4.1) Show that the Fourier integral formula

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos k(y-x)\,\mathrm{d}k\,\mathrm{d}y\,, \tag{1}$$

where f is a real function, follows from

$$f(x) = \int_{-\infty}^\infty \tilde{f}(k)e^{ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}} \quad \text{and} \quad \tilde{f}(k) = \int_{-\infty}^\infty f(x)e^{-ikx}\frac{\mathrm{d}x}{\sqrt{2\pi}} \tag{2}$$

and

$$\tilde{f}^*(k) = \int_{-\infty}^\infty \frac{\mathrm{d}x}{\sqrt{2\pi}} f(x)e^{ikx} = \tilde{f}(-k). \tag{3}$$

We know that $f^*(x) = f(x)$ since f is a real-valued function. Equivalently, $\operatorname{Im} f(x) = 0$. We can write

$$f(x) = \frac{f(x)+f^*(x)}{2}. \tag{4}$$

Thus,

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(\int_{-\infty}^\infty \tilde{f}(k)e^{ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}} + \int_{-\infty}^\infty \tilde{f}^*(k)e^{-ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}}\right) && (5)\\
&= \frac{1}{2}\left(\int_{-\infty}^\infty \tilde{f}(k)e^{ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}} + \int_{-\infty}^\infty \tilde{f}(-k)e^{-ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}}\right) && (6)\\
&= \frac{1}{2}\int_{-\infty}^\infty \left(\left[\int_{-\infty}^\infty f(y)e^{-iky}\frac{\mathrm{d}y}{\sqrt{2\pi}}\right]e^{ikx} + \left[\int_{-\infty}^\infty f(y)e^{iky}\frac{\mathrm{d}y}{\sqrt{2\pi}}\right]e^{-ikx}\right)\frac{\mathrm{d}k}{\sqrt{2\pi}} && (7)\\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty f(y)\left[\frac{1}{2}\left(e^{ik(x-y)} + e^{-ik(x-y)}\right)\right]\frac{\mathrm{d}k\,\mathrm{d}y}{2\pi} && (8)\\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty f(y)\cos k(x-y)\,\mathrm{d}k\,\mathrm{d}y && (9)\\
&= \boxed{\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos k(x-y)\,\mathrm{d}k\,\mathrm{d}y}, && (10)
\end{aligned}$$

which is the of the same form as Eq. (1) noting that cosine is odd in its argument.

4.4) By using the Fourier-transform formulas

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos kx\,\mathrm{d}k \int_0^\infty f(y)\cos ky\,\mathrm{d}y\,, \tag{11a}$$

if f is both real and even, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin kx\,\mathrm{d}k \int_0^\infty f(y)\sin ky\,\mathrm{d}y\,. \tag{11b}$$

if f is both real and odd, derive the formulas

$$e^{-\beta|x|} = \frac{2}{\pi}\int_0^\infty \frac{\beta\cos kx}{\beta^2+k^2}\,\mathrm{d}k \tag{12a}$$

and

$$\frac{x}{|x|}e^{-\beta|x|} = \frac{2}{\pi}\int_0^\infty \frac{k\sin kx}{\beta^2+k^2}\,\mathrm{d}k \tag{12b}$$

for the even and odd extensions of the exponential $\exp(-\beta|x|)$.

Let us derive the expression for $\exp(-\beta|x|)$, which is clearly a real-valued function and even, so we use Eq. (11a):

$$\boxed{e^{-\beta|x|} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos kx\, e^{-\beta y}\cos ky\,\mathrm{d}y\,\mathrm{d}k = \int_0^\infty \frac{\beta}{\beta^2+k^2}\cos kx\,\mathrm{d}k}. \tag{13}$$

Similarly, we notice that $x\exp(-\beta|x|)/|x|$ is an odd real-valued function, so

$$\boxed{\frac{x}{|x|} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin kx\, e^{-\beta y}\sin ky\,\mathrm{d}y\,\mathrm{d}k = \frac{2}{\pi}\int_0^\infty \frac{k}{\beta^2+k^2}\sin kx\,\mathrm{d}k}. \tag{14}$$

4.6) At time $t = 0$, a particle of mass m is in a gaussian superposition of momentum eigenstates centered at $p = \hbar K$

$$\psi(x,0) = \mathcal{N}\int_{-\infty}^{\infty} e^{ikx}e^{-L^2(k-K)^2}\,\mathrm{d}k\,. \tag{15}$$

a) Shift k by K and do the integral. Whare is the particle most likely to be found?

