Lecture 1

1) Estimate

$$\left|\frac{u(t+h)-u(t-h)}{2h} - u'(t)\right| \leq \ ?. \tag{1}$$

for $h > 0$ and $|\partial^3 u(s)| \leq m$, $s \in [t-h, t+h]$.

We have the expression

$$u(t+h) = u(t) + hu'(t) + \frac{h^2}{2}u''(t) + \int_t^{t+h} \frac{(t+h-s)^2}{2} u^{(3)}(s)\ \mathrm{d}s\,. \tag{2}$$

Taking $h \to -h$, Eq. (2) becomes

$$u(t-h) = u(t) - hu'(t) + \frac{h^2}{2}u''(t) - \int_{t-h}^{t} \frac{(t-h-s)^2}{2} u^{(3)}(s)\ \mathrm{d}s\,. \tag{3}$$

Subtracting the two we get

$$\begin{aligned} u(t+h) - u(t-h) = 2hu'(t) + \Bigg(&\int_t^{t+h} \frac{(t+h-s)^2}{2} u^{(3)}(s)\ \mathrm{d}s \\ &- \int_{t-h}^{t} \frac{(t-h-s)^2}{2} u^{(3)}(s)\ \mathrm{d}s \Bigg). \end{aligned} \tag{4}$$

Thus,

$$\begin{aligned} \left|\frac{u(t+h)-u(t-h)}{2h} - u'(t)\right| = \frac{1}{2h}\Bigg| &\int_t^{t+h} \frac{(t+h-s)^2}{2} u^{(3)}(s)\ \mathrm{d}s \\ &- \int_{t-h}^{t} \frac{(t-h-s)^2}{2} u^{(3)}(s)\ \mathrm{d}s \Bigg|. \end{aligned} \tag{5}$$

Using $|a \pm b| \leq |a| + |b|$, $|\int f(x)\ \mathrm{d}x\,| \leq \int |f(x)|\ \mathrm{d}x$, and the condition $|u^{(3)}| \leq m$, we can write

$$\begin{aligned} \left|\frac{u(t+h)-u(t-h)}{2h} - u'(t)\right| &\leq \frac{m}{4h}\left[\int_t^{t+h} (t+h-s)^2\ \mathrm{d}s + \int_{t-h}^{t} (t-h-s)^2\ \mathrm{d}s\right] \\ &= \frac{m}{4h}\left(\frac{h^3}{3} + \frac{h^3}{3}\right) = \boxed{\frac{mh^2}{6}}. \end{aligned} \tag{6}$$

2) Let there be positive numbers c, M such that for all $N \in \mathbb{Z}_{\geq 0}$

$$|\partial^N u(s)| \leq cM^N N!, \qquad |s-a| \leq |t-a|. \tag{7}$$

Using

$$u(t) = \sum_{n=0}^{N-1} \frac{(t-a)^n}{n!} \partial^n u(a) + \int_a^t \frac{(t-s)^{N-1}}{(N-1)!} \partial^N u(s)\,\mathrm{d}s, \tag{8}$$

show that

$$\left| u(t) - \sum_{n=0}^{N-1} \frac{(t-a)^n}{n!} \partial^n u(a) \right| \le c(M|t-a|)^N \to 0 \tag{9}$$

if $M|t-a| < 1$. In this case we obtain Taylor's series for function u.

From Eq. (8) we have (assuming $t > a$)

$$\begin{aligned} \left| u(t) - \sum_{n=0}^{N-1} \frac{(t-a)^n}{n!} \partial^n u(a) \right| &= \left| \int_a^t \frac{(t-s)^{N-1}}{(N-1)!} \partial^N u(s)\,\mathrm{d}s \right| \\ &\le \frac{cM^N N!}{(N-1)!} \int_a^t |t-s|^{N-1}\ \mathrm{d}s \\ &= cNM^N \frac{(t-s)^N}{N} \Bigg|_a^t \\ &= cM^N[-(a-s)^N] \\ &= \boxed{c\Big[M|t-a|\Big]^N}. \end{aligned} \tag{10}$$

Note that the argument is similar for $t < a$. We get a relative $-$ sign from switching the bounds of integration and another from having $t < s$ such that $|t-s| = -(t-s)$. These minus signs cancel giving us the result in Eq. (10).

