

2.4.16) Solve the diffusion equation with constant dissipation: $u_t - ku_{xx} + bu = 0$ for $-\infty < x < \infty$ with $u(x, 0) = \phi(x)$.

Consider the change of variables given by $u = e^{-bt}v$. Then $u_t = -bu + e^{-bt}v_t$ and $u_{xx} = e^{-bt}v_{xx}$, which makes the diffusion equation with constant dissipation

$$-bu + e^{-bt}v_t - ke^{-bt}v_{xx} + bu = e^{-bt}v_t - ke^{-bt}v_{xx} = 0 \Rightarrow v_t - kv_{xx} = 0. \quad (1)$$

That is, $v = e^{bt}u$ satisfies the diffusion equation with no dissipation, where the initial condition $v(x, 0) = \phi(x)$

$$v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy, \quad (2)$$

which gives u as

$$u = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy. \quad (3)$$

Notice that this form of u makes sense. It is similar to a harmonic oscillator with linear damping. That is, the solution is a product of the decaying exponential e^{-bt} , which depends on the damping parameter b , and the solution of the diffusion equation (with no damping).

2.4.17) Solve the diffusion equation with variable dissipation: $u_t - ku_x + bt^2u = 0$ for $-\infty < x < \infty$ with $u(x, 0) = \phi(x)$.

Consider the change of variables given by $u = e^{-bt^3/3}v$. Thus, $u_t = e^{-bt^3/3}v_t - bt^2u$ and $u_{xx} = e^{-bt^3/3}v_{xx}$, making the diffusion equation with variable dissipation

$$e^{-bt^3/3}v_t - bt^2u - ke^{-bt^3/3}v_{xx} + bt^2u = e^{-bt^3/3}[v_t - kv_{xx}] = 0 \Rightarrow v_t - kv_{xx} = 0. \quad (4)$$

Hence, observing that $v(x, 0) = \phi(x)$

$$v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy. \quad (5)$$

$$u = \frac{e^{-bt^3/3}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) dy, \quad (6)$$

which is similar to the form of Eq. (3), except that our solution decays faster since the dissipation becomes quadratically stronger over time.

2.4.18) Solve the heat equation with convection: $u_t - ku_{xx} + Vu_x = 0$ for $-\infty < x < \infty$ with $u(x, 0) = \phi(x)$.

We have the equation $u_t - Au = 0$ with $A = k\partial_x^2 - V\partial_x$, which has solution

$$u = e^{kt\partial_x^2} e^{-Vt\partial_x} \phi(x) = e^{-kt\partial_x^2} \phi(x - Vt), \quad (7)$$

which makes our solution (letting $y = x - Vt$)

$$u = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-z)^2/4kt} \phi(z) dz = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-z)^2/4kt} \phi(z) dz. \quad (8)$$

2.5.4) Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x, t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 c^2/4kt} u(x, s) ds. \quad (9)$$

(a) Show that $v(x, t)$ solves the diffusion equation!

We will show that v satisfies the equation $v_t - kv_{xx} = 0$, assuming that u satisfies the equation $u_{tt} = c^2 u_{xx}$. Observe that

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\frac{1}{2t} \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 c^2/4kt} u(x, s) ds + \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{s^2 c^2}{4kt^2} e^{-s^2 c^2/4kt} u(x, s) ds \\ &= \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[\frac{s^2 c^2}{4kt^2} - \frac{1}{2t} \right] e^{-s^2 c^2/4kt} u(x, s) ds \end{aligned} \quad (10)$$

Observe that

$$\frac{\partial^2}{\partial s^2} e^{-s^2 c^2/4kt} = \frac{c^2}{4k^2 t^2} [s^2 c^2 - 2kt] e^{-s^2 c^2/4kt} = \frac{c^2}{k} \left[\frac{s^2 c^2}{4kt^2} - \frac{1}{2t} \right] e^{-s^2 c^2/4kt}. \quad (11)$$

Thus, Eq. (10) becomes

$$\frac{\partial v}{\partial t} = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{k}{c^2} \frac{\partial^2}{\partial s^2} \left(e^{-s^2 c^2/4kt} \right) u(x, s) ds = \frac{k}{c^2} \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 c^2/4kt} u_{ss}(x, s) ds. \quad (12)$$

Next, we have

$$\frac{\partial^2}{\partial x^2} \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 c^2/4kt} u(x, s) ds = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 c^2/4kt} u_{xx}(x, s) ds. \quad (13)$$

Hence,

$$v_t - kv_{xx} = \frac{ck}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[\frac{1}{c^2} u_{ss}(x, s) - u_{xx}(x, s) \right] e^{-s^2 c^2/4kt} ds = 0 \quad (14)$$

(b) Show that $\lim_{t \rightarrow 0} v(x, t) = u(x, 0)$.

