The Born approximation in nonrelativistic quantum scattering

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Abstract

Scattering experiments have been one of the most successful methods for extracting information about fundamental particles and the structure of submicroscopic compmosite systems. In this work, we formulate the definition of the differential cross section in nonrelativistic quantum scattering theory and utilize the Born approximation to develop a formalism which can be used to calculate the scattering amplitude. Furthermore, we derive the Born series, discuss its interpretation, and describe the motivations that translate into the relativistic analogue of the theory.

I. INTRODUCTION

Scattering experiments are one of the most widely used tools physicists use for probing matter at scales much smaller than are visible. Generally, they have been incredibly successful [1–4]. The basic setup of many scattering experiments is the following. Consider a target at rest and a beam of particles [1], where the beam is incident on the target, and the final state of the particles, which may or may not have the same identity as the initial state particles, is measured. A primary goal of these scattering calculations is to determine the structure of the interaction potential, which would accurately reproduce or describe the experimental results.

For most of this paper, we restrict our consideration of scattering events to nonrelativistic regimes, where our moving particles have speeds much smaller than the speed of light ($v \ll c$). In this case, a full description of the dynamics in a scattering event involves solving the time dependent Schrödinger equation

$$\mathrm{H}\,|\Psi\rangle = i\hbar\frac{\partial|\Psi\rangle}{\partial t}, \tag{1}$$

where H is the Hamiltonian operator, depending on both the kinetic energy of the particles in the system and the interaction potentials between the particles. In most cases, for even the simplest potentials, it is impossible to find analytic solutions for the wave function, satisfying the boundary conditions imposed by a scattering experiment and the Schrödinger. In such cases where it is impossible to extract useful information directly from the Schrödinger equation, we must develop and utilize approximate methods that capture the physics in the regimes that we care about. In nonrelativistic quantum scattering, two primary methods are typically discussed in traditional coursework: partial wave analysis and the born approximation. Partial wave analysis essentially decomposes the incoming wave into a sum of fundamental waves and aims to determine the phase shift in each of these waves in the outgoing wave function for a given interaction potential. Although this method is useful, the purpose of this paper is to illustrate the development of a method which serves as the motivation for the perturbative calculations of scattering cross sections in relativistic quantum theories such as quantum electrodynamics (QED).

[1] Actually, we could have two particle beams – the distinction is merely a kinematic one theoretically since we can transform easily between inertial reference frames

The organization of the remainder of this paper is as follows. In Section. II, we develop the basic structure of scattering theory in the nonrelativistic quantum regime, and in Section. III we begin developing the tools needed to write down the Born series with the calculation of the nonrelativistic propagator. Finally, in Section. IV we outline the derivation of the Born series and discuss its interpretation and its implications for the development of the relativistic quantum theory calculations of scattering processes.

II. GENERAL QUANTUM SCATTERING THEORY

We want to calculate the differential scattering cross section. In classical scattering, we imagine scattering as a process by which we send a beam of concrete particles to interact with a target via some potential, which causes our incoming particles to be scattered away from the scattering center. However, in quantum mechanics, we no longer work with concrete particles. Instead, we deal with their wavefunctions, and in scattering, this means that we deal with the probability density currents defined by these wavefunctions.

Incoming particles are described by a wavefunction

$$\psi_{\text{incoming}} = e^{ikz}, \tag{2}$$

where we have selected the z-direction to be along the beam axis, and outgoing particles are described as

$$\psi_{\text{outgoing}} = f(\theta, \phi)\frac{e^{ikr}}{r}. \tag{3}$$

Observe that $\exp(ikr)$ describes an outgoing spherical wave, the $1/r$ behavior is selected such that ψ is properly normalizable, and $f(\theta, \phi)$ modulates the angular behavior of the outgoing wave which was scattered from a potential V. Generally, we want to understand the form of the wavefunction far from the potential (which is a region where the potential is negligibly small), so we can add write our wavefunction (far from the center of the potential) for our scattering particles as a superposition of incoming and outgoing waves:

$$\psi = \mathcal{N}\left[e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r}\right], \tag{4}$$

where $\mathcal{N}$ is a normalization factor. At present, the only unknown, besides the normalization factor which does not contain any physics, is the so-called scattering amplitude $f(\theta, \phi)$, and we now derive its relation to the differential scattering cross section.

