4.1) Show that the Fourier integral formula

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos k(y-x)\,\mathrm{d}k\,\mathrm{d}y\,, \tag{1}$$

where f is a real function, follows from

$$f(x) = \int_{-\infty}^\infty \tilde{f}(k)e^{ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}} \quad \text{and} \quad \tilde{f}(k) = \int_{-\infty}^\infty f(x)e^{-ikx}\frac{\mathrm{d}x}{\sqrt{2\pi}} \tag{2}$$

and

$$\tilde{f}^*(k) = \int_{-\infty}^\infty \frac{\mathrm{d}x}{\sqrt{2\pi}}f(x)e^{ikx} = \tilde{f}(-k). \tag{3}$$

We know that $f^*(x) = f(x)$ since f is a real-valued function. Equivalently, $\operatorname{Im} f(x) = 0$. We can write

$$f(x) = \frac{f(x)+f^*(x)}{2}. \tag{4}$$

Thus,

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(\int_{-\infty}^\infty \tilde{f}(k)e^{ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}} + \int_{-\infty}^\infty \tilde{f}^*(k)e^{-ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}}\right) && (5)\\
&= \frac{1}{2}\left(\int_{-\infty}^\infty \tilde{f}(k)e^{ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}} + \int_{-\infty}^\infty \tilde{f}(-k)e^{-ikx}\frac{\mathrm{d}k}{\sqrt{2\pi}}\right) && (6)\\
&= \frac{1}{2}\int_{-\infty}^\infty \left(\left[\int_{-\infty}^\infty f(y)e^{-iky}\frac{\mathrm{d}y}{\sqrt{2\pi}}\right]e^{ikx} + \left[\int_{-\infty}^\infty f(y)e^{iky}\frac{\mathrm{d}y}{\sqrt{2\pi}}\right]e^{-ikx}\right)\frac{\mathrm{d}k}{\sqrt{2\pi}} && (7)\\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty f(y)\left[\frac{1}{2}\left(e^{ik(x-y)} + e^{-ik(x-y)}\right)\right]\frac{\mathrm{d}k\,\mathrm{d}y}{2\pi} && (8)\\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty f(y)\cos k(x-y)\,\mathrm{d}k\,\mathrm{d}y && (9)\\
&= \boxed{\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos k(x-y)\,\mathrm{d}k\,\mathrm{d}y}, && (10)
\end{aligned}$$

which is the of the same form as Eq. (1) noting that cosine is odd in its argument.

4.4) By using the Fourier-transform formulas

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos kx\,\mathrm{d}k \int_0^\infty f(y)\cos ky\,\mathrm{d}y\,, \tag{11a}$$

if f is both real and even, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin kx\,\mathrm{d}k \int_0^\infty f(y)\sin ky\,\mathrm{d}y\,. \tag{11b}$$

if f is both real and odd, derive the formulas

$$e^{-\beta|x|} = \frac{2}{\pi}\int_0^\infty \frac{\beta\cos kx}{\beta^2+k^2}\,\mathrm{d}k \tag{12a}$$

and

$$\frac{x}{|x|}e^{-\beta|x|} = \frac{2}{\pi}\int_0^\infty \frac{k\sin kx}{\beta^2+k^2}\,\mathrm{d}k \tag{12b}$$

for the even and odd extensions of the exponential $\exp(-\beta|x|)$.

Let us derive the expression for $\exp(-\beta|x|)$, which is clearly a real-valued function and even, so we use Eq. (11a):

$$\boxed{e^{-\beta|x|} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos kx\, e^{-\beta y}\cos ky\,\mathrm{d}y\,\mathrm{d}k = \int_0^\infty \frac{\beta}{\beta^2+k^2}\cos kx\,\mathrm{d}k}\,. \tag{13}$$

Similarly, we notice that $x\exp(-\beta|x|)/|x|$ is an odd real-valued function, so

$$\boxed{\frac{x}{|x|} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin kx\, e^{-\beta y}\sin ky\,\mathrm{d}y\,\mathrm{d}k = \frac{2}{\pi}\int_0^\infty \frac{k}{\beta^2+k^2}\sin kx\,\mathrm{d}k}\,. \tag{14}$$

4.5) For the state $|\psi,t\rangle$ given by

$$\varphi(p) = \sqrt{\frac{a}{2\hbar}}\left(\frac{2}{\pi}\right)^{1/4} e^{-(ap)^2/(2\hbar)^2} \tag{15a}$$

and

$$e^{-iHt}\,|\psi,0\rangle = \int_{-\infty}^\infty e^{-ip^2t/2\hbar m}\,|p\rangle\,\varphi(p)\,\mathrm{d}p\,, \tag{15b}$$

find the wave function $\psi(x,t) = \langle x|\psi,t\rangle$ at time t. Then find the variance of the position operator at that time. Does it grow as time goes by? How?

