

**1.1.2)** Which of the following operators are linear?

(a)  $\mathcal{L}u = u_x + xu_y$

Linear operators of order  $n$  are of the form

$$\mathcal{L} = \sum_{m=0}^n \sum_{k=0}^m a_{mk}(x, y) \partial_x^{m-k} \partial_y^k, \quad (1)$$

where we assume that mixed derivatives can be interchanged according to Schwarz's theorem and the functions  $a_{mk}(x, y)$  serve as coefficients for the derivative terms.

For  $n = 1$ , we see that Eq. (1), reduces to

$$\mathcal{L} = a_{00}(x, y) + a_{10}(x, y)\partial_x + a_{11}\partial_y. \quad (2)$$

It is clear that  $a_{00} \equiv 0$ ,  $a_{10} = 1$  and  $a_{11} = 1$ , so this is a linear operator.

(b)  $\mathcal{L}u = u_x + uu_y$

This is not a linear operator since it involves a product of  $u$  and a derivative of  $u$ , which does not match the form of Eq. (1). We can also show that it does not satisfy the property of linear scaling ( $\mathcal{L}(\alpha u) = \alpha \mathcal{L}u$ ):

$$(\alpha u)_x + (\alpha u)(\alpha u)_y = \alpha(u_x + \alpha(uu_y)) \neq \alpha(u_x + uu_y). \quad (3)$$

(c)  $\mathcal{L}u = u_x + u_{yy}$

This is clearly a linear operator since it matches the form given in Eq. (1) for  $n = 2$  and also since any order of partial derivatives is linear.

(d)  $\mathcal{L}u = u_x + u_y + 1$

This is linear since it matches the form Eq. (1) for  $n = 1$ , explicitly listed in Eq. (2)

(e)  $\mathcal{L}u = \sqrt{1+x^2}(\cos y)u_x + -u_{yxy} - [\arctan(x/y)]u$

This is a linear operator since this is a linear combination of derivatives of  $u$  and  $u$  itself, matching the form of Eq. (1)

**1.2.9)** Solve the equation  $u_x + u_y = 1$

We can make the change of variables  $(x, y) \rightarrow (x', y')$  as follows:

$$x' = x + y \quad (4)$$

$$y' = x - y. \quad (5)$$

Thus, we have

$$u_x = \frac{\partial x'}{\partial x} u_{x'} + \frac{\partial y'}{\partial x} u_{y'} = u_{x'} + u_{y'} \quad (6)$$

$$u_y = \frac{\partial x'}{\partial y} u_{x'} + \frac{\partial y'}{\partial y} u_{y'} = u_{x'} - u_{y'}. \quad (7)$$

Substituting into the original PDE we have

$$u_x + u_y = 2u_{x'} = 1. \quad (8)$$

Solving, we find

$$u(x, y) = \frac{1}{2}x' + f(y') = \frac{1}{2}(x + y) + f(x - y), \quad (9)$$

where the constant function is the solution to the homogeneous equation  $u_x + u_y = 0$ , and the first term is a solution to the nonhomogeneous equation.

**1.3.9)** This is an exercise on the divergence theorem

$$\iiint_D \vec{\nabla} \cdot \vec{F} \, d\vec{x} = \iint_{\partial D} \vec{F} \cdot \hat{n} \, dS \quad (10)$$

valide for any bounded domain  $D$  in space with boundary surface  $\partial D$  and unit outward normal vector  $\hat{n}$ . As an exercise, verify it in the following case by calculating both sides separately:  $\vec{F} = r^2 \vec{x}$ ,  $\vec{x} = x\hat{x} + y\hat{y} + z\hat{z}$ ,  $r^2 = x^2 + y^2 + z^2$  and  $D$  = the ball of radius  $a$  and center at the origin.

We have  $D = \{\vec{x} : |\vec{x}| < a\}$  and  $\partial D = \{\vec{x} : |\vec{x}| = a\}$ . Looking at the first integral, we have

$$\begin{aligned} \iiint_D \vec{\nabla} \cdot \vec{F} \, d\vec{x} &= \iiint_D [(2x^2 + r^2) + (2y^2 + r^2) + (2z^2 + r^2)] \, d\vec{x} \\ &= \iiint_D 5r^2 \, d\vec{x} = \int_0^\pi \int_0^{2\pi} \int_0^a 5r^2(r^2 \sin \theta) \, dr \, d\phi \, d\theta \\ &= (\pi)(2\pi)a^5 = \boxed{4\pi a^5}. \end{aligned} \quad (11)$$

Turning to the second integral now

$$\begin{aligned} \iint_{\partial D} \vec{F} \cdot \hat{n} \, dS &= \int_0^\pi \int_0^{2\pi} a^3(a^2 \sin \theta) \, d\phi \, d\theta \\ &= \boxed{4\pi a^5}. \end{aligned} \quad (12)$$

Since the results of both integrals match, we have shown the divergence theorem to be valid in this case.

