Problem 1) Consider the scattering of monochromatic photons by free electrons (Compton scattering).

(a) Assuming that the electron is initially at rest and using energy and momentum conservation, show that the shift in the photon wavelength $\Delta\lambda = \lambda_f^\gamma - \lambda_i^\gamma$ is given by

$$\Delta\lambda = 2\lambda_e \sin^2 \theta_0/2 \text{ with } \lambda_e = \frac{h}{mc}, \tag{1}$$

where m is the electron mass, c is the speed of light, and θ_0 is the angle that the momentum of the scattered photon makes with the momentum of the incident photon. Determine (under the same assumption) the magnitude and direction of the recoil momentum of the electron as a function of the incident-photon energy E_i^γ and scattering angle θ_0.

(b) Assume that the electron has an initial momentum $\vec{p}_i$ parallel to the incident-photon momentum $\vec{p}_i^\gamma$. Using energy and momentum conservation, show that the wavelength shift is given in this case by

$$\Delta\lambda = 2\lambda_i^\gamma \frac{p_i^\gamma + p_i}{E_i/c - p_i} \sin^2 \theta/2, \tag{2}$$

where λ_i^γ is the wavelength of the incident photon, θ is the angle of the scattered photon, and $E_i = c^2\sqrt{p_i^2 + (mc)^2}$ is the initial energy of the electron.

(c) Show that the result in part (b) above can be derived from the expressions in part (a), where the electron is initially at rest, by a suitable Lorentz transformation.

(a) Consider the reaction $\gamma e \to \gamma e$ in Fig. .
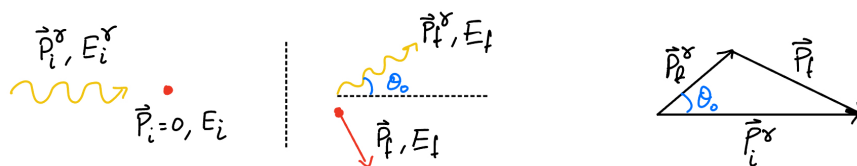

Figure 1: Sketch of the compton scattering process and momentum conservation with electron at rest in the initial state and photon deflecting at angle θ_0 relative to the beam axis.

Energy conservation gives that

$$E_i^\gamma + E_i = E_f^\gamma + E_f. \tag{3}$$

Note that the energy of the incoming/outgoing photon and electron are given as $E_{i,f}^\gamma = p_{i,f}^\gamma c$ and $E_{i,f} = c\sqrt{(mc)^2 + p_{i,f}^2}$, respectively. Using these, Eq. (3) becomes

$$p_i^\gamma c + mc^2 = p_f^\gamma c + c\sqrt{(mc)^2 + p_f^2}. \tag{4}$$

Dividing through by c, subtracting p_f^γ on both sides, and squaring, we obtain

$$(mc)^2 + p_f^2 = [(p_i^\gamma - p_f^\gamma) + mc]^2 = (p_i^\gamma - p_f^\gamma)^2 + (mc)^2 - 2(p_i^\gamma - p_f^\gamma)mc. \tag{5}$$

Now, note that momentum conservation tells us that $\vec{p}_i^\gamma = \vec{p}_f^\gamma + \vec{p}_e'$, and thus,

$$p_f^2 = (\vec{p}_i^\gamma - \vec{p}_f^\gamma)^2 = (p_i^\gamma)^2 + (p_f^\gamma)^2 - 2\vec{p}_i^\gamma \cdot \vec{p}_f^\gamma. \tag{6}$$

Using this, Eq. 5 becomes

$$-2p_i^\gamma p_f^\gamma \cos\theta_0 = -2p_i^\gamma p_f^\gamma - 2(p_i^\gamma - p_f^\gamma)mc, \tag{7}$$

and dividing through by $p_i^\gamma p_f^\gamma$ after substituting $p_{i,f}^\gamma = h/\lambda_{i,f}$ and rearranging we arrive at the desired solution:

$$\frac{1}{p_f^\gamma} - \frac{1}{p_i^\gamma} = \frac{1}{mc}(1 - \cos\theta_0)$$

$$\boxed{\Delta\lambda = \lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta_0)}. \tag{8}$$

Note that $1 - \cos\theta_0 = 1 - (\cos^2\theta_0/2 - \sin^2\theta_0/2) = 2\sin^2\theta_0/2$.

