Probability and Statistics

Based on lectures by Dr. Angshuman Bhattacharya
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.[1]

Contents

[1] This is how Dexter Chua describes his lecture notes from Cambridge. I could not have described mine in any better way.

1 Probability Space

1.1 Event space

Definition 1 ($\sigma-$field)**.** Let $\mathscr{A}$ be a collection of subsets of $\Omega(A \in \mathscr{A} \Rightarrow A \subset \Omega)$. $\mathscr{A}$ is called a $\sigma-$algebra(or, $\sigma-$field) of subsets of Ω if

1. $\mathscr{A}$ is non-empty.
2. $A \in \mathscr{A} \Rightarrow A^C \in \mathscr{A}$
3. If $A, B \in \mathscr{A}$, $A \cup B \in \mathscr{A}$ and $A \cap B \in \mathscr{A}$.

Theorem 1. A $\sigma-$algebra $\mathscr{A}$ of subsets of Ω contains Ω and $\varnothing$.

Theorem 2. The power set of $\Omega(\Omega \neq \varnothing)$ forms a $\sigma-$algebra of subsets of Ω.
This will be assumed to be the event space, unless mentioned otherwise.

Definition 2 (Event space, sample space)**.** An event space $\mathscr{A}$ of Ω is a $\sigma-$algebra of subsets of Ω. The elements of $\mathscr{A}$ are called events. Ω is called the sample space.

1.2 Probability measure

Definition 3. Let $\mathscr{A}$ be an event space of Ω. $P : \mathscr{A} \to \mathbb{R}$ is called a probability measure of $\mathscr{A}$ if

1. $P(A) \geq 0, \ \forall A \in \mathscr{A}$
2. $P(\Omega) = 1$
3. Let $A, B \in \mathscr{A}$. If $A \cap B = \varnothing$,
$$P(A \cup B) = P(A) + P(B)$$

Theorem 3. $P(A) \leq 1, \ \forall A \in \mathscr{A}$. Hence, range of P is $[0, 1]$.

Theorem 4. $P(A^C) = 1 - P(A)$

Corollary 4.1.
$$\underbrace{P\left(\bigcup_{i=1}^{\infty} A_i\right)}_{\text{at least one of the events occurs}} = 1 - \underbrace{P\left(\bigcap_{i=1}^{\infty} A_i^C\right)}_{\text{none of the events occur}}$$

Theorem 5. $A \subset B \implies P(A) \leq P(B)$

Theorem 6. $P(A \cup B) \leq P(A) + P(B)$

Theorem 7.

1. If $A_1 \subset A_2 \subset A_3 \subset \cdots$ and $A = \bigcup\limits_{n\geq 1} A_n$,
$$P(A) = \lim_{n\to\infty} P(A_n)$$
2. If $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $A = \bigcap\limits_{n\geq 1} A_n$,
$$P(A) = \lim_{n\to\infty} P(A_n)$$

Definition 4 (Probability space)**.** If Ω is a sample space, $\mathscr{A}$ is an event space of Ω and P is a probability measure of $\mathscr{A}$, $(\Omega, \mathscr{A}, P)$ is called a probability space.

2 Uniform Probability Spaces

Definition 5 (Symmetric probability space)**.** $(\Omega, \mathscr{A}, P)$ (where Ω is a finite set) is called a symmetric probability space if $P(A) = P(B)$ for all singleton sets $A, B \in \mathscr{A}$.

Theorem 8. $P(\{x\}) = \frac{1}{|\Omega|}, \ \forall x \in \Omega$. Also, $P(A) = \frac{|A|}{|\Omega|}, \ \forall A \subset \Omega$.

Definition 6 (Cardinality of uncountable set)**.** If $A \subset \mathbb{R}^n$, $|A|$ denotes the n-dimensional volume of the region A.

