a) Find the difference $c - v = \delta v$ between the velocity v of such a proton and the speed of light $c \approx 3 \times 10^8$ m/sec. Find an analytic expression for δv and only then substitute numbers.

In fact, LHC is a collider in which two protons having this energy in the laboratory frame move towards each other (along, say, x-axis).

(b) Take the frame in which one of the protons is at rest. What is the velocity v_2 of the second proton in that frame? Since v_2 is very close to the speed of light, represent it as $v_2 = c - \delta v_2$, and find δv_2. Again, find an analytic expression for δv_2 and only then substitute numbers.

(c) What is the energy of the second proton in the rest frame of the first one?

(d) Imagine that we are in a rocket that leaves the Earth with the speed v equal to the lab frame speed of the LHC protons. How far from the Earth (in light-years) would we find ourselves after spending 1 year of our life on such a rocket?

(a) For this part, we can compute γ first:

$$\gamma = \frac{E}{mc^2} \approx 7000 \gg 1. \tag{6.29}$$

Then, we have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta = \sqrt{1 - \frac{1}{\gamma^2}} = 1 - \frac{1}{2\gamma^2} + \dots, \tag{6.30}$$

so

$$\boxed{\delta v = c(1-\beta) = \frac{c}{2\gamma^2} \approx 3 \text{ m/s}}. \tag{6.31}$$

(b) Next, we can use the velocity addition rule to find

$$v_2 = \frac{2v}{1+\beta^2} \Rightarrow \boxed{\delta v_2 = c(1-\beta_2) = \frac{c}{8\gamma^4} \approx 1.5 \times 10^{-8} \text{ m/s}}. \tag{6.32}$$

(c) Here, we can simply use that

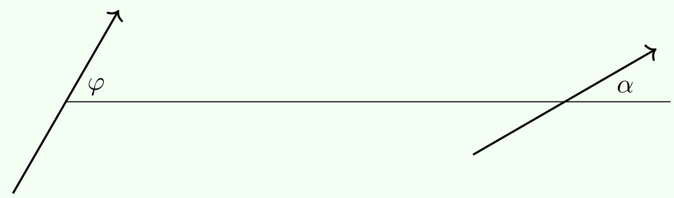
$$\boxed{E_2 = \gamma_2 mc^2 = \frac{mc^2}{\sqrt{1-\beta_2^2}} = \frac{mc^2}{\sqrt{\underbrace{(1+\beta_2)}_{\approx 2}\underbrace{(1-\beta_2)}_{=1/(8\gamma^4)}}} = 2\gamma^2 mc^2 \approx 10^8 \text{ GeV}}. \tag{6.33}$$

(d) Finally, we have that the distance the rocket travels in the frame of the earth is

$$\boxed{d = vt = \gamma v t' \approx \gamma c t' = 7000 \text{ lightyears}} \tag{6.34}$$

Problem 2.3)

Two dipoles are a certain distance apart. One is fixed at an angle φ with the line joining the 2 dipoles. The other is fixed in location but free to rotate and will be at an angle α with the same line. What is the relationship between φ and α?



The torque on the second dipole from the field of the first is just

$$\begin{aligned}\boldsymbol{N} &= \boldsymbol{p}_2 \times \boldsymbol{E}_1(\boldsymbol{r_2}) = \boldsymbol{p_2} \times \frac{3(\boldsymbol{p}_1 \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} - \boldsymbol{p}_1}{4\pi\epsilon_0 r^3} \\ &= \frac{-p_1 p_2[3\cos\varphi\sin\alpha + \sin(\varphi - \alpha)]}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{z}} = \frac{p_1 p_2[2\cos\varphi\sin\alpha + \sin\varphi\cos\alpha]}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{z}}, \end{aligned} \tag{6.35}$$

where I have chosen the z-axis to point out of the page. At equilibrium, the torque is exactly zero, which imposes the condition that

$$4\cos\varphi\sin\alpha = -\sin\varphi\cos\alpha \Rightarrow \boxed{\tan\alpha = -\frac{1}{2}\tan\varphi} \tag{6.36}$$