We have already proven that for $\delta > 0$

$$\lim_{t \rightarrow 0} \max_{\delta \leq |s| < \infty} \frac{1}{\sqrt{4\pi kt}} e^{-s^2/4kt} = 0. \quad (15)$$

In this case, we let $1/4kt \rightarrow c^2/4kt$ for this problem. Hence, this tells us that

$$\lim_{t \rightarrow 0} \frac{c}{\sqrt{4\pi kt}} e^{-s^2 c^2/4kt} = 0 \text{ if } s > 0. \quad (16)$$

Thus, we can write

$$\lim_{t \rightarrow 0} \frac{c}{\sqrt{4\pi kt}} e^{-s^2 c^2/4kt} u(x, s) = u(x, 0) \lim_{t \rightarrow 0} \frac{c}{\sqrt{4\pi kt}} e^{-s^2 c^2/4kt}. \quad (17)$$

We can now prove the claim of this problem:

$$\begin{aligned} \lim_{t \rightarrow 0} v(x, t) &= \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \frac{c}{\sqrt{4\pi kt}} e^{-s^2 c^2/4kt} u(x, s) \, ds \\ &= u(x, 0) \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \frac{c}{\sqrt{4\pi kt}} e^{-s^2 c^2/4kt} \, ds = u(x, 0) \end{aligned}$$

since $c \exp(-s^2 c^2/4kt)/\sqrt{4\pi kt}$ is a normalized gaussian for all t , meaning that

$$\int_{-\infty}^{\infty} \frac{c}{\sqrt{4\pi kt}} e^{-s^2 c^2/4kt} \, ds = 1. \quad (18)$$

12.3) Check formulas (6) and (7) from page 345:

$$\mathcal{F}\{H(a - |x|)\} = \frac{2}{k} \sin ak \quad (19)$$

$$\mathcal{F}\{e^{-a|x|}\} = \frac{2a}{a^2 + k^2} \quad \text{for } a > 0, \quad (20)$$

where $\mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$ and $H(x)$ is the Heaviside step function.

For Eq. (19), we have

$$\mathcal{F}\{H(a - |x|)\} = \int_{-\infty}^{\infty} H(a - |x|) e^{-ikx} \, dx = \int_{-a}^a e^{-ikx} \, dx = \frac{e^{-ika} - e^{ika}}{-ik} = \frac{2}{k} \sin ka, \quad (21)$$

where we have used the identity

$$\sin x = \frac{e^{ikx} - e^{-ikx}}{2i}. \quad (22)$$

Now, for Eq. (20), we have

$$\mathcal{F}\{e^{-a|x|}\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} dx = \int_{-\infty}^0 e^{(a-ik)x} dx + \int_0^{\infty} e^{-(a+ik)x} dx \quad (23)$$

$$= \frac{1}{a-ik} + \frac{1}{a+ik}, \quad (24)$$

where we have assumed $a > 0$ such that the evaluation of the integral vanishes at $\pm\infty$. Collecting terms and simplifying we find

$$\mathcal{F}\{e^{-a|x|}\} = \frac{(a+ik) + (a-ik)}{(a-ik)(a+ik)} = \frac{2a}{a^2 + k^2}. \quad (25)$$

6) Check formulas (i), (ii), and (iii) on page 346:

$$\text{i : } \mathcal{F}\left\{\frac{df}{dx}\right\} = ik\mathcal{F}\{f(x)\} \quad (26)$$

$$\text{ii : } \mathcal{F}\{xf(x)\} = i\frac{d\mathcal{F}\{f(x)\}}{dk} \quad (27)$$

$$\text{iii : } \mathcal{F}\{f(x-a)\} = e^{-iak}\mathcal{F}\{f(x)\} \quad (28)$$

For Eq. (26) we have

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{\infty} \frac{df}{dx} e^{-ikx} dx = -ike^{-ikx} f(x) \Big|_{-\infty}^{\infty} + ik \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = ik\mathcal{F}\{f(x)\}, \quad (29)$$

where we have supposed that f goes to zero at $\pm\infty$.

Next, for Eq. (27) we have

$$\mathcal{F}\{xf(x)\} = \int_{-\infty}^{\infty} xf(x) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x) \left(\frac{1}{-i}\right) \frac{\partial}{\partial k} e^{-ikx} dx \quad (30)$$

$$= i \frac{d}{dk} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = i \frac{d\mathcal{F}\{f(x)\}}{dk}. \quad (31)$$

Finally, for Eq. (28) we can make the substitution $y = x - a$ such that

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a) e^{-ikx} dx = \int_{-\infty}^{\infty} f(y) e^{-iky} e^{-ika} dy = e^{-ika} \mathcal{F}\{f(x)\}. \quad (32)$$