The differential cross section is related to the incoming and outgoing probability density currents through

$$\vec{j}_{\text{incoming}} \cdot \hat{z}\, \mathrm{d}\sigma = \vec{j}_{\text{outgoing}} \cdot \hat{r}\, r^2 \mathrm{d}\Omega, \tag{5}$$

where $\mathrm{d}\Omega$ is the solid angle that our particle scatters into and $j = -i\hbar/m(\psi^* \nabla \psi - \psi \nabla \psi^*)$. The incoming flux is $\vec{j}_{\text{incoming}} \cdot \hat{z} = \hbar k/m$, while the outgoing flux is $r^2 \vec{j}_{\text{outgoing}} \cdot \hat{r} = \hbar k |f|^2/m$. Thus we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r^2 \vec{j}_{\text{outgoing}} \cdot \hat{r}}{\vec{j}_{\text{incoming}} \cdot \hat{z}} = |f(\theta, \phi)|^2, \tag{6}$$

meaning that the differential scattering cross section is nothing but the norm-squared of the amplitude of the outgoing wave, f, also known as the scattering amplitude.

At this point, we need to build up formalism for calculating the scattering amplitude, which is an equivalent problem to solving the Schrödinger equation. There are a few tools that are used to calculate the scattering amplitude for a given potential. The first is partial wave analysis, which decomposes the outgoing wave packet into a series of contributions from different angular momentum states and phase shifts. In this paper, though, we are more interested in building up the theory which motivates much of the relativistic quantum scattering theory, so we will focus on the Green's function method and the Born series, leaving partial wave analysis for study elsewhere.

III. THE LIPPMAN-SCHWINGER EQUATION

In this section we solve the Schrödinger equation as follows. Suppose that the Hamiltonian for the scattering particle is given as $\mathrm{H} = \mathrm{H}_0 + V$, where $\mathrm{H}_0 = p^2/2m$ is the free particle Hamiltonian and V is the interaction potential. We denote the free particle states, states satisfying the free particle Schrödinger equation, as $|\phi\rangle$ such that

$$\mathrm{H}_0 |\phi\rangle = E|\phi\rangle. \tag{7}$$

Additionally, we suppose that the interaction potential effectively has a finite range, which rigorously means that $V \to 0$ as $r \to \infty$. That is, we assume that it is negligibly small far enough away from the source, although we make no restrictions here about how fast it must approach zero far away from its center.

Addressing the scattering problem, we must solve the equation

$$(\mathrm{H}_0 + V)|\psi\rangle = E|\psi\rangle, \tag{8}$$

where the energy E is the energy of the incoming beam of particles. We are free to tune this energy, at least in principle, since we suppose that $|\psi\rangle$ is a scattering state, not a bound state, meaning the energy states are continuously accessible. We can rearrange this equation a bit to read as follows:

$$(E - \mathrm{H}_0)|\psi\rangle = V|\psi\rangle. \tag{9}$$

Now, we operate on both sides of Eq. (9) with the inverse of $E - \mathrm{H}_0$, denoted by $1/(E - \mathrm{H}_0)$, and obtain

$$|\psi\rangle = \frac{1}{E - \mathrm{H}_0} V|\psi\rangle + |\phi\rangle, \tag{10}$$

where we have also added on the free particle state $|\phi\rangle$ which is the solution to the homogeneous equation ($V = 0$) corresponding to energy E. Note that the inverse of $E - \mathrm{H}_0$ is singular at the energy E, so to avoid issues with the complex integration that comes in later sections, we will move the singularity along the imaginary axis by addition (or subtraction) of a small paramater ϵ. Then,

$$|\psi\rangle = \frac{1}{E - \mathrm{H}_0 \pm i\epsilon} V|\psi\rangle + |\phi\rangle, \tag{11}$$

which is the so-called Lippman-Schwinger equation [5]. At this point we do not make a distinction between either the positive or negative shift. In Section. IV we will address the physical difference between the sign difference.

