Lecture 16 - 2) Use the recurrence relations

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{1}$$

and

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x) \tag{2}$$

to show that

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{3}$$

Let us differentiate Eq. (1):

$$(2n+1)P_n(x) + (2n+1)xP'_n(x) = (n+1)P'_{n+1}(x) + nP'_{n-1}(x). \tag{4}$$

From Eq. (2), we see that

$$2xP'_n(x) = P'_{n+1}(x) + P'_{n-1}(x) - P_n(x). \tag{5}$$

Hence, Eq. (4) becomess

$$\begin{gathered} 2(2n+1)P_n(x) + (2n+1)[P'_{n+1}(x) + P'_{n-1}(x) - P_n(x)] = 2(n+1)P'_{n+1}(x) + 2nP'_{n-1}(x) \\ [(2n+1) - (2n+2)]P'_{n+1}(x) + [(2n+1) - 2n]P'_{n-1}(x) = -[(4n+2) - (2n+1)]P_n(x) \\ -P'_{n+1}(x) + P'_{n-1}(x) = -(2n+1)P_n(x) \\ \boxed{P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)}. \end{gathered} \tag{6}$$

Lecture 16 - 3) Show that

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} \tag{7}$$

satisfies the differential equation

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x}g(x,t)\right] + t\frac{\partial^2}{\partial t^2}\left[tg(x,t)\right] = 0. \tag{8}$$

For this problem, we simply take derivatives:

$$\frac{\partial g}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} \tag{9}$$

$$\frac{\partial}{\partial x}\left(1-x^2\right)\frac{\partial g}{\partial x} = -\frac{t(2t^2x - tx^2 - 3t + 2x)}{(1-2xt+t^2)^{5/2}} \tag{10}$$

$$\begin{aligned} t\frac{\partial^2(tg)}{\partial t^2} &= t\frac{\partial}{\partial t}\left[\frac{1-xt}{(1-2xt+t^2)^{3/2}}\right] \\ &= \frac{t(2t^2x - tx^2 - 3t + 2x)}{(1-2xt+t^2)^{5/2}}. \end{aligned} \tag{11}$$

Clearly, the expressions in Eq. (10) and Eq. (11) match up to a minus sign, so $g(x,t)$ satisfies the differential equation given by Eq. (8).

Lecture 17 - 1) Check that

$$\langle Au, v\rangle = \langle u, Av\rangle \iff p(x)\big[u'v - uv'\big]\big|_a^b = 0. \tag{12}$$

The inner product on $L_{2,w}(a,b)$ (where $w(x) > 0$ for all $x \in (a,b)$)

$$\langle u, v\rangle = \int_a^b w(x)u(x)v(x)\,\mathrm{d}x\,, \tag{13}$$

and the operator A is defined such that

$$Au = -\frac{1}{w(x)}\Big[\partial_x\big(p(x)\partial_x u\big) + q(x)u\Big], \tag{14}$$

where $p(x) > 0$ and $q(x)$ is a real-valued function on (a,b).

Assuming that $\langle Au, v\rangle = \langle u, Av\rangle$ holds, then

$$\int_a^b \Big[\partial_x\big(p(x)\partial_x u(x)\big) + q(x)u(x)\Big]v(x)\,\mathrm{d}x = \int_a^b u(x)\Big[\partial_x\big(p(x)\partial_x v(x)\big) + q(x)v(x)\Big]\,\mathrm{d}x\,, \tag{15}$$

where $w(x)$ vanishes here since it is strictly positive. Simplifying, we find

$$\int_a^b \Big[p'u'v + pu''v - up'v' - upv''\Big]\,\mathrm{d}x = 0. \tag{16}$$

We can now integrate by parts on the terms containing derivatives of $p(x)$ such that

$$pu'v|_a^b - upv'|_a^b + \int_a^b \big[p\partial_x(u'v) + pu''v - p\partial_x(uv') - upv''\big]\,\mathrm{d}x = 0. \tag{17}$$

