On Some Important Added Observations Concerning the quite Fine Line That Separates Generalizations and Boundary-Case Exceptions for the Second Incompleteness Theorem under Semantic Tableau Deduction

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Abstract

BRIEF ABSTRACT (required by Electronic Archives):

The article is an extensive summary of the results from Willard's earlier year-2021 article, which appeared in the Journal of Logic and Computation. It recapitulates the latter's results in an alternate more relaxed form. (A longer 300-word formally summarized abstract-style discussion of our results appears on the next page).

CITATION INFORMATION : *ROUGH DRAFT of a planned new EVOVLING PAPER* **UNDERGOING CONSTANT REVISIONS**

KEYWORDS and PHRASES: Semantic Tableau deduction, Hilbert's Second Problem, Partial Revival of Hilbert's Consistency Program, Generalizations of the Second Incompleteness Theorem.

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A Substantially Longer 300-Word Abstract-Syle Summary of this Paper:

It is known that the semantic tableau deductive methodology, explored by Fitting and Smullyan, can prove a schema of theorems verifying all instances of the Law of the Excluded Middle. Let us have the acronym *"Tab"* denote this classic version of semantic tableaus's deductive mechanism, and let the acronym *"Xtab"* stand for an extension of the Tableau methodolgy that promotes the Law of the Excluded Middle into being a schema of logical axioms. Also, let us recall that Willard's recent paper in the LFCS-2020 Proceedings had used the term L^* to denote a langage that was essentially analogous to the usual language of Peano Arithmetic *except it treated multiplication as a 3-way predicate* (rather than as an atomic function). In this context, the main result from the previous LFCS-2020 paper was that an arithmetic formalism could prove every L^* reformatted analog of Peano Arithmetic's Π_1 theorems and *simultaneoulsy* verify its own consistency under Tab's deductive methodolgy, *but not additionally* under Xtab's deductive method ....

An alternate now-more-relaxed treatment of this subject is proposed in this second paper. Its proposal should be able to be encompassed, easily, within a roughly 3-or-4 week segment of a 2-semester introductory course on Symbolic Logic. This relaxed segment is desirable because many students will seek to learn why they choose to think and contemplate when the Second Incompleteness Theorem indicates that most conventional logics are unable to recognize their self-consistency.

1 Introduction

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2 VERY IMPORTANT - Save Forever Theorem 4.5

REMEMBER Definition 3.1 should be changed to 4.1

Theorem 4.5: In contrast, $\mathrm{IS}_{Xtab}(\ \bullet\ )$ fails to be a consistency-preserving mapping. (More specifically, $\mathrm{IS}_{Xtab}(\ \beta\ )$ is automatically inconsistent whenever β proves some conventional Π_1^* theorems stating that addition and multiplication satisfy their usual associative, commutative, distributive and identity properties. These particular Π_1^* sentences are called the **"Regulation Set"**, and they are defined in Appendix D. ???????)

$$\{\ (x=0 \vee y=0) \Rightarrow z=0\ \} \ \wedge\ \{\ (x \neq 0 \wedge y \neq 0\ ) \ \Rightarrow\ (\ \frac{z}{x} = y\ \wedge\ \frac{z-1}{x} < y\ \ )\ \} \quad (1)$$

3 Appendix D: Definition of Regulation Set

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Be CAUTIOUS with **changes** involving Group-2 scheme and/or Regultation Set

REMEMBER to change F to H in Group-1 definition !!! Also, is it a problem that H has TWO definitions?

This appendix will define the Regulation Set that was mentioned in Theorem 4.5's statement. Its purpose is to formulate a finite set of Π_1^* sentences that can encode the associative, commutative, distributive and identity properties of addition and multiplication within the language L^* . This regulation set will be called H, and Peano^H will denote the extension of Peano Arithmetic that contains H's added axioms together with our added $U-$Grounding function symbols.

This appendix's definition of the Regulation Set is lengthy. It actually does not need to be examined when the reader assumes that the Regulation Set can be formally defined. Our recommendation is that the detailed examination of this appendix be postponed until after all the main chapters of this article are first read.

Main Details: The symbols c_0 and c_1 will denote the usual constants of " 0 " and " 1 ". They will be assumed to satisfy the conventional identity properties of

$$\forall x \quad x + c_0 \quad = \quad x \tag{2}$$

$$\forall x \quad x + c_1 \quad \neq \quad x \tag{3}$$

$$\forall x \quad x + c_1 \quad \neq \quad c_0 \tag{4}$$

Also, it is assumed that the same set H of Π_1^* sentences is employed by both the Regulation Set and by $\mathrm{IS}_D(\beta)$'s Group-1 schema. (Again, we remind our readers that it doesn't matter which of several plausible sets H are employed. All that is necessary is that H meets IS_D's requirement for correctly defininining its Group-1 scheme.)

In a context where Line (1) defines multiplication as a 3-way predicate, it is obviously necessary for our Regulation scheme to recognize that no two distinctly different integers a and b can simultaneously satisfy $Mult(x, y, a)$ and $Mult(x, y, a)$. This is formalized by the following Π_1^* statement:

$$\forall\, x \;\; \forall\, y \;\; \forall\, a \;\; \forall\, b \;\; \{\, Mult(x, y, a) \wedge Mult(x, y, b) \,\} \;\; \Rightarrow \;\; a \;=\; b \;\} \tag{5}$$

Also in a context where the symbols c_0 and c_1 denote the two integers of "0" and "1", two of the Regulation Scheme's needed identities for these two constants are:

$$\forall\, x \quad Mult(\, c_0 \,,\; x \,,\; c_0 \,) \tag{6}$$

$$\forall\, x \quad Mult(\, c_1 \,,\; x \,,\; x \,) \tag{7}$$

(If one wishes to include an extra optional Π_1^* sentence defining a constant $c_2 = 2$, the relevant sentence would clearly be "$c_2 = c_1 + c_1$" .)

The relevant Π_1^* sentences for formalizing addition's commutative and associative laws, as well as multiplication's commutative law are also quite straighforward to encode. Their Π_1^* encodings are illustrated below:

$$\forall x \quad \forall y \quad x + y = y + x \tag{8}$$

$$\forall x \quad \forall y \quad \forall z \quad (x + y) + z = (x + (y + z) \tag{9}$$

$$\forall x \quad \forall y \quad \forall z \qquad Mult(x, y, z) \Leftrightarrow Mult(y, x, z) \tag{10}$$

An analog for (10)'s if-and-only-if law for the associative law of "$(r \cdot s) \cdot t = r \cdot (s \cdot t)$" is, surprisingly, more complicated to formulate. This challenge is unusual because Line (11) accruately formalizes the asscociative law, and

Peano^H can prove (11) as a theorem. The difficulty is, however, that (11) is not a Π_1^* sentence (because its variables u and v are existentially quantified). The latter is a crucial defect, as Footnote [1] explains,

$$\forall\, r \quad \forall\, s \quad \forall\, t \quad \forall\, w$$

$$\{\, \exists u \;\, Mult(r,s,u) \wedge Mult(u,t,w) \,\} \;\; \Leftrightarrow \;\; \{\, \exists v \;\, Mult(s,t,v) \wedge Mult(v,r,w) \,\} \tag{11}$$

Two nicely helpful facts are that Line (11) obviously implies (12), and the latter (combined with (10)'s law) easily implies (13). Moreover, both of (12) and (13) are Π_1^* sentences because their existential quanifiers are bounded.

$$\forall\, r \quad \forall\, s \quad \forall\, t \quad \forall\, w \quad \forall\, u \quad \exists v \,\leq\, w$$

[1] The formalism $\mathrm{IS}_{Xtab}(\mathrm{Peano}^H)$ needs Line (11) or its analog to be formalized as a Π_1^* statement so that its Group-2 axiom scheme will recognize the truth of their corresponding claim.

$$\{ \, r \neq 0 \wedge Mult(r,s,u) \wedge Mult(u,t,w) \, \} \;\Rightarrow\; \{ \; Mult(s,t,v) \wedge Mult(v,r,w) \, \} \tag{12}$$

$$\forall\, r \quad \forall\, s \quad \forall\, t \quad \forall\, w \quad \forall\, v \quad \exists u \;\leq\; w$$

$$\{ \, r \neq 0 \wedge Mult(s,t,v) \wedge Mult(v,r,w) \, \} \;\Rightarrow\; \{ \; Mult(r,s,u) \wedge Mult(u,t,w) \, \} \tag{13}$$

Thus, two pleasing aspects of Lines (12) and (13) are that

a) Their combination shows that Line (11)'s if-and-only-if statement holds true for all cases, except plausibly when $r = 0$.

b) The final remaining case (where $r = 0$) is subject to Line (7)'s Zero-Identity axiom. It therefore does not to be addressed exactly within (11)'s particlur framework, when Line (7) can correctly handle it.

