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CHAPTER 77

DAVID HILBERT AND PAUL BERNAYS, *GRUNDLAGEN DER MATHEMATIK*, FIRST EDITION (1934, 1939)

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In these two volumes, Hilbert and Bernays present systematically their proof-theoretic investigations and a wide range of current results, such as Herbrand's theorems and Gödel's incompleteness theorems. The second volume has a number of supplements, in which they discuss some specialized topics, for example, the development of mathematical analysis and the unsolvability of the decision problem.

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French translation of the second edition. Fondements des mathématiques (trans. F. Gaillard, E. Gaillard and M. Guillaume), 2 volumes, Paris: l'Harmattan, 2001.

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Related articles: Dedekind (§43), Dedekind and Peano (§47), Hilbert on geometry (§55), Whitehead and Russell (§61), Gödel (§71).

1 BACKGROUND

The two volumes of *Grundlagen der Mathematik* by David Hilbert (1862–1932) and Paul Bernays (1888–1977) are very special milestones in the development of modern mathematical logic. They were at the forefront of contemporaneous research and presented then current metamathematical results: from consistency proofs (Hilbert and Bernays had obtained in weaker forms during the 1920s) through theorems of Jacques Herbrand (1908–1931)

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and Kurt Gödel (1906–1978) to a sketch of a consistency proof for number theory found by Gerhard Gentzen (1909–1945). This material is supplemented in the second volume by a series of important appendices concerning focused topics, for example, a very elegant formal development of analysis and an incisive presentation of the undecidability of the decision problem. Indeed, the two volumes constitute an encyclopedic synthesis of metamathematical work from the preceding two decades. What is most remarkable, however, is the sheer intellectual force that structures the books: they are penetrating and systematic studies concerned with the foundations of modern mathematics as it emerged in the second half of the 19th century. That emergence was deeply influenced by C.F. Gauss, J.P.G. Dirichlet, Bernhard Riemann, and above all by Richard Dedekind (1831–1916).

Dedekind formulated abstract axiomatic theories within a general logicist framework that was articulated most explicitly in *Was sind und was sollen die Zahlen?* [1888]. His way of formulating theories was used by Hilbert in *Die Grundlagen der Geometrie* [1899] and the paper 'Über den Zahlbegriff' [1900]. Hilbert recognized, as Dedekind had done, the centrality of the consistency problem for such theories. For Dedekind this was a semantic issue, and he tried to resolve it by defining suitable models within logic. However, problematic aspects of Dedekind's broad logicist framework were noticed early by Georg Cantor (1845–1918) and formulated in letters to Hilbert in 1897. Hilbert reformulated the consistency problem as a quasi-syntactic one for his axiomatization of the arithmetic of real numbers, both in his papers (1900) and (1901), in the latter in the second of his Paris problems (§57.2). He demanded that a 'direct proof' be given to establish that no contradiction can be obtained from the axioms in a 'finite number of logical steps'. The point of such a proof was to establish the existence of a 'consistent multiplicity', i.e. a set, satisfying the axioms. At the time, Hilbert thought that a consistency proof could be given 'by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers'.

Hilbert believed, it seems, that the genetic build-up of the real numbers could be exploited to yield the blueprint for a consistency proof in Dedekind's logicist style. That is supported by Hilbert's treatment of arithmetic in other lectures from that period, but also by a more programmatic statement from the introduction to the notes for his lectures 'Elemente der Euklidischen Geometrie' (summer semester 1899). He maintains there: 'It is important to fix precisely the starting-point of our investigations: as given we consider the laws of pure logic and in particular all of arithmetic' [Toepell, 1986, 203–204]. Hilbert adds parenthetically, 'On the relation between logic and arithmetic cf. Dedekind, *Was sind und was sollen die Zahlen?*'. And, clearly, for Dedekind arithmetic is part of logic.

2 NAÏVE PROOF THEORY

In Dedekind's as well as in Hilbert's systematic developments only the mathematical parts are characterized axiomatically; logic is not given a principled formulation. That changes in 1904 with Hilbert's programmatic call for a simultaneous development of logic and mathematics. However, it is only more than a decade later that an appropriate logical frame is obtained through the careful study of *Principia mathematica* (1910–1913) by A.N. Whitehead (1862–1947) and Bertrand Russell (1872–1970). This fully formal framework is then recognized as an object of metamathematical investigation—to address the issues that arose at the beginning of the century (§61). We consider some of them now.

2.1 *Equational theories*

Hilbert changed his basic attitude towards consistency proofs only around 1903 after the discovery of the elementary contradiction of Russell and Zermelo, which convinced him that there was a *deep* problem. In early 1904 he wrote to Adolf Hurwitz and claimed: 'exactly the most important and most interesting questions [concerning the foundations of arithmetic] have not been settled by Cantor and Dedekind (and a fortiori not by Weierstrass and Kronecker)'. He announced his intention to offer in the next semester a seminar on the 'logical foundations of mathematical thought' (original in [Dugac, 1976, 271]). The lecture notes from that term contain remarks on Dedekind's achievements, but insist that fundamental difficulties remain:

> He [Dedekind] arrived at the view that the standpoint of considering the integers as obvious cannot be sustained; he recognized that the difficulties Kronecker saw in the definition of irrationals arise already for integers; furthermore, if they are removed here, they disappear there. This work [*Was sind und was sollen die Zahlen?*] was epochal, but it did not yet provide something definitive, certain difficulties remain. These difficulties are connected, as for the definition of the irrationals, above all to the concept of the infinite; [...]

All of this set the stage for the talk of August 1904 at the International Congress of Mathematicians at Heidelberg. Hilbert [1905] stresses there the programmatic goal of developing logic and mathematics, in particular arithmetic, simultaneously. His theory of arithmetic is now restricted and deals only with natural numbers; it consists of axioms for identity and Dedekind's requirements for a *simply infinite system*, except that the induction principle is not formulated. The consistency of this purely equational system is established by an inductive argument on derivations. The work has real shortcomings, as there is neither a calculus for sentential logic nor a proper treatment of quantification. In sum, Hilbert initiates an important shift from *semantic* to *syntactic* arguments, but the formal set-up is inadequate as a framework for arithmetic, and the ultimate goal of the consistency proof remains to guarantee the existence of a set, here of the 'smallest infinite'.

