Problem 1 – Chapter 3 # 4)
The time-dependent Schrödinger equation of a charged particle in an electromagnetic field reads

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{\boldsymbol{r}},t) = \left\{\frac{1}{2m}\Big[-i\hbar\vec{\boldsymbol{\nabla}} - \frac{q}{c}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)\Big]^2 + qU(\vec{\boldsymbol{r}},t)\right\}\Psi(\vec{\boldsymbol{r}},t) \tag{1}$$

where $U(\vec{\boldsymbol{r}},t)$ and $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)$ are the (real) scalar and vector potential, respectively, c is the speed of light, and q is the charge of the particle. Show that the probability density $\rho(\vec{\boldsymbol{r}},t)$ and the probability current density $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$ are given in this case by

$$\rho(\vec{\boldsymbol{r}},t) = |\Psi(\vec{\boldsymbol{r}},t)|^2 \tag{2}$$

$$\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t) = \frac{\hbar}{2mi}\Big[\Psi^*(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{\nabla}}\Psi(\vec{\boldsymbol{r}},t) - \Psi(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{\nabla}}\Psi^*(\vec{\boldsymbol{r}},t)\Big] - \frac{q}{mc}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)|\Psi(\vec{\boldsymbol{r}},t)|^2 \tag{3}$$

with

$$\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{r}},t) + \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t) = 0. \tag{4}$$

As in our previous derivations of ρ and $\vec{\boldsymbol{j}}$ we define $\boxed{\rho(\vec{\boldsymbol{r}},t) = |\Psi|^2}$, and $\vec{\boldsymbol{j}}$ such that $\frac{\partial\rho}{\partial t} = -\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{j}}$. Taking the time derivative of ρ, we have

$$\frac{\partial\rho}{\partial t} = \Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}. \tag{5}$$

Again, we get the time-derivative of the wave function from S.E. (which requires a bit more massaging than in the previous cases):

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= \frac{1}{2mi\hbar}\left[-\hbar^2\nabla^2 + \frac{i\hbar q}{c}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} + \frac{i\hbar q}{c}\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}} + \frac{q^2}{c^2}\vec{\boldsymbol{A}}^2\right]\Psi + \frac{q}{i\hbar}U\Psi \\ &= -\frac{\hbar}{2mi}\nabla^2\Psi + \frac{q}{2mc}\underbrace{\left[\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}\Psi + \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}}\Psi\right]}_{\Psi\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}+2\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}}\Psi} + \frac{q^2}{2mi\hbar c}\vec{\boldsymbol{A}}^2\Psi + \frac{q}{i\hbar}U\Psi\end{aligned} \tag{6}$$

Observe that the last two terms are purely imaginary, and therefore cancel in Eq. (5), leaving us with[1]

$$\boxed{\begin{aligned}\frac{\partial \rho}{\partial t} &= -\left\{\vec{\boldsymbol{\nabla}} \cdot \frac{\hbar}{2mi}\left[\Psi^* \vec{\boldsymbol{\nabla}} \Psi - \Psi \vec{\boldsymbol{\nabla}} \Psi^*\right] + \frac{q}{mc}\left[|\Psi|^2 \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} + \vec{\boldsymbol{A}} \cdot (\Psi^* \vec{\boldsymbol{\nabla}} \Psi + \Psi \vec{\boldsymbol{\nabla}} \Psi^*)\right]\right\} \\ &= -\vec{\boldsymbol{\nabla}} \cdot \underbrace{\left\{\left[\Psi^* \vec{\boldsymbol{\nabla}} \Psi - \Psi \vec{\boldsymbol{\nabla}} \Psi^*\right] + \frac{q}{mc} \vec{\boldsymbol{A}} |\Psi|^2\right\}}_{\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)}\end{aligned}}. \tag{7}$$

It is manifest then that

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{j}} = 0}. \tag{8}$$

Problem 2 – Chapter 3 # 8)

