March 14 Notes for "On Some Difficulties Encountered in Formalizing Set Theory and Related Systems". On 1 March 2008 while at child's fair in the Albany government plaza, I (Dan Willard) began to work out (with some careless errors) the proof of ZFC's omega inconsistency. I wrote a first darft of what I had in mind on march 2 and the morning of march 3, but did not finally decide what was proven until the evening of March 3. After notarizing the March 4 result, I further strengthened the result on the evening of March 4 to verify that ZF is also unable to prove its own inconsistency and both ZF and ZFC are inconsistent (in the context of different levels of the replacement schema present). The latter March 4 document (from which a proof can be quite easily extrapolated) was notarized in more mature forms during march 7–14. On March 18 I notatarized the current more yet polished version — whose title is planned to be be a hybrid of the titles of Gödel's and Russel's famous paper given in bibliographic references of [6, 16].

Some alternate posssible titles might be: "On the Formally Proven Inconsistency of Zermelo Fraenkel Set Theory and Related Systems" or "On Some Difficulties Encountered concerning the Formally Proven Inconsistency of Zermelo-Fraenkel Set Theory and Related Set Theoretic Systems: Part I". or "On Some Difficulties in Consistently Formalizing Zermel-Fraenkel Set Theory and Related Systems: Part I". The latter is now my favorite

Introduction

The history of axiomatic set theory is well known. Frege made a first attempt to formalize this subject in [4]. Russel and Zermelo found inconsistencies in his formalism [16, 17, 21, 22] The modern version of Set Theory is based largely on the Zermelo-Fraenkel system whose formalism can be found in many sources [2, 9, 13, 14, 19]. Gödel-Bernays Set Theory is a conservative extension of it. Gödel proved that if Zermelo-Frankel set theory (ZF) was consistent then so is the extension of it that includes the Axiom of Choice. (This theory is often abbreviated as ZFC.) Cohen [2] proved that the Axiom of Choice and Continuum hypothesis are independent of ZF Set Theory (assuming the latter is consistent), and a recent literature has explored the implications of axioms about large cardinals.

In this paper, we will prove that ZF Set Theory is actually inconsistent. After introducing some notation in Section , this article will introduce three main results. They are listed below:

1. Theorem 2 will show that ZFC can prove a theorem asserting its own inconsistency. (This initial result will be non-constructive in that Theorem 2's proof establishes the existence of an inconsistency without formally constructing an inconsistency.)

2. Theorem 3 will consist of a more elaborate version of Theorem 2 that establishes the same result for ZF Set Theory (i.e. that ZF Set Theory supports a non-constructive proof of the existence of an inconsistency internal to its formalism).

3. Finally, Corollary 2 will explain how one can use Theorem 3's non-constructive proof result *as a vital intermediate step* to construct an inconsistency internal to ZF in an explicit manner.

Technically, our discussion could have included Topics (2) and (3) without Topic (1). (This is because Topic (1)'s inconsistencies are immediate consequences of Topics (2) and (3).) However, this author suspects that a broader audience will more easily understand our core results if approximately half of our discussion will focus on Topic (1).

It is our anticipation that ZF can be modified so that it can become consistent while retaining at least most of the charms of the old inconsistent version of ZF. (It is beyond doubt that the latter contained many serious virtues when it required a full 100 years for it to be proven to be inconsistent despite the intense scrutiny of many papers that employed Set Theory during the last century in a vareity of settings). In essence, we anticipate that a new modified version of ZF will attempt to imitate the renown beauty of the old version of ZF in its fascinating ability to conceptualize highly abstract non-computable objects, within a framework that is more constricted so as to avoid housing paradoxical inconsistencies.

Notation

This section will introduce most of the notation that will be employed in this article. In most of our discussion, e will denote some arbitrary subset of the positive integers that typically is too complex to be generated by a recursive function.

The symbol α will denote some arbitrary axiom system, which often (but not always) corresponds to one of the ZF or ZFC set theories. Our notation will map the above elements e typically onto an extension of the axiom system α . Depending on the choice of e , this generic extension of axiom system α may be either consistent or inconsistent, as well as may be either complete or incomplete.

Using such a framework to study itself, Section will show that the axiom system ZFC will be forced to prove (in a non-constructive manner) a theorem that states ZFC is inconsistent. Likewise, Section will show ZF proves a non-constructive theorem declaring ZF's own inconsistency.

Our original optimistic hope at the start of this research project was that the axiom system ZFC would turn out to be an awkward axiom system that is formally consistent but proves a theorem declaring its own inconsistency. Unfortunately, this initial conjecture turned out to be overly optimistic. By the end of this paper, we will show how these non-constructive proofs of inconsistency can be used as intermediate steps for ZFC and ZF to become fully inconsistent.

The remainder of this section will be divided into two parts. It will first provide a formal list of our notation conventions. It will then give an informal explanation of the meaning of these sundry definitions. Below is the list of notation we shall employ:

1. INT will denote the set of *positive* integers (using set theory's conventional notation where each non-empty finite ordinal is mapped onto a positive integer).
2. P_1 will denote the power set of INT.
3. P_2 will denote the power set of P_1
4. Given any logical language L, the symbol Enum_L will denote a an injective function from INT into INT that enumerates all the Gödel numbers of the sentences of L . More precisely, for an integer $i \in$INT, the symbol $\text{Enum}_L(i)$ will denote the Gödel number of the i–th sentence in L.
5. The symbol $\lceil \Psi \rceil$ will have its usual meaning of denoting Ψ's Gödel number.
6. Let α denote the Gödel number of some r.e. axiom system, and e denote an element of P_1. The symbol $\text{AxSentence}_{L,\alpha}(i,e)$ will denote a Boolean value that equals TRUE when either $\text{ENUM}_L(i)$ is an axiom of α or $i \in e$. More formally in a context where AxiomSet(α) represents the set of Gödel numbers for α's axioms, $\text{AxSentence}_{L,\alpha}(i,e)$ is formally encoded by the formula:

$$\text{AxSentence}_{L,\alpha}(i,e) \;=\; [\; \text{ENUM}_L(i) \in \text{AxiomSet}(\alpha) \quad \vee \quad i \in e \;] \tag{1}$$

7. The symbol $\text{System}_{L,\alpha}(e)$ will denote the axiom system that is naturally associated with Equation (1)'s AxSentence function. It will thus contain the axiom sentence Ψ iff there exists an integer i such that:

$$\text{Enum}_L(\,i\,) \;=\; \lceil \Psi \rceil \quad \wedge \quad \text{AxSentence}_{L,\alpha}(i,e) \tag{2}$$

8. A **Boolean-4** logic will refer to a syntax structure that maps each Gödel number $\lceil \Psi \rceil$ onto one of four values of "T" (for True), "F" (for False), "B" (for proven to be "both" True and False) and "U" (for "Unknown" truth value).
9. In the context of Item 7's axiom system $\text{System}_{L,\alpha}(e)$, the function $\text{Decipher}_{L,\alpha}(i,e)$ will denote a function that maps the i–th sentence among Enum_L's list of sentences for the language L onto the Bool-4 truth value that $\text{System}_{L,\alpha}(e)$ naturally associates with it. In particular in a context where $\text{Enum}_L(i) = \lceil \Psi \rceil$, the formal value for $\text{Decipher}_{L,\alpha}(i,e)$ is defined by the following four rules

   (a) $\text{Decipher}_{L,\alpha}(i,e) =$ "T" iff there exists a proof of Ψ from $\text{System}_{L,\alpha}(e)$ and additionally $\text{System}_{L,\alpha}(e)$ is formally consistent.

