1.28) Use

$$(\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}})^2 = \theta^2 \mathbb{1}. \tag{1}$$

to derive the expression

$$\exp\left(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}}/2\right) = \cos(\theta/2)\mathbb{1} - i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}} \sin(\theta/2), \tag{2}$$

for the 2×2 rotation matrix $\exp\left(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}}/2\right)$.

We use the Taylor series of the exponential to write

$$\begin{aligned}
e^{-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}}/2} &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2}\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}})^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-\frac{i}{2}\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}})^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-\frac{i}{2}\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}})^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta^2}{4}\right)^n - \frac{i}{2}\left(\theta \cdot \vec{\boldsymbol{\sigma}}\right) \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta^2}{4}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} - \frac{i}{2}\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}} \sum_{n=0}^{\infty} \left[\frac{2}{\theta}\left(\frac{\theta}{2}\right)^{2n+1}\right] && (3) \\
&= \boxed{\cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}} \sin\left(\frac{\theta}{2}\right)}. && (4)
\end{aligned}$$

1.29) Compute the characteristic equation for the matrix $-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}$ in which the generators are $(J_k)_{ij} = -i\epsilon_{kij}$ is totally antisymmetric with $\epsilon_{123} = 1$.

Using the generator equation, we can explicitly write the matrices J_i for $i = 1, 2, 3$.

$$J_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

We can then define $\vec{\boldsymbol{\theta}} = (\theta_1, \theta_2, \theta_3)$ and $\vec{\boldsymbol{J}} = (J_1, J_2, J_3)$ so

$$\begin{aligned}
\det(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}} - \lambda\mathbb{1}) &= \begin{vmatrix} -\lambda & -\theta_3 & \theta_2 \\ \theta_3 & -\lambda & -\theta_1 \\ -\theta_2 & \theta_1 & -\lambda \end{vmatrix} \\
&= -\lambda(\lambda^2 + \theta_1^2) + \theta_3(-\theta_3\lambda - \theta_1\theta_2) + \theta_2(\theta_3\theta_1 - \theta_2\lambda) \\
&= \boxed{-\lambda(\lambda^2 + \theta^2)}. && (6)
\end{aligned}$$

1.30) Use the characteristic equation of exercise 1.29 to derive the identities

$$\begin{aligned}
\exp\left(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}\right) &= \cos\theta\mathbb{1} - i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}} \sin\theta + (1 - \cos\theta)\hat{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}})^{\mathrm{T}} \\
\exp\left(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}\right)_{ij} &= \delta_{ij}\cos\theta - \sin\theta\epsilon_{ijk}\hat{\theta}_k + (1 - \cos\theta)\hat{\theta}_i\hat{\theta}_j
\end{aligned} . \tag{7}$$

for the 3×3 real orthogonal matrix $\exp\left(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}\right)$.

Observe a few facts. First,

$$(-i\vec{\theta} \cdot \vec{\mathbf{J}})^2 = \begin{pmatrix} \theta_2^2 + \theta_3^2 & -\theta_1\theta_2 & -\theta_1\theta_3 \\ -\theta_1\theta_2 & \theta_1^2 + \theta_3^2 & -\theta_2\theta_3 \\ -\theta_1\theta_3 & -\theta_2\theta_3 & \theta_1^2 + \theta_2^2 \end{pmatrix} = -[\theta^2 \mathbb{1} - \vec{\theta}\vec{\theta}^{\mathrm{T}}]. \tag{8}$$

Next, by the Cayley-Hamilton theorem we have

$$(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^3 = -(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}). \tag{9}$$

This can be extended by using the following argument. Suppose that this result holds for any arbitrary odd integer. That is, $(-i\vec{\theta} \cdot \vec{\mathbf{J}})^{2n+1} = (-1)^n(-i\vec{\theta} \cdot \vec{\mathbf{J}})$. Then it is clear that, this formula holds for the next odd number

$$(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^{2(n+1)+1} = (-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^{2n+1}(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^2 = (-1)^n(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^3 = (-1)^{n+1}(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}). \tag{10}$$

Additionally, this leads to a result for all the even integers as well:

$$(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^{2n} = (-1)^{n-1}(-i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^2. \tag{11}$$