Let $\ell = k - K$ such that

$$\psi(x,0) = \mathcal{N}\int_{-\infty}^{\infty} e^{i(\ell+K)x}e^{-L^2\ell^2}\,\mathrm{d}\ell = e^{iKx}\mathcal{N}\int_{-\infty}^{\infty} e^{i\ell x}e^{-L^2\ell^2}\,\mathrm{d}\ell \tag{16}$$

$$= \mathcal{N}e^{iKx}\int_{-\infty}^{\infty} e^{-x^2/4L^2}e^{-L^2(\ell-ix/2L^2)^2}\,\mathrm{d}\ell \tag{17}$$

$$= \mathcal{N}e^{iKx}e^{-x^2/4L^2}\left(\frac{\sqrt{\pi}}{L}\right). \tag{18}$$

We can normalize this state as follows:

$$\int_{-\infty}^{\infty}|\psi|^2\,\mathrm{d}x = \mathcal{N}^2\left(\frac{\pi}{L^2}\right)\int_{-\infty}^{\infty} e^{-x^2/2L^2}\,\mathrm{d}x = \mathcal{N}^2\left(\frac{\pi}{L^2}\right)\left(L\sqrt{2\pi}\right) = \frac{\sqrt{2\pi^3}}{L}\mathcal{N}^2. \tag{19}$$

Thus, we require

$$\mathcal{N} = \left(\frac{L^2}{2\pi^3}\right)^{1/4} \Rightarrow \psi(x,0) = \frac{1}{(2\pi L^2)^{1/4}} e^{iKx} e^{-x^2/4L^2}. \tag{20}$$

Thus, the expected value of x at $t = 0$ is

$$\boxed{\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x,0) x \psi(x,0)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}L} \int_{-\infty}^{\infty} x e^{-x^2/2L^2}\,\mathrm{d}x = 0} \tag{21}$$

since this is an odd function integrated over an even range.

b) At time t, the wave function $\psi(x,t)$ is $\psi(x,0)$ but with ikx replaced with $ikx - i\hbar k^2 t/2m$. Shift k by K and do the integral. Where is the particle most likely to be found?

The wavefunction at an arbitrary time $t > 0$ is given by

$$\psi(x,t) = \mathcal{N} \int_{-\infty}^{\infty} e^{ikx - i\hbar k^2 t/2m} e^{-L^2(k-K)^2}\,\mathrm{d}k\,. \tag{22}$$

Again, we let $\ell = k - K$ such that

$$\psi(x,t) = \mathcal{N} \int_{-\infty}^{\infty} e^{i(\ell+K)x - i\hbar(\ell+K)^2 t/2m} e^{-L^2\ell^2}\,\mathrm{d}k \tag{23}$$

$$= \mathcal{N} e^{iKx} e^{-i\hbar K^2 t/2m} \int_{-\infty}^{\infty} e^{i\ell x} e^{-i\hbar \ell^2 t/2m} e^{-i\hbar \ell K t/m} e^{-L^2\ell^2}\,\mathrm{d}\ell \tag{24}$$

$$= \mathcal{N} e^{iKx - i\hbar K^2 t/2m} \int_{-\infty}^{\infty} e^{i(x - \hbar K t/m)\ell} e^{-(L^2 + i\hbar t/2m)\ell^2}\,\mathrm{d}\ell \tag{25}$$

$$= \mathcal{N} e^{iKx - i\hbar K^2 t/m} \sqrt{\frac{\pi}{L^2 + i\hbar t/2m}} e^{-(x - \hbar K t/2m)^2/4(L^2 + i\hbar t/2m)}. \tag{26}$$

Thus, we can normalize the wavefunction as follows:

$$\mathcal{N}^2 \frac{\pi}{\sqrt{L^4 + \hbar^2 t^2/4m^2}} \int_{-\infty}^{\infty} e^{-L^2(x - \hbar K t/m)^2/2(L^4 + \hbar^2 t^2/4m^2)}\,\mathrm{d}x = 1 \tag{27}$$

$$\mathcal{N}^2 \frac{\pi}{\sqrt{L^4 + \hbar^2 t^2/4m^2}} \sqrt{\frac{2\pi\left(L^4 + \hbar^2 t^2/4m^2\right)}{L^2}} = \mathcal{N}^2 \left[\frac{2\pi^3}{L^2}\right]^{1/2} = 1 \tag{28}$$

