Complex Variables

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.[1]

Contents

[1] This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

1 Algebra

The set of complex numbers, $\mathbb{C}$ forms a **field** with additive identity 0 and multiplicative identity 1.

Definition 1 (Multiplicative inverse)**.** The multiplicative inverse of $z = x + iy \neq 0$ is given by

$$z^{-1} := \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Definition 2 (Roots)**.** n^{th} roots of a complex number $z = re^{i\theta}$ is given by

$$z^{1/n} := r^{1/n}e^{i\frac{\theta+2k\pi}{n}}, \qquad (k \in \langle n \rangle)$$

Definition 3 (Euler form)**.**

$$e^{i\theta} := \cos\theta + i\sin\theta$$

where $\theta \in \mathbb{R}$.

2 Topology

Definition 4 (Open disk)**.** An open disk centered at $z_0 \in \mathbb{C}$ and radius $R > 0$ is the set

$$B_R(z_0) = \{z \mid |z - z_0| < R\}$$

Definition 5 ($\epsilon-$neighbourhood)**.** An $\epsilon-$neighbourhood of z_0 is the set($\epsilon > 0$) $B_\epsilon(z_0)$.

Definition 6 (Deleted $\epsilon-$neighbourhood)**.** A deleted $\epsilon-$neighbourhood of z_0 is the set($\epsilon > 0$) $B_\epsilon(z_0)\backslash\{z_0\}$.

Definition 7 (Interior, exterior and boundary points)**.** Let $S \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$.
z_0 is an interior point of S if $\exists\epsilon > 0$ such that $B_\epsilon(z_0) \subset S$.
z_0 is an exterior point of S if $\exists\epsilon > 0$ such that $B_\epsilon(z_0) \cap S = \varnothing$.
z_0 is a boundary point of S if it is neither an interior point nor an exterior point.

Definition 8 (Open sets)**.** A set is open if every element is an interior point of the set.

Definition 9 (Closed sets)**.** A set is closed if it contains all its boundary points.

Definition 10 (Closure of a set)**.** The union of the set and its boundary points.

Definition 11 (Connected set)**.** $S \subset \mathbb{C}$ is connected if every $z_1, z_2 \in S$ can be joined by a finite sequence of line segments lying inside S.

Definition 12 (Domain)**.** An open, connected set.

Definition 13 (Bounded set)**.** $S \subset \mathbb{C}$ is bounded if $\exists R > 0$ such that

$$|z| < R, \forall z \in S$$

3 Differential calculus

3.1 Limits

Definition 14 (Limit)**.** Let f be defined in some deleted $\epsilon-$neighbourhood of z_0. If $\forall\epsilon > 0, \exists\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

then we say

$$\lim_{z\to z_0} f(z) = w_0$$

Theorem 1 (Uniqueness of limit)**.** Limit of a function at a point, if exists, is unique, i.e., if we find $w_1, w_2 \in \mathbb{C}$ that satisfy the condition for the limit of a function at some $z_0 \in \mathbb{C}$, then $w_1 = w_2$.

Theorem 2. $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Theorem 3 (Multivariable limits)**.** Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. $\lim_{z\to z_0} f(z)$ exists iff the multivariable limits

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y), \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y)$$

exist. If they exist,

$$\lim_{z\to z_0} f(z) = \left(\lim_{(x,y)\to(x_0,y_0)} u(x, y)\right) + i\left(\lim_{(x,y)\to(x_0,y_0)} v(x, y)\right)$$

3.2 Continuity

Definition 15 (Continuity)**.** Let f be defined in some $\epsilon-$neighbourhood of $z_0 \in \mathbb{C}$. f is said to be continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

provided the limit exists.

Theorem 4. Composition of continuous functions(at a point) is continuous(at that point).

Theorem 5. Continuous functions are bounded in open, bounded domains. Let $R \subset \mathbb{C}$ be open and bounded. If f is continuous in R, then f is bounded in R.

