

Signals and Systems

Based on lectures by Dr. Kundan Kandhway
Notes taken by Rwik Dutta

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.¹

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¹This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

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1 What are signals?

A signal is a function of one or more independent variables. In this course, we will only work with functions of a single variable.

The signals with domain \mathbb{R} (or some union of intervals in \mathbb{R}) are called **continuous-time** signals represented by $x(t)$.

The signals with domain \mathbb{Z} (or some subset of \mathbb{Z}) are called **discrete-time** signals represented by $x[n]$.

The signals in this course will be **complex-valued** unless specified otherwise.

2 Signal energy and average power

2.1 Finite time interval

Signal energy and average power over a certain interval $t_1 \leq t \leq t_2$ or $n_1 \leq n \leq n_2$, is

$$E_t = \int_{t_1}^{t_2} |x(t)|^2 dt, \quad P_t = \frac{E_t}{t_2 - t_1}$$

or

$$E_n = \sum_{n=n_1}^{n_2} |x[n]|^2, \quad P_n = \frac{E_n}{n_2 - n_1 + 1}$$

2.2 Infinite time interval

For the energies, we can simply take the integral or sum over the entire domain $(-\infty \rightarrow \infty)$ to get E_∞ .

$$P_\infty = \lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \right)$$

or

$$P_\infty = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2 \right)$$

$$E_\infty < \infty \implies P_\infty = 0$$

3 Transformation of independent variable

3.1 Time shift

$$x(t) \rightarrow x(t - t_0)$$

$$t_0 > 0 \implies \text{Shifted right}$$

$$t_0 < 0 \implies \text{Shifted left}$$

Similarly. for discrete case.

3.2 Time reversal

$$x(t) \rightarrow x(-t)$$

Signal is reflected about y -axis. Similarly. for discrete case.

3.3 Time scaling

$$x(t) \rightarrow x(at), a > 0$$

Similarly. for discrete case.

3.4 Compound transformation

Suppose, we need to do the following transformation

$$x(t) \rightarrow x(at - t_0) \quad (a, t_0 \in \mathbb{R})$$

The simplest way to do this is

Shift \rightarrow Scale \rightarrow Reverse

Similarly, for discrete case.

4 Periodic signals

$x(t)$ is periodic with period T if $x(t+T) = x(t)$, $\forall t$. The smallest possible T is called the fundamental period of the signal.

Similarly, for discrete case.

5 Even and Odd signals

Even: $x(-t) = x(t)$

Odd: $x(-t) = -x(t)$

Theorem. Every signal is a superposition of an even and an odd signal.

$$x(t) = \underbrace{\frac{x(t) + x(-t)}{2}}_{\text{even}} + \underbrace{\frac{x(t) - x(-t)}{2}}_{\text{odd}}$$

Similary, for discrete case.

6 Continuous-time complex exponential signal

General form: $x(t) = Ce^{at}$, where $C, a \in \mathbb{C}$.

If C and a are real, we get a real exponential signal.

6.1 Periodic complex exponential

$x(t) = e^{j\omega_0 t}$ is periodic. If $\omega_0 \neq 0$, the fundamental period is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Energy of $x(t)$ over a period is given by

$$E_{\text{period}} = T_0$$

$$P_{\text{period}} = 1$$

The same over the entire domain is

$$E_{\infty} = \infty$$

$$P_{\infty} = 1$$

Theorem (Euler). $e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$

The following signals for $k \in \mathbb{Z}$ are called **harmonically related**

$$\phi_k = e^{jk\omega_0 t}$$

The general complex exponential is thus

$$x(t) = Ce^{at} = e^{a_1 t} (C_1 \cos(a_2 t) - C_2 \sin(a_2 t)) + je^{a_1 t} (C_2 \cos(a_2 t) + C_1 \sin(a_2 t))$$

where $C = C_1 + jC_2$, $a = a_1 + ja_2$.

7 Discrete-time complex exponential signal

General form: $x[n] = Ce^{\beta n} = z^n$, where $C, \beta, z \in \mathbb{C}$.

7.1 Periodic complex exponential

$x[n] = e^{j\omega_0 n}$ is periodic if $\frac{\omega}{2\pi} \in \mathbb{Q}$.

Thus, $\exists m, N \in \mathbb{N} \cup \{0\}$ such that $|\frac{\omega}{2\pi}| = \frac{m}{N}$, $\gcd(m, N) = 1$. N is the fundamental period of the signal.

Theorem. Let $x[n]$ be the finite superposition of discrete-time periodic complex exponentials with fundamental periods $\{N_k\}$. The fundamental period of $x[n]$ is $\text{lcm}(\{N_k\})$.

The following signals (for $k \in \mathbb{Z}$) are said to be **harmonically related**

$$\phi_k = e^{jk(\frac{2\pi}{N})n}$$

We have the **non-uniqueness property**

$$\phi_k[n] = \phi_{k+N}[n]$$

8 Impulse and Step signals

8.1 Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$
$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

These are related by the relations

$$\delta[n] = u[n] - u[n-1]$$
$$u[n] = \sum_{k=-\infty}^n \delta[k]$$

8.2 Continuous-time

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Observe the discontinuity at $t = 0$.

