On a 3-Way "Tripod" Styled Reply to Hilbert's Mysterious Second Problem

Dan E.Willard

State University of New York at Albany

Abstract

Hilbert's mysterious year-1900 Second Problem asked mathematicians to devise a methodology whereby Peano Arithmetic can confirm its own consistency. Gödel's famous 1931 paper showed that a fully positive reply can never be made to Hilbert's question. This article will explain how Hilbert's question is such a complicated issue that it can be better receive a 3-way styled "Tripod" reply.

We also provide substantial evidence that Gödel would likely agree with the main opinions expressed in this article.

Keywords and Phrases: Gödel's Second Incompleteness Theorem, Hilbert's Second Problem, Consistency, Smullyan-Fitting Semantic Tableau Deduction, Hilbert-Frege Deduction.

1 Introduction

This article is a continuation of a series of papers that began with the 1993 article [38] and continued until and through the year-2021 article [47]. This series, which included six papers appearing in the JSL and APAL, had focused on discussing generalizations and boundary case exceptions for the Second Incompleteness Theorem. The two goals of this paper will be to explore the underlying philosophy that motivated this series and to explain how it is related to several generalizations and boundary-case exceptions to the Second Incompleteness Theorem , including some important new results formalized by Artemov [3, 4].

Our general theme will be that Hilbert's year-1900 Second Problem is too complex an issue to receive a 1-directional or even 2-directional answer. Instead, it will require a 3-directional answer, called the "Tripod Reply" to Hilbert's question.

The first leg of this 3-part response to Hilbert's Second Problem will rest on the combination of Gödel's initial version of his Second Incompleteness Theorem and its numerous generalizations. They collectively establish that many logical formalisms, besides Peano Arithmetic (PA), are unable to corroborate their own consistency in a fully extensive respect. While these generalizations of Gödel's Second Incompleteness paradigm are indisputably important, the second section of this article will explain Hilbert and Gödel strongly doubted they constituted a full answer to Hilbert's year-1900 problem.

A second leg of a 3-part reply to Hilbert's Second Problem was formalized by Artemov in [3, 4]. He noted Peano Arithmetic (PA) can prove an infinite schema of theorems, whose collective union confirms PA's own consistency. Let us call Artemov's method a Step-By-Step Infinite-Schema approach **(SBSIS)**. This technique is an extension of the Justification Logics explored by Artemov and Beklemishev [2, 5, 6]. It was motivated by Artemov's observation that the year-1900 logic community (including Hilbert) were not aware that PA would require an infinite number of proper axioms. Thus, an SBSIS schema-driven logic approach is a valid reply to Hilbert's year-1900 second problem, although its potential significance was not well appreciated during the era when Hilbert posed his Second Problem.

Artemov's analysis [3, 4] uses Tarski's partial definitions of truth for sentences with a bounded number of quantifiers as an intermediate step. It essentially constructs a sequence of finite subsets of Peano Arithmetic $S_1 \subset S_2 \subset S_3 \subset ...$ where

1. PA $= S_1 \cup S_2 \cup S_3 \cup ...$
2. Each S_{j+1} can prove a Π_1 theorem asserting there exists no proof from S_j of 0=1, in a context where all logical axioms in the concerned proof from S_j are no more complex than Π_j or Σ_j statements. (We will henceforth call this theorem T_{j+1}.)

While Artemov's SBSIS-style response to Hilbert's year-1900 second question is intriguing, one would ideally still like access to a system that does not rely upon his infinite series $S_1\, S_2\, S_3 ...$.

A third possible leg of a proposed Tripod reply to Hilbert's second problem involves formal systems using Fixed Point axiomatic sentences that confirm their own consistency. For example, [38, 39, 41, 42, 47] examined systems strictly weaker than PA, that verified their own self-consistency, under mostly semantic tableau deduction [13, 32].

This third leg, called the **Declarative Approach**, rests on using a self-referencing *"I am consistent"* axiomatic declaration, so that a formalism can confirm its own consistency. This approach, studied in [38, 39, 41, 42, 47], will essentially be defined by the current paper's statement $\oplus$. Its advantage is that its declaration of self-consistency is compressed into one single sentence, while its **non-trivial drawback** is that its particular statement $\oplus$ (defined in §3) can be successfully used only when the surrounding formalism is sufficiently weak.

In other words, each of the three legs of a unified "Tripod" response to Hilbert's Second Problem own drawbacks, as well as significant virtues. A multi-facet Tripod reply is attractive because it straddes nicely between all three legs, simultaneously.

2 The Main Weakness of the First Leg

It is known Hilbert always suspected the Second Incompleteness Theorem, while correct, would ultimately display established exceptions. For instance, Hilbert [20] wrote:

> $*$ *" Let us admit that the situation in which we presently find ourselves with respect to paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?"*

Also, it is known Gödel's seminal 1931 paper appeared to agree with the goals of Hilbert's Consistency Program in one of its closing paragraphs:

> $**$ *"It must be expressly noted that Theorem XI"* (i.e. the Second Incompleteness Theorem) *"represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and there might conceivably be finite proofs which cannot be stated in P or in ... "*

Several biographies of Gödel [11, 17, 48] have observed Gödel's intention (prior to 1930) was to establish Hilbert's proposed objectives, before he developed his famous nearly opposing theorem. Indeed, Yourgrau's biography [48] of Gödel had traversed beyond this point. It recorded how von Neumann found it necessary during the early 1930's to *"argue against Gödel himself"* about the definitive termination of Hilbert's consistency program, which *"for several years"* after [15]'s publication, Gödel *"was cautious not to prejudge"*.