In its current form, this equation is not incredibly useful. There are a couple interesting features of this equation that we point out now. As desired, if $V \equiv 0$, then $|\psi\rangle = |\phi\rangle$, meaning that the outgoing state is the same as the incoming free particle state, which passes through all space unperturbed. Additionally, we can see hints of the born series. If we start with $|\psi\rangle$ as the free particle state and iteratively update $|\psi\rangle$ according to Eq. (11), we obtain a series

$$|\psi\rangle = \sum_{n=0}^{\infty} \left(\frac{1}{E - \mathrm{H}_0 \pm i\epsilon} V \right)^n |\phi\rangle. \tag{12}$$

We leave this series here just to give a flavor of the direction of this paper.

IV. THE BORN APPROXIMATION

Up to now, our results have referred only to the abstract states of the particles. We now write our wavefunction in the position basis as

$$\langle x|\psi\rangle = \langle x|\frac{1}{E - \mathrm{H}_0 \pm i\epsilon}V|\psi\rangle + \langle x|\phi\rangle, \tag{13}$$

and inserting two complete position states for the first term in Eq. (13)

$$\psi(\vec{x}) = \int \mathrm{d}^3\vec{y}\,\mathrm{d}^3\vec{y'}\,\langle x|\frac{1}{E - \mathrm{H}_0 \pm i\epsilon}|y\rangle\langle y|V|y'\rangle\psi(\vec{y'}) + \phi(\vec{x}). \tag{14}$$

Noting that $\langle y|V|y'\rangle = \delta^{(3)}(\vec{y} - \vec{y'})V(\vec{y'})$ and denoting $G(\vec{x},\vec{y}) = \langle x|(E - \mathrm{H}_0 \pm i\epsilon)^{-1}V|y\rangle$, then

$$\psi(\vec{x}) = \int \mathrm{d}^3\vec{y}\,G(\vec{x},\vec{y})V(\vec{y})\psi(\vec{y}) + \phi(\vec{x}). \tag{15}$$

What remains to be determined then is the form of $G(\vec{x},\vec{y})$. To do so, we insert a complete set of momentum states such that

$$G(\vec{x},\vec{y}) = \int \mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p'}\langle x|p\rangle\langle p|\frac{1}{E - \mathrm{H}_0 \pm i\epsilon}|p'\rangle\langle p'|y\rangle \tag{16}$$

We note that $\langle x|p\rangle = (2\pi\hbar)^{-3/2}\exp(i\vec{k}\cdot\vec{x})$, where $\vec{p} = \hbar\vec{k}$ and the normalization is given by $\langle p|p'\rangle = \delta^{(3)}(\vec{p} - \vec{p'})$. Putting this into Eq. (16) we find

$$\begin{aligned} G(\vec{x},\vec{y}) &= \frac{2m}{\hbar^2}\int \frac{\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p'}}{(2\pi\hbar)^3}e^{i\vec{k}\cdot\vec{x}}\delta^{(3)}(\vec{p} - \vec{p'})\frac{1}{\ell^2 - k^2 \pm i\epsilon}e^{-i\vec{k}'\cdot\vec{y}} \\ &= \frac{1}{(2\pi\hbar)^3}\frac{2m}{\hbar^2}\int \mathrm{d}^3p\,\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{(\ell - k \pm i\epsilon)(\ell + k \pm i\epsilon)}, \end{aligned} \tag{17}$$

where we used the relations $E = \hbar^2\ell^2/2m$ and $\mathrm{H}_0|p\rangle = \hbar^2k^2/2m|p'\rangle$. To further simplify the integral we switch to spherical coordinates, giving us that

$$\begin{aligned} G(\vec{x},\vec{y}) &= -\frac{1}{(2\pi)^3\hbar}\frac{2m}{\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{k^2\sin\theta e^{i|\vec{x}-\vec{y}|k\cos\theta}}{(k - (\ell \pm i\epsilon))(k + (\ell \pm i\epsilon))}\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}p \\ &= -\frac{m}{4\pi^2\hbar^2|\vec{x}-\vec{y}|i}\int_{-\infty}^{\infty}\frac{k[e^{i|\vec{x}-\vec{y}|k} - e^{-i|\vec{x}-\vec{y}|k}]}{(k - (\ell \pm i\epsilon))(k + (\ell \pm i\epsilon))}\,\mathrm{d}k = -\frac{m}{4\pi^2\hbar^2|\vec{x}-\vec{y}|i}[I_1 - I_2], \end{aligned} \tag{18}$$

where

$$I_1 = \int_{-\infty}^{\infty}\frac{ke^{ikr}}{[k - (\ell \pm i\epsilon)][k + (\ell \pm i\epsilon)]} \tag{19a}$$

and

$$I_2 = \int_{-\infty}^{\infty}\frac{ke^{-ikr}}{[k - (\ell \pm i\epsilon)][k + (\ell \pm i\epsilon)]}, \tag{19b}$$

denoting $r = |\vec{x} - \vec{y}|$.