   (b) $\text{Decipher}_{L,\alpha}(i,e) =$ "F" iff there exists a proof of $\neg\,\Psi$ from $\text{System}_{L,\alpha}(e)$ and additionally $\text{System}_{L,\alpha}(e)$ is formally consistent.

   (c) $\text{Decipher}_{L,\alpha}(i,e) =$ "U" iff there exists neither proofs of Ψ nor $\neg\,\Psi$ from $\text{System}_{L,\alpha}(e)$.

   (d) $\text{Decipher}_{L,\alpha}(i,e) =$ "B" iff there exists proofs of **BOTH** Ψ and $\neg\,\Psi$ from $\text{System}_{L,\alpha}(e)$.

1

For many typical consistent axiom formalisms $\text{System}_{L,\alpha}(e)$, the function value for $\text{Decipher}_{L,\alpha}(i,e)$ is obviously far too complex to be computable by a recursive function. However, this object is still well defined in a set theoretic sense. (Thus, theorems about its properties can be generated by the axiom systems ZF and ZFC as they prove meta-logic theorems about their own properties.)

10. The construct $\text{Reveal}_{L,\alpha}(i,e)$ will denote a formal wff that is the approximate counterpart of the functional object $\text{Decipher}_{L,\alpha}(i,e)$ when it is translated from a 4-value Boolean logic to a traditional 2-valued Boolean logic. The formal definition of $\text{Reveal}_{L,\alpha}(i,e)$ is given below:

$$\text{Reveal}_{L,\alpha}(i,e) \quad =_{\text{df}} \quad \{ \ \text{Decipher}_{L,\alpha}(i,e) \ = \ \text{“T”} \ \} \tag{3}$$

Equation (3) thus implies that $\text{Reveal}_{L,\alpha}(i,e)$ is true when $\text{Decipher}_{L,\alpha}(i,e)$ = “T”, and it is false when $\text{Decipher}_{L,\alpha}(i,e)$ represents one of the three values of “F”, “B” or “U”. Often in this paper, we will be working with special subsets e of the set of positive integers, where $\text{Decipher}_{L,\alpha}(i,e)$ always equals either “T” or “F” for arbitrary i. Under these special circumstances, $\text{Reveal}_{L,\alpha}(i,e)$ will satisfy the following important simplifying conditions:

(a) $\text{Reveal}_{L,\alpha}(i,e)$ will be true if and only if $\text{Decipher}_{L,\alpha}(i,e)$ = “T”

(b) $\text{Reveal}_{L,\alpha}(i,e)$ will be false if and only if $\text{Decipher}_{L,\alpha}(i,e)$ = “F”

11. The symbol $\text{ConsistentSyst}_{L,\alpha}(e)$ will denote a wff that is TRUE if and only if the formalism $\text{System}_{L,\alpha}(e)$ is consistent. (Using Item 9's notation, $\text{ConsistentSyst}_{L,\alpha}(e)$ is True if and only if for all integers i the quantity $\text{Decipher}_{L,\alpha}(i,e) \neq$ “B”.)
12. The symbol $\text{CompleteSyst}_{L,\alpha}(e)$ will denote a wff that is TRUE if and only if the formalism $\text{System}_{L,\alpha}(e)$ is capable of proving or disproving all sentences in L's language. (Using Item 9's notation, $\text{ConsistentSyst}_{L,\alpha}(e)$ is True if and only if for all integers i the quantity $\text{Decipher}_{L,\alpha}(i,e) \neq$ “U”.)
13. The symbol $\text{MaximalSyst}_{L,\alpha}(e)$ will denote a wff that is TRUE if and only if $\text{CompleteSyst}_{L,\alpha}(e)$ is true and additionally each sentences Ψ satisfies the condition that one of the two Gödel numbers of $\ulcorner \Psi \urcorner$ or $\ulcorner \neg\Psi \urcorner$ are formal elements among the list of Gödel numbers itemized by e. (In other words, this means that some $i \in e$ has the property that $\text{Enum}_L(i)$ equals either $\ulcorner \Psi \urcorner$ or $\ulcorner \neg\Psi \urcorner$.)
14. The symbol $\text{SupportSet}(L,\alpha)$ will will denote the set of all $e \in P_1$ satisfying all three of the conditions of $\text{ConsistentSyst}_{L,\alpha}(e)$, $\text{CompleteSyst}_{L,\alpha}(e)$ and $\text{MaximalSyst}_{L,\alpha}(e)$.
15. ZF will be abbreviations for the Gödel number for the Zermelo Fraenkel axiom system (without the Axiom of Choice). Its formal structure can be found in for example [2, 13]
16. “Choose” denotes a function whose domain is P_2 and which maps each *non-empty* $x \in P_2$ onto some $e \in P_1$ such that $e \in x$. The Axiom of Choice implies that the function “Choose” exists and thus our nomenclature is well defined. (We shall presume that $\text{Choose}(x)$ is formally undefined when x is empty.)
17. ZFC will be abbreviations for the Gödel number for the Zermelo Fraenkel axiom system with the Choice axiom added. Without loss in generality, we may assume that it contains a special function symbol added to our language for denoting the above “Choose” function. (We do not actually need the “Choose” function to have an especially named function symbol in our language, but it makes the notation in our discussion more convenient.)
18. Combining the notation from the last four items, the symbol Support-ZFC will denote the special degenerate version of Item 14's $\text{SupportSet}(L,\alpha)$ where α now represents the ZFC axiom system and L is ZFC's language. Thus, it intuitively represents the set of all “complete” and “maximal” $e \in P_1$ whose collection of itemized axiomatic sentences is consistent with ZFC.
19. Likewise, Support-ZF will denote the special degenerate version of Item 14's $\text{SupportSet}(L,\alpha)$ where α represents the ZF axiom system and L represents its language.

One has to obviously approach the above list of 19 defined objects quite carefully — because many of these entities are non-constructive in that the elements belonging to their associated sets cannot be formalized by a recursive function. Nevertheless, these 19 definitions are sufficiently explicit in a set-theoretic sense so that their meanings and implications from the frameworks of the ZF and ZFC axiom systems are quite unambiguous.

Within such a context, we will develop a new type of diagonalizatin argument that will show that ZF and ZFC are inconsistent. Our results are based on the observation that both Gödel's Completeness Theorem and the Lindenbaum Lemma imply that the axiom system α is automatically inconsistent when $\text{SupportSet}(L,\alpha)$ is empty. Thus ZF will establish a non-constructive proof of its own inconsistency iff it proves that Support-ZF is empty. Moreover, ZF (as well as ZFC) have a capacity to prove Gödel's Completeness Theorem and the Lindenbaum Lemma. *This implies that they will well understand that they have proven themselves to be inconsistent when they have proven their respective support sets are empty.*

For a fixed axiom system α that typically represents ZF or ZFC, it is useful to keep in mind that $e \in P_1$ can often formalize an entity $\text{System}_{L,\alpha}(e)$ that is too complicated to have a recursive (or otherwise simple) representation. (After all, this system will often be quite complicated when it and e are made to satisfy the three criteria of “completeness”, “consistency” and “maximality” defined by items 11, 12 and 13.) Such elements e are obviously much too complicated to be easily parsed by axiomizations of discrete mathematics, such as Peano Arithmetic. In essence, the objective of set theory (in its abstract idealized form) is to conceptualize such partially nebulous objects $e \in P_1$ in a well defined manner.

