1) Let $A = -\partial^2 : \mathcal{D} \subset L_2(-\pi, \pi) \mapsto L_2(-\pi, \pi)$, where $\mathcal{D} = \{u \in C^2[-\pi, \pi] : u_x(-\pi) = u_x(\pi) = 0, u(-\pi) = u(\pi)\}$. Find all eigenvalues and eigenvectors $Ae_n = \lambda_n e_n$. Hint: Find first the eigenvalues and eigenvectors on $D_p = \{u \in C^2[-\pi, \pi] : u_x(-\pi) = u_x(\pi), u(-\pi) = u(\pi)\}$ (i.e. periodic boundary conditions).

We first solve generally for periodic boundary conditions. It has been proven in class a few times previously that the eigenvalues of $-\partial^2$ must be positive (using integration by parts). Let us denote $\lambda = \mu^2$, where $Ae(x) = \lambda e(x)$. It has also been proven that $-\partial^2$ is a hermitian operator, meaning its eigenvalues are all real. Thus, we have the eigen-equation

$$\partial_x^2 e(x) + \mu^2 e(x) = 0. \tag{1}$$

If $\mu = 0$, then the only eigenfunction admitted is $e(x) = 1$, and if $\mu \neq 0$, then for each μ we have corresponding eigenfunctions $\cos \mu x$ and $\sin \mu x$. In order to satisfy the boundary conditions on $\mathcal{D}_p$ we must have that $\mu = n \in \mathbb{N}$.

Now, we can extend to the case for our domain $\mathcal{D}$. Observe that $\partial_x \sin nx = \cos nx = (-1)^n \neq 0$, implying that $\sin nx$ is not an eigenfunction for A. Thus, our eigenfunctions are 1 (with corresponding eigenvalue 0) and $\cos nx$ (with corresponding eigenvalues n^2 for $n = 1, 2 \ldots$). We could combine the two such that our eigenfunctions are $\cos nx$ with corresponding eigenvalues n^2 for $n = 0, 1, \ldots$.

2) Find $u(x, t)$ such that $u_t - ku_{xx} = 0$, where $x \in (0, \pi)$, $t > 0$, $u(x, 0) = \pi^2 - x^2$, $u(0, t) = 0$, and $u(\pi, t) = 0$.

We know that the equation $u_t - ku_{xx} = 0$ has solution

$$\boxed{u(x,t) = e^{-tk\partial_x^2} \sum_{n=1}^{\infty} A_n \sin nx = \sum_{n=1}^{\infty} A_n e^{-tkn^2} \sin nx}, \tag{2}$$

where

$$\boxed{A_n = \frac{2}{\pi} \int_0^{\pi} u(x, 0) \sin nx \,\mathrm{d}x = \frac{2\pi^2 n^2 - 4[(-1)^n - 1]}{\pi n^3}}. \tag{3}$$

3) Find the Fourier series of $f(x) = \pi^2 - x^2$ on the interval $(-\pi, \pi)$. Sketch the 2π-periodic extension of $f(x)$.

The Fourier series looks as follows (on the interval $[-\pi, \pi]$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right], \tag{4}$$

where the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x \quad (n = 0, 1, \ldots) \tag{5}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x \quad (n = 1, 2, \ldots). \tag{6}$$

Hence, for $f(x) = \pi^2 - x^2$ we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\cos nx \,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)\cos nx \,\mathrm{d}x \tag{7}$$

$$= -\frac{4\cos n\pi}{n^2} = -\frac{4(-1)^n}{n^2}, \tag{8}$$

and $b_n = 0$ since $f(x)$ is even. Thus,

$$\boxed{\pi^2 - x^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx}. \tag{9}$$

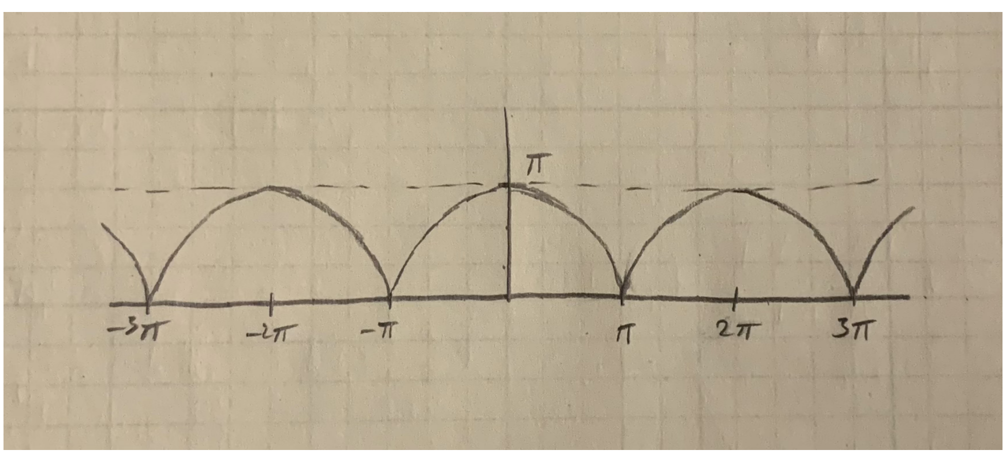
We could also have found this result from the result of problem 5:

