On the Tender Line Separating Generalizations and Boundary-Case Exceptions for the Second Incompleteness Theorem under Semantic Tableaux Deduction

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Abstract. Our previous research has studied the semantic tableaux deductive methodology, of Fitting and Smullyan, and observed that it permits boundary-case exceptions to the Second Incompleteness Theorem, when multiplication is viewed as a 3-way relation (rather than as a total function). It is known that tableaux methodologies do prove a schema of theorems, verifying all instances of the Law of the Excluded Middle. But yet we show that if one promotes this schema of theorems into formalized logical axioms, then the meaning of the pronoun "I" in our self-referencing engine changes, and our partial evasions of the Second Incompleteness Theorem come to a complete halt.

General Information: Except for using wider line-spacing, this manuscript is identical to an article that was made public on January 4, 2020 on pp. 268-286 of Volume 11972 in Springer's LNCS Series. Our talk was presented at the LFCS 2020 Conference (hekd in Deerfield, Florida).

1 Introduction

The existence of a significant chasm separating the the goals of Hilbert's consistency program from the implications of the Second Incompleteness Theorem was evident immediately after Gödel published [20]'s seminal result. We exhibited in [45–47, 49–53] a large number of articles about generalizations and boundary case exceptions to the Second Incompleteness Theorem, starting with our 1993 article [45]. These papers, which included six papers published in the JSL and APAL, showed that every extension α of Peano Arithmetic can be mapped onto an axiom system α^* that can recognize its own consistency and prove analogs of all α's Π_1 theorems (in a slightly different language, called L^*).

These formalisms were called "Self-Justifying" systems. They were able to verify their own consistency by containing a built-in self-referencing axiom which declared *"I am consistent"* (as will be explained later). In particular, our axiom systems α^* used the Fixed-Point Theorem to assure α^*'s self-referencing analogs of the pronoun "I" would enable it to refer to itself in the context of its *"I am consistent"* axiomatic declaration.

It turns out that such a self-referencing mechanism will produce unacceptable Gödel-style diagonalizing contradictions, when either α^* or its particular employed definition of consistency are too strong. This is because our methodologies *only* become contradiction-free *when* α^* uses sufficiently weak underlying structures.

These weak structures obviously have significant disadvantages. Their virtue is that their formalisms α^* can be arranged to prove more Π_1 like theorems than Peano Arithmetic, while offering *some type* of *partial* knowledge about their own consistency. We will call such formalisms **"Declarative Exceptions"** to the Second Incompleteness Theorem.

An alternative type of exception to the Second Incompleteness Theorem, which we will call an **"Infinite-Ranged Exception"**, was recently developed by Sergei Artemov [4] (It is related to the works of Beklemishev [6] and Artemov-Beklemishev [5].) Artemov has observed Peano Arithmetic can verify its own consistency, from a special infinite-ranging perspective. This means PA will generate an infinite set of theorems $T_1,\ T_2,\ T_3$ ... where each T_i shows some subset S_i of PA is unable to prove $0 = 1$ and where PA equals the formal union of these special selected S_i satisfying the inclusion property of $S_1 \subset S_2 \subset S_3 \subset \ ...$.

This perspective is also not a panacea. Thus, the abstract in [4] cautiously used the adjective of "somewhat" to describe how it sought to partially achieve the goals sought by Hilbert's Consistency Program (with an infinite collection of theorems $T_1,\ T_2,\ T_3$ ... replacing Hilbert's intended goal of finding one unifying formal consistency theorem).

Our "Declarative"' exceptions to the Second Incompleteness Theorem and Artemov's "Infinite Ranging" exceptions are rigorous results that are nicely compatible with each other. This is because each acknowledged that the Second Incompleteness Theorem is a strong result, that *will admit no full-scale exceptions.* Also, these results are of interest because Gödel conjectured that Hilbert's Consistency Program would ultimately, reach, *some levels of partial success* (see next section). We will explain, herein, how Gödel's conjecture can be *partially justified,* due to an unusual consequence of the Law of the Excluded Middle.

More specifically, we shall focus on the semantic tableaux deductive mechanisms of Fitting and Smullyan [15, 39] and their special properties from the perspective of our JSL-2005 article [49]. Each instance of the Law of the Excluded Middle has been treated by most tableaux mechanisms as a provable theorem, rather than as a built-in logical axiom. This may, at first, appear to be an insignificant distraction because most deductive methodologies do not have their consistency reversed when a theorem is promoted into becoming a logical axiom.

Our self-justifying axiom systems are *different,* however, because their built-in self-referencing *"I am consistent"* axioms have their meanings change, fundamentally, when their self-referencing concept of "I" involves promoting a schema of theorems verifying the Law of Excluded Middle *into formal logical axioms.*

This effect is counterintuitive because similar distinctions exist almost nowhere else in Logic. However some confusion, that has surrounded our prior work, can be clarified when one realizes that *an interaction* between the self-referencing concept of "I" with the Law of Excluded Middle causes the Second Incompleteness Theorem to become activated *precisely when* the Law of Excluded Middle *is promoted* into becoming a schema of logical axioms.

The intuitive reason for this unusual effect is that the transforming of derived theorems *into* logical axioms can shorten proofs under the Fitting-Smullyan semantic tableaux formalism. In a context where our special axiom systems in

Sect. 3 use a self-referencing *"I am consistent"* axiom *and view* multiplication as a 3-way relation (rather than as a total function), this compression will be capable of enacting the power of the Second Incompleteness Theorem. Moreover, the next chapter will explain how this issue is germane to central questions raised by Gödel and Hilbert about feasible boundary-case exceptions to the Second Incompleteness Effect.