Our anticipation is that after the current inconsistencies in ZF and ZFC set theory are recognized, it will be possible to repair these deficiencies under a suitably modified framework that retains most of the beauty and charms of the older and technically-inconsistent version of Set Theory that had inspired it.

Analysis of ZFC Set Theory

Our immediate goal in the current section is to present a diagonalization argument that will show that ZFC will be able to formally prove the theorem that “Support-ZFC represent the empty element belonging to P_2 ”. Since Zermelo Fraenkel Set Theory can prove Gödel's Completeness Theorem, this will imply that ZFC can prove a theorem declaring its own inconsistency. The analogous result where the ZF axiom system replaces ZFC is slightly more complicated (and it shall thus be postponed until Section)

To start our construction, we will use the what Mendelson [13] calls the Fixed Point Theorem. This theorem was first explicitly introduced into the logic literature by Carnap [1] — although both Carnap and Mendelson describe it as being implicit in Gödel's historic 1931 paper [6]. One version of its formal statement is given below:

Theorem 1 (The Carnap-Gödel Fixed Point Theorem) *Let α denote an axiom system that is an extension of Peano Arithmetic. Then for any wff $\psi(x)$ which is free in only the single variable x, it is possible to construct a sentence ϕ such that α can prove the validity of the statement:*

$$\phi \ \Leftrightarrow \ \psi(\ulcorner \phi \urcorner) \tag{4}$$

In order to review how Carnap (and also in an “implicit” form) Gödel would have us construct ϕ from $\psi(x)$, we shall use the following notation:

Subst(g, h) will denote Gödel's classic substitution formula — which yields TRUE when g is an encoding of a formula and h is an encoding of a sentence that replaces all occurrence of free variables in g with a constant representing g's Gödel number.

Also, let $\Upsilon(y)$ denote the following formula:

$$\forall z \quad \text{Subst}(y,z) \Rightarrow \psi(z) \tag{5}$$

Let N denote Equation (5)'s Gödel number. Then ϕ has been defined by the Gödel-Carnap construction to be the sentence $\Upsilon(N)$. In other words, it is the sentence defined by Equation (6).

$$\forall z \quad \text{Subst}(N,z) \Rightarrow \psi(z) \tag{6}$$

Mendelson's textbook [13] provides one example of a very nicely formulated proof showing that this particular definition for the sentence ϕ has the property that the axiom system α can verify Equation (4)'s statement. (The intuition behind this construction is that the only z satisfying $\text{Subst}(N,z)$ in Equation (6) is the formal integer quantity of $\ulcorner \phi \urcorner$.)

2

Our objective in the current paper is to employ the Fixed Point Theorem to prove the inconsistency of the axiom systems ZF and ZFC. We will do so by using the Fixed Point Theorem to construct two sentences, called Paradox-ZFC and Paradox-ZF, that enable us to formalize *non-constructive* proofs that these two respective systems are inconsistent. The final result of this paper (Corollary 2) will actually consist of a constructive proof of the inconsistency of

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these two systems. However, a very surprising aspect of the discourse in this paper is that it will *formally need* the non-constructive proofs, centering around Paradox-ZFC and Paradox-ZF, *as vital intermediate-stage steps*, to formalize the constructive contradiction proofs, that will appear at the end of Section .

Definition of Paradox-ZFC. Similar to the Liar's Paradox and its variation that had appeared in Gödel's seminal 1931 paper [6], the sentence Paradox-ZFC will be a self-referencing mathematical precept that is built with the help of Theorem 1's Fixed Point Principle. Its analog in Gödel's centennial paper consisted of a relatively simple application of the notion of self-reference that had encoded the following sentence:

* There is no proof of this sentence from the axiom system of Peano Arithmetic

The most likely reason that the mathematics literature has awaited for approximately 100 years for a proof that ZF and ZFC are inconsistent is that these proofs require a much more complicated application of the Fixed Point Theorem than the semantic object * — which Gödel had used to prove the First and Second Incompleteness Theorems. Thus, the formal definition Paradox-ZFC appears below. (A slightly more complicated version of this paradigm, which is called Paradox-ZF and which is applicable to ZF Set Theory, will appear in the next section.)

** If Support-ZFC is a nonempty set then the application of the function "Choose" to the domain element "Support-ZFC" will produce an unique $e \in$ Support-ZFC such that **this sentence** (looking at itself) is false under e's assignment of truth values under the "Reveal" formalism (defined by Item 10 of Section).

The intuitive difficulty with the sentence Paradox-ZFC is that if Support-ZFC is nonempty then it will imply that Paradox-ZFC will be simultaneously true and false (and thus enable our planned proof by method of reducto ad absurdum to reach its desired end).

Since ZFC itself can appreciate the nature of this paradox, it will be forced to conclude that Support-ZFC is empty (in order to avoid such a contradictory condition from arising). However surprisingly, this fact will, in turn, imply that ZFC can prove a theorem affirming its own inconsistency (since ZFC has the capacity of proving Gödel's Completeness Theorem — which has indicated that that ZFC is automatically inconsistent when Support-ZFC is empty).
The very short 2-sentence paragraph (above) has provided an abbreviated summary of our main result. The remainder of this section will describe it in further detail by separating its underlying formalism into two lemmas and one subsequent centralizing theorem.

Lemma 1 *It is possible to employ the machinery of Theorem 1's Fixed Point Mechanism to formally encode the sentence Paradox-ZFC, such that ZF and ZFC set theories will recognize its mathematical encoding to be the logical counterpart of the English language given in the statement* **.

Proof: Let $\psi^*(x)$ denote Equation (7)'s formula:

$$\text{If Support-ZFC} \neq \emptyset \quad \text{then} \quad [\, \exists\, e \ \text{ such that } \ e \ = \ \text{Choose(Support-ZFC)} \ \wedge \ \neg\, \text{Reveal}_{L\,,\,ZFC}(x,e)\,] \tag{7}$$

Then Theorem 1's Fixed Point mechanism enables us to construct a sentence ϕ^* which ZF Set Theory can prove satisfies the following property:

$$\phi^* \quad \Leftrightarrow \quad \psi^*(\ulcorner \phi^* \urcorner) \tag{8}$$

In particular, the proof of Theorem 1 had showed one can construct ϕ^* by utilizing the following 2-step process

1. First construct the analog of the sentence $\Upsilon(y)$ appearing in Equation (5) of Theorem 1's proof. We will call this sentence $\Upsilon^*(y)$. It is defined formally below:

$$\forall\, z \quad \text{Subst}(y,z) \ \Rightarrow \ \psi^*(z) \tag{9}$$

2. Let N^* denote Equation (9)'s Gödel number. Then ϕ^* is simply defined to be the sentence of $\Upsilon^*(\,N^*\,)$. In other words, it is the sentence defined by Equation (10).

$$\forall\, z \quad \text{Subst}(\,N^*\,,\,z\,) \quad \Rightarrow \quad \psi^*(z) \tag{10}$$

This sentence ϕ^* was shown by Theorem 1 to satisfy Equation (8)'s requirements. (In particular, Theorem 1 showed that the axiom system ZF (and hence all its extensions) can certainly prove the validity of Equation (8)'s if-and-only-if statement.) □

Definition 1 The sentence ϕ^* defined by Lemma 1's 2-step process above will be called the **Formal Encoding of Paradox-ZFC**. Also, the sentence $\psi^*(\ulcorner \phi^* \urcorner)$ on the right side of Equation (8) will be called the **Dual Encoding of Paradox-ZFC**.

