1) Which operators are linear?

a) $Au = -\partial^2 u$, $\mathcal{D} = \{u \in C^2[0,\pi] \mid u(0) = 0 \text{ and } u'(\pi) = 0\}$

This is a linear operator on the vector space $\mathcal{D}$ since

$$A(\alpha u + \beta v) = -\partial^2(\alpha u + \beta v) = \alpha(-\partial^2 u) + \beta(-\partial^2 v) = \alpha A u + \beta A v. \tag{1}$$

b) $Au = -\partial^2 u$, $\mathcal{D} = \{u \in C^2[0,\pi] \mid u(0) = 0 \text{ and } u'(\pi) = \pi\}$

The space $\mathcal{D}$ is not a proper vector space since if $u, v \in \mathcal{D}$, then $u+v \notin \mathcal{D}$ since $u'(\pi)+v'(\pi) = 2\pi$. Hence, A is not a linear operator on the space $\mathcal{D}$ since Au must be in the vector space $\mathcal{D}$ for all $u \in \mathcal{D}$.

c) $Au = u_{xx} + x^2 u$, $\mathcal{D} = C^2(\mathbb{R})$

This is clearly a linear operator since

$$A(\alpha u + \beta v) = \partial_x^2(\alpha u + \beta v) + x^2(\alpha u + \beta v) = \alpha[u_{xx} + x^2 u] + \beta[v_{xx} + x^2 v] = \alpha A u + \beta A v. \tag{2}$$

d) $Au = u_{xx} + u^2$, $\mathcal{D} = C^2(\mathbb{R})$

This is not a linear operator:

$$A(\alpha u) = \alpha u_{xx} + \alpha^2 u^2 \neq \alpha A u. \tag{3}$$

2) Solve $u_t = 3u_x + 5u$, $u(x,0) = f(x)$.

We wish to solve the equation $u_t = Au$, where $A = 3\partial_x + 5$ and $u(x,0) = f(x)$, which has solution

$$u = e^{tA} f(x) = e^{3t\partial_x} e^{5t} f(x) = e^{5t} f(x + 3t). \tag{4}$$

3) Solve $u_{tt} - u_{xx} = 0$, $x \in \mathbb{R}$, $t > 0$ for $u(x,0) = 3\sin 5x - \cos 3x$, $u_t(x,0) = 2x + 3x^2$. Find only $u(0,t)$.

The general solution to the wave equation $u_{tt} - c^2 u_{xx} = 0$ with initial conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\big[\phi(x+ct) + \phi(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s\,. \tag{5}$$

Hence, for our problem, we have the solution

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\big[3\sin[5(x+t)] - \cos[3(x+t)] + 3\sin[5(x-t)] - \cos[3(x-t)]\big] \\
&\quad + \frac{1}{2}\int_{x-t}^{x+t} (2s+3s^2)\,\mathrm{d}s \\
&= 3\cos(5t)\sin(5x) - \cos(3t)\cos(3x) \\
&\quad + \frac{1}{2}[(x+t)^2 - (x-t)^2 + (x+t)^3 - (x-t)^3] \\
&= \boxed{3\cos(5t)\sin(5x) - \cos(3t)\cos(3x) + 2t(t^2+3x^2+2x)}\,. \qquad (6)
\end{aligned}$$

4) Find $u(x,t)$ for the equation $u_t - ku_{xx} = 0$, $x \in \mathbb{R}$, $t > 0$ subject to $u(x,0) = e^{-(x+1)^2}$.

We can find $u(x,t)$ subject to the initial condition as follows:

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}e^{-(y+1)^2}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt-1}\int_{-\infty}^{\infty} e^{-y^2/4kt-y^2-2y+xy/2kt}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt-1}\int_{-\infty}^{\infty} e^{-(1+1/4kt)y^2+2(x/4kt-1)y}\,\mathrm{d}y\,. \qquad (7)
\end{aligned}$$

Denote $a = 1 + 1/4kt$ and $b = x/4kt - 1$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(1+1/4kt)y^2+2(x/4kt-1)y}\,\mathrm{d}y &= \int_{-\infty}^{\infty} e^{-a(y^2+2by/a)}\,\mathrm{d}y \\
&= e^{b^2/a}\int_{-\infty}^{\infty} e^{-a(y+b/a)^2}\,\mathrm{d}y \\
&= e^{4kt(x/4kt-1)^2/(1+4kt)}\sqrt{\frac{4\pi kt}{1+4kt}}. \qquad (8)
\end{aligned}$$

Finally, we have our solution:

$$u(x,t) = \frac{1}{\sqrt{1+4kt}}e^{-(x+1)^2/(1+4kt)}. \qquad (9)$$

5) Find the Fourier transform $\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}\omega$ for $f(x) = xe^{-x^2}$.

The fourier transform of f is given as

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} xe^{-i\omega x}e^{-x^2}\,\mathrm{d}x = \frac{i}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega}\int_{-\infty}^{\infty} e^{-i\omega x}e^{-x^2}\,\mathrm{d}\omega = \frac{i}{\omega}\frac{\mathrm{d}\mathcal{F}\{\exp(-x^2)\}}{\mathrm{d}k}. \qquad (10)$$

The fourier transform of this "gaussian" is as follows

$$\begin{aligned}\mathcal{F}\{e^{-x^2}\} &= \int_{-\infty}^{\infty} e^{-i\omega x} e^{-x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-(x^2+i\omega)}\,\mathrm{d}x \\ &= e^{-\omega^2/4}\int_{-\infty}^{\infty} e^{-(x+i\omega/2)^2}\,\mathrm{d}x = \sqrt{\pi}e^{-\omega^2/4}. \end{aligned} \tag{11}$$

Thus,

$$\widehat{f}(\omega) = \frac{i\sqrt{\pi}}{\omega}\Big(-\frac{\omega}{2}e^{-\omega^2/4}\Big) = -\frac{i\sqrt{\pi}}{2}e^{-\omega^2/4}. \tag{12}$$

6*) Find the Fourier transform for $f(x) = -H(x+1) + 2H(x) - H(x-1)$, where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

7*) Solve $u_t = 5u_x - 3xu + u$, $x \in \mathbb{R}$, $t > 0$ with $u(x,0) = f(x)$. Display the graph of $f(x)$.