3.2) Show that the Fourier series

$$f(\rho,\theta) = \sum_{n=-\infty}^{\infty} \left(\frac{\rho}{a}\right)^{|n|} \left[\int_0^{2\pi} h(\theta') \frac{e^{-in\theta'}}{\sqrt{2\pi}} \, \mathrm{d}\theta' \right] \frac{e^{in\theta}}{\sqrt{2\pi}} \tag{1}$$

obeys Laplace's equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} = 0 \tag{2}$$

and respects the boundary condition $f(a,\theta) = h(\theta)$.

→ Note that Laplace's equation can be rewritten to read

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial\theta^2} = 0. \tag{3}$$

For part differentiating with respect to θ we can write simply:

$$\frac{\partial^2 f}{\partial\theta^2} = -n^2 f(\rho,\theta). \tag{4}$$

Next, for the part with respect to ρ we have

$$\rho\frac{\partial f}{\partial\rho} = |n| f(\rho,\theta) \tag{5}$$

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) = |n|^2 f(\rho,\theta) = n^2 f(\rho,\theta). \tag{6}$$

Thus, we come to the conclusion that

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial\theta^2} = n^2 f(\rho,\theta) - n^2 f(\rho,\theta) = 0. \tag{7}$$

Finally, we check the boundary condition as follows:

$$f(a,\theta) = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\sqrt{2\pi}} \int_0^{2\pi} h(\theta') \frac{e^{-in\theta'}}{\sqrt{2\pi}} \, \mathrm{d}\theta' = h(\theta), \tag{8}$$

noting that the integral is just the coefficient h_n in the Fourier series of $h(\theta)$.

3.3) Find the forms that

$$\begin{aligned} \langle g|f\rangle &= \sum_{n=-\infty}^{\infty} \langle g|n\rangle \langle n|f\rangle = \sum_{n=-\infty}^{\infty} g_n^* f_n \\ &= \int_0^{2\pi} \langle g|x\rangle \langle x|f\rangle \, \mathrm{d}x = \int_0^{2\pi} g^*(x) f(x) \, \mathrm{d}x \end{aligned} \tag{9}$$

and

$$\langle f|f\rangle = \sum_{n=-\infty}^{\infty} |\langle n|f\rangle|^2 = \sum_{n=-\infty}^{\infty} |f_n|^2 = \int_0^{2\pi} |\langle x|f\rangle|^2 \,\mathrm{d}x = \int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x \tag{10}$$

for the inner products $\langle g|f\rangle$ and $\langle f|f\rangle$ take when one uses the asymmetrical notations

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx} \quad \text{and} \quad d_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, e^{-inx} f(x) \tag{11}$$

and

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} \,\mathrm{d}x\,. \tag{12}$$

For the first asymmetric convention $\mathbb{1} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, |x\rangle\langle x|$ and $\mathbb{1} = \sum_n |n\rangle\langle n|$, where $\langle x|n\rangle = e^{inx}$, so

$$\begin{aligned}\langle g|f\rangle &= \sum_{n=-\infty}^{\infty} \langle g|n\rangle\langle n|f\rangle = \sum_{n=-\infty}^{\infty} g_n^* f_n & (13)\\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle g|x\rangle\langle x|f\rangle \,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} g^*(x) f(x)\,\mathrm{d}x\,, & (14)\end{aligned}$$

where f_n and g_n are the coefficients as defined in Eq. (11). Hence

$$\langle f|f\rangle = \sum_{n=-\infty}^{\infty} |f_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x\,. \tag{15}$$

For the second asymmetric convention $\mathbb{1} = 2\pi\int_0^{2\pi} \mathrm{d}x\, |x\rangle\langle x|$ and $\mathbb{1} = \sum_n |n\rangle\langle n|$, where $\langle x|n\rangle = e^{inx}/2\pi$, so

$$\begin{aligned}\langle g|f\rangle &= \sum_{n=-\infty}^{\infty} \langle g|n\rangle\langle n|f\rangle = \sum_{n=-\infty}^{\infty} g_n^* f_n & (16)\\ &= 2\pi\int_0^{2\pi} \langle g|x\rangle\langle x|f\rangle \,\mathrm{d}x = 2\pi\int_0^{2\pi} g^*(x) f(x)\,\mathrm{d}x\,, & (17)\end{aligned}$$

where f_n and g_n are the coefficients as defined in Eq. (12). Hence

$$\langle f|f\rangle = \sum_{n=-\infty}^{\infty} |f_n|^2 = 2\pi\int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x\,. \tag{18}$$

3.8)

a) Show that the Fourier series for the function $|x|$ on the interval $[-\pi, \pi]$ is

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos{(2n+1)x}}{(2n+1)^2}. \tag{19}$$

Since $|x|$ is a odd real-valued function we can write

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{20}$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \,\mathrm{d}x = \pi \tag{21}$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx \,\mathrm{d}x = \frac{2}{\pi}\frac{\cos{(\pi n)} - 1}{n^2} = \begin{cases} 0 & n \equiv 0 \pmod 2 \\ -\frac{4}{\pi}\frac{1}{n^2} & n \equiv 1 \pmod 2 \end{cases} \tag{22}$$

for $n > 1$. Hence,

$$\boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos{(2n+1)x}}{(2n+1)^2}}. \tag{23}$$

b) Use this result to find a neat formula for $\pi^2/8$.