These are related by the relations

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$
$$\delta(t) = u'(t)$$

The impulse(delta) function has the properties

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$
$$\int_{-\infty}^{\infty} f(t) \delta(t - t_0) dt = f(t_0)$$

In fact, for some interval I

$$\int_I f(t) \delta(t - t_0) dt = \begin{cases} f(t_0), & t_0 \in I \\ 0, & \text{otherwise} \end{cases}$$

9 What are systems?

A system takes a signal as an input and gives an output signal. Thus, the signal, in some sense, changes or transforms when passed through a system. Thus, we say

$$x(t) \xrightarrow{\text{system}} y(t)$$

9.1 Interconnections

Multiple systems can be connected in various ways:

1. Series
2. Parallel
3. Feedback

9.2 Types of systems

9.2.1 Memoryless system

Output at certain time is only dependent on the input at that time.

$$y(t) = f(x(t))$$

Similary, for discrete case.

9.2.2 Identity system

$$y(t) = x(t)$$

Similary, for discrete case.

9.2.3 Systems with memory

Accumulator

$$y[n] = \sum_{k=-\infty}^n x[k]$$

Delay system

$$y(t) = x(t - |t_0|)$$

9.2.4 Invertible system

Distinct inputs lead to distinct outputs. An **inverse system** exists and the input signal can be distinctly obtained from the output signal.

9.2.5 Causal system

Output at any time only depends on input at present or past times.

9.2.6 Stable system

Output is bounded for bounded inputs, i.e.,

$$x(t) \leq M_x, \forall t \implies y(t) \leq M_y, \forall t$$

Similary, for discrete case.

9.2.7 Time-invariant system

$$x(t) \rightarrow y(t) \implies x(t - t_0) \rightarrow y(t - t_0)$$

Similary, for discrete case.

9.2.8 Linear system

$$x(t) \rightarrow y(t) \implies ax(t) \rightarrow ay(t) \quad (\text{Homogeneity})$$

$$x_k(t) \rightarrow y_k(t) \implies \sum_k x_k(t) \rightarrow \sum_k y_k(t) \quad (\text{Additivity})$$

Similary, for discrete case.

In this course, linear time-invariant(LTI) systems will be of importance.

10 LTI systems

10.1 Sifting

Theorem. (Sifting) Any signal can be decomposed into impulse signals.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau$$

Suppose, $\delta[n] \xrightarrow{\text{system}} h[n]$. $h[n]$ is called the **impulse response** of the system. Similar for continuous-time case.

10.2 Convolution

We define the **convolution** of two signals as

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau) d\tau$$

It has the following properties

1. Commutative
2. Associative
3. Distributive over addition

Theorem. $x(t) * \delta(t) = x(t)$

Theorem. The complete characterization of a LTI system can be done by using its impulse response.

$$\underbrace{y[n]}_{\text{output}} = \underbrace{x[n]}_{\text{input}} \underbrace{*}_{\text{convolution operator}} \underbrace{h[n]}_{\text{impulse response}}$$

i.e.,

$$y[n] = x[n] * h[n]$$

Similarly, for the continuous-time case.

Theorem. (Memoryless LTI system) A LTI system is memoryless iff the impulse response is a scaled delta function, i.e., $h[n] = k\delta[n]$ or $h(t) = k\delta(t)$.

10.3 Properties of LTI systems

We will represent systems by their impulse response.

10.3.1 Invertibility

$h(t)$ and $h_1(t)$ are inverse of one another iff

$$h(t) * h_1(t) = \delta(t)$$

10.3.2 Causality

$$h(t) = 0, \forall t < 0 \implies \text{system is causal}$$

10.3.3 Stability

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \implies \text{system is stable}$$
$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \implies \text{system is stable}$$

10.4 Unit step response

Suppose, $u[n] \xrightarrow{\text{system}} s[n]$. $s[n]$ is called the unit step response of the system. It is related to the impulse response by the following relations
Discrete-time:

$$h[n] = s[n] - s[n-1]$$
$$s[n] = \sum_{k=-\infty}^n h[k]$$

Continuous-time:

$$s(t) = \int_{-\infty}^t h(\tau) d\tau$$
$$h(t) = s'(t)$$

10.5 Complex exponential eigenfunctions

Complex exponentials are **eigenfunctions** of LTI systems.

$$e^{\omega t} \xrightarrow{LTI} g(\omega) e^{\omega t}$$

Thus, we choose them as **basis** for our signals.

$$x(t) = \sum_k a_k e^{\omega_k t} \implies y(t) = \sum_k a_k g(\omega_k) e^{\omega_k t}$$

11 Fourier series of periodic signals

Theorem. (Fourier) Let T be the fundamental period of $x(t)$. Similarly, let N be the fundamental period of $x[n]$.

Continuous:

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T})t} \quad (\text{synthesis})$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(\frac{2\pi}{T})t} dt \quad (\text{analysis})$$

Discrete:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n} \quad (\text{synthesis})$$

$$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n} \quad (\text{analysis})$$
$$a_k = a_{k+N}$$

a_0 is called the **dc component** of the signal.

