

# Complex Variables

Based on lectures by Dr. Anandateertha Mangasuli

Notes taken by Rwik Dutta

---

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.<sup>1</sup>

## Contents

<b>1</b>	<b>Algebra</b>	<b>2</b>
<b>2</b>	<b>Topology</b>	<b>2</b>
<b>3</b>	<b>Differential calculus</b>	<b>2</b>
3.1	Limits . . . . .	2
3.2	Continuity . . . . .	3
3.3	Derivative . . . . .	3
3.4	Cauchy-Riemann equations . . . . .	3
<b>4</b>	<b>Analytic Functions</b>	<b>4</b>
4.1	Analytic functions . . . . .	4
4.2	Harmonic conjugates . . . . .	4
<b>5</b>	<b>Important functions</b>	<b>5</b>
5.1	Exponential . . . . .	5
5.2	Logarithm . . . . .	5
5.3	Power . . . . .	5
5.4	Trigonometric . . . . .	5
5.5	Hyperbolic . . . . .	6
<b>6</b>	<b>Integral calculus</b>	<b>6</b>
6.1	Fundamental Theorem of Calculus . . . . .	6
6.2	Arcs and Contours . . . . .	7
6.3	Contour integral . . . . .	7
6.4	Antiderivative . . . . .	8
<b>7</b>	<b>Cauchy's Theorem</b>	<b>9</b>
7.1	Cauchy's Theorem . . . . .	9
7.2	Cauchy's Integral Formula . . . . .	9
7.3	Morera's Theorem . . . . .	10
7.4	Cauchy's Inequality . . . . .	10
7.5	Fundamental Theorem of Algebra . . . . .	10
7.6	Maximum modulus principle . . . . .	10

---

<sup>1</sup>This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

# 1 Algebra

The set of complex numbers,  $\mathbb{C}$  forms a **field** with additive identity 0 and multiplicative identity 1.

**Definition 1** (Multiplicative inverse). The multiplicative inverse of  $z = x + iy \neq 0$  is given by

$$z^{-1} := \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}$$

**Definition 2** (Roots).  $n^{th}$  roots of a complex number  $z = re^{i\theta}$  is given by

$$z^{1/n} := r^{1/n} e^{i \frac{\theta + 2k\pi}{n}}, \quad (k \in \langle n \rangle)$$

**Definition 3** (Euler form).

$$e^{i\theta} := \cos \theta + i \sin \theta$$

where  $\theta \in \mathbb{R}$ .

# 2 Topology

**Definition 4** (Open disk). An open disk centered at  $z_0 \in \mathbb{C}$  and radius  $R > 0$  is the set

$$B_R(z_0) = \{z \mid |z - z_0| < R\}$$

**Definition 5** ( $\epsilon$ -neighbourhood). An  $\epsilon$ -neighbourhood of  $z_0$  is the set  $(\epsilon > 0) B_\epsilon(z_0)$ .

**Definition 6** (Deleted  $\epsilon$ -neighbourhood). A deleted  $\epsilon$ -neighbourhood of  $z_0$  is the set  $(\epsilon > 0) B_\epsilon(z_0) \setminus \{z_0\}$ .

**Definition 7** (Interior, exterior and boundary points). Let  $S \subset \mathbb{C}$  and  $z_0 \in \mathbb{C}$ .

$z_0$  is an interior point of  $S$  if  $\exists \epsilon > 0$  such that  $B_\epsilon(z_0) \subset S$ .

$z_0$  is an exterior point of  $S$  if  $\exists \epsilon > 0$  such that  $B_\epsilon(z_0) \cap S = \emptyset$ .

$z_0$  is a boundary point of  $S$  if it is neither an interior point nor an exterior point.

**Definition 8** (Open sets). A set is open if every element is an interior point of the set.

**Definition 9** (Closed sets). A set is closed if it contains all its boundary points.

**Definition 10** (Closure of a set). The union of the set and its boundary points.

**Definition 11** (Connected set).  $S \subset \mathbb{C}$  is connected if every  $z_1, z_2 \in S$  can be joined by a finite sequence of line segments lying inside  $S$ .