Henri Poincaré (1854–1912) challenged the foundational import of Hilbert's considerations on account of the inductive character of the consistency proof [Poincaré, 1905–1906]. His incisive analysis shifted Hilbert's attention not away from foundational concerns (they are documented by lectures throughout the period from 1905 to 1917), but from the syntactic approach advocated in the Heidelberg talk. Indeed, under the impact of a detailed study of *Principia mathematica* beginning in 1913, Hilbert flirted again with logicism. What resulted from this study, very importantly as it contains the first exposition of modern mathematical logic, were the lectures 'Prinzipien der Mathematik', given in the winter semester of 1917–1918 with the assistance of Bernays. Their logicism was abandoned in the following year; a radical constructivism was adopted instead and subsequently abandoned; finally, the finitist consistency program was formulated in lectures given in the winter semester of 1921–1922.

2.2 *Quantifier-free systems*

The evolution towards this program started in the summer semester 1920, when Hilbert came back to the syntactic approach of his course of 1905. The notes from that semester contain a consistency proof for almost exactly the same fragment of arithmetic as that discussed in the Heidelberg talk; the modified argument is presented in the first part of [Hilbert, 1922] and its strategic point is made explicit there: 'Poincaré's objection, claiming that the principle of complete induction cannot be proved but by complete induction, has been refuted by my theory'.

In the second part of [Hilbert, 1922] the theory is expanded to include an appropriate logical calculus; he emphasizes that 'all formulas and statements of arithmetic can be obtained in a formal way'. The editors of his *Gesammelte Abhandlungen* mention that 'a schema for the introduction of functions by recursion equations' has to be added, if this last goal is to be reached. As to the claimed consistency result, they assert that it holds only if quantifiers are excluded and the induction axiom is replaced by the induction rule. With these modifications consistency is claimed though not proved there, for a theory that includes primitive recursive arithmetic. This work is the beginning of a genuinely new direction, which is best articulated in [Bernays, 1922] and given its principled formulation in Hilbert's Leipzig talk: the instrumental character of extensions that go beyond finitist mathematics is now emphasized.

The developments leading to a proof of the above result can be followed in contemporaneous lecture notes; the proof is only sketched in [Hilbert, 1923], but was given in detail during the winter semester of 1922–1923. The first step turns linear proofs into trees so that any formula occurrence is used at most once as a premise of an inference. That prepares the second step, namely, the elimination of all (necessarily free) variables through appropriate substitutions by a numeral. In the third step the numerical value of the closed terms and the truth-value of the formulas are determined. As all formulas in the final syntactic configuration turn out to be true, an inconsistency cannot be proved. Primitive recursively defined functions are admitted and treated in the argument. The rule of induction for quantifier-free formulas is also added, though not incorporated into the argument—it could be, as it was done already in 1921–1922.

From a contemporary perspective the arguments reveal something very important: as soon as a formal theory contains a class of finitist functions it is necessary to appeal to a wider class of functions in this kind of consistency proof. An *evaluation function* is needed to determine uniformly the numerical value of terms, and such a function is no longer in the given class. As the formal system considered in the above consistency proof includes primitive recursive arithmetic, the consistency proof goes beyond the means available in primitive recursive arithmetic. Finitist mathematics is consequently stronger than primitive recursive arithmetic at this early stage of proof theory. Indeed, as we will see, that assessment of the relative strength is clearly sustained throughout the development reported in this essay.

2.3 *Quantifiers and ε-terms*

The above proof theoretic considerations are preliminary in that they concern a theory that is *part* of finitist mathematics and thus *need* not be secured by a consistency proof. The truly expanding step involves theories with quantifiers treated according to Hilbert's *Ansatz*; that is indicated in [Hilbert, 1922] and elaborated in [Hilbert, 1923]. There, he sketches how quantifiers can be eliminated with the τ-function, the dual of the ε-operator, which replaces the τ-symbol in early 1923. The τ-function associates with every predicate $A(a)$ a particular object $\tau_x.A(x)$ or simply τA; it satisfies the *transfinite axiom* $A(\tau A) \rightarrow A(a)$ and allows the definition of the quantifiers:

$$(x)A(x) \leftrightarrow A(\tau A) \quad \text{and} \quad (Ex)A(x) \leftrightarrow A\big(\tau(\sim A)\big). \tag{1}$$

Hilbert extends the consistency argument to the 'first and simplest case' that goes beyond the finitist system and describes a particular process of eliminating instances of the transfinite axiom (later also called *epsilon axiom*, *epsilon formula* or *critical formula*).

The further development is quick and limited. Wilhelm Ackermann (1896–1962) directly continues Hilbert's proof-theoretic work in his thesis but modifies the elimination procedure for epsilon terms. His paper, based on the thesis, was submitted on 30 March 1924 and published in early 1925; it starts out in Section II with a concise review of Hilbert's considerations. That Section is entitled, tellingly, 'The consistency proof before the addition of the transfinite axioms'. At first it was believed that Ackermann [1925] had established the consistency of arithmetic and analysis; but a note was added 'in proof' restricting the result significantly. Von Neumann, whose paper 'Zur Hilbertschen Beweistheorie' was submitted on 29 July 1925, tried to clarify the extent of Ackermann's result and asserts that it covers Russell's mathematics without the axiom of reducibility or Hermann Weyl's system in his book *Das Kontinuum* (1918) [von Neumann, 1927, 46]. In his talk at the International Congress of Mathematicians held in Bologna in 1928, Hilbert [1929] stated, quite in line with von Neumann's observation, that the consistency of full number theory had been secured by the proofs of Ackermann and von Neumann; according to Bernays in his preface to the second volume, that belief was sustained until 1930. Indicating the depth of Dedekind's influence, Hilbert formulated as Problem I of his Bologna talk the consistency of the ε-axioms for function variables and commented later: 'The solution of problem I justifies also Dedekind's ingenious considerations in his essay *Was sind und was sollen die Zahlen?*'.