Definition 7 (Uniform probability space)**.** Let $\Omega \subset \mathbb{R}^n$ where $|\Omega|$ is finite. $(\Omega, \mathcal{P}(\Omega), P)$ is called an uniform probability space if $P(A) = \frac{|A|}{|\Omega|}, \ \forall A \subset \Omega$.

3 Conditional Probability

Definition 8 (Conditional probability)**.** Let A, B be two events such that $P(A) \neq 0$. The conditional probability of B given A denoted by $P(B|A)$ is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Theorem 9 (Principle of total probability)**.** Let $A_k, 1 \leq k \leq n$ be mutually disjoint events in Ω such that $\bigcup_{k=1}^{n} A_k = \Omega$ and $P(A_k) \neq 0, \forall 1 \leq k \leq n$.

$$P(B) = \sum_{k=1}^{n} P(A_k)P(B|A_k), \ \forall B \in \mathscr{A}$$

3.1 Baye's Rule

Theorem 10 (Baye's rule)**.** Let $A_k, 1 \leq k \leq n$ be mutually disjoint events in Ω such that $\bigcup_{k=1}^{n} A_k = \Omega$ and $P(A_k) \neq 0, \forall 1 \leq k \leq n$. If $B \in \mathscr{A}$ such that $P(B) \neq 0$,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)}$$

3.2 Independence

Definition 9. Two events A and B are independent iff $P(A \cap B) = P(A)P(B)$.

Theorem 11. If $P(A) > 0$, A, B are independent iff $P(B|A) = P(B)$.

4 Combinatorial Analysis

Definition 10. (Ordered sample) A random sample of size r from population S is the $r-$tuple $(x_1, x_2, \cdots, x_r), \ x_i \in S$.

It is called a random sample **without replacement** if $x_i = x_j \Rightarrow i = j (r \leq |S|)$. Otherwise, it is a random sampling **with replacement**.

Theorem 12. The number of possible random samples with replacement of size r from a population of size n is n^r.

It is the number of ways in which r balls can be picked from n distinct balls, with replacement, where the order of picking matters.

4.1 Permutations

Theorem 13 (Permutation)**.** The number of possible random samples without replacement of size r from a population of size n is

$$^{n}P_r = \frac{n!}{(n-r)!}$$

It is the number of ways in which r balls can be picked from n distinct balls, where the order of picking matters.

4.2 Combinations

Theorem 14 (Combinations)**.** The number of ways in which r balls can be picked from n distinct balls, where the order of picking does not matter is

$$^{n}C_r = \frac{n!}{(n-r)! \ r!}$$

Theorem 15. The number of $r-$tuples $(x_1, x_2, \cdots, x_r), \ x_i \in \mathbb{N} \cup \{0\}$ that satisfy$(n \in \mathbb{N})$

$$x_1 + x_2 + \cdots + x_r = n$$

is $\binom{n+r-1}{r-1}$. If $x_i \in \mathbb{N}$, we have $\binom{n-1}{r-1}$ solutions.

5 Discrete Random Variable

Definition 11 (Discrete random variable)**.** A discrete real-valued random variable X on a probability space $(\Omega, \mathscr{A}, P)$ is a function X with domain Ω and range a finite or countably infinite subset $\{x_1, x_2, \cdots\}$ of $\mathbb{R}$ such that $\{\omega : X(\omega) = x_i\}$ is an event for all i.

5.1 Discrete Density

Definition 12 (Discrete density)**.** $f_X : \mathbb{R} \to [0, 1]$,

$$f_X(x) = P(X = x)$$

is the discrete density of random variable X.

5.2 Joint Density

Definition 13 (Discrete random vector)**.** A discrete real-valued random vector $\mathbf{X}$ on a probability space $(\Omega, \mathscr{A}, P)$ is a function $\mathbf{X}$ with domain Ω and range a finite or countably infinite subset $\{\mathbf{x}_1, \mathbf{x}_2, \cdots\}$ of $\mathbb{R}^r$ such that $\{\omega : \mathbf{X}(\omega) = \mathbf{x}_i\}$ is an event for all i.

