6.1.2) Show that a function which is a power series in the complex variable $x + iy$ must satisfy the Cauchy-Riemann equations and therefore Laplace's equations.

We define a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

We can separate the sum into real and imaginary parts using the binomial series since $z = x + iy$

$$f(z) = \sum_{n=0}^{\infty} a_n (x+iy)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k \tag{2}$$

$$= \sum_{n=0}^{\infty} \left[a_n \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k} + i \sum_{k=0}^{m-o(n)} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1} \right], \tag{3}$$

where $m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$ and $o(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. If we let

$$\begin{aligned} u(x,y) &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k} \\ v(x,y) &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1}. \end{aligned} \tag{4}$$

All that remains now is to check that the Cauchy-Riemann conditions are satisfied.

$$\begin{aligned} u_x &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} \binom{n}{2k} (-1)^k (n-2k) x^{n-2k-1} y^{2k} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} (-1)^k \frac{n!}{(2k)!(n-2k-1)!} x^{n-2k-1} y^{2k} \\ v_y &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} \binom{n}{2k+1} (-1)^k (2k+1) x^{n-2k-1} y^{2k} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{m-o(n)} (-1)^k \frac{n!}{(2k)!(n-2k-1)!} x^{n-2k-1} y^{2k}. \end{aligned} \tag{5a}$$

Note that the $o(n)$ term in the upper index of the u_x series comes from the fact that when

n is even, the $k = m$ terms gives 0 when differentiated (with respect to x) since $x^{n-2m} = 1$.

$$\begin{aligned} u_y &= \sum_{n=0}^{\infty} a_n \sum_{k=1}^{m} (-1)^k \frac{n!}{(2k-1)!(n-2k)!} x^{n-2k} y^{2k-1} \\ v_x &= \sum_{n=0}^{\infty} a_n \sum_{k=1}^{m-o(n)+(o(n)-1)} (-1)^k \frac{n!}{(2k+1)!(n-2(k-1))!} x^{n-2(k-1)} y^{2k+1} \\ &= -\sum_{n=0}^{\infty} a_n \sum_{k=1}^{m} (-1)^k \frac{n!}{(2k-1)!(n-2k)!} x^{n-2k} y^{2k-1}, \end{aligned} \tag{5b}$$

where the $o(n) - 1$ term arises in the v_x term since $n - 2k - 1 = 0$ if $k = m$ whenever n is odd. Thus, we can see that any arbitrary power series in z satisfies the Cauchy-Riemann conditions.

6.1.5) Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.

The equation $u_{xx} + u_{yy} = \Delta u = 1$ is stated in cartesian coordinates. We could equivalently pose the problem in polar coordinates in terms of r and θ, which appears as follows:

$$\Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 1. \tag{6}$$

Assuming that $u(r, \theta) = u(r)$ (i.e. u is rotationally invariant), then

$$\begin{aligned} \partial_r(r\partial_r u) &= r \\ r\partial_r u &= \frac{r^2}{2} + c_1 \\ \partial_r u &= \frac{r}{2} + \frac{c_1}{r} \\ u &= \frac{r^2}{4} + c_1 \ln r + c_2. \end{aligned} \tag{7}$$

We must have $c_1 = 0$ since $\ln r \to \infty$ when $r \to 0$, and imposing the boundary condition at $r = a$ we have

$$0 = \frac{a^2}{4} + c_2 \Rightarrow c_2 = -\frac{a^2}{4}. \tag{8}$$

Hence, our solution is as follows:

$$\boxed{u = \frac{1}{4}[r^2 - a^2]}. \tag{9}$$

6.1.6) Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing both parts of the boundary $r = a$ and $r = b$.

From the last problem, we have the general solution

$$u = \frac{r^2}{4} + c_1 \ln r + c_2. \tag{10}$$

This time, though, we keep the term proportional to $\ln r$ since it is well behaved on the domain of interest. Plugging in boundary conditions we find

$$\begin{aligned} 0 &= \frac{a^2}{4} + c_1 \ln a + c_2 \\ 0 &= \frac{b^2}{4} + c_1 \ln b + c_2 \end{aligned}. \tag{11}$$

Solving the system of equations we find

$$c_1 = -\frac{b^2 - a^2}{4\ln(b/a)} \tag{12}$$

$$c_2 = \frac{b^2 \ln a - a^2 \ln b}{4\ln(b/a)}. \tag{13}$$

Thus, our solution is of the form

$$\boxed{u = \frac{1}{4}\left[r^2 - \frac{b^2 - a^2}{\ln(b/a)}\ln r + \frac{b^2 \ln a - a^2 \ln b}{\ln(b/a)}\right]}. \tag{14}$$

6.4.6) Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter $(\theta = 0, \pi)$ and $u = \pi \sin\theta - \sin 2\theta$ on $r = 1$.