**Definition 12** (Domain). An open, connected set.

**Definition 13** (Bounded set).  $S \subset \mathbb{C}$  is bounded if  $\exists R > 0$  such that

$$|z| < R, \forall z \in S$$

# 3 Differential calculus

## 3.1 Limits

**Definition 14** (Limit). Let  $f$  be defined in some deleted  $\epsilon$ -neighbourhood of  $z_0$ . If  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

then we say

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

**Theorem 1** (Uniqueness of limit). Limit of a function at a point, if exists, is unique, i.e., if we find  $w_1, w_2 \in \mathbb{C}$  that satisfy the condition for the limit of a function at some  $z_0 \in \mathbb{C}$ , then  $w_1 = w_2$ .

**Theorem 2.**  $\lim_{z \rightarrow 0} \frac{z}{z}$  does not exist.

**Theorem 3** (Multivariable limits). Let  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  and  $z_0 = x_0 + iy_0$ .  $\lim_{z \rightarrow z_0} f(z)$  exists iff the multivariable limits

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y), \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y)$$

exist. If they exist,

$$\lim_{z \rightarrow z_0} f(z) = \left( \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) \right) + i \left( \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) \right)$$

### 3.2 Continuity

**Definition 15** (Continuity). Let  $f$  be defined in some  $\epsilon$ -neighbourhood of  $z_0 \in \mathbb{C}$ .  $f$  is said to be continuous at  $z_0$  if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

provided the limit exists.

**Theorem 4.** Composition of continuous functions(at a point) is continuous(at that point).

**Theorem 5.** Continuous functions are bounded in open, bounded domains. Let  $R \subset \mathbb{C}$  be open and bounded. If  $f$  is continuous in  $R$ , then  $f$  is bounded in  $R$ .

**Theorem 6.** Let  $f$  be continuous at  $z_0$ . If  $f(z_0) \neq 0, \exists \epsilon > 0$  such that

$$f(z) \neq 0, \forall z \in B_\epsilon(z_0)$$

### 3.3 Derivative

From now onwards,  $\Omega$  represents an open subset of  $\mathbb{C}$  unless stated otherwise.

**Definition 16** (Differentiability).  $f : \Omega \rightarrow \mathbb{C}$  is differentiable at  $z_0$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is denoted by  $f'(z_0)$  and is called the derivative of  $f$  at  $z_0$ . This is equivalent to

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

**Example 1.** Find the points where  $f(z) = e^{\bar{z}}$  is differentiable.

**Theorem 7.**  $f(z) = \bar{z}$  is not differentiable anywhere.

$f(z) = |z|^2$  is only differentiable at  $z = 0$ .

**Theorem 8.** Differentiability(at a point)  $\implies$  Continuity(at that point).

### 3.4 Cauchy-Riemann equations

**Definition 17** (Cauchy-Riemann equations). Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ . The Cauchy-Riemann equations of  $u, v$  are given by

$$u_x = v_y$$

$$v_x = -u_y$$

provided the partial derivatives exist.

In polar coordinates, the equivalent set of equations are

$$ru_r = v_\theta$$

$$u_\theta = -rv_r$$

**Theorem 9.** If  $f = u + iv$  is differentiable at  $z_0 = x_0 + iy_0$ , then  $u, v$  satisfy the Cauchy-Riemann(CR) equations at  $(x_0, y_0)$ .

**Theorem 10.** If  $f$  is differentiable in an open and connected set  $\Omega$  such that  $f'(z) = 0, \forall z \in \Omega$ , then

$$f(z) = \text{constant}, \forall z \in \Omega$$

**Theorem 11.**  $f = u + iv$  is differentiable at  $z_0 = x_0 + iy_0$  if

1. The first-order partial derivatives of  $u(x, y), v(x, y)$  exist in some neighbourhood of  $(x_0, y_0)$ .
2. These partial derivatives are continuous at  $(x_0, y_0)$ .
3.  $u, v$  satisfy the CR equations at  $(x_0, y_0)$ .

