Problem 1 – Chapter 11 # 3)
The state $|\psi\rangle$ is in the subspace spanned by the eigenstates of $\hat{\mathbf{J}}^2$ having eigenvalue $2\hbar^2$. Suppose $|\psi\rangle$ is also a normalized eigenstate of $\hat{\boldsymbol{n}} \cdot \hat{\mathbf{J}}$ with eigenvalue $+\hbar$; here, $\hat{\boldsymbol{n}}$ is the unit vector with components $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Obtain $|\psi\rangle$ as a linear combination of the eigenstates $|m\rangle$ of $\hat{J}_z$ with $m = 0, \pm 1$.

We can write

$$|\psi\rangle = \sum_{m=-1}^{1} c_m |m\rangle \tag{1}$$

since $j = 1$ and hence the range of the sum is limited to $m = -1, 0, 1$ since m can only take on values between $-j$ and j separated by one unit. From the definition of the dot product, we can write

$$\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{J}} = \sin\theta\cos\phi J_1 + \sin\theta\sin\phi J_2 + \cos\theta J_3. \tag{2}$$

We can use the fact that $|\psi\rangle$ is an eigenvector of $\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{J}}$ to write

$$(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{J}})|\psi\rangle = \hbar|\psi\rangle \Rightarrow \sum_{m=-1}^{1} c_m \Big[\sin\theta\cos\phi J_1|m\rangle + \sin\theta\sin\phi J_2|m\rangle + \hbar(m\cos\theta - 1)|m\rangle\Big] = 0. \tag{3}$$

At this point, we need to know the action of J_1 and J_2 on an eigenstate of J_3. Recall the raising and lowering operators $J_\pm = J_1 \pm iJ_2$ with $J_+|m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\,|m+1\rangle$ and $J_-|m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|m-1\rangle$. Solving for the Cartesian operators in terms of the raising and lowering operators, we have

$$\begin{aligned} J_1 &= \frac{1}{2}\Big(J_+ + J_-\Big) \\ J_2 &= \frac{1}{2i}\Big(J_+ - J_-\Big). \end{aligned} \tag{4}$$

Thus,

$$\cos\phi J_1 + \sin\phi J_2 = \frac{1}{2}\Big(e^{-i\phi}J_+ + e^{i\phi}J_-\Big). \tag{5}$$

Putting this into the sum above, we have

$$\sum_{m=-1}^{1} c_m \Big[\frac{1}{2}e^{-i\phi}\sqrt{2 - m(m+1)}\,|m+1\rangle + \frac{1}{2}e^{i\phi}\sqrt{2 - m(m-1)}\,|m-1\rangle + (m\cos\theta - 1)|m\rangle\Big] = 0. \tag{6}$$

We can expand this in full glory, which gives:

$$\begin{aligned} &c_{-1}\Big[\frac{1}{\sqrt{2}}e^{-i\phi}|0\rangle - (1+\cos\theta)|-1\rangle\Big] \\ &+ c_0\Big[\frac{1}{\sqrt{2}}e^{-i\phi}|1\rangle + \frac{1}{\sqrt{2}}e^{i\phi}|-1\rangle - |0\rangle\Big] \\ &+ c_1\Big[\frac{1}{\sqrt{2}}e^{i\phi}|0\rangle + (\cos\theta - 1)|1\rangle\Big] = 0. \end{aligned} \tag{7}$$

Grouping terms and equating the coefficients in front of the eigenstates of J_3 to zero, we find

$$-(1+\cos\theta)c_{-1} + \frac{1}{\sqrt{2}}e^{i\phi}c_0 = 0 \tag{8}$$
$$\frac{1}{\sqrt{2}}e^{-i\phi}c_{-1} - c_0 + \frac{1}{\sqrt{2}}e^{i\phi}c_1 = 0 \tag{9}$$
$$\frac{1}{\sqrt{2}}e^{-i\phi}c_0 + (\cos\theta - 1)c_1 = 0. \tag{10}$$

Notice that we have three equations and three unknowns here. Per usual, though, they are redundant (i.e. linearly dependent). Let us choose our undeterimined coefficient to be c_0, giving

$$|\psi\rangle = -c_0\Big[\frac{e^{i\phi}}{\sqrt{2}(1+\cos\theta)}\,|{-1}\rangle + |0\rangle + \frac{e^{-i\phi}}{\sqrt{2}(1-\cos\theta)}\,|1\rangle\Big]. \tag{11}$$

Finally, we fix c_0 via normalization:

$$|c_0|^2\Big[\frac{1}{2(1+\cos\theta)^2} + 1 + \frac{1}{2(1-\cos\theta)^2}\Big] = |c_0|^2\frac{1+\cos^2\theta + 2\sin^4\theta}{\sin^4\theta} = 1. \tag{12}$$

