Richard Whitehill Phys 757 Test 3

1. $u_t + 2u_x - 3u = u_t + Au = 0 \quad (A = 2\partial_x - 3)$

$$\Rightarrow \boxed{u(x,t) = e^{-tA} f(x) = e^{3t} e^{-2t\partial_x} f(x) = e^{3t} f(x-2t)}$$

2. steady-state $\Rightarrow \partial_t \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u_t(x,t) = 0$

$\Rightarrow t \to \infty : \quad u_t = u_{xx} + 3 - 3x^2 = 0$

$\Rightarrow \dfrac{d^2u}{dx^2} = 3x^2 - 3 \Rightarrow \dfrac{du}{dx} = x^3 - 3x + c_1$

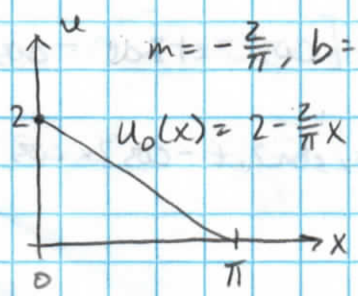
$u'(0) = c_1 = 1 \Rightarrow \dfrac{du}{dx} = x^3 - 3x + 1$

$\Rightarrow u(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + x + c_2$

$u(1) = 0 = \underbrace{\frac{1}{4} - \frac{3}{2} + 1}_{\frac{1}{4} - \frac{6}{4} + \frac{4}{4} = -\frac{1}{4}} + c_2 \Rightarrow c_2 = \frac{1}{4}$

$$\Rightarrow \boxed{\lim_{t\to\infty} u(x,t) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + x + \frac{1}{4}}$$

3. $m = -\frac{2}{\pi}, \; b = 2$



let $u(x,t) = u_0(x) + v(x,t)$

$\Rightarrow v(0,t) = u(0,t) - u_0(0) = 0$

$v(\pi,t) = u(\pi,t) - u_0(\pi) = 0$

homogeneous B.C.

$u_t - u_{xx} = v_t - \partial_x^2 u_0 - v_{xx} = v_t - v_{xx} = 0$

let $A = -\partial_x^2 \Rightarrow v_t + Av = 0 \Rightarrow v = e^{-tA} v(x,0)$

$* \; v(x,0) = u(x,0) - u_0(x) = 3\sin 2x + f(x) \quad [f(x) = \frac{2}{\pi}x - 2]$

For B.C. $v(0) = v(\pi) = 0$; $-\partial_x^2 \sin(nx) = \overset{\lambda_n}{n^2} \overset{e_n}{\sin(nx)}$ for $n = 1, 2, \ldots$

$v(x,t) = 3e^{t\partial_x^2} \sin 2x + e^{t\partial_x^2} f(x) = 3e^{-4t} \sin 2x + \sum_{n=1}^{\infty} e^{-tn^2} \hat{f}_n \sin(nx)$

$$\hat{f}_n = \frac{\langle f, e_n\rangle}{\|e_n\|^2} = \frac{2}{\pi}\int_0^\pi 2\left(\frac{x}{\pi}-1\right)\sin(nx)\,dx$$

$$= \frac{4}{\pi}\Big[\underbrace{\frac{1}{\pi}\int_0^\pi x\sin(nx)\,dx}_{I_1} - \underbrace{\int_0^\pi \sin(nx)\,dx}_{I_2}\Big]$$

$$I_1 = \int_0^\pi x\sin(nx)\,dx = -\frac{x}{n}\cos(nx)\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\,dx$$

$u = x \quad v' = \sin(nx)$ $\qquad = -\frac{\pi}{n}\cos(n\pi) + \underbrace{\frac{1}{n^2}\sin(nx)\Big|_0^\pi}_{0} = -\frac{\pi}{n}(-1)^n$

$u' = 1 \quad v = \frac{-1}{n}\cos(nx)$

$$I_2 = \int_0^\pi \sin(nx)\,dx = -\frac{1}{n}\cos(nx)\Big|_0^\pi = -\frac{1}{n}\left[(-1)^n - 1\right]$$

$$\Rightarrow \hat{f}_n = \frac{4}{\pi}\left[-\frac{1}{n}(-1)^n + \frac{1}{n}(-1)^n + \frac{1}{n}\right] = \frac{4}{\pi n}$$

$$\Rightarrow v(x,t) = 3e^{-4t}\sin 2x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}e^{-tn^2}$$

$$\Rightarrow \boxed{u(x,t) = 2\left(1-\frac{x}{\pi}\right) + 3e^{-4t}\sin 2x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}e^{-tn^2}}$$