Hence, Items (a) and (b) have explained how Lines (12) and (13) handle correctly all the nontrivial instances of (11) that do require inspection.

Finally, let us turn our attention to the distributive identity of " $(p+q) \cdot r \;=\; p \cdot r \,+\, q \cdot r$ ". It easily amounts to being equivalent to the combined implication of Lines (14) and (15).

$$\forall\, p \quad \forall\, q \quad \forall\, r \quad \forall\, s \;\; \exists\, u \leq s \quad \exists\, v \leq s$$

$$Mult(p{+}q, r, s) \quad \Rightarrow \quad \{\; Mult(p, r, u) \wedge Mult(q, r, v) \;\wedge\; s = u{+}v \;\} \tag{14}$$

$$\forall\, p \quad \forall\, q \quad \forall\, r \quad \forall\, u \quad \forall\, v \quad \forall\, s$$

$$\{\, Mult(p, r, u) \wedge Mult(q, r, v) \;\wedge\; s = u{+}v \,\} \quad \Rightarrow \quad Mult(p{+}q, r, s) \tag{15}$$

Thus, the above encoding of the Distributive Princible will accomplish our purposes in an entirely simplistic [2] manner.

Remark ????.1 summarzing the gist of this appendix: The preceeding discussion has implied that all of Lines (2) thru (15) are provable from Peano^H. (Indeed, they are provable from tiny subsets of Peano^H.) At first, this observation might appear to be problematic because (11) has failed to be a Π_1^* sentence. However, this deficiency was shown to be insignificant because Lines (12) and (13) can handle all the nontrivial cases for (11)'s pradigm (i.e. all the cases where $r \neq 0$).

Remark ????.2 commenting on the pedagogic nature of this material: The results of this appendix should be probably omitted when our results are outlined during an introductory 2-semester logic course. This is because the preced-

[2]This is because Lines (14) and (15) both formalize all cases of the distributive princible, and they own Π_1^* encodings.

ing discussion will be needed only when Appendix ??????????????? transforms Lemma 5.3's informal proof into becoming a full formalized proof.

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4 Last Paragph Section 4 REVISED

The negatavistic results of Theorems 4.5 and 4.7 will obviously serve as counterpoints to Theorem 4.4's positive results. Theorems 4.4 and 4.5 were explicitly mentioned in [61], and Theorem 4.7 was implicitly mentioned via [61]'s Example 5.1. The next chapter will actually provide a different tone than the comparable briefer discussion in [61]. It will simplify the justifications of Theorems 4.4, 4.5 and 4.7 so that its intuitive gist can be better communicated within a 3-week segment of a 2-semester introdutory Logic course. It will also better explain philosophical significance of this subject, and make it easier to appreciate our three main theorems without reading an extensive background

literature

5 Main Results

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This chapter will consist of five parts. They will:

A. Explain the underlying intuitions behind the contrasting Theorems 4.4 and 4.7.

B. Question whether the middle sentence from Hilbert's famous 3-sentence Remark ** is fully correct (as opposed to only partially valid) ?

C. Explain the underlying intuition behind Theorem 4.5's generalization of the Second Incompleteness Theorem.

D. Explain how the Group-3 *"I am consistent"* axiomatic declaration has a much clearer defintion of the pronoun "I" under Theorem 4.4's formalism than under its counterparts in Theorems 4.5 and 4.7.

E. Explain how Artemov's "Infinite Ranged" approach very neatly interfaces with [61]'s self-justification project.

This abbreviated discourse will not provide the full details behind all its proofs. These details are impractical to provide because rigorous proofs, analogous to counterparts in [50, 51, 53, 54, 56], will not fit into one moderate-sized paper.

Instead, our goal will be to sketch the underlying intuitions behind a new 3-or-4 week segment of an introductory 2-semester sequence about symbolic logic. Within this context, the "Backwards-Stepping Short-Cut" in §5.3 will simplify Theorem 4.5's proof. Along with with the Tripod theory and the WCB paradigm, it will provide a comprehensive answer to the philosophical questions that were raised by Hilbert and Gödel in their statements $*$ and $**$.

5.1 On the Contrast Between Theorems 4.4 and 4.7

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This section will study the contrast between the positive consistency preservation properties of Theorem 4.4's $\mathrm{IS}_{Tab}(\ \bullet\ )$ formalism with its antithesis under Theorem 4.7's alternate $\mathrm{IS}^{M}_{Tab}(\ \bullet\ )$ formalism. This topic had received a short survey in Chapter 5 of [61], but it will be now examined more extensively.

Essentially, [61] had only partially explored the fundamental consequences of the Second Incompleteness Theorem and Hilbert's question. Our new goal will be to explore this topic substantially more thoroughly (i.e. we will seek both detailed positive replies and negative answers to the questions raised by Hilbert's Second Problem).

This issue will be challenging because an analog of the WCB paradigm bedevils the $\text{IS}^{M}_{Tab}(\ \bullet\ )$ transformation. More specifically, a serious question arises about whether a multiplication function operator can be, fully, philosophically and metaphysically formalized. In this context, an unusual response to the famous Hilbert and Gödel statements $*$ and $**$ will be offered.

Thus, let $x_0, x_1, x_2, ...$ and $y_0, y_1, y_2, ...$ denote the particular sequences defined by:

$$x_0 \quad = \quad 2 \quad = \quad y_0 \tag{16}$$

$$x_i \quad = \quad x_{i-1} \quad + \quad x_{i-1} \tag{17}$$

$$y_i \quad = \quad y_{i-1} \quad \cdot \quad y_{i-1} \tag{18}$$

Then for any $i > 0$, let ϕ_i and ψ_i denote the sentences in (17) and (18) respectively. Also, let ϕ_0 and ψ_0 denote (16)'s sentence. Then $\phi_0,\ \phi_1,\ ...\ \phi_n$ imply $x_n = 2^{n+1}$, and $\psi_0,\ \psi_1,\ ...\ \psi_n$ imply $y_n = 2^{2^n}$. Hence, the latter sequence will grow at an exponentially faster rate than the former. It turns out that this change in growth speed causes the $\text{IS}^M_{Tab}(\ \bullet\ )$, and $\text{IS}_{Tab}(\ \bullet\ )$ to have opposite self-justification properties.

This is because the quantities $\text{Log}_2(\,y_n\,) = 2^n$ and $\text{Log}_2(\,x_n\,) = n+1$ represent the lengths for the binary codings for y_n and x_n. Thus, y_n's binary encoding will have a length 2^n, which is *much larger* than the $n+1$ steps of $\psi_0, \psi_1, \ldots \psi_n$ (used to define y_n's existence). In contrast, x_n's binary encoding will have a sharply smaller length of size $n+1$. These observations are significant because every proof establishing a variant of the Second Incompleteness Theorem involves a Gödel number z having a capacity to self-reference its own definition.

The faster growing series $y_0, y_1, , \ldots y_n$ should, intuitively, support this self-referencing capacity because y_n's binary encoding has a length 2^n that greatly exceeds the size of the $O(n)$ steps used to define its value. Leaving aside many of [51, 56]'s further details, this fast growth explains roughly why a Type-M logic, such as IS_{Tab}^{M}, satisfies the semantic tableau version of the Second Incompleteness Theorem, unlike IS_{Tab}.

