

Mathematical Methods in Physics I

Based on lectures by Dr. Ritam Mallick

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.¹

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¹This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

1 Vector Analysis

Let $\{\hat{\mathbf{e}}_i\}$ be an orthonormal basis, i.e., $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ (where δ_{ij} represents the Kronecker delta function), of our vector space. For any vector \mathbf{A} we have

$$\mathbf{A} = \sum_i A_i \hat{\mathbf{e}}_i$$

where A_i belong to the scalar field over which the vector space is defined, like \mathbb{R} or \mathbb{C} .

1.1 Products

Definition 1 (Vector Product). We define a vector product on our basis set

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_k \epsilon_{ijk} \hat{\mathbf{e}}_k$$

The R.H.S. is a vector itself, hence, the name. Using this and the fact that vector product is distributive over addition we arrive at

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \sum_i \hat{\mathbf{e}}_i \sum_{jk} \epsilon_{ijk} A_j B_k$$

where

$$C_i = \sum_{jk} \epsilon_{ijk} A_j B_k$$

Using this we can find

Theorem 1 (Scalar triple product).

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k$$

This is also represented as $[\mathbf{ABC}]$.

We can also calculate

Theorem 2 (Vector triple product).

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \sum_i \hat{\mathbf{e}}_i \sum_{jk} \epsilon_{ijk} A_j \sum_{pq} \epsilon_{kpq} B_p C_q \\ &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \end{aligned}$$

The vector product is not associative, hence the position of the parenthesis is important in the triple product.

Exercise 1. Show that

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{ABD}]\mathbf{C} - [\mathbf{ABC}]\mathbf{D}$$

1.2 Coordinate Transformation

The rotation of the 2D coordinate axes by an angle ϕ , keeping the origin fixed, leads to the

Definition 2 (Rotation transformation).

$$\mathbf{A}' = S\mathbf{A}$$

where S is the rotation transformation represented by the matrix

$$S = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

in our orthonormal basis. It is clear that $SS^T = I$, i.e., the rotation transformation is orthogonal. It takes the same form in higher dimensions, i.e., rotation of vectors is an orthogonal transformation.

This is a special property of vectors in physics. There is another kind of quantity called

Definition 3 (Pseudovectors).

$$\mathbf{A}' = |S|S\mathbf{A}$$

This is how these quantities transform under rotation, where, S is again a orthogonal transformation and $|S|$ represents the determinant of S .

1.3 Differential Calculus

We require the del operator

$$\nabla \equiv \sum_i \hat{\mathbf{e}}_i \frac{\partial}{\partial i}$$

in Cartesian coordinates. Using this we can define quantities like

Definition 4 (Gradient).

$$\text{Grad } f = \nabla f$$

A small change in a scalar field along \mathbf{r} is given by

$$df = \mathbf{r} \cdot \nabla f$$

Say, the direction of ∇f is given by some $\hat{\mathbf{r}}$. We have the most rapid increase in f along $\hat{\mathbf{r}}$ and $|\nabla f| = \frac{Df}{D\hat{\mathbf{r}}}$ is the directional derivative of f along $\hat{\mathbf{r}}$.

Definition 5 (Divergence).

$$\text{Div } \mathbf{A} = \nabla \cdot \mathbf{A}$$

This gives a measure of the accumulation or depletion of \mathbf{A} at a point. In other words, it finds how strongly a point acts as a source or sink. It is also known as the *source density* of the vector field.

Definition 6 (Curl).

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A}$$

This gives a measure of the circulation of \mathbf{A} at a point. It is also known as the *circulation density* of the vector field.

Definition 7 (Laplacian).

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

The Laplacian is also defined for a vector field as

$$\nabla^2 \mathbf{A} = \sum_i \hat{\mathbf{e}}_i \nabla^2 A_i$$

These can also be calculated in curvilinear coordinate systems as given in this Wikipedia article. Some useful properties of vector derivatives are

Theorem 3.

$$\begin{aligned}\nabla \times (\nabla f) &= 0 \\ \nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\ \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\end{aligned}$$

The last one is only valid in Cartesian coordinates.

Theorem 4 (Exact differential). $\sum_i A_i di$ is an exact differential iff $\nabla \times \mathbf{A} = 0$.