$$\mathcal{N} = \left(\frac{L^2}{2\pi^3}\right)^{1/4}, \tag{29}$$

which is the same normalization as for $\psi(x,0)$ (this is expected – actually it is required for conservation of probability). The wavefunction for all times is then given by

$$\psi(x,t) = \left(\frac{L^2}{2\pi^3}\right)^{1/4} \sqrt{\frac{\pi}{L^2 + i\hbar t/2m}} e^{iKx - i\hbar K^2 t/2m} e^{-(x - \hbar K t/m)^2/4(L^2 + i\hbar t/2m)}. \tag{30}$$

Hence, the expected position is

$$\langle x \rangle = \frac{L}{\pi\sqrt{2\pi}} \frac{\pi}{\sqrt{L^4 + \hbar^2 t^2/4m^2}} \int_{-\infty}^{\infty} x e^{-L^2(x-\hbar K t/m)^2/2(L^4+\hbar^2 t^2/4m^2)} \,\mathrm{d}x \tag{31}$$

$$= \frac{L}{\sqrt{2\pi(L^4 + \hbar^2 t^2/4m^2)}} \frac{\hbar K t}{m} \sqrt{\frac{2\pi(L^4 + \hbar^2 t^2/4m^2)}{L^2}} \tag{32}$$

$$= \boxed{\frac{\hbar K}{m} t}. \tag{33}$$

This is a very nice result, appearing as the classical relation between a particle's position and momentum ($p = \hbar K$).

c) Does the wave packet spread out like t or like $\sqrt{t}$ as in classical diffusion?

The spread of the gaussian wave packet is given by its standard deviation, where the time-dependent behavior goes like t for large time scales.

4.11) Derive

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}\right) - \vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{A}} = -\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{A}} = \mu_0 \vec{\boldsymbol{J}} \tag{34}$$

from $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ and Ampère's Law $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}}$.

Observe that

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}\right) - \vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{A}} = \mu_0 \vec{\boldsymbol{J}}. \tag{35}$$

Note that we can make $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0$ by a suitable gauge transformation. Thus, in the appropriate gauge,

$$\boxed{-\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{A}} = \mu_0 \vec{\boldsymbol{J}}}. \tag{36}$$

4.14) Use the Green's function relations

$$-\vec{\boldsymbol{\nabla}}^2 G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \int \frac{\mathrm{d}^3\vec{\boldsymbol{k}}}{(2\pi)^3} e^{i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}')} = \delta^{(3)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') \tag{37a}$$

and

$$G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \frac{1}{2\pi^2 |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|} \int_0^\infty \frac{\sin k \,\mathrm{d}k}{k} \tag{37b}$$

to show that

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\vec{\boldsymbol{x}}'\,\frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}')}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|} \tag{38}$$

satisfies Eq. (34).

Notice that the Green's function is

$$G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') = \frac{1}{4\pi|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|}. \tag{39}$$

Thus,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \mu_0\int \mathrm{d}^3\vec{\boldsymbol{x}}'\,G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}'). \tag{40}$$

Hence,

$$\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \mu_0\int \mathrm{d}^3\vec{\boldsymbol{x}}'\left[\vec{\boldsymbol{\nabla}}^2 G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')\right]J(\vec{\boldsymbol{x}}') = \mu_0\int \mathrm{d}^3\vec{\boldsymbol{x}}'\,\delta^{(3)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')J(\vec{\boldsymbol{x}}') \tag{41}$$

$$\Rightarrow \boxed{\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \mu_0\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})}. \tag{42}$$

4.16) Compute the Laplace transform of $1/\sqrt{t}$ (Hint: let $t = u^2$).

The Laplace transform of $1/\sqrt{t}$ is computed as follows

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty \frac{1}{\sqrt{t}}e^{-st}\,\mathrm{d}t\,. \tag{43}$$

If we let $t = u^2$, then $\mathrm{d}t = 2u\,\mathrm{d}u$ and

$$\boxed{\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty \frac{1}{u}e^{-su^2}2u\,\mathrm{d}u = 2\int_0^\infty e^{-su^2}\,\mathrm{d}u = \sqrt{\frac{\pi}{s}}}, \tag{44}$$

where we have assumed that $|s| > 0$ such that the integral converges.

