6.5) Do the integral

$$\oint_C \frac{\mathrm{d}z}{z^2-1} \tag{1}$$

in which the contour C is counterclockwise about the circle $|z| = 2$

We can write

$$\boxed{\oint_C \frac{\mathrm{d}z}{(z-1)(z+1)} = 2\pi i\Big(\frac{1}{z+1}\Big|_{z=1} + \frac{1}{z-1}\Big|_{z=-1}\Big) = 0}. \tag{2}$$

6.9) Use Cauchy's integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint \mathrm{d}z' \,\frac{f(z')}{(z'-z)^{n+1}} \tag{3}$$

and Rodrigues's expression

$$P_n(x) = \frac{1}{2^n n!}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n (x^2-1)^n \tag{4}$$

for Legendre's polynomial $P_n(x)$ to derive Schlaefli's formula

$$P_n(z) = \frac{1}{2^n 2\pi i}\oint \frac{(z'^2-1)^n}{(z'-z)^{n+1}}\,\mathrm{d}z'\,. \tag{5}$$

We can denote

$$f(x) = (x^2-1)^n. \tag{6}$$

Then,

$$f^{(n)(z)} = \frac{n!}{2\pi i}\oint \mathrm{d}z' \,\frac{(z'^2-1)^n}{(z'-z)^{n+1}} = 2^n n! P_n(z). \tag{7}$$

Hence,

$$\boxed{P_n(z) = \frac{1}{2^n 2\pi i}\oint \frac{(z'^2-1)^n}{(z'-z)^{n+1}}\,\mathrm{d}z'}. \tag{8}$$

6.20) Use a contour integral to evaluate the integral

$$I_a = \int_0^\pi \frac{\mathrm{d}\theta}{a+\cos\theta}, \quad a > 1. \tag{9}$$

Recall that we can write $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$. Also note that the function $(a + \cos\theta)^{-1}$ is even with respect to θ, so we can extend the integration range to be symmetric if we divide the result by 1/2. Hence,

$$I_a = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2a + e^{i\theta} + e^{-i\theta}}. \tag{10}$$

Let us define $z = e^{i\theta}$, then $e^{-i\theta} = z^{-1}$ and $\mathrm{d}\theta = \mathrm{d}z\,/iz$. Furthermore, the contour of integration is the circle $|z| = 1$. Thus,

$$I_a = \int_C \frac{1}{2a + z + z^{-1}} \frac{\mathrm{d}z}{iz} = -i \int_C \frac{\mathrm{d}z}{z^2 + 2az + 1}. \tag{11}$$

The function $f(z) = (z^2 + 2az + 1)^{-1}$ has poles at $z_\pm = -a \pm \sqrt{a^2 - 1}$. Observe that $|z_-| < -1 < |z_+|$, implying that the only pole enclosed is z_+. This means that

$$I_a = -i \int_C \frac{\mathrm{d}z}{(z - z_+)(z - z_-)} = -i(2\pi i)\Big(\frac{1}{z - z_-}\Big)\Big|_{z=z_+} \tag{12}$$

$$= 2\pi \frac{1}{(-a + \sqrt{a^2 - 1}) - (-a - \sqrt{a^2 - 1})} \tag{13}$$

$$= \boxed{\frac{\pi}{\sqrt{a^2 - 1}}}. \tag{14}$$

6.26) Show that

$$\int_0^\infty \cos ax\, e^{-x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\pi} e^{-a^2/4}. \tag{15}$$

In this problem we again use the same reasoning as in the previous problem to write

$$\begin{aligned}
I &= \frac{1}{4}\int_{-\infty}^{\infty} (e^{iax}e^{-x^2} + e^{-iax}e^{-x^2})\,\mathrm{d}x \\
&= \frac{1}{4}\int_{-\infty}^{\infty} (e^{-(x^2 - iax)} + e^{-(x^2 + iax)})\,\mathrm{d}x \\
&= \frac{1}{4}\int_{-\infty}^{\infty} \left[e^{(-ia/2)^2} e^{-(x^2 - ix + (-ia/2)^2)} + e^{(ia/2)^2} e^{-(x^2 + iax + (ia/2)^2)}\right] \mathrm{d}x \\
&= \frac{1}{4} e^{-a^2/4} \int_{-\infty}^{\infty} \left[e^{-(x - ia/2)^2} + e^{-(x + ia/2)^2}\right] \mathrm{d}x\,.
\end{aligned} \tag{16}$$

In example 6.23 it is proven that

$$\int_{-\infty}^{\infty} e^{-m^2x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-m^2(x + ic)^2}\,\mathrm{d}x\,, \tag{17}$$

so using this result here, we find

$$I = \frac{\sqrt{\pi}}{2} e^{-a^2/4}, \tag{18}$$

where the extra factor of 2 came from having two gaussian integrals in Eq. (16).

6.33) The Bessel function J_n is given by the integral

$$J_n(x) = \frac{1}{2\pi i} \oint_C e^{(x/2)(z-1/z)} \frac{\mathrm{d}z}{z^{n+1}} \tag{19}$$

along a counterclockwise about the origin. Find the generating function for these Bessel functions, that is, the function $G(x, z)$ whose Laurent series has the $J_n(x)$'s as coefficients

$$G(x, z) = \sum_{n=-\infty}^{\infty} J_n(x) z^n. \tag{20}$$

The Laurent series of $G(x, z)$ (centered at the origin) is given as

$$G(x, z) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi i} \oint_C \frac{G(x, z')}{z'^{n+1}} \, \mathrm{d}z' \right] z^n. \tag{21}$$

Thus, since we want the coefficient of this Laurent series to be the Bessel functions, we must have that

$$\boxed{G(x, z) = e^{(x/2)(z-1/z)}}. \tag{22}$$

6.34) Show that the Heaviside function $\theta(y) = (y + |y|)/(2|y|)$ is given by the integral

$$\theta(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{iyx} \frac{\mathrm{d}x}{x - i\epsilon} \tag{23}$$

in which ϵ is an infinitesimal positive number.

We can promote $x \mapsto z$ such that

$$\theta(y) = \frac{1}{2\pi i} \oint_C e^{ixz} \frac{\mathrm{d}z}{z - i\epsilon}, \tag{24}$$

where C is the contour consisting of the line from $-R$ to R (with $R \to \infty$) and a ghost contour which is a half-circle in either the upper or lower half plane (such that the integrand goes to zero along it). Observe that along the ghost contour $z = R\cos\theta + iR\sin\theta$, where $\theta \in [0, \pi]$ or $\theta \in [-\pi, 0]$, so

$$e^{iyz} = e^{iyR\cos\theta} e^{-yR\sin\theta}. \tag{25}$$

Hence, we choose the ghost contour in the upper half plane if $y > 0$ and the lower half plane if $y < 0$. Observe that the simple pole is at $z = i\epsilon$, which is in the upper half plane and $f(z) = 1/(z - i\epsilon)$ has residue 1 at its pole, so

$$\boxed{\theta(y) = \frac{1}{2\pi i}\oint_C \frac{\mathrm{d}z}{z - i\epsilon} = \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases}} \tag{26}$$

since the pole is enclosed if $y > 0$ but not if $y < 0$.