The wave function in the position basis is computed as follows:

$$\psi(x,t) = \int_{-\infty}^\infty e^{-ip^2t/2\hbar m}\frac{e^{ipx}}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{2\hbar}}\left(\frac{2}{\pi}\right)^{1/4} e^{-(ap)^2/(2\hbar)^2}\,\mathrm{d}p \tag{16}$$

$$= \left(\frac{2}{\pi}\right)^{1/4}\sqrt{\frac{a}{2\hbar}}\sqrt{\frac{4\pi\hbar^2 m}{ma^2+2i\hbar t}}e^{-m\hbar^2x^2/(ma^2+2i\hbar t)} \tag{17}$$

$$= \boxed{\left(\frac{2}{\pi}\right)^{1/4}\sqrt{\frac{2\pi\hbar ma}{ma^2+2i\hbar t}}e^{-m\hbar^2x^2/(ma^2+2i\hbar t)}}\,. \tag{18}$$

The variance of any wavefunction in space is given as $\sigma = \langle x^2 \rangle - \langle x \rangle^2$, so we could find it this way, but the form of a normalized gaussian looks as $\exp(-(x - \langle x \rangle)^2/2\sigma^2)/\sigma\sqrt{2\pi}$, where σ is the variance of the distribution. Actually the variance of the distribution is the same as the variance of the probability density given by the wavefunction, so we can find the variance as follows:

$$|\psi(x,t)|^2 = \sqrt{\frac{2}{\pi}} \frac{2\pi\hbar m a}{m^2a^4 + 4\hbar^2 t^2} e^{-2m^2a^2\hbar^2x^2/(m^2a^4+4\hbar^2t^2)}. \tag{19}$$

Thus, the variance is just

$$\boxed{\sigma^2 = \frac{m^2a^4 + 4\hbar^2t^2}{4m^2a^2\hbar^2}}. \tag{20}$$

Clearly, we can see that the variance of the wavefunction grows, meaning that the position becomes less localized in space. Specifically, the variance grows quadratically in t.

4.6) At time $t = 0$, a particle of mass m is in a gaussian superposition of momentum eigenstates centered at $p = \hbar K$

$$\psi(x,0) = \mathcal{N} \int_{-\infty}^{\infty} e^{ikx} e^{-L^2(k-K)^2}\,\mathrm{d}k\,. \tag{21}$$

a) Shift k by K and do the integral. Whare is the particle most likely to be found?

Let $\ell = k - K$ such that

$$\begin{aligned}
\psi(x,0) = \mathcal{N} \int_{-\infty}^{\infty} e^{i(\ell+K)x} e^{-L^2\ell^2}\,\mathrm{d}\ell &= e^{iKx}\mathcal{N} \int_{-\infty}^{\infty} e^{i\ell x} e^{-L^2\ell^2}\,\mathrm{d}\ell & (22)\\
&= \mathcal{N} e^{iKx} \int_{-\infty}^{\infty} e^{-x^2/4L^2} e^{-L^2(\ell - ix/2L^2)^2}\,\mathrm{d}\ell & (23)\\
&= \mathcal{N} e^{iKx} e^{-x^2/4L^2} \left(\frac{\sqrt{\pi}}{L}\right). & (24)
\end{aligned}$$

We can normalize this state as follows:

$$\int_{-\infty}^{\infty} |\psi|^2\,\mathrm{d}x = \mathcal{N}^2\left(\frac{\pi}{L^2}\right) \int_{-\infty}^{\infty} e^{-x^2/2L^2}\,\mathrm{d}x = \mathcal{N}^2\left(\frac{\pi}{L^2}\right)\left(L\sqrt{2\pi}\right) = \frac{\sqrt{2\pi^3}}{L}\mathcal{N}^2. \tag{25}$$