Also, this distinction explains why IS_{Tab}'s Type-A formalism produces boundary-case exceptions for the semantic tableau version of the Second Incompleteness Theorem. These arise because [53] showed that IS_{Tab} is unable to construct numbers z that can self-reference their own definitions (when only the *more slowly growing* addition primitive is available). In particular assuming only two bits are needed to encode each sentence in the sequence $\phi_0, \phi_1, ...\phi_n$, the length $n+1$ for x_n's binary encoding is insufficient for encoding this sequence.

Leaving aside many of [53]'s details, this short length for x_n explains the central intuition behind [53]'s evasion of the Second Incompleteness Theorem under $\mathrm{IS}_{Tab}(\beta)$. Intuitively, $\mathrm{IS}_{Tab}(\beta)$.

differs from $\mathrm{IS}^{M}_{Tab}(\beta)$, for most typical β , because of the sharp difference between the growth rates of the two sequences of $x_1, x_2, x_3...$ and $y_1, y_2, y_3...$.

We won't go into any more details here because they are provided in [50, 53, 56]. Instead, we will seek a much broader philosophical and metaphysical perspective at this juncture.

5.2 Examining Hilbert's Statement $**$ More Closely

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Our Tripod-styled response to Hilbert's Second Problem had viewed the statements $*$ and $**$ by Hilbert and Gödel in a mostly positive light. However, it is necessary to adapt a more cautious approach to the middle sentence in Hilbert's statement $*$. Its full 3-sentence statement from [28] is duplicated below:

> $*$ *" Let us admit that the situation in which we presently find ourselves with respect to paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?"*

A crucial point is that the middle sentence of $*$ refers to *"mathematics"* as a field that *"everyone"* does *" learn, teach, and use"*.

> But is it really true that literally *"everyone"* uses an arithmetic formalism with an Integer-Multiplication functional oeration ?

The WCB fable from Chapter **??** seems to suggest, at least, its mythical Brahmin, named Sissa, would be more cautious. This is because if one begins with the number "2" (written in binary) and squares it a few hundred times then the repeated doublings (in bit-length) will produce more zero-digits than there exists atoms in the universe?

Indeed, it can be plausibly argued that the notion of "I" (from an *"I am consistent"* axiom) is an ambiguous, undefined object? This is becasue its notion of"I" becomes metaphysically perplexing when it is assumed to be an entity in the real world that owns a capacity to multiply integers (and thus encode proofs) with potentially unwieldly lengths.

Our point is that the WCB paradigm would not have survived more than a millenial amount of time, if there were not several examples of *hidden* exponential explosions occuring, that were initially unrecognized ! In other words, we are suggesting that formalisms, such as $\mathrm{IS}_{Tab}(\beta)$, do have serious silent advantages over an awkward $\mathrm{IS}^{M}_{Tab}(\beta)$ formalism This is because $\mathrm{IS}_{Tab}(\beta)$, avoids entirely unrealistic exponential growths that are built intrinsically into $\mathrm{IS}^{M}_{Tab}(\beta)$'s underlying structure.

Thus, if β corresponds to Peano Arithmetic then [53]'s $\mathrm{IS}_{Tab}(\beta)$ simultaneously avoids undesirable WCB-like exponential explosions, while also proving all Peano Arithmetic's Π^*_1 theorems and verifying its own consistency.

Moreover, there is one more point that should be made about the middle sentence in Hilbert's statement $*$. It refers to *"mathematics"* as a discipline that *"everyone learns, teaches and uses"*. The question is which particular branch of mathematics does the word "*everyone*" refer to?

Thus, the physical sciences and their engineering cousins mostly rely upon a multiplication as being either a 3-way relation or an analog of a computer's floating point multiplication operation. They typically do not use integer-multiplication as an operation among integers of vast and potentially unlimited size.

Within this context, [55] demonstrated that its particular form of self-justifying logics can more nearly approximate Hilbert's objectives by treating an infinitized analog of a computer's floating point multiplication as a total function. (Please see footnote

[3] for a significant added reminder.)

For the sake of clarity, we certainly do not intend to ignore many of $\text{IS}_{Tab}(\beta)$'s obvious partial drawbacks. Thus, Self-Justification will correspond to only one leg of a broader 3-part Tripod reply to Hilbert's Second Problem.

All we request is that the reader keep an open mind when examining the remainder of this article. It also should be mentioned that $\text{IS}_{Tab}(\beta)$ will be later shown to display the same advantage over $\text{IS}_{Xtab}(\beta)$ that had prevailed over Theorem 4.7's $\text{IS}^{M}_{Tab}(\beta)$ system.

5.3 A Simplified Proof for Theorem 4.5's $\text{IS}_{Xtab}(\beta)$'s System

This section is, perhaps, the most significant chapter of the current article. It will introduce a new formalism, called the **Backwards-Stepping Short-Cut**, and explain how it sim-

[3] Obviously, the formal function of floating point multiplication was introduced by electronic computers *only many years after* Hilbert had published his statement $*$. However, these functions were implicitly used by scientific notation conventions much prior to [28]'s publication. It is within this context that [55]'s observation about the natural compatability between self-justification and floating point arithmetic is a quite significant point.

plifies much of [61]'s proof for Theorem 4.5.

This presentation will be signicantly different from [61]'s treatment. The latter used either Pudlák's initial seminal paper [39], or the Hájek-Pudlák textbook [27] or any of the closely results in [10, 18, 25, 36, 45, 46, 47, 48, 50, 52]. It then applied these materials to corroborate Theorem 4.5. In contrast, our new Backwards-Stepping Short-Cut will use only a nicely pruned subset of these methods.

This revised method is significant because its shorter proofs can be compressed into a nice 3-or-4 week segment of an introductory Logic course. Such an abbreviated presentation will be helpful because the its contrast between Theorem 4.5's negative result with Theorem 4.4's opposing negative paradigm will be enlightening. (It should stimulate a large audience to wish to learn more about this subject.)

5.3.1 Background Setting

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The crucial aspect of the Hilbert-Frege deductive methodologies (and its typical textbooks analogs in [14, 27, 33]) is that their modus ponens rules will assure that a proof of a theorem ψ from an axiom system α has a length no more than proportional to the sum of the proof-lengths used to derive ϕ and $\phi \rightarrow \psi$. This will be called the **"Linear-Sum Effect"**, and we will say a deductive apparatus D is **"Linearly-Sum Organized "** iff it satisfies this invariant.

It is well known that *Tab*-deduction fails to be so Linearly-Sum Organized (because it simply lacks a modus ponens rule). On the other hand, the *Xtab* deductive methodology (defined in Section ??????) is Linearly-Sum Organized. In particular, the Invariant 5.1 does summarize its properties:

Invariant 5.1 *The Xtab deductive apparatus is Linearly-Sum Organized. In particular, the proof of any theorem ψ from an axiom system α has a length no more than proportional to the sum of the proof-lengths used to derive ϕ and $\phi \rightarrow \psi$ from α under Xtab deduction.*

Using slightly different notation, an analog of the proof of Invariant 5.1 was presented in the third and fourth paragraphs of Section 6 from [61]. Therefore, we have placed a similar proof of Invariant 5.1 in Appendix D.

The discussion in [61] next observed that a consequence of Invariant 5.1 was the following corrolary:

$\odot$ Any axiom system $\mathcal{A}$ is automatically inconsistent whenever it satisfies the following three conditions:

- **I.** $\mathcal{A}$ can verify Successor is a total function, as is precisely formalized by Line (**??**). (In other words, $\mathcal{A}$ is a Type-S logical system.)
- **II.** $\mathcal{A}$ can prove addition and multiplication, viewed as 3-way relations and encoded with analogs of Line (1)'s 3-way formalism, satisfy their usual associative, commutative, distributive and identity-operator properties.
- **III.** $\mathcal{A}$ proves an added theorem (which turns out to be false) affirming its own consistency when the Xtab deductive apparatus is used.