The density is the function $f(\mathbf{x}) = P(\mathbf{X} = \mathbf{x})$ or

$$f_{X_1,X_2,\cdots X_r}(x_1, x_2, \cdots, x_r) = P(X_1 = x_1, X_2 = x_2, \cdots, X_r = x_r)$$

where X_i are components of $\mathbf{X}$. $f_{X_1,X_2,\cdots X_r}$ is called the **joint density** of the random variables $X_1, X_2, \cdots X_r$.

5.3 Independence

Definition 14 (Independent random variables)**.** Two random variables X, Y are independent iff

$$f_{X,Y}(x, y) = f_X(x) f_Y(y), \forall x, y \in \mathbb{R}$$

5.4 Sum

Theorem 16 (Sum of random variables)**.**

$$f_{X+Y}(z) = P(X = x, Y = z - x) = \sum_x f_{X,Y}(x, z - x)$$

5.5 Conditional Density

Definition 15 (Conditional density)**.** $P(Y = y|X = x) = \frac{P(X=x,Y=y)}{P(X=x)} = \frac{f_{X,Y}(x,y)}{f_X(x)}$

$$f_{Y|X}(y|x) := \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)}, & f_X(x) \neq 0 \\ 0, & f_X(x) = 0 \end{cases}$$

is called the conditional density of Y given X.

5.6 Probability Generating Function

Definition 16 (Probability generating function)**.** $\Phi_X : [-1, 1] \to \mathbb{R}$,

$$\Phi_X(t) = \sum_{x=0}^{\infty} f_X(x) t^x$$

5.7 Expectation

Definition 17 (Expectation of discrete random variable)**.** Let the range of X be $\{x_1, x_2, \cdots\}$. The expectation of X is

$$E(X) = \sum_{i=1}^{\infty} x_i f_X(x_i)$$

Theorem 17. Let φ be defined on the range of X and $Z = \varphi(X)$.

$$E(Z) = \sum_i \varphi(x_i) f_X(x_i)$$

Definition 18 (Conditional expectation)**.** The conditional expectation of Y, given X is

$$E(Y|X = x) = \sum_i y_i f(y|x)$$

Theorem 18 (Linearity of expectation)**.**

$$E(cX) = cE(X)$$

$$E(X+Y) = E(X) + E(Y)$$

Theorem 19. $P(X \geq Y) = 1 \implies E(X) \geq E(Y)$. Moreover, $E(X) = E(Y) \iff P(X = Y) = 1$.

Theorem 20. $|E(X)| \leq E(|X|)$

Lemma 20.1. $P(|X| \leq M) = 1 \implies |E(X)| \leq M$

Lemma 20.2. $P(|X - Y| \leq M) = 1 \implies |E(X) - E(Y)| \leq M$

Theorem 21. If X, Y are independent, $E(XY) = E(X)E(Y)$

5.8 Variance

Definition 19 (Moment)**.** Let X be a random variable with expectation μ. The r^{th} moment of X is $E(X^r)$ and the r^{th} central moment is $E((X-\mu)^r)$

Definition 20 (Variance)**.** The 2^{nd} central moment.

$$V(X) = E((X-\mu)^2) = E(X^2) - \mu^2$$

$V(X)$ is denoted by σ^2, where $\sigma \geq 0$ is called the **standard deviation** of the X.

