

Problem 1)

(a) Show that the wave function at time t for a free particle of mass m can be written as

$$\Psi(\vec{r}, t) = \int d^3\vec{r}_0 G(\vec{r} - \vec{r}_0, t - t_0) \Psi(\vec{r}_0, t_0), \quad (1)$$

where $\Psi(\vec{r}, t_0)$ is the wave function at the initial time t_0 and the function $G(\vec{r} - \vec{r}_0, t - t_0)$, known as the free-particle Green's function, reads

$$G(\vec{r} - \vec{r}_0, t - t_0) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} e^{i[\vec{p} \cdot (\vec{r} - \vec{r}_0) - E_p(t - t_0)]/\hbar}, \quad E_p = \frac{p^2}{2m}. \quad (2)$$

Hint: The free-particle wave function can be generally written as the superposition (wave packet)

$$\Psi(\vec{r}, t) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} f(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - E_p t)/\hbar}. \quad (3)$$

(b) Obtain the explicit expression for the Green's function. **Hint:** Use the following integral

$$\int_{-\infty}^{\infty} e^{-\alpha^2(x-\beta)^2} = \frac{\sqrt{\pi}}{\alpha}, \quad (4)$$

where α and β are generally complex numbers with $-\pi/4 < \arg \alpha < \pi/4$ for convergence.

(a) Plugging in the expressions for $G(\vec{r} - \vec{r}_0, t - t_0)$ and $\Psi(\vec{r}_0, t_0)$, we have

$$\begin{aligned} & \int d^3\vec{r}_0 G(\vec{r} - \vec{r}_0, t - t_0) \Psi(\vec{r}_0, t_0) \\ &= \int d^3\vec{r}_0 \left[\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} e^{i[\vec{p} \cdot (\vec{r} - \vec{r}_0) - E_p(t - t_0)]/\hbar} \right] \left[\int \frac{d^3\vec{q}}{(2\pi\hbar)^{3/2}} f(\vec{q}) e^{i(\vec{q} \cdot \vec{r}_0 - E_q t_0)/\hbar} \right] \\ &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} d^3\vec{q} \frac{d^3\vec{r}_0}{(2\pi\hbar)^3} e^{i(\vec{q} - \vec{p}) \cdot \vec{r}_0/\hbar} f(\vec{q}) e^{i(\vec{p} \cdot \vec{r} - E_p t)/\hbar} e^{i(E_p - E_q)t_0/\hbar} \\ &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} d^3\vec{q} \delta^{(3)}(\vec{q} - \vec{p}) f(\vec{q}) e^{i(\vec{p} \cdot \vec{r} - E_p t)/\hbar} e^{i(E_p - E_q)t_0/\hbar} \\ &= \boxed{\int \frac{d^3\vec{p}}{(2\pi\hbar)^{3/2}} f(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - E_p t)/\hbar} = \Psi(\vec{r}, t)}. \end{aligned} \quad (5)$$

(b) Notice that we can write

$$\begin{aligned} G(\vec{r} - \vec{r}_0, t - t_0) &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} e^{i[\vec{p} \cdot (\vec{r} - \vec{r}_0) - (\vec{p}^2/2m)(t - t_0)]/\hbar} \\ &= G(x - x_0, t - t_0)G(y - y_0, t - t_0)G(z - z_0, t - t_0), \end{aligned} \quad (6)$$

where

$$G(x - x_0, t - t_0) = \int \frac{dp_x}{2\pi\hbar} e^{i[p_x(x - x_0) - (p_x^2/2m)(t - t_0)]/\hbar} \quad (7)$$

Thus, we can solve our three-dimensional problem by splicing together three copies of the solution to a one-dimensional problem. Before moving forward, let us denote $\Delta x = x - x_0$ and $\Delta t = t - t_0$ to make the writing slightly less cumbersome (it is also slightly more illustrative). Doing the integrations, we find

$$\begin{aligned} G(x - x_0, t - t_0) &= \frac{1}{2\pi\hbar} \int dp_x e^{-i\Delta t(p_x^2 - 2m\frac{\Delta x}{\Delta t}p_x)/2m\hbar} \\ &= \frac{1}{2\pi\hbar} e^{-i\Delta t[-(m\Delta x/\Delta t)^2]/2m\hbar} \int dp_x e^{-i\Delta t(p_x - m\Delta x/\Delta t)^2/2m\hbar} \\ &= \frac{1}{2\pi\hbar} e^{im\Delta x^2/2\hbar\Delta t} \sqrt{\frac{2\pi\hbar m}{i\Delta t}} = \sqrt{\frac{m}{2i\pi\hbar\Delta t}} e^{im\Delta x^2/2\hbar\Delta t} \end{aligned} \quad (8)$$

Splicing the results in each coordinate direction, we have

$$G(\vec{r} - \vec{r}_0, t - t_0) = \left(\frac{m}{2i\pi\hbar(t - t_0)} \right)^{3/2} \exp \left[\frac{im}{2\hbar(t - t_0)} (\vec{r} - \vec{r}_0)^2 \right]. \quad (9)$$

Problem 2)

Show that the probability density and probability current density at position \vec{r}_0 can be expressed as expectation values of the operators $\rho(\vec{r}_0)$ and $\vec{j}(\vec{r}_0)$, defined as

$$\rho(\vec{r}_0) = \delta(\vec{r} - \vec{r}_0), \quad \vec{j}(\vec{r}_0) = \frac{1}{2m} [\vec{p}\delta(\vec{r} - \vec{r}_0) + \delta(\vec{r} - \vec{r}_0)\vec{p}], \quad (10)$$

where \vec{r} and \vec{p} are the position and momentum operators. Derive the expressions for these densities in both coordinate and momentum space.