2 Revisiting Some Intuitions of Gödel and Hilbert

Interestingly, neither Gödel (unequivocally) nor Hilbert (after learning about Gödel's work) would dismiss the possibility of a compromise solution, whereby a *fragment* of the goals of Hilbert's Consistency Program would remain intact. Thus, Hilbert never withdrew [26]'s statement $*$ for justifying this program:

> $*$ *" Let us admit that the situation in which we presently find ourselves with respect to paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?"*

Gödel was, also, cautious (especially during the early 1930's) not to speculate whether all facets of Hilbert's Consistency program would come to a termination. He thus inserted the following hesitant caveat into his famous 1931 paper [20]:

> $**$ *"It must be expressly noted that Theorem XI"* (e.g. the Second Incompleteness Theorem) *"represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and there might conceivably be finite proofs which cannot be stated in P or in ... "*

Several biographies of Gödel [11, 22, 55] have noted that Gödel's intention (prior to 1930) was to establish Hilbert's proposed objectives, before he formalized his famous result that led in the opposite direction. Moreover, Yourgrau's biography [55] of Gödel records how von Neumann found it necessary during the early 1930's to *"argue against Gödel himself"* about the definitive termination of Hilbert's consistency program, which *"for several years"* after [20]'s publication, Gödel *"was cautious not to prejudge"*. It is known that Gödel had hinted that the Second Incompleteness Theorem was more significant during a 1933 Vienna lecture [21].

Yet despite this endorsement, a YouTube talk by Gerald Sacks [38] explicitly recalled Gödel telling Sacks, during 1961-1962, that some type of revival of Hilbert's Consistency Program would eventually become feasible (as explained in detail by footnote [1]). This recent Year-2014 YouTube lecture by Gerald Sacks had caught many scholars by surprise because Gödel published fewer than 85 pages in his life. Thus, Gödel never explicitly recorded, during the second half of his life, his partial reluctance about the relevance of the Second Incompleteness Theorem, as Sacks did recall in [38].

The research that has followed Gödel's seminal 1931 discovery has mainly focused on studying generalizations of the Second Incompleteness Theorem (instead of also examining its boundary-case exceptions). Many of these generalizations of the Second Incompleteness Theorem [2, 3, 7–10, 13, 16, 23–25, 29, 32–36, 41, 40, 42–44, 46–48, 50] are quite beautiful. The author of this paper is especially impressed by a generalization of the Second Incompleteness Effect, arrived at by the combined work of Pudlák and Solovay together with added research by Nelson and Wilkie-Paris [31, 35, 41, 44]. These results, which also have been more recently discussed in [10, 23, 42, 46], have noted the Second Incompleteness Theorem does not require the presence of the Principle of Induction to apply to most formalisms that use a Hilbert-Frege type of deduction. (The Remark 1 of the next chapter will offer a detailed summary of this helpful generalization of the Second Incompleteness Theorem.)

[1] Some quotes from Sacks's YouTube talk [38] are that Gödel *" did not think"* the objectives of Hilbert's Consistency Program *"were erased"* by the Incompleteness Theorem, and Gödel believed (according to Sacks) it left Hilbert's program *" very much alive and even more interesting than it initially was"*.

3 Main Notation and Background Literature

Let us call an ordered pair (α, D) a **Generalized Arithmetic Configuration** (abbreviated as a **"GenAC"**) when its first and second components are defined as follows:

1. The **Axiom Basis** " α " for a GenAC is defined as its set of proper axioms.
2. The second component " D " of a GenAC, called its **Deductive Apparatus**, is defined as the union of its logical axioms " L_D" with its rules for obtaining inferences.

Example 1. This notation allows us to separate the logical axioms L_D , associated with (α, D) , from its "basis axioms" α . It also allows us to compare different deductive apparatuses from the literature. Thus, the D_E apparatus, from Enderton's textbook [12], uses only modus ponens as a rule of inference, but it deploys a complicated 4-part schema of logical axioms. This differs from the D_M and D_H apparatuses of the Mendelson [30] and Hájek-Pudlák [25] textbooks, which use a more reduced set of logical axioms but require two rules of inference (modus ponens and generalization). The D_F apparatus, from Fitting's and Smullyan's textbooks [15, 39], actually uses *no logical axioms,* but it instead employs a broader "tableaux style" rule of inference. **AN IMPORTANT POINT** is that while proofs have different lengths under different apparatuses, all the common apparatuses will produce the same set of final theorems from an initial common "axiom basis" of α (as explained in footnote [2]).

Definition 1. Let α again denote an axiom basis, D designate a deduction apparatus, and (α, D) denote their GenAC. Henceforth, (α, D)'s will be called **Self Justifying** when

[2] This is because all the common apparatuses satisfy the requirement of Gödel's Completeness Theorem.

i. one of (α, D)'s theorems (or possibly one of α's axioms) states that the deduction method D, applied to the basis system α, produces a consistent set of theorems, and

ii. the GenAC formalism (α, D) is actually, in fact, consistent.

Example 2. Using Definition 1's notation, our prior research [45, 46, 49, 50, 53] constructed GenAC pairs (α, D) that were "Self Justifying". We also proved that the Incompleteness Theorem implies specific limits beyond which self-justifying formalisms simply cannot transgress. For any (α, D), all our articles observed it was easy to construct a system $\alpha^D \supseteq \alpha$ that satisfies the Part-i condition (in an isolated context *where the Part-ii condition is not also satisfied*). In essence, α^D could consist of all of α's axioms plus the added **"SelfRef**(α, D)**"** sentence, defined below:

> $\oplus$ There is no proof (using D's deduction method) of $0 = 1$ from the *union* of the axiom system α with *this* sentence "SelfRef(α, D)" (looking at itself).