The formal statement of $\odot$ is quite analogous to the Theorem ++ , which Section **??** had credited to essentially a Pudlak-Solovay methodology [39, 45], relying upon additional methods of Nelson and Wilkie-Paris [36, 48]. The difficulty, however, is that the exact proof for $\odot$ is difficult to summarize inside the confines of a 2-semester introductory logic course.

A solution to this problem was worked out by Dan and Robert when Dan taught Robert a new version of an introductory logic course. The trick was *to be less ambitious in the first step* but still derive the same final solution. (In other words, this modification will develop an alternate simpler justification for Theorem 4.5 that can be compressed into an introductory 2-semester course.)

In order to do so, we need to develop a slight modification of the statement $\odot$ that is substantially easier to prove than

$\odot$ *but still sufficient to justify Theorem 4.5.* This modification of $\odot$ is called $\odot$ $\odot$. It is actually the same statement as $\odot$ except that its new condition I* is stronger than the earlier requiement I by treating addition as a total function (rather than as 3-way relation). Some additional minor modificantions are then made in II* and III* to take routinely into account the changes made in I* .

$\odot$ $\odot$ Any axiom system $\mathcal{A}$ is automatically inconsistent whenever it satisfies the following three conditions:

I* . $\mathcal{A}$ can verify Addition is a total function, as is formalized by Line (**??**). (Thus, $\odot$ $\odot$ is significantly different from $\odot$ by having $\mathcal{A}$ now designate a Type-A logical system.)

II*. $\mathcal{A}$ can prove its usual associative, commutative, distributive and identity-operator properties, when addition is now viewed as a total function but multiplication continues to be treated as a 3-way relation $Mult(x, y, z)$ (see footnote[4])

III*. $\mathcal{A}$ does prove an added theorem (which turns out to be false) affirming its own consistency when the Xtab deductive apparatus is applied to an axiom basis that satisfies these three conditions.

The difference between Item I in $\odot$ and its analog using Item I* of $\odot$ $\odot$ may look minor. The next section will, however, show that $\odot$ $\odot$'s moderately diluted statement leads

[4]For simplicity, we shall postpone until later discussing the meaning of this distinction.

to an actually substantially simplified proof for Theorem 4.5. (Surprisingly, the sole reason for this proof compression is that addition is treated as a total function in Item I* of Statement $\odot\ \odot$.)

5.3.2 Justification of Statement $\odot\ \odot$ and its Broad Implications

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A small amount of added notation will be needed for helping us prove the claims of statement $\odot\ \odot$.

Definition 5.2 Let α be a Type-A axiom basis that treats addition as a total function but views multiplication as a 3-way relation $Mult(x, y, z)$. Then an integer c will be called a **Natural Multiplier** for α iff α can prove:

$$\forall\ d \quad \exists\ e \quad Mult(c, d, e) \tag{19}$$

We will often use the Natural Multiplication concept in this section. Thus the symbol " NatMult(c) " will be an abbreviation for (19)'s invariant.

A subtle aspect of the NatMult(c) construct is that all the standard integers in Type-A logical systems satisfy (19)'s invariant, but non-standard integers c often do not. In this context, Lemma 5.3 will be very central for our proof of $\odot$ $\odot$.

Lemma 5.3 Let α denote a Type-A axiom basis system that treats addition as a total function, views multiplication as a 3-way relation $Mult(x, y, z)$, and also presumes these two primitives satisfy their usual associative, commutative, distributive and identity-operator properties. Then α can prove that any of its natural multipliers c will have the property c^2 is also a natural multiplier. In other words, using Definition 5.2's NatMult notation, we need to show that system α can prove:

$$\forall\, c \;\; \exists\, h \;\; \{ \;\text{NatMult}(c) \;\; \Rightarrow \;\; [\, Mult(c,c,h) \wedge \text{NatMult}(h)\,]\, \} \tag{20}$$

Informal Proof: We will provide only an informal proof of Lemma 5.3 here. Appendix E will explain how to more fully formalize it. During our informal discussion, c^2 will represent the conventional multiplicative product of $c \cdot c$. The only aspect of our "informal proof" that is actually less than perfect is that it is technically less than kosher to use either of the notations of " $c \cdot c$ " or " c^2 " when multiplication is viewed as a 3-way predicate $Mult(x, y, z)$. It turns out that this notational issue is, actually, easily fixed in Appendix E. It is, however, best to omit such details in the current first version of Lemma 5.3's proof

The lemma's assumption that c is a natural multiplier certainly implies that the integer $\mathcal{C} = c^2$ exists. Moreover, it is also beneficial to apply the NatMult(c) property on two more occasions. They imply that both the integers $D \cdot c$ and also $(D \cdot c) \cdot c$ exist whenever D exists.

But since our 3-way predicate $Mult(x, y, z)$ satisfies the natural analogs for multiplication's associative and commutative identities, it is apparent that:

$$(D \cdot c) \cdot c \;=\; D \cdot (c \cdot c) \;=\; D \cdot c^2 \;=\; D \cdot \mathcal{C} \quad (21)$$

Hence, (21), combined with the preceding paragraph, shows that $D \cdot \mathcal{C}$ can be represented as an integer, whenever D is so formulated as an integer.

Except for the annoying caveat that the preceding discussion has repeatedly used the technically undefined multiplicative function symbol of " $\cdot$ ", its analysis is exactly what is needed to justify the Line (20), which was Lemma 5.3's core claim. The more meticulous treatment in Appendix E will repair this difficulty, but such details are not recommended within a 3-week abbreviated segment of an introductory logic course.

□

Lemma 5.4 Let α again denote a Type-A axiom basis system that treats addition as a total function, views multiplication as a 3-way relation $Mult(x, y, z)$ and also presumes that the conventional associative, commutative, distributive and identity-operator properties of arithmetic do hold true. Also, let $z_0, z_1, z_2, \ldots$ denote the sequence of integers defined by:

$$z_0 \quad = \quad 2 \tag{22}$$

$$z_i \quad = \quad z_{i-1} \quad \cdot \quad z_{i-1} \tag{23}$$

Then α can use Line (20)'s identity to prove in $n+1$ steps the twin identities that $z_n = 2^{2^n}$ and that z_n satisfies Definition 5.2's NatMult(z_n) constraint.

Proof: Lemma 5.4 is essentially an easy consequence of Lemma 5.3. In particular, it is immediate that $z_0 = 2$ does satisfy the NatMult(z_0) constraint (because the hypothesis of Lemma 5.4 indicated that axiom basis α was a Type-A formalism corroborating arithmetic's conventional associative, commutative, distributive and identity-operator properties). This implies that α can verify that $z_n = 2^{2^n}$ satisfies Definition 5.2's NatMult(z_n) constraint when n additional steps are used. (Here, the $j-$th such step will use Line (20)'s identity to infer the validity of the NatMult(z_j) constraint from NatMult(z_{j-1}).) $\Box$

One should be very cautious when applying Lemma 5.4's machinery. This is because the fact NatMult(z_n)'s proof requires $n+1$ steps does not indicate that its proof has, automatically, an $O(n)$ length. For instance, if the formal encoding of its $j-th$

step has an $O(\ 2^j\ )$ length then NatMult(z_n)'s full proof will have a length proportional, obviously, to $O(\ 2^n\ )$.

At this juncture, we need to examine the distinction between Tab and $Xtab$ deduction. In particular, proofs employing Tab are often prohibitively lengthy . (This is because Tab has access to neither a modus ponens rule nor a liberal use of the Law of the Excluded Middle.) On the other hand, $Xtab$ proofs can become much more compressed. (They allow each instance of the Law of Excluded Middle to be treated as a permitted logical axiom anywhere within the domain of their proof tree.)

More precisely, the combination of Invariant 5.1 and Lemma 5.4 imply the validity of the following second invariant:

Invariant 5.5 *The statement* $\odot\ \odot$ *and (by easy inference) also Theorem 4.5 are both true.*

The justication of Invariant 5.5 is essentially a an immediate consequence of Invariant 5.1's Linear-Sum Effect, the Lemmas 5.3 and 5.4, and the preceeding two paragraphs. We actually do not need to formally prove Invariant 5.5 because it follows from the same methodology that §5.1 had used to seperate Theorem 4.4's evasion of the Second Incompleteness from Theorem 4.7's generalization of it.

