Mathematical Methods in Physics I

Based on lectures by Dr. Ritam Mallick
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.[1]

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[1]This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

1 Vector Analysis

Let $\{\hat{\mathbf{e}}_{\mathbf{i}}\}$ be an orthonormal basis, i.e., $\hat{\mathbf{e}}_{\mathbf{i}} \cdot \hat{\mathbf{e}}_{\mathbf{j}} = \delta_{ij}$(where δ_{ij} represents the Kroneker delta function), of our vector space. For any vector $\mathbf{A}$ we have

$$\mathbf{A} = \sum_i A_i \hat{\mathbf{e}}_{\mathbf{i}}$$

where A_i belong to the scalar field over which the vector space is defined, like $\mathbb{R}$ or $\mathbb{C}$.

1.1 Products

Definition 1 (Vector Product)**.** We define a vector product on our basis set

$$\hat{\mathbf{e_i}} \times \hat{\mathbf{e_j}} = \sum_k \epsilon_{ijk}\ \hat{\mathbf{e_k}}$$

The R.H.S. is a vector itself, hence, the name. Using this and the fact that vector product is distributive over addition we arrive at

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \sum_i \hat{\mathbf{e}}_{\mathbf{i}} \sum_{jk} \epsilon_{ijk} A_j B_k$$

where

$$C_i = \sum_{jk} \epsilon_{ijk} A_j B_k$$

Using this we can find

Theorem 1 (Scalar triple product)**.**

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k$$

This is also represented as $[\mathbf{ABC}]$.

We can also calculate

Theorem 2 (Vector triple product)**.**

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \sum_i \hat{\mathbf{e}}_{\mathbf{i}} \sum_{jk} \epsilon_{ijk} A_j \sum_{pq} \epsilon_{kpq} B_p C_q \\ &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \end{aligned}$$

The vector product is not associative, hence the position of the parenthesis is important in the triple product.

Exercise 1. Show that

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{ABD}]\mathbf{C} - [\mathbf{ABC}]\mathbf{D}$$

1.2 Coordinate Transformation

The rotation of the 2D coordinate axes by an angle ϕ, keeping the origin fixed, leads to the

Definition 2 (Rotation transformation).

$$\mathbf{A}' = S\mathbf{A}$$

where S is the rotation transformation represented by the matrix

$$S = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

in our orthonormal basis. It is clear that $SS^T = I$, i.e., the rotation transformation is orthogonal. It takes the same form in higher dimensions, i.e., rotation of vectors is an orthogonal transformation.

This is a special property of vectors in physics. There is another kind of quantity called

Definition 3 (Pseudovectors).

$$\mathbf{A}' = |S|S\mathbf{A}$$

This is how these quantities transform under rotation, where, S is again a orthogonal transformation and $|S|$ represents the determinant of S.

1.3 Differential Calculus

We require the del operator

$$\boldsymbol{\nabla} \equiv \sum_i \hat{\mathbf{e}}_{\mathbf{i}} \frac{\partial}{\partial i}$$

in Cartesian coordinates. Using this we can define quantities like

Definition 4 (Gradient).

$$\text{Grad } f = \boldsymbol{\nabla} f$$

A small change in a scalar field along $\mathbf{r}$ is given by

$$df = \mathbf{r} \cdot \boldsymbol{\nabla} f$$

Say, the direction of $\boldsymbol{\nabla} f$ is given by some $\hat{\mathbf{r}}$. We have the most rapid increase in f along $\hat{\mathbf{r}}$ and $|\boldsymbol{\nabla} f| = \frac{Df}{D\hat{\mathbf{r}}}$ is the directional derivative of f along $\hat{\mathbf{r}}$.

Definition 5 (Divergence).

$$\text{Div } \mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A}$$

This gives a measure of the accumulation or depletion of $\mathbf{A}$ at a point. In other words, it finds how strongly a point acts as a source or sink.

Definition 6 (Curl).

$$\text{Curl } \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}$$

This gives a measure of the circulation of $\mathbf{A}$ at a point.

Definition 7 (Laplacian).

$$\nabla^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f)$$

The Laplacian is also defined for a vector field as

$$\nabla^2 \mathbf{A} = \sum_i \hat{\mathbf{e}}_{\mathbf{i}} \nabla^2 A_i$$

These can also be calculated in curvilinear coordinate systems as given in this Wikipedia article.
Some useful properties of vector derivatives are

Theorem 3.

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) &= 0 \\ \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) &= 0 \\ \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\end{aligned}$$

The last one is only valid in Cartesian coordinates.

Theorem 4 (Exact differential)**.** $\sum_i A_i \; di$ is an exact differential iff $\boldsymbol{\nabla} \times \mathbf{A} = 0$.

1.4 Integral Calculus

A very useful quantity for integration is

Theorem 5 (Infintesimal length)**.**

$$\begin{aligned}d\mathbf{l} &= dx \; \hat{\mathbf{e_x}} + dy \; \hat{\mathbf{e_y}} + dz \; \hat{\mathbf{e_z}} \\ &= dr \; \hat{\mathbf{e_r}} + r \; d\theta \; \hat{\mathbf{e}_\theta} + r \sin\theta \; d\phi \; \hat{\mathbf{e}_\phi} \\ &= ds \; \hat{\mathbf{e_s}} + s \; d\phi \; \hat{\mathbf{e}_\phi} + dz \; \hat{\mathbf{e_z}}\end{aligned}$$

Now, we may define some of the most important vector integrals in physics.

Definition 8 (Line integral)**.**

$$\int_C \mathbf{A} \cdot d\mathbf{l}$$

Definition 9 (Surface integral)**.**

$$\iint_S \mathbf{A} \cdot d\boldsymbol{\sigma}$$

The contour enclosing a surface S is represented by ∂S.

Definition 10 (Volume integral)**.**

$$\iiint_V \mathbf{A} \; d\tau$$

The surface enclosing a volume V is represented by ∂V.

Some important theorems that will help simplify integrals are

Theorem 6 (Conservation of gradient)**.**

$$\int_C (\boldsymbol{\nabla} f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

where $\mathbf{a}, \mathbf{b}$ are the endpoints of C(which is directed from $\mathbf{a}$ to $\mathbf{b}$).

Theorem 7 (Green)**.**

$$\oint_{\partial S} P(x,y) \; dx + Q(x,y) \; dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \; dA$$

Theorem 8 (Gauss)**.**

$$\oiint_{\partial V} \mathbf{A} \cdot d\boldsymbol{\sigma} = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{A} \; d\tau$$

Theorem 9 (Stokes)**.**

$$\oint_{\partial S} \mathbf{A} \cdot d\mathbf{l} = \iint_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\boldsymbol{\sigma}$$

Note that C, S, V need to be "sufficiently nice" for these to be valid, which is often the case in physics.