Lemma 2 *Let us assume (as will be proven later in this section) that the axiom system ZFC will prove a theorem indicating that the formal set called Support-ZFC is empty. Then ZFC will prove a theorem stating that ZFC is inconsistent.*

Proof Sketch. It is well known that the axiom system ZF (and therefore also ZFC) are able to prove Gödel's Completeness and Compactness theorems. The contrapositive forms of these theorems easily imply that if Support-ZFC is empty then ZFC is inconsistent. □

We are almost ready to prove the first of our three key theorems. (It will state that ZFC can prove a theorem declaring its own inconsistency). Most traditional proofs of inconsistency theorems have relied upon Gödel's Incompleteness Theorem to prove impossibility results — rather than using his more positively oriented Completeness Theorem. However, Lemma 2 has illustrated how the Completeness Theorem can also be applied to establish impossibility results (if we follow the paradigm illustrated by the proof of Lemma 2 in applying the Completeness Theorem in its contrapositive form).

Theorem 2 *The axiom system ZFC can prove a theorem declaring its own inconsistency. (The same property also applies to the axiom system ZF, but we will postpone proving this stronger result until the next section of this paper.)*

Proof. From Lemma 2, we may infer that ZFC will know that if Support-ZFC represents the empty-set property then ZFC must be inconsistent. Hence to prove Theorem 2, we must merely establish that ZFC will prove the theorem $\mho$ below

$\mho \quad =_{df} \quad$ *The formal set called Support-ZFC contains no elements*

In summary form, ZFC's proof of $\mho$ will be a proof by contradiction that uses Gödel-like diagonalization methods to show that if $\mho$ was false then Paradox-ZFC's statement of ** would be forced to be simultaneously true and false.
We will now explain in greater detail the formal structure of this proof-by-contradiction for the assertion $\mho$. The formal assertion of the statement $\neg\,\mho$ is given below. The proof of $\mho$ from ZFC will temporarily assume that $\neg\,\mho$ is valid and derive a contradiction from this assumption.

$$\neg\,\mho \quad =_{df} \quad [\ \text{Support-ZFC} \neq \emptyset\ ] \tag{11}$$

3

The statement $\neg\,\mho$ will imply that Item 16's function "Choose" will map the domain element Support-ZFC onto a particular unique member of Support-ZFC, which we shall now call e. This element e will, in turn, have the following properties:

A. There must be a model of ZFC that is compatible with e's logical framework (formalized by the "Decipher" function from Item 9 of Section). This is because every element of Support-ZFC, when viewed as an axiom system, is required to be consistent with ZFC. Moreover, ZFC *itself* must know that e has this property — since ZFC can prove Gödel's Completeness Theorem. (The footnote [1] clarifies one point in this regards that may otherwise confuse some readers.)

F Since every element of Support-ZFC (including e) must be complete, consistent and maximal (using the terminology form items 11 – 13 of Section), it follows that every sentence θ in the language L must have the property that either θ or $\neg\theta$ has a truth-value of "T" (true) under e's formalization of truth-values (defined by the "Decipher" function from Item 9 of Section).

We will now show how facts (A) and (B) enable ZFC to obtain the needed proof that Equation (11)'s statement $\neg\mho$ cannot be valid. The first point is that Item (B) implies that both Definition 1's "Formal Encoding of Paradox-ZFC" (denoted as ϕ^*) and its "Dual Encoding of Paradox-ZFC" (denoted as $\psi^*(\lceil\phi^*\rceil)$) must contain formal Boolean values of either "T" (true) of "F" (false) under under e's formal framework for defining truth-values using the "Decipher" function from Item 9 of Section . (In other words, it precludes the Decipher function from assigning these predicates a psuedo-Boolean values of "B" or "U" under e's framework for assigning truth values.) In order to complete our proof by contradiction, we need to show that there exists no pair of Boolean values that can be assigned to these two encodings for Paradox-ZFC under e that does not result in a contradictory circumstance from arising. In particular, there are two cases that need to be considered to justify this claim:

Case I. *The sentences ϕ^* and $\psi^*(\lceil\phi^*\rceil)$ have opposite Boolean values:* (i.e. one is represented under e's "Decipher" formalism by the Boolean value of "T" (for True) and the other by its negation of "F"') : This case is infeasible because it would violate Equation (8)'s invariant (which indicates that the two concerned predicates are logically equivalent). *Moreover Equation (8) is proveable under ZFC's logic, and Item (A) explicitly indicated that e's interpretation of the truth is consistent with ZFC's model of the truth.* Hence, Equation (8)'s invariant must also be seen as valid under e's interpretation of truth, which is formally denoted as $\text{System}_{L\,,\,ZFC}(e)$. Thus, an unavoidable contradiction arises in this case.

Case II. *The sentences ϕ^* and $\psi^*(\lceil\phi^*\rceil)$ have the same Boolean values:* This case is infeasible because the syntactic structure of $\psi^*(\lceil\phi^*\rceil)$'s statement (defined by the combination of Equations (7), (9) and (10)) automatically causes ϕ^* and $\psi^*(\lceil\phi^*\rceil)$ to have opposite Boolean values *(whenever Support-ZFC $\neq\emptyset$, as is this case here* on account of our initial assumption (see footnote [2]) that $\neg\mho$ does hold). More precisely, this syntactic opposition can be verified under ZFC's logic. It thus extends also to e's interpretation of the truth because Item (A) had indicated that $\text{System}_{L\,,\,ZFC}(e)$ was consistent with ZFC. It thus thereby forces a contradiction to once again arise.

The point is that the Cases I and II above show that it is impossible for the element e = Choose(Support-ZFC) to own a formal definition of Truth consistent with ZFC. In many respects, Tarski's theorem [20] about the inability to define arithmetic truth within the language of arithmetic is analogous to preceding paragraph's paradigm. However, the ZFC paradigm contains one intriguing aspect, that has no analog in Tarski's earlier 1936 paper [20]. It is that the element e = Choose(Support-ZFC) can be formally proven to be well defined whenever Support-ZFC is nonempty.

Moreover what adds further complexity to the paradigm described herein is that ZFC (itself) can prove all the just mentioned facts .... Thus, ZFC is forced to conclude that an unavoidable contradiction will arise if Support-ZFC is nonempty.

Hence, ZFC will contain a *purely non-constructive proof* of the statement $\mho$ (which had formally asserted that Support-ZFC was empty). By the force of Lemma 2, this means that ZFC will also own a likewise *non-constructive* proof that ZFC (itself) is self-contradictory. □

The presence of a non-constructive proof of ZFC's inconsistency of course does not automatically mean that ZFC is actually inconsistent. For example, let PA stand for Peano Arithmetic. Consider the system PA + Inconsistency(PA). Then Gödel's seminal paper [6] presumed that this system was formally consistent — although PA + Inconsistency(PA) can prove a theorem declaring its own inconsistency(since its axiomatic structure is a superset of PA).

