4.2.2) Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$ (Neumann at the left, Dirichlet at the right).

a) Show that the eigenfunctions are $\cos\left[\left(n+\frac{1}{2}\right)\pi x/l\right]$

We want to solve the equation $u_{tt} - Au = 0$, where the linear operator $A = -c^2\partial_x^2$. Hence, the eigenequation for the operator A is

$$-c^2\frac{\partial^2 f}{\partial x^2} = \lambda f(x) \Leftrightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\lambda}{c^2} f(x) = 0, \tag{1}$$

which has solution $f(x) = C_1\cos(\omega x) + C_2\sin(\omega x)$, where $\lambda = \omega^2 c^2$. Now, we plug in our boundary conditions:

$$\frac{\partial f}{\partial x} = C_1\omega\sin(\omega x) - C_2\omega\cos(\omega x) = 0 \Rightarrow C_2 = 0 \tag{2}$$

$$f(l) = C_1\cos(\omega l) = 0. \tag{3}$$

Now, $f(x) \not\equiv 0$, so we can satisfy the latter equation of the two above by imposing

$$\omega l = \frac{(2n+1)\pi}{2} \Leftrightarrow \omega = \frac{\left(n+\frac{1}{2}\right)\pi}{l}, \tag{4}$$

where $n \in \mathbb{N}$ (note: $\lambda > 0$ for ∂_x^2). Hence, our eigenfunctions of A are just

$$\boxed{f_n(x) = \cos\left[\left(n+\frac{1}{2}\right)\pi x/l\right]}, \tag{5}$$

where we dropped the coefficient since scaling the eigenfunction also produces a valid eigenfunction (albeit not a linearly independent one!).

b) Write the series expansion for a solution $u(x,t)$.

Observe that the equation $u_{tt} - Au = 0$ has solution $u(t) = C_1\cos(\sqrt{A}t) + C_2\sin(\sqrt{A}t)$ and $g(A)f_n(x) = g(\lambda)f_n(x)$, assuming that g is a function with a valid, convergent Taylor expansion, which both $\sin x$ and $\cos x$ do. Hence, our solution

$$\boxed{u(x,t) = \sum_{n\in\mathbb{N}}\left[C_{1,n}\cos(\omega_n ct) + C_{2,n}\sin(\omega_n ct)\right] f_n(x)}, \tag{6}$$

where ω_n is given by Eq. (4) and $f_n(x)$ is given by Eq. (5).

4.2.3) Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.

Schrödinger's equation reads $u_t + ikAu = 0$, where the linear operator $A = -\partial_x^2$. This equation has the simple solution $u = \exp\left(-i\frac{t}{k}A\right)$, and the operator A has eigenfunction

$f(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$ From the boundary conditions, we find that $C_1 = 0$ and $\sqrt{\lambda_n} = (n + \frac{1}{2})\pi/l$ $(n \in \mathbb{N})$, as in the previous problem. The general solution to the Schrödinger equation for the boundary conditions in this problem is given as

$$\boxed{u(x,t) = \sum_{n\in\mathbb{N}} e^{-i\frac{t}{k}A} C_n \cos\sqrt{\lambda_n}x = \sum_{n\in\mathbb{N}} C_n e^{-i\lambda_n t/k} \cos\sqrt{\lambda_n}x}, \tag{7}$$

where $\lambda_n = (n + \frac{1}{2})^2\pi^2/l^2$.

5.1.2) Let $\phi(x) \equiv x^2$ for $0 \leq x \leq 1 = l$.

a) Calculate its Fourier sine series.

We write

$$x^2 = \sum_{n=1}^{\infty} A_n \sin(n\pi x), \tag{8}$$

in the interval $[0, 1]$, where

$$A_n = 2\int_0^1 x^2 \sin(n\pi x)\,\mathrm{d}x = \frac{2[(2-\pi^2n^2)(-1)^n - 2]}{\pi^3 n^3} = \begin{cases} -\frac{1}{\pi n}, & n \equiv 0 \mod 2 \\ \frac{\pi^2n^2-4}{\pi^3n^3}, & n \equiv 1 \mod 2 \end{cases}. \tag{9}$$

Hence,

$$\boxed{x^2 = \sum_{n=1}^{\infty} \frac{2(2-\pi^2n^2)(-1)^n - 4}{\pi^3n^3} \sin(n\pi x)}. \tag{10}$$

Interestingly, this sequence converges point-wise to x^2 everywhere except $x = 1$, to which it converges to 0 trivially.