If these conditions are satisfied, we have

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

**Corollary 11.1.**  $f = u + iv$  is differentiable at  $z_0 = r_0 e^{i\theta_0}$  if

1. The first-order partial derivatives of  $u(r, \theta), v(r, \theta)$  exist in some neighbourhood of  $(r_0, \theta_0)$ .
2. These partial derivatives are continuous at  $(r_0, \theta_0)$ .
3.  $u, v$  satisfy the CR equations at  $(r_0, \theta_0)$ .

If these conditions are satisfied, we have

$$f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0))$$

## 4 Analytic Functions

### 4.1 Analytic functions

**Definition 18** (Analytic function).  $f$  is analytic at  $z_0$  if it is differentiable in some neighbourhood of  $z_0$ .

**Definition 19** (Entire function). A function is entire if it is analytic everywhere in  $\mathbb{C}$ .

**Definition 20** (Singular point).  $z_0$  is a singular point of  $f$  if

1.  $f$  is not analytic at  $z_0$
2.  $f$  is analytic in some deleted neighbourhood of  $z_0$ .

**Theorem 12.** If  $f$  is analytic in  $\Omega$ , then it is continuous in  $\Omega$ .

**Theorem 13** (Rational functions). A polynomial is an entire function. A rational function is analytic everywhere in its domain.

**Theorem 14.** If  $f = u + iv$  is analytic at  $z_0 = x_0 + iy_0$ , then  $u, v$  have continuous, partial derivatives of all orders at  $(x_0, y_0)$ .

**Theorem 15.**  $f = u + iv$  is analytic in a domain  $\mathcal{D}$  iff

1. The first-order partial derivatives of  $u, v$  exist in  $\mathcal{D}$ .
2. These partial derivatives are continuous in  $\mathcal{D}$ .
3.  $u, v$  satisfy the CR equations in  $\mathcal{D}$ .

### 4.2 Harmonic conjugates

**Definition 21** (Laplace equation). Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . The Laplacian of  $f$  at  $(x_0, y_0)$  is defined as

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

evaluated at  $(x_0, y_0)$  and is denoted by  $\nabla^2 f(x_0, y_0)$  or  $\Delta f(x_0, y_0)$ .

The Laplace equation of  $f$  is given by

$$\Delta f = 0$$

**Definition 22** (Harmonic function).  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  is harmonic in a domain  $\mathcal{D}$  if

1. First and second order partial derivative of  $H$  exist in  $\mathcal{D}$ .
2.  $H$  satisfies Laplace equation in  $\mathcal{D}$ .

**Theorem 16.** If  $f = u + iv$  is analytic in  $\mathcal{D}$ , then  $u, v$  are harmonic in  $\mathcal{D}$ .

**Definition 23** (Harmonic conjugates). Let  $u, v$  be harmonic in  $\mathcal{D}$ . If they satisfy the CR equations in  $\mathcal{D}$ ,  $v$  is called the harmonic conjugate of  $u$  in  $\mathcal{D}$ .

**Theorem 17.** Let  $v(x, y)$  be a harmonic conjugate of  $u(x, y)$ . The set of all harmonic conjugates of  $u$  is given by

$$\{v(x, y) + k \mid k \in \mathbb{R}\}$$

**Definition 24** (Level curve). Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a multivariable function.  $f(x, y) = c, c \in \mathbb{R}$  is called a level curve of  $f$ .

**Example 2.** Sketch the family of level curves of the real and imaginary parts of  $f(z) = \frac{1}{z-1}$ .

**Theorem 18.** If  $v$  is a harmonic conjugate of  $u$ , then their level curves always intersect orthogonally in the  $xy$ -plane.

**Theorem 19.**  $f = u + iv$  is analytic in  $\mathcal{D}$  iff  $v$  is a harmonic conjugate of  $u$  in  $\mathcal{D}$ .

