1) Which operators are linear?

a) $Au = -\partial^2 u$, $\mathcal{D} = \{u \in C^2[0,\pi] \;\big|\; u(0) = 0 \text{ and } u'(\pi) = 0\}$

This is a linear operator on the vector space $\mathcal{D}$ since

$$A(\alpha u + \beta v) = -\partial^2(\alpha u + \beta v) = \alpha(-\partial^2 u) + \beta(-\partial^2 v) = \alpha Au + \beta Av. \tag{1}$$

b) $Au = -\partial^2 u$, $\mathcal{D} = \{u \in C^2[0,\pi] \;\big|\; u(0) = 0 \text{ and } u'(\pi) = \pi\}$

The space $\mathcal{D}$ is not a proper vector space since if $u, v \in \mathcal{D}$, then $u+v \notin \mathcal{D}$ since $u'(\pi)+v'(\pi) = 2\pi$. Hence, A is not a linear operator on the space $\mathcal{D}$ since Au must be in the vector space $\mathcal{D}$ for all $u \in \mathcal{D}$.

c) $Au = u_{xx} + x^2 u$, $\mathcal{D} = C^2(\mathbb{R})$

This is clearly a linear operator since

$$A(\alpha u + \beta v) = \partial_x^2(\alpha u + \beta v) + x^2(\alpha u + \beta v) = \alpha[u_{xx} + x^2 u] + \beta[v_{xx} + x^2 v] = \alpha Au + \beta Av. \tag{2}$$

d) $Au = u_{xx} + u^2$, $\mathcal{D} = C^2(\mathbb{R})$

This is not a linear operator:

$$A(\alpha u) = \alpha u_{xx} + \alpha^2 u^2 \neq \alpha Au. \tag{3}$$

2) Solve $u_t = 3u_x + 5u$, $u(x,0) = f(x)$.

We wish to solve the equation $u_t = Au$, where $A = 3\partial_x + 5$ and $u(x,0) = f(x)$, which has solution

$$\boxed{u = e^{tA} f(x) = e^{3t\partial_x} e^{5t} f(x) = e^{5t} f(x+3t)}. \tag{4}$$

3) Solve $u_{tt} - u_{xx} = 0$, $x \in \mathbb{R}$, $t > 0$ for $u(x,0) = 3\sin 5x - \cos 3x$, $u_t(x,0) = 2x + 3x^2$. Find only $u(0,t)$.

The general solution to the wave equation $u_{tt} - c^2 u_{xx} = 0$ with initial conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\big[\phi(x+ct) + \phi(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s\,. \tag{5}$$

Hence, at $x = 0$, we have

$$\begin{aligned} u(0,t) &= \frac{1}{2}\left[3\sin(5t) - \cos(3t) - 3\sin(-5t) + \cos(-3t)\right] + \frac{1}{2}\int_{-t}^{t}(2s + 3s^2)\,\mathrm{d}s \\ &= 3\sin(5t) + \int_0^t 3s^2\,\mathrm{d}s \\ &= \boxed{3\sin(5t) + t^3}. \end{aligned} \tag{6}$$

4) Find $u(x,t)$ for the equation $u_t - ku_{xx} = 0$, $x \in \mathbb{R}$, $t > 0$ subject to $u(x,0) = e^{-(x+1)^2}$.

We can find $u(x,t)$ subject to the initial condition as follows:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}e^{-(y+1)^2}\,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt-1}\int_{-\infty}^{\infty} e^{-y^2/4kt-y^2-2y+xy/2kt}\,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt-1}\int_{-\infty}^{\infty} e^{-(1+1/4kt)y^2+2(x/4kt-1)y}\,\mathrm{d}y\,. \end{aligned} \tag{7}$$

Denote $a = 1 + 1/4kt$ and $b = x/4kt - 1$, then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(1+1/4kt)y^2+2(x/4kt-1)y}\,\mathrm{d}y &= \int_{-\infty}^{\infty} e^{-a(y^2+2by/a)}\,\mathrm{d}y \\ &= e^{b^2/a}\int_{-\infty}^{\infty} e^{-a(y+b/a)^2}\,\mathrm{d}y \\ &= e^{4kt(x/4kt-1)^2/(1+4kt)}\sqrt{\frac{4\pi kt}{1+4kt}}. \end{aligned} \tag{8}$$

Finally, we have our solution:

$$\boxed{u(x,t) = \frac{1}{\sqrt{1+4kt}}e^{-(x+1)^2/(1+4kt)}}. \tag{9}$$

5) Find the Fourier transform $\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,\mathrm{d}\omega$ for $f(x) = xe^{-x^2}$.