In a context where Gödel published fewer than 85 pages during his career, historians will probably never fully characterize the nature of Gödel's finished philosophical position about his Second Incompleteness Theorem. From informal notes taken during a 1933 Vienna lecture [16], it is known Gödel was more positive about its implications in 1933 than he was in 1931.

On the other hand, Gerald Sacks recorded a stunning 80-minute YouTube recollection about his interactions with Gödel in the year 2007. These recordings explicitly state that Gödel held *"contrarious"* opinions about some of his most famous results. Also Sacks explicitly recalled (see footnote [1]) Gödel communicating private opinions that were *"almost the opposite of what every one else would have expected".*

These remarks by Sacks deserve to be taken seriously, given that Sacks interacted twice with Gödel at the Institute of Advanced Studies (once in 1959-1960 and a second time during the 1970s). Moreover, Sacks went beyond the preceding quoted remarks. Other memorable comments from [31]'s YouTube lecture are that:

a) Gödel *"did not think"* the objectives of Hilbert's Consistency Program *"were erased"* by the Incompleteness Theorem,

b) Gödel believed (according to Sacks) it left Hilbert's program *"very much alive and even more interesting than it initially was".*

Also, Nerode [24] indicated in private communications that Stanley Tennenbaum shared similar conversations with Gödel, as those remembered by Sacks. These conversations likewise noted the need for some type of revival of Hilbert's Consistency Program.

Thus, the last two paragraphs, have summarized the quandary that Symbolic Logic has faced since 1931. They suggest some type of Tripod-like response to Hilbert's Second Problem may become necessary to formulate a more comprehensive response to Hilbert's question.

3 Starting Notation with the WCB & USEGR Paradigms

We will summarize the contents of the prior articles [42, 47] so it will be unnecessary to read these papers. The particular annotated paragraphs in [47] had used decimal reference numbers such as "Example 3.1" or "Definition 3.2". A non-decimal paragraph notation will be used in the current paper, thus having its first annotated paragraph instead called "Definition 1". This type of adjusted notation should cross-reference the prior paper [47] in a manner that avoids any ambiguity and/or confusion.

Definition 1 An ordered pair (α, D) is called a *Generalized Arithmetic Configuration* (abbreviated as a **"GenAC"**) when its first and second components are defined as follows:

1. The **Axiom Basis** " α " for a GenAC is defined as its set of proper axioms.
2. The second component " D " of a GenAC, called its **Deductive Apparatus**, is defined as the union of its logical axioms " L_D" with its rules for obtaining inferences.

The Example 3.1 from [47] provided several examples of GenAC formalisms. Its Definition 3.2 defined a GenAC (α, D) to be **"Self-Justifying"** when:

[1]These two quotations can be found in the 7-th and and 9-th minutes of the YouTube recording [31] that Gerald Sacks had made about Kurt Gödel.

i. one of (α, D)'s theorems (or possibly one of α's axioms) states that the deduction method D, applied to the basis system α, produces a consistent set of theorems, and

ii. the GenAC formalism (α, D) is actually, in fact, consistent.

Example 2 For any (α, D), it is easy to construct a system $\alpha^D \supseteq \alpha$ that satisfies the Part-i condition in an isolated context where the Part-ii condition is not also satisfied. Thus, α^D can consist of all of α's axioms plus the added **"SelfRef**(α, D)**"** sentence, defined below:

> $\oplus$ There is no proof (using D's deduction method) of $0 = 1$ from the *union* of the axiom system α with *this* sentence "SelfRef(α, D)" (looking at itself).

Each of Kleene and Rogers [21, 30] noticed how to encode analogs of SelfRef(α, D)'s above statement, which we will often call an **"I AM CONSISTENT" axiom.** The catch is α^D may be inconsistent (e.g. violate Part-ii of self-justification's definition despite the assertion in SelfRef(α, D)'s particular statement). This is because if the pair (α, D) is too strong then a quite conventional Gödel-style diagonalization argument can be applied to the axiom basis of $\alpha^D = \alpha +$ SelfRef(α, D), where the added presence of the statement SelfRef(α, D) will cause this extended version of α, ironically, to become automatically inconsistent. Thus, while an encoding for statement $\oplus$'s *"I am consistent"* declaration is a routine consequence of the Fixed Point Theorem, its computational implementation is potentially devastating.

Definition 3 Let $Add(x, y, z)$ and $Mult(x, y, z)$ denote two 3-way predicates specifying $x + y = z$ and $x * y = z$. Let us say a formalized system of α recognizes successor, addition and multiplication as **Total Functions** iff it can prove all of (1) - (3) as theorems:

$$\forall x \; \exists z \quad Add(x, 1, z) \tag{1}$$

$$\forall x \; \forall y \; \exists z \quad Add(x, y, z) \tag{2}$$

$$\forall x \; \forall y \; \exists z \quad Mult(x, y, z) \tag{3}$$

Also a GenAC system (α, D) will be called **Type-M** formalism iff it proves (1) - (3) as theorems, **Type-A** if it proves only (1) and (2), and it will be called **Type-S** if it proves only (1) as a theorem. Furthermore, (α, D) will be called **Type-NS** iff it proves none of (1) - (3).

BROADER OUTLOOK: The remainder of this chapter will step backwards roughly 1,500 years in time, and explore the relationship between Definitions 1 and 3 with India's famous "Wheat-and-Chess-Board" paradigm (often denoted as the **"WCB"** phenomenon.)

This paradigm will help clarify the persistent confusion that has surrounded Gödel's Second Incompleteness Theorem. It will also help us formulate a new interpretation for the Remarks $*$ and $**$ that Hilbert and Gödel had made.

According to Wikipedia, scholars are uncertain about exactly when the WCB fable had emerged in ancient India. (It may possibly have originated as early as 400-600 AD.)