Alternatively, one could observe that the angle α is the same as that which the electric field makes with the axis at equilibrium, which is given by

$$\cos\alpha = \frac{2\cos\varphi}{\sqrt{3\cos^2\varphi + 1}}. \tag{6.37}$$

One can check that the results are equivalent (although there are some subtlelties related to the range of inverse trigonometric functions to be cautious of). Also note that $\boldsymbol{p}$ can be either parallel to antiparallel to $\boldsymbol{E}$, so if α is a solution to either of the above equations, then $\alpha + n\pi$ also is for $n = \pm 1, 2, \ldots$.

Problem 2.4)

Show by explicit calculation that the energy of the classical system particles + electromagnetic field given by (in Gaussian units)

$$H = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2(t) + \frac{1}{8\pi}\int \mathrm{d}^3\boldsymbol{r}\left[\boldsymbol{E}^2(\boldsymbol{r},t) + \boldsymbol{B}^2(\boldsymbol{r},t)\right]$$

is a constant of motion, namely

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0.$$

Here $\dot{\boldsymbol{r}}(t)$ is the velocity of the particle. Assume that the $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ fields vanish as $|\boldsymbol{r}| \to \infty$.

Hints: You may need the expression for the current density given by

$$\boldsymbol{j}(\boldsymbol{r}, t) = \sum_i q_i \dot{\boldsymbol{r}}_i(t) \delta[\boldsymbol{r} - \boldsymbol{r}_i(t)],$$

where q_i is the charge of particle i.

We will need Maxwell's equations in Gaussian units, which read as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho \tag{6.38}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{6.39}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{6.40}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{6.41}$$

Let us now take the derivative of the Hamiltonian:

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i + \frac{1}{4\pi}\int \mathrm{d}^3\boldsymbol{r} \left[\boldsymbol{E} \cdot \dot{\boldsymbol{E}} + \boldsymbol{B} \cdot \dot{\boldsymbol{B}}\right] \\
&= \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i + \frac{1}{4\pi}\int \mathrm{d}^3\boldsymbol{r} \left\{\boldsymbol{E} \cdot c\left[(\boldsymbol{\nabla} \times \boldsymbol{B}) - \frac{4\pi}{c}\boldsymbol{j}\right] - \boldsymbol{B} \cdot c(\boldsymbol{\nabla} \times \boldsymbol{E})\right\} \\
&= \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i + \frac{c}{4\pi}\int \mathrm{d}^3\boldsymbol{r} \left[\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \times \boldsymbol{E})\right] - \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{E} \cdot \boldsymbol{j} \\
&= \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i + \frac{c}{4\pi}\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) - \sum_i q_i \dot{\boldsymbol{r}}_i \cdot \boldsymbol{E}(\boldsymbol{r}_i).
\end{aligned} \tag{6.42}$$

Observe that

$$Home\boldsymbol{F}_i = m_i \ddot{\boldsymbol{r}}_i = q_i\left[\boldsymbol{E}(\boldsymbol{r}_i) + \frac{\dot{\boldsymbol{r}}_i}{c} \times \boldsymbol{B}(\boldsymbol{r}_i)\right] \Rightarrow m_i \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i = q_i \dot{\boldsymbol{r}}_i \cdot \boldsymbol{E}(\boldsymbol{r}_i), \tag{6.43}$$

so the first and last terms in our expansion cancel. Thus,

$$\boxed{\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{c}{4\pi}\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \oint \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = 0} \tag{6.44}$$

since the fields go to zero at infinity.

Problem 3.1)

A conducting sphere of radius R_1 carries charge Q. A second, initially uncharged conducting sphere of radius R_2 is placed at large distance $R \gg R_1, R_2$ and then connected to the first sphere by a long thin wire with large resistance. After a long time, the system of two conducting spheres reaches equilibrium.

