1.2) Derive the cyclicity (1.24) of the trace from Eq. (1.23).

Using the definition of the trace, we can write

$$\mathrm{Tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} A_{k\ell} B_{\ell k}. \tag{1}$$

Note that this sum is symmetric in the indices $\{k, \ell\}$, and since $A_{k\ell}$ and $B_{\ell k}$ are just scalars

$$\boxed{\mathrm{Tr}(AB) = \sum_{\ell=1}^{n} \sum_{k=1}^{n} B_{\ell k} A_{k\ell} = \mathrm{Tr}(BA)}. \tag{2}$$

1.3) Show that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$, which is Eq. (1.26).

Recall the definition of a matrix transpose: $(A^{\mathrm{T}})_{ij} = A_{ji}$, so

$$(AB)^{\mathrm{T}}_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1} B^{\mathrm{T}}_{ik} A^{\mathrm{T}}_{kj} = (B^{\mathrm{T}} A^{\mathrm{T}})_{ij}. \tag{3}$$

Hence,

$$\boxed{(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}}. \tag{4}$$

1.5) Show that $(AB)^\dagger = B^\dagger A^\dagger$, which is Eq. (1.29).

Recall that the hermitian adjoint of a matrix is just $A^\dagger = (A^{\mathrm{T}})^*$, so

$$(AB)^\dagger = ((AB)^{\mathrm{T}})^* = (B^{\mathrm{T}} A^{\mathrm{T}})^*. \tag{5}$$

Observe that

$$(AB)^*_{ij} = \left(\sum_{k=1}^{n} A_{ik} B_{kj} \right)^* = \sum_{k=1}^{n} A^*_{ik} B^*_{kj} = (A^* B^*)_{ij}, \tag{6}$$

so Eq. (5) becomes

$$\boxed{(AB)^\dagger = (B^{\mathrm{T}} A^{\mathrm{T}})^* = (B^{\mathrm{T}})^* (A^{\mathrm{T}})^* = B^\dagger A^\dagger}. \tag{7}$$

1.7) Show that the two 4×4 matrices (1.46) satisfy Grassman's algebra (1.11) for $n = 2$.

The matrices in Eq. (1.46) are given by

$$\theta_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Using the kronecker delta, notice that we can write the components of θ_1 and θ_2 as follows:

$$(\theta_1)_{ij} = \delta_{i1}\delta_{j3} - \delta_{i2}\delta_{j4} \tag{9}$$

$$(\theta_2)_{ij} = \delta_{i1}\delta_{j2} + \delta_{i3}\delta_{j4}. \tag{10}$$

Thus,

$$(\theta_1\theta_2)_{ij} = \sum_{k=1}^{n} (\theta_1)_{ik}(\theta_2)_{kj} \tag{11}$$

$$= \sum_{k=1}^{n} \delta_{i1}\delta_{j2}\delta_{k3}\delta_{k1} + \delta_{i1}\delta_{j4}\delta_{k3}\delta_{k3} - \delta_{i2}\delta_{j2}\delta_{k4}\delta_{k1} - \delta_{i2}\delta_{j4}\delta_{k4}\delta_{k3} \tag{12}$$

$$= \delta_{i1}\delta_{j4}. \tag{13}$$

In the same way,

$$(\theta_2\theta_1)_{ij} = \sum_{k=1}^{n} (\theta_2)_{ik}(\theta_1)_{kj} \tag{14}$$

$$= \sum_{k=1}^{n} \delta_{i1}\delta_{j3}\delta_{k2}\delta_{k1} - \delta_{i1}\delta_{j4}\delta_{k2}\delta_{k2} + \delta_{i3}\delta_{j3}\delta_{k4}\delta_{k1} - \delta_{i3}\delta_{j4}\delta_{k4}\delta_{k2} \tag{15}$$

$$= -\delta_{i1}\delta_{j4}. \tag{16}$$

Thus, since addition is defined component wise, we have the anticommutator

$$(\{\theta_1, \theta_2\})_{ij} = \{(\theta_1)_{ij}, (\theta_2)_{ij}\} = \delta_{i_1}\delta_{j_4} + (-\delta_{i1}\delta_{j4}) = 0 \tag{17}$$

$$\Rightarrow \boxed{\{\theta_1, \theta_2\} = 0}. \tag{18}$$

1.12) Show that the Minkowski product $(x, y) = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}} - x^0 y^0$ of two 4-vectors x and y is an inner product obeying the rules (1.78,1.79,1.84).

The three properties we check for this inner product are as follows:

$$(f, g) = (g, f)^* \tag{19a}$$

$$(f, zg + wh) = z(f, g) + w(f, h) \tag{19b}$$

$$(f,g)=0 \text{ for all } f\in V \Rightarrow g=0. \tag{19c}$$

For the following, we assume that x, y are real 4-vectors. That is, their components are real numbers.