Under one variant of the WCB fable, a Brahmin named Sissa ibn Dahir invented a type of Indian predecessor to the game of Chess. At some subsequent juncture, the king of Sissa's province offered to award him a prize for the game that Sissa had invented. Sissa replied that it would be sufficient to place one grain of wheat on the first square of a Chess Board, two grains on its second square, four grains on its third square, etc.

There are several versions of the Sissa fable, and no one knows which (if any ?) is accurate. One variant involves the king executing Sissa after discovering the last chess-board square would require 2^{63} grains of wheat. Another variant concludes with the king smiling upon realizing the nature of Sissa's puzzle and declaring that this puzzle is more fascinating and stimulating than the game of chess (itself).

While many precise details have been lost over time, the underlying message of the WCB fable has survived for roughly 1,500 years. This is because it is thought to convey an instructive lesson about the unsustainable nature of an exponential growth.

For instance, the 64 squares of a Chess Board has been found to require an amount of wheat that is a factor of 1,000 greater than the world's full year-2019 production. Moreover, a direct analog of the WCB paradigm, involving a few hundred doubling operations, will require more wheat than there are atoms in the Universe.

Our point is that this fable would not have survived the millennial test of time if there were not many examples in human history where individuals have accidentally got involved in exponential growth processes that started at a gradual rate but then ran wild. One of these examples seems to concern Hilbert's statement $*$ (i.e. see Section 2). It appears to ask mathematicians to perform an analog of Sissa's unsustainable WCB task.

This is because if one seeks to perform several hundred squarings of the number 2 than its binary encoding will own more zero-digits than there are atoms in the universe. This can be appreciated by comparing the sequences $x_0,\ x_1,\ x_2,\ ...$ and $y_0,\ y_1,\ y_2, ...$ defined below:

$$x_0 \quad = \quad 2 \quad = \quad y_0 \tag{4}$$

$$x_i \quad = \quad x_{i-1} \ + \ x_{i-1} \tag{5}$$

$$y_i \quad = \quad y_{i-1} \ * \ y_{i-1} \tag{6}$$

For $i > 0$, let ϕ_i and ψ_i denote the sentences in (5) and (6) respectively. Then if ϕ_0 and ψ_0 denote (4)'s sentence, it is apparent that $\phi_0,\ \phi_1,\ ...\ \phi_n$ imply $x_n = 2^{n+1}$, while $\psi_0,\ \psi_1,\ ...\ \psi_n$ imply $y_n = 2^{2^n}$. Thus, the latter sequence will grow at an exponentially faster rate than the former. (E.g. the respective quantities of $\mathrm{Log}_2(\, y_n\,) = 2^n$ and $\mathrm{Log}_2(\, x_n\,) = n+1$ represent the lengths for the binary codings for y_n and x_n .)

In other words, y_n 's binary encoding will have a length 2^n, roughly analogous to the n-th square belonging to the WCB's chess board paradigm. Leaving aside technical details that were largely explored in [47] and shall be briefly reviewed later, we are suggesting that Hilbert's statement $*$ is partially analogous to Sissa's WCB paradigm.

The latter is not meant to deny that are some intriguing interpretations of the Hilbert and Gödel statements $*$ and $**$ that should draw one's partial sympathy. This is because there are other types of amended axiomatizations of arithmetic that avoid Line (6)'s malignant assumption that multiplication is a total function.

The theme of this article is largely that one needs to be more flexible when approaching the Hilbert and Gödel statements $*$ and $**$. Our proposed "Tripod Reply" to Hilbert's Second

Problem will be a gentle hybridized reply, that accepts these statements half-way and views Hilbert's Second Problem from the three different perspectives summarized in Section 1.

Definition 4 During the remainder of this article, the acronym **"USEGR"** will refer to an *Unsustainable Exponential Growth Rate*, similar to the dizzying number of zero digits that was produced by Line (6)'s sequence of $y_0,\ y_1,\ y_2,\ \ldots$. We will argue that such sequences resemble Sissa's WCB paradigm, and a mathematical realist should avoid them as much as feasible.

For instance, a plain 1-way response to Hilbert's Second Open Problem was observed to be questionable by Sacks, when his remarks (a) and (b) (in Section 2) noted Gödel thought the Second Incompleteness Theorem was essentially an excessively pessimistic reply to Hilbert's second problem. Likewise, a broader 2-way response is troubling because the various self-justifying formalisms that Willard developed between the times of [38]'s initial 1993 presentation and that of [47]'s recent LFCS-2020 paper were much weaker than would be ideally preferred.

Artemov's third type of response uses his elegant infinite ranged SBSIS-styled exceptions [3, 4] to the Second Incompleteness Theorem. It is almost ideal. Yet, it also has drawbacks (see footnote[2]) because its SBSIS approach is a mixture of tempting positive results and compromises. This is the reason the current article seeks to examine this topic from several perspectives simultaneously, using a 3-way hybridized reply to Hilbert's 120 year-old problem.

4 Characterizations of Several Base Languages

Sissa's 1,500-year old "WCB" paradigm, as well as Definition 4's "USEGR" effect have similar themes. This is because both involve unsustainable exponential growth processes, having often unwelcome and unanticipated effects. In particular, their circumstances can be made applicable to essentially all the known variants of the Second Incompleteness Theorem [1, 2, 7, 8, 9, 10, 12, 14, 18, 19, 22, 25, 26, 27, 28, 29, 33, 34, 35, 36, 37, 39, 40, 44, 42, 46]. One of its most interesting variants arose during a 1994 private telephone conversation with Robert Solovay [34]. Its formalism (due to Solovay) can be thought of as a generalization of a methodology that was initially developed by Pudlák in [28] and which was further explored by Nelson and Wilkie-Paris in [23, 37]. It essentially amounts to the following observation:

Theorem ++ : *(Solovay's modification [34] of Pudlák [28]'s formalism using some of Nelson and Wilkie-Paris [23, 37]'s methods)* : Let (α, D) denote a Type-S GenAC system which assures the successor operation will satisfy both $x' \neq 0$ and $x' = y' \Leftrightarrow x = y$. Then (α, D) cannot verify its own consistency whenever simultaneously D is some type of a Hilbert-Frege deductive apparatus and α treats addition and multiplication as 3-way relations, satisfying their usual associative, commutative, distributive and identity axioms.