Consider a particle in a potential $V(\vec{\boldsymbol{r}})$ with associated wave function satisfying the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{\boldsymbol{r}}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{\boldsymbol{r}})\right] \Psi(\vec{\boldsymbol{r}}, t). \tag{9}$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \vec{\boldsymbol{r}}(t) \rangle = \int \mathrm{d}^3\vec{\boldsymbol{r}}\; \vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}, t), \tag{10}$$

where $\langle \vec{\boldsymbol{r}}(t) \rangle$ is the average position of the particle (notation as in notes) and $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}, t)$ is the probability current density. Using the definition of $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}, t)$, show that the equation above can also be written as

$$m \frac{\mathrm{d}}{\mathrm{d}t} \langle \vec{\boldsymbol{r}}(t) \rangle = \langle \vec{\boldsymbol{p}}(t) \rangle. \tag{11}$$

(b) Show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \langle \vec{\boldsymbol{p}}(t) \rangle &= -\left\langle \vec{\boldsymbol{\nabla}} V \right\rangle = -\int \mathrm{d}^3\vec{\boldsymbol{r}}\; \Psi^*(\vec{\boldsymbol{r}}, t)[\vec{\boldsymbol{\nabla}} V(\vec{\boldsymbol{r}})] \Psi(\vec{\boldsymbol{r}}, t) \\ &= \int \mathrm{d}^3\vec{\boldsymbol{r}}\; \Psi^*(\vec{\boldsymbol{r}}, t) \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \Psi(\vec{\boldsymbol{r}}, t),\end{aligned} \tag{12}$$

where we have introduced the force $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$.

[1] We use the fact that $\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} |\Psi|^2 = |\Psi|^2 \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} + \vec{\boldsymbol{A}} \cdot \vec{\nabla} |\Psi|^2 = |\Psi|^2 \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} + \vec{\boldsymbol{A}} \cdot (\Psi^* \vec{\boldsymbol{\nabla}} \Psi + \Psi \vec{\boldsymbol{\nabla}} \Psi^*)$.

Hint: Consider, say, the x-component and, by using the Schrödinger equation and its complex conjugate, obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle p_x(t)\rangle = &-\frac{\hbar^2}{2m}\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[(\nabla^2\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\nabla^2\frac{\partial\Psi}{\partial x}\right] \\ &+\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial(V\Psi)}{\partial x}\right],\end{aligned} \tag{13}$$

where the dependence on $\vec{\boldsymbol{r}}$ and t on the r.h.s. has been suppressed for brevity. Next examine these two terms.
(c) The equation above looks like Newton's second law but for average values. As a matter of fact if $\left\langle \vec{\boldsymbol{F}}\right\rangle = \vec{\boldsymbol{F}}(\langle\vec{\boldsymbol{r}}\rangle)$, then $\langle\vec{\boldsymbol{r}}(t)\rangle$ changes in time as the position of a classical particle under the action of the force $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$. Under what condition(s) can this happen? Obtain $\langle\vec{\boldsymbol{r}}(t)\rangle$ and $\langle\vec{\boldsymbol{p}}(t)\rangle$ for a particle in a harmonic potential

$$V(\vec{\boldsymbol{r}}) = \frac{m\omega^2}{2}\vec{\boldsymbol{r}}^2. \tag{14}$$

(a) Observe that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}(t)\rangle = \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{r}},t) = -\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)}, \tag{15}$$

where we have used the continuity equation for the probability density and current density.

Using $\vec{\boldsymbol{j}} = (\hbar/2mi)[\Psi^*\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Psi^*]$, we can also rewrite Eq. (15) as

$$\boxed{m\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}(t)\rangle = \frac{1}{2}\int \mathrm{d}^3\vec{\boldsymbol{r}}\,[\Psi\vec{\boldsymbol{p}}\Psi^* + \Psi^*\vec{\boldsymbol{p}}\Psi] = \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Psi^*\vec{\boldsymbol{p}}\Psi = \langle\vec{\boldsymbol{p}}(t)\rangle}. \tag{16}$$

Note that we have used the fact that $\vec{\boldsymbol{p}}$ is hermitian to rewrite $\Psi\vec{\boldsymbol{p}}\Psi^* = \Psi^*\vec{\boldsymbol{p}}\Psi$ under the integral sign.