Our initial hope was that ZFC would be likewise consistent — albeit capable of supporting a non-constructive proof of its own inconisitency.... Unfortunately, this is not the case. Section will thus use Theorem 2's intermediate results, as a vital mediating mechanism, to formally prove that both ZFC and ZF are actually inconsistent (in a fully constructive sense).

Analysis of ZF Set Theory

This section will have two goals. The first will be to show that Theorem 2 generalizes for ZF Set Theory (i.e. that ZF will possess a non-constructive proof of an inconsistency from itself). Our second objective will be to show that ZF is actually formally inconsistent. An unusually pleasing aspect of this section's results is that it will show how a non-constructive proof of the existence of an inconsistency in ZF can be used as *a vital intermediate step* needed to show that it is also inconsistent (in a fully constructive sense).

In addition to using the 19 notational precepts introduced in Section , our current discussion will also use the following three added notational precepts:

20. Let us recall that Item 2 from Section has defined P_1 as the power set over the set of positive integers. The symbol **ORD** will denote the natural total ordering over this power set that uses the symbol of " $\prec$ ". In particular, each element $e \in P_1$ can be associated with a bit sequence β_1 , β_2 , β_2 , ... such that β_i belongs to Sequence(e) iff and only if the integer $i \in e$. In this context, two elements e^A and e^B of P_1 will satisfy the condition $e^A \prec e^B$ under the ordering of "ORD" if and only if there exists an integer $k \geq 1$ such that:

- **a.** $\beta_k^A = 0 \;\wedge\; \beta_k^B = 1$
- **b.** Every $j < k$ satisfies $\beta_j^A = \beta_j^B$

21. Let s be a non-empty subset of P_1 . (Thus it is an element of the power set of P_1 , which is denoted as P_2 .) . We will say s is **Top-Good** iff there exists some $e \in s$ where e is a maximal element of s under Item 20's ordering of ORD.

22. The symbol **Choose*** will denote a partial function over the set P_2 which maps each $s \in P_2$ onto its maximal element under the ordering of ORD — if such an element exists (as it will when s is "Top-Good"). Otherwise, Choose*(s) will be undefined.

Lemma 3 *The following statement $+$ can be formally proven by ZF Set Theory.*

$+$ Let α denote an arbitrary recursively defined axiom system, and L denote the language α uses. (The concerned axiom system can be ZF Set Theory but it does not need to be.) *If SupportSet(L,α) is non-empty then SupportSet(L,α) will be Top-Good and Choose*[SupportSet(L,α)] will consequently be a well defined member of SupportSet(L,α).*

4

Clarifying Comment: For the sake of clarity, it is technically immaterial in this paper whether the formal statement $+$ will turn out to be true or false. Rather all what is technically needed is that $+$ is proven as a formal theorem by ZF Set Theory. This fact, by itself, will be shown to be sufficent to establish that ZF Set Theory is formally inconsistent.

[1] A potential point that could initially confuse some readers is that ZFC cannot know that ZFC possesses a model, since it is presumed not to know whether or not it is consistent. However, the point is that our proof by contradiction (carried out within ZFC) has begun with the temporary assumption that statement $\mho$ is false. As the negation of the statement $\mho$ implies (in ZFC's formalism) that ZFC is consistent, the proof-by-contradiction that we are discussing is allowed to temporarily entertain the hypothesis that ZFC is consistent.

[2] The syntactic oppostion that arises when Support-ZFC $\neq\emptyset$ is the single most subtle point of this paper. It appears to have no analog in the prior literature. If a reader asked me to insert a footnote into this article flagging the single paragraph that is most responsible for producing the proof that ZF is inconsistent, then that flag would be placed precisely here ! In other words, the core gist of our argument is based on the logical oppositon between ϕ^* and $\psi^*(\lceil\phi^*\rceil)$ that is unavoidable when Support-ZFC $\neq\emptyset$.

State of New York
County of Albany
Sworn before me this 18th day of March, 2008.

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Notary Public - State of New York
No. 01-SL6171549
Qualified in Schenectady County
My Commission Exp. July 23, 2011

Proof. The justification of Lemma 3's claim is fairly easy. The required proof is similar to Mendelson's proof [13] of the Lindenbaum Lemma. Thus to prove the existence of $e = \text{Choose}^*[\ \text{SupportSet}(L,\alpha)\ ]$, ZF Set Theory will employ a sequence of elements e_0, e_1, e_2, ... whose limiting state converges upon e and which satisfies the condition of $e_0 \prec e_1 \prec e_2 \prec \ldots$. In a context where $\text{System}_{L,\alpha}(e)$ was defined by Item 7 of Section , ZF Set Theory will employ the following 2-step method for constructing a sequence e_0, e_1, e_2, ... (which is intended to converge upon e).

1. The element $e_0 \in P_2$ will be set equal to the empty set. (Thus $\text{System}_{L,\alpha}(e_0)$ will be automatically be the same as the axiom system α — which the hypothesis of the statement $+$ presumed was consistent on account of the fact that α's $\text{SupportSet}(L,\alpha)$ was non-empty.)

2. Using the terminology form Item 4 of Section , let Enum_L again denote an enumeration of all the sentences in the language of L. Let Ψ_i denote its i–th sentence. Then $\text{System}_{L,\alpha}(e_i)$ is defined by the following two rules:

   (a) If $\text{System}_{L,\alpha}(e_{i-1}) \cup \Psi_i$ is inconsistent then $e_i = e_{i-1}$ (thereby causing $\text{System}_{L,\alpha}(e_i) = \text{System}_{L,\alpha}(e_{i-1})$).

   (b) Otherwise, $e_i = e_{i-1} \cup i$ (thereby causing $\text{System}_{L,\alpha}(e_i)$ to essentially be the formalism of $\text{System}_{L,\alpha}(e_{i-1}) \cup \Psi_i$.)

The point of this inductive construction is that it guarantees that all the element of the sequence e_0, e_1, e_2, ... will be associated with a $\text{System}_{L,\alpha}(e_i)$ that is consistent. Also, the elements of this sequence will have the further properties that $e_0 \prec e_1 \prec e_2 \prec \ldots$. Moreover by reasoning that is similar to what was used to prove Lindenbaum's Lemma and Gödel's Compactness and Completeness theorems, it follows that the sequence e_0, e_1, e_2, ... converges upon an element e that is maximal under ORD's ordering of the elements of $\text{SupportSet}(L,\alpha)$ and which has the property that $\text{System}_{L,\alpha}(e)$ is consistent.
□

We are now ready to define the sentence Paradox-ZF. As with its analog in Section , Paradox-ZF will be used to prove that ZF *must at least support a non-constructive proof of its own inconsistency.* The approximate English language wording of Paradox-ZF is given by the statement *** below. Its wording is identical to Paradox-ZFC (from statement ** of Section) except that the Choose function is replaced by Choose* and references to ZFC are naturally replaced by ZF.

> *** If Support-ZF is a nonempty set then the application of the partial function Choose* to the domain element "Support-ZF" will produce an unique $e \in$Support-ZF such that **this sentence** (looking at itself) is false under e's assignment of truth values under the "Reveal" formalism (defined by Item 10 of Section).