Thus,

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{1+\cos^2\theta+2\sin^4\theta}}\Big[\frac{1-\cos\theta}{\sqrt{2}}e^{i\phi}\,|{-1}\rangle + \sin^2\theta\,|0\rangle + \frac{1+\cos\theta}{\sqrt{2}}e^{-i\phi}\,|1\rangle\Big]}. \tag{13}$$

Notice that the result above is consistent with the following sanity check. If $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$, then $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{J}} = J_3$ and $|\psi\rangle = |1\rangle$, and the above formula yields this result as needed (with an irrelevant phase factor that we can simply toss away in this special case).

Problem 2 – Chapter 11 # 4)
The eigenstates of the orbital angular momentum satisfy the eigenvalue equations

$$\hat{\mathbf{L}}^2\,|\psi_{lm}\rangle = l(l+1)\hbar^2\,|\psi_{lm}\rangle\,, \quad \hat{L}_z\,|\psi_{lm}\rangle = m\hbar\,|\psi_{lm}\rangle\,, \tag{14}$$

where $l = 0, 1, 2, \ldots$ and $m = -l, \ldots, l$. In the r-representation, the corresponding wave functions are the spherical harmonics,

$$Y_{lm}(\theta,\phi) = \langle\vec{\boldsymbol{r}}|\psi_{lm}\rangle\,. \tag{15}$$

(a) Using the expression of L_z as a differential operator, solve the differential equation implied by the (second) eigenvalue equation above to show that the ϕ dependence of $Y_{lm}(\theta,\phi)$ is proportional to $e^{im\phi}$.

(b) Using the condition $L_+ |\psi_{ll}\rangle = 0$ and the fact that $Y_{ll}(\theta, \phi) = F_l(\theta)e^{il\phi}$, show that

$$Y_{l,l}(\theta, \phi) = c_l \sin^l \theta e^{il\phi}, \tag{16}$$

where c_l is a normalization factor.
(c) Assume that the orbital angular momentum could also take on half-integer values, say $l = 1/2$. Construct the "spherical harmonic $Y_{1/2,-1/2}(\theta, \phi)$" by (i) applying the lowering operator L_- to $Y_{1/2,1/2}(\theta, \phi) \propto \sin^{1/2} \theta e^{i\phi/2}$ and (ii) by solving the differential equation resulting from $L_- Y_{1/2,-1/2}(\theta, \phi) = 0$. Show that the two procedures are problematic and yield contradictory results. Thus, half-integer values cannot occur for the orbital angular momentum operator.

(a) The eigenvalue equation for the z-component of the orbital angular momentum has the form

$$L_z f(\phi) = -i\hbar \frac{\partial f(\phi)}{\partial \phi} = \hbar m f(\phi) \tag{17}$$

in coordinate space, which has the solution we all know and love:

$$f(\phi) \propto e^{im\phi}. \tag{18}$$

Note that the spherical harmonics are factorized into azimuthal and polar functions as $Y_{lm}(\theta, \phi) = c_{lm} P(\theta) f(\phi)$, where c_{lm} is a normalizing constant.

(b) For fixed l, we can construct the spherical harmonics using the properties of the raising and lowering operators, which take the form

$$L_\pm = \hbar e^{\pm i\phi} \Big[\pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \Big]. \tag{19}$$

If we apply L_+ on Y_{ll}, we get zero:

$$\hbar e^{i\phi} \Big(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \Big) P(\theta) e^{il\phi} = \hbar e^{i(l+1)\phi} \Big(\frac{\mathrm{d}P(\theta)}{\mathrm{d}\theta} - l \cot \theta P(\theta) \Big) = 0. \tag{20}$$

We thus have a first-order separable differential equation for θ, which has solution

$$P(\theta) \propto \sin^l \theta, \tag{21}$$

meaning that

$$Y_{ll}(\theta, \phi) = c_l \sin^l \theta e^{il\phi}. \tag{22}$$

(c) Suppose that $l = 1/2$, that is that l can take on half-integer values. Then

$$Y_{1/2,1/2} = c_l \sin^{1/2} \theta e^{i\phi/2}. \tag{23}$$

Thus,

$$\begin{aligned} Y_{1/2,-1/2} &\propto e^{-i\phi}\Big(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Big)\sqrt{\sin\theta}e^{i\phi/2} \\ &= e^{-i\phi/2}\Big(-\frac{1}{2}\cot\theta\sqrt{\sin\theta} - \frac{1}{2}\cot\theta\sqrt{\sin\theta}\Big) \\ &= -\cot\theta\sqrt{\sin\theta}e^{-i\phi/2}. \end{aligned} \tag{24}$$

Next, let's find the form of $Y_{1/2,-1/2}$ from $L_{-}Y_{1/2,-1/2} = 0$:

$$-\frac{\mathrm{d}P}{\mathrm{d}\theta} + \frac{1}{2}\cot\theta P = 0. \tag{25}$$

Thus we find from this that

$$Y_{1/2,-1/2} \propto \sqrt{\sin\theta}e^{-i\phi/2}, \tag{26}$$

which contradicts our findings from the method above, meaning that in fact l cannot be 1/2.