(b) We can do this as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{p}}(t)\rangle &= -i\hbar\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\frac{\partial}{\partial t}\Psi^*\vec{\boldsymbol{\nabla}}\Psi \\
&= -i\hbar\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[\frac{\partial\Psi^*}{\partial t}\vec{\boldsymbol{\nabla}}\Psi + \Psi^*\vec{\boldsymbol{\nabla}}\frac{\partial\Psi}{\partial t}\right] \\
&= -i\hbar\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[-\frac{1}{i\hbar}\Big(-\frac{\hbar^2}{2m}\nabla^2\Psi^* + V\Psi^*\Big)\vec{\boldsymbol{\nabla}}\Psi + \frac{1}{i\hbar}\Psi^*\vec{\boldsymbol{\nabla}}\Big(\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi\Big)\right] \\
&= \int \mathrm{d}^3\vec{\boldsymbol{r}}\left\{-\frac{\hbar^2}{2m}\Big[(\nabla^2\Psi^*)(\vec{\boldsymbol{\nabla}}\Psi) - \Psi^*\vec{\boldsymbol{\nabla}}\big(\nabla^2\Psi\big)\Big] + \Big[V\Psi^*\vec{\boldsymbol{\nabla}}\Psi - \Psi^*\vec{\boldsymbol{\nabla}}(V\Psi)\Big]\right\} \\
&= \boxed{\Big\langle -\vec{\boldsymbol{\nabla}}V\Big\rangle = \Big\langle\vec{\boldsymbol{F}}\Big\rangle}.
\end{aligned} \tag{17}$$

Notice that the integral

$$\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\big[(\nabla^2\Psi^*)(\vec{\boldsymbol{\nabla}}\Psi) - \Psi^*\vec{\boldsymbol{\nabla}}\big(\nabla^2\Psi\big)\big] = \sum_{i=1}^{3}\hat{\boldsymbol{x}}_i\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Big[(\nabla^2\Psi^*)\frac{\partial\Psi}{\partial x_i} - \Psi^*\frac{\partial}{\partial x_i}\big(\nabla^2\Psi\big)\Big] \tag{18}$$

Let us focus on only the x-component since the result there applies also in the y and z components:

$$\begin{aligned}
\int \mathrm{d}^3\vec{\boldsymbol{r}}\Big[(\nabla^2\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}\big(\nabla^2\Psi\big)\Big] &= \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Big[(\nabla^2\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\nabla^2\frac{\partial\Psi}{\partial x}\Big] \\
&= \int \mathrm{d}^3\vec{\boldsymbol{r}}\;\;\vec{\boldsymbol{\nabla}}\cdot\Big[(\vec{\boldsymbol{\nabla}}\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\vec{\boldsymbol{\nabla}}\frac{\partial\Psi}{\partial x}\Big] \\
&= \int \mathrm{d}S_\infty\,\Big[(\vec{\boldsymbol{\nabla}}\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\vec{\boldsymbol{\nabla}}\frac{\partial\Psi}{\partial x}\Big] = 0,
\end{aligned} \tag{19}$$

where we have used the divergence theorem to rewrite the volume integral as a surface integral. Note that S_∞ denotes the surface at infinity. It should be clear that $\Psi|_{S_\infty} = \partial_x\Psi|_{S_\infty} = 0$.

(c) The condition $\Big\langle\vec{\boldsymbol{F}}\Big\rangle = \vec{\boldsymbol{F}}(\langle\vec{\boldsymbol{r}}\rangle)$ when $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_0 + F_1\vec{\boldsymbol{r}}$, where $\vec{\boldsymbol{F}}_0$ and F_1 are a constant vector and scalar, respectively.

