The Born Approximation in Nonrelativistic Quantum Scattering

Richard Whitehill

Why is scattering interesting?

Scattering has been one of the most fruitful methods by which we learn about fundamental particles and composite systems

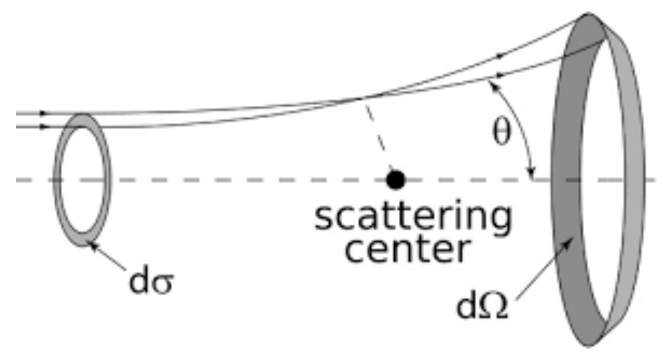
- Rutherford (basic atomic structure)
- Deep inelastic scattering (confirmation of existence of quarks, nuclear structure)
- Higgs (discovery of Higgs boson – mass in Standard model)

There are several facilities internationally dedicated to scattering experiments (large structures, expensive)

Motivation

Desired quantity: (differential) cross section

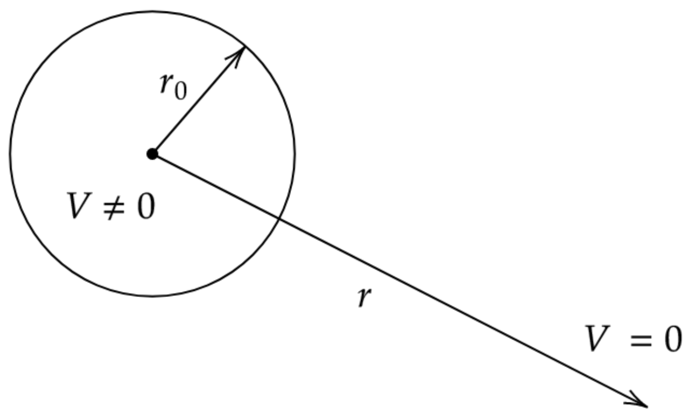
Classical: concrete particles interacting via potential

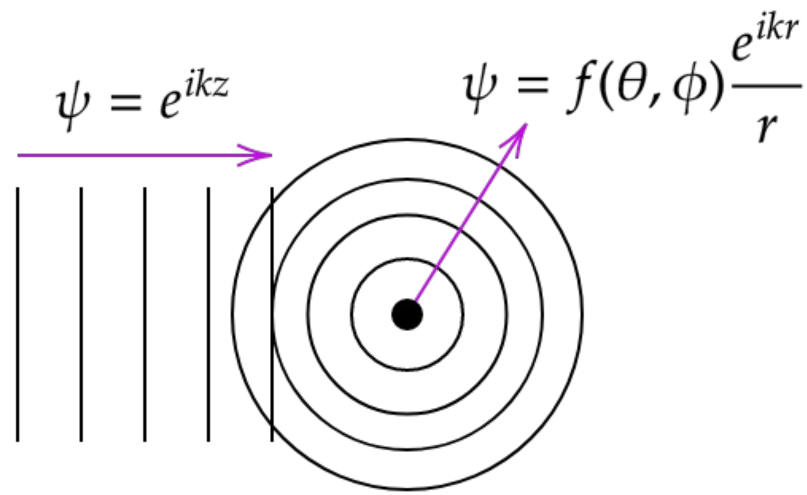


Quantum: wavefunction formalism

$$\mathrm{H}\left|\psi\right\rangle = i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t}$$

Wavefunction setup





$$\psi = \mathcal{N}\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right]$$

Wavefunction to differential cross section

$$\vec{J}_{\text{incoming}} \cdot \hat{z}\, \mathrm{d}\sigma = \vec{J}_{\text{outgoing}} \cdot \hat{r}\ r^2\, \mathrm{d}\Omega$$

$$\vec{J} = \frac{-i\hbar}{m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

$$\vec{J}_{\text{incoming}} \cdot \hat{z} = \frac{\hbar k}{m} \qquad \vec{J}_{\text{outgoing}} \cdot \hat{r} = \frac{\hbar k |f(\theta,\phi)|^2}{m r^2}$$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\phi)|^2}$$