More precisely, Line (17)'s exponentially growing sequence $x_0, x_1, x_2, ...$ was transformed into (18)'s super-exponentially growing sequence $y_0, y_1, y_2, ...$ when §5.1's self-reflecting formalism was changed from $\mathrm{IS}_{Tab}(\beta)$ into $\mathrm{IS}^M_{Tab}(\beta)$. It may initially surprise some readers that a change of merely the preceding subscipt of "Tab" to "$Xtab$" will cause $\mathrm{IS}_{Xtab}(\beta)$ to behave more analogously to $\mathrm{IS}^M_{Tab}(\beta)$ (than to $\mathrm{IS}_{Tab}(\beta)$). However, Line (23)'s $z_0, z_1, z_2, ...$ sequence was shown by Lemma 5.4 to grow at a double-exponentially rate, similar to (18)'s super-exponentially growing rate of $y_0, y_1, y_2, ...$.

The remaining parts of our proof for both Invariant 5.5 and the resulting Theorem 4.5 can now be easily summarized. The fundamental point is that Lines (18) and (23) will display the exactly same double-exponential growth rates, since $y_n = z_n = 2^{2^n}$. This will cause both statement $\odot \odot$ and the related Theorem 4.5 to each hold true, as a straightforward generalization [5] of Theorem 4.7's particular version of the Second Incompleteness Theorem. $\square$

[5]Omitting some straightforward details, the central point is that §5.1's justification of Theorem 4.7 will easily generalize to prove both $\odot \odot$ and the related Theorem 4.5.

5.4 Philosophical Questions about Circuit-Breakers and Ultrafinitism

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This section will examine the preceding $\mathrm{IS}^{M}_{Tab}(\ \beta\ )$ and $\mathrm{IS}_{Xtab}(\ \beta\ )$ basis systems in a context where β = Peano Arithmetic (PA). It was already shown in Sections 5.1 and 5.3 that both $\mathrm{IS}^{M}_{Tab}(\ \mathrm{PA}\ )$ and $\mathrm{IS}_{Xtab}(\ \mathrm{PA}\ )$ were inconsistent, while $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ was consistent. This was essentially because $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ employed **"circuit-breakers"**, that prevented its Group-3 axiom from becoming so excessively strong as to make $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ become inconsistent (as footnote [6] explains).

This circuit breaking capacity is related to the ultrafinistic philosophy, that has been subscribed to by both Parikh [**?**] and Nelson [36]. It is also related to Chapter **??**'s WCB paradigm.

More specifically, Section 5.2 explained that if one increased

[6] In other words, this "circuit-breaking" precludes $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ from becoming so strong that its knowledge about its own consistency will render it helplessly inconsistent, similar to its analogs of $\mathrm{IS}^{M}_{Tab}(\ \mathrm{PA}\ )$ and $\mathrm{IS}_{Xtab}(\ \mathrm{PA}\ )$.

the size of Sissa's chess-board from 64 to roughly 640 squares, then Sissa's exponential increase would prodouce more grains of Wheat than there exist atoms in the universe. This fact led §5.2 to question whether the pronoun "I" in $\text{IS}^{M}Tab$(PA)'s Group-3 *"I am consistent"* axiom had failed to be a fully well defined concept? (A similar challenge was applicable also to IS_{Xtab}(PA)'s Group-3 statement, although the latter is more complex to analyze.) There are also other questions that arise about WCB-like exponential growth rates

For instance, let $\text{Flat}^{M}_{Tab}(\text{PA})$ and $\text{Flat}_{Xtab}(\text{PA})$ denote the two degenerate versions of the $\text{IS}^{M}_{Tab}(\text{PA})$ and $\text{IS}_{Xtab}(\text{PA})$ formalisms that have their Group-3 *"I am consistent"* axiom-sentence removed. Both these two formal systems are obviously consistent theories that have exponential growths and related side-effects embedded inside them. (See footnote [7] for an ex-

[7] Some readers may be intially surprised that $\text{Flat}_{Xtab}(\text{PA})$ dis-

planation about how $\mathrm{Flat}_{Xtab}(\mathrm{PA})$ has an actual exponential growth property internalized within it). A fundamental question is whether $\mathrm{Flat}_{Xtab}(\mathrm{PA})$ and $\mathrm{Flat}^{M}_{Tab}(\mathrm{PA})$ are both tempting mankind to speculate, perhaps recklessly (?), outside the finite-sized universe, in which God had placed us in.

Remark 5.6 The preceding paragraphs were not intended to either promote or dismiss Ultrafinitism. Thus both Ultrafinitism and its antitheses seem to retain certain advantages. Thus, the utility of the circuit-breaking analysis from Sections

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plays a growth rate analogous to $\mathrm{Flat}^{M}_{Tab}(\mathrm{PA})$, when only the latter system contains a multiplication function symbol. The point is that the combination of Lemmas 5.3 and 5.4 enable $\mathrm{Flat}_{Xtab}(\mathrm{PA})$ to generate an $Xtab$ style proof that simulates a cousin of the multiplication operation, which these lemmas call "NatMult". This primitive will enable a Sissa-like exponential explosion to generalize from $\mathrm{Flat}^{M}_{Tab}(\mathrm{PA})$ to $\mathrm{Flat}_{Xtab}(\mathrm{PA})$ (essentially because the latter's loss of a multiplication function symbol is offset by its resulting more efficient $Xtab$ deductive apparatus having an ability to simulate "NatMult".).

5.1 and 5.3 was clear. It had provided a purely mathematical demonstration that both $\mathrm{IS}^M_{Tab}(\mathrm{\ PA\ })$ and $\mathrm{IS}_{Xtab}(\mathrm{\ PA\ })$ were inconsistent, while $\mathrm{IS}_{Tab}(\mathrm{\ PA\ })$ is surprisingly consistent.

Remark 5.7 Let $\mathrm{IS}^S_{Xtab}(\mathrm{\ PA\ })$ denote a weakenned version of $\mathrm{IS}_{Xtab}(\mathrm{\ PA\ })$ that recognizes only Succesor as a total function (and thus treats both addition and multiplication as 3-way relations). Also, let $\mathrm{IS}^S_{Hilb}(\mathrm{\ PA\ })$ denote its obvious analog, that recognizes its self consistency under a Hilbert-style deductive apparatus. Both these formalisms are unfortunately inconsistent, similar to the $\mathrm{IS}^M_{Tab}(\mathrm{\ PA\ })$ and $\mathrm{IS}_{Xtab}(\mathrm{\ PA\ })$ basis systems. For the sake of brevity, we will not delve into details here, but Footnote [8] roughly outlines how all four of these formalisms

[8] We remind the reader that Chapter **??** indicated that some versions of WCB-like exponential explosions are much more difficult to identify than others. Thus, a key difference between $\mathrm{IS}^M_{Tab}(\mathrm{\ PA\ })$ and $\mathrm{IS}_{Xtab}(\mathrm{\ PA\ })$ was that the latter employed one additional level of subtle detail (embodied by Lemmas 5.3 and 5.4) to formalalise an exponential explosion triggered by the "NatMult" primitive. The analogous and more complicated exponential growth properties of $\mathrm{IS}^S_{Hilb}(\mathrm{\ PA\ })$ and

become inconsistent, when the combination of their Group-3 axioms and WCB-like exponential growths activate the Second Incompleteness Theorem.