Problem 1 – Chapter 11 # 6)
Let a_r and $a_r^\dagger$ with $r = 1, 2$ be the annihilation and creation operators of a two-dimensional harmonic oscillator, satisfying

$$[a_r, a_s] = [a_r^\dagger, a_s^\dagger] = 0, \quad [a_r, a_s^\dagger] = \delta_{rs}. \tag{27}$$

We define

$$S = \frac{1}{2}\Big(a_1^\dagger a_1 + a_2^\dagger a_2\Big), \tag{28}$$

and

$$\hat{J}_1 = \frac{1}{2}\Big(a_2^\dagger a_1 + a_1^\dagger a_2\Big), \quad \hat{J}_2 = \frac{i}{2}\Big(a_2^\dagger a_1 - a_1^\dagger a_2\Big), \quad \hat{J}_3 = \frac{1}{2}\Big(a_1^\dagger a_1 - a_2^\dagger a_2\Big), \tag{29}$$

and the $\hat{J}_i$ may be considered as the Cartesian components of a certain vector operator.

(a) Show that the components of $\vec{\boldsymbol{J}}$ as defined above satisfy the commutation relations characteristic of an angular momentum up to factors of $\hbar$, that is,

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k \tag{30}$$

and that

$$\vec{\boldsymbol{J}}^2 = S(S + \mathbb{1}). \tag{31}$$

(b) Hereafter $\vec{\boldsymbol{J}}$ will be considered to be the angular momentum of the system. We denote the eigenvalues of $\vec{\boldsymbol{J}}^2$ and J_3 by $j(j+1)$ and m, respectively. Show that $\vec{\boldsymbol{J}}^2$ and

J_3 form a complete set of commuting observables, and that j may take all integral or half-integral values ≥ 0.
(c) Show that the states

$$\frac{1}{\sqrt{(j+m)!(j-m)!}}(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}\left|0,0\right\rangle \tag{32}$$

form a basis of common eigenstates of $\vec{\boldsymbol{J}}^2$ and J_3.

(a) We will determine the commutation relations via brute force:

$$\begin{aligned}
[J_1, J_2] &= \frac{i}{4}\left[\left(a_2^\dagger a_1 + a_1^\dagger a_2\right)\left(a_2^\dagger a_1 - a_1^\dagger a_2\right) - \left(a_2^\dagger a_1 - a_1^\dagger a_2\right)\left(a_2^\dagger a_1 + a_1^\dagger a_2\right)\right] \\
&= \frac{i}{2}\left[a_1^\dagger a_1 a_2 a_2^\dagger - a_2^\dagger a_2 a_1 a_1^\dagger\right] \\
&= \frac{i}{2}\left[a_1^\dagger a_1 - a_2^\dagger a_2\right] = iJ_3
\end{aligned} \tag{33}$$

$$\begin{aligned}
[J_2, J_3] &= \frac{i}{4}\left[\left(a_2^\dagger a_1 - a_1^\dagger a_2\right)\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) - \left(a_1^\dagger a_1 - a_2^\dagger a_2\right)\left(a_2^\dagger a_1 - a_1^\dagger a_2\right)\right] \\
&= \frac{i}{2}\left[a_2^\dagger a_1 + a_1^\dagger a_2 + (a_2^\dagger a_2 a_2^\dagger - a_2^\dagger a_2^\dagger a_2)a_1\right] = iJ_1
\end{aligned} \tag{34}$$

$$\begin{aligned}
[J_3, J_1] &= \frac{1}{4}\left[\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)\left(a_2^\dagger a_1 + a_1^\dagger a_2\right) - \left(a_2^\dagger a_1 + a_1^\dagger a_2\right)\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)\right] \\
&= \frac{1}{2}\left[a_1^\dagger a_2 - a_2^\dagger a_1\right] = i\frac{i}{2}\left[a_2^\dagger a_1 - a_1^\dagger a_2\right] = iJ_2.
\end{aligned} \tag{35}$$

Finally, we bake in the antisymmetric nature of the commutation relations, we can write generically $[J_i, J_j] = i\epsilon_{ijk}J_k$.

