

Problem 1 – Ch. 1 #3)

Consider the Rutherford model of the atom: an electron of electric charge $-e$ orbiting a point-like nucleus (much heavier, and hence at rest) of electric charge Ze in a circular orbit of radius R . Knowing that the electron radiates energy away at the rate dE/dt given by

$$\frac{dE}{dt} = -\frac{2}{3} \frac{e^2 |\vec{a}(t)|^2}{c^3}, \quad (1)$$

where $\vec{a}(t)$ is the acceleration and c is the speed of light, show that it will take a time

$$\tau = \frac{m^2 c^3 R^3}{Ze^4} \frac{1}{4} \quad (2)$$

for the electron to spiral into the nucleus. Assume that τ is much larger than the revolution period. By taking $Z = 1$ and $R \approx 10^{-8}$ cm appropriate for the hydrogen atom, justify this assumption *a posteriori* by comparing τ with the revolution period.

The energy of this system is

$$E = \frac{1}{2}mv^2 - \frac{Ze^2}{r}, \quad (3)$$

when the electron is at a distance r from the nucleus. Since the acceleration is instantaneously centripetal¹ we can write

$$ma = m \frac{v^2}{r} = \frac{Ze^2}{r^2} \Rightarrow T = -\frac{1}{2}V. \quad (4)$$

Using this, we find

$$E = -\frac{1}{2} \frac{Ze^2}{r}. \quad (5)$$

From here, we can differentiate this expression of E giving

$$\frac{dE}{dt} = \frac{1}{2} \frac{Ze^2}{r^2} \frac{dr}{dt} = -\frac{2e^2}{3c^3} \frac{1}{m^2} \left(\frac{Ze^2}{r^2} \right)^2. \quad (6)$$

solving for dr/dt gives a differential equation for r as

$$\frac{dr}{dt} = -\frac{4e^2}{3c^3} \frac{1}{m^2} \frac{Ze^2}{r^2}. \quad (7)$$

We can solve this as follows

$$\int_R^0 r^2 dr = -\frac{R^3}{3} = -\frac{4Ze^4}{3m^2c^3} \tau. \quad (8)$$

¹This is approximately true if we assume that τ is much larger than the revolution period. Under this assumption, the energy lost per revolution is much smaller than the energy of the system, and thus, that the electron's tangential component of acceleration is very small.

And finally we rearrange to find

$$\tau = \frac{m^2 c^3}{Z e^4} \frac{R^3}{4} \approx \frac{1}{Z} \times 10^{-10} \text{ s}. \quad (9)$$

Note that a distance R from the center one orbital period is

$$T = 2\pi \sqrt{\frac{R^3 m}{Z e^2}} \approx \frac{1}{\sqrt{Z}} (4 \times 10^{-16} \text{ s}), \quad (10)$$

and hence, the proportion of one period to the time of the electron's orbital decay is given as

$$\frac{T}{\tau} \approx 4\sqrt{Z} \times 10^{-6}, \quad (11)$$

which is quite tiny for any relevant Z (including $Z = 1$ for the hydrogen atom) and at least tells us that our results are consistent with our assumptions.

The statement that the acceleration is instantaneously centripetal is always true: the force is always central. It is that the motion is instantaneously circular that allows us to write $F = mv^2/r$

Problem 2 – Ch. 1 #7)

In addition to its failures in explaining the blackbody radiation spectrum, the photo-electric effect, and the stability of atoms and spectral lines, classical physics could also not explain the heat capacity of a solid.

(a) Assume a solid of volume V with N atoms (or molecules) can be modeled as a set of $3N$ independent one-dimensional harmonic oscillators of frequency ν_0 (that is, all oscillators have the same frequency ν_0). Making use of the equipartition theorem, calculate the total average energy E of the solid at temperature T and derive the Dulong-Petit law for the heat capacity (at constant volume)

$$c_V = \left(\frac{\partial E}{\partial T} \right)_V = 3Nk_B, \quad (12)$$

where k_B is the Boltzmann's constant.

