Problem 1 – Chapter 9 # 1)
Prove the relation

$$[\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{L}}, \vec{\boldsymbol{V}}] = i\hbar \vec{\boldsymbol{V}} \times \hat{\boldsymbol{n}}, \tag{1}$$

where $\hat{\boldsymbol{n}}$ is a unit vector and $\vec{\boldsymbol{V}}$ is a vector operator.

We will use the Einstein summation convention for brevity in our notation while still maintaining clarity: repeated indices in products are implicitly summed over unless otherwise specified. The commutator

$$[\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{L}}, \vec{\boldsymbol{V}}] = [n_i L_i, \hat{\boldsymbol{e}}_j V_j] = n_i \hat{\boldsymbol{e}}_j [L_i, V_j] = n_i \hat{\boldsymbol{e}}_j i\hbar \epsilon_{ijk} V_k = i\hbar \hat{\boldsymbol{e}}_j \epsilon_{jki} V_k n_i = i\hbar \vec{\boldsymbol{V}} \times \hat{\boldsymbol{n}}. \tag{2}$$

Problem 2 – Chapter 9 # 4)
A particle of mass μ is under the influence of a central potential $V(r)$. Its wave function is given by

$$\psi(\vec{\boldsymbol{r}}) = (x + y + 3z) f(r). \tag{3}$$

(a) Is $\psi(\vec{\boldsymbol{r}})$ an eigenfunction of $\vec{\boldsymbol{L}}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain if $\vec{\boldsymbol{L}}^2$ is measured?
(b) What are the probabilities for the particle to be found in various m states?
(c) Suppose it is known that $\psi(\vec{\boldsymbol{r}})$ above is an energy eigenfunction with eigenvalue E. Indicate how we may determine the potential $V(r)$.

(a) We know that the eigenstates of $\vec{\boldsymbol{L}}^2$ are the spherical harmonics $Y_{lm}(\theta, \phi)$ with corresponding eigenvalue $\hbar^2 l(l+1)$. Notice that

$$\begin{aligned} x + y + 3z &= r[\sin\theta(\cos\phi + \sin\phi) + 3\cos\theta] \\ &= r\left[\frac{1-i}{2}\sin\theta e^{-i\phi} + \frac{1+i}{2}\sin\theta e^{i\phi} + 3\cos\theta\right] \\ &= r\left[(1+i)\sqrt{\frac{2\pi}{3}}Y_{1,-1} - (1-i)\sqrt{\frac{2\pi}{3}}Y_{1,1} + 2\sqrt{3\pi}Y_{1,0}\right]. \end{aligned} \tag{4}$$

We have a linear combination of spherical harmonics with $l = 1$ and $m = -1, 0, 1$. For fixed l, the different m-states are degenerate, meaning that $\psi(\vec{\boldsymbol{r}})$ is in fact an eigenstate of $\vec{\boldsymbol{L}}^2$ with eigenvalue $2\hbar^2$.

(b) If a measurement of L_z were performed, then we could obtain

$$\begin{aligned} m = -1 \leftrightarrow P(-1) &= \frac{1}{11} \\ m = 0 \leftrightarrow P(0) &= \frac{9}{11} \\ m = 1 \leftrightarrow P(1) &= \frac{1}{11}. \end{aligned} \tag{5}$$

(c) Since ψ is an energy eigenstate, we can write

$$H\psi = \frac{\vec{\boldsymbol{p}}^2}{2m}\psi + V(r)\psi = E\psi. \tag{6}$$

We know how $\vec{\boldsymbol{p}}^2$ and $\vec{\boldsymbol{L}}^2$ are related:

$$\vec{\boldsymbol{p}}^2 = -\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\vec{\boldsymbol{L}}^2}{r^2}. \tag{7}$$

Since $\psi(\vec{\boldsymbol{r}})$ is an eigenstate of $\vec{\boldsymbol{L}}^2$, we can determine the action of the derivatives in the first term on ψ and rearrange the S.E. to isolate $V(r)$.