As we know now and as was recognized in 1931, Ackermann and von Neumann had established only the consistency of arithmetic with quantifier-free induction. In late 1933 Gödel attributed the most far-reaching partial result in the pursuit of Hilbert's program still to Herbrand, who in [Herbrand, 1931] had extended the Ackermann/von Neumann result by allowing a larger class of finitist functions that included, in particular, the non-primitive recursive Ackermann function. By then, Herbrand knew of Gödel's incompleteness theorems and agreed with von Neumann's related assertion: 'If there is a finitist consistency proof at all, then it can be formalized. Thus, Gödel's proof implies the impossibility of a consistency proof'. The historical development as sketched above is actually reflected in the structure of *Grundlagen der Mathematik*, whose systematic metamathematical content is to be described in the next two sections.

3 THE FIRST VOLUME

According to the preface of this volume, a presentation of proof theory had almost been completed, when the publication of papers by Herbrand and Gödel in 1931 produced a deeply changed situation for proof theory. This resulted in an extension of the scope of the work and its division into two volumes. The volumes were completed in early 1934 and early 1939; though both volumes use much material from joint work in the 1920s, the actual writing of the volumes was done by Bernays. The contents of the volumes are summarized in Table 1.

The eight chapters of Volume I can be divided roughly into three parts: Chapters 1 and 2 introduce the central foundational issues, Chapters 3 to 5 develop systematically the logical framework of first-order logic (with identity) and Chapters 6 to 8 investigate the consistency problem and other metamathematical questions for a variety of (sub-) systems

Table 1. Summary by Chapters of *Grundlagen der Mathematik.* Titles translated.

Chapter; pp.	**Chapter title**
I.1; 19	The consistency problem in axiomatics as a logical decision problem.
I.2; 23	Elementary number theory. Finitist inference and its limits.
I.3; 23	The formalization of logical inference I; the propositional calculus.
I.4; 79	The formalization of logical inference II; the predicate calculus.
I.5; 46	Inclusion of identity. Completeness of the one-place predicate calculus.
I.6; 78	The consistency of infinite domains of individuals. Beginnings of number theory.
I.7; 97	Recursive definitions.
I.8; 76	The concept "that, which" and its eliminability.
II.1; 48	The method of elimination of bound variables by means of Hilbert's ε-symbol.
II.2; 82	Proof theoretic investigation of number theory by means of methods connected with the ε-symbol.
II.3; 75	Application of the ε-symbol for the investigation of the logical formalism.
II.4; 48	The method of the arithmetization of metamathematics applied to the predicate calculus.
II.5; 120	The reason for extending of the methodological frame for proof theory.
II.Supp. I;16	Overview of the predicate calculus and connected formalisms.
II.Supp. II; 29	A sharpening of the concept of calculable function and Church's theorem on the decision problem.
II.Supp. III; 58	On certain parts of the propositional calculus and their deductive demarcation by means of schemata.
II.Supp. IV; 44	Formalisms for the deductive development of analysis.

of number theory. Volume I focuses on the development of proof theory without use of the ε-operator.

3.1 Existential axiomatics

Chapter 1 begins with a general discussion of axiomatics, at the center of which is a distinction between *contentual* and *formal axiomatic theories*. This distinction occurs under different formulations throughout Hilbert and Bernays's writings. Contentual axiomatic theories (examples of which include Euclid's geometry, Newton's mechanics and Clausius's thermodynamics) draw on experience for the introduction of their fundamental concepts and basic principles, which are understood contentually. By contrast, formal axiomatic theories such as Hilbert's axiomatization of geometry abstract away such intuitive content; they begin with the assumption of a fixed system of things (or several such systems), which is delimited from the outset and constitutes a 'domain of individuals for all predicates from which the statements of the theory are built up' (p. 2). The assumption of the existence of such a domain of individuals constitutes an 'idealizing assumption that joins the assumptions formulated in the axioms' (p. 3). Hilbert and Bernays elsewhere refer to this approach as *existential axiomatics*. While they clearly consider formal axiomatics to be a sharpening of contentual axiomatics, nonetheless they are quite explicit that these two types of axiomatics complement each other and are both necessary.

Through a general discussion of the consistency problem for formal axiomatic theories, they are led to conclude that the consistency of a formal axiomatic theory with a finite domain can be established by the exhibition of a model satisfying that system; however, one cannot proceed in this fashion for formal axiomatic theories with infinite domains. Consistency proofs for such theories present a special problem, because 'reference to non-mathematical objects cannot settle the question whether an infinite manifold exists; the question must be solved within mathematics itself' (p. 17). One must treat, they argue, the consistency problem for a formal axiomatic theory F with an infinite domain as a logical problem. This involves i) the formalization of principles of logical reasoning for F, and ii) a proof that from F one cannot derive (using these principles) both a formula and its negation. In short, one must treat the consistency problem from a proof theoretic perspective.

Such a proof need not be given individually for each F. Instead, one need only carry out such a proof for some axiom system F that 1) has a structure that is sufficiently *surveyable* to make a consistency proof for the system plausible, and 2) has a rich enough structure so that by assuming the existence of a system S of things and relations satisfying F, one can derive the satisfiability of axiom systems for the branches of physics and geometry. The satisfiability of an axiom system from those subjects is to be accomplished by representing its objects by individuals (or complexes of individuals) of S and its basic relationships by predicates constructed from those of S by logical operations. Hilbert and Bernays identify arithmetic (including number theory and analysis) as a candidate for such an F.

3.2 Finitist considerations

For such a consistency argument to be foundationally significant, it must avoid the idealizing existence assumptions made by formal axiomatic theories. But if a proof-theoretic justification of arithmetic by elementary means should be possible, might it not be possible to give a direct development of arithmetic free from non-elementary assumptions (and thus not requiring any additional foundational justification)?