[2] Section 1 explained how [3, 4] also had some hidden difficulties. They are that Peano Arithmetic (PA) can verify only the consistency of finite subsets of itself. Thus, [3, 4]'s perspective is useful, but it does not fully explain how PA can formalize the consistency of its infinitely drawn expanse. In this context, we shall argue a hybridized Tripod-style response is a more comprehensive reply to Hilbert's Second Problem.

Essentially, Solovay [34] had privately communicated to Willard an approximate analog of Theorem ++. (This communication was similar to several other often-privately-communicated comments that the literature [10, 19, 23, 26, 28, 37] has often attributed to Robert Solovay's unpublished observations.) It also should be mentioned that partial analogs of ++'s statement were explored subsequently by Buss-Ignjatović, Hájek and Švejdar in [10, 18, 35], as well as in Appendix A of the paper [39] and in [42].

Furthermore, we stress that Pudlák's initial 1985 article [28] had captured the majority of ++'s implications, chronologically before Solovay's observations. Also, Friedman did nicely related work as early as 1979 in [14].

In order to explain how Sissa's 1,500-year old "WCB" paradigm and Definition 4's similar "USEGR" effect are related to this material, it will be useful to employ Definition 5's notation. (Its predecessor can be found in [42], but the latter did not discuss the "USEGR" effect.)

Definition 5 A function $F(a_1, \ldots, a_j)$ is called a **Non-Growth** operation when $F(a_1, \ldots, a_j) \leq Maximum(a_1, \ldots, a_j)$ holds. Seven examples of non-growth functions are:

1. *Integer Subtraction* (where $x - y$ is defined to equal zero when $x \leq y$),
2. *Integer Division* (where $x \div y$ equals x when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise),
3. $Maximum(x, y)$,
4. $Log_{\spadesuit}(x)$ which is an abbreviation for $\lceil \text{Log}_2(\ x + 1\ ) \rceil$ under the conventional notation.
5. $Root(x, y) = \lceil x^{1/y} \rceil$
6. $Count(x, j)$ designating the number of physical "1" bits stored among x's rightmost j bits.
7. $Bit(x, i)$ designating the $i-$th rightmost bit of the string x (as explained by footnote [3] .)

These function were called **Grounding Functions** in [42]. Also, L^0 will denote a quite weak language built from the Grounding functions, together with three constant symbols c_0, c_1 and c_2 (representing 0, 1 and 2) and the "$=$" and "$\leq$" primitives.

Since L^0 contains absolutely no growth functions, it is so weak that it cannot even formalize the integer 3 as an isolated term. Thus, [42] found it necessary to strengthen L^0's language with the two additional Type-NS 3-way predicates $Add(x, y, z)$ and $Mult(x, y, z)$ (that formalize the concepts of " $x + y = z$ " and " $x * y = z$ "). Their two formal definitions are given below:

$$z - x = y \quad \wedge \quad z \geq x \tag{7}$$

$$\{\ (x = 0 \vee y = 0) \Rightarrow z = 0\ \} \quad \wedge \quad \{\ (x \neq 0 \wedge y \neq 0\ ) \Rightarrow (\ \frac{z}{x} = y \ \wedge\ \frac{z-1}{x} < y\ \ )\ \} \tag{8}$$

More precisely to help Type-NS self-justifying axiom systems encode integers distinctly larger than 2, [42] introduced some additional constant symbols, of a_2, a_3, a_4 .... and b_2, b_3, b_4 .... into L^0 's language, satisfying the following three constraints:

[3] The $Bit(x,i)$ operator is technically unnecessary because it can be encoded as: $\text{Bit}(x,i) = \text{Count}(x,i) - \text{Count}(x,i-1)$.

$$a_2 \quad = \quad b_2 \quad = \quad c_2 \quad = \quad 2 \tag{9}$$

$$a_{j+1} \quad = \quad a_j \; + \; a_j \qquad \text{for } j \geq 2 \tag{10}$$

$$b_{j+1} \quad = \quad b_j \; * \; b_j \qquad \text{for } j \geq 2 \tag{11}$$

In particular, Line (10) was called an **"Additive Naming Convention" (ANC)** in [42], and Line (11) was called an **"Multiplicative Naming Convention" (MNC)**.

The precise encoding for Lines (10) and (11) in [42] is delicate because the earlier Theorem ++ (of Pudlák and Solovay) implied a Self-Justifying axiom system cannot treat either addition or multiplications as total functions. This difficulty was resolved by applying the $Add(x, y, z)$ and $Mult(x, y, z)$ 3-way predicates (from Lines (7) and (8)). Thus, reworded forms of Lines (10) and (11), evading ++'s generalization of the Second Incompleteness Theorem, appear below:

$$ADD(a_j, a_j, a_{j+1}) \qquad \text{for any } j \geq 2 \tag{12}$$

$$Mult(b_j, b_j, b_{j+1}) \qquad \text{for any } j \geq 2 \tag{13}$$

Definition 6 The term **Additive Naming Language** refers to the extension of the L^0 language that defines the constant symbols $a_2, \; a_3, \; a_4 \; ....$ with the formalisms from Lines (9) and (12). This revised version of L^0 's language will be denoted as L^{ANC}. Likewise the phrase **Multiply-Additive Naming Language** refers to the revision of the L^0 's language that defines all the constant symbols of $b_2, \; b_3, \; b_4 \; ....$ and $a_2, \; a_3, \; a_4 \; .... ,$ using the formalisms of Lines (9), (12) and (13). Its language is called L^{MNC}.

