Mathematical Methods in Physics I

Based on lectures by Dr. Ritam Mallick
Notes taken by Rwik Dutta

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.[1]

Contents

[1]This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

1 Vector Analysis

Let $\{\hat{\mathbf{e_i}}\}$ be an orthonormal basis, i.e., $\hat{\mathbf{e_i}} \cdot \hat{\mathbf{e_j}} = \delta_{ij}$(where δ_{ij} represents the Kroneker delta function), of our vector space. For any vector $\mathbf{A}$ we have

$$\mathbf{A} = \sum_i A_i \hat{\mathbf{e_i}}$$

where A_i belong to the scalar field over which the vector space is defined, like $\mathbb{R}$ or $\mathbb{C}$.

1.1 Products

Definition 1 (Vector Product)**.** We define a vector product on our basis set

$$\hat{\mathbf{e_i}} \times \hat{\mathbf{e_j}} = \sum_k \epsilon_{ijk}\ \hat{\mathbf{e_k}}$$

The R.H.S. is a vector itself, hence, the name. Using this and the fact that vector product is distributive over addition we arrive at

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \sum_i \hat{\mathbf{e_i}} \sum_{jk} \epsilon_{ijk} A_j B_k$$

where

$$C_i = \sum_{jk} \epsilon_{ijk} A_j B_k$$

Using this we can find

Theorem 1 (Scalar triple product)**.**

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k$$

This is also represented as $[\mathbf{ABC}]$.

We can also calculate

Theorem 2 (Vector triple product)**.**

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \sum_i \hat{\mathbf{e_i}} \sum_{jk} \epsilon_{ijk} A_j \sum_{pq} \epsilon_{kpq} B_p C_q \\ &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})\end{aligned}$$

The vector product is not associative, hence the position of the parenthesis is important in the triple product.

Here we have used

Theorem 3 (Product of levi-cevitas)**.**

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right).\end{aligned}$$

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}$$

Exercise 1. Show that

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{ABD}]\mathbf{C} - [\mathbf{ABC}]\mathbf{D}$$

1.2 Rotation Transformation

The rotation of the 2D coordinate axes by an angle ϕ, keeping the origin fixed, leads to the

Definition 2 (Rotation transformation)**.**

$$\mathbf{A}' = S\mathbf{A}$$

where S is the rotation transformation represented by the matrix

$$S = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

in our orthonormal basis. It is clear that $SS^T = I$, i.e., the rotation transformation is orthogonal. It takes the same form in higher dimensions, i.e., rotation of vectors is an orthogonal transformation.

This is a special property of vectors in physics. There is another kind of quantity called

Definition 3 (Pseudovectors)**.**

$$\mathbf{A}' = |S|S\mathbf{A}$$

This is how these quantities transform under rotation, where, S is again an orthogonal transformation and $|S|$ represents the determinant of S.

1.3 Differential Calculus

We require the del operator

$$\boldsymbol{\nabla} \equiv \sum_i \hat{\mathbf{e}}_{\mathbf{i}} \frac{\partial}{\partial i}$$

in Cartesian coordinates. Using this we can define quantities like

Definition 4 (Gradient)**.**

$$\text{Grad } f = \boldsymbol{\nabla} f$$

A small change in a scalar field along $\mathbf{r}$ is given by

$$df = \mathbf{r} \cdot \boldsymbol{\nabla} f$$

Say, the direction of $\boldsymbol{\nabla} f$ is given by some $\hat{\mathbf{r}}$. We have the most rapid increase in f along $\hat{\mathbf{r}}$ and $|\boldsymbol{\nabla} f| = \frac{Df}{D\hat{\mathbf{r}}}$ is the directional derivative of f along $\hat{\mathbf{r}}$.

Definition 5 (Divergence)**.**

$$\text{Div } \mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A}$$

This gives a measure of the accumulation or depletion of $\mathbf{A}$ at a point. In other words, it finds how strongly a point acts as a source or sink. It is also known as the *source density* of the vector field.

Definition 6 (Curl)**.**

$$\text{Curl } \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}$$

This gives a measure of the circulation of $\mathbf{A}$ at a point. It is also known as the *circulation density* of the vector field.

Definition 7 (Laplacian)**.**

$$\nabla^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f)$$

The Laplacian is also defined for a vector field as

$$\nabla^2 \mathbf{A} = \sum_i \hat{\mathbf{e}}_{\mathbf{i}} \frac{\partial^2 A_i}{\partial i^2}$$

These can also be calculated in curvilinear coordinate systems as given in this Wikipedia article.
Some useful properties of vector derivatives are

Theorem 4.

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) &= 0 \\ \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) &= 0 \\ \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned}$$

The last one is only valid in Cartesian coordinates.

Theorem 5 (Exact differential)**.** $\sum_i A_i \ di$ is an exact differential iff $\boldsymbol{\nabla} \times \mathbf{A} = 0$.

