1.2) Derive the cyclicity (1.24) of the trace from Eq. (1.23).

Using the definition of the trace, we can write

$$\mathrm{Tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} A_{k\ell} B_{\ell k}. \tag{1}$$

Note that this sum is symmetric in the indices $\{k, \ell\}$, and since $A_{k\ell}$ and $B_{\ell k}$ are just scalars

$$\boxed{\mathrm{Tr}(AB) = \sum_{\ell=1}^{n} \sum_{k=1}^{n} B_{\ell k} A_{k\ell} = \mathrm{Tr}(BA)}. \tag{2}$$

1.3) Show that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$, which is Eq. (1.26).

Recall the definition of a matrix transpose: $(A^{\mathrm{T}})_{ij} = A_{ji}$, so

$$(AB)^{\mathrm{T}}_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1} B^{\mathrm{T}}_{ik} A^{\mathrm{T}}_{kj} = (B^{\mathrm{T}} A^{\mathrm{T}})_{ij}. \tag{3}$$

Hence,

$$\boxed{(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}}. \tag{4}$$

1.5) Show that $(AB)^\dagger = B^\dagger A^\dagger$, which is Eq. (1.29).

Recall that the hermitian adjoint of a matrix is just $A^\dagger = (A^{\mathrm{T}})^*$, so

$$(AB)^\dagger = ((AB)^{\mathrm{T}})^* = (B^{\mathrm{T}} A^{\mathrm{T}})^*. \tag{5}$$

Observe that

$$(AB)^*_{ij} = \left(\sum_{k=1}^{n} A_{ik} B_{kj} \right)^* = \sum_{k=1}^{n} A^*_{ik} B^*_{kj} = (A^* B^*)_{ij}, \tag{6}$$

so Eq. (5) becomes

$$\boxed{(AB)^\dagger = (B^{\mathrm{T}} A^{\mathrm{T}})^* = (B^{\mathrm{T}})^* (A^{\mathrm{T}})^* = B^\dagger A^\dagger}. \tag{7}$$

1.7) Show that the two 4×4 matrices (1.46) satisfy Grassman's algebra (1.11) for $n = 2$.

The matrices in Eq. (1.46) are given by

$$\theta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Using the kronecker delta, notice that we can write the components of θ_1 and θ_2 as follows:

$$(\theta_1)_{ij} = \delta_{i1}\delta_{j3} - \delta_{i2}\delta_{j4} \tag{9}$$
$$(\theta_2)_{ij} = \delta_{i1}\delta_{j2} + \delta_{i3}\delta_{j4}. \tag{10}$$

Thus,

$$\begin{aligned} (\theta_1\theta_2)_{ij} &= \sum_{k=1}^{n} (\theta_1)_{ik}(\theta_2)_{kj} && (11)\\ &= \sum_{k=1}^{n} \delta_{i1}\delta_{j2}\delta_{k3}\delta_{k1} + \delta_{i1}\delta_{j4}\delta_{k3}\delta_{k3} - \delta_{i2}\delta_{j2}\delta_{k4}\delta_{k1} - \delta_{i2}\delta_{j4}\delta_{k4}\delta_{k3} && (12)\\ &= \delta_{i1}\delta_{j4}. && (13) \end{aligned}$$

In the same way,

$$\begin{aligned} (\theta_2\theta_1)_{ij} &= \sum_{k=1}^{n} (\theta_2)_{ik}(\theta_1)_{kj} && (14)\\ &= \sum_{k=1}^{n} \delta_{i1}\delta_{j3}\delta_{k2}\delta_{k1} - \delta_{i1}\delta_{j4}\delta_{k2}\delta_{k2} + \delta_{i3}\delta_{j3}\delta_{k4}\delta_{k1} - \delta_{i3}\delta_{j4}\delta_{k4}\delta_{k2} && (15)\\ &= -\delta_{i1}\delta_{j4}. && (16) \end{aligned}$$

Thus, since addition is defined component wise, we have the anticommutator

$$(\theta_1, \theta_2)_{ij} = (\theta_1)_{ij}, (\theta_2)_{ij} = \delta_{i_1}\delta_{j_4} + (-\delta_{i1}\delta_{j4}) = 0 \tag{17}$$
$$\Rightarrow \boxed{\{\theta_1, \theta_2\} = 0}. \tag{18}$$

1.12) Show that the Minkowski product $(x, y) = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}} - x^0 y^0$ of two 4-vectors x and y is an inner product obeying the rules (1.78,1.79,1.84).