Kleene [28] was the first to show how to encode analogs of SelfRef(α, D)'s above statement, which we often call an **"I AM CONSISTENT" axiom.** Each of Kleene, Rogers and Jeroslow [28, 37, 27] emphasized α^D may be inconsistent (e.g. violate Part-ii of self-justification's definition *despite* the assertion in SelfRef(α, D)'s particular statement). This is because if the pair (α, D) is too strong then a quite conventional Gödel-style diagonalization argument can be applied to the axiom basis of $\alpha^D = \alpha +$ SelfRef(α, D), where the added presence of the statement SelfRef(α, D) will cause this extended version of α, ironically, to become automatically inconsistent. Thus, an encoding for "SelfRef(α, D)" is relatively easy, via an application of the Fixed Point Theorem, but this sentence is *potentially devastating.*

Definition 2. Let $Add(x, y, z)$ and $Mult(x, y, z)$ denote two 3-way predicates, specifying $x + y = z$ and $x * y = z$, for which the associative, commutative, identity and distributive properties have Π_1 style encodings provable under an axiom system of α. Then we will say that α **recognizes** successor, addition and multiplication as **Total Functions** iff it can prove all of (1) - (3) as theorems:

$$\forall x \; \exists z \quad Add(x,1,z) \tag{1}$$
$$\forall x \; \forall y \; \exists z \quad Add(x,y,z) \tag{2}$$
$$\forall x \; \forall y \; \exists z \quad Mult(x,y,z) \tag{3}$$

We will call the GenAC system (α, D) a **Type-M** formalism iff it includes (1) - (3) as theorems, **Type-A** if it includes only (1) and (2) as theorems, and it will be called **Type-S** if it contains only (1) as a theorem. Also, (α, D) will be called **Type-NS** iff it can prove none of (1) - (3).

Remark 1. The separation of GenAC systems into the categories of Type$-$NS, Type-S, Type-A and Type-M systems helps summarize the prior literature about generalizations and boundary-case exceptions for the Second Incompleteness Theorem. This is because:

i. The combined research of Pudlák, Solovay, Nelson and Wilkie-Paris $[31, 35, 41, 44]$, as formalized by Theorem $++$, implies that no natural Type$-$S system (α, D) can recognize its own consistency (and thereby be self-justifying) when D is one of Example 1's three examples of Hilbert-Frege style deduction operators of D_E, D_H or D_M. It thus establishes the following result:

> $++$ *(Solovay's modification [41] of Pudlák [35]'s formalism using some of Nelson and Wilkie-Paris [31, 44]'s methods)* : Let (α, D) denote a Type-S GenAC system which assures the successor operation will provably satisfy both $x' \neq 0$ and $x' = y' \Leftrightarrow x = y$. Then (α, D) cannot verify its own consistency whenever simultaneously D is some type of a Frege-Hilbert deductive apparatus and α treats addition and multiplication as 3-way relations, satisfying their usual associative, commutative distributive and identity axioms.

Essentially, Solovay [41] privately communicated to us in 1994 an analog of theorem $++$. Many authors have noted Solovay has been reluctant to publish his nice privately communicated results on many occasions [10, 25, 31, 33, 35, 44]. Thus, approximate analogs of ++ were explored subsequently by Buss-Ignjatović, Hájek and Švejdar in [10, 23, 42], as well as in Appendix A of our paper [46] and in [48] Also, Pudlák's initial 1985 article [35] captured the majority of ++'s essence, chronologically before Solovay's observations, Also, Friedman did some closely related work in [16].

ii. Part of what makes ++ interesting is that [46, 49, 50] presented two cases of self-justifying GenAC systems, whose natural hybrid is precluded by ++. Specifically, these results involve using Example 2's self-referencing *"I am consistent"* axiom (from statement $\oplus$). Thus, they established that some (not all) Type-NS systems [46, 50] can verify their own consistency under a Hilbert-Frege style deductive apparatus [3], and some (not all) Type-A systems [45, 46, 49, 51] can, likewise, corroborate their consistency under a more restrictive semantic tableaux apparatus. Also, we observed in [47, 52] how one could refine ++ with Adamowicz-Zbierski's methods [2] to show most Type-M systems cannot recognize their semantic tableaux consistency.

Remark 2. . Several of our papers, starting with our 1993 article [45], have used Example 2's *"I am consistent"* axiomatic declaration $\oplus$ for evading the Second Incompleteness Effect. Other possible types of evasions rest on the cut-free methods of Gentzen and Kreisel-Takeuti [19, 29], an interpretational approach (such as what Adamowicz, Bigorajska, Friedman, Nelson, Pudlák and Visser had applied in [1, 17, 31, 35, 43]), or Artemov's Infinite-Range perspective [4] (where an infinite schema of theorems replaces one single unified consistency theorem). We encourage the reader to examine all these articles, each of which has their own separate virtues. Our focus, in this paper, will be primarily on the next section's Theorem 1 and 2. They will show that some types of partial *(and not full)* evasions of the Second Incompleteness Effect are possible under a semantic tableaux deductive apparatus.

[3] The Example 1 had provided three examples of Hilbert-Frege style deduction operators, called D_E , D_H and D_M . It explained how these deductive operators differ from a tableaux-style deductive apparatus by containing a modus ponens rule.

4 Main Theorems and Related Notation

A function F is called **Non-Growth** when $F(a_1, \ldots, a_j) \leq Maximum(a_1, \ldots a_j)$ does hold. Six examples of non-growth functions are *Integer Subtraction* (where $x - y$ is defined to equal zero when $x \leq y$), *Integer Division* (where $x \div y$ equals x when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise), $Maximum(x, y)$, $Logarithm(x)$, $Root(x, y) = \lceil x^{1/y} \rceil$ and $Count(x, j)$ (which designates the number of physical "1" bits that are stored among x's rightmost j bits). The term **U-Grounding Function** will refer to either one of one of these six functions or the *growth-oriented* Addition and $Double(x) = x + x$ operations. Our language L^*, introduced in [49], was built out of these eight functions plus the primitives of "0", "1", "$=$" and "$\leq$".

This language L^* differs from a conventional arithmetic by **EXCLUDING** a formal multiplication function symbol. It CAN VIEW multiplication as a 3-way relation (via the use of its Division primitive.) This revised notation will lead us to a surprisingly strong evasion of the Second Incompleteness Effect.

Let t be any term. The v-based quantifiers used by the wffs $\forall\, v \leq t \;\; \Psi(v)$ and $\exists\, v \leq t \;\; \Psi(v)$ will be called *bounded quantifiers*. Any formula in our language L^*, all of whose quantifiers are similarly bounded, will be called a Δ_0^* formula. The Π_n^* and Σ_n^* formulae are defined by usual rules except they **DO NOT** contain multiplication function symbols. These rules are that:

1. Every Δ_0^* formula will also be a " Π_0^* " and " Σ_0^* " formula.
2. A wff will be called Π_n^* when it is encoded as $\forall v_1$ ... $\forall v_k$ Φ with Φ being Σ_{n-1}^*.
3. A wff will be called Σ_n^* when it is encoded as $\exists v_1 .. \exists v_k$ Φ, with Φ being Π_{n-1}^*.

