2.4.16) Solve the diffusion equation with constant dissipation: $u_t - ku_{xx} + bu = 0$ for $-\infty < x < \infty$ with $u(x,0) = \phi(x)$.

Consider the change of variables given by $u = e^{-bt}v$. Then $u_t = -bu + e^{-bt}v_t$ and $u_{xx} = e^{-bt}v_{xx}$, which makes the diffusion equation with constant dissipation

$$-bu + e^{-bt}v_t - ke^{-bt}v_{xx} + bu = e^{-bt}v_t - ke^{-bt}v_{xx} = 0 \Rightarrow v_t - kv_{xx} = 0. \tag{1}$$

That is, $v = e^{bt}u$ satisfies the diffusion equation with no dissipation, where the initial condition $v(x,0) = \phi(x)$

$$v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, \mathrm{d}y\,, \tag{2}$$

which gives u as

$$\boxed{u = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, \mathrm{d}y}\,. \tag{3}$$

This form of u makes sense. It is similar to a harmonic oscillator with linear damping. That is, the solution is a product of the decaying exponential e^{-bt}, which depends on the damping parameter b, and the solution of the diffusion equation (with no damping).

2.4.17) Solve the diffusion equation with variable dissipation: $u_t - ku_x + bt^2u = 0$ for $-\infty < x < \infty$ with $u(x,0) = \phi(x)$.

Consider the change of variables given by $u = e^{-bt^3/3}v$. Thus, $u_t = e^{-bt^3/3}v_t - bt^2u$ and $u_{xx} = e^{-bt^3/3}v_{xx}$, making the diffusion equation with variable dissipation

$$e^{-bt^3/3}v_t - bt^2u - ke^{-bt^3/3}v_{xx} + bt^2u = e^{-bt^3/3}[v_t - kv_{xx}] = 0 \Rightarrow v_t - kv_{xx} = 0. \tag{4}$$

Hence, observing that $v(x,0) = \phi(x)$

$$v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, \mathrm{d}y\,. \tag{5}$$

$$\boxed{u = \frac{e^{-bt^3/3}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, \mathrm{d}y}\,, \tag{6}$$

which is similar to the form of Eq. (3), except that our solution decays faster since the dissipation becomes quadratically stronger over time.

2.4.18) Solve the heat equation with convection: $u_t - ku_{xx} + Vu_x = 0$ for $-\infty < x < \infty$ with $u(x,0) = \phi(x)$.

We have the equation $u_t - Au = 0$ with $A = k\partial_x^2 - V\partial_x$, which has solution

$$u = e^{kt\partial_x^2} e^{-Vt\partial_x}\phi(x) = e^{-kt\partial_x^2}\phi(x - Vt), \tag{7}$$

which means that $\phi(x-Vt)$ satisfies the diffusion equation (without convection). This makes our solution (letting $y = x - Vt$)

$$\boxed{u = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-z)^2/4kt}\phi(z)\,\mathrm{d}z = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-Vt-z)^2/4kt}\phi(z)\,\mathrm{d}z}. \tag{8}$$

2.5.4) Here is a direct relationship between the wave and diffusion equations. Let $u(x,t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\,\mathrm{d}s\,. \tag{9}$$

(a) Show that $v(x,t)$ solves the diffusion equation!

We will show that v satisfies the equation $v_t - kv_{xx} = 0$, assuming that u satisfies the equation $u_{tt} = c^2u_{xx}$. Observe that

$$\begin{aligned}\frac{\partial v}{\partial t} &= -\frac{1}{2t}\frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\,\mathrm{d}s + \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{s^2c^2}{4kt^2} e^{-s^2c^2/4kt} u(x,s)\,\mathrm{d}s \\ &= \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[\frac{s^2c^2}{4kt^2} - \frac{1}{2t}\right] e^{-s^2c^2/4kt} u(x,s)\,\mathrm{d}s\end{aligned} \tag{10}$$

Observe that

$$\frac{\partial^2}{\partial s^2} e^{-s^2c^2/4kt} = \frac{c^2}{4k^2t^2}\left[s^2c^2 - 2kt\right] e^{-s^2c^2/4kt} = \frac{c^2}{k}\left[\frac{s^2c^2}{4kt^2} - \frac{1}{2t}\right] e^{-s^2c^2/4kt}. \tag{11}$$

Thus, Eq. (10) becomes

$$\frac{\partial v}{\partial t} = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{k}{c^2}\frac{\partial^2}{\partial s^2}\left(e^{-s^2c^2/4kt}\right) u(x,s)\,\mathrm{d}s = \frac{k}{c^2}\frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u_{ss}(x,s)\,\mathrm{d}s\,. \tag{12}$$

Next, we have

$$\frac{\partial^2}{\partial x^2}\frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\,\mathrm{d}s = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u_{xx}(x,s)\,\mathrm{d}s\,. \tag{13}$$

Hence,

$$\boxed{v_t - kv_{xx} = \frac{ck}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[\frac{1}{c^2}u_{ss}(x,s) - u_{xx}(x,s)\right] e^{-s^2c^2/4kt}\,\mathrm{d}s = 0} \tag{14}$$

(b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

We have already proven that for $\delta > 0$

$$\lim_{t\to 0} \max_{\delta\le|s|<\infty} \frac{1}{\sqrt{4\pi kt}} e^{-s^2/4kt} = 0. \tag{15}$$