Setting up the problem

Hamiltonian = free particle + interaction potential as perturbation

$$\mathrm{H}\left|\psi\right\rangle = \mathrm{H}_0\left|\psi\right\rangle + V\left|\psi\right\rangle = V\left|\psi\right\rangle \Leftrightarrow (E - \mathrm{H}_0)\left|\psi\right\rangle = V\left|\psi\right\rangle$$

Solution (kind of): Lippman-Schwinger equation

$$\left|\psi\right\rangle = \frac{1}{E - \mathrm{H}_0 \pm i\epsilon} V\left|\psi\right\rangle + \left|\phi\right\rangle \quad (\mathrm{H}_0\left|\phi\right\rangle = E\left|\phi\right\rangle)$$

Wavefunction in position space

Insert complete set of position states (twice)

$$\psi(x) = \phi(x) + \int \mathrm{d}^3\vec{y}\,\mathrm{d}^3\vec{y}'\,\langle x|\,(E - \mathrm{H}_0 \pm i\epsilon)^{-1}\,|y\rangle\,\langle y|\,V\,|y'\rangle\,\psi(\vec{y}')$$

$$= \phi(x) + \int \mathrm{d}^3\vec{y}\,G(x,y)V(y)\psi(y)$$

We now need determine the function G (potential is specified)

Determining the Green's function (part 1)

$$G(x,y) = \int \mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}'\, \langle x|p\rangle \langle p| \left(E - \mathrm{H}_0\right)^{-1} |p'\rangle \langle p'|y\rangle$$

$$= \frac{2m}{\hbar^2} \int \frac{\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}'}{(2\pi\hbar)^3} e^{i\vec{k}\cdot\vec{x}} \delta^{(3)}(\vec{p}-\vec{p}') \frac{1}{\ell^2 - k^2 \pm i\epsilon} e^{-i\vec{k}'\cdot\vec{y}}$$

$$= \frac{1}{(2\pi\hbar)^3} \frac{2m}{\hbar^2} \int \mathrm{d}^3\vec{p}\, \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{(\ell - k \pm i\epsilon)(\ell + k \pm i\epsilon)}$$

Note: $\langle x|p\rangle = (2\pi\hbar)^{-3/2} e^{i\vec{k}\cdot\vec{x}},\ \mathrm{H}_0 |p\rangle = \frac{p^2}{2m} |p\rangle = \frac{\hbar^2 k^2}{2m} |p\rangle,\ E = \frac{\hbar^2 \ell^2}{2m}$

Determining the Green's function (part 2)

$$G(x,y) = -\frac{2m}{(2\pi)^3\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{k^2\sin\theta e^{i|\vec{x}-\vec{y}|k\cos\theta}}{[k-(\ell\pm i\epsilon)][\ell+(\ell\pm i\epsilon)]}\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}k$$

$$= \frac{im}{4\pi^2\hbar^2|\vec{x}-\vec{y}|}\int_{-\infty}^\infty\frac{k[e^{i|\vec{x}-\vec{y}|k}-e^{-i|\vec{x}-\vec{y}|k}]}{[k-(\ell\pm i\epsilon)][k+(\ell\pm i\epsilon)]}\,\mathrm{d}k$$

$$I_1 = \int_{-\infty}^\infty\frac{ke^{ikr}}{[k-(\ell\pm i\epsilon)][k+(\ell\pm i\epsilon)]}\qquad I_2 = \int_{-\infty}^\infty\frac{ke^{-ikr}}{[k-(\ell\pm i\epsilon)][k+(\ell\pm i\epsilon)]}$$

Determining the Green's function (part 3)