Also, the sentence Ψ will be called a **Rank-1*** statement iff it can be encoded in either a Π_1^* or Σ_1^* format. (The reader is reminded that our definitions for Π_1^* or Σ_1^* formulae differ from Arithmetic's counterparts by excluding multiplication function symbols.)

There will be three variants of formal deductive apparatus methods, which we will now compare. The first is *semantic tableaux*. It will receive the abbreviated name of **"Tab"** and correspond to Fitting's textbook formalism from [15]. (It is also summarized by us in the attached Appendix A.) Thus, a Tab-proof for a theorem Ψ, from an axiom basis α, is a tree-like structure that begins with the sentence $\neg\,\Psi$ stored inside the tree's root and whose every root−to−leaf path establishes a contradiction by containing some pair of contradictory nodes that will "close" its path. The rules for generating internal nodes, along each root−to−leaf path, are that each node must be *either* a proper axiom of α *or* a deduction from an ancestor node via one of the Appendix A 's six stated "elimination" rules for the $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, and $\exists$ symbols.

Our second explored deductive apparatus is called *Extended Tableaux*, and shall be abbreviated as **"Xtab"**. Its definition is identical to **Tab-**deduction, except that for any sentence ϕ in our language L^*, the sentence $\phi \vee \neg\phi$ is allowed as an internal node in an Xtab proof tree. (In other words, *Xtab*-deduction differs from *Tab*-deduction by allowing all instances of the Law of Excluded Middle to appear as permitted logical axioms. In contrast, *Tab*-deduction will view these instances only as derived theorems.)

Our third deductive apparatus was called **Tab-1** in [49]. It is, essentially, a compromise between Tab and Xtab, where a "Tab-1" proof for Ψ from an axiom basis α corresponds to a set of ordered pairs (p_1,ϕ_1), (p_2,ϕ_2), $..(p_k,\phi_k)$ where

1. $\phi_k \;=\; \Psi$
2. Each p_j is a Tab-proof of what we have called a Rank-1* sentence ϕ_j from the union of α with the preceding Rank-1* sentences of $\phi_1,\ \phi_2,\ ..\ \phi_{j-1}$.

We emphasize Tab-1 deduction can be *substantially* less efficient than Xtab because *the former requires* ϕ_j be a Rank-1* sentence, while Xtab *does not impose* a similar Rank-1* constraint upon ϕ, when it invokes its permitted axiom of $\phi \vee \neg\phi$.

Let us say that an axiom system α owns a **Level-1** appreciation of its own self-consistency (under a deductive apparatus D) iff it can verify that D produces no two simultaneous proofs for a Π_1^* sentence and its negation. Within this context, where β denotes any basis axiom system using L^*'s U-Grounding language, $\text{IS}_D(\beta)$ was defined in [49] to be an axiomatic formalism capable of recognizing all of β's Π_1^* theorems and corroborating its own Level-1 consistency under D's deductive apparatus. It consisted of the following four groups of axioms:

Group-Zero: Two of the Group-zero axioms will define the constant-symbols, $\bar{c}_0$ and $\bar{c}_1$, designating the integers of 0 and 1. The Group-zero axioms will also define the growth functions of Addition and $Double(x) = x + x$. (They will enable our formalism to define any integer $n \geq 2$ using fewer than $3 \cdot \lceil \text{Log } n \rceil$ logic symbols.)

Group-1: This axiom group will consist of a finite set of Π_1^* sentences, denoted as F, which can prove any Δ_0^* sentence that holds true under the standard model of the natural numbers. (Any finite set of Π_1^* sentences F, with this property, may be used to define Group-1, as [49] had noted.)

Group-2: Let $\ulcorner \Phi \urcorner$ denote Φ's Gödel Number, and $\text{HilbPrf}_\beta(\ulcorner \Phi \urcorner, p)$ denote a Δ_0^* formula indicating that p is a Hilbert-Frege styled proof of theorem Φ from axiom system β. For each Π_1^* sentence Φ, the Group-2 schema will contain the below axiom (4). (Thus $\text{IS}_D(\beta)$ can trivially prove all β's Π_1^* theorems.)

$$\forall\, p \quad \{ \ \text{HilbPrf}_\beta(\ulcorner \Phi \urcorner, p) \quad \Rightarrow \quad \Phi \ \} \tag{4}$$

Group-3: The final part of $\text{IS}_D(\beta)$ will be a self-referencing Π_1^* axiom, that indicates $\text{IS}_D(\beta)$ is "Level-1 consistent" under D's deductive apparatus. It thus amounts to the following declaration:

> # *No two proofs exist for a Π_1^* sentence and its negation, when D's deductive apparatus is applied to an axiom system, consisting of the union of Groups 0, 1 and 2 with* **this sentence** *(looking at itself).*

One encoding for # as a self-referencing Π_1^* axiom, had appeared in [49]. Thus, (5) is a Π_1^* representation for # where: 1) $\text{Prf}_{\text{IS}_{D(\beta)}}(a,b)$ is a Δ_0^* formula indicating that b is a proof of a theorem a from the axiom basis $\text{IS}_D(\beta)$ under D's deductive apparatus, and 2) $\text{Pair}(x,y)$ is a Δ_0^* formula indicating that x is a Π_1^* sentence and y represents x's negation.

$$\forall\, x\, \forall\, y\, \forall\, p\, \forall\, q \quad \neg\ [\ \text{Pair}(x,y)\ \wedge\ \text{Prf}_{\,\text{IS}_{D(\beta)}}(x,p)\ \wedge\ \text{Prf}_{\,\text{IS}_{D(\beta)}}(y,q)\ ] \tag{5}$$

For the sake of brevity, we will not provide the exact details, here, about how (5) can be encoded via the Fixed Point Theorem. Adequate details were provided by us in [46, 49].