Theorem 3 *The axiom system ZF does prove a theorem (using a non-constructive proof methodology) declaring its own inconsistency.*

Proof. The proof of Theorem 3 is similar to the proof of Theorem 2, except that it needs Lemma 3 to show that the partial function Choose* (whose existence does not depend on the Axiom of Choice) has the property that ZF Set Theory can prove the statement $+$.
Once the preceding is done, we may complete Theorem 3's proof by essentially copying verbatim Section 's proofs of its Lemma 1 and 2 and its Theorem 2 — except that all references to ZFC, Choose and Paradox-ZFC are changed to referrals to ZF, Choose* and Paradox-ZF. □

Corollary 1 *Let us recall that the axiom system ZF contains an infinite number of different instances of its Replacement Axiom schemata. Let ZF_k denote the reduced version of ZF that contains only the first k instances of the Replacement scheme. Then there exists some fixed integer m such that for all $k > m$, ZF_k can prove a theorem affirming its own inconsistency.*

Proof . The proof of Corollary 1 is very easy. It is an entirely trivial generalization of Theorem 3. This is because an inspection of Theorem 3's proof (when tediously worked out in meticulous and extreme formal detail) shows that it requires usage of only a finite number of instances of the Replacement schema. Thus if m is chosen to be large enough to include all the needed instances of the Replacement scheme, it follows easily that that for any $k > m$, ZF_k can prove a theorem affirming its own inconsistency by a straightforward generalization of the preceding proof of Theorem 3 (by esentially doing an exact routine copying of the preceding proof). □

Our next result is called a "corollary" because it is another trivial and fairly obvious consequence of Theorems 2 and 3. (Thus, Theorems 2 and 3 and their diagonalizaton proofs did the hard part of the work by containing the most surprising mathematical aspects of our results through their complicated constructions of proofs by contradiction.)

Corollary 2 *The axiom system ZF is inconsistent in a fully constructive sense (in that one can identify a formal sentence Ψ such that ZF proves both Ψ and $\neg\Psi$.)*

Comment: The reason that Corollary 2 is significant is because its formal statement (unlike the intermediate results given in Theorems 2 and 3) is a fully constructive and explicit statement.

Proof. The following statement does not appear explicitly in Chapter 1 of Takeuti's textbook on Proof Theory [18]. However, it is an ~~easy~~ exercise to extrapolate it from the Gentzen Sequent Calculus formalism, as was summarized in Chapter 1 of Takeuti's textbook on [18]:

> ++ For each integer k there exists a $n > k$ such that ZF_n can prove the consistency of ZF_k.

5

For the convenience of those readers who are unacquainted about how to prove ++ , we provide a brief summary of its proof in the attached appendix. (We suggest that the reader postpone examining this appendix until after the remainder of this paper is first completed.)
The reason that the statement ++ is helpful is that ZF is an extension of ZF_n . Thus for *any fixed integer* k , the statement ++ certainly also implies ZF can prove the consistency of ZF_k
But Corollary 1 had showed that for sufficiently large k , ZF_k can also prove ZF_k's own inconsistency. Hence, it certainly also follows that ZF (which is stronger than ZF_k) can also prove a theorem stating that ZF_k is inconsistent.
Combining these results, we obtain that ZF can simultaneously prove that ZF_k is consistent and that it is inconsistent (for at least some fixed constant k). Thus, if Ψ is the statement that *"ZF_k is inconsistent"*, then we have established that ZF will prove both both Ψ and $\neg\Psi$. Hence ZF is inconsistent in the conventional formal sense of this construct (where it proves two statements of the form Ψ and $\neg\Psi$) . □
Remark 1: One of the fascinating aspects about the formalism of Corollary 2 and of Theorems 2 and 3 is the shortness of their combined proofs in resolving a 100 year-old standing open question. A clarifying comment is that the proof of ZF's inconsistency would be probably be much more difficult and longer if it was carried out entirely from first principles. However, a pleasing short cut method for proving ZF's inconsistency was revealed in this paper. It consisted of first providing a non-constructive proof of the existence of an inconsistency in ZF. Then the further machineries of Corollaries 1 and 2 can be used to extend this result so that we can obtain that ZF will prove two mutually incompatible results, thus rendering it to be inconsistent (in a fully conventional and constructive sense).
Remark 2. It is presumably possible to also prove the inconsistency of ZF without the usage of our Choose and Choose* functions. However, these functions played an important role in shortening our proofs. This is because they made it easier to encode Paradox-ZF in a much more simple and terse form.

Intuition Behind Core Results

The purpose of this section is to explain the intuition behind the inconsistencies hidden in ZF Set Theory. Many readers will find this discussion helpful because diagonalization proofs have been notorious for being hard to understand because of their dizzying heights.

Thus, many readers may find it helpful to set aside, temporarily, the proofs executed in the last three sections of this paper — and to instead focus upon the underlying intuitions that shall be vented in the current section.

A crucial point is that ZF's inconsistencies arise not from the fault of any one of ZF's axioms. Rather it occurs due to interaction between two of its axioms, often called the Replacement schemata and Power Set. As this section shall explain, the formal inconsistencies revealed by Theorems 2 and 3 would disappear if the Replacement schema was weakened so that it was applicable only to sets of countable cardinality.

Our discussion will use the formal definition of ZF that had appeared on Page 288 of Mendelson's textbook. Equality is thus defined extensionally, i.e. $x = y$ stands for $\forall z \quad z \in x \leftrightarrow z \in y$. In this context, ZF's power set axiom, and its replacement and selection axiom schemata were defined as follows in Mendelson's textbook:

1. *Power Set:* $\forall x \; \exists y \forall u \; (\; u \in y \; \Leftrightarrow \; u \subseteq x \;)$

2. *Replacement:* Let Fun(Ψ) indicate that the formula $\Psi(x,y)$ represents a partial function in the sense that Fun(Ψ) stands for:

$$\forall x \; \forall a \; \forall b \; [\Psi(x,a) \wedge \Psi(x,b)] \; \Rightarrow \; a = b] \tag{12}$$

Then for any wff $\Psi(x,y)$ (which as footnote [3] explains may contain some additional free variables), Equation (13) is called Ψ 's instance of the Replacement schemata:

$$\text{Fun}(\Psi) \; \Rightarrow \; \{ \; \forall z \, \exists w \, \forall v \; [\; v \in w \; \Leftrightarrow \; \exists u \, (\, u \in z \; \wedge \; \Psi(u,v) \;] \; \} \tag{13}$$

3. *Selection:* For any wff $\Phi(u)$ that does not contain the variable y , the Selection schemata will contain any axiom of the form:

$$\forall z \; \exists y \; \forall v \quad v \in y \; \Leftrightarrow \; (\; v \in z \; \wedge \; \Phi(v) \;) \tag{14}$$

The ZF axiom formalism also contains five further axiom sentences, in addition to the power set, replacement and selection axioms. These other five axioms, called T, P, N, U, and I in Mendelson's textbook [13], are not directly relevant to our current discussion. However, they are listed below for the reader's convenience.