(b) The observed heat capacity of a solid is not a constant independent of T , but rather vanishes as T^3 at low temperature and approaches the classical prediction (the Dulong-Petit law) at high temperature. Einstein (1907) proposed that the energy of each harmonic oscillator is quantized and that its average energy at temperature T is

given by

$$\langle E \rangle = \frac{h\nu_0}{\exp(h\nu_0/k_B T) - 1}, \quad (13)$$

see class notes. Show that the heat capacity is now found to be

$$c_V = 3Nk_B \frac{x_0^2 e^{x_0}}{(e^{x_0} - 1)^2}, \quad x_0 = \frac{h\nu_0}{k_B T}. \quad (14)$$

Does c_V vanish as $T \rightarrow 0$? What happens at high temperature?

(c) Einstein's theory predicts that c_V vanishes exponentially at low temperature, a result that is at variance with experimental observations. A more realistic model of a solid is that it consists of $3N$ independent harmonic oscillators (modes) with a distribution in frequency $g(\nu)$ given by

$$g(\nu) = 4\pi \frac{V}{v_S^3} \nu^2, \quad (15)$$

where v_S is the sound velocity in the solid, such that

$$\text{number of modes} = \int_0^{\nu_{\max}} d\nu g(\nu) = 3N. \quad (16)$$

This condition fixes ν_{\max} as a function of the density N/V . Introduce the parameter T_D (the Debye temperature) defined as $T_D = h\nu_{\max}/k_B$ and show that the total average energy at temperature T is given by

$$E = 9N h \nu_{\max} \left(\frac{T}{T_D} \right)^4 \int_0^{T_D/T} dx \frac{x^3}{e^x - 1}, \quad (17)$$

and that the constant-volume heat capacity is proportional to T^3 for low T , in agreement with experiment. Show that it also satisfies the Dulong-Petit law at high temperatures.

(a) The Hamiltonian for a 1D harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} q^2, \quad (18)$$

where p and q are the generalized momentum and spatial variables of the oscillator and $\omega_0 = 2\pi\nu_0$ is the angular frequency of the oscillator. The equipartition theorem states that for a system of N particles with 2 quadratic degrees of freedom per particle

$$E = \sum_{\text{particles}} \sum_{\text{d.o.f.}} \frac{1}{2} k_B T = 3Nk_B T. \quad (19)$$

From this, we find

$$c_V = \left(\frac{\partial E}{\partial T} \right)_V = 3Nk_B, \quad (20)$$

which is constant with temperature.

(b) We can derive the average energy for an oscillator at temperature T using some basic results from statistical mechanics. Namely, the energy of an ensemble is given by

$$\langle E \rangle = -\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \beta} = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}, \quad (21)$$

where \mathcal{Z} is the partition function of our ensemble and defined as

$$\mathcal{Z} = \sum_i e^{-\beta E_i}, \quad (22)$$

where $\beta^{-1} = k_B T$. In our case, the energy of an oscillator with frequency ν_0 is quantized (per Einstein's postulate) as $E_n = n h \nu_0$ (for $n = 0, 1, \dots$). Putting this into the partition function definition, we find

$$\mathcal{Z} = \sum_n (e^{-\beta h \nu_0})^n = \frac{1}{1 - e^{-\beta h \nu_0}}. \quad (23)$$

Hence, the average energy of an oscillator is

$$\begin{aligned} \langle E_o \rangle &= -(1 - e^{-\beta h \nu_0}) \frac{\partial}{\partial \beta} \left[\frac{1}{1 - e^{-\beta h \nu_0}} \right] \\ &= -(1 - e^{-\beta h \nu_0}) [-(1 - e^{-\beta h \nu_0})^{-2} (h \nu_0) e^{-\beta h \nu_0}] \\ &= \frac{h \nu_0}{e^{\beta h \nu_0} - 1} \end{aligned} \quad (24)$$

which is the result quoted above.