The answer to this question involves elementary presentations of parts of number theory and formal algebra; these presentations simultaneously serve to introduce the *finitist standpoint*. The finitist deliberations take here their *purest form*, i.e. the form of '*thought experiments* involving objects assumed to be *concretely given*' (p. 20). The word 'finitist' is intended to convey the idea that a consideration, a claim or definition respects that objects are to be representable in principle, and that processes are to be executable in principle (p. 32).

Having given finitist presentations of elementary number theory and formal algebra, Hilbert and Bernays remark that one cannot obtain a direct, elementary justification for all of mathematics, because already in number theory and analysis one uses non-finitist principles. While it is conceivable one could circumvent the use of such principles in number theory (where one only assumes the existence of the domain of integers), the case is different for analysis. There one assumes in addition the existence of real numbers, that is, infinite sets of integers, and applies the principle of the excluded middle also to these extended domains.

Thus one is led back to the strategy of proceeding in an indirect fashion, i.e., of using proof theory as a tool to secure the consistency of mathematics. As part of this strategy, Hilbert and Bernays adopt the methodological requirement that proof theory be finitist. This requirement ensures that the sought after consistency proof for arithmetic will avoid making idealizing existential assumptions which, after all, are in need of justification. This requirement that proof theory be finitist is relaxed only at the end of the second volume when 'extensions of the methodological framework of proof theory' are considered.

The first stage of this endeavor, the formulation of an appropriate logical formalism, occupies Chapters 3–5. The logical systems they develop are so close to contemporary ones that we do not discuss them in detail; they can actually be traced back to the lectures given in 1917–1918 and are presented already in [Hilbert and Ackermann, 1928]. The systematic development of logical formalisms is accompanied by their proof theoretic investigation. For instance, these chapters contain a number of normal form results as well as a proof of the completeness of the monadic predicate calculus with identity.

3.3 *Consistency proofs*

The second stage, in Chapters 6 and 7, involves the formulation and investigation of subsystems of number theory, which can be arranged into two groups. The first group of systems consists of weak fragments of arithmetic containing first-order quantification but few, if any, function symbols. These formalisms extend the predicate calculus with equality by mathematical axioms for 0, successor and $<$; some of them also involve quantifier-free induction. Hilbert and Bernays explore relations between them and establish independence, as well as consistency results. The main technique for giving consistency proofs is that discussed in section 1.2. However, since the formalisms contain quantifiers, an additional procedure is required here, namely a reduction procedure that assigns quantifier-free formulas, *reducts* acting as witnesses, to formulas containing quantifiers. The method underlying this procedure is due to Herbrand and to Emil Presburger. Additionally, the procedure for the replacement of free variables now must also handle free formula variables.

A further difference is that the consistency results are inferred from more general results involving the notion of *verifiability*, which is an extension of the notion of truth to certain formulas containing free variables, bound variables, and recursively defined function signs. More precisely, letting A be a formula of the formalism F,

i) if A is a numeric formula (that is, if it is composed of equalities and inequalities between numerals by means of sentential connectives), then it is verifiable if it is true;

ii) if A contains free numeric variables (but no formula variables or bound variables), then it is verifiable if one can show by finitist means that the substitution of arbitrary numerals for variables (followed by the evaluation of all function-expressions and their replacement through their numerical values) yields a true numeric formula;

iii) if A contains bound variables but no formula variables, then it is verifiable if its reduct is verifiable (according to i) and ii)).

In order to establish the consistency of a formalism F, one proves now that every formula not containing formula variables is verifiable, if it is derivable in F. Since $0 \neq 0$ is not verifiable, it is not derivable in F; it follows that F is consistent.

The second group of subsystems of number theory contains formalisms arising from the elementary calculus with free variables (the quantifier-free fragment of the predicate calculus) through the addition of functions defined by primitive recursion. Hilbert and Bernays start Chapter 7 with a discussion of the formalization of the principle of definition by recursion. They take the simplest schema of recursion to be

$$f(a, \ldots, k, 0) = \mathrm{a}(a, \ldots, k), \tag{2}$$

$$f(a, \ldots, k, n') = \mathrm{b}\big(a, \ldots, k, n, f(a, \ldots, k, n)\big), \tag{3}$$

where a and b denote previously defined functions and $a, \ldots, k, n$ are numerical variables. After discussing this definitional principle, they prove a

GENERAL CONSISTENCY THEOREM. *Let F be a formalism extending the elementary calculus with free variables by verifiable axioms* (*that may contain recursively defined functions whose defining equations are taken as axioms*) *and the schema of quantifier free induction, then every derivable formula of F is verifiable.*

They explicitly take this theorem to establish the consistency of a number of formalisms including that of recursive number theory, which they develop at length in order to illustrate the strength of recursive definitions. As their notion of recursive number theory is equivalent to primitive recursive arithmetic, finitist mathematics here goes beyond primitive recursive arithmetic. Following this development they discuss formalisms arising from the extension of the recursion and induction schemas and remark that their previous consistency results are easily extended to these systems as well; these remarks push the bounds of finitist mathematics still further.

3.4 Full number theory

The third stage of the development carried out in the first volume occurs towards the end of Chapter 7 and in Chapter 8. Here one finds a third group of formalisms that are each equivalent to full Peano Arithmetic. The first of these is the formalism of the axiom system (Z); call this formalism Z. When arriving at Z, Hilbert and Bernays comment that the techniques used in their previous consistency proofs for fragments of number theory cannot be generalized to Z. The problem is that any reduction procedure for Z would provide a decision procedure for Z and thus would allow one to solve all problems of number theory. They leave the possibility of such a procedure as an open problem (whose solution, if it exists, is a long way off) and focus on showing that Z provides the means for the formalization of full number theory.

With this end in mind, Hilbert and Bernays prove in Chapter 8 that all recursive functions are representable in Z. This proof involves establishing three separate claims: 1) that the least number operator μ can be explicitly defined in terms of Russell and Whitehead's ι-symbol; 2) that any recursive definition (a notion that they leave unanalyzed) can be explicitly defined in Z_μ (that is, Z extended by defining axioms for the μ-operator); and 3) that the addition of the ι-rule to Z is a conservative extension of Z. After the discussion of some additional results, such as the general eliminability of function symbols using predicate symbols, the first volume concludes with the remark that the above results entail the consistency of Z_μ relative to that of Z, but that none of the results or methods considered so far suffice to show that Z is consistent.