(a) Find the electrostatic force between two spheres.

(b) Find the ohmic heat dissipated in the wire and the spheres. Neglect effects of radiation.

(a) Using Ohm's law, we see that at equilibrium no current flows and therefore that the potential difference between the spheres is zero:

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{R_1} - \frac{Q_2}{R_2}\right] = 0, \tag{6.45}$$

where we have used the assumption that $R \gg R_1, R_2$ to neglect the contributions to the potential from the other sphere at the surface of the other one. Thus, we have the following system of equations at equilibrium:

$$\begin{cases} Q_1/R_1 - Q_2/R_2 = 0 \\ Q_1 + Q_2 = Q \end{cases} \Rightarrow \begin{cases} Q_1 = Q[R_1/(R_1+R_2)] \\ Q_2 = Q[R_2/(R_1+R_2)]. \end{cases} \tag{6.46}$$

From this, we calculate simply the magnitude of the force (clearly it is repulsive) between spheres using Coulomb's law:

$$\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^2}\frac{R_1 R_2}{(R_1+R_2)^2}}. \tag{6.47}$$

(b) Finally, we can compute the heat dissipated while the spheres were approaching equilibrium as the difference between the energies of the configurations before the wire was connected and after equilibrium is achieved. Observe that for a sphere with total charge q and radius r the energy stored in its field is

$$U = \frac{\epsilon_0}{2}\int \mathrm{d}^3\boldsymbol{r}'\, \boldsymbol{E}^2 = \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}\int \mathrm{d}\Omega' \int_r^\infty \frac{\mathrm{d}r'}{r'^2} = \frac{q^2}{2(4\pi\epsilon_0)r}. \tag{6.48}$$

Thus, the energy dissipated to achieve equilibrium is

$$\begin{aligned}\mathcal{E} &= \frac{Q^2}{2(4\pi\epsilon_0)R_1} - \left[\frac{Q_1^2}{2(4\pi\epsilon_0)R_1} + \frac{Q_2^2}{2(4\pi\epsilon_0)R_2}\right] \\ &= \frac{Q^2}{2(4\pi\epsilon_0)}\left[\frac{1}{R_1} - \left(\frac{R_1}{(R_1+R_2)^2} + \frac{R_2}{(R_1+R_2)^2}\right)\right] \\ &= \boxed{\frac{Q^2}{2(4\pi\epsilon_0)}\frac{R_2}{R_1(R_1+R_2)}}. \end{aligned} \tag{6.49}$$

Observe three limiting cases: $R_2 \ll R_1$ yields $\mathcal{E} \to 0$; $R_1 = R_2$ yields $\mathcal{E} = U_0/2$ (where U_0 is the energy of the initial configuration); and finally, $R_2 \gg R_1$ yields $\mathcal{E} \to U_0$.

Problem 3.2)

Two identical electric dipoles rotate with a circular frequency ω in the same direction in the xy-plane. The moment $\boldsymbol{d}_2(t)$ of the second dipole makes a constant angle α with $\boldsymbol{d}_1(t)$ of the first dipole, and $|\boldsymbol{d}_1(t)| = |\boldsymbol{d}_2(t)| = d_0$. Calculate the net power of radiation P_ω as functions of ω and α if the spacing between the dipoles in the xy-plane is much smaller than the wavelength $2\pi c/\omega$ of radiation. Find the angles α_1 and α_2 at which P_ω is minimum and maximum, respectively.

Hint: The net power of dipole radiation is $P_\omega = 2|\ddot{\boldsymbol{d}}|^2/(3c^2)$.