T: $\forall x \forall y \forall z \quad x = y \; \Rightarrow \; (\; x \in z \; \Leftrightarrow \; y \in z \;)$

P: $\forall x \forall y \exists z \; \forall u \; \{ \; u \in z \; \Leftrightarrow \; (\; u = x \; \vee \; u = y \;) \; \}$

N: $\exists x \; \forall y \quad \neg \, y \in x$

U: $\forall x \exists y \; \forall u \quad \{ \; u \in y \; \Leftrightarrow \; [\; \exists v \; (\; u \in v \; \wedge \; v \in x \;) \;] \; \}$

I: $\exists x \quad \emptyset \in x \; [\; \forall z \; (\; z \in x \; \Rightarrow \; z \cup \{z\} \in x \;) \;]$

Zermelo's original 1908 version of Set Theory included all of the preceding axioms except for the Replacement schemata. Fraenkel discovered the Replacement schemata was necessary for axiomatic set theory to gain adequate strength. The current formal coding of this schemata was subsequently done by Skolem. The infinite number of axioms belonging to the Replacement schemata is strictly stronger than the Selection schemata (which can be viewed as a degenerate and strictly weaker form of replacement).

The surprising aspect of Theorem 3 is that it establishes that there is a very fundamental difficulty with the Replacement's and Selection's schemata at a surprisingly low level. In particular, Theorem 3's proof (unlike Theorem 2's proof) did not even require the formalized construction of the second-level power set of P_2 . It merely required that one take P_1 (which is the smallest conceivable power set with an infinite number of elements) and apply the Selection schemata to construct Support-ZF as a formal set. The contradictory behavior of Support-ZF was then used to show ZF was laden with inconsistencies.

These difficulties could be avoided if one used a variant of Replacement's and Selection's schemata that forbidded their application to sets z (in Equations (13) and (14)) which have cardinality equal to that of P_1 or higher.

At first, some researchers might find it counter-intuitive that two axiom schemata, as seemingly straightforward and simple as Replacement and Selection, would be capable of producing inconsistencies. However, the point is that an infinite set of "uncountably large size" is such a daunting construct, due to its uncountable nature, that one's initial intuitions should not necessarily be presumed to be correct. (After all, such a set includes more than merely an infinite number of elements ..... Its cardinality is also of the *dizzying* size of an "uncountable" infinity — at which point all *common sense* intuitions and reckonings may just simply terminate !)

All the difficulties that we have found with ZF Set Theory would be obviated if the Replacement and Selection schemata in Equations (13) and (14) were revised to additionally contain a predicate requiring that their base sets z had countable size. (Indeed instead, it might feasible to merely require that the cardinality of z be strictly less than the cardinality of P_1 in these axioms.)

Let us use the term Low-ZF to refer to such a weaker version of ZF. The formalism Low-ZF is probably too weak for it to be thought of as a serious variant of set theory. For example, common set manipulation operations such as set-subtraction, set-intersection, function-composition and the construction of the cross product of two sets require usage of a version of the Replacement and/or Selection schemata that are not available in Low-ZF. (Also, the definition of the set of non-negative integers as the minimal-sized infinite set satisfying Axiom I's "infinity" condition requires usage of the Selection axiom.) One would therefore wish to add to Low-ZF a series of other proper axioms so that these functional operations and objects would be retained.

Our intuition is that some starting construct, similar Low-ZF, is needed as a base for repairing ZF's inconsistencies. However, a further difficulty is that Low-ZF may also be too strong (especially when it is united with new axioms for defining the set of integers and the operations of intersection, cross-product, function-composition and set-subtraction). The difficulty is that the unlimited usage of arbitrary quantified variables within the formulae of $\Psi(x,y)$ and $\Phi(u)$ in the Replacement and/or Selection axiom schemata is potentially troublesome (and could potentially cause Low-ZF to be inconsistent).

After all, Russel's paradox concerning the conception of the notion of the *"set of all sets such that ..."* has a framework that has an uncomfortable analogy with the arbitrary use of quantifiers within the formulae of $\Psi(x,y)$ and $\Phi(u)$ in the Replacement and/or Selection axiom shemeta.... *Should this be permitted ?* .... Or alternatively should each set quantified variable in $\Psi(x,y)$ and $\Phi(u)$ be a "bounded set quantifier" of the approximate form of " $\forall p \in q$ " and " $\exists p \in q$ " , where q is some pre-specified fixed set?

One is tempted to follow the analogy of Russel's Paradox and to wonder whether some type of bounded quantifiers should be required to appear in $\Psi(x,y)$ and $\Phi(u)$ .... But it is not 100% clear?

The confusing aspect is that we were already implicitly using bounded quantifiers in Section 's definition of Support-ZF because all the intermediate variables used to define the notions of completeness, consistency and maximality can be formalized by quantifiers ranging over the set of positive integers (or their collection of associated Gödel numbers). *Thus even with this added constraint in place about quantifier ranges* ..... , Theorem 3's vexing inconsistency result continues to remain in force ***unless one forbids*** the base sets, called z in the Replacement and Selection schemata of Equations (13) and (14), from having a cardinality as large as that of P_1.

In a context where P_1 is a set whose infinite cardinality has a well-known "uncountable" dizzying nature, this restriction is unlikely to affect many results in Applied Mathematics, Computer Science or in the numerous concrete facets of Theoretical Mathematics.

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[3]We follow Mendelson's notation convention that $\Psi(x,y)$ in Equation (13) may implicitly contain the additional free variables of say t_1 , t_2 ,... t_k . Then if $\Upsilon(t_1 , t_2 ,... t_k)$ is the formal statement in Equation (13), Mendelson's method of couching Replacement would be able to infer $\forall t_1 \; \forall t_2 \; ... \; \forall t_k \;\; \Upsilon(\, t_1 , t_2 ,... t_k \,)$ from the preceding formula via k applications of the generalization rule. Obviously, an alternate style for couching ZF's Replacement axiom would be to write it formally as " $\forall t_1 \; \forall t_2 \; ... \; \forall t_k \;\; \Upsilon(\, t_1 , t_2 ,... t_k \,)$ " , as Cohen does in [2].

Concluding Remarks

The prior chapter of this paper had clearly indicated that one possible method to repair ZF Set Theory would be to simply drop its power set axiom. Then the P_1 power set would no longer be available to interact with the Replacement axioms. In our opinion, this option would be too radical — in that it would force one to depart from the majestic foundational formalism that Hilbert had called "Cantor's Paradise".

A better solution is to weaken ZF's Replacement axiom schemata (which was not part of the initial somewhat informally specified Cantor scheme). In that case, ZF's Replacement axiom schemata would still retain an infinite number of instances, but it would not be as broad as the current schemata.

Our hope and anticipation is that most of the renown beauty of Set Theory to conceptualize highly abstract objects would be retained within such a revised framework — while the inconsistencies that arise from the excesses of the current version of ZF would be singularly removed. In such a context, a new revised version of ZF Set Theory would presumably support all the predictions of Applied Mathematics and most of the formalisms of Theoretical Mathematics — while being protected from inconsistencies.

The author of this article plans to accompany this paper with a second article, which outlines our proposals for revising ZF's formalism. We deliberately do not include those proposals in this paper. This is because any efforts to revise Set Theory at a short notice would be speculative (because it might fail to be sufficiently far-reaching on account of its weakness ..... or alternatively it could be inconsistent on account of its undue strength). We would thus prefer our proposals on how to reconstruct Set Theory to appear in a separate manuscript — so that the community of readers could not possibly confuse the speculative part of this latter aspect of our research project from the firmly derived results concerning the inconsistency of ZF, given in this paper.