Definition 3. *Let "D" denote any one of the Tab, Xtab or Tab-1 deductive apparatus. Then we will say that the resulting mapping of* $IS_D(\ \bullet\ )$ *is* **Consistency Preserving** *iff* $IS_D(\beta)$ *is automatically consistent whenever all the axioms of* β *hold true under the standard model of the natural numbers.*

The preceding definition raises questions about whether the mappings of $\text{IS}_{Tab}(\ \bullet\ )$, $\text{IS}_{Tab-1}(\ \bullet\ )$, and $\text{IS}_{Xtab}(\ \bullet\ )$ are consistency preserving. It turns out that Theorem 1 will show the first two of these mappings are consistency preserving, while Theorem 2 explores how the Law of the Excluded Middle conflicts with $\text{IS}_{Xtab}(\ \bullet\ )$'s Group-3 axiom.

Theorem 1. *The* $IS_{Tab-1}(\ \bullet\ )$ *and* $IS_{Tab}(\ \bullet\ )$ *mapping are consistency preserving. (I.e. the axiom systems* $IS_{Tab-1}(\beta)$ *and* $IS_{Tab}(\beta)$ *are automatically consistent whenever all* β*'s axioms hold true under the standard model of the Natural Numbers.)*

Theorem 2. *In contrast,* $IS_{Xtab}(\ \bullet\ )$ *fails to be consistency-preserving mappings. (More specifically,* $IS_{Xtab}(\ \beta\ )$ *is automatically inconsistent whenever* β *proves some conventional* Π_1^* *theorems stating that addition and multiplication satisfy their usual associative, commutative, distributive and identity properties.)*

The proofs of Theorems 1 and 2 would be quite lengthy, if they were derived from first principles. Fortunately, it is unnecessary for us to do so here because we gave a detailed justification of Theorems 1's result for $\text{IS}_{Tab-1}(\ \bullet\ )$ in [49], and one can incrementally modify the Remark 1's special Invariant of ++ to justify Theorem 2. Thus, it will be possible for the next two sections of this paper to adequately summarize the intuition behind Theorems 1 and 2, without delving into all the formal details.

Part of the reason Theorems 1 and 2 are of interest is because of their surprising contrast. Thus, some historians have wondered whether Hilbert and Gödel were entirely incorrect when their statements $*$ and $**$ suggested some form of the Consistency Program should likely be viable. Moreover Gerald Sacks's Year-2014 YouTube lecture [38] has reinforced this question by recording how Gödel had repeated analogs of $**$'s speculation during 1961-1962. The contrast between Theorems 1 and 2 will thus provide a plausible suggestion that some portion of what Hilbert and Gödel advocated may be part-way feasible.

This extended abstract will not have the page space to go into all the details. However, the next three sections will, be sufficient to communicate the main gist behind the proofs of Theorems 1 and 2,

5 Intuition Behind Theorem 1

Let us recall the acronym **"Tab"** stands for semantic tableaux deduction. This was defined by Fitting [14, 15] to be a tree-like proof of a theorem Ψ from an axiom basis α , whose root consists of the temporary negated assumption of $\neg\,\Psi$ and whose every root$-$to$-$leaf path establishes a contradiction by containing some pair of contradictory nodes that "closes" its path. Each internal node along these paths must *either* be a proper axiom of α *or be* a deduction from an ancestor node via one of the "elimination" rules associated with the logic symbols of $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, or $\exists$ (that are itemized in the attached Appendix A.)

Example 3. Let $\mathrm{IS}^{M}_{Tab}(\ \bullet\ )$ denote a mapping transformation identical to Theorem 1's formalism of $\mathrm{IS}_{Tab}(\ \bullet\ )$, *except that* IS^{M}_{Tab} shall contain a further multiplication function operation and, accordingly, have its Group-3 "I am consistent" axiom statements *updated* to recognize multiplication as a total function. It turns out this change will cause $\mathrm{IS}^{M}_{Tab}(\ \bullet\ )$ to stop satisfying the consistency-preservation property, which Theorem 1 attributed to $\mathrm{IS}_{Tab}(\ \bullet\ )$.

The intuition behind this change can be roughly summarized if we let $x_0, x_1, x_2, ...$ and $y_0, y_1, y_2, ...$ denote the sequences defined by:

$$x_0 \quad = \quad 2 \quad = \quad y_0 \tag{6}$$

$$x_i \quad = \quad x_{i-1} \ + \ x_{i-1} \tag{7}$$

$$y_i \quad = \quad y_{i-1} \ * \ y_{i-1} \tag{8}$$

For $i > 0$, let ϕ_i and ψ_i denote the sentences in (7) and (8) respectively. Also, let ϕ_0 and ψ_0 denote (6)'s sentence. Then $\phi_0, \phi_1, ... \phi_n$ imply $x_n = 2^{n+1}$, and $\psi_0, \psi_1, ... \psi_n$ imply $y_n = 2^{2^n}$. Thus, the latter sequence shall grow at an exponentially faster rate than the former. It turns out that this change in growth speed will cause the $\mathrm{IS}^{M}_{Tab}(\ \bullet\ )$, and $\mathrm{IS}_{Tab}(\ \bullet\ )$ to have opposite self-justification properties.

In particular, let the quantities $\mathrm{Log}(\,y_n\,) = 2^n$ and $\mathrm{Log}(\,x_n\,) = n+1$ represent the lengths for the binary codings for y_n and x_n. Thus, y_n's coding will have a length 2^n, which is *much larger* than the $n+1$ steps that $\psi_0, \psi_1, ... \psi_n$ uses to define y_n's existence. In contrast, x_n's binary encoding will have a smaller length of $n+1$. These observations are helpful because every proof of the Incompleteness Theorem involves a Gödel number z encoding a capacity to self-reference its own definition.