The full dipole moment is just the sum $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$. Since the dipole moments are rotating with frequency ω, we have $\ddot{\boldsymbol{d}} = \omega^2\boldsymbol{d}$ (to see this create a physical dipole where the charges are rotating and see what this implies about the dipole moment's time derivative). Thus

$$\boxed{P_\omega = \frac{2\omega^4|\boldsymbol{d}_1 + \boldsymbol{d}_2|^2}{3c^2} = \frac{2\omega^4\left[|\boldsymbol{d}_1|^2 + 2|\boldsymbol{d}_1\cdot\boldsymbol{d}_2| + |\boldsymbol{d}_2|^2\right]}{3c^2} = \frac{4d_0^2\omega^4[1+\cos\alpha]}{3c^2}}. \tag{6.50}$$

Observe that if we restrict $\alpha \in [0, \pi]$, then a minimum in the net dipole radiation occurs when $\alpha_1 = \pi$ (i.e. when the dipoles cancel). On the other hand, a maximum in the net dipole radiation occurs when $\alpha_2 = 0$ (i.e. when the dipoles sum to twice their strength).

Quantum Mechanics

Problem 3.3)

Consider a spin-1/2 particle in one dimension subject to the spin-dependent interaction given by

$$V(x) = V_0 \sigma_x \delta(x), \quad V_0 > 0,$$

where $\delta(x)$ is the δ-function at the origin and σ_x is the Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume the particle approaches the interaction region from the far left ($x = -\infty$) and has energy $E > 0$ and spin projection $+\hbar/2$ in the $\hat{\boldsymbol{z}}$-direction. What is the probability that the particle has spin projection $-\hbar/2$ relative to the $\hat{\boldsymbol{z}}$-direction after it has traversed the interaction region and is at the far right ($x = \infty$)?

Hint: In the basis of eigenstates of σ_x the Schrödinger equation for spin up and spin down along the $\hat{\boldsymbol{x}}$-direction decouple.

The relevant energy eigenvalue equation reads

$$\frac{\mathrm{d}^2 |\Psi\rangle}{\mathrm{d}x^2} + \Big[k^2 - v_0 \sigma_x \delta(x)\Big] |\Psi\rangle \,, \tag{6.51}$$

where $v_0 = 2mV_0/\hbar^2$ and $E = \hbar^2 k^2/(2m) > 0$. Notice that if we consider the regions $x < 0$ and $x > 0$ separately, the particle is free

$$|\Psi\rangle = \begin{cases} e^{ikx} |+\rangle + e^{-ikx}(A\,|+\rangle_x + B\,|-\rangle_x) & x < 0 \\ e^{ikx}(C\,|+\rangle_x + D\,|-\rangle_x) & x > 0. \end{cases} \tag{6.52}$$

Note that $|+\rangle = (|+\rangle_x + |-\rangle_x)/\sqrt{2}$.

We have two boundary conditions:

$$\Big|\Psi(x=0^-)\Big\rangle = \Big|\Psi(x=0^+)\Big\rangle \Rightarrow \begin{cases} \frac{1}{\sqrt{2}} + A = C \\ \frac{1}{\sqrt{2}} + B = D \end{cases} \tag{6.53}$$

$$\frac{\mathrm{d}\,|\Psi(x=0^+)\rangle}{\mathrm{d}x} - \frac{\mathrm{d}\,|\Psi(x=0^-)\rangle}{\mathrm{d}x} = v_0 \sigma_x\, |\Psi(x=0)\rangle \Rightarrow \begin{cases} ik[C - \frac{1}{\sqrt{2}} + A] = v_0 C \\ ik[D - \frac{1}{\sqrt{2}} + B] = -v_0 D. \end{cases} \tag{6.54}$$

Observe that each boundary condition yields two equations since $\langle + | - \rangle = 0$. Solving these yields

$$A = -\frac{i}{\sqrt{2}(2\alpha + i)}, \quad C = \frac{\sqrt{2}\alpha}{2\alpha + i}, \quad B = \frac{i}{\sqrt{2}(2\alpha - i)}, \quad D = \frac{\sqrt{2}\alpha}{2\alpha - i}, \tag{6.55}$$

where $\alpha = k/v_0$.