An examination of these two naming conventions was the core topic in [42]. Roughly summarized, the Theorem 3 from [42] showed that the language L^{ANC} can house robust forms of self-justifying systems, and its Theorem 4 showed that the similar evasions of the Second Incompleteness Theorem are impossible under L^{MNC}.

What was missing from [42] was an intuitive explanation about why L^{MNC} 's failure did not out-shine the partial (but limited) success that L^{ANC} achieved. In other words, one needs to ask: *Which of these two languages provide a better standpoint for approximating conventional theorem-proving?*

Our reply to the preceding question is that the language L^{ANC} is preferable because Definition 4's awkwardly exponential "USEGR" growth suggests one must look merely at the 600-th item in the sequence $b_2, \; b_3, \; b_4 \; ....$ to find an element whose binary encoding-length exceeds the number of atoms lying inside the universe.

Remark 7 More specifically, we are suggesting the reader keep in mind this dichotomy when the next section compares the Parts (a) and (b) of its Propositions 11 and 12. This is because the negativistic Propositions 11.b and 12.b will appear to be divorced from reality, when their excessive USEGR growth rates are noticed to resemble Sissa's quite outlandish exponential rates of growth.

5 More Details

The primary goal in [42] was to determine when self-justification was possible under either (12)'s "Additive" or (13)'s "Multiplicative" naming schema. Some added notation is needed to briefly review [42]'s main results. In a context where t denotes any term in L^{ANC}'s language that does not contain the variable symbol of " v ", the quantifiers in $\forall\, v \leq t \;\; \Psi(v)$ and $\exists\, v \leq t \;\; \Psi(v)$ will be called L^{ANC}'s **" v-Restricting Quantifiers "**. Then Definition 8 formalizes the analogs of conventional arithmetic's Δ_0, Π_n and Σ_n formulae in L^{ANC}'s language:

Definition 8 Any formula in L^{ANC}'s language will be called Δ_0^{ANC} iff all its quantifiers variables v meet the preceding v-Restricting constraint. The Π_n^{ANC} and Σ_n^{ANC}. sentences of L^{ANC} are then defined by 1-3. (They are analogous to conventional arithmetic's counterparts):

1. Every Δ_0^{ANC} formula is also a Π_0^{ANC} and an Σ_0^{ANC} expression.
2. A formula is called Π_n^{ANC} when it can be encoded as $\forall v_1 \; ... \; \forall v_k \;\; \Phi$ where Φ is Σ_{n-1}^{ANC}.
3. A formula is called Σ_n^{ANC} when it can be encoded as $\exists v_1 \; ... \; \exists v_k \;\; \Phi$, where Φ is Π_{n-1}^{ANC}.

Given an initial axiom system β, the Theorem 3 of [42] did formalize a self-justifying logic, called **ISCE**(β), that could prove all β's Π_1^{ANC} theorems and which could also verify its own consistency under any Hilbert-Frege style deductive apparatus. The axiom basis for ISCE(β) was comprised, formally, of the following four distinct groups of axioms:

GROUP-ZERO: This schema will formally represent the integers of 0,1 and 2 with the defined constant-symbols of c_0, c_1 and c_2. It will also apply Line (12)'s Additive Naming schema to formally define the further constants of $a_2, \; a_3, \; a_4, \; ...$

GROUP-1: This axiom group will consist of a finite set of Π_1^{ANC} sentences, whose union with Group-zero axioms is consistent and can prove every Δ_0^{ANC} sentence that holds true in the standard model. (It was explained in [42] that any set H, meeting these conditions, may formalize the Group-1 axioms.)

GROUP-2: Let $\ulcorner \Phi \urcorner$ denote Φ's Gödel number, and $\text{HilbPrf}_\beta(x, y)$ denote a Δ_0^{ANC} formula indicating y is a Hilbert-styled proof from axiom system β of the theorem x. For each Π_1^{ANC} sentence Φ, the Group-2 schema will contain an encoding for (14)'s Π_1^{ANC} axiom:

$$\forall\, y \quad \{ \; \text{HilbPrf}_{\,\beta} \; (\; \ulcorner\Phi\urcorner \; , \; y \;) \quad \Rightarrow \quad \Phi \; \} \tag{14}$$

GROUP-3: This last part of [42]'s ISCE(β) formalism is a single self-referencing Π_1^{ANC} sentence stating:

> $\oplus\oplus$ "There exists no Hilbert-style proof of 0=1 from the union of the Group-0, 1 and 2 axioms with *THIS SENTENCE* (referring to itself)".

More details about $\oplus\oplus$'s exact formal encoding "fixed point" encoding will not be mentioned here because our earlier articles [39, 41, 42] discussed this topic quite thoroughly.