If we plug in $x = 0$, we find

$$\boxed{0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}}. \tag{24}$$

3.16) Suppose we wish to approximate the real square-integrable function $f(x)$ by the Fourier series with N terms

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx). \tag{25}$$

Then the error

$$E_N = \int_0^{2\pi}\big[f(x) - f_N(x)\big]^2 \,\mathrm{d}x\,. \tag{26}$$

will depend upon the $2N+1$ coefficients a_n and b_n. The best coefficients minimize this error and satisfy the conditions

$$\frac{\partial E_N}{\partial a_n} = \frac{\partial E_N}{\partial b_n} = 0. \tag{27}$$

By using these conditions, find the best coefficients.

For the derivative of E_N with respect to any parameter, say η,

$$\frac{\partial E_N}{\partial \eta} = -2\int_0^{2\pi} \big[f(x) - f_N(x)\big]\frac{\partial f_N}{\partial \eta}\,\mathrm{d}x = 0. \tag{28}$$

Hence,

$$\begin{aligned}
\frac{\partial E_N}{\partial a_0} &= -2\int_0^{2\pi} \frac{1}{2}\big[f(x) - f_N(x)\big]\,\mathrm{d}x = 0 \\
\frac{\partial E_N}{\partial a_n} &= -2\int_0^{2\pi} \cos nx\big[f(x) - f_N(x)\big]\,\mathrm{d}x = 0. \\
\frac{\partial E_N}{\partial b_n} &= -2\int_0^{2\pi} \sin nx\big[f(x) - f_N(x)\big]\,\mathrm{d}x = 0
\end{aligned} \tag{29}$$

For the following, it will be useful to recall the following properties:

$$\int_0^{2\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = 0 \tag{30}$$

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,\mathrm{d}x = \pi\delta_{nm} \tag{31}$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,\mathrm{d}x = \pi\delta_{nm}, \tag{32}$$

where $n, m \geq 1$. Plugging in the definition of $f_N(x)$ for the first equation, we have

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{j=1}^{N}\big[a_j \cos jx + b_j \sin jx\big]\right) = \pi a_0 \tag{33}$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x}. \tag{34}$$

For the a_n $(n > 1)$ equations we have

$$\int_0^{2\pi} \cos nx f(x)\,\mathrm{d}x = \int_0^{2\pi} \cos nx\Big(\frac{a_0}{2} + \sum_{j=1}^{N}\big[a_j \cos jx + b_j \sin jx\big]\Big) \tag{35}$$

$$= \sum_{j=1}^{N}\left[a_j \int_0^{2\pi} \cos nx \sin jx\,\mathrm{d}x + b_j \int_0^{2\pi} \cos nx \sin jx\,\mathrm{d}x\right] \tag{36}$$

$$= \sum_{j=1}^{N} \pi a_j \delta_{nj} = \pi a_n \tag{37}$$

$$\Rightarrow \boxed{a_n = \int_0^{2\pi} \cos nx f(x)\,\mathrm{d}x}. \tag{38}$$

By a similar line of reasoning we have

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} \sin nx f(x)\,\mathrm{d}x}. \tag{39}$$

3.19) Use the commutation relation $[q, p] = i\hbar$ to show that the annihilation and creation operators satisfy the commutation relation $[a, a^\dagger] = 1$.

The creation and annihilation operators are defined as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\Big(q + i\frac{p}{m\omega}\Big) \tag{40}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\Big(q - i\frac{p}{m\omega}\Big). \tag{41}$$

Thus, the commutator

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = \frac{m\omega}{2\hbar}\left[\Big(q + i\frac{p}{m\omega}\Big)\Big(q - i\frac{p}{m\omega}\Big) - \Big(q - i\frac{p}{m\omega}\Big)\Big(q + i\frac{p}{m\omega}\Big)\right]$$

$$= \frac{m\omega}{2\hbar}\left[q^2 - \frac{i}{m\omega}qp + \frac{i}{m\omega}pq + \frac{p^2}{m^2\omega^2} - q^2 - \frac{i}{m\omega}qp + \frac{i}{m\omega}pq - \frac{p^2}{m^2\omega^2}\right]$$

$$= \frac{1}{i\hbar}[q, p] = \boxed{1}. \tag{42}$$

3.25)

a) Find the Fourier series for the function $f(x) = x^2$ on the interval $[-\pi, \pi]$.

Note that x^2 is an odd real-valued function so we can write

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{43}$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \,\mathrm{d}x = \frac{2\pi^3}{3} \tag{44}$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \,\mathrm{d}x = \frac{4\cos(\pi n)}{n^2} = \frac{4}{n^2}(-1)^n. \tag{45}$$

Thus,

$$x^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}. \tag{46}$$

b) Use your result at $x = \pi$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{47}$$

which is the value of Riemann's zeta function $\zeta(x)$ at $x = 2$

If we plug in $x = \pi$ to our Fourier expansion above, we find

$$\pi^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\pi)}{n^2} = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}. \tag{48}$$