The faster growing series $y_0, y_1, , ... y_n$ should, intuitively, have this self-referencing capacity because y_n's binary encoding has a 2^{n+1} length that greatly exceeds the size of the $O(n)$ steps used to define its value. Leaving aside many of [47, 52]'s further details, this fast growth explains roughly why a Type-M logic, such as IS^{M}_{Tab}, satisfies the semantic tableaux version of the Second Incompleteness Theorem, unlike IS_{Tab}.

Our paradigm also explains why IS_{Tab}'s Type-A formalisms produce boundary-case exceptions to the semantic tableaux version of the Second Incompleteness Theorem. This is because [49] showed that it was unable to construct numbers z that can self-reference their own definitions (when only the *more slowly growing* addition primitive is available). In particular assuming only two bits are needed to encode each sentence in the sequence $\phi_0, \phi_1, ...\phi_n$, the length $n+1$ for x_n's binary encoding is insufficient for encoding this sequence.

Leaving aside many of [49]'s details, this short length for x_n explains the core intuition behind [49]'s evasion of the Second Incompleteness Theorem under IS_{Tab} . It arises essentially because of the difference between the growth rates of the sequences $x_1, x_2, x_3...$ and $y_1, y_2, y_3...$.

There is obviously insufficient space for this extended abstract to provide more details, here. A full detailed proof of Theorem 1 can be found in [49]. It establishes (see [4]) that Peano Arithmetic can prove β's consistency implies *both* the consistency and also the self-justifying properties for $\mathrm{IS}_{Tab-1}(\beta)$. Our more modest goal, within the present abbreviated paper, has been *merely to* summarize the intuition behind Theorem 1's surprising evasion of the Second Incompleteness effect.

It arises, intuitively, because of the difference in growth rates between the $x_1, x_2, x_3...$ and $y_1, y_2, y_3...$ series.

[4] The *exact* meaning of this implication is subtle. This is because Peano Arithmetic (PA) cannot know whether β is consistent when $\beta = PA$. Thus, *unlike* the quite different formalism of $\mathrm{IS}_{Tab-1}(PA)$, the system of PA shall linger in a state of self-doubt, about whether both PA and $\mathrm{IS}_{Tab-1}(PA)$ are consistent.

6 Summary of the Justification for Theorem 2

The proof of Theorem 2 is complex, but it can be nicely summarized because it is related to the justification for Invariant ++ , which Remark 1 had credited to the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris. The crucial aspect of the Frege-Hilbert methodologies, explored by ++ , is that modus ponens assures that a proof of a theorem ψ from an axiom system α has a length no greater than the sum of the proof-lengths needed to derive ϕ and $\phi \rightarrow \psi$ from α. This **"Linear-Sum Effect"** does not apply, actually, also to *Tab*-deduction because it owns no analog of a modus ponens rule (for assuring that ψ's proof length is bounded by the sum of the proof lengths for ϕ and $\phi \rightarrow \psi$) .

The *Xtab* methodology, however, differs from *Tab*-deduction by allowing any node of its proof-tree to store a sentence of the form $\phi \vee \neg \phi$, as an application of its allowed use of the Law of Excluded Middle. This added feature will allow an *Xtab* proof for ψ to have a length proportional to the sum of the proof lengths for ϕ and $\phi \rightarrow \psi$ (i.e. it can roughly simulate the actions of modus ponens). In particular, the relevant *Xtab* proof for ψ will consist of the following four steps:

1. The root of an Xtab proof for ψ will be the usual temporary negated hypothesis of $\neg\psi$ (which the remainder of the proof tree will show is impossible to hold).
2. The child of this root node will be an allowed invocation of the Law of the Excluded Middle of the *particular* form $\phi \vee \neg \phi$.
3. The relevant Xtab proof tree will next employ the Appendix A's branching rule for allowing the two sibling nodes of ϕ and $\neg \phi$ to descend from Item 2's node.
4. Finally, our Xtab proof will insert below (3)'s left sibling node of ϕ a subtree that is no longer than a proof for $\phi \rightarrow \psi$, and likewise insert a proof for ϕ below (3)'s right sibling of $\neg \phi$.

The point is that the last step of the above 4-part proof has a length no greater than the sum of the two proof lengths for ϕ and $\phi \rightarrow \psi$ (similar to the proof compressions resulting from a modus ponens operation). Its first three steps shall produce inconsequential effects that increase the overall proof length by no more than a *very tiny* amount proportional to the length of the particular sentence " $\phi \ \rightarrow \ \psi$ ".

We can apply the preceeding "Linear-Sum Effect" to construct an analog of Remark 1 's earlier Theorem ++ that now applies to Xtab deduction. Saving several details for a longer article, the intuition behind this analog is that modus ponens *is the only rule of inference* used by [12]'s classic textbook-style first-order logic system, and Xtab can apply its Linear-Sum Effect to essentially simulate modus ponens. Our natural analog of ++ will, thus, assure that any axiom system $\mathcal{A}$, using Xtab deduction, is *automatically inconsistent* when:

- **I.** $\mathcal{A}$ can verify Successor is a total function (as is formalized by Line (1)).
- **II.** $\mathcal{A}$ can prove addition and multiplication (viewed as 3-way relations) satisfy their usual associative, commutative, distributive and identity-operator properties.
- **III.** $\mathcal{A}$ proves an added theorem (which turns out to be false) affirming its own consistency when the Xtab deductive apparatus is used.

The preceding paragraph has nicely captured the essence of Theorem 2's proof. It has noted $\text{IS}_{Xtab}(\beta)$ *is automatically inconsistent* when it satisfies analogs of the above three requirements (see footnote [5]). This is intuitively because $\text{IS}_{Xtab}(\beta)$ can roughly simulate the power of modus ponens, whose crucial Linear-Sum Effect from modus ponens will be imitated by $\text{IS}_{Xtab}(\beta)$ (when the latter views the Law of Excluded Middle as a shema of logical axioms). Thus, an analog of Remark 1's invariant ++ shall apply, consequently, to $\text{IS}_{Xtab}(\beta)$.

