1.1.2) Which of the following operators are linear?

(a) $\mathcal{L}u = u_x + xu_y$

Linear operators of order n are of the form

$$\mathcal{L} = \sum_{m=0}^{n} \sum_{k=0}^{m} a_{mk}(x,y)\partial_x^{m-k}\partial_y^k, \tag{1}$$

where we assume that mixed derivatives can be interchanged according to Schwarz's theorem and the functions $a_{mk}(x,y)$ serve as coefficients for the derivative terms.

For $n = 1$, we see that Eq. (1), reduces to

$$\mathcal{L} = a_{00}(x,y) + a_{10}(x,y)\partial_x + a_{11}\partial_y. \tag{2}$$

It is clear that $a_{00} \equiv 0$, $a_{10} = 1$ and $a_{11} = 1$, so this is a linear operator.

(b) $\mathcal{L}u = u_x + uu_y$

This is not a linear operator since it involves a product of u and a derivative of u, which does not match the form of Eq. (1). We can also show that it does not satisfy the property of linear scaling ($\mathcal{L}(\alpha u) = \alpha\mathcal{L}u$):

$$(\alpha u)_x + (\alpha u)(\alpha u)_y = \alpha\left(u_x + \alpha(uu_y)\right) \neq \alpha(u_x + uu_y). \tag{3}$$

(c) $\mathcal{L}u = u_x + u_{yy}$

This is clearly a linear operator since it matches the form given in Eq. (1) for $n = 2$ and also since any order of partial derivatives is linear.

(d) $\mathcal{L}u = u_x + u_y + 1$

This is linear since it matches the form Eq. (1) for $n = 1$, explicitly listed in Eq. (2)

This is not linear because of the $+1$ term in $\mathcal{L}u$. The superposition property does not work: $\mathcal{L}(u+v) = u_x + u_y + v_x + v_y + 1 \neq u_x + u_y + v_x + v_y + 2 = \mathcal{L}u + \mathcal{L}v$.

(e) $\mathcal{L}u = \sqrt{1+x^2}(\cos y)u_x + -u_{yxy} - [\arctan(x/y)]u$

This is a linear operator since this is a linear combination of derivatives of u and u itself, matching the form of Eq. (1)

1.2.9) Solve the equation $u_x + u_y = 1$

We can make the change of variables $(x, y) \to (x', y')$ as follows:

$$x' = x + y \tag{4}$$
$$y' = x - y. \tag{5}$$

Thus, we have

$$u_x = \frac{\partial x'}{\partial x} u_{x'} + \frac{\partial y'}{\partial x} u_{y'} = u_{x'} + u_{y'} \tag{6}$$
$$u_y = \frac{\partial x'}{\partial y} u_{x'} + \frac{\partial y'}{\partial y} u_{y'} = u_{x'} - u_{y'}. \tag{7}$$

Substituting into the original PDE we have

$$u_x + u_y = 2u_{x'} = 1. \tag{8}$$

Solving, we find

$$\boxed{u(x, y) = \frac{1}{2} x' + f(y') = \frac{1}{2}(x + y) + f(x - y)}, \tag{9}$$

where the constant function is the solution to the homogeneous equation $u_x + u_y = 0$, and the first term is a solution to the nonhomogeneous equation.

1.3.9) This is an exercise on the divergence theorem

$$\iiint_D \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{F}} \, \mathrm{d}\vec{\boldsymbol{x}} = \iint_{\partial D} \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S \tag{10}$$

valide for any bounded domain D in space with boundary surface ∂D and unit outward normal vector $\hat{\boldsymbol{n}}$. As an exercise, verify it in the following case by calculating both sides separately: $\vec{\boldsymbol{F}} = r^2 \vec{\boldsymbol{x}}$, $\vec{\boldsymbol{x}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, $r^2 = x^2 + y^2 + z^2$ and $D =$ the ball of radius a and center at the origin.

We have $D = \{\vec{\boldsymbol{x}} \; : \; |\vec{\boldsymbol{x}} < a|\}$ and $\partial D = \{\vec{\boldsymbol{x}} \; : \; |\vec{\boldsymbol{x}}| = a\}$. Looking at the first integral, we have

$$\begin{aligned}
\iiint_D \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{F}} \, \mathrm{d}\vec{\boldsymbol{x}} &= \iiint_D [(2x^2 + r^2) + (2y^2 + r^2) + (2z^2 + r^2)] \, \mathrm{d}\vec{\boldsymbol{x}} \\
&= \iiint_D 5r^2 \, \mathrm{d}\vec{\boldsymbol{x}} = \int_0^\pi \int_0^{2\pi} \int_0^a 5r^2 (r^2 \sin\theta) \, \mathrm{d}r \, \mathrm{d}\phi \, \phi\theta \\
&= (\pi)(2\pi)a^5 = \boxed{4\pi a^5}.
\end{aligned} \tag{11}$$

Turning to the second integral now

$$\begin{aligned}
\iint_{\partial D} \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} \Big|_{\partial D} \mathrm{d}S &= \int_0^\pi \int_0^{2\pi} a^3 (a^2 \sin\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= \boxed{4\pi a^5}.
\end{aligned} \tag{12}$$

Since the results of both integrals match, we have shown the divergence theorem to be valid in this case.