We may now tackle Eq. (7) using the Taylor expansion of the exponential function:

$$\begin{aligned} e^{-i\vec{\theta}\cdot\vec{\mathbf{J}}} &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!}(-i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^n \\ &= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}(-i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^{2n} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}(-i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^{2n+1} \\ &= \mathbb{1} - (-i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} + (-i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}}) \sum_{n=0} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} \\ &= \mathbb{1} + (\mathbb{1} - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}^{\mathrm{T}})(\cos\theta - 1) - i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}} \sin\theta \\ &= \boxed{\cos\theta \mathbb{1} - i\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}} \sin\theta + (1 - \cos\theta)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}^{\mathrm{T}}}. \end{aligned} \tag{12}$$

Finally, we can translate Eq. (12) into an expression relating the elements of the matrices on the right and left side of the equation as follows:

$$\begin{aligned} \left(e^{-i\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{J}}}\right)_{ij} &= \cos\theta \mathbb{1}_{ij} - i(\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{J}})_{ij} \sin\theta + (1 - \cos\theta)\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}^{\mathrm{T}}\right)_{ij} \\ &= \delta_{ij}\cos\theta - i\sin\theta\theta_k(J_k)_{ij} + (1 - \cos\theta)\hat{\theta}_i\hat{\theta}_j \\ &= \boxed{\delta_{ij}\cos\theta - \sin\theta\epsilon_{ijk}\theta_k + (1 - \cos\theta)\hat{\theta}_i\hat{\theta}_j}. \end{aligned} \tag{13}$$

1.32) Consider the 2×3 matrix A

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix}. \tag{14}$$

Perform the singular value decomposition $A = USV^{\mathrm{T}}$, where V^{T} the transpose of V. Use Matlab or another program to find the singular values and the real orthogonal matrices U and V.

Notice that

$$A^{\mathrm{T}}A = VS^{\mathrm{T}}SV^{\mathrm{T}}. \tag{15}$$

so the eigenvalues of $A^{\mathrm{T}}A$ are the squares of the singular values, and the eigenvectors form the columns of V. Computing the eigenvalues of $A^{\mathrm{T}}A$, we solve the characteristic equation for $A^{\mathrm{T}}A$:

$$\det\left(A^{\mathrm{T}}A - \lambda\mathbb{1}\right) = \begin{vmatrix} 10-\lambda & 2 & 0 \\ 2 & 4-\lambda & 6 \\ 0 & 6 & 10-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 14,\ 10,\ 0. \tag{16}$$

Hence, our singular values are $\sigma_1 = \sqrt{14}$ and $\sigma_2 = \sqrt{10}$. Solving for the column vectors of V, we solve $(A^{\mathrm{T}}A - \lambda_i\mathbb{1})v_i$ for $i = 1,\ 2$ and normalize the eigenvectors.

$$\lambda_1 = 14: \quad \begin{pmatrix} -4 & 2 & 0 \\ 2 & -10 & 6 \\ 0 & 6 & -14 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix} \tag{17}$$

$$\lambda_2 = 10: \quad \begin{pmatrix} 0 & 2 & 0 \\ 2 & -6 & 6 \\ 0 & 6 & 0 \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} -3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{pmatrix}. \tag{18}$$

Finally, we find v_3 by simply requiring that $\{v_1, v_2, v_3\}$ is an orthonormal set:

$$\begin{cases} v_1 \cdot v_3 = 0 \\ v_2 \cdot v_3 = 0 \\ |v_3| = 1 \end{cases} \Rightarrow v_3 = \begin{pmatrix} 1/\sqrt{35} \\ -5/\sqrt{35} \\ 3/\sqrt{35} \end{pmatrix}. \tag{19}$$

Now, we can determine the column vectors of U by solving $u_i = \frac{1}{\sigma_i}Av_i$:

$$u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{20}$$

$$u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} \\ 0 \\ 3/\sqrt{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{21}$$

Therefore, we have our SVD matrices as

$$\boxed{U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{14} & -3/\sqrt{10} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \\ 3/\sqrt{14} & 1/\sqrt{10} & 3/\sqrt{35} \end{pmatrix}}. \tag{22}$$

Using Wolfram, the matrices for the SVD are given as

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{14} & -3/\sqrt{10} & 1/\sqrt{35} \\ \sqrt{2/7} & 0 & \sqrt{5/7} \\ 3/\sqrt{14} & 1/\sqrt{10} & 3/\sqrt{35} \end{pmatrix}, \tag{23}$$

which is the result given in Eq. (22).