Now, we turn our attention to the harmonic potential $V(\vec{\boldsymbol{r}}) = m\omega^2 r^2/2$. Observe that $F = -\vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}}) = -m\omega^2\vec{\boldsymbol{r}}$, which is linear in $\vec{\boldsymbol{r}}$ and falls under the special case we outlined above. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}\rangle = \frac{1}{m}\langle\vec{\boldsymbol{p}}\rangle \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{p}}(t)\rangle = \langle F(\vec{\boldsymbol{r}})\rangle = -m\omega^2\langle\vec{\boldsymbol{r}}\rangle\,. \tag{21}$$

Notice that this is a set of coupled first-order differential equations. We can "decouple" these by differentiating Eq. (20), which gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\vec{\boldsymbol{r}}\rangle = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle = -\omega^2\langle\vec{\boldsymbol{r}}\rangle\,, \tag{22}$$

which is as expected the equation for a particle undergoing harmonic motion and has solution

$$\boxed{\langle\vec{\boldsymbol{r}}\rangle = \vec{\boldsymbol{A}}\cos\omega t + \vec{\boldsymbol{B}}\sin\omega t}, \tag{23}$$

We also have then

$$\boxed{\langle\vec{\boldsymbol{p}}\rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}\rangle = m\omega\Big(\vec{\boldsymbol{B}}\cos\omega t - \vec{\boldsymbol{A}}\sin\omega t\Big)}. \tag{24}$$

Note that $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are constants of integration determined by initial or boundary conditions on $\langle\vec{\boldsymbol{r}}\rangle$ and $\langle\vec{\boldsymbol{p}}\rangle$. For example, if $\langle\vec{\boldsymbol{r}}(0)\rangle$ and $\langle\vec{\boldsymbol{p}}(0)\rangle$, which are the expectation values of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{p}}$ at $t=0$ in the state Ψ, are given as initial conditions

$$\vec{\boldsymbol{A}} = \langle\vec{\boldsymbol{r}}(0)\rangle \text{ and } \vec{\boldsymbol{B}} = \frac{\langle\vec{\boldsymbol{p}}(0)\rangle}{m\omega}. \tag{25}$$

Problem 3 – Chapter 4 # 1)
Consider the problem of a particle in an attractive δ-function potential given by

$$V(x) = -V_0\delta(x) \quad V_0 > 0. \tag{26}$$

(a) Obtain the energy and wave-function of the bound state. Sketch the wave function and provide an estimate for Δx.
(b) Calculate the probability $\mathrm{d}P(p)$ that a measurement of the momentum in this bound state will give a result included between p and $p+\mathrm{d}p$. For what value of p is this probability largest? Provide an estimate for Δp and an order of magnitude for $\Delta x\Delta p$.

(a) The Schrödinger equation under this potential reads

$$\psi''(x) = -[v_0\delta(x) - |\varepsilon|]\psi(x), \tag{27}$$

where $v_0 = 2mV_0/\hbar^2$ and $\varepsilon = 2mE/\hbar^2$. Note that we have explicitly written $|\varepsilon|$ since bound states may only exist for $E<0$. This admits a solution of the form

$$\psi(x) = \Theta(-x)[A_-e^{\kappa x} + B_-e^{-\kappa x}] + \Theta(x)[A_+e^{\kappa x} + B_+e^{-\kappa x}], \tag{28}$$

where $\Theta(x)$ denotes the Heaviside step function and $\kappa = \sqrt{|\epsilon|}$.

At this point, we need to determine the constants $A_\pm$, $B_\pm$ which satisfy relevant boundary conditions. Firstly, we must have that the wave function is normalizable, which implies $\psi(x \to \pm\infty) = 0$ and therefore $A_+ = B_- = 0$. Secondly, we have the conditions on the wave function and its derivative at $x = 0$:

$$\psi(0^-) = \psi(0^+) \Rightarrow A_- = B_+ \tag{29}$$

$$\psi'(0^+) - \psi'(0^-) = v_0\psi(0) \Rightarrow \kappa[-B_+ - A_-] = -v_0 A_-, \tag{30}$$

where we use the notation $\psi(0^\pm) = \lim_{x\to 0^\pm} \psi(x)$. Also, note that Eq. (30) comes from integrating the Eq. (27) in an infinitesimal region around $x = 0$. The first condition tells us that the wave function is symmetric about $x = 0$:

$$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}. \tag{31}$$

The second gives the energy of our bound state: $\epsilon = -(v_0/2)^2$.