Theorem 22.

$$V(X+b) = V(x)$$

$$V(aX) = a^2 V(X)$$

Theorem 23.

$$\mu = \Phi'_X(1)$$

$$E(X^2) = \Phi''_X(1) + \Phi'_X(1)$$

5.9 Covariance and Correlation

Theorem 24 (Variance of sum)**.**

$$V(X+Y) = V(X) + V(Y) + 2E[(X - E(X))(Y - E(Y))]$$

Definition 21 (Covariance)**.** The covariance of X, Y is given by

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

Theorem 25. X, Y are independent $\implies Cov(X,Y) = 0$

Definition 22 (Correlation coefficient)**.** The correlation coefficient of X, Y is

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}}$$

5.10 Schwartz and Chebyshev's Inequalities

Theorem 26 (Schwartz inequality)**.** $[E(XY)]^2 \leq E(X^2)E(Y^2)$

Corollary 26.1. $[Cov(X,Y)]^2 \leq V(X)V(Y)$. This means $|\rho(X,Y)| \leq 1$.

Theorem 27 (Chebyshev's inequality)**.**

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}, \forall t > 0$$

6 Important Discrete Densities

6.1 Uniform

$$\text{DiscreteUnif}(a,b) \sim f(x) = \begin{cases} \frac{1}{b-a+1}, & a \leq x \in \mathbb{Z} \leq b \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}$$

$$\sigma^2 = \frac{(a+b-1)^2 - 1}{12}$$

6.2 Bernoulli

$$\text{Ber}(p) \sim f(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = p$$

$$\sigma^2 = p - p^2$$

6.3 Binomial

$$\text{Binom}(n;p) \sim f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = np$$

$$\sigma^2 = np(p-1)$$

6.4 Geometric

$$\text{Geom}(p) \sim f(x) = \begin{cases} p(1-p)^x, & x = 0, 1, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = \frac{1-p}{p}$$

$$\sigma^2 = \frac{1-p}{p^2}$$

6.5 Negative binomial

$$\text{NegBinom}(\alpha;p) \sim f(x) = \begin{cases} \binom{\alpha+x-1}{x} p^\alpha (1-p)^x, & x = 0, 1, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = \frac{p\alpha}{1-p}$$

$$\sigma^2 = \frac{p\alpha}{(1-p)^2}$$

6.6 Hypergeometric

$$\text{HyperGeom}(r, r_1; n) \sim f(x) = \begin{cases} \frac{\binom{r_1}{x}\binom{r-r_1}{n-x}}{\binom{r}{n}}, & x = 0, 1, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = \frac{nr_1}{r}$$

$$\sigma^2 = \frac{nr_1(r-r_1)(r-n)}{r^2(r-1)}$$

Consider a population of r objects, of which r_1 are of one type and $r_2 = r - r_1$ are of a second type. Suppose a random sample of size n is drawn from the pouplation. Let X be the number of objects of the first type in the sample.

6.7 Poisson

$$\text{Poisson}(\lambda) \sim f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\mu = \lambda$$

$$\sigma^2 = \lambda$$

$$\Phi(z) = e^{\lambda(z-1)}$$

6.8 Multinomial

$$\text{Multinom}(n; p_1, p_2, \cdots, p_n) \sim f(x_1, x_2, \cdots, x_n) \begin{cases} \frac{n!}{x_1!\ x_2! \cdots x_r!} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}, & x_i \in \mathbb{N} \cup \{0\},\ x_1 + x_2 + \cdots + x_r = n \\ 0, & \text{otherwise} \end{cases}$$

We have, $X_i \sim \text{Binom}(n; p_i)$.

Theorem 28. Let $X_1, X_2, \cdots, X_r$ be pair-wise independent. Let $Y = X_1 + X_2 + \cdots + X_r$.

- $X_i \sim \text{Binom}(n_i; p) \implies Y \sim \text{Binom}(\sum_i n_i; p)$
- $X_i \sim \text{NegBinom}(\alpha_i; p) \implies Y \sim \text{NegBinom}(\sum_i \alpha_i; p)$
- $X_i \sim \text{Poisson}(\lambda) \implies Y \sim \text{Poisson}(\sum_i \lambda_i)$
- $X_i \sim \text{Geom}(p) \implies Y \sim \text{NegBinom}(r; p)$.