Theorem 6. Let f be continuous at z_0. If $f(z_0) \neq 0, \exists \epsilon > 0$ such that

$$f(z) \neq 0, \forall z \in B_\epsilon(z_0)$$

3.3 Derivative

From now onwards, Ω represents an open subset of $\mathbb{C}$ unless stated otherwise.

Definition 16 (Differentiability)**.** $f : \Omega \to \mathbb{C}$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is denoted by $f'(z_0)$ and is called the derivative of f at z_0. This is equivalent to

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Example 1. Find the points where $f(z) = e^{\bar{z}}$ is differentiable.

Theorem 7. $f(z) = \bar{z}$ is not differentiable anywhere.
$f(z) = |z|^2$ is only differentiable at $z = 0$.

Theorem 8. Differentiability(at a point) $\implies$ Continuity(at that point).

3.4 Cauchy-Riemann equations

Definition 17 (Cauchy-Riemann equations)**.** Let $u : \mathbb{R}^2 \to \mathbb{R}, v : \mathbb{R}^2 \to \mathbb{R}$. The Cauchy-Riemann equations of u, v are given by

$$u_x = v_y$$
$$v_x = -u_y$$

provided the partial derivatives exist.

In polar coordinates, the equivalent set of equations are

$$ru_r = v_\theta$$

$$u_\theta = -rv_r$$

Theorem 9. If $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$, then u, v satisfy the Cauchy-Riemann(CR) equations at (x_0, y_0).

Theorem 10. If f is differentiable in an open and connected set Ω such that $f'(z) = 0,\ \forall z \in \Omega$, then

$$f(z) = \text{constant}, \forall z \in \Omega$$

Theorem 11. $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$ if

1. The first-order partial derivatives of $u(x, y), v(x, y)$ exist in some neighbourhood of (x_0, y_0).
2. These partial derivatives are continuous at (x_0, y_0).
3. u, v satisfy the CR equations at (x_0, y_0).

If these conditions are satisfied, we have

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Corollary 11.1. $f = u + iv$ is differentiable at $z_0 = r_0 e^{i\theta_0}$ if

1. The first-order partial derivatives of $u(r, \theta), v(r, \theta)$ exist in some neighbourhood of (r_0, θ_0).
2. These partial derivatives are continuous at (r_0, θ_0).
3. u, v satisfy the CR equations at (r_0, θ_0).

If these conditions are satisfied, we have

$$f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0))$$

4 Analytic Functions

4.1 Analytic functions

Definition 18 (Analytic function)**.** f is analytic at z_0 if it is differentiable in some neighbourhood of z_0.

Definition 19 (Entire function)**.** A function is entire if it is analytic everywhere in $\mathbb{C}$.

Definition 20 (Singular point)**.** z_0 is a singular point of f if

1. f is not analytic at z_0
2. f is analytic in some deleted neighbourhood of z_0.

Theorem 12. If f is analytic in Ω, then it is continuous in Ω.

Theorem 13 (Rational functions)**.** A polynomial is an entire function. A rational function is analytic everywhere in its domain.

Theorem 14. If $f = u + iv$ is analytic at $z_0 = x_0 + iy_0$, then u, v have continuous, partial derivatives of all orders at (x_0, y_0).

Theorem 15. $f = u + iv$ is analytic in a domain $\mathscr{D}$ iff

1. The first-order partial derivatives of u, v exist in $\mathscr{D}$.
2. These partial derivatives are continuous in $\mathscr{D}$.
3. u, v satisfy the CR equations in $\mathscr{D}$.

4.2 Harmonic conjugates

Definition 21 (Laplace equation)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$. The Laplacian of f at (x_0, y_0) is defined as

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

evaluated at (x_0, y_0) and is denoted by $\nabla^2 f(x_0, y_0)$ or $\Delta f(x_0, y_0)$.

The Laplace equation of f is given by

$$\Delta f = 0$$

Definition 22 (Harmonic function)**.** $H : \mathbb{R}^2 \to \mathbb{R}$ is harmonic in a domain $\mathscr{D}$ if

1. First and second order partial derivative of H exist in $\mathscr{D}$.
2. H satisfies Laplace equation in $\mathscr{D}$.