If $M|t-a| < 1$, then in the limit $N \to \infty$, we have $c[M|t-a|]^N \to 0$.

3) Let there be positive numbers c, M such that for all $N \in \mathbb{Z}_{\ge 0}$

$$|\partial^N u(s)| \le cM^N (N!)^\alpha, \qquad \alpha < 1, \quad s \in \mathbb{R}. \tag{11}$$

Show that in this case we have convergent Taylor series for all $t, a \in \mathbb{R}$.

Note that

$$|\partial^N u(s)| \le cM^N (N!)^\alpha = \frac{cM^N N!}{(N!)^{1-\alpha}}. \tag{12}$$

Thus, we can alter Eq. (10) to read

$$\left| u(t) - \sum_{n=0}^{N-1} \frac{(t-a)^n}{n!} \partial^n u(a) \right| \le \frac{c[M|t-a|]^N}{(N!)^{1-\alpha}} \to 0, \quad \text{as } N \to \infty, \tag{13}$$

noting that factorials grow faster than powers of N.

Lecture 2

1) Check that

$$v(t) = \int_0^t \frac{\sin[(t-\tau)\sqrt{A}]}{\sqrt{A}} f(\tau)\ \mathrm{d}\tau\,. \tag{14}$$

satisfies $v(0) = \partial_t v(0) = 0$ and $(\partial_t^2 + A)v = f(t)$ for $t > 0$.

It is clear that $v(0) = 0$ since the upper bound is the same as the lower bound at $t = 0$. Next,

$$\begin{aligned}
\partial_t v(t) &= \frac{\sin[(t-\tau)\sqrt{A}]}{\sqrt{A}} f(\tau)\Bigg|_{\tau=t} + \int_0^t \partial_t\Big(\frac{\sin[(t-\tau)\sqrt{A}]}{\sqrt{A}}\Big) f(\tau)\ \mathrm{d}\tau \\
&= \int_0^t \cos[(t-\tau)\sqrt{A}] f(\tau)\ \mathrm{d}\tau \Rightarrow \boxed{\partial_t v(0) = 0}.
\end{aligned} \tag{15}$$

Finally,

$$\begin{aligned}
\partial_t^2 v(t) &= \partial_t \int_0^t \cos[(t-\tau)\sqrt{A}] f(\tau)\ \mathrm{d}\tau \\
&= f(\tau) - \int_0^t \sqrt{A}\sin[(t-\tau)\sqrt{A}] f(\tau)\ \mathrm{d}\tau \\
&= f(\tau) - Av(t) \\
&\Rightarrow \boxed{(\partial_t^2 - A)v(t) = f(\tau)}
\end{aligned} \tag{16}$$

2) Check

$$\begin{aligned}
u(t) = \frac{\sin[(T-t)\sqrt{A}]}{\sin(T\sqrt{A})} g &+ \frac{\sin(t\sqrt{A})}{\sin(T\sqrt{A})}\Big[h - \int_0^T \frac{\sin[(T-\tau)\sqrt{A}]}{\sqrt{A}} f(\tau)\ \mathrm{d}\tau\Big] \\
&+ \int_0^t \frac{\sin[(t-\tau)\sqrt{A}]}{\sqrt{A}} f(\tau)\ \mathrm{d}\tau\,.
\end{aligned} \tag{17}$$

satisfies the boundary conditions $u(0) = g$ and $u(T) = h$.

First, we check at $t = 0$:

$$u(0) = \frac{\sin(T\sqrt{A})}{\sin(T\sqrt{A})} g = g. \tag{18}$$

Next, we check at $t = T$:

$$u(T) = h - \int_0^T \frac{\sin[(T-\tau)\sqrt{A}]}{\sqrt{A}} f(\tau)\ \mathrm{d}\tau + \int_0^T \frac{\sin[(T-\tau)\sqrt{A}]}{\sqrt{A}} f(\tau)\ \mathrm{d}\tau = h. \tag{19}$$