1.35) Consider the hamiltonian $H = \frac{1}{2}\hbar\omega\sigma_3$ where σ_3 is defined in (1.453). The entropy S of this system at temperature T is $S = -k\mathrm{Tr}[\rho\ln(\rho)]$ in which the density operator ρ is

$$\rho = \frac{e^{-H/(kT)}}{\mathrm{Tr}\left[e^{-H/(kT)}\right]}. \tag{24}$$

Find expressions for the density operator ρ and its entropy S.

We have $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so

$$e^{-\frac{\hbar\omega\sigma_3}{kT}} = \begin{pmatrix} e^{-\frac{\hbar\omega}{2kT}} & 0 \\ 0 & e^{\frac{\hbar\omega}{2kT}} \end{pmatrix}, \tag{25}$$

so

$$\boxed{\rho = \frac{1}{e^{-\frac{\hbar\omega}{2kT}} + e^{\frac{\hbar\omega}{2kT}}} \begin{pmatrix} e^{-\frac{\hbar\omega}{2kT}} & 0 \\ 0 & e^{\frac{\hbar\omega}{2kT}} \end{pmatrix} = \frac{2}{\cosh\left(\frac{\hbar\omega}{2kT}\right)} \begin{pmatrix} e^{-\frac{\hbar\omega}{2kT}} & 0 \\ 0 & e^{\frac{\hbar\omega}{2kT}} \end{pmatrix}}. \tag{26}$$

Additionally,

$$\ln(\rho) = \begin{pmatrix} \ln[\frac{2e^{-\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)}] & 0 \\ 0 & \ln[\frac{2e^{\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)}] \end{pmatrix}, \tag{27}$$

giving

$$\begin{aligned} S &= -k\mathrm{Tr}[\rho\ln\rho] \\ &= -\frac{2k}{\cosh(\hbar\omega/2kT)}\left[e^{\hbar\omega/2kT}\ln\left[\frac{2e^{\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)}\right] + e^{-\hbar\omega/2kT}\ln\left[\frac{2e^{-\hbar\omega/2kT}}{\cosh(\hbar\omega/2kT)}\right]\right] \\ &= \boxed{-2k\left[\frac{\hbar\omega}{2kT}\tanh\left(\hbar\omega/2kT\right) + 2\ln\left[\frac{2}{\cosh(\hbar\omega/2kT)}\right]\right]}. \end{aligned} \tag{28}$$

1.37) A system that has three fermionic states has three creation operators $a_i^\dagger$ and three annihilation operators a_k which satisfy the anticommutation relations $\{a_i, a_k^\dagger\} = \delta_{ik}$ and $\{a_i, a_k\} = \{a_i^\dagger, a_k^\dagger\} = 0$ for $i, k = 1, 2, 3$. The eight states of the system are $|t, u, v\rangle \equiv (a_1^\dagger)^t (a_2^\dagger)^u (a_3^\dagger)^v |0, 0, 0\rangle$. We can represent them by eight 8-vectors each of which has seven 0's with a 1 in position $4t + 2u + v + 1$. How big should the matrices that represent the creation and annihilation operators be? Write down the three matrices that represent the three creation operators.

The matrices should be 8×8 such that operating on a vector gives a vector in the same space. For just one state, the creation operator is just

$$a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{29}$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\,|0\rangle + b\,|1\rangle\,. \tag{30}$$

The creation operator for the first state is the tensor product $a_1^\dagger = a^\dagger \otimes \mathbb{1} \otimes \mathbb{1}$ such that

$$\boxed{a_1^\dagger = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{pmatrix}}\,. \tag{31}$$

Similarly,

$$\boxed{a_2^\dagger = \mathbb{1} \otimes a^\dagger \otimes \mathbb{1} = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix}} \tag{32}$$

and

$$\boxed{a_3^\dagger = \mathbb{1} \otimes \mathbb{1} \otimes a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}. \tag{33}$$