

Lecture 16 - 2) Use the recurrence relations

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x) \quad (1)$$

and

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x) \quad (2)$$

to show that

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n + 1)P_n(x). \quad (3)$$

Let us differentiate Eq. (1):

$$(2n + 1)P_n(x) + (2n + 1)xP'_n(x) = (n + 1)P'_{n+1}(x) + nP'_{n-1}(x). \quad (4)$$

From Eq. (2), we see that

$$2xP'_n(x) = P'_{n+1}(x) + P'_{n-1}(x) - P_n(x). \quad (5)$$

Hence, Eq. (4) becomess

$$\begin{aligned} 2(2n + 1)P_n(x) + (2n + 1)[P'_{n+1}(x) + P'_{n-1}(x) - P_n(x)] &= 2(n + 1)P'_{n+1}(x) + 2nP'_{n-1}(x) \\ [(2n + 1) - (2n + 2)]P'_{n+1}(x) + [(2n + 1) - 2n]P'_{n-1}(x) &= -[(4n + 2) - (2n + 1)]P_n(x) \\ -P'_{n+1}(x) + P'_{n-1}(x) &= -(2n + 1)P_n(x) \\ \boxed{P'_{n+1}(x) - P'_{n-1}(x) = (2n + 1)P_n(x)}. \end{aligned} \quad (6)$$

Lecture 16 - 3) Show that

$$g(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}} \quad (7)$$

satisfies the differential equation

$$\frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial}{\partial x} g(x, t) \right] + t \frac{\partial^2}{\partial t^2} [tg(x, t)] = 0. \quad (8)$$

For this problem, we simply take derivatives:

$$\frac{\partial g}{\partial x} = \frac{t}{(1 - 2xt + t^2)^{3/2}} \quad (9)$$

$$\frac{\partial}{\partial x} (1 - x^2) \frac{\partial g}{\partial x} = -\frac{t(2t^2x - tx^2 - 3t + 2x)}{(1 - 2xt + t^2)^{5/2}} \quad (10)$$

$$\begin{aligned} t \frac{\partial^2 (tg)}{\partial t^2} &= t \frac{\partial}{\partial t} \left[\frac{1 - xt}{(1 - 2xt + t^2)^{3/2}} \right] \\ &= \frac{t(2t^2x - tx^2 - 3t + 2x)}{(1 - 2xt + t^2)^{5/2}}. \end{aligned} \quad (11)$$

Clearly, the expressions in Eq. (10) and Eq. (11) match up to a minus sign, so $g(x, t)$ satisfies the differential equation given by Eq. (8).

Lecture 17 - 1) Check that

$$\langle Au, v \rangle = \langle u, Av \rangle \iff p(x)[u'v - uv'] \Big|_a^b = 0. \quad (12)$$

The inner product on $L_{2,w}(a, b)$ (where $w(x) > 0$ for all $x \in (a, b)$)

$$\langle u, v \rangle = \int_a^b w(x)u(x)v(x) \, dx, \quad (13)$$

and the operator A is defined such that

$$Au = -\frac{1}{w(x)} \left[\partial_x(p(x)\partial_x u) + q(x)u \right], \quad (14)$$

where $p(x) > 0$ and $q(x)$ is a real-valued function on (a, b) .

Assuming that $\langle Au, v \rangle = \langle u, Av \rangle$ holds, then

$$\int_a^b \left[\partial_x(p(x)\partial_x u(x)) + q(x)u(x) \right] v(x) \, dx = \int_a^b u(x) \left[\partial_x(p(x)\partial_x v(x)) + q(x)v(x) \right] \, dx, \quad (15)$$

where $w(x)$ vanishes here since it is strictly positive. Simplifying, we find

$$\int_a^b \left[p'u'v + pu''v - up'v' - upv'' \right] \, dx = 0. \quad (16)$$

We can now integrate by parts on the terms containing derivatives of $p(x)$ such that

$$pu'v|_a^b - upv'|_a^b + \int_a^b \left[p\partial_x(u'v) + pu''v - p\partial_x(uv') - upv'' \right] \, dx = 0. \quad (17)$$

Upon taking derivatives and simplifying, we realize that the integrand in the second term is identically zero, leaving us with

$$p(x)[u'(x)v(x) - u(x)v'(x)] \Big|_a^b = 0. \quad (18)$$

Now we assume that Eq. (18) holds true. Thus,

$$\begin{aligned} \langle Au, v \rangle &= \int_a^b \left[\partial_x(p(x)\partial_x u(x)) + q(x)u(x) \right] v(x) \, dx \\ &= \int_a^b \left[p'u'v + pu''v + qvu \right] \, dx \\ &= pu'v|_a^b + \int_a^b \left[-pu''v - pu'v' + pu''v + qvu \right] \, dx \end{aligned} \quad (19)$$

$$\begin{aligned} &= pu'v|_a^b - puv'|_a^b + \int_a^b \left[p'v' + pv'' + qvu \right] \, dx \\ &= \langle u, Av \rangle, \end{aligned} \quad (20)$$

which concludes the second portion of the proof, showing a necessary and sufficient condition for operator A to be symmetric.