Next, we find

$$\begin{aligned}
\vec{\boldsymbol{J}}^2 &= J_1^2 + J_2^2 + J_3^2 \\
&= \frac{1}{4}\Big[a_1 a_1 a_2^\dagger a_2^\dagger + a_1^\dagger a_1^\dagger a_2 a_2 + a_1 a_1^\dagger a_2^\dagger a_2 + a_1^\dagger a_1 a_2 a_2^\dagger \\
&\quad - a_1 a_1 a_2^\dagger a_2^\dagger - a_1^\dagger a_1^\dagger a_2 a_2 + a_1 a_1^\dagger a_2^\dagger a_2 + a_1^\dagger a_1 a_2 a_2^\dagger \\
&\quad + a_1^\dagger a_1 a_1^\dagger a_1 + a_2^\dagger a_2 a_2^\dagger a_2 - 2a_1^\dagger a_1 a_2^\dagger a_2\Big] \\
&= \frac{1}{4}\left[2a_1 a_1^\dagger a_2^\dagger a_2 + 2a_1^\dagger a_1 a_2 a_2^\dagger + a_1^\dagger a_1 a_1^\dagger a_1 + a_2^\dagger a_2 a_2^\dagger a_2 - 2a_1^\dagger a_1 a_2^\dagger a_2\right] \\
&= \frac{1}{4}\left[2a_2^\dagger a_2 + 2a_1^\dagger a_1 a_2^\dagger a_2 + 2a_1^\dagger a_1 + (a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2\right] \\
&= \left[S + S^2\right] = S(S + \mathbb{1}).
\end{aligned} \tag{36}$$

(b) Let us now consider $\vec{\boldsymbol{J}}$ to be the angular momentum of the system, denoting the eigenvalue of J_3 by m, S by j and therefore $\vec{\boldsymbol{J}}^2$ by $j(j+1)$. It is clear that $\vec{\boldsymbol{J}}^2$ and J_3 form a set of commuting observables (since they obey the fundamental commutation relations of angular momentum operators sans a factor $\hbar$, which is irrelevant for the commutator $[\vec{\boldsymbol{J}}^2, J_3]$). Thus, we can find a common orthonormal eigenbasis of $\vec{\boldsymbol{J}}^2$ and J_3. The next question is whether any state of this two-dimensional harmonic oscillator can be expanded in the common eigenbasis of $\vec{\boldsymbol{J}}^2$ and J_3, or equivalently, if

$$\sum_{j,m} |j,m\rangle \langle j,m| \stackrel{?}{=} \mathbb{1}. \tag{37}$$

We know that the eigenstates of the Hamiltonians for particles 1 and 2, H_1 and H_2 respectively, form a complete set of commuting observables, or equivalently, the number operators N_1 and N_2 form a complete set of commuting observables since the eigenstates of N_1 and H_1 and also N_2 and H_2 are the same, respectively. Thus,

$$\sum_{n_1,n_2} |n_1,n_2\rangle \langle n_1,n_2| = \mathbb{1}. \tag{38}$$

Now, observe that the eigenstates $|n_1,n_2\rangle$ are also eigenstates of $\vec{\boldsymbol{J}}^2$ and J_3:

$$\begin{aligned} \vec{\boldsymbol{J}}^2 |n_1,n_2\rangle &= S(S+1)|n_1,n_2\rangle = \frac{1}{2}(n_1+n_2)\left[\frac{1}{2}(n_1+n_2)+1\right]|n_1,n_2\rangle \\ J_3|n_1,n_2\rangle &= \frac{1}{2}(n_1-n_2)|n_1,n_2\rangle . \end{aligned} \tag{39}$$

That is, $|n_1,n_2\rangle$ corresponds to $j=(n_1+n_2)/2$ and $m=(n_1-n_2)/2$. Hence, j can take on any half-integer or integral value, and m can range from $-j$ to j in integer steps, and thus the completeness statement for the states $|n_1,n_2\rangle$ is also a completeness statement for $|j,m\rangle$.

(c) From our work in part (b), we found $n_1 = j+m$ and $n_2 = j-m$. We know that

$$|n_1,n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0,0\rangle , \tag{40}$$

so

$$|n_1,n_2\rangle = \frac{1}{\sqrt{(j+m)!(j-m)!}}(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}|0,0\rangle . \tag{41}$$

These states forms a basis of common eigenstates of $\vec{\boldsymbol{J}}^2$ and J_3 since any combination of n_1 and n_2 specifies a corresponding unique combination of j and m.