Theorem 16. If $f = u + iv$ is analytic in $\mathscr{D}$, then u, v are harmonic in $\mathscr{D}$.

Definition 23 (Harmonic conjugates)**.** Let u, v be harmonic in $\mathscr{D}$. If they satisfy the CR equations in $\mathscr{D}$, v is called the harmonic conjugate of u in $\mathscr{D}$.

Theorem 17. Let $v(x, y)$ be a harmonic conjugate of $u(x, y)$. The set of all harmonic conjugates of u is given by

$$\{v(x, y) + k \mid k \in \mathbb{R}\}$$

Definition 24 (Level curve)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a multivariable function. $f(x, y) = c, c \in \mathbb{R}$ is called a level curve of f.

Example 2. Sketch the family of level curves of the real and imaginary parts of $f(z) = \frac{1}{z-1}$.

Theorem 18. If v is a harmonic conjugate of u, then their level curves always intersect orthogonally in the $xy-$plane.

Theorem 19. $f = u + iv$ is analytic in $\mathscr{D}$ iff v is a harmonic conjugate of u in $\mathscr{D}$.

Example 3. Let $u(x, y) = 2x(1 - y),\ v(x, y) = x^2 - y^2 + 2y$. Show that v is a harmonic conjugate of u.

Example 4. Let $u(x, y) = \cos x \cosh y,\ v(x, y) = -\sin x \sinh y$. Show that v is a harmonic conjugate of u.(*Hint:* $\cos z$ is analytic, Definition 29Cosine functiondefn.29)

5 Important functions

5.1 Exponential

Definition 25 (Exponential function)**.**

$$\exp(x + iy) := e^x e^{iy}$$

where e^x is the real exponential function.
Domain: $\mathbb{C}$
Range: $\mathbb{C}\backslash\{0\}$
Analytic: $\mathbb{C}$

5.2 Logarithm

Definition 26 (Logarithm multi-valued function)**.** Let $z = re^{i\theta}$.

$$\log(z) := \ln r + i\theta$$

where ln is the real logarithm function.
Domain: $\mathbb{C}\backslash\{0\}$
Range: $\mathbb{C}$
When we put in the restriction $-\pi < \theta \leq \pi$, we get the principle logarithm $\mathrm{Log}(z)$.

Definition 27 (Branch of logarithm)**.** As the complex logarithm is multi-valued, we often restrict θ to intervals of length 2π. These functions are called branches of the logarithm function. We have already defined the principle branch.

Theorem 20. A branch of the logarithm function is analytic in $\mathbb{C}\backslash\{x \in \mathbb{R} \mid x \leq 0\}$.

Corollary 20.1. $\mathrm{Log}(z)$ is analytic if $-\pi < \mathrm{Arg}(z) < \pi$.

Theorem 21. $\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}(z) = \frac{1}{z}, -\pi < \mathrm{Arg}(z) < \pi$

5.3 Power

Definition 28 (Power function)**.**

$$z^c := \exp(c \log z)$$

This is also a multi-valued function and its branches are defined by the branch of the logarithm in the function.
Domain: $\mathbb{C}\backslash\{0\}$
Range: $\mathbb{C}\backslash\{0\}$
Analytic: Any branch is analytic in its domain

Theorem 22. $\frac{\mathrm{d}}{\mathrm{d}z} z^c = cz^{c-1}$

5.4 Trigonometric

Definition 29 (Cosine function)**.**

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

Domain: $\mathbb{C}$
Range: $\mathbb{C}$
Analytic: $\mathbb{C}$

Definition 30 (Sine function)**.**

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2}$$

Domain: $\mathbb{C}$
Range: $\mathbb{C}$
Analytic: $\mathbb{C}$

Using these, we may define $\tan, \sec$, cosec. These are analytic in their domains. The complex trigonometric functions satisfy most of the relations satisfied by the real trigonometric functions.