From a pedagogical perspective, one obvious drawback to this type of justification for Theorem 2 is that it assumes the reader is familiar with either the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris in [31, 35, 41, 44], or

[5] $\text{IS}_{Xtab}(\beta)$ actually satisfies a requirement stronger than Item I because it recognizes addition as total.

related work (also mentioned in Remark 1) by Buss-Ignjatović, Friedman, Hájek, Švejdar [10, 16, 23, 42] and in Appendix A of [46] (or in Willard's closely related paper of [48]).

Many readers will, of course be unfamiliar with any of this material because it relies upon a much more sophisticated version of the Second Incompleteness Theorem than does appear in most introductory logic textbooks. We have therefore inserted a brief Appendix B into this article. It explains the main idea behind the Theorem ++, which the Remark 1 credited to a recurring theme appearing in [10, 16, 23, 25, 31, 35, 41, 42, 44, 46, 48].

This appendix will not be sufficiently detailed to formulate the precise nature of our reductionistic argument for justifying Theorem 2. (The latter will be saved for a longer version of this article.) Our Appendix B will, however be adequate to provide the reader with a rough intuitive grasp of the type of extension of Theorem ++ , which is needed to establish Theorem 2's generalization of the Second Incompleteness Effect.

7 On the Significance of Theorems 1 and 2

The main topic of this paper is surprising because it is quite unusual for an initially consistent formalism α *to become inconsistent* when its initial schema of theorems (establishing the universal validity of the Law of the Excluded Middle) is essentially transformed into becoming a formal schema of logical axioms.

The reason for this unusual effect is that the meaning of a Group-3 *"I am consistent"* axiom changes, *quite substantially,* when theorems are transformed into logical axioms. This is because unacceptable diagonalizing contradictions can occur (as summarized by footnote [6]) when such a transition *significantly alters* the meaning of an *"I am consistent"* axiom.

[6] The point is that proofs are compressed when theorems are transformed into logical axioms, and such compressions can produce diagonalizing contradictions under some Type-A logics using *"I am consistent"* axioms.

The resulting contrast between Theorems 1 and 2 is helpful in explaining how Hilbert and Gödel could simultaneously fully appreciate the significance of the Second Incompleteness Theorem, but yet also allow their statements $*$ and $**$ to question whether its paradigm could be *partially* evaded. Moreover, Gödel's remark $**$ should not be ignored when Gerald Sacks's year-2014 YouTube lecture [38] has recalled how a middle-aged 55-year-old Kurt Gödel had repeated analogs of his 1931 remark $**$ during the 1961-1962 period. (It is also noteworthy that Harvey Friedman recorded a You-Tube lecture [18] in 2014 where he indicated he was tentatively open to the possibility that the Second Incompleteness Theorems might permit some type of limited forms of partial exceptions to it.)

Thus, while there is no doubt that the Second Incompleteness Theorem will, certainly, always be remembered for its seminal impact on 20th century Logic, its part-way exceptions should also be seen as significant. This is because futuristic high-tech computers will better understand their self-capacities if they own some *partial* awareness about their own consistency.

There is no page space to go into all the details here. However, the distinction between the initial "IS(A)" system from our 1993 and 2001 papers [45, 46] with the more sophisticated $\text{IS}_{\mathit{Tab-1}}(\beta)$ formalism in our year-2005 article [49] should, also, be briefly mentioned. Our older "IS(A)" formalism was actually simpler, but it was substantially weaker because it only recognized the non-existence of a proof of $0 = 1$ from itself. In contrast, $\text{IS}_{\mathit{Tab-1}}(\beta)$'s Group-3 axiom can corroborate that *no two simultaneous proofs* exist for a Rank-1* sentence and its negation. This is an important distinction, because the First Incompleteness Theorem indicates no decision procedure can exist for separating all true from false Rank-1* sentences. (See also [50, 51, 53, 54] for other particular refinements for our "IS(A)" formalism.)

In summary, the main purpose of this article has been to explore the contrast between the opposing Theorems 1 and 2. The latter theorem, thus, provides *another helpful reminder* about the millenial importance of Gödel's seminal Second Incompleteness Theorem. Yet at the same time, Theorem 1 illustrates how some *partial exceptions* to Gödel's result do arise, as Hilbert and Gödel had predicted both in their statements $*$ and $**$ and in Gödel's private communications with Gerald Sacks [38].

The 2-way contrast between Theorems 1 and 2 may be as significant as their individual actual results. This is because the Second Incompleteness Theorem is fundamental to Logic. Many historians have, thus, been quite perplexed by the *partial* reluctance that Hilbert and Gödel had expressed about it in $*$ and $**$. A partial reason for this reluctance is, perhaps, significantly related to the contrast between these two opposing theorems.

ACKNOWLEDGMENT: I thank Seth Chaiken for several helpful comments about how to improve the presentation.

APPENDIX A: Formal Definition of a Tableaux Proof

Our definition of a semantic tableaux proof is similar to analogs in the textbooks by Fitting and Smullyan [15, 39]. A tableaux proof of a theorem Ψ from a set of proper axioms, denoted as α, will be a tree structure whose root contains the temporary contradictory assumption of $\neg\Psi$ and whose every descending root-to-leaf branch affirms a contradiction by containing both some sentence ϕ and its negation $\neg\phi$. Each internal node in this tree will be either a proper axiom of α or a deduction from a higher ancestor in this tree using one of the following six elimination rules for the logical connective symbols of $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$ and $\exists$. These rules use a notation where "$\mathbf{A} \implies \mathbf{B}$" is an abbreviation for a sentence $\mathbf{B}$ being an allowed deduction from its ancestor of $\mathbf{A}$.

1. $\Upsilon \wedge \Gamma \implies \Upsilon$ and $\Upsilon \wedge \Gamma \implies \Gamma$.
2. $\neg\neg\Upsilon \implies \Upsilon$. Other rules for the "$\neg$" symbol are: $\neg(\Upsilon \vee \Gamma) \implies \neg\Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \rightarrow \Gamma) \implies \Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \wedge \Gamma) \implies \neg\Upsilon \vee \neg\Gamma$, $\neg\exists v\,\Upsilon(v) \implies \forall v \neg\Upsilon(v)$ and $\neg\forall v\,\Upsilon(v) \implies \exists v\,\neg\Upsilon(v)$
3. A pair of sibling nodes Υ and Γ is allowed when their ancestor is $\Upsilon \vee \Gamma$.
4. A pair of sibling nodes $\neg\Upsilon$ and Γ is allowed when their ancestor is $\Upsilon \rightarrow \Gamma$.

