6.1.2) Show that a function which is a power series in the complex variable $x + iy$ must satisfy the Cauchy-Riemann equations and therefore Laplace's equations.

We define a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

We can separate the sum into real and imaginary parts using the binomial series since $z = x + iy$

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (x+iy)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k && (2) \\ &= \sum_{n=0}^{\infty} \left[a_n \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k} + i \sum_{k=0}^{m-o(n)} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1} \right], && (3) \end{aligned}$$

where $m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$ and $o(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. If we let

$$\begin{aligned} u(x,y) &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k} \\ v(x,y) &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1}. \end{aligned} \tag{4}$$

All that remains now is to check that the Cauchy-Riemann conditions are satisfied.

$$\begin{aligned} u_x &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} \binom{n}{2k} (-1)^k (n-2k) x^{n-2k-1} y^{2k} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} (-1)^k \frac{n!}{(2k)!(n-2k-1)!} x^{n-2k-1} y^{2k} \\ v_y &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} \binom{n}{2k+1} (-1)^k (2k+1) x^{n-2k-1} y^{2k} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} (-1)^k \frac{n!}{(2k)!(n-2k-1)!} x^{n-2k-1} y^{2k}. \end{aligned} \tag{5a}$$

Note that the $o(n)$ term in the upper index of the u_x series comes from the fact that when

n is even, the $k = m$ terms gives 0 when differentiated (with respect to x) since $x^{n-2m} = 1$.

$$\begin{aligned}
u_y &= \sum_{n=0}^{\infty} a_n \sum_{k=1}^{m} (-1)^k \frac{n!}{(2k-1)!(n-2k)!} x^{n-2k} y^{2k-1} \\
v_x &= \sum_{n=0}^{\infty} a_n \sum_{k=1}^{m-o(n)+(o(n)-1)} (-1)^k \frac{n!}{(2k+1)!(n-2(k-1))!} x^{n-2(k-1)} y^{2k+1} \\
&= -\sum_{n=0}^{\infty} a_n \sum_{k=1}^{m} (-1)^k \frac{n!}{(2k-1)!(n-2k)!} x^{n-2k} y^{2k-1},
\end{aligned} \tag{5b}$$

where the $o(n) - 1$ term arises in the v_x term since $n - 2k - 1 = 0$ if $k = m$ whenever n is odd. Thus, we can see that any arbitrary power series in z satisfies the Cauchy-Riemann conditions.

6.1.5) Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.

6.1.6) Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing both parts of the boundary $r = a$ and $r = b$.

6.4.6) Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter $(u = 0, \pi)$ and $u = \pi \sin\theta - \sin 2\theta$ on $r = 1$.

Lecture 15 - 1) Show that $\partial z^n = nz^{n-1}$ and $\overline{\partial}\overline{z}^n = n\overline{z}^{n-1}$ for $n = 0, 1, 2, \ldots$.

Lecture 15 - 2) Find $\overline{\partial}|z|^2$, $\partial|z|^2$, and $\Delta|z|^2$.

Lecture 15 - 3) For an analytic function f in the open set Ω, let $u(x, y) = \operatorname{Re} f(z)$ and $v(x, y) = \operatorname{Im} f(z)$. Show that $\Delta u = \Delta v = 0$ and find $\nabla u \cdot \nabla v$.

An analytic function satisfies the equation $\overline{\partial} f$, where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Since $f = u + iv$, where u, v are real-valued functions (i.e. $u^*(x, y) = u(x, y)$ and similarly for v), we can write

$$\overline{\partial} f = \frac{1}{2}(\partial_x + i\partial_y)(u + iv) = \frac{1}{2}(\partial_x u + i\partial_y u + \partial_x v + i\partial_y v). \tag{6}$$

Lecture 15 - 4) Using the divergence theorem rewrite the integral

$$\int_\Omega \overline{\partial} f(z)\, \mathrm{d}x\, \mathrm{d}y\,, \qquad f \in C^1(\Omega_1),\ \Omega_1 \supset \Omega \tag{7}$$

as an integral over the boundary $\partial\Omega$ of the domain Ω.

Lecture 15 - 5) For harmonic conjugate functions u and v from Exercise 3 above show that $u_x = v_y$ and $u_y = -v_y$ (Caucy-Riemann equations).