11.1 Dirichlet conditions

A continuous-time signal must be **piecewise continuous** and periodic to have a Fourier series. If Dirichlet conditions are satisfied by a signal $x(t)$, then it is guaranteed to be equal to its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous.

1. $\int_T |x(t)| < \infty$
2. There are finite number of maxima and minima in one period.
3. There are only finite number of discontinuities in one period and all of them are finite.

12 Fourier transform of non-periodic signals

Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega)e^{j\omega n} d\omega$$

$|X(j\omega)|$ vs ω is called the **magnitude spectrum** and $\text{Arg}(X(j\omega))$ vs ω is called the **phase spectrum**.

Continuous-time fourier transform exists if the following(Dirichlet) conditions are satisfied:

1. $\int_{-\infty}^{\infty} |x(t)| < \infty$ (absolutely integrable)
2. There are finite number of maxima and minima over any finite interval.
3. There are only finite number of discontinuities in any finite interval and all of them are finite.

For discrete-time:

1. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$
2. $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ (finite energy)

Theorem. (Parseval's relation)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Theorem. The normalized sinc function($\text{sinc}(\theta) = \frac{\sin \theta}{\theta}$) is the FT of the rectangular function.

Let $X(j\omega) = \begin{cases} 1, & -W \leq \omega \leq W \\ 0, & \text{otherwise} \end{cases}$. Then we have

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} X(j\omega)$$

Theorem. $\int_{-\infty}^{\infty} \text{sinc}(x) dx = \pi$

12.1 Properties of Fourier Transform

12.1.1 Linearity

$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega)$$

$$ax[n] + by[n] \xleftrightarrow{\mathcal{F}} aX(e^{j\omega}) + bY(e^{j\omega})$$

12.1.2 Time shifting

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

$$x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$$

12.1.3 Conjugation

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

$$x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

12.1.4 Differentiation

$$x'(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} (1 - e^{-j\omega})X(e^{j\omega})$$

12.1.5 Integration

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi\delta(\omega)X(0)$$

$$\sum_{k=-\infty}^n x[k] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(1) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

12.1.6 Time-scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

12.1.7 Time-reversal

$$x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

12.1.8 Convolution

$$h(t) * x(t) \xleftrightarrow{\mathcal{F}} H(j\omega)X(j\omega)$$

$$h[n] * x[n] \xleftrightarrow{\mathcal{F}} H(e^{j\omega})X(e^{j\omega})$$

12.1.9 Multiplication

$$s(t)p(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

13 Laplace transform

Laplace transform can be used for analysis of unstable systems too for which FT does not exist.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

When $s = \sigma + j\omega$ is purely imaginary, this becomes the Fourier transform.

13.1 Zeroes and Poles

Suppose, the Laplace transform $X(s)$ is rational and $X(s) = \frac{N(s)}{D(s)}$.

Zeroes: Roots of $N(s)$

Poles: Roots of $D(s)$

If degree of $N(s) >$ degree of $D(s)$ we get poles at infinity.

If degree of $N(s) <$ degree of $D(s)$ we get zeroes at infinity.

13.2 Region of convergence

The range of values for which the Laplace integral converges is called its region of convergence(ROC). It has the following properties:

1. ROC consists of the points where the FT of $x(t)e^{-\sigma t}$ converges, i.e.,

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$$

2. ROC consists of strips parallel to the $j\omega$ -axis of the s -plane.
3. For rational LT, the ROC does not contain any poles. The ROC is bounded by the poles or extends to infinity.

4. If $x(t)$ is of finite duration (i.e., non-zero over a finite interval) and absolutely integrable, then ROC is the entire s -plane.
5. Let $x(t)$ be a **right-sided signal**, i.e., $\exists T$ such that $x(t) = 0, \forall t < T$. If the line $\text{Re}(s) = \sigma_0$ is in the ROC of $X(s)$ then all s for which $\text{Re}(s) > \sigma_0$ are also in the ROC.
6. Let $x(t)$ be a **left-sided signal**, i.e., $\exists T$ such that $x(t) = 0, \forall t > T$. If the line $\text{Re}(s) = \sigma_0$ is in the ROC of $X(s)$ then all s for which $\text{Re}(s) < \sigma_0$ are also in the ROC.
7. Let $x(t)$ be a two-sided signal. If the line $\text{Re}(s) = \sigma_0$ is in the ROC of $X(s)$ then the ROC is a strip including that line.
8. Let the LT of $x(t)$ be rational. If $x(t)$ is right-sided, the ROC lies to right of the rightmost pole. If $x(t)$ is left-sided, the ROC lies to the left of the leftmost pole. If $x(t)$ is two sided, ROC lies in the region between the rightmost and the leftmost poles.

14 Z-transform

Z-transform is the discrete-time counterpart of the Laplace transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

This reduces to the discrete-time FT when $|z| = 1$.

Analogous to the Laplace transform, we have ROC, zeroes and poles.

Z-transform converges when the FT of $x[n]|z|^{-n}$ converges, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$