Lecture 17 - 2) Check that for operator A given by Eq. (14)

$$\langle Au, u \rangle = -p(x)u'u|_a^b + \int_a^b [p(x)|u'(x)|^2 - q(x)|u|^2] dx. \quad (21)$$

We can steal the form of Eq. (19) and substitute $v = u$, which gives us that

$$\langle Au, u \rangle = p(x)u'(x)u(x)|_a^b - \int_a^b [p(x)|u'(x)|^2 - q(x)|u(x)|^2] dx. \quad (22)$$

Lecture 17 - 3) For all $u, v \in C^2(\bar{\Omega})$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ (Ω is bounded), check the so-called Green's identities

$$\langle Au, v \rangle - \langle u, Av \rangle = \int_{\Omega} p \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS, \quad (23)$$

where $\partial_n = \hat{n} \cdot \nabla$ is the directional derivative along the direction normal to the surface of Ω , where \hat{n} is the normal vector to $\partial\Omega$,

$$\langle Au, u \rangle = - \int_{\partial\Omega} pu \frac{\partial u}{\partial n} dS + \int_{\Omega} [p|\nabla u|^2 - q|u|^2] d^3x. \quad (24)$$

Hint for Ex. 3. $\nabla \cdot (p \nabla u) v = \sum_j \partial_j (p \partial_j u) v =$
 $= \sum_j \partial_j (p v \partial_j u) - \sum_j p \partial_j v \partial_j u = \nabla \cdot (p v \nabla u) -$
 $- p \nabla v \cdot \nabla u = \nabla \cdot (p v \nabla u) - p \nabla u \cdot \nabla v$

The multi-dimensional operator A is defined in analogy with the one-dimensional version such that

$$Au = -\frac{1}{w(x)} \left[\nabla \cdot (p \nabla u) + qu \right], \quad (25)$$

and the inner product is defined in analogy with the one-dimensional version as

$$\langle u, v \rangle = \int_{\Omega} w(x) u(x) v(x) d^3x. \quad (26)$$

Checking the first identity is quite simple then:

$$\langle Au, v \rangle - \langle u, Av \rangle = \int_{\Omega} [v \nabla \cdot (p \nabla u) - u \nabla \cdot (p \nabla v)] d^3x \quad (27)$$

$$= \int_{\Omega} [(\nabla \cdot (pv \nabla u) - p \nabla u \cdot \nabla v) - (\nabla \cdot (pu \nabla v) - p \nabla v \cdot \nabla u)] d^3x \quad (28)$$

$$= \int_{\Omega} \nabla \cdot p [u \nabla v - v \nabla u] d^3x = \boxed{\int_{\partial\Omega} p \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS}. \quad (29)$$

For the second, we follow a similar sequence of work:

$$\langle Au, u \rangle = - \int_{\Omega} [\nabla \cdot pu \nabla u - p \nabla u \cdot \nabla u + q|u|^2] d^3x \quad (30)$$

$$= \boxed{- \int_{\partial\Omega} pu \frac{\partial u}{\partial n} dS + \int_{\Omega} [p|\nabla u|^2 - q|u|^2] d^3x}. \quad (31)$$

Lecture 17 - 4) Find eigenvectors and eigenvalues for operator A , defined such that $Au = -\Delta u$ (defined in Example 2) in the following cases.

For the Laplace operator we have the eigenfunctions as $e_{nm}(x, y) = e_n(x)e_m(y)$ with corresponding eigenvalues $\lambda_{mn} = \lambda_m + \lambda_n$.

$$4.1) \mathcal{D}(A) = \{u \in C^2(\overline{\Omega}) : u(x, 0) = u(x, H) = 0, u_x(0, y) = u_x(L, y) = 0\}$$

For these boundary conditions we have $e_n(x) = \cos(n\pi x/L)$ ($\lambda_n = (n\pi/L)^2$) for $n = 0, 1, \dots$, and $e_m(y) = \sin(n\pi y/H)$ ($\lambda_m = (m\pi/H)^2$) for $m = 1, 2, \dots$

$$4.2) \mathcal{D}(A) = \{u \in C^2(\overline{\Omega}) : u(0, y) = u(L, y), u_x(0, y) = u_x(L, y), u(x, 0) = u(x, L), u_y(x, 0) = u_y(x, H)\}$$

The periodic boundary conditions for x are $e_n(0) = e_n(L)$ and $e'_n(0) = e'_n(L)$, which gives eigenfunctions $e_0(x) = 1$ ($\lambda_0 = 0$), $e_n(x) = \cos(n\pi x/L)$ and $\sin(n\pi x/L)$ ($\lambda_n = (n\pi/L)^2$) for $n \geq 1$. Similarly, the eigenfunctions $e_m(y)$ are $e_0(y) = 1$ ($\lambda_0 = 0$) and $e_m(y) = \cos(m\pi y/H)$ and $\sin(m\pi y/H)$ ($\lambda_m = (m\pi/H)^2$) for $m \geq 1$.