From this assumption, Eq. (19a) follows trivally

$$\boxed{(x,y)=\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{y}}-x^0y^0=\vec{\boldsymbol{y}}\cdot\vec{\boldsymbol{x}}-y^0x^0=(y,x)}. \tag{20}$$

Additionally, Eq. (19b) is satisfied as follows:

$$\begin{aligned}(x,\alpha y+\beta z)&=\vec{\boldsymbol{x}}\cdot(\alpha\vec{\boldsymbol{y}}+\beta\vec{\boldsymbol{z}})-x^0(\alpha y^0+\beta z^0)\\ &=\alpha\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{y}}+\beta\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{z}}-\alpha x^0y^0-\beta x^0z^0\\ &=\alpha\left(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{y}}-x^0y^0\right)+\beta\left(\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{z}}-x^0z^0\right)\\ &=\boxed{\alpha(x,y)+\beta(x,z)}.\end{aligned} \tag{21}$$

Finally, we prove Eq. (19c) as follows. Suppose that $(x,y)=0$ for all x but $y\neq 0$. Then, there is some component of y, say y^i such that $y^i\neq 0$. Obviously then, we could choose $x^i=1$ and the rest of the components of x to be zero, meaning that

$$(x,y)=\begin{cases}y^i & \text{if } i\neq 0\\ -y^i & \text{if } i=0\end{cases}. \tag{22}$$

That is, it must be the case that $y\equiv 0$, otherwise there is always at least one choice of x such that (x,y) is nonzero, which contradicts the original assumption.

1.18) Use the Gram-Schmidt method to find orthonormal linear combinations of the three vectors

$$\vec{\mathbf{s}}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \vec{\mathbf{s}}_1=\begin{pmatrix}1\\1\\0\end{pmatrix},\quad \vec{\mathbf{s}}_3=\begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{23}$$

Notice that $\hat{\boldsymbol{u}}_1=\vec{\boldsymbol{s}}_1$ is already normalized, so we can find an orthonormal vector to $\hat{\boldsymbol{u}}_2$ from $\vec{\boldsymbol{s}}_2$ as

$$\vec{\boldsymbol{u}}_2=\hat{\boldsymbol{s}}_2-(\hat{\boldsymbol{u}}_1,\vec{\boldsymbol{s}}_2)\hat{\boldsymbol{u}}_1 \tag{24}$$

$$=\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix} \tag{25}$$

$$=\begin{pmatrix}0\\1\\0\end{pmatrix}. \tag{26}$$

Noting that $\vec{\boldsymbol{u}}_2$ is normalized, we have $\hat{\boldsymbol{u}}_2 = \vec{\boldsymbol{u}}_2$. Lastly, we can repeat the process for the third vector.

$$\begin{aligned}
\vec{\boldsymbol{u}}_3 &= \vec{\boldsymbol{s}}_3 - (\hat{\boldsymbol{u}}_1, \vec{\boldsymbol{s}}_3)\hat{\boldsymbol{u}}_1 - (\hat{\boldsymbol{u}}_2, \vec{\boldsymbol{s}}_3)\hat{\boldsymbol{u}}_2 && (27)\\
&= \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} && (28)\\
&= \begin{pmatrix}0\\0\\1\end{pmatrix}. && (29)
\end{aligned}$$

Again, $\vec{\boldsymbol{u}}_3 = \hat{\boldsymbol{u}}_3$ since $\vec{\boldsymbol{u}}_3$ is already normalized. Notice that these are just the standard basis vectors for a three dimensional vector space, so the orthonormal linear combinations of $\vec{\boldsymbol{s}}_1$, $\vec{\boldsymbol{s}}_2$, and $\vec{\boldsymbol{s}}_3$ are

$$\boxed{\hat{\boldsymbol{u}}_1 = \vec{\boldsymbol{s}}_1, \qquad \hat{\boldsymbol{u}}_2 = \vec{\boldsymbol{s}}_1 - \vec{\boldsymbol{s}}_2, \qquad \hat{\boldsymbol{u}}_3 = \vec{\boldsymbol{s}}_1 - \vec{\boldsymbol{s}}_2 - \vec{\boldsymbol{s}}_3}. \tag{30}$$

1.21) Show that a linear operator A that is represented by a hermitian matrix (1.167) in an orthonormal basis satisfies $(g, Af) = (Ag, f)$.

Observe that

$$(g, Af) = g^\dagger A f, \tag{31}$$

where g, f are column vectors and A is represented by a matrix. Recalling that $(AB)^\dagger = B^\dagger A^\dagger$ and that $A^\dagger = A$ since A is hermitian, Eq. (31) becomes

$$\boxed{(g, Af) = (A^\dagger g)^\dagger f = (Ag)^\dagger f = (Ag, f)}. \tag{32}$$