Definition 9 The symbols Π_n^{MNC}, Σ_n^{MNC} and Δ_0^{MNC} will denote the obvious analogs of Definition 8's Π_n^{ANC}, Σ_n^{ANC} and Δ_0^{ANC} sentences, when the broader language L^{MNC} of the Multiplicative Naming Convention replaces the initial language of L^{ANC}. Also, the symbol $\mathrm{ISCE}^{\mathrm{MNC}}(\beta)$ will denote the intended generalization of the preceding ISCE(β) system where all references to the language L^{ANC} and its Additive Naming Convention are replaced by statements about the language L^{MNC}, along with its Multiplicative Naming Convention, its associated Π_n^{MNC} and Δ_0^{MNC} sentences and a natural Π_1^{MNC} counterpart of ISCE(β)'s particular Group-3 *"I am consistent"* axiom. (It turns out that Proposition 11 will formalize that $\mathrm{ISCE}^{\mathrm{MNC}}(\beta)$ typically differs from ISCE(β) by being inconsistent.)

Definition 10 Let $I(\ \bullet\ )$ denote an operation that maps an initial axiom basis β onto an alternate system $I(\beta)$. (One example of such an operation is the ISCE($\bullet$) framework, that maps an initial axiom basis of β onto the alternate formalism of ISCE(β).) Such an operation $I(\ \bullet\ )$ is called **Consistency Preserving** iff $I(\beta)$ is consistent whenever the union of β with the Groups 0 and 1 axiom schemas is consistent.

Proposition 11 The $\mathrm{ISCE}^{(}\ \bullet\ )$ and $\mathrm{ISCE}^{\mathrm{MNC}}(\ \bullet\ )$ transformations have different properties, insofar as only the former is Consistency Preserving. In particular, if PA^{Ground} denotes the extension of Peano Arithmetic that includes both the conventional arithmetic operators and the seven Grounding functions, then the following two invariants do hold:

a. ISCE(PA^{Ground}) is a consistent system.

b. $\mathrm{ISCE}^{\mathrm{MNC}}$($\mathrm{PA}^{Ground}$) fails to be consistent.

It is unnecessary to prove Propositions 11.a and 11.b because they both are easy consequences of the Theorems 3 and 4 from [42]. Instead, we will discuss in this paper how Proposition 11 and some of its generalizations are related to a needed Tripod-style response to Hilbert's second problem.

It firstly should be noted that Definition 4's very fast "USEGR" growth is essential for appreciating the sharp contrast between Proposition 11.a and Proposition 11.b. This is because the generalization of the Second Incompleteness Theorem, in Proposition 11.b, arises because the MNC supports an unfortunately fatal USEGR rate of exponential growth among numbers.

It next should be remarked that some half-analogs of Proposition 11's twin 2-part statement for semantic tableau deduction were established by [39, 40, 41, 45]. Moreover, the recent year-2021 article [47] summarized and extended these articles. Thus, let $\mathrm{IS}_{\mathrm{Tab}}(\ \beta\ )$ be [47]'s 4-part axiom system, whose relationship to the ISCE(β) system is as follows:

i. The Group-1 and Group-2 schemes for $\mathrm{IS}_{\mathrm{Tab}}(\ \beta\ )$ and ISCE(β) are essentially identical.

ii. The Group-Zero scheme for $\mathrm{IS}_{\mathrm{Tab}}(\ \beta\ )$ is stronger than that of ISCE(β) because it replaces the Additive Naming Convention with a stronger more compact "Type-A" statement declaring that the operations of Addition and Double(x) $=\ x+x$ are total functions.

iii. The Group-3 scheme for $\mathrm{IS}_{\mathrm{Tab}}(\ \beta\ )$ is, however, weaker that its analog under ISCE(β) because it only recognizes its self-consistency under a semantic tableau form of deduction.

In essence, $\mathrm{IS}_{\mathrm{Tab}}(\ \beta\ )$ is a self-justifying formalism that avoids being inconsistent by using a different type of trade-off than ISCE(β), where its Group-Zero schema is stronger while its Group-3 statement uses a weaker deductive methodology.

Also, we will employ the Example 5.1 from [47], where $\mathrm{IS}^{M}_{\mathrm{Tab}}(\ \beta\ )$ denotes the natural modification of $\mathrm{IS}_{\mathrm{Tab}}(\ \beta\ )$ wherein:

1. The Group-Zero axiom further recognizes integer-multiplication as a total function.
2. The Group-3 axiom is the same as that described in (iii) except that its Group-3 *"I am consistent"* statement has been adjusted to recognize that the Group-Zero formalism now treats integer-multiplication as a total function.

Then Proposition 12 is [47]'s counterpart to Proposition 11. (We again remind the reader about Remark 7's warnings about excessively demanding USEGR growth rates. It is the intuitive reason behind the two inconsistency results appearing in both the Parts (b) of the Propositions 11 and 12.)

Proposition 12 The $\mathrm{IS}_{\mathrm{Tab}}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ and $\mathrm{IS}^{M}_{\mathrm{Tab}}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ systems differ insofar as:

a. $\mathrm{IS}_{\mathrm{Tab}}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ is consistent.

b. $\mathrm{IS}^{M}_{\mathrm{Tab}}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ fails to be consistent.

Formal analogs of Propositions 12.a and 12.b were established in [47], and we shall not repeat that discussion here. Instead, our discussion will focus on the two consistency results established by Propositions 11.a and 12.a. The Remark 13 will explain how there exists a quite natural interface between these two results with the SBSIS-styled approach that Artemov had advocated in [3, 4].

Remark 13 (about how the Second and Third Legs of a "Tripod" Reply to Hilbert's Second Problem will interact Quite Nicely with Each Other) : Let us recall that Step 2 of Artemov's SBSIS-styled [3, 4] constructions (summarized earlier in §1) constructed a class of Π_1-like theorems, $T_1,\ T_2,\ T_3,\ \ldots$, which confirmed the consistency of the respective $S_0,\ S_1,\ S_2,\ \ldots$ subsets of PA. It turns out that the validity of these T_j theorems is known to both the preceding $\mathrm{ISCE}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ and $\mathrm{IS}_{\mathrm{Tab}}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ formalisms from Propositions 11.a and 12.a (see Footnote [4]). The remainder of this section will explain how this subtle interconnection causes the Second and Third legs of a proposed Tripod Reply to Hilbert's Second Open Question to interact quite eloquently with each other.