Thus, we require

$$\mathcal{N} = \left(\frac{L^2}{2\pi^3}\right)^{1/4} \Rightarrow \psi(x,0) = \frac{1}{(2\pi L^2)^{1/4}} e^{iKx} e^{-x^2/4L^2}. \tag{26}$$

Thus, the expected value of x at $t = 0$ is

$$\boxed{\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x,0) x \psi(x,0)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}L} \int_{-\infty}^{\infty} x e^{-x^2/2L^2}\,\mathrm{d}x = 0} \tag{27}$$

since this is an odd function integrated over an even range.

b) At time t, the wave function $\psi(x,t)$ is $\psi(x,0)$ but with ikx replaced with $ikx - i\hbar k^2 t/2m$. Shift k by K and do the integral. Where is the particle most likely to be found?

The wavefunction at an arbitrary time $t > 0$ is given by

$$\psi(x,t) = \mathcal{N} \int_{-\infty}^{\infty} e^{ikx - i\hbar k^2 t/2m} e^{-L^2(k-K)^2} \, \mathrm{d}k \,. \tag{28}$$

Again, we let $\ell = k - K$ such that

$$\begin{aligned}
\psi(x,t) &= \mathcal{N} \int_{-\infty}^{\infty} e^{i(\ell+K)x - i\hbar(\ell+K)^2 t/2m} e^{-L^2\ell^2} \, \mathrm{d}k & (29)\\
&= \mathcal{N} e^{iKx} e^{-i\hbar K^2 t/2m} \int_{-\infty}^{\infty} e^{i\ell x} e^{-i\hbar \ell^2 t/2m} e^{-i\hbar \ell K t/m} e^{-L^2\ell^2} \, \mathrm{d}\ell & (30)\\
&= \mathcal{N} e^{iKx - i\hbar K^2 t/2m} \int_{-\infty}^{\infty} e^{i(x - \hbar K t/m)\ell} e^{-(L^2 + i\hbar t/2m)\ell^2} \, \mathrm{d}\ell & (31)\\
&= \mathcal{N} e^{iKx - i\hbar K^2 t/m} \sqrt{\frac{\pi}{L^2 + i\hbar t/2m}} e^{-(x - \hbar K t/2m)^2/4(L^2 + i\hbar t/2m)}. & (32)
\end{aligned}$$

Thus, we can normalize the wavefunction as follows:

$$\mathcal{N}^2 \frac{\pi}{\sqrt{L^4 + \hbar^2 t^2/4m^2}} \int_{-\infty}^{\infty} e^{-L^2(x - \hbar K t/m)^2/2(L^4 + \hbar^2 t^2/4m^2)} \, \mathrm{d}x = 1 \tag{33}$$

$$\mathcal{N}^2 \frac{\pi}{\sqrt{L^4 + \hbar^2 t^2/4m^2}} \sqrt{\frac{2\pi\left(L^4 + \hbar^2 t^2/4m^2\right)}{L^2}} = \mathcal{N}^2 \left[\frac{2\pi^3}{L^2}\right]^{1/2} = 1 \tag{34}$$

$$\mathcal{N} = \left(\frac{L^2}{2\pi^3}\right)^{1/4}, \tag{35}$$

which is the same normalization as for $\psi(x,0)$ (this is expected – actually it is required for conservation of probability). The wavefunction for all times is then given by

$$\psi(x,t) = \left(\frac{L^2}{2\pi^3}\right)^{1/4} \sqrt{\frac{\pi}{L^2 + i\hbar t/2m}} e^{iKx - i\hbar K^2 t/2m} e^{-(x - \hbar K t/m)^2/4(L^2 + i\hbar t/2m)}. \tag{36}$$

Hence, the expected position is

$$\begin{aligned}
\langle x \rangle &= \frac{L}{\pi\sqrt{2\pi}} \frac{\pi}{\sqrt{L^4 + \hbar^2 t^2/4m^2}} \int_{-\infty}^{\infty} x e^{-L^2(x - \hbar K t/m)^2/2(L^4 + \hbar^2 t^2/4m^2)} \, \mathrm{d}x & (37)\\
&= \frac{L}{\sqrt{2\pi(L^4 + \hbar^2 t^2/4m^2)}} \frac{\hbar K t}{m} \sqrt{\frac{2\pi(L^4 + \hbar^2 t^2/4m^2)}{L^2}} & (38)\\
&= \boxed{\frac{\hbar K}{m} t}. & (39)
\end{aligned}$$

This is a very nice result, appearing as the classical relation between a particle's position and momentum ($p = \hbar K$).

c) Does the wave packet spread out like t or like $\sqrt{t}$ as in classical diffusion?