5.5 The Third Leg of the Tripod and its Philosphical Signficance

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The preceeding discusion summarized the first two legs of our Tripod reply to Hilbert's Second Problem. Thus, Theorems 4.5 and 4.7 presented generalizations of the Second Incmpleteness Theorem showing that $\mathrm{IS}_{Xtab}(\ \bullet\ )$ and $\mathrm{IS}^{M}_{Tab}(\ \bullet\ )$ failed to be consistency preseving transformations. In contrast, Theorem

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$\mathrm{IS}^{S}_{Xtab}(\ \mathrm{PA}\ )$ are yet-more challenging to recognize (because they require much of the metamathematical machinery from Section III-3 of the Hájek-Pudlá textbook [27]). Once their common WCB-like exponential growth properties are recognized, it is evident that all four of $\mathrm{IS}^{M}_{Tab}(\ \mathrm{PA}\ )$, $\mathrm{IS}_{Xtab}(\ \mathrm{PA}\ )$, $\mathrm{IS}^{S}_{Hilb}(\ \mathrm{PA}\ )$ and $\mathrm{IS}^{S}_{Xtab}(\ \mathrm{PA}\ )$ become inconsistent for essentially the same reason. (This is because each of their Group-3 *"I am consistent"* axiomatic declarations activate the Second Incompleteness Theorem, essentially automatically, when a WCB-like growth property is present.)

4.4 showed $\text{IS}_{Tab}(\text{ PA })$ was, indeed, a boundary-case exception to the Second Incompleteness Theorem.

Our goal in this section will be to summarize how Artemov's results in [4] can bridge the gap between these two results. It will formulate a Third Leg of a Tripod Reply to Hilbert's second question, using Artemov's strategy. Thus, [4] sought to resist the temptation of trying to find a means whereby PA (or some other type of partially analogous system) can prove a theorem about its own consistency under a Hilbert-like deductive apparatus. Instead, Artemov's approach was to construct a sequence of subsets of Peano Arithmetic called $S_1 \subset S_2 \subset S_3 \subset ....$ where

A. Each S_{j+1} can prove a statement $\text{Cons}(S_j)$, which asserts that there exists no Hilbert-style proof of $0 = 1$ from S_j .

B. Peano Arithmetic's set of formal axioms will consist of the union of all the axioms sentences belonging to the infintized seqence of $S_1 \cup S_2 \cup S_3 \cup$ .... .

Artemov's main result in [4] was to show it is possible to construct a sequence S_1 , S_2 , S_3 , ... that meets the preceding two conditions.

We will call Artemov's approach the **"Infinite Ranged"** methodology. The full details behind this technique will not be described here because they were described by Artemov very nicely in [4]. Instead, our goal will be to summarize how [4]'s result can nicely interface with other facets of the Tripod Reply to Hilbert's question.

A major point is that Theorem 4.4's $\text{IS}_{Tab}($ PA $)$ system can prove fully all the Π_1^* theorems of Peano Arithmetic via its Group-2 axiom schema. This implies $\text{IS}_{Tab}($ PA $)$ can prove the validity of each sentence of the form Cons(S_j) where S_j lies in Item A's sequence of S_1 , S_2 , S_3 , ... (because all of the relevant sentences Cons(S_j) have Π_1^* encodings).

A further point is that IS_{Tab}(PA) can also prove the more ambitious statement Cons(PA) (itself). This is because one of IS_{Tab}(PA)'s Group-2 axioms is Line (24)'s particular declaration (as footnote [9] explains). Hence, IS_{Tab}(PA) can formally prove Cons(PA) by combining its knowledge of $0 \neq 1$ with (24)'s declaration.

$$\forall \, p \quad \{ \text{ HilbPrf}_{\text{PA}}(\ulcorner \, 0 = 1 \, \urcorner, p) \quad \Rightarrow \quad 0 = 1 \quad \} \tag{24}$$

In other words, Artemov's proofs from [4] can be naturally incorporated into IS_{Tab}(PA)'s knowledge base as a direct application of the Group-2 axiom scheme. An advantage of viewing Artemov's results from IS_{Tab}(PA)'s perspective is that the latter's Group-3 *"I am consistent"* axiom allows IS_{Tab}(PA) to

[9] Line (24) is an instance of a Group-2 axiom simply because " $0 = 1$ is an example of a Π_1^* sentence.

possess a type of knowledge about its own self-consistency.

Moreover, there is actually a 2-way feedback loop between $\text{IS}_{Tab}(\text{ PA })$ and Peano Arithmetic. Thus while Peano Arithmetic cannot prove the statement Cons$\{\text{ IS}_{Tab}(\text{ PA })\ \}$, it can prove the following conditional provability statement

$$\text{Cons(PA)} \quad \Longrightarrow \quad \text{Cons}\{\ \text{IS}_{Tab}(\text{ PA })\ \} \tag{25}$$

More specifically, Peano Arithmetic's specific proof of (25) arises because Theorem 4.4's consistency-preservation result can be proven by PA.

Remark 5.8 This Tripod-styled answer to Hilbert's Second Problem is somewhat unexpected because each of Items (a)-(c) shall simultaneously hold true under it:

a) PA will prove Line (25)'s invaraint,

b) $\text{IS}_{Tab}(\text{ PA })$ will conversely establish Cons(PA) , and

c) The Second Incompleteness Theorem will, nevertheless, imply that no strong robust axiom system can formally justify its own consistency.

This ironic contrast between the technically compatable Items (a)-(c) is, perhaps, partially related to what Hilbert and Gödel were seeking in their famous statements $*$ and $**$. In any case, it shows that certainly a fragment of what they were seeking is feasible.

6 Concluding Remarks

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All the core results in this article were mentioned, at least in some implicit form, during [61]'s alternate briefer presentation. For instance, Theorem 4.7's counterpart in [61], was called Example 5.1, and the analog of the current "Tripod" reply to Hilbert's second question was divided into several separate observations in [61].

Aside from providing many more technical details, the current article differs from [61] by proposing how this material can be compressed into a 3-week segment of a 2-semester introductory logic course, as well as speculating how this subject can be made more comprehensible to logicians working outside the precise subfield of proof theory, called metamathematics.

We have already noted that our proposed 2-semester sequence would be analogous to a Calculus I & II sequence in Mathematics or a Physics I & II. It would cover, thus, all of the four main branches of Logic (e.g. proof theory, model theory, computability theory and set theory), as well as briefly examine Theorems 4.4, 4.5 and 4.7.

One must be prudently cautious, however, when discussing a 2-semester sequence in Logic, analogous to Calculus I & II or a Physics I & II. This is because these two introductory sequences are not, actually, fully similar to each other. Thus, Calculus is intended to be a clear presentation for students who have read the introductory Calculus textbooks. On the other hand, a Physics I & II starting sequence provides only an informal introduction to the Quantum Mechanincs and Relativity constructs (with a gentle but abbreviated introduction).

Our proposed Logic I & II sequence is intended to be a compromise between these two approaches. This is because there is simply inadequate time to cover Theorems 4.4, 4.5 and 4.7 in full detail. Instead, an abbreviated overview of these three topics, similar to what is done during a Physics-style overview, seems preferable. This overview should, obviously, be accompanied with a much more detailed summary of Logic's four main branches of model theory, computability theory, set theory and proof theory.

Within such a framework (assuming the Completeness and Incompleteness theorems are covered with adequate detail), a broad spectrum of students will likely find their cursiousity adequately aroused, so that many will likely wish to take four or five additional courses. In particular, the counter-intuitive aspects of the Second Incompleteness Theorem should be mentioned (but not be discussed in full tedious detail).

Lastly, it is obviously difficult to exactly speculate, but we do suspect that the preceeding hybridized Tripod methodology will meet the approximate expectations that Hilbert and Gödel were advancing in their famous statements of $*$ and $**$.

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APPENDIX E: More Detailed Justification for Lemma 5.3.2

The short 4-paragraph proof for Lemma 5.3 in §5.3.2 would be an ideally brief presentation, if it did not contain one significant drawback. This is that Section 5.3.2's "informal" proof used an abbreviated multiplication function symbol, in a context where multiplication was rigorously treated as a 3-way relation $Mult(x, y, z)$ by the lemma.

While this inaccurate notation is harmless in the context of an abbreviated proof summary, there should be, certainly, a documented account about how the informal proof for Lemma 5.3 can be transformed into a more formalized structure. Such will be outlined in this appendix.

It is harmless to skip (or possibly skim) this appendix during one's first pass though this article. Its material is not meant to be read line-by-line, until all other sections of this article have been very meticulously examined. (Please see footnote [10] .)