4. $u_{tt} - c^2u_{xx} = 0 \quad x>0,\ t>0$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 3\sin(2s)\,ds = -\frac{3}{4c}\left[\cos(2x+2ct) - \cos(2x-2ct)\right]$$

$$= -\frac{3}{4c}\left[\cos 2x\cos 2ct - \sin 2x\sin 2ct - \cos 2x\cos 2ct - \sin 2x\sin 2ct\right]$$

$$\boxed{u(x,t) = \frac{3}{2c}\sin 2x\sin 2ct}$$

5. let $u(x,t) = \text{Im}\, v(x,t)$

$$v(x,t) = e^{i4t} e^{\xi x}$$

$$\Rightarrow v_t - K v_{xx} = 0$$

$$4i v - K\xi^2 v = 0 \Rightarrow \xi^2 = \frac{4i}{K} \Rightarrow \xi = \sqrt{\tfrac{4}{k}} \cdot \sqrt{i}$$

note $i = e^{i(\frac{\pi}{2} + 2\pi n)} \Rightarrow \sqrt{i} = e^{i\frac{\pi}{4}}, e^{i(\frac{\pi}{4}+\pi)} = e^{i\frac{5\pi}{4}}$

$$\Rightarrow \xi_1 = \frac{2}{\sqrt{k}}\left[\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right] = \sqrt{\tfrac{2}{k}}(1+i)$$

$$\xi_2 = \frac{2}{\sqrt{k}}\left[\cos\left(\tfrac{5\pi}{4}\right) + i\sin\left(\tfrac{5\pi}{4}\right)\right] = -\sqrt{\tfrac{2}{k}}(1+i)$$

$$v_1(x,t) = e^{i(4t + \frac{x}{\sqrt{2k}})} e^{\frac{x}{\sqrt{2k}}} \rightarrow u_1(x,t) = e^{\sqrt{\frac{2}{k}}x} \sin\left(4t + \sqrt{\tfrac{2}{k}}x\right)$$

$$v_2(x,t) = e^{i(4t - \frac{x}{\sqrt{2k}})} e^{-\frac{x}{\sqrt{2k}}} \rightarrow u_2(x,t) = e^{-\sqrt{\frac{2}{k}}x} \sin\left(4t - \sqrt{\tfrac{2}{k}}x\right)$$

since we need $|u| < \infty$ choose $u = u_2$

$$\Rightarrow \boxed{u(x,t) = e^{-\sqrt{\frac{2}{k}}x} \sin\left(4t - \sqrt{\tfrac{2}{k}}x\right)}$$

6. $\Box u = \partial_t^2 u - c^2 \Delta u = 0$

since initial conditions only depend on $r = |x|$

$\Rightarrow u(x,t) = u(r,t)$

$$\partial_t^2 u - c^2 \frac{1}{r^2} \partial_r (r^2 \partial_r u) = 0$$

let $u = \frac{1}{r} v \Rightarrow \partial_r u = -\frac{1}{r^2} v + \frac{1}{r} v'$

$\Rightarrow \partial_r (r^2 \partial_r u) = \partial_r [-v + r v'] = -v' + v' + r v'' = r v''$

$$\Rightarrow \frac{1}{r} \partial_t^2 v - c^2 \frac{1}{r^2} (r \partial_r^2 v) = 0$$

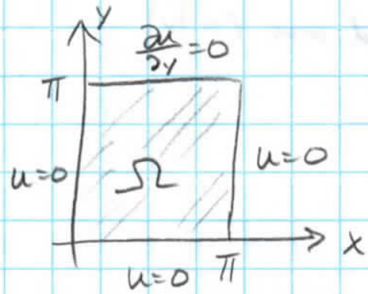
$$\Rightarrow \partial_t^2 v - c^2 \partial_r^2 v = 0$$

$$v(r,0) = \frac{r}{1+r^2}, \quad v_t(r,0) = 0$$

$$\Rightarrow v(x,t) = \frac{1}{2} \left[\frac{r+ct}{1+(r+ct)^2} + \frac{r-ct}{1+(r-ct)^2} \right]$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2r} \left[\frac{r+ct}{1+(r+ct)^2} + \frac{r-ct}{1+(r-ct)^2} \right]}$$

7. $Au = -\Delta u + 3u \qquad \Omega = \{(x,y) \mid x,y \in (0,\pi)\}$



$e_{nm}(x,y) = e_n(x)\,e_m(y)$

$e_n(x) = \sin(nx)$, $\|e_n\|^2 = \frac{\pi}{2}$, $\lambda_n = n^2$

$e_m(y) = \sin[(m+\frac{1}{2})y]$, $\|e_m\|^2 = \frac{\pi}{2}$, $\lambda_m = (m+\frac{1}{2})^2$

$* \; n = 1,2,\ldots \; ; m = 0,1,\ldots$

$$\boxed{\begin{aligned} e_{nm}(x,y) &= \sin(nx)\sin[(m+\tfrac{1}{2})y] \\ \|e_{nm}\|^2 &= \|e_n\|^2\|e_m\|^2 = \frac{\pi^2}{4} \\ \lambda_{nm} &= \lambda_n + \lambda_m + 3 = n^2 + (m+\tfrac{1}{2})^2 + 3 \end{aligned}}$$