To perform these integrals, we must now choose whether the denominator is of the form $E - \mathrm{H}_0 + i\epsilon$ or $E - \mathrm{H}_0 - i\epsilon$. This choice determines where the poles $k = \ell \pm i\epsilon$ and $k = -\ell \pm i\epsilon$ are located along the imaginary k axis, which will influence our choice of ghost contour when performing the integration for G. That is, since our we have simple poles in the complex k plane, we can use Cauchy's integral formula

$$\oint_C \frac{f(z)}{z - z_0}\, \mathrm{d}z = 2\pi i f(z_0), \tag{20}$$

where C is any closed path such that z_0 is an interior point to the region bounded by C. For integrals of the form

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, \mathrm{d}x \tag{21}$$

it is conventional to take $x \mapsto z$, $x_0 \mapsto z_0 \pm i\epsilon$ (where we take $\epsilon \to 0$ after integrating), and take C to be the semi-circle enclosed along the real axis either in the upper or lower half plane, such that $f(z) \to 0$ along the semi-circle as we take the radius of the semi-circle to infinity and such that the pole $z_0 \pm i\epsilon$ is enclosed by C.

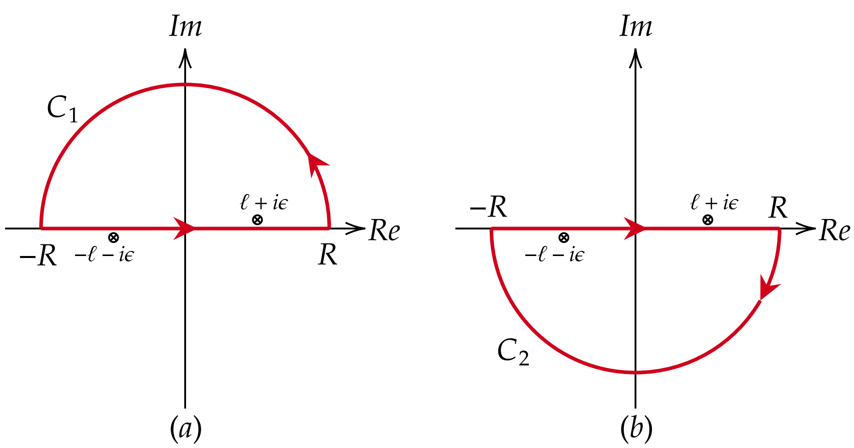

FIG. 1. Illustration of complex integration contours and poles for I_1 and I_2 taking $+i\epsilon$ for the form of the pole shift.

Choosing $+i\epsilon$ for the form of our poles we obtain

$$I_1 = \oint_{C_1} \left[\frac{z e^{izr}}{z + (\ell + i\epsilon)}\right] \frac{1}{z - (\ell + i\epsilon)}\, \mathrm{d}z = 2\pi i \left[\frac{\ell e^{i\ell r}}{2\ell}\right] = i\pi e^{i\ell r}, \tag{22}$$

where C_1 is the contour shown in Fig. 1. Similarly, using the contour C_2 in Fig. 1, we find that

$$I_2 = \oint_{C_2} \left[\frac{z e^{-izr}}{z - (\ell + i\epsilon)} \right] \frac{1}{z + (\ell + i\epsilon)} \, \mathrm{d}z = 2\pi i \left[- \frac{\ell e^{i\ell r}}{2\ell} \right] = -i\pi e^{i\ell r} \tag{23}$$

Putting these results into Eq. (18), we obtain

$$G(\vec{x}, \vec{y}) = -\frac{m}{2\pi\hbar^2 |\vec{x} - \vec{y}|} e^{i\ell|\vec{x}-\vec{y}|}. \tag{24}$$