Finally, we want the probability that a transmitted particle at $x = \infty$ is measured to have spin $-\hbar/2$, given by

$$\boxed{T_{\text{flip}} = \left|\frac{C-D}{\sqrt{2}}\right|^2 = \left|\frac{\alpha}{2\alpha+i} - \frac{\alpha}{2\alpha-i}\right|^2 = \frac{4\alpha^2}{(1+4\alpha^2)^2}}. \tag{6.56}$$

Problem 3.4)

Consider two identical spin-1/2 fermions each with mass m confined to move along a line of length L. The confining potential energy is zero for $L > x > 0$ and infinite elsewhere.

(a) What is the ground-state energy of the combined system? Write down the normalized ground-state wave function accounting for spatial and spin-state symmetry.

(b) What is the first excited energy level and its degeneracy? Write down the corresponding normalized wave functions.

(a) The Hamiltonian of our system reads

$$H = H_1 + H_2, \tag{6.57}$$

where H_1 and H_2 are the single infinite potential well Hamiltonians for particles 1 and 2, respectively. The eigenstates of this combination are simply the states $|\Psi_{n_1 n_2, sm}\rangle = \Psi_{n_1,n_2}(x)\,|sm\rangle$ (corresponding to energies $E = E_{n_1} + E_{n_2}$), where $\Psi_{n_1,n_2}(x)$ is an eigenstate of our two-particle infinite potential well system, and $|sm\rangle$ is the combined spin state of the fermionic system and is one of the below states:

$$\text{triplet}: \begin{cases} |11\rangle = |{+}{+}\rangle \\ |10\rangle = [|{+}{-}\rangle + |{-}{+}\rangle]/\sqrt{2} \\ |1\ -1\rangle = |{-}{-}\rangle \end{cases} \tag{6.58}$$

$$\text{singlet}: \quad |00\rangle = [|{+}{-}\rangle - |{-}{+}\rangle]/\sqrt{2}. \tag{6.59}$$

Recall that the overall state $|\Psi\rangle$ must be antisymmetric under exchange of our identical fermions, so the position or spin state must be antisymmetric under this exchange (not both though!). Thus, the ground state must simply be

$$\boxed{|\Psi_{11,00}\rangle = \psi_1(x_1)\psi_2(x_2)\,|00\rangle}. \tag{6.60}$$

(b) Continuing our arguments from part (a), we find the first excited states as

$$\boxed{\begin{aligned} |\Psi_{12,1m}\rangle &= \frac{1}{\sqrt{2}}\Big[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\Big]\,|1m\rangle \\ |\Psi_{12,00}\rangle &= \frac{1}{\sqrt{2}}\Big[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\Big]\,|1m\rangle \end{aligned}}, \tag{6.61}$$

which implies a four-fold degeneracy.

Problem 4.1)

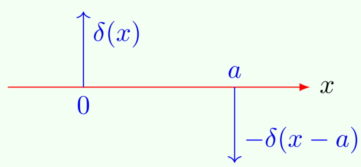
In a one-dimensional quantum scattering problem, the potential barrier is given by

$$U(x) = \alpha[\delta(x) - \delta(x-a)],$$

(see Figure below).

(a) Find the reflection coefficient for particles moving from left to right and having momentum k.

(b) Find the momenta k for which particles are not reflected by the potential barrier.