Upon taking derivatives and simplifying, we realize that the integrand in the second term is identically zero, leaving us with

$$p(x)\big[u'(x)v(x) - u(x)v'(x)\big]\big|_a^b = 0. \tag{18}$$

Now we assume that Eq. (18) holds true. Thus,

$$\begin{aligned}
\langle Au, v\rangle &= \int_a^b \Big[\partial_x\big(p(x)\partial_x u(x)\big) + q(x)u(x)\Big]v(x)\,\mathrm{d}x \\
&= \int_a^b \big[p'u'v + pu''v + qvu\big]\,\mathrm{d}x \\
&= pu'v|_a^b + \int_a^b \big[-pu''v - pu'v' + pu''v + qvu\big]\,\mathrm{d}x \qquad (19)\\
&= pu'v|_a^b - puv'|_a^b + \int_a^b \big[p'v' + pv'' + qvu\big]\,\mathrm{d}x \\
&= \langle u, Av\rangle, \qquad (20)
\end{aligned}$$

which concludes the second portion of the proof, showing a necessary and sufficient condition for operator A to be symmetric.

Lecture 17 - 2) Check that for operator A given by Eq. (14)

$$\langle Au, u\rangle = -p(x)u'u\Big|_a^b + \int_a^b [p(x)|u'(x)|^2 - q(x)|u|^2]\,\mathrm{d}x\,. \tag{21}$$

We can steal the form of Eq. (19) and substitute $v = u$, which gives us that

$$\boxed{\langle Au, u\rangle = p(x)u'(x)u(x)\Big|_a^b - \int_a^b \left[p(x)|u'(x)|^2 - q(x)|u(x)|^2\right]\mathrm{d}x}\,. \tag{22}$$

Lecture 17 - 3) For all $u, v \in C^2(\overline{\Omega})$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ (Ω is bounded), check the so-called Green's identities

$$\langle Au, v\rangle - \langle u, Av\rangle = \int_\Omega p\Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,\mathrm{d}S\,, \tag{23}$$

where $\partial_n = \hat{n}\cdot\nabla$ is the directional derivative along the direction normal to the surface of Ω,

$$\langle Au, u\rangle = -\int_{\partial\Omega} pu\frac{\partial u}{\partial n}\,\mathrm{d}S + \int_\Omega \left[p|\nabla u|^2 - q|u|^2\right]\mathrm{d}^3x\,. \tag{24}$$

Hint for Ex. 3. $\underline{\nabla\cdot(p\nabla u)v} = \sum_j \partial_j(p\partial_j u)v =$

$= \sum_j \partial_j(pv\partial_j u) - \sum_j p\partial_j v\,\partial_j u = \nabla\cdot(pv\nabla u) -$

$- p\nabla v\cdot\nabla u = \underline{\nabla\cdot(pv\nabla u) - p\nabla u\cdot\nabla v}$

The multi-dimensional operator A is defined in analogy with the one-dimensional version such that

$$Au = -\frac{1}{w(x)}\Big[\nabla\cdot(p\nabla u) + qu\Big], \tag{25}$$

and the inner product is defined in analogy with the one-dimensional version as

$$\langle u, v\rangle \int_\Omega w(x)u(x)v(x)\,\mathrm{d}^3x\,. \tag{26}$$

Checking the first identity is quite simple then:

$$\langle Au, v\rangle - \langle u, Av\rangle = \int_a^b \Big[v\nabla \cdot (p\nabla u) - u\nabla \cdot (p\nabla v)\Big] \, \mathrm{d}^3x \,. \tag{27}$$

Lecture 17 - 4) Find eigenvectors and eigenvalues for operator A, defined such that $Au = -\Delta u$ (defined in Example 2) in the following cases.

4.1) $\mathcal{D}(A) = \{u \in C^2(\overline{\Omega}) \,:\, u(x,0) = u(x,H) = 0,\, u_y(0,y) = u_y(L,y) = 0\}$

4.2) $\mathcal{D}(A) = \{u \in C^2(\overline{\Omega}) \,:\, u(0,y) = u(L,y),\, u(x,0) = u(x,H),\, u_y(0,y) = u_y(L,y),$
$u_x(x,0) = u_x(x,H)\}$