Note that the choice between $\pm i\epsilon$ also has different physical interpretations. With our choice, we obtained an outgoing spherical wave, which matches the form above in Section. II which we desired. If we picked $-i\epsilon$ we would have obtained the outgoing spherical wave

$$G(\vec{x}, \vec{y}) = -\frac{m}{2\pi\hbar^2 |\vec{x} - \vec{y}|} e^{-i\ell|\vec{x}-\vec{y}|}. \tag{25}$$

Finally, we obtain our solution ψ by inserting Eq. (24) into Eq. (15):

$$\psi(\vec{x}) = \phi(\vec{x}) - \frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\vec{y} \, \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x} - \vec{y}|} V(\vec{y})\psi(\vec{y}). \tag{26}$$

As a last note before moving onto the Born approximations, observe that the function G is a Green's function for the Helmholtz equation. If we write the Schrödinger equation explicitly in position space, we have

$$(\nabla^2 + k^2)\psi = \frac{2m}{\hbar^2} V\psi, \tag{27}$$

where $k = \sqrt{2mE}/\hbar$. If we consider the right side to be an arbitrary function $h(\vec{x})$, we have recovered the Helmholtz equation. The function G then satisfies the equation

$$(\nabla^2 + k^2) G(\vec{x}) = \delta^{(3)}(\vec{x}). \tag{28}$$

The right side of Eq. (27) depends explicitly on ψ, which means that in this case the Green's function method only transforms our differential equation into an integral equation for ψ.

Since $V(\vec{x})$ is localized around $\vec{y}$ and we care about ψ far from the interaction potential, we can write

$$|\vec{x} - \vec{y}| \approx \sqrt{x^2 + y^2 - 2\vec{x} \cdot \vec{y}} \approx x - \hat{x} \cdot \vec{y} \tag{29}$$

If we let $\vec{\ell} = \ell\hat{x}$, then we can rewrite Eq. (26) as follows:

$$\psi(\vec{x}) = \phi(\vec{x}) + \left[-\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\vec{y} \, e^{-i\vec{\ell}\cdot\vec{y}} V(\vec{y})\psi(\vec{y}) \right] \frac{e^{i\ell x}}{x}. \tag{30}$$

In the denominator of the Green's function we made the stronger approximation $|\vec{x}-\vec{y}| \approx x$, which is the first term in the Taylor series expansion of $(x - 2\hat{x}\cdot\vec{y})^{-1}$.

Observe that Eq. (30) matches the form of Eq. (4) with

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\vec{y}\, e^{-i\vec{\ell}\cdot\vec{y}} V(\vec{y})\psi(\vec{y}). \tag{31}$$

We have not made any significant approximation yet. Actually, the only approximations we have made thus far come from Taylor expansions, which are essentially exact in the limit that we care about.

Now comes the Born approximation. Suppose that the potential is weak enough such that the incoming waves are not significantly changed upon interaction with the potential. Then, we may write

$$\psi(\vec{x}) \approx \phi(\vec{x}) = e^{i\ell z} = e^{i\vec{\ell'}\cdot\vec{x}}, \tag{32}$$

where we have denoted $\vec{\ell'} = \ell\hat{z}$, which allows us to write

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\vec{y}\, e^{i\vec{q}\cdot\vec{y}/\hbar} V(\vec{y}), \tag{33}$$

where $\vec{q} = \hbar(\vec{\ell'} - \vec{\ell})$ is the change in momentum from scattering off the potential.

A. An application of the Born approximation

In this section, I will to briefly apply the first Born approximation developed above to reproduce the results of Rutherford for the scattering of charged objects, which lead to the modern picture of the atom with a dense, point-like nucleus. To do so, we solve the more general problem of scattering from a Yukawa potential ($V(r) = a\exp(-br)/r$ where a and $b > 0$ are free parameters) [6], which is interesting in its own right.