We can solve this problem by dividing the x-axis into three portions, within which the particle is free:

$$\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x < 0 \\ Be^{ikx} + Ce^{-ikx} & 0 < x < a \\ De^{ikx}. & \end{cases} \tag{6.62}$$

The coefficients are determined via boundary conditions:

$$\psi(0^-) = \psi(0^+) \Rightarrow 1 + A = B + C \tag{6.63}$$

$$\psi'(0^+) - \psi(0^-) = \frac{2m\alpha}{\hbar^2}\psi(0) \Rightarrow ik\Big[(B-C) - (1-A)\Big] = \frac{2m\alpha}{\hbar^2}(1+A) \tag{6.64}$$

$$\psi(a^-) = \psi(a^+) \Rightarrow Be^{ika} + Ce^{-ika} = De^{ika} \tag{6.65}$$

$$\psi'(0^+) - \psi(0^-) = -\frac{2m\alpha}{\hbar^2}\psi(0) \Rightarrow ik\Big[De^{ika} - (Be^{ika} - Ce^{-ika})\Big] = -\frac{2m\alpha}{\hbar^2}De^{ika}. \tag{6.66}$$

The problem asks for the reflection coefficient, which is given as $R = |A|^2$, so we care only to solve the system for A. Unfolding the system of equations and defining $\lambda = 2m\alpha/(\hbar^2 k)$, we have

$$
\begin{aligned}
D &= \frac{1}{1+i\lambda}(B - Ce^{-2ika}) \\
C &= -\frac{i\lambda}{2+i\lambda}e^{2ika}B \\
B &= \frac{2+i\lambda}{2+i\lambda(1+e^{2ika})}[(1-i\lambda) - (1+i\lambda)A] \\
A &= -\frac{i\lambda[(2+i\lambda)e^{2ika} + (2-i\lambda)]}{(2+i\lambda)^2 + \lambda^2 e^{2ika}}.
\end{aligned} \tag{6.67}
$$

Thus, the reflection coefficient

$$
\boxed{R = \frac{2[(4+\lambda^2) + (4-\lambda^2)\cos(2ka) - 4\lambda\sin(2ka)]}{\lambda^4 + (4+\lambda^2)^2 + 2\lambda^2[(4-\lambda^2)\cos(2ka) + 4\lambda\sin(2ka)]}}. \tag{6.68}
$$

(b) Finally, particles are not reflected when

$$
\begin{gathered}
(4+\lambda^2) + (4-\lambda^2)\cos(2ka) - 4\lambda\sin(2ka) = 0 \\
4 - (4-\lambda^2)\sin^2(ka) - 2\lambda\sin(2ka) = 0 \\
4\cos^2(ka) - 4\lambda\sin(ka)\cos(ka) + \lambda^2\sin^2(ka) = 0 \\
(2\cos(ka) - \lambda\sin(ka))^2 = 0 \\
\boxed{\tan(ka) = \frac{\hbar^2 k}{m\alpha}}.
\end{gathered} \tag{6.69}
$$

This is a transcendental equation for k and thus there does not exist a closed form for k, but we could define $z = ka$ and solve numerically the equation

$$
\tan(z) = z/z_0, \tag{6.70}
$$

where $z_0 = ma\alpha/\hbar^2$. Observe that there will be an infinite number of solutions, all within the intervals $[n\pi, (n+1/2)\pi]$ (where $n = 0, 1, \ldots$), and as $k \to \infty$ we have $z \to (2n+1/2)\pi$.

Problem 4.2)

The purpose of this exercise is to prove what is known as the Hellmann-Feynman theorem. That theorem relates the derivative of the total energy with respect to a parameter to the expectation value of the derivative of the Hamiltonian with respect to the same parameter.

Consider a time-independent system where: (1) $\hat{H}_\lambda$ is a Hamiltonian depending upon a continuous parameter λ, (2) $|\Psi_\lambda\rangle$ is an eigenstate of the Hamiltonian $\hat{H}_\lambda$, and (3) E_λ

is the energy of the state $|\Psi_\lambda\rangle$, i.e. $\hat{H}_\lambda |\Psi_\lambda\rangle = E_\lambda |\Psi_\lambda\rangle$.