Problem 3 – Chapter 9 # 5)
Consider a particle in three dimensions with Hamiltonian given by

$$H = \frac{\vec{\boldsymbol{p}}^2}{2m} + V(\vec{\boldsymbol{r}}). \tag{8}$$

Show that the time derivative of the expectation value of the orbital angular momentum operator $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\,\vec{\boldsymbol{L}}\,|\psi(t)\rangle = -\langle\psi(t)|\,\vec{\boldsymbol{r}} \times \vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}})\,|\psi(t)\rangle. \tag{9}$$

Does this equation have a classical counterpart?

We have already proven Ehrenfest's theorem, which states that for any operator A

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\,A\,|\psi(t)\rangle = i\hbar\,\langle\psi(t)|\,[H,A]\,|\psi(t)\rangle + \langle\psi(t)|\,\frac{\partial A}{\partial t}\,|\psi(t)\rangle. \tag{10}$$

Note that $\vec{\boldsymbol{L}}$ is a time-independent operator, so we only need to determine the first term. The commutator is a vector operator, and it is easier to look at only a single component and generalize to the vector itself. Observe that

$$[H, L_i] = \epsilon_{ijk}[H, r_j p_k] = \epsilon_{ijk}\Big\{[H, r_j]p_k + r_j[H, p_k]\Big\}. \tag{11}$$

We therefore have two commutators to determine. The first is as follows:

$$\begin{aligned}[H, r_j] &= \frac{1}{2m}[\vec{\boldsymbol{p}}^2, r_j] = \frac{1}{2m}[p_l p_m \delta_{lm}, r_j] = \frac{1}{2m}\delta_{lm}(p_l[p_m, r_j] + [p_l, r_j]p_m) \\ &= \frac{1}{2m}(-2i\hbar p_j) = -\frac{i\hbar}{m}p_j.\end{aligned} \tag{12}$$

The second commutator is as follows:

$$[H, p_k] = [V(\vec{\boldsymbol{r}}), p_k] = -p_k V(\vec{\boldsymbol{r}}) = i\hbar\partial_k V(\vec{\boldsymbol{r}}). \tag{13}$$

Putting these together, we have

$$[H, L_i] = -\frac{i\hbar}{m}\epsilon_{ijk}p_j p_k + i\hbar\epsilon_{ijk}r_j\partial_k V(\vec{\boldsymbol{r}}) = i\hbar[\vec{\boldsymbol{r}} \times \vec{\boldsymbol{\nabla}}]_i V(\vec{\boldsymbol{r}}). \tag{14}$$

Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\,\vec{\boldsymbol{L}}\,|\psi(t)\rangle = -\langle\psi(t)|\,\vec{\boldsymbol{r}} \times \vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}})\,|\psi(t)\rangle\,. \tag{15}$$

This is the quantum analogue of the classical definition of torque.

Problem 4 – Chapter 9 # 6)
Show that the following properties relating to the orbital angular momentum operator $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ are satisfied:
(a) $\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{L}} \cdot \vec{\boldsymbol{r}}$, and similarly $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{L}} \cdot \vec{\boldsymbol{p}}$, are null operators;
(b) $\vec{\boldsymbol{L}}^2 = -\vec{\boldsymbol{r}} \cdot [\vec{\boldsymbol{p}}(\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}) - \vec{\boldsymbol{p}}^2\vec{\boldsymbol{r}}]$ (pay attention to the order of the operators); next show

$$[\vec{\boldsymbol{r}}, \vec{\boldsymbol{p}}^2] = 2i\hbar\vec{\boldsymbol{p}}, \quad \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} - \vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}} = 3i\hbar, \tag{16}$$

and hence obtain

$$\vec{\boldsymbol{L}}^2 = r^2\vec{\boldsymbol{p}}^2 + i\hbar\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} - (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}})^2, \tag{17}$$

(c) By direct calculation show that in spherical coordinates

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} = -i\hbar r\frac{\partial}{\partial r}, \tag{18}$$

and using the result in part (b) above obtain

$$\vec{\boldsymbol{L}}^2 = r^2\vec{\boldsymbol{p}}^2 + \hbar^2\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}. \tag{19}$$

It is easy to see that

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{L}} = \epsilon_{ijk}r_i r_j p_k = 0 \tag{20}$$

since we can permute the indices j and k and pick up a minus sign of the same sum. Similarly,

$$\vec{\boldsymbol{L}} \cdot \vec{\boldsymbol{p}} = \epsilon_{ijk} r_j p_k p_i = \epsilon_{jki} r_j p_k p_i = 0. \tag{21}$$