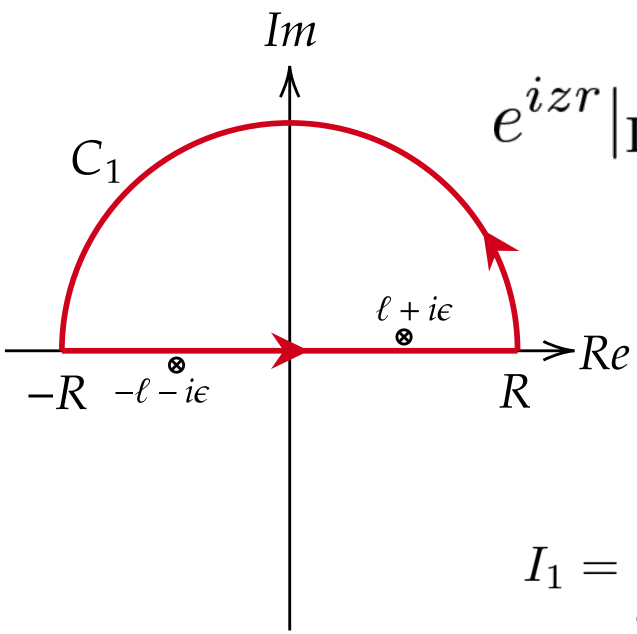
At this point we have to choose how we want to shift the poles

- Mathematically: influences what integration contour we choose
  - Need to enclose pole and choose proper ghost contour

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\,\mathrm{d}x = \int_L \frac{f(z)}{z - z_0 \pm i\epsilon}\,\mathrm{d}z + \int_\Gamma \frac{f(z)}{z - z_0 \pm i\epsilon}\,\mathrm{d}z$$
$$= \int_C \frac{f(z)}{z - z_0 \pm i\epsilon}\,\mathrm{d}z = 2\pi i f(z_0)$$

- Physically: "+" corresponds to outgoing waves and "-" corresponds to incoming waves
  - We care about outgoing waves so we choose "+"

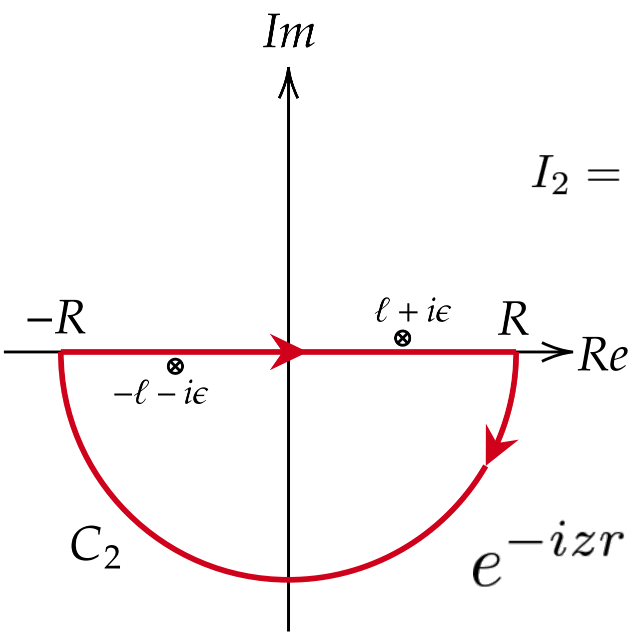
Determining the Green's function (part 4)



$$e^{izr}|_\Gamma = e^{irR\cos\theta} e^{-rR\sin\theta}$$

$$I_1 = \oint_{C_1} \left[\frac{ze^{izr}}{z + (\ell + i\epsilon)} \right] \frac{1}{z - (\ell + i\epsilon)} \, \mathrm{d}z = 2\pi i \left[\frac{\ell e^{i\ell r}}{2\ell} \right] = i\pi e^{i\ell r}$$

Determining the Green's function (part 5)



$$I_2 = \oint_{C_2} \left[\frac{z e^{-izr}}{z - (\ell + i\epsilon)} \right] \frac{1}{z + (\ell + i\epsilon)} \, \mathrm{d}z = 2\pi i \left[\frac{-\ell e^{i\ell r}}{2\ell} \right] = -i\pi e^{i\ell r}$$

$$e^{-izr}|_\Gamma = e^{-irR\cos\theta} e^{rR\sin\theta}$$

Determining the Green's function (part 6)

$$G(x,y) = -\frac{m}{2\pi\hbar^2}\frac{e^{i\ell|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}$$

Final notes before the Born approximation

Having found the Green's function we can write the integral form of the Schrodinger equation

$$\psi(\vec{x}) = \phi(\vec{x}) - \frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\vec{y}\,\frac{e^{i\ell|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}V(\vec{y})\psi(\vec{y})$$

This is still not incredibly useful – we can only calculate the wavefunction with this if we know what the wavefunction is *a priori* !

