

3.2) Show that the Fourier series

$$f(\rho, \theta) = \sum_{n=-\infty}^{\infty} \left(\frac{\rho}{a}\right)^{|n|} \left[\int_0^{2\pi} h(\theta') \frac{e^{-in\theta'}}{\sqrt{2\pi}} d\theta' \right] \frac{e^{in\theta}}{\sqrt{2\pi}} \quad (1)$$

obeys Laplace's equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} = 0 \quad (2)$$

and respects the boundary condition $f(a, \theta) = h(\theta)$.

→ Note that Laplace's equation can be rewritten to read

$$\rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial^2 f}{\partial \theta^2} = 0. \quad (3)$$

For part differentiating with respect to θ we can write simply:

$$\frac{\partial^2 f}{\partial \theta^2} = -n^2 f(\rho, \theta). \quad (4)$$

Next, for the part with respect to ρ we have

$$\rho \frac{\partial f}{\partial \rho} = |n| f(\rho, \theta) \quad (5)$$

$$\rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) = |n|^2 f(\rho, \theta) = n^2 f(\rho, \theta). \quad (6)$$

Thus, we come to the conclusion that

$$\rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial^2 f}{\partial \theta^2} = n^2 f(\rho, \theta) - n^2 f(\rho, \theta) = 0. \quad (7)$$

Finally, we check the boundary condition as follows:

$$f(a, \theta) = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\sqrt{2\pi}} \int_0^{2\pi} h(\theta') \frac{e^{-in\theta'}}{\sqrt{2\pi}} d\theta' = h(\theta), \quad (8)$$

noting that the integral is just the coefficient h_n in the Fourier series of $h(\theta)$.

3.3) Find the forms that

$$\begin{aligned} \langle g|f \rangle &= \sum_{n=-\infty}^{\infty} \langle g|n \rangle \langle n|f \rangle = \sum_{n=-\infty}^{\infty} g_n^* f_n \\ &= \int_0^{2\pi} \langle g|x \rangle \langle x|f \rangle dx = \int_0^{2\pi} g^*(x) f(x) dx \end{aligned} \quad (9)$$

and

$$\langle f|f \rangle = \sum_{n=-\infty}^{\infty} |\langle n|f \rangle|^2 = \sum_{n=-\infty}^{\infty} |f_n|^2 = \int_0^{2\pi} |\langle x|f \rangle|^2 dx = \int_0^{2\pi} |f(x)|^2 dx \quad (10)$$

for the inner products $\langle g|f \rangle$ and $\langle f|f \rangle$ take when one uses the asymmetrical notations

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx} \quad \text{and} \quad d_n = \frac{1}{2\pi} \int_0^{2\pi} dx e^{-inx} f(x) \quad (11)$$

and

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \quad (12)$$

3.8)

a) Show that the Fourier series for the function $|x|$ on the interval $[-\pi, \pi]$ is

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}. \quad (13)$$

Since $|x|$ is a odd real-valued function we can write

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad (14)$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \pi \quad (15)$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2 \cos(\pi n) - 1}{\pi n^2} = \begin{cases} 0 & n \equiv 0 \pmod{2} \\ -\frac{4}{\pi n^2} & n \equiv 1 \pmod{2} \end{cases} \quad (16)$$

for $n > 1$. Hence,

$$\boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}}. \quad (17)$$

b) Use this result to find a neat formula for $\pi^2/8$.

If we plug in $x = 0$, we find

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}. \quad (18)$$

3.16) Suppose we wish to approximate the real square-integrable function $f(x)$ by the Fourier series with N terms

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx). \quad (19)$$

Then the error

$$E_N = \int_0^{2\pi} [f(x) - f_N(x)]^2 dx. \quad (20)$$

will depend upon the $2N+1$ coefficients a_n and b_n . The best coefficients minimize this error and satisfy the conditions

$$\frac{\partial E_N}{\partial a_n} = \frac{\partial E_N}{\partial b_n} = 0. \quad (21)$$

By using these conditions, find the best coefficients.

For the derivative of E_N with respect to any parameter, say η ,

$$\frac{\partial E_N}{\partial \eta} = -2 \int_0^{2\pi} [f(x) - f_N(x)] \frac{\partial f_N}{\partial \eta} dx = 0. \quad (22)$$

Hence,

$$\begin{aligned} \frac{\partial E_N}{\partial a_0} &= -2 \int_0^{2\pi} \frac{1}{2} [f(x) - f_N(x)] dx = 0 \\ \frac{\partial E_N}{\partial a_n} &= -2 \int_0^{2\pi} \cos nx [f(x) - f_N(x)] dx = 0. \\ \frac{\partial E_N}{\partial b_n} &= -2 \int_0^{2\pi} \sin nx [f(x) - f_N(x)] dx = 0 \end{aligned} \quad (23)$$

3.19) Use the commutation relation $[p, q] = i\hbar$ to show that the annihilation and creation operators satisfy the commutation relation $[a, a^\dagger] = 1$.

The creation and annihilation operators are defined as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(q + i \frac{p}{m\omega} \right) \quad (24)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(q - i \frac{p}{m\omega} \right) \quad (25)$$

$$. \quad (26)$$

Thus, the commutator

$$\begin{aligned}
 [a, a^\dagger] &= aa^\dagger - a^\dagger a = \frac{m\omega}{2\hbar} \left[\left(q + i\frac{p}{m\omega} \right) \left(q - i\frac{p}{m\omega} \right) - \left(q - i\frac{p}{m\omega} \right) \left(q + i\frac{p}{m\omega} \right) \right] \\
 &= \frac{m\omega}{2\hbar} \left[q^2 - \frac{i}{m\omega} qp + \frac{i}{m\omega} pq + \frac{p^2}{m^2\omega^2} - q^2 - \frac{i}{m\omega} qp + \frac{i}{m\omega} pq - \frac{p^2}{m^2\omega^2} \right] \\
 &= \frac{1}{i\hbar} [q, p] = \boxed{1}.
 \end{aligned} \tag{27}$$

3.25)

a) Find the Fourier series for the function $f(x) = x^2$ on the interval $[-\pi, \pi]$.

Note that x^2 is an odd real-valued function so we can write

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{28}$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^3}{3} \tag{29}$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{4 \cos(\pi n)}{n^2} = \frac{4}{n^2} (-1)^n. \tag{30}$$

Thus,

$$x^2 = \frac{\pi^3}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}. \tag{31}$$

b) Use your result at $x = \pi$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{32}$$

which is the value of Riemann's zeta function $\zeta(x)$ at $x = 2$

If we plug in $x = \pi$ to our Fourier expansion above, we find

$$\begin{aligned}
 \pi^2 &= \frac{\pi^3}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\pi)}{n^2} = \frac{\pi^3}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \\
 &\quad \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}.
 \end{aligned} \tag{33}$$