4 THE SECOND VOLUME

The second volume picks up where the first left off. It presents in Chapters 1 and 2 Hilbert's proof theoretic 'Ansätze' based on the ε-symbol as well as related consistency proofs; this is the first main topic. The methods used there open a simple approach to Herbrand's theorem, which is at the center of Chapter 3. The discussion of the decision problem at the end of that chapter leads, after a thorough discussion of the 'method of the arithmetization of metamathematics', in the next chapter to a proof theoretic sharpening of Gödel's completeness theorem. The remainder of the volume is devoted to the second main topic, the examination of the fact, which is the basis for the necessity to expand the frame of the contentual inference methods, which are admitted for proof theory, beyond the earlier delimitation of the 'finitist standpoint'. Of course, Gödel's incompleteness theorems are at the center of that discussion.

4.1 Limited results

The consistency proofs in Section 7.a) of the first volume were given for quantifier-free systems. Now these theories are embedded in the system of full predicate logic together with the ε-axioms, which have the form $A(a) \rightarrow A(\varepsilon_x.A(x))$; the ε-terms $\varepsilon_x.A(x)$ represent individuals having the property expressed by $A(a)$, if the latter holds of any individual at all. The crucial task is to eliminate all references to bound variables from proofs of theorems that do not contain them; axioms used in these proofs must not contain bound variables either. In the formulation of Hilbert and Bernays, the consistency of a system of proper axioms relative to the predicate calculus together with the ε-axioms is to be reduced to the consistency of the system relative to the elementary calculus (with free variables) (p. 33). The consistency of the latter system is recognized on account of a suitable finitist interpretation. Thus, Hilbert and Bernays emphasize that operating with the ε-symbol can be viewed as 'merely an auxiliary calculus, which is of considerable advantage for many metamathematical considerations' (pp. 12–13).

In the framework of the extended calculus, bound variables can be seen to be associated really only with terms, as the quantifiers can be defined in a way dual to that shown earlier for the τ-symbol. The initial elimination result is the

FIRST ε-THEOREM. *If the axioms $A_1, \ldots, A_k$ and the conclusion of a proof do not contain bound individual variables or (free) formula variables, then all bound variables can be eliminated from the proof.*

The argument can be extended to cover proofs of purely existential formulas, but the formal proofs then yield as their conclusion a suitable disjunction of instances of the existential formula. Based on this extension Hilbert and Bernays prove their

CONSISTENCY THEOREM. *If the axioms $A_1, \ldots, A_k$ are verifiable, then* i) *any provable formula containing at most free individual variables is verifiable, and* ii) *for any provable, purely existential formula $(Ex_1)\cdots(Ex_n)\ A(x_1, \ldots, x_n)$ (with only the variables shown) there are variable-free terms $t_1, \ldots, t_n$ such that $A(t_1, \ldots, t_n)$ is true.*

This theorem is applied to establish the consistency i) of Euclidean and Non-Euclidean geometry without continuity assumptions in section 1.4, and ii) of arithmetic with recursive definitions, but only quantifier-free induction as in sections 2.1 and 2.2. In essence then, the consistency theorem from [Herbrand, 1931] has been reestablished in a subtly more general way, as is emphasized on p. 52: Hilbert and Bernays allow the introduction of a larger class of recursive functions. We can put the result also in a different historical context and see that the consistency proof of 1923 for the quantifier-free system of primitive recursive arithmetic has been extended to cover that system's expansion by full classical quantification theory.

The remainder of Chapter 2 discusses the difficulty of extending the elimination procedure (in the proof of the first ε-theorem) to a system with full induction and examines Hilbert's original *Ansatz* for eliminating ε-symbols. (As to the character of the original and the later version of the elimination method and Ackermann's work see pp. 21, 29, 92ff, the note on p. 121, as well as Bernays's preface.) The next two chapters investigate the formalism for predicate logic, beginning in Chapter 3 with a proof of the

SECOND ε-THEOREM. *If the axioms and the conclusion of a proof (in predicate logic with identity) do not contain ε-symbols, then all ε-symbols can be eliminated from the proof.*

Then Herbrand's theorem is obtained as well as a variety of criteria for the refutability of formulas in predicate logic; proofs of the Löwenheim–Skolem theorem and of Gödel's completeness theorem are also given. These considerations are used to establish results concerning the decision problem, and solvable cases as well as reduction classes are discussed. In Chapter 4 Gödel's method of the 'arithmetization of metamathematics' is presented in great detail and applied to obtain a fully formalized proof of the completeness theorem.

Here is one standard formulation of the completeness theorem: consistency of an axiom system relative to the calculus of predicate logic coincides with satisfiability of the system by an arithmetic model. The formalized proof is intended to establish a kind of finitist equivalent (p. 205) to a consequence of this formulation, namely, that the consistency relative to the predicate calculus guarantees consistency in an open contentual sense ('im unbegrenzten inhaltlichen Sinne'). The finitist equivalent is formulated in terms of irrefutability roughly as follows: if a formula is irrefutable in predicate logic, then it remains irrefutable in 'every consistent number theoretic formalism', that is, in every formalism that is consistent and remains consistent when the axioms of Z_μ and possibly also verifiable formulas are added (p. 253). That fact can be interpreted as expressing a deductive closure of the predicate calculus, but obviously only if Z_μ is consistent. Thus, there is an additional reason for establishing the consistency of this number theoretic formalism.

4.2 *Incompleteness*

The discussion of Gödel's incompleteness theorems (§71) begins with a thorough investigation of semantic paradoxes. However, this investigation does not try to 'solve' the paradoxes in the case of natural languages, but focuses on the question under what conditions analogous situations can occur in the case of *formalized languages*. These conditions are formulated quasi-axiomatically for general deductive formalisms F taking for granted that there is a bijection between the expressions of F and natural numbers, a 'Gödel-numbering'. The formalism F and the numbering are required to satisfy roughly two *representability conditions*: R1) primitive recursive arithmetic is 'contained in' F; and R2) the syntactic properties and relations of F's expressions, as well as the processes that can be carried out on such expressions, are given by primitive recursive predicates and functions.