The spread of the gaussian wave packet is given by its standard deviation, where the time-dependent behavior goes like t for large time scales.

4.11) Derive

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = -\vec{\nabla}^2 \vec{A} = \mu_0 \vec{J} \tag{40}$$

from $\vec{B} = \vec{\nabla} \times \vec{A}$ and Ampère's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$.

Observe that

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}. \tag{41}$$

Note that we can make $\vec{\nabla} \cdot \vec{A} = 0$ by a suitable gauge transformation. Thus, in the appropriate gauge,

$$\boxed{-\vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}}. \tag{42}$$

4.14) Use the Green's function relations

$$-\vec{\nabla}^2 G(\vec{x} - \vec{x}') = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = \delta^{(3)}(\vec{x} - \vec{x}') \tag{43a}$$

and

$$G(\vec{x} - \vec{x}') = \frac{1}{2\pi^2 |\vec{x} - \vec{x}'|} \int_0^\infty \frac{\sin k \, \mathrm{d}k}{k} \tag{43b}$$

to show that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3\vec{x}' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{44}$$

satisfies Eq. (40).

Notice that the Green's function is

$$G(\vec{x} - \vec{x}') = \frac{1}{4\pi |\vec{x} - \vec{x}'|}. \tag{45}$$

Thus,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \mu_0 \int \mathrm{d}^3\vec{\boldsymbol{x}}'\, G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}'). \tag{46}$$

Hence,

$$\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \mu_0 \int \mathrm{d}^3\vec{\boldsymbol{x}}' \left[\vec{\boldsymbol{\nabla}}^2 G(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}')\right] J(\vec{\boldsymbol{x}}') = \mu_0 \int \mathrm{d}^3\vec{\boldsymbol{x}}'\, \delta^{(3)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}') J(\vec{\boldsymbol{x}}') \tag{47}$$

$$\Rightarrow \boxed{\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = \mu_0 \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})}. \tag{48}$$

4.16) Compute the Laplace transform of $1/\sqrt{t}$ (Hint: let $t = u^2$).

The Laplace transform of $1/\sqrt{t}$ is computed as follows

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty \frac{1}{\sqrt{t}} e^{-st}\,\mathrm{d}t\,. \tag{49}$$

If we let $t = u^2$, then $\mathrm{d}t = 2u\,\mathrm{d}u$ and

$$\boxed{\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty \frac{1}{u} e^{-su^2} 2u\,\mathrm{d}u = 2\int_0^\infty e^{-su^2}\,\mathrm{d}u = \sqrt{\frac{\pi}{s}}}, \tag{50}$$

where we have assumed that $|s| > 0$ such that the integral converges.

4.17) Show that the commutation relations

$$[a(\vec{\boldsymbol{k}}), a^\dagger(\vec{\boldsymbol{k}}')] = \delta(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}') \quad \text{and} \quad [a(\vec{\boldsymbol{k}}), a(\vec{\boldsymbol{k}}')] = [a^\dagger(\vec{\boldsymbol{k}}), a^\dagger(\vec{\boldsymbol{k}}')] = 0 \tag{51}$$

of the annihilation and creation operators imply the equal-time commutation relations

$$[\phi(\vec{\boldsymbol{x}}, t), \pi(\vec{\boldsymbol{y}}, t)] = i\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}) \quad \text{and} \quad [\phi(\vec{\boldsymbol{x}}, t), \phi(\vec{\boldsymbol{y}}, t)] = [\pi(\vec{\boldsymbol{x}}, t), \pi(\vec{\boldsymbol{y}}, t)] = 0 \tag{52}$$

for the field ϕ and its conjugate momentum π.

