

1.28) Use

$$(\vec{\theta} \cdot \vec{\sigma})^2 = \theta^2 \mathbb{1}. \quad (1)$$

to derive the expression

$$\exp\left(-i\vec{\theta} \cdot \vec{\sigma}/2\right) = \cos(\theta/2)\mathbb{1} - i\hat{\theta} \cdot \vec{\sigma} \sin(\theta/2), \quad (2)$$

for the  $2 \times 2$  rotation matrix  $\exp\left(-i\vec{\theta} \cdot \vec{\sigma}/2\right)$ .

We use the Taylor series of the exponential to write

$$\begin{aligned} e^{-i\vec{\theta} \cdot \vec{\sigma}/2} &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(-\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}\right)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(-\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}\right)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta^2}{4}\right)^n - \frac{i}{2} (\theta \cdot \vec{\sigma}) \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta^2}{4}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} - \frac{i}{2} \vec{\theta} \cdot \vec{\sigma} \sum_{n=0}^{\infty} \left[ \frac{2}{\theta} \left(\frac{\theta}{2}\right)^{2n+1} \right] \end{aligned} \quad (3)$$

$$= \boxed{\cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\hat{\theta} \cdot \vec{\sigma} \sin\left(\frac{\theta}{2}\right)}. \quad (4)$$

1.29) Compute the characteristic equation for the matrix  $-i\vec{\theta} \cdot \vec{J}$  in which the generators are  $(J_k)_{ij} = i\epsilon_{ijk}$  is totally antisymmetric with  $\epsilon_{123} = 1$ .

Using the generator equation, we can explicitly write the matrices  $J_i$  for  $i = 1, 2, 3$ .

$$J_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5)$$

We can then define  $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$  and  $\vec{J} = (J_1, J_2, J_3)$  so

$$\det(-i\vec{\theta} \cdot \vec{J} - \lambda \mathbb{1}) = \begin{vmatrix} -\lambda & -\theta_3 & -\theta_2 \\ \theta_3 & -\lambda & \theta_1 \\ \theta_2 & -\theta_1 & -\lambda \end{vmatrix} \quad (6)$$

$$= -\lambda(\lambda^2 + \theta_1^2) + \theta_3(-\theta_3\lambda - \theta_1\theta_2) - \theta_2(-\theta_3\theta_1 + \theta_2\lambda) \quad (7)$$

$$= \boxed{-\lambda(\lambda^2 + \theta^2)}. \quad (8)$$

1.30) Use the characteristic equation of exercise 1.29 to derive the identities

$$\begin{aligned} \exp\left(-i\vec{\theta} \cdot \vec{J}\right) &= \cos \theta \mathbb{1} - i\hat{\theta} \cdot \vec{J} \sin \theta + (1 - \cos \theta) \hat{\theta}(\hat{\theta})^T \\ \exp\left(-i\vec{\theta} \cdot \vec{J}\right)_{ij} &= \delta_{ij} \cos \theta - \sin \theta \epsilon_{ijk} \hat{\theta}_k + (1 - \cos \theta) \hat{\theta}_i \hat{\theta}_j \end{aligned} \quad (9)$$

for the  $3 \times 3$  real orthogonal matrix  $\exp(-i\vec{\theta} \cdot \vec{J})$ .

**1.32)** Consider the  $2 \times 3$  matrix  $A$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix}. \quad (10)$$

Perform the singular value decomposition  $A = USV^T$ , where  $V^T$  the transpose of  $V$ . Use Matlab or another program to find the singular values and the real orthogonal matrices  $U$  and  $V$ .

Notice that

$$A^T A = V \Sigma^T \Sigma V^T. \quad (11)$$

so the eigenvalues of  $A^T A$  are the squares of the singular values, and the eigenvectors form the columns of  $V$ . Computing the eigenvalues of  $A^T A$ , we solve the characteristic equation for  $A^T A$ :

$$\det(A^T A - \lambda \mathbb{1}) = \begin{vmatrix} 10 - \lambda & 2 & 0 \\ 2 & 4 - \lambda & 6 \\ 0 & 6 & 10 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 14, 10, 0. \quad (12)$$

Hence, our singular values are  $\sigma_1 = \sqrt{14}$  and  $\sigma_2 = \sqrt{10}$ . Solving for the column vectors of  $V$ , we solve  $(A^T A - \lambda_i \mathbb{1})v_i$  for  $i = 1, 2$  and normalize the eigenvectors.

$$\begin{aligned} \lambda_1 = 14 : \quad & \begin{pmatrix} -4 & 2 & 0 \\ 2 & -10 & 6 \\ 0 & 6 & -14 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix} \\ \lambda_2 = 10 : \quad & \begin{pmatrix} 0 & 2 & 0 \\ 2 & -6 & 6 \\ 0 & 6 & 0 \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} -3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{pmatrix}. \end{aligned}$$