For the consideration of the first incompleteness theorem the second representability condition is made more specific. It now requires that the *substitution function* (yielding the number of the expression obtained from an expression with number $\mathbf{k}$, when every occurrence of the number variable a is replaced by a numeral $\mathbf{l}$) is given primitive recursively by a binary function $s(k,l)$ and the *proof predicate* by a binary relation $B(m,n)$ (holding when $\mathbf{m}$ is the number of a sequence of formulas constituting an F-derivation of the formula with number $\mathbf{n}$). Consider, as Gödel did, the formula $\sim B(m, s(a,a))$; according to the first representability condition this is a formula of the formalism F and has a number, say $\mathbf{p}$. Because of the defining property of $s(k,l)$, the value of $s(\mathbf{p},\mathbf{p})$ is then the number $\mathbf{q}$ of the formula $\sim B(m, s(\mathbf{p},\mathbf{p}))$. The equation $s(\mathbf{p},\mathbf{p}) = \mathbf{q}$ is provable in F; thus, $\sim B(m, s(\mathbf{p},\mathbf{p}))$ is actually equivalent to $\sim B(m,\mathbf{q})$ and expresses that 'the formula with number $\mathbf{q}$ is not provable in F'. As $\mathbf{q}$ is the number of $\sim B(m, s(\mathbf{p},\mathbf{p}))$, this formula consequently expresses (via the equivalence) its own underivability. The argument adapted from that for the liar paradox leads, from the assumption that this formula is provable, directly to a contradiction in F. But instead of encountering a paradox, we infer now that the formula is not provable, if the formalism F is consistent.

Hilbert and Bernays discuss—following Gödel and assuming the ω-consistency of F—the unprovability of the sentence $\sim(x)\sim B(m, \mathbf{q})$. Then they establish the Rosser version of the first incompleteness theorem, i.e., the independence of a formula R from F assuming just F's consistency. Thus, a 'sharpened version' of the theorem can be formulated for deductive formalisms satisfying certain conditions: 'One can always determine a unary primitive recursive function f, such the equation $f(m) = 0$ is not provable in F, while for each numeral $\mathbf{l}$ the equation $f(\mathbf{l}) = 0$ is true and provable in F; neither the formula $(x)f(x) = 0$ nor its negation is provable in F' (p. 279). This sharpened version of the theorem asserts that every sufficiently expressive, sharply delimited, and consistent formalism is deductively incomplete. An important consequence of this result is discussed in section 5.1.

4.3 *Unprovability of consistency*

For a formalism F that is consistent and satisfies the restrictive conditions, the proof of the first incompleteness theorem shows the formula $\sim B(m, \mathbf{q})$ to be unprovable. However, it also shows that the sentence $\sim B(\mathbf{m}, \mathbf{q})$ holds and is provable in F, for each numeral $\mathbf{m}$. The second incompleteness theorem is obtained by formalizing these considerations, i.e. by proving in F the formula $\sim B(m, \mathbf{q})$ from the formal expression C of F's consistency. That is possible, however, only if F satisfies certain additional conditions, the so-called *derivability conditions*. Hilbert and Bernays conclude immediately 'in case the formalism F is consistent no formalized proof of this consistency, i.e. no derivation of that formula C, can exist in F' (p. 284).

The formalized argument makes use of the representability conditions R1) and R2), where the second condition now requires also that there is a unary primitive recursive function e, which when applied to the number $\mathbf{n}$ of a formula yields as its value the number of the negation of the formula. These then are the derivability conditions: D1) If there is a derivation of a formula with number $\mathbf{l}$ from a formula with number $\mathbf{k}$, then the formula $(Ex)B(x, \mathbf{k}) \to (Ex)B(x, \mathbf{l})$ is provable in F; D2) The formula $(Ex)B(x, e(k)) \to (Ex)B(x, e(s(k, l)))$ is provable in F; and D3) If $f(m)$ is a primitive recursive term with m as its only variable and if $\mathbf{r}$ is the number of the equation $f(a) = 0$, then the formula $f(m) = 0 \to (Ex)B(x, s(\mathbf{r}, m))$ is provable in F. Consistency is formally expressed by $(Ex)B(x, n) \to \sim(Ex)B(x, e(n))$; starting with that assumption, the formula $\sim B(m, \mathbf{q})$ is obtained in F by a rather direct argument on pp. 286–288.

There are two brief remarks with which we want to complement this metamathematical discussion of the incompleteness theorems. The first simply states that verifying the representability conditions and the derivability conditions is the central mathematical work that has to be done; Hilbert and Bernays accomplish this for the formalism Z_μ (starting on p. 293) and for Z (beginning on p. 324). Thus, the second volume of *Grundlagen der Mathematik* contains the first full argument for the second incompleteness theorem; after all, Gödel's paper contains only a minimal sketch of a proof. However, it has to added—and that is the second brief remark—that the considerations are not fully satisfactory for a *general* formulation of the theorems, as there is no argument given why deductive formalisms should satisfy the particular restrictive conditions on their syntax. This added observation points to one of the general methodological issues discussed next.

5 PHILOSOPHICAL AND MATHEMATICAL ISSUES

The existential formal axiomatics that emerged in the second half of the 19th century and found its remarkable expression in Hilbert's *Grundlagen der Geometrie* (§55) constituted the major pressing issue for the various Hilbert programs during the period from 1899 to 1934, the date of the publication of the first volume of *Grundlagen der Mathematik*. The finitist consistency program that began to be pursued in 1922 is the intellectual thread holding the investigations in both volumes together. The general programmatic direction was formulated clearly in the first volume and presented above in section 2.1. The ultimate goal of proof theoretic investigations, as Hilbert formulated it in the preface to volume I, is to recognize the usual methods of mathematics, without exception, as consistent. Hilbert continued, 'With respect to this goal I would like to emphasize the following: the view, which temporarily arose and maintained that certain recent results of Gödel imply the infeasibility of my program, has been shown to be erroneous'. How is the program affected by those results? Is it indeed the case, as Hilbert expressed it also in 1934, that the Gödel theorems just force proof theorists to exploit the finitist standpoint in a sharper way?