5.5 Hyperbolic

Definition 31 (Hyperbolic cosine function)**.**

$$\cosh(z) := \frac{e^{z} + e^{-z}}{2}$$

Domain: $\mathbb{C}$
Range: $\mathbb{C}$
Analytic: $\mathbb{C}$

Definition 32 (Hyperbolic sine function)**.**

$$\sinh(z) := \frac{e^{z} - e^{-z}}{2}$$

Domain: $\mathbb{C}$
Range: $\mathbb{C}$
Analytic: $\mathbb{C}$

Using these, we may define $\tanh, \mathrm{sech}$, cosech. These are analytic in their domains. The complex hyperbolic functions satisfy most of the relations satisfied by the real hyperbolic functions.

6 Integral calculus

6.1 Fundamental Theorem of Calculus

Definition 33. Let $f : [a, b] \to \mathbb{C}$ such that $f(t) = u(t) + iv(t)$.

$$\int_a^b f(t)\ dt := \int_a^b u(t)\ dt + i \int_a^b v(t)\ dt$$

Theorem 23.

$$\left| \int_a^b f(t)\ dt \right| \leq \int_a^b |f(t)|\ dt$$

Example 5. Let $x \in [-1, 1], \theta \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$. Show that

$$\left| \frac{1}{\pi} \int_0^\pi (x + i\sqrt{1 - x^2} \cos\theta)^n\ d\theta \right| \leq 1$$

Theorem 24 (Fundamental Theorem of Calculus)**.**

1. Let $f : [a, b] \to \mathbb{C}$ be a continuous function.

$$F(x) = \int_a^x f(t)\ dt \implies F'(x) = f(x), \forall x \in [a, b]$$

2. Let $f : [a, b] \to \mathbb{C}$ be a continuous function.

$$F'(t) = f(t), \forall t \in [a, b] \implies \int_a^b f(t)\ dt = F(b) - F(a)$$

Example 6. Evaluate(using a complex-valued function)

$$\int_0^\pi e^{2t} \cos t\ dt$$

6.2 Arcs and Contours

Definition 34 (Arc)**.** Let $z : [a, b] \to \mathbb{C}$. The set

$$C = \{z(t) \mid a \leq t \leq b\}$$

is called an arc in $\mathbb{C}$.

C is a **simple arc** in [a,b] if $\forall t_1, t_2 \in [a, b]$,

$$t_1 \neq t_2 \implies z(t_1) \neq z(t_2)$$

C is a simple, closed arc or a **Jordan curve** in $[a, b]$ if

1. C is simple in (a, b).
2. $z(a) = z(b)$

C is a **differentiable arc** if z is continuously differentiable in $[a, b]$.
C is a **smooth arc** if it is differentiable and $z'(t) \neq 0, \forall t \in [a, b]$.

Definition 35 (Orientation of arc)**.** Let C be an arc as defined earlier. If $z(t)$ moves in the counter-clockwise direction in the complex plane as t increases, C is said to be positively oriented. Otherwise, it is negatively oriented.

Definition 36 (Reparameterization)**.** Let C be the arc

$$C : z(t), t \in [a, b]$$

Let $\varphi : [c, d] \to [a, b]$ be a continuously differentiable, strictly increasing bijective mapping such that

$$t = \varphi(s)$$

We get a reparametrization of C

$$C : Z(s), s \in [c, d]$$

where $Z(s) = z(\varphi(s))$.

Definition 37 (Length of arc)**.** Let $C : z(t), t \in [a, b]$ be a differentiable arc.

$$\text{length}(C) := \int_a^b |z'(t)| \ dt$$

Definition 38 (Contour)**.** A piecewise smooth curve.

Contours in $\mathbb{C}$ are analogous to intervals in $\mathbb{R}$, in a sense that we will integrate functions along contours.