**Example 3.** Let  $u(x, y) = 2x(1 - y)$ ,  $v(x, y) = x^2 - y^2 + 2y$ . Show that  $v$  is a harmonic conjugate of  $u$ .

**Example 4.** Let  $u(x, y) = \cos x \cosh y$ ,  $v(x, y) = -\sin x \sinh y$ . Show that  $v$  is a harmonic conjugate of  $u$ . (*Hint:  $\cos z$  is analytic, Definition 29*)

## 5 Important functions

### 5.1 Exponential

**Definition 25** (Exponential function).

$$\exp(x + iy) := e^x e^{iy}$$

where  $e^x$  is the real exponential function.

Domain:  $\mathbb{C}$

Range:  $\mathbb{C} \setminus \{0\}$

Analytic:  $\mathbb{C}$

### 5.2 Logarithm

**Definition 26** (Logarithm multi-valued function). Let  $z = re^{i\theta}$ .

$$\log(z) := \ln r + i\theta$$

where  $\ln$  is the real logarithm function.

Domain:  $\mathbb{C} \setminus \{0\}$

Range:  $\mathbb{C}$

When we put in the restriction  $-\pi < \theta \leq \pi$ , we get the principle logarithm  $\text{Log}(z)$ .

**Definition 27** (Branch of logarithm). As the complex logarithm is multi-valued, we often restrict  $\theta$  to intervals of length  $2\pi$ . These functions are called branches of the logarithm function. We have already defined the principle branch.

**Theorem 20.** A branch of the logarithm function is analytic in  $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ .

**Corollary 20.1.**  $\text{Log}(z)$  is analytic if  $-\pi < \text{Arg}(z) < \pi$ .

**Theorem 21.**  $\frac{d}{dz} \text{Log}(z) = \frac{1}{z}$ ,  $-\pi < \text{Arg}(z) < \pi$

### 5.3 Power

**Definition 28** (Power function).

$$z^c := \exp(c \log z)$$

This is also a multi-valued function and its branches are defined by the branch of the logarithm in the function.

Domain:  $\mathbb{C} \setminus \{0\}$

Range:  $\mathbb{C} \setminus \{0\}$

Analytic: Any branch is analytic in its domain

**Theorem 22.**  $\frac{d}{dz} z^c = cz^{c-1}$

### 5.4 Trigonometric

**Definition 29** (Cosine function).

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

Domain:  $\mathbb{C}$

Range:  $\mathbb{C}$

Analytic:  $\mathbb{C}$

**Definition 30** (Sine function).

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2}$$

Domain:  $\mathbb{C}$   
Range:  $\mathbb{C}$   
Analytic:  $\mathbb{C}$

Using these, we may define  $\tan$ ,  $\sec$ ,  $\operatorname{cosec}$ . These are analytic in their domains. The complex trigonometric functions satisfy most of the relations satisfied by the real trigonometric functions.

## 5.5 Hyperbolic

**Definition 31** (Hyperbolic cosine function).

$$\cosh(z) := \frac{e^z + e^{-z}}{2}$$

Domain:  $\mathbb{C}$   
Range:  $\mathbb{C}$   
Analytic:  $\mathbb{C}$

**Definition 32** (Hyperbolic sine function).

$$\sinh(z) := \frac{e^z - e^{-z}}{2}$$

Domain:  $\mathbb{C}$   
Range:  $\mathbb{C}$   
Analytic:  $\mathbb{C}$

Using these, we may define  $\tanh$ ,  $\operatorname{sech}$ ,  $\operatorname{cosech}$ . These are analytic in their domains. The complex hyperbolic functions satisfy most of the relations satisfied by the real hyperbolic functions.