(b) If the electron is moving collinearly and in the same direction as γ before the interaction, energy conservation reads

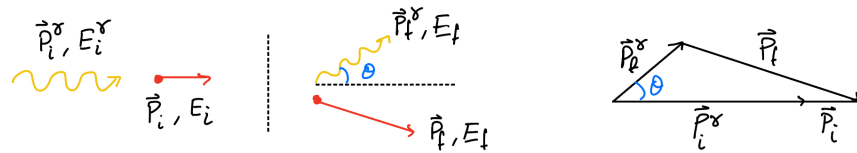$$p_i^\gamma + \frac{E_i}{c} = p_f^\gamma + \sqrt{(mc)^2 + p_f^2}. \tag{9}$$

Figure 2: Sketch of compton scattering process with electron with momentum p_i in the initial state and photon deflecting at angle θ_0 relative to the beam axis.

The manipulations are much the same since p_i is a known parameter (and also therefore

E_i).

$$(mc)^2 + p_f^2 = (p_i^\gamma - p_f^\gamma)^2 + (mc)^2 + p_i^2 + 2\left(\frac{E_i}{c}\right)(p_i^\gamma - p_f^\gamma)$$

$$[\vec{\boldsymbol{p}}_i + \vec{\boldsymbol{p}}_i^\gamma - \vec{\boldsymbol{p}}_f^\gamma]^2 = (p_i^\gamma - p_f^\gamma)^2 + p_i^2 - 2\left(\frac{E_i}{c}\right)(p_f^\gamma - p_i^\gamma)$$

$$\vec{\boldsymbol{p}}_i \cdot \vec{\boldsymbol{p}}_i^\gamma - \vec{\boldsymbol{p}}_i \cdot \vec{\boldsymbol{p}}_f^\gamma - \vec{\boldsymbol{p}}_i^\gamma \cdot \vec{\boldsymbol{p}}_f^\gamma = -p_i^\gamma p_f^\gamma - \frac{E_i}{c}(p_f^\gamma - p_i^\gamma)$$

$$p_i p_i^\gamma - p_i p_f^\gamma \cos\theta - p_i^\gamma p_f^\gamma \cos\theta = -p_i p_f^\gamma - \frac{E_i}{c}(p_f^\gamma - p_i^\gamma)$$

$$\frac{E_i}{c}\left(\frac{1}{p_f^\gamma} - \frac{1}{p_i^\gamma}\right) = \frac{p_i}{p_f^\gamma} - \frac{p_i}{p_i^\gamma} + (1 - \cos\theta) \tag{10}$$

$$\left(\frac{E_i}{c} - p_i\right)\left(\frac{1}{p_f^\gamma} - \frac{1}{p_i^\gamma}\right) = 2\left(\frac{p_i}{p_i^\gamma} + 1\right)\sin^2\theta/2$$

$$\frac{1}{p_f^\gamma} - \frac{1}{p_i^\gamma} = 2\left(\frac{1}{p_i^\gamma}\right)\frac{p_i + p_i^\gamma}{E_i/c - p_i}\sin^2\theta/2$$

$$\boxed{\Delta\lambda = 2\lambda_i^\gamma \frac{p_i + p_i^\gamma}{E_i/c - p_i}\sin^2\theta/2}\ .$$

(c) We can transform from the result Eq. (8) in part (a), where the electron is initially at rest, to the result of part (b), given by Eq. (10), via a suitable Lorentz transformation.
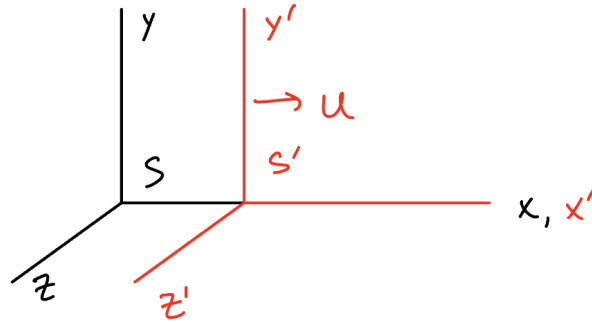

Figure 3: Skeach of Lorentz transformation between frames S and S' moving with relative velocity u.