**1.3.10)** If  $\vec{f}(\vec{x})$  is continuous and  $|\vec{f}(\vec{x})| \leq 1/(|\vec{x}|^3 + 1)$  for all  $\vec{x}$ , show that

$$\iiint_{\mathbb{R}^3} \vec{\nabla} \cdot \vec{f} \, d\vec{x} = 0. \quad (13)$$

We can write

$$\int_{\mathbb{R}^3} \vec{\nabla} \cdot \vec{f}(\vec{x}) \, d\vec{x} = \lim_{a \rightarrow \infty} \int_{B(a)} \vec{\nabla} \cdot \vec{f}(\vec{x}) \, d\vec{x} = \lim_{a \rightarrow \infty} \int_{\partial B(a)} \vec{f}(\vec{x}) \cdot \hat{r} \Big|_{\partial B(a)} a^2 \, d\Omega, \quad (14)$$

where  $B(a)$  is an open sphere of radius  $a$  and  $\partial B(a)$  is the surface of the sphere of radius  $a$ , and  $d\Omega = \sin \theta \, d\phi \, d\theta$ .

Using some basic integral and dot product inequalities we find that

$$\left| \int_{\partial B(a)} \vec{f} \cdot \hat{r} \, d\Omega \right| \leq \int_{\partial B(a)} |\vec{f} \cdot \hat{r}| \, d\Omega \leq \int_{\partial B(a)} |\vec{f}| |\hat{r}| \, d\Omega. \quad (15)$$

Using the assumption that  $\vec{f}$  is bounded at each  $\vec{x}$  in the manner specified above, we find

$$\int_{\partial B(a)} |\vec{f}|_{\partial B(a)} \, d\Omega \leq \int_0^\pi \int_0^{2\pi} \frac{1}{a^3 + 1} \sin \theta \, d\phi \, d\theta = \frac{4\pi}{a^3 + 1}. \quad (16)$$

Hence,

$$\left| \int_{\mathbb{R}^3} \vec{\nabla} \cdot \vec{f}(\vec{x}) \, d\vec{x} \right| \leq \lim_{a \rightarrow \infty} \frac{4\pi a^2}{a^3 + 1} = 0 \Rightarrow \int_{\mathbb{R}^3} \vec{\nabla} \cdot \vec{f}(\vec{x}) \, d\vec{x} = 0. \quad (17)$$

**2.1.1)** Solve  $u_{tt} = c^2 u_{xx}$ ,  $u(x, 0) = e^x$ ,  $u_t(x, 0) = \sin x$ .

We know the general solution of the wave equation with initial conditions  $u(x, 0) = \phi(x)$  and  $u_t(x, 0) = \psi(x)$  is

$$u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds. \quad (18)$$

In this case, we have  $\phi(x) = e^x$  and  $\psi(x) = \sin x$ , so

$$\begin{aligned} u(x, t) &= \frac{1}{2} [e^{x+ct} + e^{x-ct}] + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin s \, ds \\ &= \frac{1}{2} e^x [e^{ct} + e^{-ct}] - \frac{1}{2c} [\cos(x + ct) - \cos(x - ct)] \\ &= e^x \cosh ct + \frac{1}{c} \sin x \sin ct, \end{aligned} \quad (19)$$

where we have used the definition of  $\cosh$  and the relation  $\cos(a+b) - \cos(a-b) = -2 \sin a \sin b$  to simplify our result.

**2.1.8)** A *spherical wave* is a solution of the three-dimensional wave equation of the form  $u(r, t)$ , where  $r$  is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right). \quad (20)$$

(a) Change variables  $v = ru$  to get the equation for  $v$ :  $v_{tt} = c^2 v_{rr}$ .

If we make the change of variables  $v = ru$ , we obtain

$$u_r = \left( \frac{v}{r} \right)_r = -\frac{1}{r^2} v + \frac{1}{r} v_r \quad (21)$$

$$u_{rr} = \frac{2}{r^3} v - \frac{2}{r^2} v_r + \frac{1}{r} v_{rr}. \quad (22)$$

Plugging this into Eq. (20) and simplifying we find

$$\begin{aligned} \left( \frac{v}{r} \right)_{tt} &= \frac{1}{r} v_{tt} = c^2 \left[ \frac{2}{r^3} v - \frac{2}{r^2} v_r + \frac{1}{r} v_{rr} - \frac{2}{r^3} v + \frac{2}{r^2} v_r \right] \\ v_{tt} &= c^2 v_{rr}, \end{aligned} \quad (23)$$

which is the linear 1D wave equation for  $v$ .

(b) Solve for  $v$  using  $v(x, t) = f(x + ct) + g(x - ct)$  and thereby solve the spherical wave equation.

The solution to Eq. (23) is just

$$\boxed{\begin{aligned} v &= f(x + ct) + g(x - ct) \\ u &= \frac{1}{r} [f(x + ct) + g(x - ct)] \end{aligned}}. \quad (24)$$

(c) Use

$$v(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, \quad (25)$$

to solve it with initial conditions  $u(r, 0) = \phi(r)$ ,  $u_t(r, 0) = \psi(r)$ , taking both  $\phi(r)$  and  $\psi(r)$  to be even functions of  $r$ .

The initial conditions on  $u$  give the initial conditions  $v(r, 0) = r\phi(r)$  and  $v_t(r, 0) = r\psi(r)$  so

$$\begin{aligned} v(r, t) &= \frac{1}{2} [(r + ct)\phi(r + ct) + (r - ct)\phi(r - ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} s\psi(s) \, ds \\ u(r, t) &= \frac{1}{2r} [(r + ct)\phi(r + ct) + (r - ct)\phi(r - ct)] + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi(s) \, ds \end{aligned} \quad (26)$$