Two special kinds of vector fields that often appear in physics are

Definition 8 (Solenoidal).

$$\nabla \cdot \mathbf{A} = 0$$

Definition 9 (Irrotational).

$$\nabla \times \mathbf{A} = 0$$

Vector fields can be decomposed into solenoidal and irrotational components using

Theorem 5 (Helmholtz). Suppose, $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ vanish at infinity and \mathbf{F} is smooth. $\exists f, \mathbf{A}$ such that

$$\mathbf{F} = \underbrace{-\nabla f}_{\text{irrotational}} + \underbrace{\nabla \times \mathbf{A}}_{\text{solenoidal}}$$

1.4 Integral Calculus

A very useful quantity for integration is

Theorem 6 (Infinitesimal length).

$$\begin{aligned}d\mathbf{l} &= dx \hat{\mathbf{e}}_x + dy \hat{\mathbf{e}}_y + dz \hat{\mathbf{e}}_z \\ &= dr \hat{\mathbf{e}}_r + r d\theta \hat{\mathbf{e}}_\theta + r \sin \theta d\phi \hat{\mathbf{e}}_\phi \\ &= ds \hat{\mathbf{e}}_s + s d\phi \hat{\mathbf{e}}_\phi + dz \hat{\mathbf{e}}_z\end{aligned}$$

Now, we may define some of the most important vector integrals in physics.

Definition 10 (Line integral).

$$\int_C \mathbf{A} \cdot d\mathbf{l}$$

Definition 11 (Surface integral).

$$\iint_S \mathbf{A} \cdot d\boldsymbol{\sigma}$$

The contour enclosing a surface S is represented by ∂S .

Definition 12 (Volume integral).

$$\iiint_V \mathbf{A} \, d\tau$$

The surface enclosing a volume V is represented by ∂V .

Some important theorems that will help simplify integrals are

Theorem 7 (Conservation of gradient).

$$\int_C (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

where \mathbf{a}, \mathbf{b} are the endpoints of C (which is directed from \mathbf{a} to \mathbf{b}).

Theorem 8 (Green).

$$\oint_{\partial S} P(x, y) \, dx + Q(x, y) \, dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

Theorem 9 (Gauss).

$$\oiint_{\partial V} \mathbf{A} \cdot d\boldsymbol{\sigma} = \iiint_V \nabla \cdot \mathbf{A} \, d\tau$$

Theorem 10 (Stokes).

$$\oint_{\partial S} \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma}$$

Note that C, S, V need to be “sufficiently nice” for these to be valid, which is often the case in physics.

1.5 Orthogonal Curvilinear Coordinate Systems

Let $\hat{\mathbf{q}}_i$ be the unit vectors of our generalized system. We would like to transform among coordinate systems.

Note. A vector \mathbf{A} can be written as $\sum_i A_i \hat{\mathbf{q}}_i$ in the generalized coordinate system. However, the position $\mathbf{r} \neq \sum_i q_i \hat{\mathbf{q}}_i$, in general though the equality holds in Cartesian system.

The square of the infinitesimal arc length is given by $d\mathbf{r} \cdot d\mathbf{r} = ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates. This arc length is invariant of our coordinate system.

$$ds^2 = \sum_{ij} g_{ij} \, dq_i \, dq_j$$

The g_{ij} depend on the geometry of the coordinate system and are given by

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j}$$

For orthogonality,

$$g_{ij} = 0, \quad i \neq j$$

$$\hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j = \delta_{ij}$$

We can take $g_{ii} = h_i^2 > 0$. Now, to represent $d\mathbf{r}$ in our generalized orthogonal coordinate system, we use

Theorem 11 (Scale factors).

$$ds^2 = \sum_i (h_i \, dq_i)^2$$

$$d\mathbf{r} = \sum_i h_i \, dq_i \hat{\mathbf{q}}_i$$

$$\frac{\partial \mathbf{r}}{\partial q_i} = h_i \hat{\mathbf{q}}_i$$

Thus, we can define our differential operators as

Definition 13 (Gradient).

$$\nabla \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial q_i} \hat{\mathbf{q}}_i$$

Definition 14 (Divergence).

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \sum_{ijk} \epsilon_{ijk} \frac{\partial A_i}{\partial q_j} h_j h_k$$

Definition 15 (Curl).

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{q}}_1 & h_2 \hat{\mathbf{q}}_2 & h_3 \hat{\mathbf{q}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$