The three properties we check for this inner product are as follows:

$$(f, g) = (g, f)^* \tag{19a}$$

$$(f, zg + wh) = z(f, g) + w(f, h) \tag{19b}$$

$$(f, g) = 0 \text{ for all } f \in V \Rightarrow g = 0. \tag{19c}$$

For the following, we assume that x, y are real 4-vectors. That is, their components are real numbers.

From this assumption, Eq. (19a) follows trivally

$$\boxed{(x, y) = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} - x^0 y^0 = \vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{x}} - y^0 x^0 = (y, x)}. \tag{20}$$

Additionally, Eq. (19b) is satisfied as follows:

$$\begin{aligned}(x, \alpha y + \beta z) &= \vec{\boldsymbol{x}} \cdot (\alpha \vec{\boldsymbol{y}} + \beta \vec{\boldsymbol{z}}) - x^0(\alpha y^0 + \beta z^0) \\ &= \alpha \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} + \beta \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{z}} - \alpha x^0 y^0 - \beta x^0 z^0 \\ &= \alpha \left(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} - x^0 y^0\right) + \beta \left(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{z}} - x^0 z^0\right) \\ &= \boxed{\alpha(x, y) + \beta(x, z)}. \end{aligned} \tag{21}$$

Finally, we prove Eq. (19c) as follows. Suppose that $(x, y) = 0$ for all x but $y \neq 0$. Then, there is some component of y, say y^i such that $y^i \neq 0$. Obviously then, we could choose $x^i = 1$ and the rest of the components of x to be zero, meaning that

$$(x, y) = \begin{cases} y^i & \text{if } i \neq 0 \\ y^i & \text{if } i = 0 \end{cases}. \tag{22}$$

That is, it must be the case that $y \equiv 0$, otherwise there is always at least one choice of x such that (x, y) is nonzero, which contradicts the original assumption.

1.18) Use the Gram-Schmidt method to find orthonormal linear combinations of the three vectors

$$\vec{\mathbf{s}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{s}}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{s}}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{23}$$

Notice that $\hat{\boldsymbol{u}}_1 = \vec{\boldsymbol{s}}_1$ is already normalized, so we can find an orthonormal vector to $\hat{\boldsymbol{u}}_2$ from $\vec{\boldsymbol{s}}_2$ as

$$\vec{\boldsymbol{u}}_2 = \hat{\boldsymbol{s}}_2 - (\hat{\boldsymbol{u}}_1, \vec{\boldsymbol{s}}_2)\hat{\boldsymbol{u}}_1 \tag{24}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{25}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{26}$$

Noting that $\vec{\boldsymbol{u}}_2$ is normalized, we have $\hat{\boldsymbol{u}}_2 = \vec{\boldsymbol{u}}_2$. Lastly, we can repeat the process for the third vector.

$$\begin{aligned}
\vec{\boldsymbol{u}}_3 &= \vec{\boldsymbol{s}}_3 - (\hat{\boldsymbol{u}}_1, \vec{\boldsymbol{s}}_3)\hat{\boldsymbol{u}}_1 - (\hat{\boldsymbol{u}}_2, \vec{\boldsymbol{s}}_3)\hat{\boldsymbol{u}}_2 &\quad (27)\\
&= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &\quad (28)\\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. &\quad (29)
\end{aligned}$$

Again, $\vec{\boldsymbol{u}}_3 = \hat{\boldsymbol{u}}_3$ since $\vec{\boldsymbol{u}}_3$ is already normalized. Notice that these are just the standard basis vectors for a three dimensional vector space, so the orthonormal linear combinations of $\vec{\boldsymbol{s}}_1$, $\vec{\boldsymbol{s}}_2$, and $\vec{\boldsymbol{s}}_3$ are

$$\boxed{\hat{\boldsymbol{u}}_1 = \vec{\boldsymbol{s}}_1, \qquad \hat{\boldsymbol{u}}_2 = \vec{\boldsymbol{s}}_1 - \vec{\boldsymbol{s}}_2, \qquad \hat{\boldsymbol{u}}_3 = \vec{\boldsymbol{s}}_1 - \vec{\boldsymbol{s}}_2 - \vec{\boldsymbol{s}}_3}. \tag{30}$$

1.21) Show that a linear operator A that is represented by a hermitian matrix (1.167) in an orthonormal basis satisfies $(g, Af) = (Ag, f)$.

Observe that

$$(g, Af) = g^\dagger A f, \tag{31}$$

where g, f are column vectors and A is represented by a matrix. Recalling that $(AB)^\dagger = B^\dagger A^\dagger$ and that $A^\dagger = A$ since A is hermitian, Eq. (31) becomes

$$\boxed{(g, Af) = (A^\dagger g)^\dagger f = (Ag)^\dagger f = (Ag, f)}. \tag{32}$$