The fields

$$\phi(x) = \int \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}} - \omega_k t)} a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}} - \omega_k t)} a^\dagger(\vec{\boldsymbol{k}})\right] \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2\omega_k}} \tag{53a}$$

and

$$\pi(x) = -i \int \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}} - \omega_k t)} a(\vec{\boldsymbol{k}}) - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}} - \omega_k t)} a^\dagger(\vec{\boldsymbol{k}})\right] \sqrt{\frac{\omega_k}{2(2\pi^3)}}\,\mathrm{d}^3 k\,. \tag{53b}$$

Hence,

$$[\phi(\vec{\boldsymbol{x}},t),\pi(\vec{\boldsymbol{y}},t)] = \phi(\vec{\boldsymbol{x}},t)\pi(\vec{\boldsymbol{y}},t) - \pi(\vec{\boldsymbol{y}},t)\phi(\vec{\boldsymbol{x}},t) \tag{54}$$

$$= -i\iint \Bigg[\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]$$

$$- \left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\Bigg]\times$$

$$\frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\sqrt{\frac{\omega_{k'}}{2(2\pi)^3}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{55}$$

$$= i\iint \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}),a^\dagger(\vec{\boldsymbol{k}}')\right] + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}'),a^\dagger(\vec{\boldsymbol{k}})\right]\right]\times$$

$$\frac{1}{2(2\pi)^3}\sqrt{\frac{\omega_{k'}}{\omega_k}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{56}$$

$$= i\int \left[e^{i\vec{\boldsymbol{k}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} + e^{-i\vec{\boldsymbol{k}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\right]\frac{\mathrm{d}^3k}{(2\pi)^3} = \frac{i}{2(2\pi)^3}2(2\pi)^3\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}) = \boxed{i\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}. \tag{57}$$

Similarly,

$$[\phi(\vec{\boldsymbol{x}},t),\phi(\vec{\boldsymbol{y}},t)] = \phi(\vec{\boldsymbol{x}},t)\phi(\vec{\boldsymbol{y}},t) - \phi(\vec{\boldsymbol{y}},t)\phi(\vec{\boldsymbol{x}},t) \tag{58}$$

$$= \iint \Bigg[\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') + e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]$$

$$- \left[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') + e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\right]\left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) + e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\right]\Bigg]\times$$

$$\frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\frac{1}{\sqrt{(2\pi)^3 2\omega_{k'}}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{59}$$

$$= \iint \left[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}),a^\dagger(\vec{\boldsymbol{k}}')\right] - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\left[a(\vec{\boldsymbol{k}}'),a^\dagger(\vec{\boldsymbol{k}})\right]\right]\times$$

$$\frac{1}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{60}$$

$$= \int \left[e^{i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} - e^{-i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\right]\frac{\mathrm{d}^3k}{(2\pi)^3\omega_k} = \boxed{0}, \tag{61}$$

and

$$[\pi(\vec{\boldsymbol{x}},t),\pi(\vec{\boldsymbol{y}},t)] = \pi(\vec{\boldsymbol{x}},t)\pi(\vec{\boldsymbol{y}},t) - \pi(\vec{\boldsymbol{y}},t)\pi(\vec{\boldsymbol{x}},t) \tag{62}$$

$$= -\iint \Bigg[\Big[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\Big]\Big[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\Big]$$
$$- \Big[e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a(\vec{\boldsymbol{k}}') - e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}a^\dagger(\vec{\boldsymbol{k}}')\Big]\Big[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a(\vec{\boldsymbol{k}}) - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}a^\dagger(\vec{\boldsymbol{k}})\Big]\Bigg]\times$$
$$\sqrt{\frac{\omega_k}{2(2\pi)^3}}\sqrt{\frac{\omega_{k'}}{2(2\pi)^3}}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{63}$$

$$= \iint \Bigg[e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{-i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\Big[a(\vec{\boldsymbol{k}}),a^\dagger(\vec{\boldsymbol{k}}')\Big] - e^{-i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega_k t)}e^{i(\vec{\boldsymbol{k}}'\cdot\vec{\boldsymbol{y}}-\omega_{k'} t)}\Big[a(\vec{\boldsymbol{k}}'),a^\dagger(\vec{\boldsymbol{k}})\Big]\Bigg]\times$$
$$\frac{\sqrt{\omega_k\omega_{k'}}}{2(2\pi)^3}\,\mathrm{d}^3k\,\mathrm{d}^3k' \tag{64}$$

$$= \int \Big[e^{i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} - e^{-i\vec{\boldsymbol{k}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\Big]\frac{\omega_k\,\mathrm{d}^3k}{(2\pi)^3} = \boxed{0}. \tag{65}$$