## 6 Integral calculus

### 6.1 Fundamental Theorem of Calculus

**Definition 33.** Let  $f : [a, b] \rightarrow \mathbb{C}$  such that  $f(t) = u(t) + iv(t)$ .

$$\int_a^b f(t) dt := \int_a^b u(t) dt + i \int_a^b v(t) dt$$

**Theorem 23.**

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$$

**Example 5.** Let  $x \in [-1, 1], \theta \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$ . Show that

$$\left| \frac{1}{\pi} \int_0^\pi (x + i\sqrt{1-x^2} \cos \theta)^n d\theta \right| \leq 1$$

**Theorem 24** (Fundamental Theorem of Calculus).

1. Let  $f : [a, b] \rightarrow \mathbb{C}$  be a continuous function.

$$F(x) = \int_a^x f(t) dt \implies F'(x) = f(x), \forall x \in [a, b]$$

2. Let  $f : [a, b] \rightarrow \mathbb{C}$  be a continuous function.

$$F'(t) = f(t), \forall t \in [a, b] \implies \int_a^b f(t) dt = F(b) - F(a)$$

**Example 6.** Evaluate (using a complex-valued function)

$$\int_0^\pi e^{2t} \cos t dt$$

## 6.2 Arcs and Contours

**Definition 34** (Arc). Let  $z : [a, b] \rightarrow \mathbb{C}$ . The set

$$C = \{z(t) \mid a \leq t \leq b\}$$

is called an arc in  $\mathbb{C}$ .

$C$  is a **simple arc** in  $[a, b]$  if  $\forall t_1, t_2 \in [a, b]$ ,

$$t_1 \neq t_2 \implies z(t_1) \neq z(t_2)$$

$C$  is a simple, closed arc or a **Jordan curve** in  $[a, b]$  if

1.  $C$  is simple in  $(a, b)$ .
2.  $z(a) = z(b)$

$C$  is a **differentiable arc** if  $z$  is continuously differentiable in  $[a, b]$ .

$C$  is a **smooth arc** if it is differentiable and  $z'(t) \neq 0, \forall t \in [a, b]$ .

**Definition 35** (Orientation of arc). Let  $C$  be an arc as defined earlier. If  $z(t)$  moves in the counter-clockwise direction in the complex plane as  $t$  increases,  $C$  is said to be positively oriented. Otherwise, it is negatively oriented.

**Definition 36** (Reparameterization). Let  $C$  be the arc

$$C : z(t), t \in [a, b]$$

Let  $\varphi : [c, d] \rightarrow [a, b]$  be a continuously differentiable, strictly increasing bijective mapping such that

$$t = \varphi(s)$$

We get a reparametrization of  $C$

$$C : Z(s), s \in [c, d]$$

where  $Z(s) = z(\varphi(s))$ .

**Definition 37** (Length of arc). Let  $C : z(t), t \in [a, b]$  be a differentiable arc.

$$\text{length}(C) := \int_a^b |z'(t)| dt$$

**Definition 38** (Contour). A piecewise smooth curve.

Contours in  $\mathbb{C}$  are analogous to intervals in  $\mathbb{R}$ , in a sense that we will integrate functions along contours.

The **negative of a contour** is defined as

$$C : z(t), t \in [a, b]$$

$$-C : z(-t), -t \in [-b, -a]$$

## 6.3 Contour integral

**Definition 39** (Contour integral). Let

$$f : \Omega \rightarrow \mathbb{C}$$

$$C : z(t), t \in [a, b]$$

$f(z(t))$  be a piecewise continuous function of  $t$ .

$$\int_C f(z) dz := \int_a^b f(z(t)) z'(t) dt$$

**Theorem 25.**

$$\begin{aligned} \int_{C_1+C_2} f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ \int_{-C} f(z) dz &= - \int_C f(z) dz \end{aligned}$$

**Example 7.** Let  $C$  be the contour consisting of the semicircle  $e^{i\theta}, \theta \in [0, \pi]$  along with the part of the real axis  $x, x \in [-1, 1]$ . Let

$$f(z) = f(re^{i\theta}) = \sqrt{r}e^{i\frac{\theta}{2}} \quad (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2})$$