Now, we can derive the heat capacity of a solid by setting the energy of the solid to be $E = 3N \langle E_o \rangle$. Then

$$\begin{aligned} c_V &= 3N \left(\frac{\partial \langle E \rangle}{\partial T} \right)_V = 3N \left(-k_B \beta^2 \right) \frac{\partial \langle E \rangle}{\partial \beta} \\ &= 3N k_B \frac{(\beta h \nu_0)^2 e^{\beta h \nu_0}}{(e^{\beta h \nu_0} - 1)^2}. \end{aligned} \quad (25)$$

If we rewrite our result in terms of the dimensionless variable $x_0 = \beta h \nu_0$, we have

$$c_V = 3N k_B \frac{x_0^2 e^{x_0}}{(e^{x_0} - 1)^2}. \quad (26)$$

From this expression, we can evaluate the extreme behavior of the heat capacity. First, as $T \rightarrow 0$, $x_0 \rightarrow \infty$ and $c_V \rightarrow 0$ since the term e^{2x_0} in the denominator dominates the expression at large x_0 , meaning that c_V vanishes exponentially fast at small temperatures. Additionally, for large temperatures T (and thus $x_0 \approx 0$), we find

$$c_V = 3Nk_B \left[1 - \frac{x_0^2}{12} + \mathcal{O}(x_0^4) \right] \rightarrow 3Nk_B \text{ as } T \rightarrow \infty, \quad (27)$$

recovering the Dulong-Petit law.

(c) To correct Einstein's prediction, we allow the modes to oscillate at independent frequencies $\nu \in [0, \nu_{\max}]$ with a distribution $g(\nu) = 4\pi(V/v_S^3)\nu^2$. We can solve for ν_{\max} as follows:

$$4\pi \frac{V}{v_S^3} \int_0^{\nu_{\max}} d\nu \nu^2 = \frac{4\pi}{3} \frac{V}{v_S^3} \nu_{\max}^3 = 3N \Rightarrow \nu_{\max} = \left[\frac{9}{4\pi} \left(\frac{N}{V} \right) \right]^{1/3} v_S. \quad (28)$$

The average energy of the solid is given simply as

$$E = \int_0^{\nu_{\max}} g(\nu) \frac{h\nu}{e^{\beta h\nu} - 1} d\nu. \quad (29)$$

Switching to a dimensionless integration variable $x = \beta h\nu$ we find

$$\begin{aligned} E &= \frac{9N}{\nu_{\max}^3} \int_0^{T_D/T} \left(\frac{x}{\beta h} \right)^2 \frac{1}{\beta} \frac{x}{e^x - 1} \left(\frac{1}{\beta h} \right) dx \\ &= \frac{9N}{\nu_{\max}^3 \beta^4 h^3} \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx \\ &= 9Nh\nu_{\max} \left(\frac{1}{\beta h\nu_{\max}} \right)^4 \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx, \\ &= \boxed{9Nh\nu_{\max} \left(\frac{T}{T_D} \right)^4 \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx} \end{aligned} \quad (30)$$

where we have expressed ν_{\max} in terms of T_D and $g(\nu) = 9N\nu^2/\nu_{\max}^3$. The heat capacity of a solid is then given by

$$c_V = 9Nh\nu_{\max} \frac{\partial}{\partial T} \left[\left(\frac{T}{T_D} \right)^4 \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx \right]. \quad (31)$$

Note that the temperature dependence is contained in the derivative of the factor in brackets, so for our analysis we only consider this term, which we denote \tilde{c}_V . Carrying out the arithmetic, we have

$$\begin{aligned} \tilde{c}_V &= \frac{4}{T_D} \left(\frac{T}{T_D} \right)^3 \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx + \left(\frac{T}{T_D} \right)^4 \left(-\frac{T_D}{T^2} \right) \frac{(T_D/T)^3}{e^{T_D/T} - 1} \\ &= T^3 \left[\frac{4}{T_D^4} \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx - \frac{1}{T^4 (e^{T_D/T} - 1)} \right]. \end{aligned} \quad (32)$$

If we consider $T \rightarrow 0$, then

$$\tilde{c}_V \approx T^3 \left[\frac{4}{T_D^4} \int_0^\infty \frac{x^3}{e^x - 1} dx \right], \quad (33)$$

which has the desired behavior.