The **negative of a contour** is defined as

$$C : z(t), t \in [a, b]$$

$$-C : z(-t), -t \in [-b, -a]$$

6.3 Contour integral

Definition 39 (Contour integral)**.** Let
$f : \Omega \to \mathbb{C}$
$C : z(t), t \in [a, b]$
$f(z(t))$ be a piecewise continuous function of t.

$$\int_C f(z) \ dz := \int_a^b f(z(t))z'(t) \ dt$$

Theorem 25.

$$\int_{C_1+C_2} f(z) \ dz = \int_{C_1} f(z) \ dz + \int_{C_2} f(z) \ dz$$

$$\int_{-C} f(z) \ dz = -\int_C f(z) \ dz$$

Example 7. Let C be the contour consisting of the semicircle $e^{i\theta}, \theta \in [0, \pi]$ along with the part of the the real axis $x, x \in [-1, 1]$. Let

$$f(z) = f(re^{i\theta}) = \sqrt{r}e^{i\frac{\theta}{2}} \qquad (r > 0, \tfrac{-\pi}{2} < \theta < \tfrac{3\pi}{2})$$

Find $\int_C f(z)\ dz$.(*Hint:* Set $f(0) = 0$)

Theorem 26 (Bound of a contour integral)**.** Let
$f : \Omega \to \mathbb{C}$
$C : z(t), t \in [a, b]$ be a contour of length L
$f(z(t))$ be a piecewise continuous function of t, i.e. it has an upper bound(Theorem 5thm.5) $M > 0$ in $[a, b]$. We have,

$$\left| \int_C f(z)\ dz \right| \leq ML$$

Example 8. Let C be the right-triangle formed by the two axes and the line segment joining -4 and $3i$, oriented anticlockwise. Show that

$$\left| \int_C (e^z - \bar{z})\ dz \right| \leq 60$$

Example 9. Let C be the circle $|z| = R, R > 1$, oriented anticlockwise. Show that

$$\left| \int_C \frac{\operatorname{Log}(z)}{z^2}\ dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

6.4 Antiderivative

Definition 40 (Antiderivative)**.** Let f and F be defined in Ω such that

$$F'(z) = f(z), \forall z \in \Omega$$

F is called the anti-derivative of f in Ω.

Theorem 27. Antiderivatives, if exist, are analytic.

Theorem 28. If F_1, F_2 are antiderivatives of f in Ω, $\exists k \in \mathbb{C}$ such that

$$F_2(z) = F_1(z) + k, \forall z \in \Omega$$

Theorem 29. Let f be a continuous function defined on Ω. The following statements are equivalent:

- f has an antiderivative F in Ω.
- Let C_1, C_2 be two contours in Ω with the same end-points z_1, z_2 and same orientation $z_1 \to z_2$.

  $$\int_{C_1} f(z)\ dz = \int_{C_2} f(z)\ dz$$

  In fact, this integral is equal to $F(z_2) - F(z_1)$.
- If C is a closed contour in Ω,

  $$\int_C f(z)\ dz = 0$$

Example 10. Let C be the unit circle centered at 0 and $f(z) = \frac{1}{z}$. Show that f does not have an antiderivative in its domain.

Can we find an antiderivative of f in some open subset of its domain?(*Hint:* Theorem 21thm.21)

Example 11. Let C be any simple contour with end-points $-3, 3$ lying below the real axis, oriented $-3 \to 3$.

$$f(z) = \exp\left(\frac{1}{2} \log z \right) \qquad (0 < \arg z < 2\pi)$$

Find $\int_C f(z)\ dz$.

7 Cauchy's Theorem

7.1 Cauchy's Theorem

Definition 41 (Simply-connected domain)**.** A domain $\mathscr{D}$ is simply-connected if every simple, closed contour within it encloses only points in $\mathscr{D}$.

Theorem 30. If C is a simple, closed contour, the set of points inside C form a simply-connected domain.

