**Problem 1 – Chapter 3 # 4)**
The time-dependent Schrödinger equation of a charged particle in an electromagnetic field reads

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{\boldsymbol{r}},t) = \left\{\frac{1}{2m}\Big[-i\hbar\vec{\boldsymbol{\nabla}} - \frac{q}{c}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)\Big]^2 + qU(\vec{\boldsymbol{r}},t)\right\}\Psi(\vec{\boldsymbol{r}},t) \tag{1}$$

where $U(\vec{\boldsymbol{r}},t)$ and $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)$ are the (real) scalar and vector potential, respectively, c is the speed of light, and $q$ is the charge of the particle. Show that the probability density $\rho(\vec{\boldsymbol{r}},t)$ and the probability current density $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$ are given in this case by

$$\rho(\vec{\boldsymbol{r}},t) = |\Psi(\vec{\boldsymbol{r}},t)|^2 \tag{2}$$

$$\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t) = \frac{\hbar}{2mi}\Big[\Psi^*(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{\nabla}}\Psi(\vec{\boldsymbol{r}},t) - \Psi(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{\nabla}}\Psi^*(\vec{\boldsymbol{r}},t)\Big] - \frac{q}{mc}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t)|\Psi(\vec{\boldsymbol{r}},t)|^2 \tag{3}$$

with

$$\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{r}},t) + \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t) = 0. \tag{4}$$

**Problem 2 – Chapter 3 # 8)**
Consider a particle in a potential $V(\vec{\boldsymbol{r}})$ with associated wave function satisfying the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{\boldsymbol{r}},t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{\boldsymbol{r}})\Big]\Psi(\vec{\boldsymbol{r}},t). \tag{5}$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}(t)\rangle = \int \mathrm{d}^3\vec{\boldsymbol{r}}\;\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t), \tag{6}$$

where $\langle\vec{\boldsymbol{r}}(t)\rangle$ is the average position of the particle (notation as in notes) and $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$ is the probability current density. Using the definition of $\vec{\boldsymbol{j}}(\vec{\boldsymbol{r}},t)$, show that the equation above can also be written as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{r}}(t)\rangle = \langle\vec{\boldsymbol{p}}(t)\rangle. \tag{7}$$

(b) Show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\boldsymbol{p}}(t)\rangle &= -\left\langle\vec{\boldsymbol{\nabla}}V\right\rangle = -\int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Psi^*(\vec{\boldsymbol{r}},t)[\vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}})]\Psi(\vec{\boldsymbol{r}},t)\\ &= \int \mathrm{d}^3\vec{\boldsymbol{r}}\,\Psi^*(\vec{\boldsymbol{r}},t)\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})\Psi(\vec{\boldsymbol{r}},t)\end{aligned}, \tag{8}$$

where we have introduced the force $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$.
**Hint**: Consider, say, the $x$-component and, by using the Schrödinger equation and its complex conjugate, obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle p_x(t)\rangle = &-\frac{\hbar^2}{2m}\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[(\nabla^2\Psi^*)\frac{\partial\Psi}{\partial x} - \Psi^*\nabla^2\frac{\partial\Psi}{\partial x}\right] \\ &+\int \mathrm{d}^3\vec{\boldsymbol{r}}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial(V\Psi)}{\partial x}\right],\end{aligned} \tag{9}$$

where the dependence on $\vec{\boldsymbol{r}}$ and $t$ on the r.h.s. has been suppressed for brevity. Next examine these two terms.
(c) The equation above looks like Newton's second law but for average values. As a matter of fact if $\left\langle\vec{\boldsymbol{F}}\right\rangle = \vec{\boldsymbol{F}}(\langle\vec{\boldsymbol{r}}\rangle)$, then $\langle\vec{\boldsymbol{r}}(t)\rangle$ changes in time as the position of a classical particle under the action of the force $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$. Under what condition(s) can this happen? Obtain $\langle\vec{\boldsymbol{r}}(t)\rangle$ and $\langle\vec{\boldsymbol{p}}(t)\rangle$ for a particle in a harmonic potential

$$V(\vec{\boldsymbol{r}}) = \frac{m\omega^2}{2}\vec{\boldsymbol{r}}^2. \tag{10}$$

**Problem 3 – Chapter 4 # 1)**
Consider the problem of a particle in an attractive $\delta$-function potential given by

$$V(x) = -V_0\delta(x) \quad V_0 > 0. \tag{11}$$

(a) Obtain the energy and wave-function of the bound state. Sketch the wave function and provide an estimate for $\Delta x$.
(b) Calculate the probability $\mathrm{d}P(p)$ that a measurement of the momentum in this bound state will give a result included between $p$ and $p + \mathrm{d}p$. For what value of $p$ is this probability largest? Provide an estimate for $\Delta p$ and an order of magnitude for $\Delta x \Delta p$.

**Problem 4 – Chapter 4 # 5)**
Consider a particle in the one-dimensional potential $V(x)$, such that $V(x) = \infty$ for $x < 0$ and

$$V(x) = -V_0\,\delta(x - a) \text{ for } x > 0 \tag{12}$$

where $V_0 > 0$. Determine whether this potential admits any bound states.