1) Let $A = -\partial^2 : \mathcal{D} \subset L_2(-\pi,\pi) \mapsto L_2(-\pi,\pi)$, where $\mathcal{D} = \{u \in C^2[-\pi,\pi] : u_x(-\pi) = u_x(\pi) = 0, u(-\pi) = u(\pi)\}$. Find all eigenvalues and eigenvectors $Ae_n = \lambda_n e_n$. Hint: Find first the eigenvalues and eigenvectors on $D_p = \{u \in C^2[-\pi,\pi] : u_x(-\pi) = u_x(\pi), u(-\pi) = u(\pi)\}$ (i.e. periodic boundary conditions).

2) Find $u(x,t)$ such that $u_t - ku_{xx} = 0$, where $x \in (0,\pi)$, $t > 0$, $u(x,0) = \pi^2 - x^2$, $u(0,t) = 0$, and $u(\pi,t) = 0$.

3) Find the Fourier series of $f(x) = \pi^2 - x^2$ on the interval $(-\pi,\pi)$. Sketch the 2π-periodic extension of $f(x)$.

The Fourier series looks as follows (on the interval $[-\pi,\pi]$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos nx + b_n \sin nx\big], \tag{1}$$

where the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x \quad (n = 0,1,\ldots) \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x \quad (n = 1,2,\ldots). \tag{3}$$

Hence, for $f(x) = \pi^2 - x^2$ we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\cos nx \,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)\cos nx \,\mathrm{d}x \tag{4}$$

$$= -\frac{4\cos n\pi}{n^2} = -\frac{4(-1)^n}{n^2}, \tag{5}$$

and $b_n = 0$ since $f(x)$ is even. Thus,

$$\boxed{\pi^2 - x^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx}. \tag{6}$$

We could also have found this result from the result of problem 5:

$$\frac{3x^2 - \pi^2}{12} = -\frac{(\pi^2 - x^2)}{4} + \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \tag{7}$$

$$\Rightarrow \pi^2 - x^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx, \tag{8}$$

which is the result we found above in Eq. (6).

4) Check the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx = \frac{1}{12}\big(3x^2 - 6\pi x + 2\pi^2\big), \tag{9}$$

where $0 \leq x \leq 2\pi$.

The cosine fourier series on the interval $[0, \pi]$ for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos{(nx)}, \tag{10}$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos{(nx)} \,\mathrm{d}x\,. \tag{11}$$

Observe that $\sin^2{(nx/2)} = (1 - \cos nx)/2$ for any x. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos{(nx)} = \sum_{n=1}^{\infty} \frac{2}{n^2}(1 - \sin^2{(nx/2)}). \tag{12}$$

5) Check the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = \frac{1}{12}(3x^2 - \pi^2), \tag{13}$$

where $x \in [-\pi, \pi]$.

The fourier series for $f(x) = (3x^2 - \pi^2)/12$ is given by

$$\frac{3x^2 - \pi^2}{12} = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx, \tag{14}$$

where the sine terms vanish since $f(x)$ is an even function and

$$a_0 = \frac{1}{6\pi} \int_0^{\pi} (3x^2 - \pi^2) \,\mathrm{d}x = 0 \tag{15}$$

$$a_{n>0} = \frac{1}{6\pi} \int_0^{\pi} (3x^2 - \pi^2) \cos nx \,\mathrm{d}x = \frac{(-1)^n}{n^2}. \tag{16}$$

Hence,

$$\boxed{\frac{1}{12}(3x^2 - \pi^2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx}. \tag{17}$$

6) Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the result of problem 4 or 5.

Using $x = 0$ with Eq. (9), we find

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12} = \frac{\pi^2}{6}}. \tag{18}$$

7) Find $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ using the result of problem 5 or 4.

Using $x = 0$ with Eq. (13), we find

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}}. \tag{19}$$