(a) Show that:

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \langle\Psi_\lambda| \frac{\mathrm{d}\hat{H}_\lambda}{\mathrm{d}\lambda} |\Psi_\lambda\rangle .$$

(b) For a general time-dependent wave function satisfying the time-dependent Schrödinger equation the Hellman-Feynman theorem is not valid. However show that the following identity holds:

$$\langle\Psi_\lambda| \frac{\mathrm{d}\hat{H}_\lambda}{\mathrm{d}\lambda} |\Psi_\lambda\rangle = i\hbar\frac{\partial}{\partial t}\left\langle \Psi_\lambda(t) \middle| \frac{\mathrm{d}\Psi_\lambda(t)}{\mathrm{d}t} \right\rangle$$

(a) An energy eigenstate of the Hamiltonian satisfies the equation $H|\psi\rangle = E|\psi\rangle$, where the dependence on λ is implied for brevity. Notice then that

$$E = \langle\psi|H|\psi\rangle \, , \tag{6.71}$$

so the derivative

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}\lambda} &= \frac{\mathrm{d}\langle\psi|}{\mathrm{d}\lambda} H|\psi\rangle + \langle\psi|\frac{\mathrm{d}H}{\mathrm{d}\lambda}|\psi\rangle + \langle\psi| H\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}\lambda} \\ &= E_\lambda \underbrace{\frac{\mathrm{d}}{\mathrm{d}\lambda}\langle\psi|\psi\rangle}_{=0} + \langle\psi|\frac{\mathrm{d}H}{\mathrm{d}\lambda}|\psi\rangle = \langle\psi|\frac{\mathrm{d}H}{\mathrm{d}\lambda}|\psi\rangle \end{aligned} \tag{6.72}$$

(b) In this part, the relevant equation is the time-dependent Schrödinger equation: $i\hbar\,\partial|\Psi\rangle/\partial t = H|\Psi\rangle$. Thus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\langle\Psi| i\hbar\frac{\partial|\Psi\rangle}{\partial t}\right] &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle\Psi|H|\Psi\rangle \\ i\hbar\frac{\mathrm{d}\langle\Psi|}{\mathrm{d}\lambda}\frac{\partial|\Psi\rangle}{\partial t} + i\hbar\langle\Psi|\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial|\Psi\rangle}{\partial t} &= \frac{\mathrm{d}\langle\Psi|}{\mathrm{d}\lambda}H|\Psi\rangle + \langle\Psi|\frac{\mathrm{d}H}{\mathrm{d}\lambda}|\Psi\rangle + \langle\Psi| H\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}\lambda} \\ i\hbar\langle\Psi|\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial|\Psi\rangle}{\partial t} &= \langle\Psi|\frac{\mathrm{d}H}{\mathrm{d}\lambda}|\Psi\rangle - i\hbar\frac{\partial\langle\Psi|}{\partial t}\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}\lambda} \\ \langle\Psi|\frac{\mathrm{d}H}{\mathrm{d}\lambda}|\Psi\rangle &= i\hbar\frac{\partial}{\partial t}\langle\Psi|\frac{\mathrm{d}|\Psi\rangle}{\mathrm{d}\lambda} \end{aligned} \tag{6.73}$$

Problem 4.3)

The spin component of an electron along the z-axis is determined to be $+1/2$. Another axis z', makes an angle θ with z. What is,

(a) the probability that a projection of the spin along z' is $+1/2$ or $-1/2$ and,

(b) the mean value of the spin component along z' axis?

