Problem 1 – Chapter 3 # 4)
The time-dependent Schrödinger equation of a charged particle in an electromagnetic field reads

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{\boldsymbol{r}},t) = \left\{\frac{1}{2m}\Big[-i\hbar\vec{\boldsymbol{\nabla}} - \frac{q}{c}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)\Big]^2 + qU(\vec{\boldsymbol{r}},t)\right\}\Psi(\vec{\boldsymbol{r}},t) \tag{1}$$

where $U(\vec{\boldsymbol{r}},t)$ and $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)$ are the (real) scalar and vector potential, respectively, c is the speed of light, and q is the charge of the particle. Show that the probability density $\rho(\vec{\boldsymbol{r}},t)$ and the probability current density $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$ are given in this case by

$$\rho(\vec{\boldsymbol{r}},t) = |\Psi(\vec{\boldsymbol{r}},t)|^2 \tag{2}$$

$$\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t) = \frac{\hbar}{2mi}\Big[\Psi^*(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{\nabla}}\Psi(\vec{\boldsymbol{r}},t) - \Psi(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{\nabla}}\Psi^*(\vec{\boldsymbol{r}},t)\Big] - \frac{q}{mc}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)|\Psi(\vec{\boldsymbol{r}},t)|^2 \tag{3}$$

with

$$\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{r}},t) + \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t) = 0. \tag{4}$$

As in our previous derivations of ρ and $\vec{\boldsymbol{j}}$ we define $\boxed{\rho(\vec{\boldsymbol{r}},t) = |\Psi|^2}$, and $\vec{\boldsymbol{j}}$ such that $\frac{\partial\rho}{\partial t} = -\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{j}}$. Taking the time derivative of ρ, we have

$$\frac{\partial\rho}{\partial t} = \Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}. \tag{5}$$

Again, we get the time-derivative of the wave function from S.E. (which requires a bit more massaging than in the previous cases):

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= \frac{1}{2mi\hbar}\left[-\hbar^2\nabla^2 + \frac{i\hbar q}{c}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} + \frac{i\hbar q}{c}\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}} + \frac{q^2}{c^2}\vec{\boldsymbol{A}}^2\right]\Psi + \frac{q}{i\hbar}U\Psi \\ &= -\frac{\hbar}{2mi}\nabla^2\Psi + \frac{q}{2mc}\underbrace{\Big[\Psi\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} + \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}}\Psi\Big]}_{\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}\Psi} + \frac{q^2}{2mi\hbar c}\vec{\boldsymbol{A}}^2\Psi + \frac{q}{i\hbar}U\Psi\end{aligned} \tag{6}$$

Observe that the last two terms are purely imaginary, and therefore cancel in Eq. (5), leaving us with[1]

$$\boxed{\begin{aligned}\frac{\partial \rho}{\partial t} &= -\left\{\vec{\boldsymbol{\nabla}}\cdot\frac{\hbar}{2mi}\left[\Psi^*\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Psi^*\right] + \frac{q}{2mc}\left[\Psi^*\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\Psi^*\right]\right\} \\ &= -\vec{\boldsymbol{\nabla}}\cdot\underbrace{\left\{\left[\Psi^*\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Psi^*\right] + \frac{q}{mc}\vec{\boldsymbol{A}}|\Psi|^2\right\}}_{\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)}\end{aligned}}. \tag{7}$$

It is then manifestly obvious that

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{j}} = 0}. \tag{8}$$

Problem 2 – Chapter 3 # 8)
Consider a particle in a potential $V(\vec{\boldsymbol{r}})$ with associated wave function satisfying the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{\boldsymbol{r}},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{\boldsymbol{r}})\right]\Psi(\vec{\boldsymbol{r}},t). \tag{9}$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vec{\boldsymbol{r}}(t)\right\rangle = \int \mathrm{d}^3\vec{\boldsymbol{r}}\;\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t), \tag{10}$$

where $\langle \vec{\boldsymbol{r}}(t)\rangle$ is the average position of the particle (notation as in notes) and $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$ is the probability current density. Using the definition of $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$, show that the equation above can also be written as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vec{\boldsymbol{r}}(t)\right\rangle = \left\langle \vec{\boldsymbol{p}}(t)\right\rangle. \tag{11}$$

(b) Show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vec{\boldsymbol{p}}(t)\right\rangle &= -\left\langle \vec{\boldsymbol{\nabla}}V\right\rangle = -\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Psi^*(\vec{\boldsymbol{r}},t)[\vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}})]\Psi(\vec{\boldsymbol{r}},t) \\ &= \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Psi^*(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})\Psi(\vec{\boldsymbol{r}},t),\end{aligned} \tag{12}$$

where we have introduced the force $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$.

[1]We use the fact that $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}|\Psi|^2 = |\Psi|^2\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} + \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}}|\Psi|^2 = |\Psi|^2\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}} + \vec{\boldsymbol{A}}\cdot(\Psi^*\vec{\boldsymbol{\nabla}}\Psi + \Psi\vec{\boldsymbol{\nabla}}\Psi^*)$ and $\Psi^*\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\Psi = \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}|\Psi|^2 - \Psi\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}}\Psi^*$.