At large temperatures (i.e. $T \rightarrow \infty$), we have

$$\begin{aligned} \tilde{c}_V &\approx \frac{4T^3}{T_D^4} \int_0^{T_D/T} [x^2 + \mathcal{O}(x^3)] dx - \frac{1}{T[(1 + T_D/T + \mathcal{O}(T_D^2/T^2)) - 1]}, \\ &\approx \frac{4}{3T_D} - \frac{1}{T_D} = \frac{1}{3T_D} = \frac{1}{3} \frac{k_B}{h\nu_{\max}} \end{aligned} \quad (34)$$

and therefore

$$c_V = 3Nk_B. \quad (35)$$

In retrospect, since we were only concerned here with the behavior of c_V as $T \rightarrow 0, \infty$, it is likely simpler to define $\tilde{E} = E/(9Nh\nu_{\max})$, consider the limiting behavior of $\tilde{E}(T)$ at leading order in T (or $1/T$), and let $\tilde{c}_V = (\partial_T \tilde{E})_V$. From this approach, we only have to keep track of one term instead of two, which is certainly cleaner.

Problem 3 – Ch. 2 # 2)

A beam of neutrons having energy E is incident on a linear chain of atomic nuclei. The chain is oriented perpendicularly to the beam. The nuclei are located along the chain at regular intervals separated by a distance d , which is much larger than the radius R of the nucleus. A neutron detector is positioned far away from the chain and in a direction making an angle θ with the direction of the incident beam. The mass of the neutron is $M = 1.67 \times 10^{-24}$ g.

(a) Describe qualitatively the phenomenon observed at the detector as the energy E of the incident neutrons is changed (**Hint**: think of Bragg's reflection and obtain the condition for its occurrence).

(b) The counting rate (or intensity), as a function of E , presents a resonance (peak) when $E = E_1$. Knowing that there are no other resonances for $E < E_1$, show that the separation distance d between contiguous nuclei can be determined. Calculate d when $\theta = 30^\circ$ and $E_1 = 1.3 \times 10^{-13}$ erg.

(c) At about what value of E must we be taking the nuclear size into account. Take $R \approx 10^{-13}$ cm.

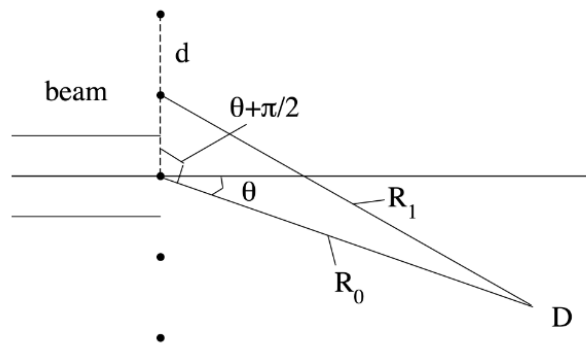


Figure 1: A beam of neutrons incident on a linear chain of nuclei.

(a) At very low energy, the wavelength of the neutrons would be very large, and hence would be almost like classical particles, meaning that the intensity distribution for small E would be quite flat. At higher energy scales, the neutrons wavelength would become comparable to the scale d of the separation distance between the atomic nuclei. Hence at this energy scale, the wave-like nature of the neutrons becomes important, as the neutrons interfere with each other creating a more rich intensity distribution from scattering off the linear chain of atomic nuclei. Similar to Bragg reflection and the two slit experiment, these interference effects can be constructive or destructive (or some intermediate between the two). If the path length $|R_1 - R_0| = n\lambda$, where λ is the wavelength of the neutron and $n \in \mathbb{Z}$, then we expect the neutron waves to be in phase and to therefore interfere constructively, and if $|R_1 - R_0| = (n + 1/2)\lambda$, then the waves destructively interfere. Since $\lambda \propto 1/\sqrt{E}$, these intensity maxima and nodes become more closely packed since $|R_1 - R_0| \approx \frac{n}{\sqrt{E}}$ for the maxima (and similarly for the nodes)².