$$\frac{3x^2 - \pi^2}{12} = -\frac{(\pi^2 - x^2)}{4} + \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \tag{10}$$

$$\Rightarrow \pi^2 - x^2 = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \tag{11}$$

which is the result we found above in Eq. (9).

Additionally, we can sketch the 2π-periodic continuation of $f(x)$ as below. Note that this is only a sketch. These are meant to be parabolas (not half circles).



4) Check the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx = \frac{1}{12}\big(3x^2 - 6\pi x + 2\pi^2\big), \tag{12}$$

where $0 \leq x \leq 2\pi$.

Observe that $\sin^2(nx/2) = (1 - \cos nx)/2$ for any x. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) = \sum_{n=1}^{\infty} \frac{1}{n^2}(1 - 2\sin^2(nx/2)) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(nx/2). \tag{13}$$

Notice that

$$2\sum_{n=1}^{\infty} \frac{\sin^2(nx/2)}{n^2} = \sum_{n=-\infty}^{\infty} \frac{\sin^2(nx/2)}{n^2} - \Big(\frac{x}{2}\Big)^2, \tag{14}$$

where the $n = 0$ term is the limiting value of $(\sin(nx/2)/n)^2$ as $n \to 0$, which is just $(x/2)^2$. Hence,

$$\boxed{\begin{aligned}\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} &= \frac{\pi^2}{6} + \frac{x^2}{4} - \sum_{n=-\infty}^{\infty} \frac{\sin^2(nx/2)}{n^2} = \frac{\pi^2}{6} + \frac{x^2}{4} - \frac{\pi x}{2} \\ &= \frac{1}{12}\big(3x^2 - 6\pi x + 2\pi^2\big)\end{aligned}}, \tag{15}$$

where we have used the result that

$$\pi u = \sum_{n=-\infty}^{\infty} \frac{\sin^2(nu)}{n^2}, \tag{16}$$

on the interval $u \in (0, \pi)$.

This is proven using Parseval's theorem with the function $f(x) = \begin{cases} \pi & |x| < u \\ 0 & |x| > u \end{cases}$. Then, on the interval $(-u, u)$ we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{17}$$

where

$$c_n = \frac{1}{2\pi} \int_{-u}^{u} \pi e^{inx}\, \mathrm{d}x = \frac{\sin nu}{n}. \tag{18}$$

Parseval's theorem then states that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} \frac{\sin^2(nu)}{n^2} = \frac{1}{2\pi} \int_{-u}^{u} |f(x)|^2\, \mathrm{d}x = \pi u. \tag{19}$$

5) Check the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = \frac{1}{12}(3x^2 - \pi^2), \tag{20}$$

where $x \in [-\pi, \pi]$.

The fourier series for $f(x) = (3x^2 - \pi^2)/12$ is given by

$$\frac{3x^2 - \pi^2}{12} = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx, \tag{21}$$

where the sine terms vanish since $f(x)$ is an even function and

$$a_0 = \frac{1}{6\pi} \int_0^\pi (3x^2 - \pi^2)\, \mathrm{d}x = 0 \tag{22}$$

$$a_{n>0} = \frac{1}{6\pi} \int_0^\pi (3x^2 - \pi^2) \cos nx\, \mathrm{d}x = \frac{(-1)^n}{n^2}. \tag{23}$$

Hence,

$$\boxed{\frac{1}{12}(3x^2 - \pi^2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx}. \tag{24}$$

6) Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the result of problem 4 or 5.

Using $x = 0$ with Eq. (12), we find

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12} = \frac{\pi^2}{6}}. \tag{25}$$

7) Find $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ using the result of problem 5 or 4.

Using $x = 0$ with Eq. (20), we find

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}}. \tag{26}$$