Hint: Consider, say, the x-component and, by using the Schrödinger equation and its complex conjugate, obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle p_x(t)\rangle = -\frac{\hbar^2}{2m}\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[(\nabla^2\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\nabla^2\frac{\partial\Psi}{\partial x}\right] \\ + \int \mathrm{d}^3\vec{\boldsymbol{r}}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial(V\Psi)}{\partial x}\right],\end{aligned} \tag{13}$$

where the dependence on $\vec{\boldsymbol{r}}$ and t on the r.h.s. has been suppressed for brevity. Next examine these two terms.

(c) The equation above looks like Newton's second law but for average values. As a matter of fact if $\left\langle\vec{\boldsymbol{F}}\right\rangle = \vec{\boldsymbol{F}}(\langle\vec{\boldsymbol{r}}\rangle)$, then $\langle\vec{\boldsymbol{r}}(t)\rangle$ changes in time as the position of a classical particle under the action of the force $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$. Under what condition(s) can this happen? Obtain $\langle\vec{\boldsymbol{r}}(t)\rangle$ and $\langle\vec{\boldsymbol{p}}(t)\rangle$ for a particle in a harmonic potential

$$V(\vec{\boldsymbol{r}}) = \frac{m\omega^2}{2}\vec{\boldsymbol{r}}^2. \tag{14}$$

(a) Observe that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}(t)\rangle = \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{r}},t) = -\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)}, \tag{15}$$

where we have used the continuity equation for the probability density and current density.

Using $\vec{\boldsymbol{j}} = (\hbar/2mi)[\Psi^*\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Psi^*]$, we can also rewrite Eq. (15) as

$$\boxed{m\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}(t)\rangle = \frac{1}{2}\int \mathrm{d}^3\vec{\boldsymbol{r}}\,[\Psi\vec{\boldsymbol{p}}\Psi^* + \Psi^*\vec{\boldsymbol{p}}\Psi] = \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Psi^*\vec{\boldsymbol{p}}\Psi = \langle\vec{\boldsymbol{p}}(t)\rangle}. \tag{16}$$

Note that we have used the fact that $\vec{\boldsymbol{p}}$ is hermitian to rewrite $\Psi\vec{\boldsymbol{p}}\Psi^* = \Psi^*\vec{\boldsymbol{p}}\Psi$ under the integral sign.

(b) We can do this as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{p}}(t)\rangle &= -i\hbar\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\frac{\partial}{\partial t}\Psi^*\vec{\boldsymbol{\nabla}}\Psi \\ &= -i\hbar\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[\frac{\partial\Psi^*}{\partial t}\vec{\boldsymbol{\nabla}}\Psi+\Psi^*\vec{\boldsymbol{\nabla}}\frac{\partial\Psi}{\partial t}\right] \\ &= -i\hbar\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[-\frac{1}{i\hbar}\Big(-\frac{\hbar^2}{2m}\nabla^2\Psi^*+V\Psi^*\Big)\vec{\boldsymbol{\nabla}}\Psi+\frac{1}{i\hbar}\Psi^*\vec{\boldsymbol{\nabla}}\Big(\frac{-\hbar^2}{2m}\nabla^2\Psi+V\Psi\Big)\right] \quad (17) \\ &= \int \mathrm{d}^3\vec{\boldsymbol{r}}\left\{-\frac{\hbar^2}{2m}\Big[(\nabla^2\Psi^*)(\vec{\boldsymbol{\nabla}}\Psi)-\Psi^*\vec{\boldsymbol{\nabla}}\big(\nabla^2\Psi\big)\Big]+\Big[V\Psi^*\vec{\boldsymbol{\nabla}}\Psi-\Psi^*\vec{\boldsymbol{\nabla}}(V\Psi)\Big]\right\} \\ &= \int \mathrm{d}^3\vec{\boldsymbol{r}}\ \ ?+\Big\langle-\vec{\boldsymbol{\nabla}}V\Big\rangle.\end{aligned}$$

Problem 3 – Chapter 4 # 1)
Consider the problem of a particle in an attractive δ-function potential given by

$$V(x) = -V_0\delta(x) \quad V_0 > 0. \tag{18}$$

(a) Obtain the energy and wave-function of the bound state. Sketch the wave function and provide an estimate for Δx.
(b) Calculate the probability $\mathrm{d}P(p)$ that a measurement of the momentum in this bound state will give a result included between p and $p+\mathrm{d}p$. For what value of p is this probability largest? Provide an estimate for Δp and an order of magnitude for $\Delta x\Delta p$.

Problem 4 – Chapter 4 # 5)
Consider a particle in the one-dimensional potential $V(x)$, such that $V(x)=\infty$ for $x<0$ and

$$V(x) = -V_0\,\delta(x-a) \text{ for } x>0 \tag{19}$$

where $V_0>0$. Determine whether this potential admits any bound states.