(b) Let's make the previous discussion more quantitative. If we assume that the path along R_1 is approximately parallel to R_0 ³ then we can write $|R_1 - R_0| \approx d \sin \theta = n\lambda = nh/\sqrt{2ME}$. Thus, (taking $n = 1$) at $\theta = 30^\circ$ and $E_1 = 1.3 \times 10^{-13}$ erg, we find

$$d = \frac{2h}{\sqrt{2(1.67 \times 10^{-24} \text{ g})(1.3 \times 10^{-13} \text{ erg})}} \approx 2 \times 10^{-8} \text{ cm} . \quad (36)$$

(c) The “rule of thumb” for when the target size must be considered is that the wavelength of the incoming particles is on the same scale as the size of the target particles, which in this case is about 1 fm or 10^{-13} cm. Doing the quick estimations, we find

$$\lambda = \frac{h}{\sqrt{2ME}} \approx R \Rightarrow E \approx \frac{h}{2MR^2} \approx 1.2 \times 10^{-3} \text{ erg or } 0.7 \times 10^9 \text{ eV}. \quad (37)$$

²Actually, this rationale gives us the more “classical” expectation for small E since if $E \rightarrow 0$, then $|R_1 - R_0| \rightarrow \infty$, implying that the intensity changes slowly as a function of θ .

³This is valid since we assume $d \ll R_0, R_1$

This estimation may not be quite correct, though. Observe that the classical speed of such a particle is $v = \sqrt{2E/m} \approx 1.25c$, which is a clear violation of a fundamental postulate of relativity. In this high energy regime, we should write $E = c\sqrt{(Mc)^2 + p^2}$, which means that the relation between a relativistic particle and its energy is given by

$$E = \sqrt{1 + \frac{1}{\lambda^2} \left(\frac{h}{Mc} \right)^2} Mc^2. \quad (38)$$

Taking $\lambda \approx R$, we have the energy at which the nuclear size becomes relevant to be

$E \approx 2.4 \times 10^{-3}$ erg or 1.5×10^9 eV, which is about double the prediction using the classical formulae.

Problem 4 – Ch. 2 #5)

Consider a wave packet in three dimensions

$$\Psi(\vec{r}, t) = \int \frac{d^3\vec{k}'}{(2\pi)^{3/2}} f(\vec{k}') e^{i(\vec{k}' \cdot \vec{r} - \omega' t)}, \quad (39)$$

and assume that the dependence of ω' on \vec{k}' is not specified; in other words, assume that $\omega' = \omega'(\vec{k}')$.

(a) Suppose $f(\vec{k}') \neq 0$ only for \vec{k}' in a small region around \vec{k} . By expanding ω' in Taylor series around $\vec{k}' = \vec{k}$ up to linear terms in $\vec{k}' - \vec{k}$, show that

$$\Psi(\vec{r}, t) \approx e^{-i(\omega - \vec{k} \cdot \vec{\nabla}_{\vec{k}} \omega) t} \Psi(\vec{r} - \vec{\nabla}_{\vec{k}} \omega t, 0), \quad (40)$$

where we have defined $\omega = \omega'(\vec{k})$ and $\vec{\nabla}_{\vec{k}} \omega = \vec{\nabla}_{\vec{k}'} \omega'|_{\vec{k}' = \vec{k}}$. Consider the shape of the wave packet $|\Psi(\vec{r}, t)|$. What does the relation above imply about the change of this shape with time?