A sketch of the wavefunction is given in Fig. 1. From the form of the wavefunction, we can estimate $\Delta x \sim 1/\sqrt{\kappa}$[2].

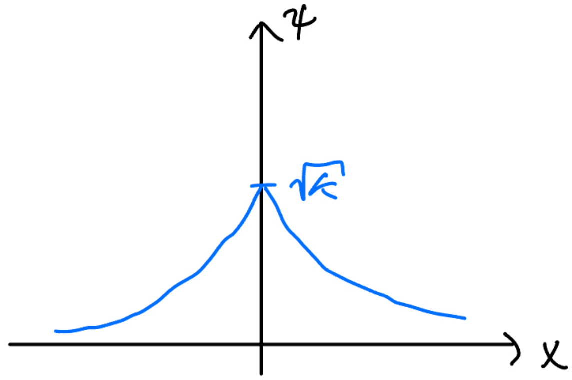

Figure 1: Sketch of bound state wave function in the potential $V(x) = -V_0\delta(x)$. Excuse the non-smooth behavior of the sketch, which demonstrates why I did not pursue any career requiring a steady hand!

(b) We now calculate the probability density of the momentum of a function in this potential by taking its Fourier transform:

$$\begin{aligned}\tilde{\psi}(p) &= \sqrt{\kappa}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} e^{-\kappa|x|} \\ &= \sqrt{\frac{\kappa}{2\pi\hbar}}\Big[\int_{-\infty}^{0} \mathrm{d}x\, e^{(\kappa - ip/\hbar)x} + \int_{0}^{\infty} \mathrm{d}x\, e^{-(\kappa+ip/\hbar)x}\Big] \\ &= \sqrt{\frac{\kappa}{2\pi\hbar}}\Big[\frac{1}{\kappa - ip/\hbar} + \frac{1}{\kappa + ip/\hbar}\Big] = \sqrt{\frac{2\kappa}{\pi\hbar}}\,\frac{\kappa}{\kappa^2 + (p/\hbar)^2}.\end{aligned} \tag{32}$$

[2]Upon first glance, it may be tempting to estimate $\Delta x \sim 1/\kappa$, but $\Delta x = \sqrt{\langle x^2\rangle} = \sqrt{[\int_{-\infty}^{\infty} e^{-|u|}\,\mathrm{d}u]/\kappa}$

The probability to find the particle with momentum in the interval $[p, p + \mathrm{d}p]$ is then given as

$$\tilde{\rho}(p) = |\tilde{\psi}(p)|^2 = \frac{2}{\pi\hbar}\frac{\kappa^3}{[\kappa^2 + (p/\hbar)^2]^2} = \frac{2}{\pi\hbar\kappa}\frac{1}{[1 + (p/\kappa\hbar)^2]^2}. \tag{33}$$

Observe that the maximum of this probability density is peaked at $p = 0$ with a value of $2/(\pi\hbar\kappa)$.

From the form of the momentum-space wavefunction, we can also estimate that $\Delta p \sim \hbar\sqrt{\kappa}$[3]. Hence, we have $\boxed{\Delta x \Delta p \sim \hbar}$.

Problem 4 – Chapter 4 # 5)
Consider a particle in the one-dimensional potential $V(x)$, such that $V(x) = \infty$ for $x < 0$ and

$$V(x) = -V_0\,\delta(x - a) \text{ for } x > 0 \tag{34}$$

where $V_0 > 0$. Determine whether this potential admits any bound states.

This potential does have the ability to admit bound states since $\min_{x\in\mathbb{R}} V(x) = -\infty$. For $E < 0$, the wave function would be of the form

$$\psi(x) = \begin{cases} 0 & x < 0 \\ Ae^{\kappa x} + Be^{-\kappa x} & 0 < x < a \\ Ce^{-\kappa x} & x > a, \end{cases} \tag{35}$$

where $\kappa^2 = 2m|E|/\hbar^2$. Note that we have used the fact that $\psi \to 0$ as $x \to \infty$ to rule out the solution $e^{\kappa x}$ in the region $x > a$.