Find  $\int_C f(z) dz$ . (Hint: Set  $f(0) = 0$ )

**Theorem 26** (Bound of a contour integral). Let

$$f : \Omega \rightarrow \mathbb{C}$$

$C : z(t), t \in [a, b]$  be a contour of length  $L$

$f(z(t))$  be a piecewise continuous function of  $t$ , i.e. it has an upper bound (Theorem 5)  $M > 0$  in  $[a, b]$ . We have,

$$\left| \int_C f(z) dz \right| \leq ML$$

**Example 8.** Let  $C$  be the right-triangle formed by the two axes and the line segment joining  $-4$  and  $3i$ , oriented anticlockwise. Show that

$$\left| \int_C (e^z - \bar{z}) dz \right| \leq 60$$

**Example 9.** Let  $C$  be the circle  $|z| = R, R > 1$ , oriented anticlockwise. Show that

$$\left| \int_C \frac{\text{Log}(z)}{z^2} dz \right| < 2\pi \left( \frac{\pi + \ln R}{R} \right)$$

## 6.4 Antiderivative

**Definition 40** (Antiderivative). Let  $f$  and  $F$  be defined in  $\Omega$  such that

$$F'(z) = f(z), \forall z \in \Omega$$

$F$  is called the anti-derivative of  $f$  in  $\Omega$ .

**Theorem 27.** Antiderivatives, if exist, are analytic.

**Theorem 28.** If  $F_1, F_2$  are antiderivatives of  $f$  in  $\Omega$ ,  $\exists k \in \mathbb{C}$  such that

$$F_2(z) = F_1(z) + k, \forall z \in \Omega$$

**Theorem 29.** Let  $f$  be a continuous function defined on  $\Omega$ . The following statements are equivalent:

- $f$  has an antiderivative  $F$  in  $\Omega$ .
- Let  $C_1, C_2$  be two contours in  $\Omega$  with the same end-points  $z_1, z_2$  and same orientation  $z_1 \rightarrow z_2$ .

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

In fact, this integral is equal to  $F(z_2) - F(z_1)$ .

- If  $C$  is a closed contour in  $\Omega$ ,

$$\int_C f(z) dz = 0$$

**Example 10.** Let  $C$  be the unit circle centered at 0 and  $f(z) = \frac{1}{z}$ . Show that  $f$  does not have an antiderivative in its domain.

Can we find an antiderivative of  $f$  in some open subset of its domain? (Hint: Theorem 21)

**Example 11.** Let  $C$  be any simple contour with end-points  $-3, 3$  lying below the real axis, oriented  $-3 \rightarrow 3$ .

$$f(z) = \exp\left(\frac{1}{2} \log z\right) \quad (0 < \arg z < 2\pi)$$

Find  $\int_C f(z) dz$ .



## 7 Cauchy's Theorem

### 7.1 Cauchy's Theorem

**Definition 41** (Simply-connected domain). A domain  $\mathcal{D}$  is simply-connected if every simple, closed contour within it encloses only points in  $\mathcal{D}$ .

**Theorem 30.** If  $C$  is a simple, closed contour, the set of points inside  $C$  form a simply-connected domain.

**Theorem 31** (Cauchy). Let  $f$  be analytic in a simply-connected domain  $\mathcal{D}$ . For any simple, closed contour  $C \subset \mathcal{D}$ ,

$$\int_C f(z) dz = 0$$

**Example 12.** Let  $C$  be the unit circle. Find

$$\int_C z|z|^4 dz$$

**Corollary 31.1.** If  $f$  is analytic in a simply-connected domain  $\mathcal{D}$ , it has an antiderivative in  $\mathcal{D}$ . (Theorem 29)  
Let  $F$  be the antiderivative of  $f$  in  $\mathcal{D}$  and  $C$  be any contour in  $\mathcal{D}$  with endpoints  $z_1, z_2$  and orientation  $z_1 \rightarrow z_2$ .

$$\int_C f(z) dz = F(z_2) - F(z_1)$$

**Corollary 31.2.** If  $f$  is analytic on and inside a simple, closed contour  $C$ ,

$$\int_C f(z) dz = 0$$

### 7.2 Cauchy's Integral Formula

**Theorem 32** (Cauchy's integral formula). Let  $f$  be analytic inside and on a simple, closed, positive contour  $C$ . If  $z_0$  is a point inside  $C$ ,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Hence, the value of the function inside  $C$  is completely defined by its value on  $C$ .