Some partial speculations about the likely shape of a new version of set theory had appeared in Section . Our anticipation is that such a revised formalism is unlikely to cause major changes in Discrete Mathematics, in Computer Science, in the concrete facets of Theoretical Mathematics or in Applied Mathematics because the major part of these formalism can be presumably couched in terms that do not require ascending very far into the heights of uncountably large infinite sets.

The author of this article is not adequately familiar with the literature about large cardinals to make any firm comments about it. It is plausible that large cardinal numbers may play a quite significant role in some new type of axiomization for set theory. This is because if the Replacement Schemata is weakened then theorems about large cardinals may play a very helpful role as intermediate results for understanding the property of smaller sets (with conventional cardinalities) that occur in every-day reality.

The essential point is that ZF Set Theory has had a remarkable success record over the last 100 years despite its technical inconsistency. Otherwise, it would not have been the subject of so much research attention during the last century and defied prior efforts to find any inconsistency embedded within it. In such a context, we are quite certain that a revised form of Set Theory is feasible and will continue to do the magic that Hilbert had called "Cantor's Paradise".

In essence, some type of revised form of Set Theory is needed to explain how a language of logic can conceptualize and formalize the many non-recursively defined entities that appear in the world of mathematics, computer science, philosophy and every-day reality in a formally consistent and helpful manner.

The future utility of set theory is likely to belong as much to philosophy and to general models of every-day reality, as it is to the formalization of the ultimate foundations of mathematics and to computer science.

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Appendix: Summary of the Proof of the Statement ++

This appendix will briefly summarize how one can use the Gentzen Sequent Calculus formalism, as was summarized in Chapter 1 of Takeuti's textbook in [18], to extrapolate a proof of the following statement:

++ For each integer k there exists a $n > k$ such that ZF_n can prove the consistency of ZF_k.

We need one definition in order to summarize how ++ may be proven.

Definition 2. Given any axiom system α containing a *strictly finite number* of proper axioms, the following notation will be used

a. The symbol Empty(α) will denote the Gentzen style sequent which (using Takeuti's notation [18]) enumerates all of α's axioms on the left side of its turn-style symbol and which contains the empty set on its right side.

b. The symbol InconsCF(α) will denote that there is a cut-free sequent calculus proof of Empty(α) using the Gentzen sequent calculus formalism called **LK** in Chapter 1.2 of Takeuti's textbook.

c. The symbol InconsHilb(α) will denote that there is a Hilbert-style proof of α's inconsistency.

d. The symbol ConsCF(α) will denote the negation of the sentence InconsCF(α). Thus it will designate that there exists no cut-free sequent calculus proof of Empty(α) using the Gentzen Cut-Free sequent calculus formalism.

In order to formally prove ++ , one needs to employ the following facts:

1. Gentzen's Cut Elimination Theorem (which has a very nice proof in Takeuti's textbook [18]) implies that InconsCF(α) and InconsHilb(α) are logically equivalent to each other. (Moreover for some fixed constant c_0 this proof can be carried out within ZF_m for all $m > c_0$. Thus if we choose n to be large enough in the statement ++ , then the knowledge of this effect will be available to ZF_n .)

2. For any fixed k , it is easy to choose a large enough $n > k$ such that ZF_n can prove the statement ConsCF(ZF_k). This is because each formula in a proof of Empty(ZF_n) will contain no more than than a fixed number of quantifiers, denoted by some number L_k , where the value of the constant L_k depends only on k . Thus if we choose n to be large enough, ZF_n will be capable of constructing a model that houses all the proper axioms of ZF_k and thereby shows that it is impossible to construct a cut-free proof of Empty(ZF_n).

The combination of Items (1) and (2) imply the validity of ++ . This is because Item (2) implies ZF_n can prove ConsCF(ZF_k) (when n is large enough), and Item (1) implies that ZF_n knows the latter to be equivalent to the Hilbert consistency of ZF_k. (See footnote [4] to explore one significant point that may otherwise potentially confuse some readers.) □

It was by deliberate intention that I put the proof of ++ in an appendix section of this paper, rather than in one of the five main chapters of this article. This is because I am quite convinced that ++ is already known in the literature.

Thus, I conceived of this theorem 15 years ago when reading Chapter 1 of Takeuti's textbook [18]. My somewhat hazy memory is that after proving ++ during my reading of Takeuti's textbook in 1993, I became convinced that someone else had proved ++ earlier. The first author who proved an analog of this result was probably Mostowski in connection with what is called "reflective" axiom systems, but I am not 100 % sure who did what and when?

It was for this reason that I thought it was safest to put the proof of ++ in an appendix section of this article. The result is relatively easy to prove, and the correct citations about who proved it first can can be inserted into a later draft of this paper before publication takes place.

Don't forget to check section numbes and theorem numbers. One mistake found on Mar 17 and others may exist!

Old Section 5 to be removed

The preceding difficulties could be avoided by a new system. WZF (with the W for Willard) where the Replacement Axiom's base formalae are required to have "bounded set quantifiers". These would force the $\forall$ and $\exists$ quantifiers to select elements from a prespecfied sets defined by earlier stages of a proof, called say $S_1, S_2, ...$ where each set S_i is a countable set. It may be necessary to add some other set algebraic operatons to WZF. For example, if set subtraction and intersection cannot be encoded in this theory, then they should probably be added as new operations.

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[4] This construction does not imply that ZF can prove its own consistency because the fact that it can prove every finite subset of it to be consistent does not imply a similar argument applies to their infinite union. For example, it is known that Peano Arithmetic can prove every finite subset of its axioms is consistent, but it cannot prove their infinite union is consistent (assuming as we do that Peano Arithmetic is consistent).

Under the formalism I envision, a proof would construct a series of new sets in its first k stages that could be called say C_1 , C_2 , C_3 , ... The new replacement axiom could use any of these sets when it contructs a new set called say C_k . However, the only quantifiers that are allowed to vary over an infinite number of sets would have their range restricted to either the set of positive integers or a countable set. (These two notions are equivalent because a countable set is generated by I and a pre-constructed function.)

It is possible that the above may even be too much. Perhaps one should allow only one universal or existential quantified variable (called X) to formally range over the set of integers. All other quantifiers ranging over the set of intgers should be bounded quantifiers of the forms $\forall v \leq X$ or $\exists v \leq X$ where X is the integer generated by the initial unbounded quantifier. Somehow, I find this idea especially appealing.

Other Remarks Gödel's Completenss and Compactness Theorem and the Lindenbaum Lemma are likely to be invalid under WZF. It may (?) be possible to add other rules to WZF where one defines the notion of quasi-set, which can be perhaps be defined by stronger versions of the Replacement axiom but whose application in the logic is somehow limited. In such a respect (which is currently ambiguous ?), one might be able to prove a theorem that is analogous to Gödel's Completeness Theorem (but involves quasi sets rather than sets being models of consistent axiom systems).

My guess is that Gödel's Completeness Theorem can be partially reconstructed in such a diluted form. For example, the way one can partially escape this whole dilemma is that the quasi-sets might be defined so that they are not a subset of any power set. Then one might be able to define a quasi-set Q, all of whose members are positive integers, but which is not an element of the Section 's power set P_1.

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Back side notarize