Let S be the frame for part (b), where the electron has momentum p_i in the initial state, and S' be the frame for part (a), where the electron is at rest in the initial state. The Lorentz transformation of any 4-vector a from S' to S is given as

$$\begin{cases} ca^0 = \gamma(ca'^0 + \beta a'^x) \\ a^x = \gamma(a'^x + \beta a'^0) \\ a^{y,z} = a'^{y,z} \end{cases} \tag{11}$$

where the Lorentz factor $\gamma = [1 - \beta^2]^{-1/2}$, where β is the velocity of the electron in S (normalized to c). Note that $\gamma^2 = 1 + [p_i/(mc)]^2$

Using Eq. (11) for the energy of the photon we find

$$E_i^\gamma = \gamma(E_i'^\gamma + \beta p_{i,x}'^\gamma) = \gamma E_i'^\gamma(1 + \beta) \tag{12}$$

$$E_f^\gamma = \gamma E_f'^\gamma(1 + \beta \cos\theta_0). \tag{13}$$

Now, we can write $E_{i,f}^\gamma = cp_{i,f}^\gamma = hc/\lambda_{i,f}^\gamma$. Hence, Eqs. (12)–(13) tell us how the wavelengths of the initial and final state photon transform between the S and S'[1]:

$$\frac{1}{\lambda_i^\gamma} = \frac{\gamma}{\lambda_i'^\gamma}(1 + \beta) \tag{14}$$

$$\frac{1}{\lambda_f^\gamma} = \frac{\gamma}{\lambda_f'^\gamma}(1 + \beta \cos\theta_0). \tag{15}$$

All that remains is to transform the factor $\cos\theta_0$ from S to S'. By definition,

$$p_{f,x}'^\gamma = p_f'^\gamma \cos\theta_0 = \frac{E_f'^\gamma}{c} \cos\theta_0. \tag{16}$$

Using this equation, we can use the Lorentz transformation to find

$$\gamma(p_{f,x}^\gamma - \frac{\beta}{c}E_f^\gamma) = \frac{\gamma E_f^\gamma(1 - \beta\cos\theta)}{c} \cos\theta_0$$
$$\cos\theta_0 = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}. \tag{17}$$

Plugging the expressions for $\lambda_{i,f}^\gamma$ and $\cos\theta_0$ into Eq. (8) and allowing a CAS do the algebra we find

$$\gamma\left[\lambda_f^\gamma\left(1 + \beta\frac{\cos\theta - \beta}{1 - \beta\cos\theta}\right) - \lambda_i^\gamma(1 + \beta)\right] = \frac{h}{mc}\left(1 - \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\right)$$

$$\boxed{\Delta\lambda = \frac{p_i^\gamma + p_i}{E_i/c - p_i}\lambda_i^\gamma(1 - \cos\theta)} \tag{18}$$

as shown in part (b).

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[1] As a sanity check these should reduce to the doppler shift.

Problem 2) After the discovery of the electron by Thomson in 1897, it was believed that "atoms were like puddings, with negatively charged electrons stuck in like raisins in a smooth background of positive charge" (S. Weinberg). This picture was drastically changed by experiments performed by Rutherford and collaborators, who scattered α particles (^{4}He nuclei, which we now know consist of two protons and two neutrons bound together by the nuclear force, having electric charge $2e$) off a thin foil of gold. They observed α particles scattered at large backward angles. This was totally unexpected, since electrons are much lighter than α particles.

(a) Consider a particle of mass M and velocity v hitting a particle of mass m at rest and continuing along the same line with velocity v'. Show that, for a given v, energy and momentum conservation lead to two possible solutions for v'. If a certain condition is satisfied, one of these solutions corresponds to the case in which particle M inverts its direction of motion. What is this condition?

(b) Suppose the α particles (which were in fact emitted by a radium source in Rutherford's experiment) have velocity $v \approx 2.1 \times 10^9 \,\mathrm{cm\,s^{-1}}$, and that the target particles (much heavier than the α particles) have each charge Ze. If the α's and target particles interact via the Coulomb repulsion, what is the distance of closest approach? Show that this distance is of the order $3Z \times 10^{-14}$ cm, and therefore (even for $Z \approx 100$) much smaller than atomic radii.

Problem 3)
Consider the Rutherford model of the atom: an electron of electric charge $-e$ orbiting a point-like nucleus (much heavier, and hence at rest) of electric charge Ze in a circular orbit of radius R. Knowing that the electron radiates energy away at the rate $\mathrm{d}E\,/\,\mathrm{d}t$ given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2}{3}\frac{e^2|\vec{a}(t)|^2}{c^3}, \tag{19}$$

where $\vec{a}(t)$ is the acceleration and c is the speed of light, show that it will take a time

$$\tau = \frac{m^2c^3}{Ze^4}\frac{R^3}{4} \tag{20}$$

for the electron to spiral into the nucleus. Assume that τ is much larger than the revolution period. By taking $Z = 1$ and $R \approx 1 \times 10^{-8}$ cm appropriate for the hydrogen atom, justify this assumption *a posteriori* by comparing τ with the revolution period.