(b) From the relation obtained in part (a) deduce that the wave packet travels with the velocity – the so-called group velocity \vec{v}_{gr} – given by $\vec{\nabla}_{\vec{k}} \omega$. Identifying this velocity as the velocity $\hbar \vec{k}/m$ of the particle, where m is its mass, show that

$$\omega = \frac{\hbar k^2}{2m} + \text{constant}. \quad (41)$$

If we assume that $|f(\vec{k}')|$ is strongly localized in a neighborhood of $\vec{k}' = \vec{k}$, we can linearize ω' as

$$\omega' \approx \omega + (\vec{k}' - \vec{k}) \cdot \vec{\nabla}_{\vec{k}} \omega. \quad (42)$$

In this case, we can then write Ψ as

$$\Psi(\vec{r}, t) \approx \int \frac{d^3 \vec{k}'}{(2\pi)^{3/2}} f(\vec{k}') e^{i(\vec{k}' \cdot \vec{r} - [\omega + (\vec{k}' - \vec{k}) \cdot \vec{\nabla}_{\vec{k}} \omega] t)}. \quad (43)$$

We can then take out any exponential factors which are independent of \vec{k}' , which gives

$$\boxed{\Psi(\vec{r}, t) \approx e^{-i(\omega - \vec{k} \cdot \vec{\nabla}_{\vec{k}} \omega) t} \int \frac{d^3 \vec{k}'}{(2\pi)^{3/2}} f(\vec{k}') e^{i\vec{k}' \cdot (\vec{r} - t \vec{\nabla}_{\vec{k}} \omega)} = e^{-i(\omega - \vec{k} \cdot \vec{\nabla}_{\vec{k}} \omega) t} \Psi(\vec{r} - t \vec{\nabla}_{\vec{k}} \omega, 0)} \quad (44)}$$

Note that the factor $\vec{\nabla}_{\vec{k}} \omega = \vec{\nabla}_{\vec{k}'} \omega'|_{\vec{k}' = \vec{k}}$ is in fact independent of \vec{k}' since we make the replacement $\vec{k}' = \vec{k}$ after taking the gradient with respect to \vec{k}' .

Consider $\vec{r} = \vec{r}_0 + t \vec{\nabla}_{\vec{k}} \omega$ then at time t , $\Psi(\vec{r}, t) \approx \exp(-i[\omega - \vec{k} \cdot \vec{\nabla}_{\vec{k}} \omega]) \Psi(\vec{r}_0, 0)$. Essentially, the wavefunction changes in time by translating in space an amount $t \vec{\nabla}_{\vec{k}} \omega$ and by a global phase, although this has a norm of unity and therefore does not manifest itself in a physically meaningful way. That is, $|\Psi(\vec{r}, t)| = |\Psi(\vec{r} - t \vec{\nabla}_{\vec{k}} \omega, 0)| \approx |\Psi(\vec{r}_0, 0)|$.

(b) From part (a) we know that the wave packet travels at the group velocity

$$\vec{v}_{\text{gr}} = \frac{d}{dt} (\vec{r}_0 + t \vec{\nabla}_{\vec{k}} \omega) = \vec{\nabla}_{\vec{k}} \omega. \quad (45)$$

Intuitively, since $|\Psi|^2$ represents the spatial probability density of the particle, then it is reasonable to conclude that the group velocity, which is the velocity of the center of the wave packet, is equal to that of the particle. That is

$$\vec{v}_{\text{gr}} = \frac{\vec{p}}{m} = \frac{\hbar \vec{k}}{m} = \vec{\nabla}_{\vec{k}} \omega. \quad (46)$$

Solving this equation for ω we find

$$\boxed{\omega = \frac{\hbar k^2}{2m} + \text{constant}}, \quad (47)$$

observing that $\vec{\nabla}(r^2 + \text{constant}) = \vec{\nabla}(\vec{r} \cdot \vec{r}) = 2\vec{r} \cdot \vec{\nabla} \vec{r} = 2\vec{r}$.