Detailed Formalized Jusitification for Lemma 5.3:

It is helpul to begin by considering how the the commutative and associative multiplation principles of $r \cdot s = s \cdot r$ and $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ need to be revised in a technically rigorous setting. In this case, Line (26) offers a straightforward translation of the commutative identity when Line ???????'s 3-way $Mult(x, y, z)$ predicate symbol replaces the multiplication function symbol.

$$\forall\, r \quad \forall\, s \quad \forall\, z \quad \forall\, a \neq z$$

$$\{ \; Mult(r,s,z) \;\Leftrightarrow\; Mult(s,r,z) \; \} \quad \wedge \quad \{ \; Mult(r,s,z) \;\Rightarrow\; \neg\, Mult(r,s,a) \; \} \tag{26}$$

Likewise, the associative identity of $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ translates into:

[10] The material in this appendix is too complicated to appear in an introductory logc course.

$$\forall r \quad \forall s \quad \forall t \quad \forall w$$

$$\{\, \exists u \ \ Mult(r,s,u) \wedge Mult(u,t,w) \,\} \ \Leftrightarrow \ \{\, \exists v \ \ Mult(s,t,v) \wedge Mult(v,r,w) \,\} \tag{27}$$

A nicelxyyyyy helpful fact is that (27) obviously implies (28). It turns out that we will need only (28)'s rightway-pointing implication to achieve our goal of proving Lemma 5.3.

$$\forall r \quad \forall s \quad \forall t \quad \forall w$$

$$\{\, \exists u \ \ Mult(r,s,u) \wedge Mult(u,t,w) \,\} \ \Rightarrow \ \{\, \exists v \ \ Mult(s,t,v) \wedge Mult(v,r,w) \,\} \tag{28}$$

Moreover *any* deductive apparatus that satisfies Gödel's Completeness Theorem can trivially[11] prove Line (28) is equivalent to the statement (**??**) below:

$$\forall r \quad \forall s \quad \forall t \quad \forall w \quad \forall u$$

[11] This trivial equivalence follows because Lines (28) and (29) are identical statements except for the allowed change of the " $\exists\, u$ " into a " $\forall\, u$ " quantifier after the precise physical position of u's quantifier is moved

$$\{\, Mult(r,s,u) \wedge Mult(u,t,w) \,\} \;\Rightarrow\; \{\, \exists v \;\; Mult(s,t,v) \wedge Mult(v,r,w) \,\} \tag{29}$$

Indeed any common deductive apparatus can show that Line (30) is also equivalent to both of these two preceding lines:

$$\forall\, r \quad \forall\, s \quad \forall\, t \quad \forall\, w \quad \forall\, u \quad\; \exists v$$

$$\{\, Mult(r,s,u) \wedge Mult(u,t,w) \,\} \;\Rightarrow\; \{\; Mult(s,t,v) \wedge Mult(v,r,w) \,\} \tag{30}$$

Unfortunately, we need Π_1^* sentence to express multiplication's associative property, and none of the preceeding three lines were Π_1^* sentences. (Thus, Line (30) is a Π_2^* sentence, and (28) and (29) were expressions that that are equivalent to Π_2^* formats, after being normalized into exactly (30)'s specific form.)

Fortuntately, there is available an easy avenue for repairing this gap. Thus, Line (12) is a Π_1^* sentence that is technically different from (30) (although they are mostly quite similar).

$$\forall\, r \quad \forall\, s \quad \forall\, t \quad \forall\, w \quad \forall\, u \quad \exists v \,\leq\, w$$

$$\{\, r \neq 0 \wedge Mult(r,s,u) \wedge Mult(u,t,w)\,\} \;\Rightarrow\; \{\; Mult(s,t,v) \wedge Mult(v,r,w)\,\} \tag{31}$$

We will call Line (12) the **Revised Π_1^* Restatement of the Multiplication's Associative Princible**. It differs from (30) by containing a bound on v's allowed value and also requiring $r \neq 0$. (It turns out this relatively minor constraint on r will be ultimately harmless because Line ???????'s definition of $Mult(x,y,z)$ implies $\forall\, a \;\; Mult(0,a,0)$.)

The curious aspect of Lines (30) and (12) is that they are **NOT** logically equivalent to each other (in the absence of any proper axioms). Nevertheless, they can be proven to be equivalent from either the prespective of Peano Arithmetic (or even some astonishingly tiny subset of it).

The relatively minor distinction between Lines (30) and (12) may, possibly strike many readers as being almost trivial. It turns out, however, these distinctions are needed by Lemma 5.3 and its mainapplications. This is because our restatement of multiplication's associative princible will be required to have exactly a Π_1^* encoding.

It is next should be mentioned that two possible different notational symbols can be used to designate the mulitplicative product of two integers x and y. The first notation of " $x \cdot y$ " had been used during §5.3.2's informal proof for Lemma 5.3. It will be avoided anywhere in the remainder of this appendix to emphasize that multipliplication is not presumed to be a total function. A second alternate notation of " $x * y$ " will be (occasionally) allowed. This latter notation will be employed only when the germane axiom system α has proven the following sentence to be true:

$$\exists \ z \quad Mult(x, y, z) \tag{32}$$

Thus in this case, " $x \ * \ y$ " will be an informal (but well-defined) abbreviation for the term z .

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We will now use the preceding notation to provide a formal proof for Lemma 5.3. Its core claim was stated in its Line (20). This statement is dupicated below and called $***$.

$$*** \quad \forall \ c \ \ \exists \ h \ \ \{ \ \mathrm{NatMult}(c) \ \ \Rightarrow \ \ [\ Mult(c, c, h) \ \wedge \ \mathrm{NatMult}(h) \ ] \ \}$$

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Our goal will be to construct five terms, denoted by the upper case letters of R, S, T, W, and U, that lie in 1-to-1 correspondence with the five universally quantified variables appearing in the first part of (**??**)'s statement. They are defined by the following rules

i) R can designate any integer quantity satisfying $R \neq 0$.

Thus R will satisfy the needs of the inequality in Line (12)'s left curly bracket expression.0000

ii) S and T will equal to each other and represent the quantity c satisfying $***$'s requirment of NatMult(c).

iii) U will designate the quantity $R * S$, and W will designate the quantity $U * T$. (The footnote [12] provides a thorough explanation for the meticulous readers about exactly why this notation can be used legally.)

pppppppppppppp x and y . a and b . a and b .

denote six fixed objests that correspond

So, let us begin our goal of proving Lemma 5.3's by assuming that NatMult(c) (on the left side of Line (20)) is true. Let the six upper case letters of R, S, T, W, U and V denote six fixed

[12]

objests that correspond to their six lower case counterparts from Line (28). We will also use the following conventions:

i) The constants S and T will correspond to the quantity c from Line (20)'s left side. This allows us to presume from Lemma 5.3's Line (20) that both NatMult(S) and NatMult(T) hold true. Hence, the left-side curly bracket expression in Line (28) must also hold true for such $S = T = c$ (since multiplication satisfies the commutative princible). Thus for such $S = T = c$, Line (20) trivially implies

$$\forall\, r \;\exists\, u \;\exists\, w \;\{\; Mult(r,S,u) \;\wedge\; Mult(u,T,w) \;\} \quad (33)$$

ii) Next let us set $V = c^2$. Since the hypothesis of Lemma 5.3's allows us to presume that that its axiom system α knows that NatMult(c) is true and that every integer $x \neq c$ fails to satisfy $Mult(c, c, x)$

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ii) The constant V will correspond to the quantity h on the right side of Line (20). (This means our proof of (20)'s claim will be done when we have shown simultaneously that $V = c^2$ and V satisfies NatMult(V)

iii) From Definition 5.2, we know NatMult(V) is true only if the following identity holds:

$$\forall\, d \quad \exists\, e \quad Mult(V, d, e) \tag{34}$$

In this context, R will now correspond to d in Line (34), and W will correspond to e . Moreover since Item 2 used the assumption $V = c^2$, this means our proof of Lemma 5.3 will need the chain of equalities of $W = e = c^2 \cdot d = c^2 \cdot R$. to hold true on the right half of Line (28).

iv) The final point is that if we let U correspond to the quantity of $c \cdot R$ on the left side of Line (34) then the

" $\exists\, u$ " claim in its curly-bracked expression will certainly be satisfied when $W = c^2 \cdot R$. More specifically, the last line in Item 3 showed this equality must both hold true and shall be able to be confirmed by the basis axiom system α via the combination of Items 1-3.