Observe that the Yukawa potential is rotationally symmetric. For potentials of the form $V(\vec{r}) = V(r)$ we can write $\vec{q}\cdot\vec{y} = qy\cos\theta$ since $f = f(\theta)$. Thus,

$$\begin{aligned} f(\theta) &= -\frac{m}{2\pi\hbar^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^2 \sin\theta e^{iqr\cos\theta/\hbar} V(r) \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r \\ &= -\frac{2m}{\hbar q} \int_0^\infty r \sin\left(\frac{qr}{\hbar}\right) V(r)\, \mathrm{d}r. \end{aligned} \tag{34}$$

The scattering amplitude for scattering from a Yukawa potential is then

$$f(\theta) = -\frac{2ma}{\hbar q} \int_0^\infty \sin\left(\frac{qr}{\hbar}\right) e^{-br} \mathrm{d}r = -\frac{2ma}{q^2 + \hbar^2 b^2}. \tag{35}$$

The potential that we are interested in for Rutherford scattering is just the electromagnetic potential $V(r) = Q_1Q_2/(4\pi\epsilon_0 r)$, which is just the Yukawa potential with $a = Q_1Q_2/(4\pi\epsilon_0)$ and $b = 0$. Hence, the scattering amplitude for Rutherford scattering is

$$f(\theta) = -\frac{2mQ_1Q_2}{4\pi\epsilon_0 q^2}, \tag{36}$$

and the differential scattering cross section is the norm-square:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{2mQ_1Q_2}{4\pi\epsilon_0 q^2}\right)^2. \tag{37}$$

Observing that $q = \hbar|\vec{\ell}' - \vec{\ell}| = 2\hbar\ell\sin(\theta/2)$ and using the relation $\hbar^2k^2 = 2mE$ the differential cross section can be written in terms of the incoming particle energy and deflection angle as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Q_1Q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)^2. \tag{38}$$

The angular dependence of the differential cross section is plotted in Fig. 2 over a range of scattering angles. It is clear from the analytic formula and the plot that the probability of scattering away from the incoming beam axis is significantly suppressed. With the plum pudding model of the atom, the cross section is expected to follow a shape similar to a Gaussian, which drops off slowly for scattering angles close to the beam axis, corresponding to small deflections, but much more steeply far from the axis.

B. The Born Series

At this point, we have built up a formalism to calculate the scattering amplitude, and therefore the cross section, from the interaction potential using Eq. (31). Performing this integral analytically may prove difficult or impossible, however, which gives us reason to develop another approach for scattering calculations.

We begin with the formula from Eq. (15). Using the method of successive approximations, applying the same reasoning used to derive Eq. (12), we find

$$\psi(\vec{x}) = \sum_{n=0}^{\infty}\left(\int \mathrm{d}^3\vec{y}\, GV\right)^n \phi. \tag{39}$$

The interpretation of this Born series is where the physics lies. We now outline some terminology associated with this series, where the graphical picture of Eq. (39) is displayed in

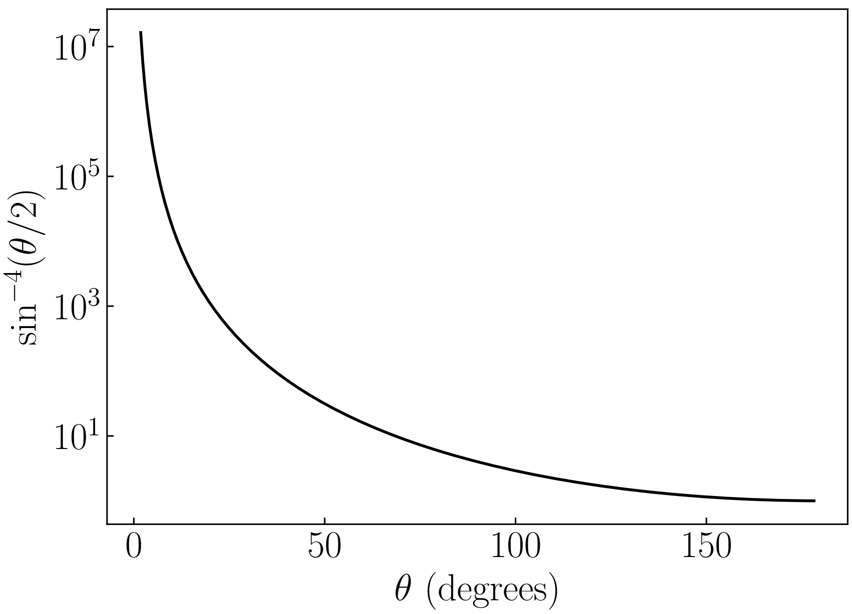

FIG. 2. Plot of the angular dependence of the differential scattering cross section for the electromagnetic scattering of two charged objects.