The fourier transform of f is given as

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} xe^{-i\omega x}e^{-x^2}\,\mathrm{d}x = \frac{i}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega}\int_{-\infty}^{\infty} e^{-i\omega x}e^{-x^2}\,\mathrm{d}\omega = \frac{i}{\omega}\frac{\mathrm{d}\mathcal{F}\{\exp(-x^2)\}}{\mathrm{d}k}. \tag{10}$$

The fourier transform of this “gaussian” is as follows

$$\begin{aligned}\mathcal{F}\{e^{-x^2}\} &= \int_{-\infty}^{\infty} e^{-i\omega x} e^{-x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-(x^2+i\omega)}\,\mathrm{d}x \\ &= e^{-\omega^2/4}\int_{-\infty}^{\infty} e^{-(x+i\omega/2)^2}\,\mathrm{d}x = \sqrt{\pi}e^{-\omega^2/4}. \end{aligned} \tag{11}$$

Thus,

$$\widehat{f}(\omega) = \frac{i\sqrt{\pi}}{\omega}\Big(-\frac{\omega}{2}e^{-\omega^2/4}\Big) = \boxed{-\frac{i\sqrt{\pi}}{2}e^{-\omega^2/4}}. \tag{12}$$

6*) Find the Fourier transform for $f(x) = -H(x+1) + 2H(x) - H(x-1)$, where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

Our function

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

Therefore,

$$\begin{aligned}\widehat{f}(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)\,\mathrm{d}x = -\int_{-1}^{0} e^{-ikx}\,\mathrm{d}x + \int_{0}^{1} e^{-ikx}\,\mathrm{d}x \\ &= \frac{1-e^{ik}}{ik} - \frac{1-e^{-ik}}{ik} = \frac{e^{-ik}-e^{ik}}{ik} = \boxed{-\frac{2\sin k}{k}}. \end{aligned} \tag{14}$$

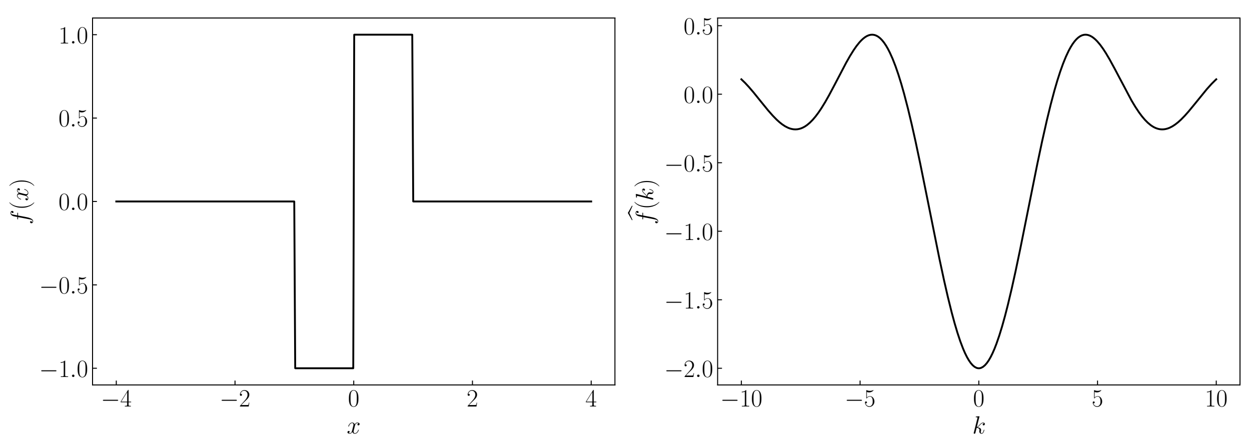

Figure 1: Plot of $f(x)$ and the fourier transform $\widehat{f}(k)$.

7*) Solve $u_t = 5u_x - 3xu + u$, $x \in \mathbb{R}$, $t > 0$ with $u(x,0) = f(x)$.

We perform the change of variables $u = e^{\lambda x^2} v$, where λ is a variable to be determined such that the equation becomes one that we know how to solve easily. Then, the time and spatial derivatives are

$$u_t = e^{\lambda x^2} v_t \tag{15}$$

$$u_x = e^{\lambda x^2}(v_x + 2\lambda x v). \tag{16}$$

Our equation then becomes

$$\big[v_t - 5v_x - 10\lambda x v + 3xv - v\big] e^{\lambda x^2} = 0. \tag{17}$$

To simplify our calculation, we pick $\lambda = 3/10$, which gives $u = e^{3x^2/10} v$ and

$$v_t - 5v_x - v = v_t - Av = 0, \tag{18}$$

where $A = 5\partial_x + 1$. This equation has solution

$$v = e^{t(5\partial_x + 1)} e^{-3x^2/10} f(x) = e^t e^{5t\partial_x} e^{-3x^2/10} f(x) = e^t e^{-3(x+5t)^2/10} f(x+5t), \tag{19}$$

meaning

$$\boxed{u = e^{3x^2/10} e^t e^{-3(x+5t)^2/10} f(x+5t) = e^{-t(6x+15t-2)/2} f(x+5t)}. \tag{20}$$