7 Continuous Random Variable

Definition 23 (Continuous random varibale)**.** A continuous random variable X on a probability space $(\Omega, \mathscr{A}, P)$ is a real-valued function $X : \Omega \to \mathbb{R}$ such that for $x \in \mathbb{R}$, $\{\omega | X(\omega) \leq x\}$ is an event.

$$P(X = x) = 0, \forall x$$

7.1 Continuous Distribution

Definition 24 (Continuous distribution)**.** $F_X : \mathbb{R} \to [0, 1]$

$$F_X(x) = P(X \leq x)$$

is called the distribution of continuous random varibale X.

Theorem 29. A continuous distribution is right-continuous and non-decreasing.

Corollary 29.1. If F is a continuous distribution,

$$\lim_{x \to -\infty} F(x) = 0$$

$$\lim_{x \to +\infty} F(x) = 1$$

7.2 Continuous Density

Definition 25 (Continuous density function)**.** $f_X : \mathbb{R} \to \mathbb{R}$

$$\int_{-\infty}^{\infty} f_X(x)\ dx = 1$$

$$F(x) = \int_{-\infty}^{x} f(t)\ dt,\ \forall x \in \mathbb{R}$$

If such a density f_X exists, X and F_X are called absolutely continuous.

Theorem 30. If F_X is continuously differentiable(or, f is continuous) at x_0,

$$f(x_0) = F'(x_0)$$

7.3 Change of Variable

Theorem 31 (Change of variable)**.** Let $Y = \varphi(X)$, where φ is differentiable and strictly increasing or decreasing in an interval I. Let $f_X(x) = 0, \forall x \notin I$.

$$f_Y(y) = \begin{cases} f\left(\varphi^{-1}(y)\right) \left| \frac{d}{dy}\left(\varphi^{-1}(y)\right) \right|, & \forall y \in \varphi(I) \\ 0, & \text{otherwise} \end{cases}$$

7.4 Joint distribution

Definition 26 (Joint distribution)**.** $F : \mathbb{R}^2 \to [0, 1]$

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

Theorem 32. Probability in a rectangular region is given by

$$P(a < X \leq b, c < Y \leq d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$$

7.5 Joint Density

Definition 27 (Joint density)**.** $f : \mathbb{R}^2 \to \mathbb{R}$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy = 1$$

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\ dv\ du$$

Theorem 33. If f is continuous at (x_0, y_0),

$$f(x_0, y_0) = F_{xy}(x_0, y_0)$$

7.6 Independence

Definition 28 (Independence)**.** X, Y are independent iff

$$F(x, y) = F_X(x)F_y(y), \forall x, y \in \mathbb{R}$$

or,

$$f(x, y) = f_X(x)f_Y(y), \forall x, y \in \mathbb{R}$$

Theorem 34. Let $X_1, X_2, \cdots, X_n$ be pair-wise independent. Suppose, $Y = \varphi(X_1, X_2, \cdots, X_n)$ and $Z = \psi(X_{m+1}, X_{m+2}, \cdots, X_n)$ where $1 \leq m < n$. Y, Z are independent.

7.7 Sum

Theorem 35 (Sum of continuous random variables)**.**

$$F_{X+Y}(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{X,Y}(x, v - x)\ dx\ dv$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\ dx = f_X(z) * f_Y(z)$$

7.8 Conditional Density

Definition 29 (Conditional density)**.** The conditional density of Y given X is defined as

$$f_{Y|X}(y|x) := \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)}, & 0 < f_X(x) < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$P(a \leq Y \leq b | X = x) = \int_a^b f_{Y|X}(y|x)\ dy$$

Theorem 36 (Baye's rule)**.**

$$f_{X|Y}(x|y) = \frac{f_X(x)f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(x)f_{Y|X}(y|x)\ dx}$$