Two special kinds of vector fields that often appear in physics are

Definition 8 (Solenoidal)**.**

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

Definition 9 (Irrotational)**.**

$$\boldsymbol{\nabla} \times \mathbf{A} = 0$$

Vector fields can be decomposed into solenoidal and irrotational components using

Theorem 6 (Helmholtz)**.** Suppose, $\boldsymbol{\nabla} \cdot \mathbf{F}$ and $\boldsymbol{\nabla} \times \mathbf{F}$ vanish at infinity and $\mathbf{F}$ is smooth. $\exists f, \mathbf{A}$ such that

$$\mathbf{F} = \underbrace{-\boldsymbol{\nabla} f}_{\text{irrotational}} + \underbrace{\boldsymbol{\nabla} \times \mathbf{A}}_{\text{solenoidal}}$$

1.4 Integral Calculus

A very useful quantity for integration is

Theorem 7 (Infintesimal length)**.**

$$\begin{aligned} d\mathbf{l} &= dx\ \hat{\mathbf{e_x}} + dy\ \hat{\mathbf{e_y}} + dz\ \hat{\mathbf{e_z}} \\ &= dr\ \hat{\mathbf{e_r}} + r\ d\theta\ \hat{\mathbf{e}_\theta} + r\sin\theta\ d\phi\ \hat{\mathbf{e}_\phi} \\ &= ds\ \hat{\mathbf{e_s}} + s\ d\phi\ \hat{\mathbf{e}_\phi} + dz\ \hat{\mathbf{e_z}} \end{aligned}$$

Now, we may define some of the most important vector integrals in physics.

Definition 10 (Line integral)**.**

$$\int_C \mathbf{A} \cdot d\mathbf{l}$$

Definition 11 (Surface integral)**.**

$$\iint_S \mathbf{A} \cdot d\boldsymbol{\sigma}$$

The contour enclosing a surface S is represented by ∂S.

Definition 12 (Volume integral)**.**

$$\iiint_V \mathbf{A}\ d\tau$$

The surface enclosing a volume V is represented by ∂V.

Some important theorems that will help simplify integrals are

Theorem 8 (Conservation of gradient)**.**

$$\int_C (\boldsymbol{\nabla} f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

where $\mathbf{a}, \mathbf{b}$ are the endpoints of C(which is directed from $\mathbf{a}$ to $\mathbf{b}$).

Theorem 9 (Green)**.**

$$\oint_{\partial S} P(x,y)\ dx + Q(x,y)\ dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

Theorem 10 (Gauss)**.**

$$\oiint_{\partial V} \mathbf{A} \cdot d\boldsymbol{\sigma} = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{A}\ d\tau$$

Theorem 11 (Stokes)**.**

$$\oint_{\partial S} \mathbf{A} \cdot d\mathbf{l} = \iint_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\boldsymbol{\sigma}$$

Note that C, S, V need to be "sufficiently nice" for these to be valid, which is often the case in physics.

1.5 Orthogonal Curvilinear Coordinate Systems

Let $\hat{\mathbf{q}}_\mathbf{i}$ be the unit vectors of our generalized system. We would like to transform among coordinate systems.

Note. A vector $\mathbf{A}$ can be written as $\sum_i A_i \hat{\mathbf{q}}_\mathbf{i}$ in the generalized coordinate system. However, the position $\mathbf{r} \neq \sum_i q_i \hat{\mathbf{q}}_\mathbf{i}$, in general though the equality holds in Cartesian system.

The square of the infinitesimal arc length is given by $d\mathbf{r} \cdot d\mathbf{r} = ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates. This arc length is invariant of our coordinate system.

$$ds^2 = \sum_{ij} g_{ij}\ dq_i\ dq_j$$

The g_{ij} depend on the geometry of the coordinate system and are given by

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j}$$

For orthogonality,

$$g_{ij} = 0, \ i \neq j$$

$$\mathbf{\hat{q}_i} \cdot \mathbf{\hat{q}_j} = \delta_{ij}$$

We can take $g_{ii} = h_i^2 > 0$. Now, to represent $d\mathbf{r}$ in our generalized orthogonal coordinate system, we use

Theorem 12 (Scale factors)**.**

$$ds^2 = \sum_i (h_i \ dq_i)^2$$

$$d\mathbf{r} = \sum_i h_i \ dq_i \mathbf{\hat{q}_i}$$

$$\frac{\partial \mathbf{r}}{\partial q_i} = h_i \mathbf{\hat{q}_i}$$

Thus, we can define our differential operators as

Definition 13 (Gradient)**.**

$$\boldsymbol{\nabla} \phi = \sum_i \frac{1}{h_i} \frac{\partial \phi}{\partial q_i} \mathbf{\hat{q}_i}$$

Definition 14 (Divergence)**.**

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \sum_{ijk} \epsilon_{ijk} \frac{\partial A_i}{\partial q_i} h_j h_j$$

Definition 15 (Curl)**.**

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{\hat{q}_1} & h_2 \mathbf{\hat{q}_2} & h_3 \mathbf{\hat{q}_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$