Problem 4)
In order to solve the stability problem raised in the previous problem, Niels Bohr proposed in 1913 that the atom can exist only in certain states having energies $E_1 < E_2 < \ldots$, that is, atomic energies are quantized. To obtain these energies, Bohr assumed that the angular momentum of an electron of mass m and electric charge $-e$ in a *stable circuilar orbit* of radius r around a nucleus of eletric charge Ze is an integer multiple n of the PLanck constant $\hbar = h/(2\pi)$. Following Bohr, calculate the energies E_n.

Problem 5)
In the old quantum theory, one assumes that the particles follow the laws of classical mechanics, but postulates further that, of all the solutions of the equations of motion, one must only retain those which satisfy certain *ad hoc* quantization rules. One therefore selects a discontinuous family of motions; these are, by hypothesis, the only motions which are realized in nature. The discontinuous sequence of energy values thus obtained constitutes the spectrum of quantized energy levels.

For a one dimensional periodic motion, the quantization rule – known as the Bohr-Sommerfield quantization rule – is

$$\oint_E \mathrm{d}q\, p = nh \quad n = 1, 2, \ldots, \tag{21}$$

where h is Planck's constant and the symbol $\oint_E$ means that one must integrate over a complete period of the motion corresponding to the energy E. Here q and p are the position and momentum variables, respectively. The integral is known as the action integral. Apply this rule to the case of the one-dimensional harmonic oscillator for which

$$E = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2. \tag{22}$$

Calculate the energy, period, and amplitude of the quantized trajectories.

Problem 6)
Quantize the *circular* electronic orbits of the hydrogen atom by applying the Bohr-Sommerfield rule introduced in the previous problem. Determine the energy, period, and radius of the quantized orbits. Calculate specifically the numerical values of the energy, period, and radius of the lowest orbit. Use $mc^2 \approx 0.51 \times 10^6$ eV and $\hbar c / e^2 \approx 137$.

Problem 7)
In addition to its failures in explaining the blackbody radiation spectrum, the photo-electric effect, and the stability of atoms and spectral lines, classical physics could also not explain the heat capacity of a solid.

(a) Assume a solid of volume V with N atoms (or molecules) can be modeled as a set of $3N$ independent one-dimensional harmonic oscillators of frequency ν_0 (that is, all oscillators have the same frequency ν_0). Making use of the equipartition theorem, calculate the total average energy E of the solid at temperature T and derive the Dulong-Petit law for the heat capacity (at constant volume)

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V = 3Nk_B, \tag{23}$$

where k_B is the Boltzmann's constant.

(b) The observed heat capacity of a solid is not a constant independent of T, but rather vanishes as T^3 at low temperature and approaches the classical prediction (the Dulong-Petit law) at high temperature. Einstein (1907) proposed that the energy of each harmonic oscillator is quantized and that its average energy at temperature T is given by

$$\langle E \rangle = \frac{h\nu_0}{\exp\{h\nu_0/k_BT\} - 1}, \tag{24}$$

see class notes. Show that the heat capacity is now found to be

$$c_V = 3Nk_B\frac{x_0^2 e^{x_0}}{(e^{x_0} - 1)^2}, \quad x_0 = \frac{h\nu_0}{k_BT}. \tag{25}$$

Does c_V vanish as $T \to 0$? What happens at high temperature?

(c) Einstein's theory predicts that c_V vanishes exponentially at low temperature, a result that is at variance with experimental observations. A more realistic model of a solid is that it consists of $3N$ independent harmonic oscillators (modes) with a distribution in frequency $g(\nu)$ given by

$$g(\nu) = 4\pi\frac{V}{v_S^3}\nu^2, \tag{26}$$

where v_S is the sound velocity in the solid, such that

$$\text{number of modes} = \int_0^{\nu_{\text{max}}} d\nu\, g(\nu) = 3N. \tag{27}$$

This condition fixes ν_{max} as a function of the density N/V. Introduce the parameter T_D (the Debye temperature) defined as $T_D = h\nu_{\text{max}}/k_B$ and show that the total average energy at temperature T is given by

$$E = 9Nh\nu_{\text{max}}\left(\frac{T}{T_D}\right)^4\int_0^{T_D/T} dx\, \frac{x^3}{e^x - 1}, \tag{28}$$