1) Provide proof that in cylindrical coordinates that the position vector can be written as $\vec{\boldsymbol{r}} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}$

We would like to change from the coordinates (x, y, z) to (ρ, ϕ, z) subject to the relations $x = \rho\cos\phi$, $y = \rho\sin\phi$, where the inverse relations are $\rho = \sqrt{x^2 + y^2}$, $\tan\phi = y/x$. Thus, to rewrite the position vector in cylindrical coordinates we need to rewrite the part dependent on $x,\ y$ in terms of $\rho,\ \phi$.

Note that we can write an arbitrary unit vector $\hat{\boldsymbol{u}}$ in a curvilinear coordinate system as

$$\hat{\boldsymbol{u}} = \frac{1}{h_u}\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}u}, \tag{1}$$

where $h_u = \left|\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}u}\right|$. Hence,

$$\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}\rho} = \frac{\mathrm{d}x}{\mathrm{d}\rho}\hat{\boldsymbol{x}} + \frac{\mathrm{d}y}{\mathrm{d}\rho}\hat{\boldsymbol{y}} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}, \tag{2}$$

which is incidentally already a unit vector. Repeating the process for ϕ we find

$$\frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}\phi} = -\rho\sin\phi\hat{\boldsymbol{x}} + \rho\cos\phi\hat{\boldsymbol{y}} \Rightarrow \hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}. \tag{3}$$

Now, we can take dot products to obtain $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ in terms of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$:

$$\hat{\boldsymbol{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}} \tag{4}$$

$$\hat{\boldsymbol{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}. \tag{5}$$

Observe that the vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ coincide with the new unit vectors in a system rotated by angle ϕ about the z-axis.

Finally, we can rewrite the position vector:

$$\begin{aligned}\vec{\boldsymbol{r}} &= \rho\cos\phi(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}) + \rho\sin\phi(\sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}) + z\hat{\boldsymbol{z}} \\ &= \rho(\cos^2\phi + \sin^2\phi)\hat{\boldsymbol{\rho}} + \rho(-\cos\phi\sin\phi + \sin\phi\cos\phi)\hat{\boldsymbol{\phi}} + z\hat{\boldsymbol{z}} \\ &= \boxed{\rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}}.\end{aligned} \tag{6}$$