1) $u_t - 3u_x + 2u = u_t + Au = 0 \quad (A = -3\partial_x + 2)$

$$\Rightarrow \boxed{u = e^{-tA} f(x) = e^{-2t} e^{3t\partial_x} f(x) = e^{-2t} f(x+3t)}$$

2) $u_t = u_{xx} + x^2(1-x)$

steady-state: $\lim_{t\to\infty} u_t(x,t) = 0$

$$\Rightarrow \frac{d^2u}{dx^2} = x^3 - x^2 \Rightarrow \frac{du}{dx} = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C_1$$

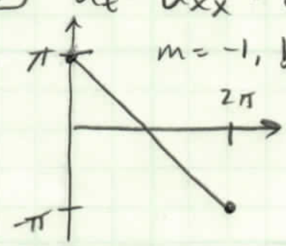
$$u'(0) = 0 = C_1 \Rightarrow \frac{du}{dx} = \frac{1}{4}x^4 - \frac{1}{3}x^3$$

$$\Rightarrow u(x) = \frac{1}{20}x^5 - \frac{1}{12}x^4 + C_2$$

$$u(1) = \frac{1}{20} - \frac{1}{12} + C_2 = 1 \Rightarrow C_2 = \frac{31}{30}$$

$$\Rightarrow \boxed{u(x, t\to\infty) = \frac{x^5}{20} - \frac{x^4}{12} + \frac{31}{30}}$$

3) $u_t - u_{xx} = 0 \qquad u(0,t) = \pi;\ u(2\pi, t) = -\pi$



$m = -1,\ b = \pi \Rightarrow u_0(x,t) = -x + \pi$

let $u(x,t) = u_0(x,t) + v(x,t)$

$\Rightarrow v(0,t) = \pi - \pi = 0,\ v(2\pi, t) = -\pi + \pi = 0$

$$u_t - u_{xx} = v_t - v_{xx} = 0$$

$$\Rightarrow v_t + Av = 0 \quad (A = -\partial_x^2 \text{ with } e_n(x) = \sin(\tfrac{nx}{2}),\ \lambda_n = (\tfrac{n}{2})^2,\ n = 1, 2, \dots)$$

$$v = e^{-tA} v(x,0) = e^{-tA}[u(x,0) - u_0(x,0)]$$

$$= 2e^{-tA} \sin 3x - \sum_{n=1}^{\infty} e^{-t(\frac{n}{2})^2} \hat{u}_0 \sin(\tfrac{nx}{2})$$

$$= 2e^{-9t} \sin 3x - \sum_{n=1}^{\infty} e^{-t(\frac{n}{2})^2} \hat{u}_0 \sin(\tfrac{nx}{2})$$

$$\hat{u}_0 = \frac{\langle u_0, e_n \rangle}{\|e_n\|^2} = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin\left(\tfrac{nx}{2}\right) dx = \frac{1}{2\pi}\left[I_1 - I_2\right]$$

$$I_1 = \pi\int_0^{2\pi} \sin\left(\tfrac{nx}{2}\right) dx = -\frac{2\pi}{n}\cos\left(\tfrac{nx}{2}\right)\Big|_0^{2\pi} = \frac{2\pi}{n}\left[\cos(n\pi) - 1\right]$$

$$= \frac{2\pi}{n}\left[1 - (-1)^n\right]$$

$$I_2 = \int_0^{2\pi} x\sin\left(\tfrac{nx}{2}\right) dx = -\frac{2}{n}x\cos\left(\tfrac{nx}{2}\right)\Big|_0^{2\pi} + \frac{2}{n}\underbrace{\int_0^{2\pi}\cos\left(\tfrac{nx}{2}\right)dx}_{0}$$

$$\left.\begin{array}{ll} u = x & v = -\frac{2}{n}\cos\left(\frac{nx}{2}\right) \\ u' = 1 & v' = \sin\left(\frac{nx}{2}\right) \end{array}\right| = -\frac{4\pi}{n}(-1)^n$$

$$\hat{u}_0 = \frac{1}{\pi}\left(\frac{2\pi}{n}\right)\left[1 - (-1)^n + 2(-1)^n\right] = \frac{2(1 + (-1)^n)}{n} = \frac{1}{n}\begin{cases} 4 & \text{if } n = \text{even} \\ 0 & \text{if } n = \text{odd} \end{cases}$$

$$\Rightarrow v(x,t) = 2e^{-9t}\sin 3x - 2\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n}\sin\left(\tfrac{nx}{2}\right)e^{-t\left(\frac{n}{2}\right)^2}$$

$$\Rightarrow \boxed{u(x,t) = \pi - x + 2e^{-9t}\sin 3x - 2\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n}\sin\left(\tfrac{nx}{2}\right)e^{-t\left(\frac{n}{2}\right)^2}}$$

4) Neumann condition on half-line $\Rightarrow$ even continuation

$f(x) = \cos x \to f_{even} = \cos x$, $g(x) = 1 \to g_{even} = 1$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\cos(x+ct) + \cos(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} ds$$

$$= \frac{1}{2}\left[\cos x\cos ct \underline{- \sin x\sin ct} + \cos x\cos ct \underline{+ \sin x \sin ct}\right]$$

$$+ \frac{1}{2c}\left[(x+ct) - (x-ct)\right]$$

$$\Rightarrow \boxed{u(x,t) = \cos x\cos ct + t}$$

5) $\Box u = \partial_t^2 u - c^2 \Delta u = 0$

$u(x,0) = u(r,0) = e^{-r^2}$ $(r = |x|$ for $x \in \mathbb{R}^3)$

$\Rightarrow \Box u = \partial_t^2 u - c^2 \frac{1}{r^2}\partial_r(r^2 \partial_r u)$

let $u = \frac{1}{r} v \Rightarrow r^2\partial_r u = -v + rv'$

$\Rightarrow \frac{1}{r^2}\partial_r(r^2 \partial_r u) = \frac{1}{r^2}\left[-v' + v' + rv''\right] = \frac{v''}{r}$