Theorem 31 (Cauchy)**.** Let f be analytic in a simply-connected domain $\mathscr{D}$. For any simple, closed contour $C \subset D$,

$$\int_C f(z) = 0$$

Example 12. Let C be the unit circle. Find

$$\int_C z|z|^4 \; dz$$

Corollary 31.1. If f is analytic in a simply-connected domain $\mathscr{D}$, it has an antiderivative in $\mathscr{D}$.(Thoerem 29thm.29) Let F be the antiderivative of f in $\mathscr{D}$ and C be any contour in $\mathscr{D}$ with endpoints z_1, z_2 and orientation $z_1 \to z_2$.

$$\int_C f(z) \; dz = F(z_2) - F(z_1)$$

Corollary 31.2. If f is analytic on and inside a simple, closed contour C,

$$\int_C f(z) \; dz = 0$$

7.2 Cauchy's Integral Formula

Theorem 32 (Cauchy's integral formula)**.** Let f be analytic inside and on a simple, closed, positive contour C. If z_0 is a point inside C,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \; dz$$

Hence, the value of the function inside C is completely defined by its value on C.

Example 13. Let C be the square whose sides lie along $x = \pm 2, y = \pm 2$, oriented anticlockwise. Find

$$\int_C \frac{\cos z}{z(z^2 + 8)} \; dz$$

Example 14. Let C be the circle $|z| = 3$, oriented anticlockwise.

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} \; dz$$

Find $g(2)$ and $g(4)$.

Corollary 32.1 (Gauss' mean value theorem)**.** Let f be analytic in $B_r(z_0)$.

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \; d\theta$$

Lemma 32.1. Let f be analytic inside and on a simple, closed, positive contour C. If z_0 is a point inside C, f is infinitely differentiable at z_0. The n^{th} derivative of f at z_0 is given by

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \; dz \qquad (n = 1, 2, \cdots)$$

Example 15. Let C be the circle $|z - i| = 2$, oriented anticlockwise. Find

$$\int_C \frac{dz}{(z^2 + 4)^2}$$

(*Hint*: $z^2 + 4 = (z + 2i)(z - 2i)$)

Theorem 33. If f is analytic at z_0 then its derivatives of all orders exist and are analytic at z_0.

Corollary 33.1. Theorem 14thm.14

7.3 Morera's Theorem

Theorem 34 (Morera)**.** Let f be continuous in Ω and C be a closed contour in Ω.

$$\int_C f(z)\ dz = 0, \forall C \implies f \text{ is analytic in } \Omega$$

Theorem 35. Let $\mathscr{D}$ be a simply-connected domain and f be continuous in $\mathscr{D}$. Let C represent a closed contour in $\mathscr{D}$.

$$\int_C f(z)\ dz = 0, \forall C \iff f \text{ is analytic in } \Omega$$

7.4 Cauchy's Inequality

Theorem 36 (Cauchy's inequality)**.** Let

$$D_R(z_0) = \{z \mid |z - z_0| \leq R\}$$

$$C_R(z_0) = \{z \mid |z - z_0| = R\}$$

f is an analytic function in $D_R(z_0)$ such that M_R is the maximum value of $|f(z)|$ on $C_R(z_0)$.

$$|f^n(z_0)| \leq \frac{n!\ M_R}{R^n} \qquad (n = 1, 2, \cdots)$$

Example 16. f is an entire function such that

$$|f(z)| \leq A|z|, \forall z \in \mathbb{C} \qquad (A \in \mathbb{R})$$

Show that $f(z) = az, a \in \mathbb{C}$.

Lemma 36.1 (Liouville)**.** A bounded, entire function is a constant function.

7.5 Fundamental Theorem of Algebra

Theorem 37 (Fundamental theorem of algebra)**.** Every non-constant polynomial has at least one zero.

7.6 Maximum modulus principle

Theorem 38 (Maximum modulus principle)**.** If f is a non-constant, analytic function in $\mathscr{D}$, $|f(z)|$ has no maximum value in $\mathscr{D}$.

Corollary 38.1. Let f be a non-constant, analytic function in a bounded domain Ω such that it is continuous on $\overline{\Omega}$(closure of Ω, Definition 10Closure of a setdefn.10). The maximum value of $|f(z)|$ in $\overline{\Omega}$ exists and lies on the boundary of Ω.