Finally, we find  $v_3$  by simply requiring that  $\{v_1, v_2, v_3\}$  is an orthonormal set:

$$\begin{cases} v_1 \cdot v_3 = 0 \\ v_2 \cdot v_3 = 0 \\ |v_3| = 1 \end{cases} \Rightarrow v_3 = \begin{pmatrix} 1/\sqrt{35} \\ -5/\sqrt{35} \\ 3/\sqrt{35} \end{pmatrix}. \quad (13)$$

Now, we can determine the column vectors of  $U$  by solving  $u_i = \frac{1}{\sigma_i} A v_i$ :

$$u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (14)$$

$$u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (15)$$

Therefore, we have our SVD matrices as

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} \quad V = \begin{pmatrix} 1/\sqrt{14} & -3/\sqrt{10} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \\ 3/\sqrt{14} & 1/\sqrt{10} & 3/\sqrt{35} \end{pmatrix}. \quad (16)$$

Using Wolfram, the matrices for the SVD are given as

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} \quad V = \begin{pmatrix} 1/\sqrt{14} & -3/\sqrt{10} & 1/\sqrt{35} \\ \sqrt{2/7} & 0 & \sqrt{5/7} \\ 3/\sqrt{14} & 1/\sqrt{10} & 3/\sqrt{35} \end{pmatrix}, \quad (17)$$

which is the result given in Eq. (16).

**1.35)** Consider the hamiltonian  $H = \frac{1}{2}\hbar\omega\sigma_3$  where  $\sigma_3$  is defined in (1.453). The entropy  $S$  of this system at temperature  $T$  is  $S = -k\text{Tr}[\rho \ln(\rho)]$  in which the density operator  $\rho$  is

$$\rho = \frac{e^{-H/(kT)}}{\text{Tr}[e^{-H/(kT)}]}. \quad (18)$$

Find expressions for the density operator  $\rho$  and its entropy  $S$ .

We have  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , so

$$e^{-\frac{\hbar\omega\sigma_3}{kT}} = \begin{pmatrix} e^{-\frac{\hbar\omega}{2kT}} & 0 \\ 0 & e^{\frac{\hbar\omega}{2kT}} \end{pmatrix}, \quad (19)$$

so

$$\rho = \frac{1}{e^{-\frac{\hbar\omega}{2kT}} + e^{\frac{\hbar\omega}{2kT}}} \begin{pmatrix} e^{-\frac{\hbar\omega}{2kT}} & 0 \\ 0 & e^{\frac{\hbar\omega}{2kT}} \end{pmatrix} = \frac{2}{\cosh(\frac{\hbar\omega}{2kT})} \begin{pmatrix} e^{-\frac{\hbar\omega}{2kT}} & 0 \\ 0 & e^{\frac{\hbar\omega}{2kT}} \end{pmatrix}. \quad (20)$$

Additionally,

$$\ln(\rho) = \begin{pmatrix} \ln\left[\frac{2e^{-\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)}\right] & 0 \\ 0 & \ln\left[\frac{2e^{\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)}\right] \end{pmatrix}, \quad (21)$$

giving

$$\begin{aligned} S &= -k\text{Tr}[\rho \ln \rho] \\ &= -\frac{2k}{\cosh(\hbar\omega/2kT)} \left[ e^{\hbar\omega/2kT} \ln \left[ \frac{2e^{\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)} \right] + e^{-\hbar\omega/2kT} \ln \left[ \frac{2e^{-\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)} \right] \right] \\ &= \boxed{-2k \left[ \frac{\hbar\omega}{kT} \tanh(\hbar\omega/2kT) + 2 \ln \left[ \frac{2}{\cosh(\hbar\omega/2kT)} \right] \right]}. \end{aligned} \quad (22)$$

**1.37)** A system that has three fermionic states has three creation operators  $a_i^\dagger$  and three annihilation operators  $a_k$  which satisfy the anticommutation relations  $\{a_i, a_k^\dagger\} = \delta_{ik}$  and  $\{a_i, a_k\} = \{a_i^\dagger, a_k^\dagger\} = 0$  for  $i, k = 1, 2, 3$ . The eight states of the system are  $|t, u, v\rangle \equiv (a_1^\dagger)^t (a_2^\dagger)^u (a_3^\dagger)^v |0, 0, 0\rangle$ . We can represent them by eight 8-vectors each of which has seven 0's with a 1 in position  $4t + 2u + v + 1$ . How big should the matrices that represent the creation and annihilation operators be? Write down the three matrices that represent the three creation operators.

The matrices should be  $8 \times 8$  such that operating on a vector gives a vector in the same space. For just one state, the creation operator is just

$$a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (23)$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle. \quad (24)$$

The creation operator for the first state is just  $a^\dagger \otimes \mathbb{1} \otimes \mathbb{1}$  such that

$$a_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (25)$$

Similarly,

$$a_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (26)$$

and

$$a_3^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (27)$$