Fig. 3 and is akin to the Feynman diagrams used in quantum field theory scattering calculations. The function $V(\vec{x})$ is called the vertex factor because it represents the interaction between the particle and the target, and the function $G(\vec{x}, \vec{y})$ is called the propagator because it represents the likelihood, or rather the probability amplitude, for a particle to travel between points $\vec{x}$ and $\vec{y}$. The product of integrals in front of ϕ essentially represent then the number of interactions the particle had with the potential V and the propagation of the particle between each successive interaction with V, giving rise to a new wave function.

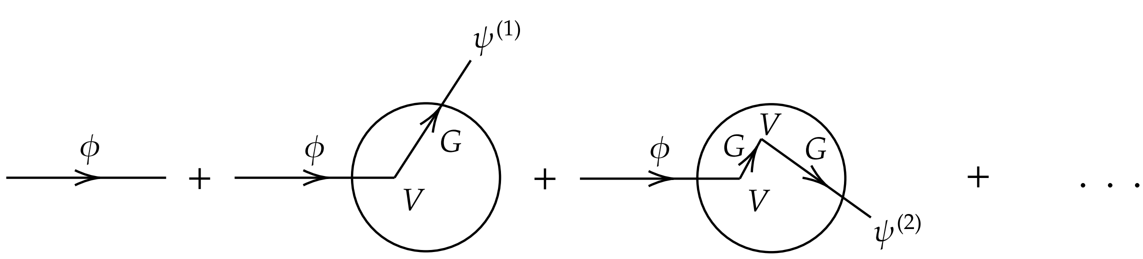

FIG. 3. Diagramatic representation of born series calculated in Eq. (39) as Feynman diagrams with interaction vertices and propagators. Note that the interior region of the circle in each term indicates the interaction region where $V(\vec{r})$ is nonzero.

At this point we make a brief note of the convergence of the series. The propagator G and initial state preparation ϕ are fixed in the series, meaning that only the vertex factor V can affect the convergence of the Born series. A sufficient condition for the convergence of the Born series is that the potential depends on some multiplicative constant that is small, meaning that each term in the series becomes successively smaller. We can thus terminate the series after a finite number of terms to obtain a decent approximation to the scattered wave function far away from the interaction region.

As an example consider the Coulomb potential. The overall factor in front of the $1/r$ behavior is $Q_1Q_2/(4\pi\epsilon_0)$. Noting that Q_1, Q_2 are proportional to the fundamental unit of charge e, the absolute value of the multiplicative constant is proportional to $e^2/(4\pi\epsilon_0) = \hbar c\alpha$, where $\alpha \approx 1/137$ is the dimensionless fine structure constant. Clearly α is a small parameter, which suppresses higher order terms in electromagnetic scattering processes.

V. CONCLUSIONS

The problem of solving the Schrödinger equation for scattering states from a potential proves generally to be a difficult problem analytically. Throughout this paper we have built up the formalism for scattering theory, identifying the relevant quantities which are related to the differential scattering cross section, which is the primary observable of interest in both theoretical and experimental contexts. In particular, we formulated the integral form of the Schrödinger equation by computing the Green's function for the Helmholtz equation from the Lippman-Schwinger equation. By considering the form of the Green's function far from the interaction region and utilizing the Born approximation, which is that the incoming plane wave is not significantly altered by the interaction potential, we derived an expression for the scattering amplitude, which we utilized to calculate the Rutherford cross section for electromagnetic scattering between charged objects in the nonrelativistic limit. The Born series was also derived, which is a separate calculational trick, assuming that the potential depends on a small overall multiplicative constant, that allows us to write the wavefunction as a series of scattering events. Ultimately, this series serves as much of the foundation of the modern perturbative calculations in relativistic quantum field theories when the coupling constant of the theory is small enough to neglect Feynman diagrams with a larger number of vertices. Further work includes building up the relativistic quantum theory of scattering, which is essential since most of the physics being studied at modern facilities require high energy probes. Much of the work is similar in spirit to what was observed here, albeit with some complications arising from the need to mesh together a theory which is relativistic and quantum in nature, meaning that the dynamics are governed by different equations. Therefore the content of this paper serves as a simpler foundation on which to build a more comprehensive scattering theory.

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