12.1.5) Verify, directly from the definition of a distribution, that the discontinuous function $u(x,t) = H(x-ct)$ is a weak solution of the wave equation.

We must check that u satisfies the wave equation in the sense that $\langle u_{tt} - c^2 u_{xx}, \phi\rangle = 0$, where ϕ is any test function. We have

$$\langle u, \phi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{ct}^{\infty} \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{x/c} \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x\,. \quad (1)$$

Note that the last two integrations are equivalent with only the order of integration swapped. Thus,

$$\langle u_{tt}, \phi\rangle = \langle u, \phi_{tt}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{x/c} \phi_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty} \phi_t(x, x/c)\,\mathrm{d}x\,, \quad (2)$$

where we have used the fact that the test functions satisfy $\lim_{x\to\pm\infty}\phi(x) = 0$ (for the one-dimensional case). Now we will introduce the substitution $y = x/c$ to rewrite the integral above

$$\langle u_{tt}, \phi\rangle = \int_{-\infty}^{\infty} c\phi_t(cy, y)\,\mathrm{d}y\,. \quad (3)$$

Similarly,

$$\langle u_{xx}, \phi\rangle = \langle u, \phi_{xx}\rangle = \int_{-\infty}^{\infty}\int_{ct}^{\infty} \phi_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{-\infty}^{\infty} \phi_x(ct, t)\,\mathrm{d}t\,. \quad (4)$$

Hence,

$$\boxed{\langle u_{tt} - c^2 u_{xx}, \phi\rangle = \int_{-\infty}^{\infty}[c\phi_t(cy,y) + c^2\phi_x(cy,y)]\,\mathrm{d}y = \int_{-\infty}^{\infty} c\frac{\mathrm{d}\phi(cy,y)}{\mathrm{d}y}\,\mathrm{d}y = 0}\,, \quad (5)$$

where y depends implicitly on t.

12.3.4) Prove the following properties of the convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y\,. \quad (6)$$

a) $f * g = g * f$

We make a change of variables in the integration $z = x - y$, which gives

$$\boxed{f * g = -\int_{\infty}^{-\infty} f(z)g(x-z)\,\mathrm{d}z = g * f}\,. \quad (7)$$

b) $(f * g)' = f' * g = f * g'$, where $'$ denotes the derivative in one variable.

The derivative of the convolution

$$\boxed{\begin{aligned}(f * g)' &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y = \int_{-\infty}^{\infty}\Big[\frac{\mathrm{d}f}{\mathrm{d}(x-y)}\frac{\mathrm{d}(x-y)}{\mathrm{d}x}\Big]g(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} f'(x-y)g(y)\,\mathrm{d}y = f' * g \\ &= (g * f)' = g' * f = f * g'\end{aligned}}. \tag{8}$$

12.3.5)

a) Show that $\delta * f = f$ for any distribution f, where δ is the delta function.

Since f is a distribution, we show this in the sense that $\langle \delta * f, \phi\rangle = \langle f, \phi\rangle$, where ϕ is a test function. The work is as follows:

$$\boxed{\begin{aligned}\langle \delta * f, \phi\rangle &= \int_{-\infty}^{\infty}(\delta * f)(x)\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\delta(x-y)f(y)\,\mathrm{d}y\right]\phi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x = \langle f, \phi\rangle\end{aligned}}. \tag{9}$$

b) Show that $\delta' * f = f'$ for any distribution f, where $'$ is the derivative.

Observe that

$$\boxed{\delta' * f = (\delta * f)' = f'}. \tag{10}$$

12.3.6) Let $f(x)$ be a continuous function defined for $-\infty < x < \infty$ such that its Fourier transform $F(k)$ satisfies $F(k) = 0$ for $|k| > \pi$. Such a function is said to be *band-limited*

a) Show that

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\left[\pi(x-n)\right]}{\pi(x-n)}. \tag{11}$$

Thus $f(x)$ is completely determined by its values at the integers! We say that $f(x)$ is *sampled* at the integers.

We can write

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{ikx}\frac{\mathrm{d}k}{2\pi} = \int_{-\pi}^{\pi} F(k)e^{ikx}\frac{\mathrm{d}k}{2\pi}. \tag{12}$$

Utilizing the Fourier series, we can rewrite $F(k)$ as

$$F(k) = \sum_{n=-\infty}^{\infty} \hat{F}_n e^{-ink}. \tag{13}$$

Note that the coefficients

$$\hat{F}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(k)e^{-ink}\,\mathrm{d}k = f(n), \tag{14}$$

which is the inverse fourier transform of f evaluated at $x = n$. Hence,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{2\pi}\int_{-\pi}^{\pi} e^{-ink}e^{ikx}\,\mathrm{d}k\,. \tag{15}$$

Performing the integral we find

$$\int_{-\pi}^{\pi} e^{i(x-n)k}\,\mathrm{d}k = \frac{2\sin\left[\pi(x-n)\right]}{x-n}, \tag{16}$$

and inserting this back into our expression above for $f(x)$ we obtain

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\left[\pi(x-n)\right]}{\pi(x-n)}}. \tag{17}$$

b) Let $F(k) = 1$ in the interval $(-\pi, \pi)$ and $F(k) = 0$ outside this interval. Calculate both sides of (a) directly to verify that they are equal.

We can calculate f from the inverse fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,\mathrm{d}k = \frac{\sin(\pi x)}{\pi x} =, \tag{18}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi n)}{\pi n}\frac{\sin\left[\pi(x-n)\right]}{\pi(x-n)} = \frac{\sin(\pi x)}{\pi x}, \tag{19}$$

observing that $\sin(\pi n) = 0$ for $n \in \mathbb{Z}$. The saving factor at $n = 0$ is that $\sin(\pi n)/\pi n$ is defined to take on the value 1, which is the limit of this expression.