$\Rightarrow \Box v = \frac{1}{r}\partial_t^2 v - c^2 \frac{v''}{r} = 0 \Rightarrow \partial_t^2 v - c^2 \partial_x^2 v = 0$

note: $v(r,0) = re^{-r^2}$, $v_t(r,0) = 0$

$\Rightarrow v(r,t) = \frac{1}{2}\left[(r+ct)e^{-(r+ct)^2} + (r-ct)e^{-(r-ct)^2}\right]$

$$\Rightarrow \boxed{u(r,t) = \frac{1}{2r}\left[(r+ct)e^{-(r+ct)^2} + (r-ct)e^{-(r-ct)^2}\right]}$$

6) $Ae_{nm}(x,y) = (\lambda_{nm}^{\Delta} + 5)e_{nm}$

$e_{nm}(x,y) = e_n(x)e_m(y)$, $\lambda_{nm}^{\Delta} = \lambda_n^{\Delta} + \lambda_m^{\Delta}$, $\|e_{nm}\|^2 = \|e_n\|^2 \times \|e_m\|^2$

$\rightarrow e_n(x) = \cos\left[(n+\frac{1}{2})x\right]$, $\lambda_n^{\Delta} = (n+\frac{1}{2})^2$ for $n = 0, 1, 2, \ldots$

$\rightarrow e_m(y) = \cos(my)$, $\lambda_m^{\Delta} = m^2$ for $m = 0, 1, 2, \ldots$

$$\Rightarrow \left.\begin{array}{l} e_{nm}(x,y) = \cos\left[(n+\frac{1}{2})x\right]\cos(my) \\ \lambda_{nm} = (n+\frac{1}{2})^2 + m^2 + 5 \\ \|e_{nm}\|^2 = \left(\frac{\pi}{2}\right)^2 \end{array}\right\} \begin{array}{l} n = 0, 1, 2, \ldots \\ m = 0, 1, 2, \ldots \end{array}$$

7) let $v(x,t) = e^{i4t} e^{\xi x}$, $u(x,t) = \mathrm{Re}\, v(x,t)$

$$\Rightarrow v_t - K v_{tt} = [4i - K\xi^2] v = 0$$

$$\Rightarrow 4i - K\xi^2 = 0 \Rightarrow \xi = \frac{2}{\sqrt{K}}\sqrt{i} \Rightarrow \begin{array}{l} \xi_1 = \frac{2}{\sqrt{K}} e^{i\frac{\pi}{4}} = \sqrt{\frac{2}{K}}(1+i) \\ \xi_2 = \frac{2}{\sqrt{K}} e^{i(\frac{\pi}{4}+\pi)} = -\sqrt{\frac{2}{K}}(1+i) \end{array}$$

$$V_1 = e^{i4t} e^{\sqrt{\frac{2}{K}}(1+i)x}, \quad V_2 = e^{i4t} e^{-\sqrt{\frac{2}{K}}(1+i)x}$$

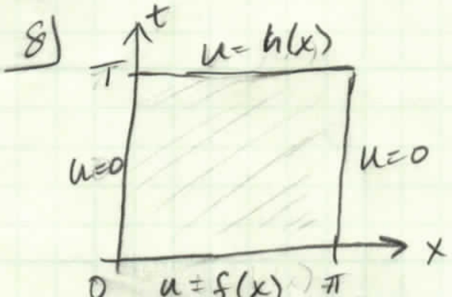
since $|u| < \infty \Rightarrow V = A V_2 = A e^{i(4t - \sqrt{\frac{2}{K}}x)} e^{-\sqrt{\frac{2}{K}}x}$

$$\Rightarrow u(x,t) = A e^{-\sqrt{\frac{2}{K}}x} \cos(4t - \sqrt{\tfrac{2}{K}}x)$$

$$u(0,t) = A\cos(4t) = \cos(4t) \Rightarrow A = 1$$

$$\Rightarrow \boxed{u(x,t) = e^{-\sqrt{\frac{2}{K}}x} \cos(4t - \sqrt{\tfrac{2}{K}}x)}$$

8)



$$u_{tt} + Au = 0$$

$A = -\partial_x^2$ w/ $e_n(x) = \sin(nx)$, $\lambda_n = n^2$

$* \; n = 1, 2, \ldots$

$$\Rightarrow u(x,t) = \frac{\sin[(T-t)\sqrt{A}]}{\sin(T\sqrt{A})} f + \frac{\sin(t\sqrt{A})}{\sin(T\sqrt{A})} h$$

$$= \sum_{n=1}^{\infty} \left[\frac{\sin[(T-t)n]}{\sin(nT)} \hat{f}_n + \frac{\sin(nt)}{\sin(nT)} \hat{h}_n \right] \sin(nx)$$

for this to be defined $T \neq m\pi$, otherwise $\sin(nT) = 0$

$m \in \mathbb{N}$