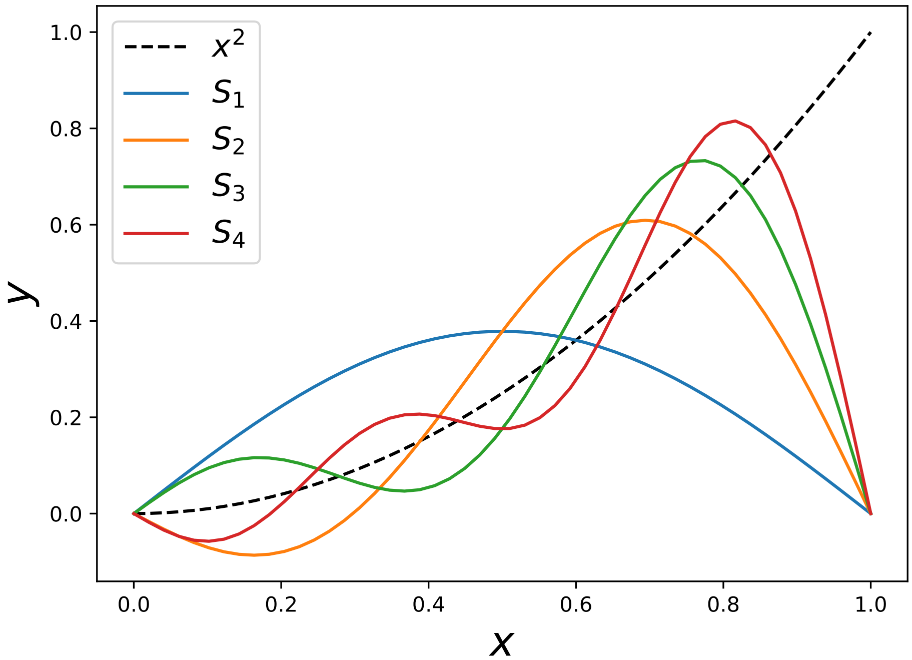

Figure 1: Plot of x^2 and partial sums of Fourier sine series $S_N = \sum_{n=1}^{N} A_n \sin(n\pi x)$.

b) Calculate its Fourier cosine series.

The cosine series is

$$x^2 = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x), \tag{11}$$

where

$$A_0 = 2\int_0^1 x^2 \, \mathrm{d}x = \frac{2}{3} \tag{12}$$

and

$$A_n = 2\int_0^1 x^2 \cos(n\pi x) = \frac{2(-1)^n}{\pi^2 n^2}. \tag{13}$$

This gives us the sum

$$\boxed{x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi^2 n^2} \cos(n\pi x)}. \tag{14}$$

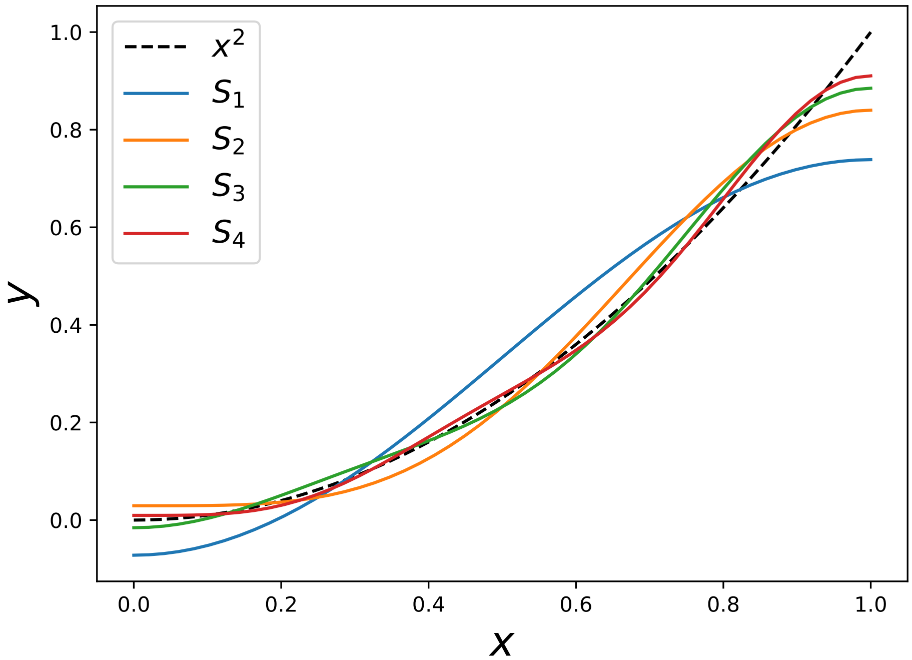

Figure 2: Plot of x^2 and partial sums of Fourier cosine series $S_N = \frac{1}{3} + \sum_{n=1}^{N} A_n \cos(n\pi x)$.

5.1.4) Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}. \tag{15}$$

Observe that $\sin x > 0$ for $x \in [0, \pi]$, so we can write

$$\sin x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx, \tag{16}$$

where

$$A_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, \mathrm{d}x = \frac{4}{\pi} \tag{17}$$

and

$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, \mathrm{d}x = \frac{2[1 - (-1)^n]}{\pi(1 - n^2)} = \begin{cases} 0, & n \equiv 1 \mod 2 \\ -\frac{4}{\pi(n^2-1)}, & n \equiv 0 \mod 2 \end{cases}. \tag{18}$$

Hence,

$$\sin x = \frac{4}{\pi} \left[\frac{1}{2} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \right]. \tag{19}$$

Actually, this is the expansion for $|\sin x|$ on $(-\pi, \pi)$ since it is an even function and cos is even. We can then say at $x = 0$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}}. \tag{20}$$

Next, if we set $x = \pi/2$

$$1 = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}\right], \tag{21}$$

and rearranging gives

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4} = \frac{2 - \pi}{4}}. \tag{22}$$