Let us construct the z' axis such that $\hat{\boldsymbol{z}}' = \sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{z}}$ and therefore

$$S_{z'} = \sin\theta S_x + \cos\theta S_z = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\sin\theta\end{pmatrix}. \tag{6.74}$$

Diagonalizing, we find eigenvalues $\lambda_\pm = \pm 1$ and eigenvectors

$$\chi_\pm = \frac{\sin\theta}{\sqrt{2(1\mp\cos\theta)}}\begin{pmatrix}1 \\ (-\cos\theta \pm 1)/\sin\theta\end{pmatrix}. \tag{6.75}$$

The probability of measuring $\pm 1/2$ for S_z' is just

$$\boxed{P(\pm 1/2) = |\langle\chi_\pm|+\rangle|^2 = \left|\frac{\sin\theta}{\sqrt{2(1\mp\cos\theta)}}\right|^2 = \frac{\sin^2\theta}{2(1\mp\cos\theta)}}. \tag{6.76}$$

(b) The expectation value can be computed in two ways. First we can write

$$\langle S_{z'}\rangle = \langle +|S_{z'}|+\rangle = \frac{\hbar}{2}\cos\theta. \tag{6.77}$$

Second, we also have

$$\langle S_{z'}\rangle = \frac{\sin^2\theta}{2(1-\cos\theta)} - \frac{\sin^2\theta}{2(1+\cos\theta)} = \boxed{\frac{\hbar}{2}\cos\theta} \tag{6.78}$$

Problem 4.4)

A particle of mass m is in a 1d infinite potential well of width L so that $U(x) = 0$ at $0 < x < L$ and $U(x) = \infty$ outside the well. The initial state of the particle at $t = 0$ is described by the normalized wave function:

$$\psi_0(x) = \frac{\sqrt{30}}{L^{5/2}}(L-x)x, \quad 0 < x < L,$$

and $\psi_0(x) = 0$ outside the well.

(a) Write down the wave function $\psi(x,t)$ at $t > 0$.

(b) Calculate the probabilities w_n or measuring different energies E_n of the particle in the well and show that $\sum_{n=1}^{\infty} w_n = 1$.

(c) Calculate the expectation value of energy.

Hint: You may need the formulas:

$$\int_0^1 z(1-z)\sin(\pi n z)\,\mathrm{d}z = \frac{2}{\pi^3 n^3}[1-(-1)^n], \quad n = 1, 2, 3, \ldots$$
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} = \frac{\pi^6}{960}, \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$$

(a) Recall the spectrum of the infinite square well:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}. \tag{6.79}$$

We can expand any function in the basis of eigenstates as follows

$$\Psi(x,0) = \sum_n c_n \psi_n(x), \tag{6.80}$$

where

$$\begin{aligned} c_n &= \int_0^L \mathrm{d}x\, \psi_n^*(x)\Psi(x,0) \\ &= \sqrt{\frac{2}{L}}\frac{\sqrt{30}}{L^{5/2}}\int_0^L \mathrm{d}x\, x(L-x)\sin\left(\frac{n\pi x}{L}\right) \\ &= \sqrt{60}\int_0^1 \mathrm{d}z\, z(1-z)\sin(n\pi z) = \frac{2\sqrt{60}}{\pi^3 n^3}[1-(-1)^n], \end{aligned} \tag{6.81}$$

which is zero for even n. From the expansion at $t = 0$

$$\boxed{\Psi(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar} = \frac{8}{\pi^3}\sqrt{\frac{30}{L}}\sum_{n=\text{odd}}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)e^{-iE_nt/\hbar}}. \tag{6.82}$$

(b) The weights are simply

$$\boxed{w_n = |\langle\psi_n|\Psi\rangle|^2 = |c_n|^2 = \frac{960}{\pi^2 n^6}}. \tag{6.83}$$

Using this, we have

$$\sum_{n=\text{odd}} = \frac{960}{\pi^2}\sum_{n=\text{odd}}\frac{1}{n^6} = 1, \tag{6.84}$$

from the hint above.

(c) Finally, the expected value of energy

$$\boxed{E_n = \sum_n w_n E_n = \frac{\pi^2\hbar^2}{2mL^2}\frac{960}{\pi^6}\sum_{n=\text{odd}}\frac{1}{n^4} = \frac{5\hbar^2}{mL^2} = \frac{10}{\pi^2}E_1} \tag{6.85}$$