Problem 1)
(a) Show that the wave function at time t for a free particle of mass m can be written as

$$\Psi(\vec{\boldsymbol{r}}, t) = \int \mathrm{d}^3\vec{\boldsymbol{r}}_0\, G(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0, t - t_0)\Psi(\vec{\boldsymbol{r}}_0, t_0), \tag{1}$$

where $\Psi(\vec{\boldsymbol{r}}, t_0)$ is the wave function at the initial time t_0 and the function $G(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0, t - t_0)$, known as the free-particle Green's function, reads

$$G(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0, t - t_0) = \int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^3} e^{i[\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_0) - E_p(t-t_0)]/\hbar}, \quad E_p = \frac{p^2}{2m}. \tag{2}$$

Hint: The free-particle wave funcction can be generally written as the superposition (wave packet)

$$\Psi(\vec{\boldsymbol{r}}, t) = \int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^{3/2}} f(\vec{\boldsymbol{p}}) e^{i(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}} - E_p t)/\hbar}. \tag{3}$$

(b) Obtain the explicit expression for the Green's function. **Hint**: Use the following integral

$$\int_{-\infty}^{\infty} e^{-\alpha^2(x-\beta)^2} = \frac{\sqrt{\pi}}{\alpha}, \tag{4}$$

where α and β are generally complex numbers with $-\pi/4 < \arg\alpha < \pi/4$ for convergence.

(a) Plugging in the expressions for $G(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0, t - t_0)$ and $\Psi(\vec{\boldsymbol{r}}_0, t_0)$, we have

$$\begin{aligned}
&\int \mathrm{d}^3\vec{\boldsymbol{r}}_0\;\; G(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0, t - t_0)\Psi(\vec{\boldsymbol{r}}_0, t_0) \\
&\qquad = \int \mathrm{d}^3\vec{\boldsymbol{r}}_0 \left[\int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^3} e^{i[\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_0) - E_p(t-t_0)]/\hbar}\right]\left[\int \frac{\mathrm{d}^3\vec{\boldsymbol{q}}}{(2\pi\hbar)^{3/2}} f(\vec{\boldsymbol{q}}) e^{i(\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}_0 - E_q t_0)/\hbar}\right] \\
&\qquad = \int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^{3/2}}\, \mathrm{d}^3\vec{\boldsymbol{q}}\, \frac{\mathrm{d}^3\vec{\boldsymbol{r}}_0}{(2\pi\hbar)^3} e^{i(\vec{\boldsymbol{q}}-\vec{\boldsymbol{p}})\cdot\vec{\boldsymbol{r}}_0/\hbar} f(\vec{\boldsymbol{q}}) e^{i(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}} - E_p t)/\hbar} e^{i(E_p - E_q)t_0/\hbar} \\
&\qquad = \int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^{3/2}}\, \mathrm{d}^3\vec{\boldsymbol{q}}\, \delta^{(3)}(\vec{\boldsymbol{q}} - \vec{\boldsymbol{p}}) f(\vec{\boldsymbol{q}}) e^{i(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}} - E_p t)/\hbar} e^{i(E_p - E_q)t_0/\hbar} \\
&\qquad = \boxed{\int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^{3/2}} f(\vec{\boldsymbol{p}}) e^{i(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}} - E_p t)/\hbar} = \Psi(\vec{\boldsymbol{r}}, t)}.
\end{aligned} \tag{5}$$

(b) Notice that we can write

$$\begin{aligned} G(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0, t - t_0) &= \int \frac{\mathrm{d}^3\vec{\boldsymbol{p}}}{(2\pi\hbar)^3} e^{i[\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_0)-(\vec{\boldsymbol{p}}^2/2m)(t-t_0)]/\hbar} \\ &= G(x - x_0, t - t_0)G(y - y_0, t - t_0)G(z - z_0, t - t_0), \end{aligned} \tag{6}$$

where

$$G(x - x_0, t - t_0) = \int \frac{\mathrm{d}p_x}{2\pi\hbar} e^{i[p_x(x-x_0)-(p_x^2/2m)(t-t_0)]/\hbar} \tag{7}$$

Thus, we can solve our three-dimensional problem by splicing together three copies of the solution to a one-dimensional problem. Before moving forward, let us denote $\Delta x = x - x_0$ and $\Delta t = t - t_0$ to make the writing slightly less cumbersome (it is also slightly more illustrative). Doing the integrations, we find

$$G(x - x_0, t - t_0) = \frac{1}{2\pi\hbar} \int \mathrm{d}p_x \, e^{-i\Delta t(p_x^2 - 2m\frac{\Delta x}{\Delta t}p_x)/2m}. \tag{8}$$

Problem 2)
Show that the probability density and probability current density at position $\vec{\boldsymbol{r}}_0$ can be expressed as expectation values of the operators $\rho(\vec{\boldsymbol{r}}_0)$ and $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}_0)$, defined as

$$\rho(\vec{\boldsymbol{r}}_0) = \delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0), \quad \vec{\boldsymbol{j}}(\vec{\boldsymbol{r}}_0) = \frac{1}{2m}[\vec{\boldsymbol{p}}\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0) + \delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)\vec{\boldsymbol{p}}], \tag{9}$$

where $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{p}}$ are the position and momentum operators. Derive the expressions for these densities in both coordinate and momentum space.