[4]This is because the Group-2 schemes of $\mathrm{ISCE}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ and $\mathrm{IS}_{\mathrm{Tab}}(\ \mathrm{PA}^{\mathrm{Ground}}\ )$ both formally recognize the validity of all of PA's Π_1 theorems. Thus using a direct analog of Artemov's proposed methodology in [3, 4], they can prove each of [3, 4]'s theorems of $T_1,\ T_2,\ T_3,\ \ldots$.

The central point is that both Artemov's formalisms in [3, 4] and Willard's approach in [38]-[47] own distinct partial drawbacks. It turns out that one can essentially overcome these deficiencies by using Remark 13's theoretical hybrid methodology. In particular, the partial drawbacks which need to be addressed are listed below:

a. The partial drawback to Artemov's SBSIS-styled method [3, 4] is that it does not produce one single theorem, asserting a formalism's overall consistency. (Instead, it produces a sequence of theorems T_1, T_2, T_3, ... , whose approximate union comprises the desired *"I am consistent"* statement.)

b. The comparable drawback in Willard's self-justifying formalisms in [38]-[47]'s 28-year long series of articles is that none of these papers involve a system as powerful as Peano Arithmetic declaring its own self-consistency.

The nice aspect of Remark 13's hybridization of these two methods is that its methodology views the two formalisms from Propositions 11.a and 12.a as declaring their self-consistencies under certain unifying perspectives, while owning a formalized awareness about Artemov's broader expanding series of theorems T_1, T_2, T_3, ... This paradigm overcomes the main challenges that were posed by Items (a) and (b).

For the sake of clarity, Remark 13's observations certainly should not be viewed as being a full panacea. Thus, the Second Incompleteness Theorem certainly imposes severe restrictions upon any attempts to evade it. Thus while not fully embracing the philosophical positions of Hilbert's and Gödel's statements of $*$ and $**$, our perspective in Remark 13 certainly conveys definite strong partial support for Hilbert's and Gödel's general approach.

In other words, a 3-way "Tripod" reply for Hilbert's Second Open Problem appears to be attractive because a more compressed one-or-two leg reply to Hilbert's foundational Second Problem appears to be essentially over-simplistic.

6 Additional Curious Aspects about Self-Justification

Contrary to the impression that may have been conveyed by the 28-year-long literature about self-justification [38]-[47], these papers *were not actually* intended to completely reject the assumption that integer-multiplication is a total function. This is because several *near-cousins* of integer-multiplication are allowed within the languages of [38]-[47].

This point was first made in [38]'s initial 1993 paper. Its Item (viii) on page 326 suggested using a function called "AndMultiply(x, y, z)" The latter function first computes an integer v , derived by taking the multiplicative product of x and y , and it then computes the Bit-Wise-AND of v and z so that its final output is produced.

This "AndMultiply(x, y, z)" operation can be viewed as a non-growth operator because AndMultiply$(x, y, z) \;\leq\;$ Max(x, y, z). Thus, it can be easily incorporated into the Type-A Self-Justifying formalisms of [39, 41, 47]. This raises the following tantalizing issue:

+++ *Could the main mysteries about a formalism's inability to simultaneously recognize its semantic tableau consistency and multiplication as a total function be philosophically resolved by letting "AndMultiply" replace the classic Integer-Multiply operation?*

Furthermore, if "AndAddition(x, y, z)" designates the straightforward analog of the primitive "AndMultiply(x, y, z)" for Integer-Addition, then the AndAddition(x, y, z) operator can be easily incorporated into the Type-NS Grounding-language "ISCE" system (which did appear in both this article and in its year-2006 predecessor [42]).

These two AndAddition(x, y, z) and AndMultiply(x, y, z) operations are not as powerful as the more conventional classic Integer-Addition and Integer-Multiplication operations. However, their potential in several pragmatic engineering environments should not be under-estimated.

Also, it should be mentioned that the results in [38]-[47] can be further amplified by using [43]'s Floating-Point-With-Rounding Multiplication operation (FPWRM). This FPWRM primitive suggests [43] that floating point arithmetics, unlike classic integer arithmetics, are compatible with self-justifying logics using semantic tableau deduction.

7 Overall Perspective

The preceding discussion has focused mainly on only one of the three legs of a more elaborate Tripod Response to Hilbert's Second Problem. The other two legs, involving generalizations of Gódel's Second Incompleteness Theorem and Artemov's step-by-step SBSIS-like approach [3, 4], have been surveyed only very much more briefly. Together, these three legs formulate a triple reply to Hilbert's Second Problem, which is significantly more far-reaching than a more isolated type of one-or-two leg response.

Collectively, these three legs clarify the nature of the statements $*$ and $**$, which Hilbert and Gödel had articulated. Moreover, the Question +++ (in §6) is quite tantalizing.

In essence, Hilbert's Second Problem is so complex that an isolated one-or-two legged reply is insufficient to answer Hilbert's fascinating question. Moreover, Gerald Sacks has recalled Gödel expressing opinions that were *"almost the opposite of what every one else would have expected"* him to make (see again §2). Thus, he would presumably quite directly approve the philosophical positions that Remark 13 and Section 6 have formulataed, since Gödel repeated several analogs of his often-quoted controversial 1931 remark $**$ during numerous private conversations Gödel shared with Gerald Sacks [31].

Acknowledgment: I thank Seth Chaiken for helping improve the presentation.