The first Born approximation

We care about the potential far from the interaction region

$$|\vec{x}-\vec{y}| = \sqrt{x^2+y^2-2\vec{x}\cdot\vec{y}} \approx x - \hat{x}\cdot\vec{y}$$

$$\psi(\vec{x}) = \phi(\vec{x}) + \left[-\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\vec{y}\, e^{-i\vec{\ell}\cdot\vec{y}} V(\vec{y})\psi(\vec{y})\right]\frac{e^{i\ell x}}{x}$$

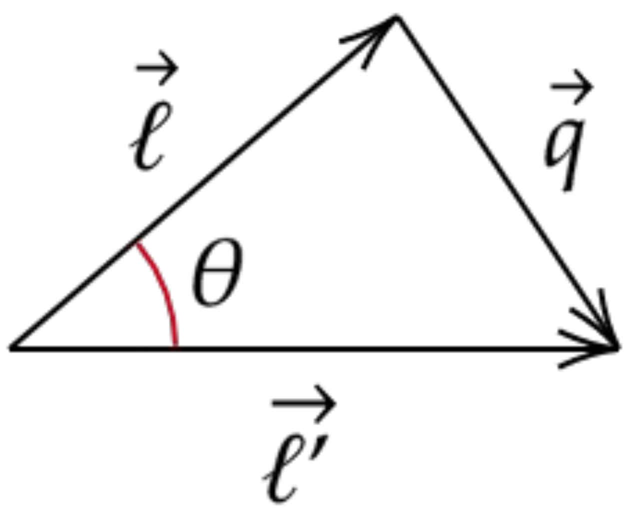
Now for the approximation: assume that the incoming wavefunction is not significantly altered after interacting with the target

$$\psi \approx \phi \Rightarrow \psi(\vec{x}) \approx \phi(\vec{x}) + \underbrace{\left[-\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\vec{y}\, e^{i\vec{q}\cdot\vec{y}} V(\vec{y})\right]}_{f(\theta,\phi)}\frac{e^{i\ell x}}{x}$$

What is *q*?

Momentum transfer

- Remember: wavenumber not changed but direction of propagation is



$$\vec{q} = \hbar(\vec{\ell'} - \vec{\ell}) = \hbar\ell(\hat{z} - \hat{r})$$

Interlude: Rutherford scattering (part 1)

Scattering amplitude for spherically symmetric potentials:

$$\begin{aligned} f(\theta) &= -\frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} r^2\sin\theta e^{i\ell r\cos\theta}V(r)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\ &= -\frac{2m}{\hbar q}\int_0^\infty r\sin\left(\frac{qr}{\hbar}\right)V(r)\,\mathrm{d}r \end{aligned}$$

And now for the Coulomb potential

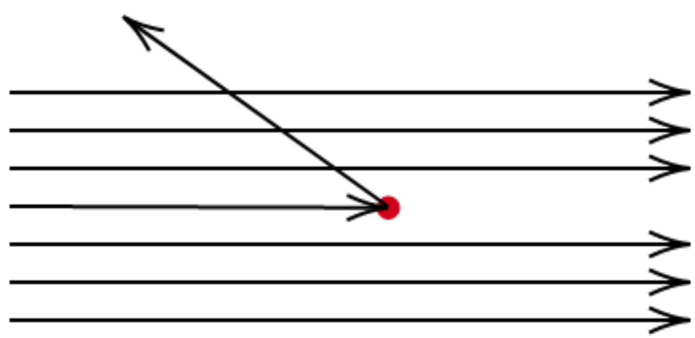
$$f(\theta) = -\frac{2mQ_1Q_2}{\hbar q}\int_0^\infty \sin\left(\frac{qr}{\hbar}\right)\mathrm{d}r$$