In this case, we let $1/4kt \to c^2/4kt$ for this problem. Hence, this tells us that

$$\lim_{t\to 0} \frac{c}{\sqrt{4\pi kt}} e^{-s^2c^2/4kt} = 0 \text{ if } s > 0. \tag{16}$$

Furthermore, it is trivial to see that if $s = 0$, then

$$\lim_{t\to 0} v(0,t) = \lim_{t\to 0} \frac{c}{\sqrt{4\pi kt}} = \infty. \tag{17}$$

Thus, we can write

$$\lim_{t\to 0} \frac{c}{\sqrt{4\pi kt}} e^{-s^2c^2/4kt} u(x,s) = u(x,0) \lim_{t\to 0} \frac{c}{\sqrt{4\pi kt}} e^{-s^2c^2/4kt}. \tag{18}$$

We can now prove the claim of this problem:

$$\boxed{\begin{aligned}\lim_{t\to 0} v(x,t) &= \lim_{t\to 0} \int_{-\infty}^{\infty} \frac{c}{\sqrt{4\pi kt}} e^{-s^2c^2/4kt} u(x,s)\,\mathrm{d}s \\ &= u(x,0) \lim_{t\to 0} \int_{-\infty}^{\infty} \frac{c}{\sqrt{4\pi kt}} e^{-s^2c^2/4kt}\,\mathrm{d}s = u(x,0)\end{aligned}},$$

since $c\exp(-s^2c^2/4kt)/\sqrt{4\pi kt}$ is a normalized gaussian for all $t > 0$. That is,

$$\int_{-\infty}^{\infty} \frac{c}{\sqrt{4\pi kt}} e^{-s^2c^2/4kt}\,\mathrm{d}s = 1. \tag{19}$$

12.3) Check formulas (6) and (7) from page 345:

$$\mathcal{F}\{H(a-|x|)\} = \frac{2}{k}\sin ak \tag{20}$$

$$\mathcal{F}\{e^{-a|x|}\} = \frac{2a}{a^2+k^2} \quad \text{for } a > 0, \tag{21}$$

where $\mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$ and $H(x)$ is the Heaviside step function.

For Eq. (20), we have

$$\boxed{\mathcal{F}\{H(a-|x|)\} = \int_{-\infty}^{\infty} H(a-|x|)e^{-ikx}\,\mathrm{d}x = \int_{-a}^{a} e^{-ikx}\,\mathrm{d}x = \frac{e^{-ika}-e^{ika}}{-ik} = \frac{2}{k}\sin ka}, \tag{22}$$

where we have used the identity

$$\sin x = \frac{e^{ikx} - e^{-ikx}}{2i}. \tag{23}$$

Now, for Eq. (21), we have

$$\mathcal{F}\{e^{-a|x|}\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx}\,\mathrm{d}x = \int_{-\infty}^{0} e^{(a-ik)x}\,\mathrm{d}x + \int_{0}^{\infty} e^{-(a+ik)x}\,\mathrm{d}x \tag{24}$$

$$= \frac{1}{a-ik} + \frac{1}{a+ik}, \tag{25}$$

where we have assumed $a > 0$ such that the evaluation of the integral vanishes at $\pm\infty$. Collecting terms and simplifying we find

$$\boxed{\mathcal{F}\{e^{-a|x|}\} = \frac{(a+ik)+(a-ik)}{(a-ik)(a+ik)} = \frac{2a}{a^2+k^2}}. \tag{26}$$

6) Check formulas (i), (ii), and (iii) on page 346:

$$\text{i}: \ \mathcal{F}\left\{\frac{\mathrm{d}f}{\mathrm{d}x}\right\} = ik\mathcal{F}\{f(x)\} \tag{27}$$

$$\text{ii}: \ \mathcal{F}\{xf(x)\} = i\frac{\mathrm{d}\mathcal{F}\{f(x)\}}{\mathrm{d}k} \tag{28}$$

$$\text{iii}: \ \mathcal{F}\{f(x-a)\} = e^{-iak}\mathcal{F}\{f(x)\} \tag{29}$$

For Eq. (27) we have

$$\boxed{\mathcal{F}\left\{\frac{\mathrm{d}f}{\mathrm{d}x}\right\} = \int_{-\infty}^{\infty} \frac{\mathrm{d}f}{\mathrm{d}x} e^{-ikx}\,\mathrm{d}x = -ike^{-ikx}f(x)\Big|_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x = ik\mathcal{F}\{f(x)\}}, \tag{30}$$

where we have supposed that f goes to zero at $\pm\infty$.

Next, for Eq. (28) we have

$$\mathcal{F}\{xf(x)\} = \int_{-\infty}^{\infty} xf(x)e^{-ikx}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{-i}\right)\frac{\partial}{\partial k}e^{-ikx}\,\mathrm{d}x \tag{31}$$

$$= \boxed{i\frac{\mathrm{d}}{\mathrm{d}k}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x = i\frac{\mathrm{d}\mathcal{F}\{f(x)\}}{\mathrm{d}k}}. \tag{32}$$

Finally, for Eq. (29) we can make the substitution $y = x - a$ such that

$$\boxed{\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-ikx}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(y)e^{-iky}e^{-ika}\,\mathrm{d}y = e^{-ika}\mathcal{F}\{f(x)\}}. \tag{33}$$