The repeated substitutions in the preceeding 4-part argument may initially look complicated, but the intuition behind its procedure is actually very simple (almost bordering on the trivial). Lemma 5.3's proof is essentially an extension of an almost routine paradigm, made to become quite dizzying and complicated only because we insist on viewing multiplication as a 3-way relation (rather than as a total function).

In any case while Line (27)'s notation is tedious, the remainder of Lemma 5.3's proof shall only be a tiny two paragraphs long. It will corroborate Line (20)'s claim by showing that if

c satisfies NatMult(c) then $h = c^2$ will exist and satisfy NatMult(h).

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We saw there that Sissa's king was required to

's formalisms then a decision to shun them as self-justifying formalism has philosophical justifications (aside from their other intrinsic problems).

Finally, we return to the point that some Sissa-like exponential explosions are much more subtle to detect than others. For example, Theorem 4.5's $\mathrm{IS}_{Xtab}(\text{ PA })$ basis system does not include the axiom that multiplication is a total function (analogous to the case of

Yet nevertheless, the pronoun "I" is as difficult to define under $\mathrm{IS}_{Xtab}(\text{ PA })$ as it would be under $\mathrm{IS}^{M}_{Tab}(\text{ PA })$. (This is because Lemmas 5.3 and 5.4 exhibit complicated counterparts of multiplication's squaring functionality available to $\mathrm{IS}_{Xtab}(\mathrm{PA})$.)

The same request would have violated the laws of Physics, indeed, if the chess board had

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some readers may wonder about the inherent weaknesses of Theorem 4.4's

self-justifying formalism.

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how a careful analysis of the effects of the Group-3 *"I am consistent"* axioms is revealing.

formalism with its antithesis under Theorem 4.7's alternate $\mathrm{IS}^{M}_{Tab}(\bullet)$

7 STUPID More about the Basic Nature of Self-Referencing

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PROBABLY omit **HIDDEN** PARAGAPH BELOW

In a context where Theorems 4.5 and 4.7 showed that both $\mathrm{IS}_{Xtab}(\ \beta\ )$ and $\mathrm{IS}^{M}_{Tab}(\ \beta\ )$ were inconsistent when $\beta\ =$ Peano Arithmetic (PA), some readers may, perhaps, wonder whether Theorem 4.4's self-justifying $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ system may rest, excessibely, on its inherently weak underlying structure?

This is, of course, a legitimate concern. After all, $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ was able to evade the Second Incompleteness Effect only by it instituting sharp circuit breakers, that make it operate substantially different (and much more weaker) than both $\mathrm{IS}_{Xtab}(\ \mathrm{PA}\ )$ and $\mathrm{IS}^{M}_{Tab}(\ \mathrm{PA}\ )$. It turns out, however, that this weakness in $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$ is not as awkward, as it might first initially appear. This is because $\mathrm{IS}_{Tab}(\ \mathrm{PA}\ )$'s exact formal structure also has a second *very different* quality.

It arises because the Group-3 *"I am consistent"* axiom declarations, utilized by each of $\text{IS}_{Tab}(\text{ PA })$, $\text{IS}^{M}_{Tab}(\text{ PA })$ and $\text{IS}_{Xtab}(\text{ PA })$, become well-defined concepts *only when* the pronoun "I" (itself) is exactly well defined.

It turns out that the pronoun "I" is *much more awkward* to define under $\text{IS}^{M}_{Tab}(\text{ PA })$ and $\text{IS}_{Xtab}(\text{ PA })$ than it was for $\text{IS}_{Tab}(\text{ PA })$.

The source of the difficulty harks back to Chapter **??**'s WCB paradigm. We saw there that Sissa's king was required to walk away from Sissa's repeated doubling request when it simply involved a chess board with 64 squares. The same request would have violated the laws of Physics, indeed, if the chess board had a few hundred squares (perhaps 640 squares) Then it would, obvioulsy, required more grains of wheat than there are atoms in the entire universe.

Our point, firstly, is that the pronoun "I" in $\text{IS}^{M}_{Tab}(\text{ PA })$'s

Group-3 *"I am consistent"* axiom can be viewed almost as a non-existent object when it produces (via a few hundred iterations) an integer whose binary encoding has more zero-digits than there are atoms in the universe. (This is quite different from the ISTab(PA)'s Group-3 axiom that produces a much more polite and modest version of the pronoun "I", when only addition is viewed as a total function.)

In other words, one may legitimately wonder whether the two versions of the pronoun "I" are actually similar in the Group-3 axioms of $\mathrm{IS}_{Tab}^{M}(\mathrm{~PA~})$ and $\mathrm{IS}_{Tab}(\mathrm{~PA~})$?

Our suggestion is that they are qualitatively different, and they should not be confused as being analogous. This is because $\mathrm{IS}_{Tab}^{M}(\mathrm{~PA~})$'s variant presumes the existence of a *quite unrealistic exponential growth process*, that can easily produce numbers whose binary lengths are larger than the number of atoms in the universe !

Another way of stating the same opinion is that the 1,500 year old fable about the Brahmin Sissa would not have withstood the test of time, without many similar difficulties about impermissilbe exponential growths having occured repeatedly. Perhaps the assumption that pronoun "I" can be defined with equal ease by both the Group-3 axioms for $\mathrm{IS}^{M}_{Tab}(\text{ PA })$ and $\mathrm{IS}_{Tab}(\text{ PA })$ is, thus, simply incorrect? In other words, its underlying assumption is trespassing upon Sissa's domain?

Finally, we return to the point that some Sissa-like exponential explosions are much more subtle to detect than others. For example, Theorem 4.5's $\mathrm{IS}_{Xtab}(\text{ PA })$ basis system does not include the axiom that multiplication is a total function (analogous to the case of $\mathrm{IS}^{M}_{Tab}(\text{ PA })$). Yet nevertheless, the pronoun "I" is as difficult to define under $\mathrm{IS}_{Xtab}(\text{ PA })$ as it would be under $\mathrm{IS}^{M}_{Tab}(\text{ PA })$. (This is because Lemmas 5.3 and 5.4 exhibit complicated counterparts of multiplication's squaring functionality available to $\mathrm{IS}_{Xtab}(\text{ PA })$.)

Thus in a context where the first and third legs of our proposed Tripod reply to Hilbert's Second Problem are quite helpful, its second leg should, also, not be ignored. This is because a careful analysis of Theorems 4.4, 4.5 and 4.7 reveals that self-justification becomes feasible at exactly the precise juncture where the meaning of the pronoun "I" does become *quite realisticly* manageable....

Again, it should be remembered that Theorem 4.4 is not a small result because Section 5.5 explained how there is a fascinating connection beteween Theorem 4.4 and Artemov's Infinite Ranged methological techniques from [4].

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the paper [61] defined a second axiom basis system, called $\mathrm{IS}_D(\beta)$, also employing L^* 's language.

Unlike β which is presumed to be consistent, this second basis system, $\mathrm{IS}_D(\beta)$, will be treated by Theorems 4.4 and 4.5

as entities that can be either consistent or inconsistent. A 4-paragraph description of $\mathrm{IS}_D(\beta)$ as a union of four subcomponents can be found in Section 4 of [61] under the separate headings of "Group-zero", "Group-1", "Group-2" and "Group-3".

From the top-level perspective taken in this article, the first three components of $\mathrm{IS}_D(\beta)$'s system will assure it does prove all of β's Π_1^* theorem, and its final Group-3 component shall consist of one single precise sentence, asserting the overall consistency of $\mathrm{IS}_D(\beta)$'s self-justifying formalism.