**Example 13.** Let  $C$  be the square whose sides lie along  $x = \pm 2, y = \pm 2$ , oriented anticlockwise. Find

$$\int_C \frac{\cos z}{z(z^2 + 8)} dz$$

**Example 14.** Let  $C$  be the circle  $|z| = 3$ , oriented anticlockwise.

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} dz$$

Find  $g(2)$  and  $g(4)$ .

**Corollary 32.1** (Gauss' mean value theorem). Let  $f$  be analytic in  $B_r(z_0)$ .

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

**Lemma 32.1.** Let  $f$  be analytic inside and on a simple, closed, positive contour  $C$ . If  $z_0$  is a point inside  $C$ ,  $f$  is infinitely differentiable at  $z_0$ . The  $n^{th}$  derivative of  $f$  at  $z_0$  is given by

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n = 1, 2, \dots)$$

**Example 15.** Let  $C$  be the circle  $|z - i| = 2$ , oriented anticlockwise. Find

$$\int_C \frac{dz}{(z^2 + 4)^2}$$

(Hint:  $z^2 + 4 = (z + 2i)(z - 2i)$ )

**Theorem 33.** If  $f$  is analytic at  $z_0$  then its derivatives of all orders exist and are analytic at  $z_0$ .

**Corollary 33.1.** Theorem 14

### 7.3 Morera's Theorem

**Theorem 34** (Morera). Let  $f$  be continuous in  $\Omega$  and  $C$  be a closed contour in  $\Omega$ .

$$\int_C f(z) dz = 0, \forall C \implies f \text{ is analytic in } \Omega$$

**Theorem 35.** Let  $\mathcal{D}$  be a simply-connected domain and  $f$  be continuous in  $\mathcal{D}$ . Let  $C$  represent a closed contour in  $\mathcal{D}$ .

$$\int_C f(z) dz = 0, \forall C \iff f \text{ is analytic in } \Omega$$

### 7.4 Cauchy's Inequality

**Theorem 36** (Cauchy's inequality). Let

$$D_R(z_0) = \{z \mid |z - z_0| \leq R\}$$

$$C_R(z_0) = \{z \mid |z - z_0| = R\}$$

$f$  is an analytic function in  $D_R(z_0)$  such that  $M_R$  is the maximum value of  $|f(z)|$  on  $C_R(z_0)$ .

$$|f^n(z_0)| \leq \frac{n! M_R}{R^n} \quad (n = 1, 2, \dots)$$

**Example 16.**  $f$  is an entire function such that

$$|f(z)| \leq A|z|, \forall z \in \mathbb{C} \quad (A \in \mathbb{R})$$

Show that  $f(z) = az, a \in \mathbb{C}$ .

**Lemma 36.1** (Liouville). A bounded, entire function is a constant function.

### 7.5 Fundamental Theorem of Algebra

**Theorem 37** (Fundamental theorem of algebra). Every non-constant polynomial has at least one zero.

### 7.6 Maximum modulus principle

**Theorem 38** (Maximum modulus principle). If  $f$  is a non-constant, analytic function in  $\mathcal{D}$ ,  $|f(z)|$  has no maximum value in  $\mathcal{D}$ .

**Corollary 38.1.** Let  $f$  be a non-constant, analytic function in a bounded domain  $\Omega$  such that it is continuous on  $\overline{\Omega}$  (closure of  $\Omega$ , Definition 10). The maximum value of  $|f(z)|$  in  $\overline{\Omega}$  exists and lies on the boundary of  $\Omega$ .