4.17) Show that the commutation relations

$$[a(\vec{\boldsymbol{k}}), a^\dagger(\vec{\boldsymbol{k}}')] = \delta(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}') \quad \text{and} \quad [a(\vec{\boldsymbol{k}}), a(\vec{\boldsymbol{k}}')] = [a^\dagger(\vec{\boldsymbol{k}}), a^\dagger(\vec{\boldsymbol{k}}')] = 0 \tag{45}$$

of the annihilation and creation operators imply the equal-time commutation relations

$$[\phi(\vec{\boldsymbol{x}}, t), \pi(\vec{\boldsymbol{y}}, t)] = i\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}) \quad \text{and} \quad [\phi(\vec{\boldsymbol{x}}, t), \phi(\vec{\boldsymbol{y}}, t)] = [\pi(\vec{\boldsymbol{x}}, t), \pi(\vec{\boldsymbol{y}}, t)] = 0 \tag{46}$$

for the field ϕ and its conjugate momentum π.

The fields

$$\phi(x) = \int \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{47a}$$

and

$$\pi(x) = -i\int \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\sqrt{\frac{\omega_k}{2(2\pi^3)}}\,\mathrm{d}^3k\,. \tag{47b}$$

Hence,

$$[\phi(\vec{\boldsymbol{x}},t),\pi(\vec{\boldsymbol{y}},t)] = \phi(\vec{\boldsymbol{x}},t)\pi(\vec{\boldsymbol{y}},t) - \pi(\vec{\boldsymbol{y}},t)\phi(\vec{\boldsymbol{x}},t) \tag{48}$$

$$= -i\iint\Bigg[\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]$$
$$-\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\Bigg]\times$$
$$\frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\sqrt{\frac{\omega_{k'}}{2(2\pi)^3}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{49}$$

$$= i\iint\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}),a^\dagger(\vec{\boldsymbol{k}}')\right] + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}'),a^\dagger(\vec{\boldsymbol{k}})\right]\right]\times$$
$$\frac{1}{2(2\pi)^3}\sqrt{\frac{\omega_{k'}}{\omega_k}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{50}$$

$$= i\int\left[e^{i\vec{\boldsymbol{k}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} + e^{-i\vec{\boldsymbol{k}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\right]\frac{\mathrm{d}^3k}{(2\pi)^3} = \frac{i}{2(2\pi)^3}2(2\pi)^3\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}) = \boxed{i\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}. \tag{51}$$

Similarly,

$$[\phi(\vec{\boldsymbol{x}},t),\phi(\vec{\boldsymbol{y}},t)] = \phi(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{y}},t) - \phi(\vec{\boldsymbol{y}},t)\phi(\vec{\boldsymbol{x}},t) \tag{52}$$

$$= \iint\Bigg[\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') + e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]$$
$$-\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') + e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\Bigg]\times$$
$$\frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\frac{1}{\sqrt{(2\pi)^3 2\omega_{k'}}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{53}$$

$$= \iint\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}),a^\dagger(\vec{\boldsymbol{k}}')\right] - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}'),a^\dagger(\vec{\boldsymbol{k}})\right]\right]\times$$
$$\frac{1}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{54}$$

$$= \int\left[e^{i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} - e^{-i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\right]\frac{\mathrm{d}^3k}{(2\pi)^3\omega_k} = \boxed{0}, \tag{55}$$

and

$$[\pi(\vec{\boldsymbol{x}},t),\pi(\vec{\boldsymbol{y}},t)] = \pi(\vec{\boldsymbol{x}},t)\pi(\vec{\boldsymbol{y}},t) - \pi(\vec{\boldsymbol{y}},t)\pi(\vec{\boldsymbol{x}},t) \tag{56}$$

$$= -\iint \left[\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]\right.$$
$$\left. - \left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\right] \times$$
$$\sqrt{\frac{\omega_k}{2(2\pi)^3}}\sqrt{\frac{\omega_{k'}}{2(2\pi)^3}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{57}$$

$$= \iint \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}),a^\dagger(\vec{\boldsymbol{k}}')\right] - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}'),a^\dagger(\vec{\boldsymbol{k}})\right]\right] \times$$
$$\frac{\sqrt{\omega_k\omega_{k'}}}{2(2\pi)^3}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{58}$$

$$= \int \left[e^{i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} - e^{-i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\right]\frac{\omega_k\,\mathrm{d}^3k}{(2\pi)^3} = \boxed{0}. \tag{59}$$