5.1 Issue of completeness

The second question is raised prima facie only through the second incompleteness theorem. However, Hilbert and Bernays discuss also the effect of the first incompleteness theorem and ask quite explicitly (p. 280), whether the deductive completeness of formalisms is a necessary feature for the consistency program to make sense. They touched on this very issue already in pre-Gödel publications, Hilbert in his Bologna Lecture of 1928 and Bernays in his penetrating article [1930]. He formulated in his lecture the question of the syntactic completeness for number theory and analysis as Problem III, and concluded the discussion by suggesting that 'in höheren Gebieten' (higher than number theory) it is thinkable that a system of axioms could be consistently extended by a statement S, but also by its negation $\sim S$; the acceptance of one of the statements is then to be justified by 'systematic advantages (principle of the permanence of laws, possibilities of further developments etc.)'.

Hilbert conjectured that number theory is deductively complete (p. 59). That is reiterated in [Bernays, 1930] and followed by the remark that 'the problem of a real proof for this is completely unresolved'. The problem becomes even more difficult, Bernays continues, when we consider systems for analysis or set theory. However, this 'Problematik' is not to be taken as an objection against the standpoint presented (p. 59):

> We only have to realize that the [syntactic] formalism of statements and proofs we use to represent our conceptions does not coincide with the [mathematical] formalism of the structure we intend in our thinking. The [syntactic] formalism

> suffices to formulate our ideas of infinite manifolds and to draw the logical consequences from them, but in general it [the syntactic formalism] cannot combinatorially generate the manifold as it were out of itself.

That is also the central point in the general discussion of the first incompleteness theorem (p. 280). Indeed, Hilbert and Bernays emphasize there that in formulating the problems and goals of proof theory they avoided from the very beginning 'to introduce the idea of a total system for mathematics with a philosophically principled significance'. It suffices for their purposes to characterize the actual systematic structure of analysis and set theory in such a way that it provides an appropriate frame for (the reducibility of) the geometric and physical disciplines.

From these reflective remarks it follows that the first incompleteness theorem for the central formalisms F (of number theory, analysis, and set theory) does not directly undermine Hilbert's program. Nevertheless, it raises in its sharpened form a peculiar issue: any finitist consistency proof for F would yield a finitist proof of a statement in recursive number theory—that is not provable in F. Finitist methods would thus go beyond those of analysis and set theory, even for the proof of number theoretic statements. This is a 'paradoxical' situation, in particular, as Hilbert and Bernays quite unambiguously state in the first volume (p. 42), 'finitist methods are included in the usual arithmetic'. Consequently, even the first theorem forces us to address two general tasks, namely, i) to explore the extent of finitist methods, and ii) to demarcate appropriately the methodological standpoint for proof theory.

5.2 *The extent of finitist methods*

Tasks i) and ii) are usually associated with the second incompleteness theorem, which, as emphasized at the end of Section 4.3, allows us to infer directly and sharply that a finitist consistency proof for a formalism F (satisfying the representability and derivability conditions) cannot be carried out in F. Hilbert and Bernays explore the extent of finitist methods in Section 5.3.a) by first trying to answer the question, in which formalism their various finitist investigations can actually be carried out. The immediate claim is that most considerations can be formalized, perhaps with a great deal of effort, in primitive recursive arithmetic (p. 340). But then they assert: 'At various places this formalism is admittedly no longer sufficient for the desired formalization. However, in each of these cases the formalization is possible in Z_μ'. They point to the more general recursion principles from Chapter 7 of the first volume as an example of 'procedures of finitist mathematics' that cannot be captured in primitive recursive arithmetic, but can be formalized in Z_μ.

In the remainder of Section 5.3.a) they discuss 'certain other typical cases', in which the boundaries of primitive recursive arithmetic are too narrow to allow a formalization of their prior finitist investigations. There is, first of all, the issue of an evaluation function that is needed for the consistency proof of primitive recursive arithmetic (already in volume I) but cannot be defined by primitive recursion (p. 341). Secondly, there is the general concept of a calculable function (p. 342); that concept is used (p. 189) to formulate a finitistically sharpened notion of satisfiability, i.e. *effective satisfiability*, in finitist treatments of solvable cases of the decision problem. Thirdly, they discuss the principle of induction for universally quantified formulas used in consistency proofs (p. 344). The issue surrounding this principle is settled metamathematically, as we now know, by later proof theoretic work: the system of elementary number theory with this induction principle is conservative over primitive recursive arithmetic.

As to ii), some remarks concerning supplement II are relevant in the above context, as the notion of a calculable function has to be sharpened in such a way that it can be formalized. The presentation in that supplement of the negative solution of the decision problem is preceded by a conceptual analysis of the concept 'reckonable function', i.e. of a function whose values can be calculated according to rules. The latter rather vague notion is sharpened, in a way that is methodologically very similar to the analysis of the incompleteness theorems, namely by formulating *recursiveness conditions* for deductive formalisms that allow equational reasoning. The central condition requires the proof predicate to be primitive recursive. It is then shown that the functions calculable in formalisms satisfying the recursiveness conditions are exactly the general recursive ones. The latter notion can be defined in the language of number theory as is necessary for the formalization in ii). Though the conceptual analysis is not fully satisfactory for the reason mentioned in Section 4.3, it is nevertheless a major and concluding step in the analysis of effectively calculable functions as pursued in the mid-1930s by Gödel, Alonzo Church, Stephen Kleene, and others.

5.3 *Beyond finitism?*

The examination of their own proof-theoretic practice leads Hilbert and Bernays to the conclusion that some considerations require means that go beyond primitive recursive arithmetic, but can be formally captured in Z_μ. It is at exactly this point that the second incompleteness theorem provides, as the title of Chapter 5 states, the 'reason for extending the methodological frame for proof theory'. Already on p. 253, as a transition from Chapter 4 to Chapter 5, Hilbert and Bernays state specifically that consequences of the theorem force us to view the domain of the contentual inference methods used for the investigations of proof theory more broadly 'than it corresponds to our development of the finitist standpoint so far'.