We need to have $\Delta u = 0$. In polar coordinates

$$\Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u = 0. \tag{15}$$

If we let $A = -\partial_\theta^2$ on $(0, \pi)$, where $u|_{\theta=0} = u|_{\theta=\pi} = 0$, then A has eigenfunctions $e_n(\theta) = \sin n\theta$ $(||e_n||^2 = \pi/2)$ with corresponding eigenvalues n^2. In class we have solve the remaining equation $r^2\partial_r^2 u + r\partial_r u - Au = 0$:

$$u = r^{\sqrt{A}}c_1 + r^{-\sqrt{A}}c_2. \tag{16}$$

Since we need u to be bounded on the domain, it follows that $c_2 = 0$. Plugging in our boundary conditions we find

$$u(1) = c_1 = \pi \sin\theta - \sin 2\theta = g. \tag{17}$$

Hence, our solution is given as

$$u(r, \theta) = r^{\sqrt{A}} g = \sum_{n=1}^{\infty} r^n \hat{g}_n \sin n\theta, \tag{18}$$

where

$$\hat{g}_n = \frac{2}{\pi}\int_0^{\pi}(\pi\sin\theta - \sin 2\theta)\sin n\theta\,\mathrm{d}\theta = \pi\delta_{1n} - \delta_{2n}. \tag{19}$$

Note that δ_{nm} is the kronecker-delta symbol. Hence, our solution simply becomes

$$\boxed{u(r,\theta) = \pi r\sin\theta - r^2\sin 2\theta}. \tag{20}$$

Lecture 15 - 1) Show that $\partial z^n = nz^{n-1}$ and $\bar{\partial}\bar{z}^n = n\bar{z}^{n-1}$ for $n = 0, 1, 2, \ldots$.

We write

$$\begin{align} \partial z^n &= \frac{1}{2}(\partial_x - i\partial_y)(x+iy)^n = \frac{1}{2}[\partial_x(x+iy)^n - i\partial_y(x+iy)^n] \tag{21}\\ &= \frac{1}{2}[n(x+iy)^{n-1} + n(x+iy)^{n-1}] = n(x+iy)^{n-1} = nz^{n-1}. \tag{22} \end{align}$$

Similarly

$$\begin{align} \bar{\partial}\bar{z}^n &= \frac{1}{2}(\partial_x + i\partial_y)(x-iy)^n = \frac{1}{2}[\partial_x(x-iy)^n + i\partial_y(x-iy)^n] \tag{23}\\ &= \frac{1}{2}[n(x-iy)^{n-1} + n(x-iy)^{n-1}] = n(x-iy)^{n-1} = n\bar{z}^{n-1}. \tag{24} \end{align}$$

Lecture 15 - 2) Find $\bar{\partial}|z|^2$, $\partial|z|^2$, and $\Delta|z|^2$.

We have

$$\boxed{\bar{\partial}|z|^2 = \frac{1}{2}(\partial_x + i\partial_y)(x^2+y^2) = \frac{1}{2}(2x+2iy) = x+iy = z}. \tag{25}$$

Similarly

$$\boxed{\partial|z|^2 = \frac{1}{2}(\partial_x - i\partial_y)(x^2+y^2) = x - iy = \bar{z}}. \tag{26}$$

Thus,

$$\Delta|z|^2 = 4\partial\bar{\partial}|z|^2 = 4\partial z = 4. \tag{27}$$

Lecture 15 - 3) For an analytic function f in the open set Ω, let $u(x,y) = \operatorname{Re} f(z)$ and $v(x,y) = \operatorname{Im} f(z)$. Show that $\Delta u = \Delta v = 0$ and find $\nabla u \cdot \nabla v$. Functions u and v are called harmonic conjugate functions.

Since f is analytic $\bar{\partial} f = 0$, and furthermore $\Delta f = 4\partial\bar{\partial} f = 0$. We know that the Laplace operator is linear, which implies that

$$\Delta f = \Delta u + i\Delta v = 0. \tag{28}$$

Thus, in order to have this equation hold generally, we must have its real and imaginary parts be zero separately, giving us

$$\boxed{\Delta u = \Delta v = 0}. \tag{29}$$

For the second expression we will use the Cauchy-Riemanan conditions (which is a necessary and sufficient for a function to be analytic). We write

$$\boxed{\nabla u \cdot \nabla v = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0}. \tag{30}$$

Lecture 15 - 4) Using the divergence theorem rewrite the integral

$$\int_\Omega \bar{\partial} f(z)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad f \in C^1(\Omega_1),\ \Omega_1 \supset \Omega \tag{31}$$

as an integral over the boundary $\partial\Omega$ of the domain Ω.

We can write $f(z) = u(x,y) + iv(x,y)$, meaning

$$\begin{aligned}
\int_\Omega \bar{\partial} f(z)\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{2}\int_\Omega (\partial_x + i\partial_y) f\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_\Omega [\partial_x f - \partial_y[-if]]\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_{\partial\Omega} [-if\,\mathrm{d}x + f\,\mathrm{d}y] \\
&= \frac{1}{2i}\int_{\partial\Omega} f[\mathrm{d}x + i\,\mathrm{d}y] \\
&= \boxed{\frac{1}{2i}\int_{\partial\Omega} f(z)\,\mathrm{d}z}.
\end{aligned} \tag{32}$$

If f is analytic, then $\bar{\partial} f = 0$, and the integral is zero. That is,

$$\int_\Omega \bar{\partial} f(z)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2i}\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0. \tag{33}$$

Lecture 15 - 5) For harmonic conjugate functions u and v from Exercise 3 above show that $u_x = v_y$ and $u_y = -v_y$ (Caucy-Riemann equations).

An analytic function $f = u + iv$ satisfies $\bar{\partial} f = 0$, which implies that

$$\frac{1}{2}(\partial_x + i\partial_y)(u + iv) = \frac{1}{2}([\partial_x u - \partial_y v] + i[\partial_y u + \partial_x v]) = 0. \tag{34}$$

For this equality to hold true we must have that the real and imaginary parts of the middle expression are identically zero individually, meaning that

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}. \tag{35}$$