7.9 Expectation

Definition 30 (Expectation of continuous random variable)**.**

$$E(X) = \int_{-\infty}^{\infty} xf(x)\ dx$$

Theorem 37. Let $X_1, X_2, \cdots, X_n$ have joint density f and $Z = \varphi(X_1, \cdots, X_n)$.

$$E(Z) = \int \varphi(x_1, \cdots, x_n) f(x_1, \cdots, x_n)\ d\mathbf{x}$$

Definition 31 (Conditional expectation)**.**

$$E[Y|X = x] = \int_{-\infty}^{\infty} yf(y|x)\ dy = \frac{\int_{-\infty}^{\infty} yf(x, y)\ dy}{f_X(x)}$$

is called the conditional expectation of Y given X or the regression function of Y on X.

7.10 Moments

Moments, variance, covariance, correlation are defined like the discrete case.

8 Important Continuous Densities

8.1 Symmetric

f is symmetric if $f(-x) = f(x), \forall x$. If X and $-X$ have the same distribution, X is called a symmetric random variable.

8.2 Standard Normal

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$
$$\mu = 0$$
$$\sigma^2 = 1$$

Φ is symmetric. The density and distribution have the same function.

8.3 Normal

For $\sigma > 0$,

$$n(\mu, \sigma^2) \sim f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$
$$F(x) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

$X \sim \Phi \implies \mu + \sigma X \sim n(\mu, \sigma^2)$

8.4 Exponential

$$\mathrm{Exp}(\lambda) \sim f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$
$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$
$$\mu = \frac{1}{\lambda}$$
$$\sigma^2 = \frac{1}{\lambda^2}$$

8.5 Gamma

$$\Gamma(\alpha, \lambda) \sim f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\ dt, x > 0$.

$$\mu = \frac{\alpha}{\lambda}$$
$$\sigma^2 = \frac{\alpha}{\gamma^2}$$

$\text{Exp}(\lambda) \sim \gamma(1, \lambda)$

8.6 Cauchy

$$\text{Cauchy}(\alpha, \gamma) \sim f(x) = \frac{1}{\pi\gamma \left[1 + \left(\frac{x-\alpha}{\gamma}\right)^2\right]}$$
$$F(x) = \frac{1}{\pi} \tan^{-1}\left(\frac{x-\alpha}{\gamma}\right) + \frac{1}{2}$$

8.7 Standard Bivariate Normal

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

Theorem 38. Let $X \sim \text{Exp}(\lambda)$ and $a, b \geq 0$.

1. $P(X > a + b) = P(X > a)P(X > b)$
2. $P(X > a + b | X > a) = P(x > b)$

Both these statements are equivalent. If they hold for some X, X is either exponentially distributed or $P(X > 0) = 0$.

Theorem 39. $X \sim n(0, \sigma^2) \implies X^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$

Theorem 40. $X_i \sim \Gamma(\alpha_i, \lambda) \implies \sum_i X_i \sim \Gamma(\sum_i \alpha_i, \lambda)$

Theorem 41. $X_i \sim n(\mu_i, \sigma_i^2) \implies \sum_i X_i \sim n(\sum_i \mu_i, \sum_i \sigma_i^2)$

Theorem 42. Here are some important results that may be useful in computations:

- $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\ dx = \sqrt{2\pi}$
- $\int_0^\infty x^{\alpha-1} e^{-\lambda x}\ dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$
- $\Gamma(x + 1) = x\Gamma(x)$
- $\Gamma(1) = 1,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(n) = (n - 1)!,\ \forall n \in \mathbb{N}$
- $\Gamma\left(\frac{n}{2}\right) = \frac{\sqrt{\pi}(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}, \forall$ odd natural number n

9 Central Limit Theorem

Theorem 43 (Central limit theorem)**.** Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with mean μ and variance σ^2. Let $S_n = \sum_i X_i$.

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) = \Phi(x),\ \forall x \in \mathbb{R}$$

Corollary 43.1. For very large n, $\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim \Phi$.