1.3.10) If $\vec{\boldsymbol{f}}(\vec{\boldsymbol{x}})$ is continuous and $|\vec{\boldsymbol{f}}(\vec{\boldsymbol{x}})| \leq 1/(|\vec{\boldsymbol{x}}|^3 + 1)$ for all $\vec{\boldsymbol{x}}$, show that

$$\iiint_{\mathbb{R}^3} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{f}} \, \mathrm{d}\vec{\boldsymbol{x}} = 0. \tag{13}$$

We can write

$$\int_{\mathbb{R}^3} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}) \, \mathrm{d}\vec{\boldsymbol{x}} = \lim_{a\to\infty} \int_{B(a)} \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}) \, \mathrm{d}\vec{\boldsymbol{x}} = \lim_{a\to\infty} \int_{\partial B(a)} \vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}) \cdot \hat{\boldsymbol{r}} \Big|_{\partial B(a)} a^2 \, \mathrm{d}\Omega \, , \tag{14}$$

where $B(a)$ is an open sphere of radius a and $\partial B(a)$ is the surface of the sphere of radius a, and $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta$.

Using some basic integral and dot product inequalities we find that

$$\left| \int_{\partial B(a)} \vec{\boldsymbol{f}} \cdot \hat{\boldsymbol{r}} \, \mathrm{d}\Omega \right| \leq \int_{\partial B(a)} |\vec{\boldsymbol{f}} \cdot \hat{\boldsymbol{r}}| \, \mathrm{d}\Omega \leq \int_{\partial B(a)} |\vec{\boldsymbol{f}}||\hat{\boldsymbol{r}}| \, \mathrm{d}\Omega \, . \tag{15}$$

Using the assumption that $\vec{\boldsymbol{f}}$ is bounded at each $\vec{\boldsymbol{x}}$ in the manner specified above, we find

$$\int_{\partial B(a)} |\vec{\boldsymbol{f}}|_{\partial B(a)} \, \mathrm{d}\Omega \leq \int_0^\pi \int_0^{2\pi} \frac{1}{a^3+1} \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta = \frac{4\pi}{a^3+1}. \tag{16}$$

Hence,

$$\boxed{\left| \int_{\mathbb{R}^3} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}) \, \mathrm{d}\vec{\boldsymbol{x}} \right| \leq \lim_{a\to\infty} \frac{4\pi a^2}{a^3+1} = 0 \Rightarrow \int_{\mathbb{R}^3} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}) \, \mathrm{d}\vec{\boldsymbol{x}} = 0}. \tag{17}$$

2.1.1) Solve $u_{tt} = c^2 u_{xx}, \; u(x,0) = e^x, \; u_t(x,0) = \sin x$.

We know the general solution of the wave equation with initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s) \, \mathrm{d}s \, . \tag{18}$$

In this case, we have $\phi(x) = e^x$ and $\psi(x) = \sin x$, so

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[e^{x+ct} + e^{x-ct}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s \, \mathrm{d}s \\ &= \frac{1}{2}e^x\left[e^{ct} + e^{-ct}\right] - \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right] \\ &= e^x \cosh ct + \frac{1}{c}\sin x \sin ct, \end{aligned} \tag{19}$$

where we have used the definition of cosh and the relation $\cos(a+b) - \cos(a-b) = -2\sin a \sin b$ to simplify our result.

2.1.8) A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right). \tag{20}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.

If we make the change of variables $v = ru$, we obtain

$$u_r = \left(\frac{v}{r}\right)_r = -\frac{1}{r^2}v + \frac{1}{r}v_r \tag{21}$$

$$u_{rr} = \frac{2}{r^3}v - \frac{2}{r^2}v_r + \frac{1}{r}v_{rr}. \tag{22}$$

Pluggin this into Eq. (20) and simplifying we find

$$\begin{aligned}\left(\frac{v}{r}\right)_{tt} = \frac{1}{r}v_{tt} &= c^2\left[\frac{2}{r^3}v - \frac{2}{r^2}v_r + \frac{1}{r}v_{rr} - \frac{2}{r^3}v + \frac{2}{r^2}v_r\right] \\ v_{tt} &= c^2 v_{rr}, \end{aligned} \tag{23}$$

which is the linear 1D wave equation for v.

(b) Solve for v using $v(x,t) = f(x+ct) + g(x-ct)$ and thereby solve the spherical wave equation.

The solution to Eq. (23) is just

$$\boxed{\begin{aligned} v &= f(x+ct) + g(x-ct) \\ u &= \frac{1}{r}[f(x+ct) + g(x-ct)] \end{aligned}}. \tag{24}$$

(c) Use

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s\,, \tag{25}$$

to solve it with initial conditions $u(r,0) = \phi(r)$, $u_t(r,0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

The initial conditions on u give the initial conditions $v(r,0) = r\phi(r)$ and $v_t(r,0) = r\psi(r)$ so

$$\boxed{\begin{aligned} v(r,t) &= \frac{1}{2}\left[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\,\mathrm{d}s \\ u(r,t) &= \frac{1}{2r}\left[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right] + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\,\mathrm{d}s \end{aligned}}. \tag{26}$$