Next, we find

$$\vec{\boldsymbol{L}} \cdot \vec{\boldsymbol{r}} = \epsilon_{ijk} r_j p_j r_k = \epsilon_{ijk} r_j r_k p_j + \epsilon_{ijk} r_j \underbrace{[p_j, r_k]}_{-i\hbar\delta_{jk}} = 0. \tag{22}$$

Similarly,

$$\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{L}} = 0. \tag{23}$$

(b) Notice that we can write

$$\begin{aligned}
\vec{\boldsymbol{L}}^2 &= L_i L_i = \epsilon_{ijk}\epsilon_{ilm} r_j p_k r_l p_m = [\delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}] r_j p_k r_l p_m \\
&= r_j p_k r_j p_k - r_j p_k r_k p_j \\
&= -r_j [p_k r_k p_j - p_k r_j p_k] \\
&= -r_j [p_j p_k r_k + p_k [r_k, p_j] - p_k p_k r_j - p_k [r_j, p_k]] \\
&= -r_j [p_j (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}) - \vec{\boldsymbol{p}}^2 r_j] \\
&= \vec{\boldsymbol{r}} \cdot [\vec{\boldsymbol{p}}(\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}}) - \vec{\boldsymbol{p}}^2 \vec{\boldsymbol{r}}].
\end{aligned} \tag{24}$$

Next, we have

$$[\vec{\boldsymbol{r}}, \vec{\boldsymbol{p}}^2] = \hat{\boldsymbol{e}}_i [r_i, p_j p_j] = \hat{\boldsymbol{e}}_i ([r_i, p_j] p_j + p_j [r_i, p_j]) = \hat{\boldsymbol{e}}_i [i\hbar\delta_{ij} p_j + p_j i\hbar\delta_{ij}] = 2i\hbar\vec{\boldsymbol{p}}, \tag{25}$$

and

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} - \vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{r}} = r_i p_i - p_i r_i = r_i p_i - r_i p_i + [p_i, r_i] = 3i\hbar. \tag{26}$$

From these two relations, we can write

$$\begin{aligned}
\vec{\boldsymbol{L}}^2 &= -\vec{\boldsymbol{r}} \cdot [\vec{\boldsymbol{p}}(\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}}) - 3i\hbar\vec{\boldsymbol{p}} + 2i\hbar\vec{\boldsymbol{p}} + \vec{\boldsymbol{r}}\vec{\boldsymbol{p}}^2] \\
&= r^2 \vec{\boldsymbol{p}}^2 + i\hbar \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} - (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}})^2.
\end{aligned} \tag{27}$$

(c) Observe the following

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} = -i\hbar\, \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}}. \tag{28}$$

We can write $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$. We write explictly the transformation from cartesian to spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2},\ \phi = \arctan\left(\frac{y}{x}\right),\ \theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right). \tag{29}$$

We will need the following table of derivatives for the chain rules:

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi \quad \frac{\partial r}{\partial z} = \cos\theta \tag{30}$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r} \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r} \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r} \tag{31}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta} \quad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \quad \frac{\partial \phi}{\partial z} = 0. \tag{32}$$

Note that the derivatives transform from cartesian to spherical coordinates as

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{33}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{34}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \tag{35}$$

It follows then that

$$\begin{aligned}
\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{\nabla}} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\
&= r\Bigg\{\Big[\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\Big]\frac{\partial}{\partial r} \\
&\quad + \frac{1}{r}\Big[\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta\Big]\frac{\partial}{\partial\theta} \\
&\quad + \frac{1}{r}\Big[-\sin\phi\cos\phi + \cos\phi\sin\phi\Big]\frac{\partial}{\partial\phi}\Bigg\} \\
&= r\frac{\partial}{\partial r},
\end{aligned} \tag{36}$$

so finally, we arrive at

$$\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{p}} = -i\hbar r\frac{\partial}{\partial r} \tag{37}$$

and

$$\vec{\boldsymbol{L}}^2 = r^2\vec{\boldsymbol{p}}^2 + \hbar^2 r\frac{\partial}{\partial r} + \hbar^2 r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} = r^2\vec{\boldsymbol{p}}^2 + \hbar^2\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}. \tag{38}$$