The following boundary conditions must be respected:

$$\psi(0) = 0 \tag{36}$$

$$\psi(a^-) = \psi(a^+) \tag{37}$$

$$\psi'(a^+) - \psi'(a^-) = -\frac{2mV_0}{\hbar^2}\psi(a). \tag{38}$$

Translating this into an statement in terms of A, B, C, and κ, we have

$$A + B = 0 \tag{39}$$

$$Ae^{\kappa a} + Be^{-\kappa a} = Ce^{-\kappa a} \tag{40}$$

$$-Ce^{-\kappa a} - (Ae^{\kappa a} - Be^{-\kappa a}) = -\frac{v_0}{\kappa}Ce^{-\kappa a}, \tag{41}$$

[3] Again, this comes from the fact that $\Delta p = \sqrt{\langle p^2\rangle} = \sqrt{\hbar^2\kappa[(2/\pi)\int_{-\infty}^{\infty} u^2/(1+u^2)^2\,\mathrm{d}u]}$

where we have defined $v_0 = 2mV_0/\hbar^2$. Remember that we also have the normalization condition, which imposes an additional constraint on these constants. Thus, for a non-trivial solution $\{A, B, C\}$, Eqs. (39)–(41) must be linearly dependent.

$$\begin{vmatrix} \sinh \kappa a & -e^{-\kappa a} \\ \cosh \kappa a & (1 - v_0/\kappa)e^{-\kappa a} \end{vmatrix} = e^{-\kappa a}[(1 + v_0/\kappa) \sinh \kappa a + \cosh \kappa a] = 0 \tag{42}$$

$$\tanh \kappa a = -\frac{1}{1 - v_0/\kappa} = \frac{1}{v_0/\kappa - 1}, \tag{43}$$

where we have used $B = -A$ to reduce the dimensionality (and therefore some complexity) of the problem. Also note that we have redefined $2A \to A$.

This is a transcendental equation for κ, so we cannot solve it directly, but notice that $\kappa a > 0$. Thus, $\tanh \kappa a > 0$, meaning that this equation may only has a solution if $v_0/\kappa > 1$, and hence, a bound state only under this condition. Additionally, $\tanh \kappa a < 1$ for all κa. We thus have a second condition on v_0/κ:

$$\frac{v_0}{\kappa} < 2, \tag{44}$$

and therefore, generally a bound state exists if and only if

$$\boxed{1 < \frac{v_0}{\kappa} = \frac{V_0}{E} < 2 \Leftrightarrow \frac{V_0}{2} < E < V_0}. \tag{45}$$

The above analysis is not entirely correct. We can rewrite Eq. (43) to read

$$\tanh z = \frac{z}{z_0 - z}, \tag{46}$$

where $z = \kappa a$ and $z_0 = v_0 a$. In this case then, we have something like $f(z) = g(z)$. Clearly then, whether there is a solution depends on the value of z_0.

Let us observe the following:

1. $z > 0$ (since $\kappa, a > 0$)

2. $0 < \tanh z < 1$

3. $z/(z_0 - z) > 0$ for $z \in [0, z_0)$

4. for $z \in [0, z_0)$, the function $z/(z_0 - z)$ is monotonically increasing

5. $\tanh 0 = 0$ and $0/(z_0 - 0) = 0$

Clearly, these functions always intersect at $E = 0$, but this is not interesting for this problem since ψ is not normalizable then and hence not a bound state. We can, however, state whether or not the transcendental equation has a solution (only one if any!). Since $z/(z_0 - z)$ is monotonically increasing, for a bound state to exist, we must have

$$\frac{\mathrm{d}}{\mathrm{d}z} \tanh z|_{z=0} > \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{z_0 - z}|_{z=0} \Rightarrow z_0 > 1. \tag{47}$$

If this condition is met, though, we must resort to some numerical scheme to determine the energy of the bound state.