2.4.3) Use

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) \, \mathrm{d}y \,. \tag{1}$$

to solve the diffusion equation if $\phi(x) = e^{3x}$.

Plugging ϕ into Eq. (1), we have

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt+3y} \, \mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x^2-2xy+y^2-12kty)/4kt} \, \mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt} \int_{-\infty}^{\infty} e^{-(y^2-2(6kt+x)y+(6kt-x)^2-(6kt-x)^2)/4kt} \, \mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt} e^{(x-6kt)^2/4kt} \int_{-\infty}^{\infty} e^{-[y-(6kt+x)]^2/4kt} \, \mathrm{d}y \\
&= \boxed{e^{3x+9kt}} \,.
\end{aligned} \tag{2}$$

2.4.6) Compute $\int_0^\infty e^{-x^2} \, \mathrm{d}x$.

Let

$$I = \int_0^\infty e^{-x^2} \, \mathrm{d}x \,, \tag{3}$$

then

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, \mathrm{d}x \, \mathrm{d}y = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, \mathrm{d}x \, \mathrm{d}y \,. \tag{4}$$

Changing to polar coordinates, we have $\mathrm{d}x \, \mathrm{d}y = r \, \mathrm{d}r \, \mathrm{d}\phi$, where $r \in [0, \infty]$ and $\phi \in [0, \pi/2]$ and $r = \sqrt{x^2+y^2}$. Thus, Eq. (4) becomes

$$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\phi \,. \tag{5}$$

If we make the substitution $u = r^2$, then

$$I^2 = \frac{\pi}{4} \int_0^\infty e^{-u} \, \mathrm{d}u = \frac{\pi}{4}, \tag{6}$$

leaving us with

$$\boxed{I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}} \,. \tag{7}$$

2.4.7) Use Exercise 6 to show that $\int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}p = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x,t)\,\mathrm{d}x = 1. \tag{8}$$

Notice that e^{-p^2} is an even function in p, so

$$\int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}p = 2\int_{0}^{\infty} e^{-p^2}\,\mathrm{d}p = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}. \tag{9}$$

It is given in the text that $S(x,t) = \exp(-x^2/4kt)/\sqrt{4\pi kt}$, so

$$\int_{-\infty}^{\infty} S(x,t)\,\mathrm{d}x = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4kt}}\,\mathrm{d}x\,. \tag{10}$$

If we make the change of variables $p = x/\sqrt{4kt}$, then $p \in (-\infty,\infty)$ and $\mathrm{d}x = \sqrt{4kt}\,\mathrm{d}p$, meaning

$$\boxed{\int_{-\infty}^{\infty} S(x,t)\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2} = 1}. \tag{11}$$

2.4.8) Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta\le|x|<\infty} S(x,t) \to 0 \quad \text{as } t \to 0. \tag{12}$$

Observe that $S(x,t)$ is a monotonically decreasing function for $x > 0$ at every t. Thus, the absolute maximum of $S(x,t)$ is taken on whenever $x = \delta$ in the interval $[\delta,\infty)$. That is,

$$\lim_{t\to 0}\max_{\delta\le|x|<\infty} S(x,t) = \lim_{t\to 0} S(\delta,t) = \lim_{t\to 0}\frac{1}{\sqrt{4\pi kt}}e^{-\delta^2/4kt}. \tag{13}$$

We make the change of variables $s = 1/4kt$, which makes Eq. (13)

$$\boxed{\lim_{t\to 0}\frac{e^{-\delta^2/4kt}}{\sqrt{4\pi kt}} = \frac{1}{\sqrt{\pi}}\lim_{s\to\infty}\frac{\sqrt{s}}{e^{\delta^2 s}} = \frac{1}{2\delta^2\sqrt{\pi}}\lim_{s\to\infty}\frac{1}{\sqrt{s}e^{\delta^2 s}} = 0}, \tag{14}$$

where L'Hopital's rule was used.

2.4.9) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness $u_{xxx} \equiv 0$. Integrating this result thrice, obtain

$u(x,t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

Notice

$$(u_{xxx})_t = (u_t)_{xxx} = (ku_{xx})_{xxx} = k(u_{xxx})_{xx}. \tag{15}$$

Hence, u_{xxx} is a solution to the diffusion equation and satisfies $u_{xxx}(x,0) = \partial_x^3(x^2) = 0$. By uniqueness, it immediately follows that $u_{xxx} \equiv 0$, and upon integration, we obtain

$$u = A(t)x^2 + B(t)x + C. \tag{16}$$

We can then solve for A, B, and C by plugging our expression for u into the diffusion equation.

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t). \tag{17}$$

Matching the coefficients in x, we have

$$\begin{cases} A' = 0 \\ B' = 0 \\ C' = 2kA \end{cases}. \tag{18}$$

Thus, $A = c_1$, $B = c_2$, and $C = 2kc_1t + c_3$, giving

$$u(x,t) = c_1x^2 + c_2x + 2kc_1t + c_3. \tag{19}$$

Finally, we can solve for our coefficients by plugging in our initial condition

$$u(x,0) = x^2 = c_1x^2 + c_2x + c_3, \tag{20}$$

which makes our solution

$$\boxed{u(x,t) = x^2 + 2kt}. \tag{21}$$

2.4.15) Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x,t) \quad \text{for } 0 < x < l, t > 0 \tag{22}$$
$$u(x,0) = \phi(x) \;\; u_x(0,t) = g(t) \;\; u_x(l,t) = h(t). \tag{23}$$

by the energy method.

Suppose that there are two distinct solutions to the diffusion problem satisfying Neumann boundary conditions: u and v. We may define $w = u - v$, which is a solution to the diffusion problem with the following boundary conditions: $w(x,0) = 0$, $w_x(0,t) = 0$, and $w_x(l,t)0$.

Now, consider the following quantity

$$E(t) = \int_0^l w^2(x,t)\,\mathrm{d}x\,. \tag{24}$$

We may differentiate this expression, yielding

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^l 2ww_t\,\mathrm{d}x = \int_0^l 2w(kw_{xx})\,\mathrm{d}x\,. \tag{25}$$

We may integrate by parts, noticing that the boundary term vanishes since w_x vanishes at the boundaries, giving

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2k\int_0^l w_x^2\,\mathrm{d}x \leq 0, \tag{26}$$

since the integrand is positive-definite. This, equation says that $E(t)$ is a strictly decreasing function of t, and since $E(0) = 0$, it follows that $E(t) \leq 0$. We also know by inspection of Eq. (24), that $E(t) \geq 0$ for all t. Hence, both inequalities may only be satisfied simultaneously if $E(t) = 0$, which implies that the integrand must be identically zero and

$$\boxed{u \equiv v}\,. \tag{27}$$