The question is, whether there are any methods that can still be called properly 'finitist' and yet go beyond Z_μ. Hilbert and Bernays argue that this is not a precise question, as 'finitist' is not a sharply delimited notion, but rather indicates methodological guidelines that enable us to recognize some considerations as definitely finitist and others as definitely non-finitist. The limits of finitist considerations are to be 'loosened' (vol. 2, 348), and two possibilities of such loosenings are considered that are quickly seen to be 'conservative'. Which further loosenings are 'admissible, if we want to adhere to the fundamental tendencies of proof theory?' Against this background two then recent results are examined: the reduction of classical arithmetic Z to the system $\mathscr{Z}$ of arithmetic with just minimal logic, and Gentzen's consistency proof for a version of $\mathscr{Z}$ (and thus of Z) using a special form of transfinite induction.

The reductive result that Hilbert and Bernays formulate is a slightly stronger one than that obtained by Gödel and, independently, by Gentzen. The proof showing that Z is consistent relative to $\mathscr{Z}$ is an elementary finitist one. Thus, the obstacle for obtaining a finitist consistency proof for Z does not lie in the fact that it contains the typically non-finitist logical principles like *tertium non datur*! The obstacle appears already when one tries to give a finitist consistency proof for $\mathscr{Z}$. The consistency of Z would be established on the basis of any assumptions, 'which suffice to give a verifying interpretation of the restricted formalism' (p. 357). Such a contentual verification, based on interpretations of A.N. Kolmogoroff and Arend Heyting, is then examined with the conclusion that it involves the intuitionistic understanding of negation as absurdity.

The question is raised, whether—in a proof of the consistency of Z—the systematic use of absurdity could be avoided, as well as the appeal to an interpretation of the formalism (in contrast to its direct proof theoretic examination). It is claimed that Gentzen's consistency proof addresses both these issues. After a thorough discussion of the details of the system of ordinal notation and the (justification of the) principle of transfinite induction, but only the briefest indication of the structure of Gentzen's proof, the main body of the book concludes with some extremely general remarks about the significance of Gentzen's proof: it provides a perspective for the proof theoretic investigation also of stronger formalisms, when one clearly has to countenance the use of larger and larger ordinals. The volume concludes with the sentence: 'If this perspective should prove its value, then Gentzen's consistency proof would open a new phase of proof theory'. In this way, it seems, Bernays sees Gentzen's approach as overcoming 'the temporary fiasco of proof theory' he discussed in the introduction to volume II and attributed to 'exaggerated methodological demands put on the theory'.

No explicit final and definitive judgment on the (non-)finitist character of these two consistency proofs is actually articulated in the book. However, in the first volume (p. 43), intuitionism is viewed as a proper extension of finitist mathematics. That view is also expressed in contemporaneous papers by Bernays and in many later comments, perhaps most dramatically in his article on Hilbert, where the above relative consistency proof for Z is seen as the reason for the recognition 'that intuitionistic reasoning is not identical with finitist reasoning, contrary to the prevailing views at the time' [Bernays, 1967, 502]. As to Gentzen's consistency proof, Bernays states in the introduction to the second edition of volume II that the transfinite induction principle used in it is 'a non-finitist tool'.

5.4 Demarcation

In the introduction of the first edition and the detailed discussion there is perhaps an ambiguity; whether the extension of the finitist standpoint necessitated by the incompleteness theorems still is essentially the finitist standpoint as articulated in the first two chapters of volume I, or whether it is a proper extension compatible with the broader strategic considerations underlying proof theory. We think the ambiguity should be resolved in the latter sense; after all, the considerations in Chapter 5 come under the heading 'Transcending the former methodological standpoint of proof theory—Consistency proofs for the full number theoretic formalism'.

However, there is not even a broad demarcation of a new, wider methodological standpoint for proof theory; a reason for this lack is perhaps implicit in the remarks connecting the consistency proof for Z relative to intuitionistic arithmetic with Gentzen's consistency proof (p. 360). It is claimed, first of all, that it is 'unsatisfactory from the standpoint of proof theory' to have only a consistency proof for Z that 'rests mainly on an interpretation of a formalism'. It is observed, secondly, that the only method of going beyond the formalism Z has been the formulation of truth definitions: a classical truth definition is given for Z on pp. 329–340, and the formalization of the consistency proof based on an intuitionistic interpretation would amount to using a truth definition. Thirdly and finally, it is argued that a consistency proof is desirable that rests on 'the direct treatment of the formalism itself'; that is seen in analogy for obtaining the consistency of primitive recursive arithmetic, where Hilbert and Bernays were not satisfied with the possibility of a finitist interpretation, but rather convinced themselves of the consistency by specific proof theoretic methods. Where in this discussion is even an opening for a broader demarcation?

6 CONCLUDING REMARKS

The free and open way in which Hilbert and Bernays joined in the 1920s a number of different tendencies into a sharply focused program with a special mathematical and philosophical perspective is remarkable. The program has been transformed, in accord with the broad strategy underlying Hilbert's proposal, to a *general reductive* one; here one tries to give consistency proofs for strong classical theories relative to 'appropriate constructive' ones. The expanding development of proof theory is one effect of Hilbert's broad view on foundational problems and of his sharply articulated questions. Another effect is visible in the rich and varied results of Hilbert, Bernays, and other members of the Hilbert School (Ackermann, Gentzen, Kurt Schütte); finally, we have to consider the stimulus his approach and questions provided to contemporaries outside the school (von Neumann, Herbrand, Gödel, Church, and Alan Turing). Indeed, there is no foundational enterprise with a more profound and far-reaching effect on the emergence and development of mathematical logic. What Ackermann [1934] formulated in his review of just the first volume, holds even more for the complete two-volume work, namely, that it 'is to be viewed in a line with the great publications of Frege, Peano, and Russell–Whitehead'.

NOTE

Ravaglia wrote section 3, Sieg the rest.

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