5. $\forall v\, \Upsilon(v) \implies \Upsilon(t)$ where t may denote any term.
6. $\exists v\, \Upsilon(v) \implies \Upsilon(p)$ where p is a newly introduced parameter symbol.

A minor additional comment about our notation is that we treat " $\forall\, v \leq s \;\; \Phi(v)$ " as an abbreviation for $\forall v \;\; \{\; v \leq s \;\; \rightarrow \;\; \Phi(v) \;\}$ and likewise " $\exists\, v \leq s \;\; \Phi(v)$ " as an abbreviation for $\exists v \;\; \{\; v \leq s \;\; \wedge \;\; \Phi(v) \;\}$. In our year-2005 article [49], we thus applied Rules 5 and 6 to derive the following further hybrid rules for processing the bounded universal and also the bounded existential quantifiers:

a. $\forall v \leq s\, \Upsilon(v) \implies t \leq s \rightarrow \Upsilon(t)$ where t may be any arithmetic term.
b. $\exists v \leq s \;\; \Upsilon(v) \implies p \leq s \;\wedge\; \Upsilon(p)$ where p is a new parameter symbol.

APPENDIX B: More Details About Theorem 2's Proof

The most surprising aspect of Theorem 2 is the sharp contrast between its result with the opposing property of Theorem 1. Our goal in this appendix will be to intuitively explain why the Invariant ++ (from Remark 1) ushers in a machinery that applies only to Theorem 2.

During our discussion, we will employ our U-Grounding language L^* that treats multiplication as a 3-way relation (rather than as a functional operation). Its 3-way predicate *Mult*(x,y,z), for formalizing multiplication, is defined as follows:

$$[\;(x=0 \vee y=0) \Rightarrow z=0\;] \;\wedge\; [\;(x \neq 0 \wedge y \neq 0\;) \;\Rightarrow\; (\;\frac{z}{x} = y \;\wedge\; \frac{z-1}{x} < y \;\;)\;] \quad (9)$$

We will say that an axiom basis α is **Regular** iff

1. It presumes all the U-Grounding operations are total functions (including the Addition and Doubling primitives).
2. It can prove all true Δ_0^* sentences, and α is also consistent.
3. It can prove a Π_1^* theorem showing addition and multiplication, viewed as 3-way relations, satisfy their usual associative, commutative, distributive and identity-operator properties.

Also in this appendix, we will employ a notation where for any $j \geq 0$, the symbol $\omega_j(x)$ will be recursively defined by the following rules:

1. $\omega_0(x) \;=\; x^2$.
2. $\omega_{j+1}(x) \;=\; 2^{\omega_j(\,2 \cdot Log_2\,(x+1)\,)}$

These two rules imply that $\omega_{j+1}(x) \;>\; \omega_j(x)$ and $\omega_1(x) \;\geq\; x^x$.

Clarification about Notation: Since our U-Grounding language L^* does not permit using any function symbols to grow as fast as multiplication, it does not technically allow us to use any of the ω_j primitive symbols. One can, however, use techniques from [25]'s textbook to construct a Δ_0^* formula $\psi_j(x, y)$ that satisfies (10)'s invariant for all standard numbers. It will, thus, capture most of ω_j's salient features.

$$\forall\, x \;\; \forall\, y \qquad \psi_j(x, y) \;\;\Leftrightarrow\;\; \omega_j(x) = y \tag{10}$$

Definition 4. A formula $\Phi(x)$ will be called **Locally-J-Closed** relative to the axiom basis α iff α can prove the following three assertions about $\Phi(x)$:

A. All of $\Phi(0)$, $\Phi(1)$ and $\Phi(2)$ are true.

B. The predicate $\Phi(x)$ is operationally closed under the growth operation ω_j. (Line (11) formally encodes this closure condition, using the preceding paragraph's notation.)

$$\forall\, x \quad \forall\, y \quad \{ \quad [\; \psi_j(x, y) \;\wedge\; \Phi(x) \;\;] \;\;\Rightarrow\;\; \Phi(y) \;\; \} \tag{11}$$

C. The predicate $\Phi(x)$ is also closed under (12)'s decrement operation.

$$\forall\, x \;\; \forall\, y < x \;\; \{ \;\; \Phi(x) \quad \Rightarrow \quad \Phi(y) \;\; \} \tag{12}$$

Theorem 3. *For each regular axiom basis* α *(that is consistent) and for each fixed integer* $J \geq 1$ *, there exists a corresponding formula* $\Phi(x)$ *where* α *can prove that* $\Phi(x)$ *is Locally-J-Closed.*

Due to a lack of page space, a formal proof of Theorem 3 will be postponed until a longer version of this article. Theorem 3 is related to various intermediate results that were used to establish Remark 1's Invariant ++ and [10, 16, 23, 25, 31, 35, 41, 42, 44, 46, 48]'s closely related results.

The fascinating feature of Theorem 3 is that it can explain why Theorems 1 and 2 display nearly opposite effects with regards to Hilbert's Second Open Question. This is because the needed diagonalization for producing Theorem 2's variations of the Second Incompleteness Effect become feasible only [7] when $\text{IS}_{Xtab}(\beta)$'s Linear-Sum Effect is applied to the intermediate results produced by its possible derived theorems (which include the formalisms that are illustrated by lines (11) and (12)). On the other hand, no such similar types of nicely compressed constructed proofs are available under Theorem 1's $\text{IS}_{Tab-1}(\beta)$ formalism (because all instances of the Law of Excluded Middle are excluded by it from becoming logical axioms). This is the intuitive reason that Theorems 1 and 2 display such sharply contrasting results.

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[7] Actually, we will only need the "Locally 1-Closure" property to prove that $\text{IS}_{Xtab}(\beta)$. cannot possibly be self-justifying.

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