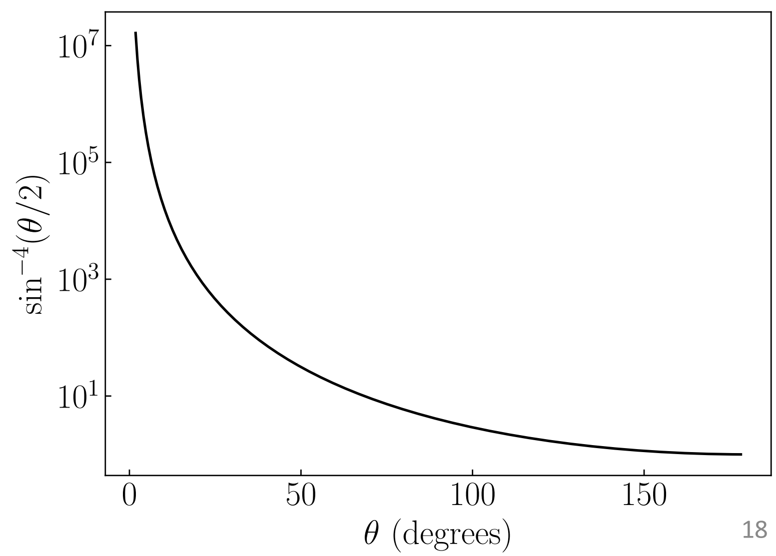
Interlude: Rutherford scattering (part 2)

Since that integral diverges, let's play a clever mathematical trick

$$f(\theta) = -\frac{2mQ_1Q_2}{\hbar q} \lim_{\lambda \to 0} \int_0^\infty e^{-\lambda r} \sin\left(\frac{qr}{\hbar}\right) \mathrm{d}r = -\frac{-2mQ_1Q_2}{4\pi\epsilon_0 q^2}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Q_1Q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right)^2$$

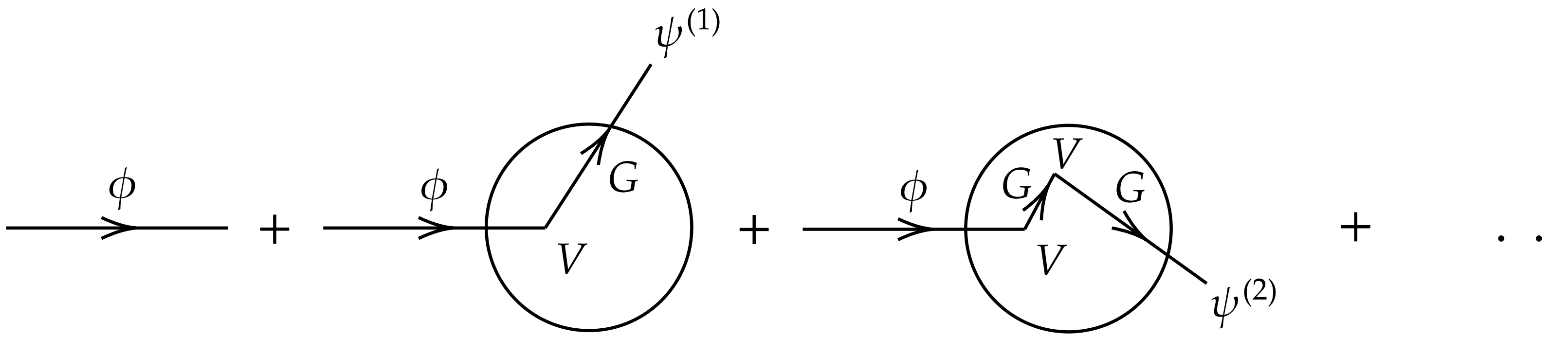




The Born series

Method of successive approximations

$$\psi = \sum_{n=0}^{\infty} \left(\int \mathrm{d}^3\vec{y}\ G\,V \right)^n \phi$$



Final remarks

Born approximation provides way to calculate scattering amplitudes for cases where incoming particle energy is large, potential is weak

Continued succession of approximation leads to Born series

Analogous calculations in relativistic QFT similar in spirit/flavor (different because the governing equation is not the Schrodinger equation)