References

[1] Adamowicz, Z., Zbierski, P.: On Herbrand consistency in weak theories. *Archive for Mathematical Logic* 40(6): 399-413 (2001).

[2] Artemov, S.: Explicit provablity and constructive semantics. *Bulletin of Symbolic Logic* 7(1): 1-36 (2001).

[3] Artemov, S.: The provability of consistency. *Cornell Archives arXiv Report* 1902.07404v5 (2020).

[4] Artemov, S.: Missing Proofs and the Provability of Consistency. Video presentation at the *2021 Tübingen conference "Celebrating 90 Years of Gödel's Incomppleteness Theorem".* For the video recording see: https://www.youtube.com/watch?v=Nw4nERAqi2U

[5] Artemov, S., Beklemishev, L. D.: Provability logic. In *Handbook of Philosophical Logic, Second Edition*, pp. 189-360 (2005).

[6] Beklemishev, L. D.: Reflection principles and provability algebras in formal arithmetic. *Russian Mathematical Surveys* 60(2): 197-268 (2005).

[7] Beklemishev, L. D.: Positive provability for uniform reflection principles. *Annals Pure & Applied Logic* 165(1): 82-105 (2014).

[8] Bezboruah, A., Shepherdson, J. C.: Gödel's second incompleteness theorem for Q. *Journal of Symbolic Logic* 41(2): 503-512 (1976).

[9] Buss, S. R.: *Bounded Arithmetic.* Studies in Proof Theory, Lecture Notes 3, disseminated by Bibliopolis as revised version of Ph. D. Thesis (1986).

[10] Buss, S. R., Ignjatović, A.: Unprovability of consistency statements in fragments of bounded arithmetic. *Annals Pure & Applied Logic* 74(3): 221-244 (1995).

[11] Dawson, J. W.: *Logical Dilemmas: The Life and Work of Kurt Gödel*, AKPeters Press (1997).

[12] Feferman, S.: Arithmetization of mathematics in a general setting. *Fundmenta Mathematicae* 49: 35-92 (1960).

[13] Fitting, M.: *First Order Logic and Automated Theorem Proving.* Springer-Verlag (1996).

[14] Friedman, H. M.: On the consistency, completeness and correctness problems. Ohio State Tech Report (1979). See also Pudlák [29]'s summary of this result.

[15] Gödel, K.: Über formal unentscheidbare Sätze der Principia Mathematica und verwandte Systeme I. *Monatshefte für Mathematik und Physik* 38: 349-360 (1931).

[16] Gödel, K.: The present situation in the foundations of mathematics. In *Collected Works Volume III: Unpublished Essays and Lectures* (eds. Feferman, S., Dawson, J. W., Goldfarb, W., Parsons, C., Solovay, R. M.), pp. 45–53, Oxford University Press (2004).

[17] Goldstein, R.: *Incompleteness: The Proof and Paradox of Kurt Gödel.* Norton Press (2005).

[18] Hájek, P.: Towards metamathematics of weak arithmetics over fuzzy logics. *Logic Journal of the IPL* 19: 467-475 (2011).

[19] Hájek, P., Pudlák, P.: *Metamathematics of First Order Arithmetic.* Springer (1991).

[20] Hilbert, D.: Über das unendliche. *Mathematische Annalen* 95: 161-191 (1926).

[21] Kleene, S. C.: On notation for ordinal numbers. *Journal of Symbolic Logic* 3(1): 150-15 (1938).

[22] Kreisel, G., Takeuti, G.: Formally self-referential propositions for cut-free classical analysis. *Dissertationes Mathematicae* 118: 1-50 (1974).

[23] Nelson, E.: *Predicative Arithmetic.* Mathematics Notes, Princeton University Press (1986)

[24] Nerode, A: Some useful comments by A. Nerode on Janaury 7, 2020 when he heard Willard's conference talk at the LFCS 2020 conference. Nerode said he could reinforce Gerald Sacks's YouTube presentation [31] (because Nerode heard Stanley Tennenbaum make similar comments about Gödel).

[25] Parikh, R.: Existence and feasibility in arithmetic. *Journal of Symbolic Logic* 36(3): 494-508 (1971).

[26] Paris, J. B., Dimitracopoulos, C.: A note on the undefinability of cuts. *Journal of Symbolic Logic* 48(3): 564-569 (1983).

[27] Parsons, C.: On $n-$quantifier elimination. *Journal of Symbolic Logic* 37(3): 466-482 (1972).

[28] Pudlák, P.: Cuts, consistency statements and interpretations. *Journal of Symbolic Logic* 50(2): 423-441 (1985).

[29] Pudlák, P.: On the lengths of proofs of consistency. *Collegium Logicum: 1996 Annals of the Kurt Gödel Society* (Volume 2, pp 65-86). Springer-Wien-NewYork (1996).

[30] Rogers, H. A.: *Theory of Recursive Functions and Effective Computibility.* McGrawHill (1967).

[31] Sacks, G.: Reflections on Gödel. A YouTube recorded talk delivered on 11 April 2007 as Lecture 3 in *The Thomas and Yvonne Williams Symposia for the Advancement of Logic*. The easiest way to find a video recording of the Upenn-Sacks lecture is to do a google-search on the keywords of *"Gerald Sacks, Reflections on Goedel, YouTube"*. This video is also among several recordings stored at the deposit site of:
https://itunes.apple.com/us/itunes-u/lectures-events-williams-lecture/id431294044

[32] Smullyan, R.: *First Order Logic*. Dover Books (1995).

[33] Solovay, R. M.: Injecting Inconsistencies into models of PA. *Annals Pure & Applied Logic* 44(1-2): 102-132 (1989).

[34] Solovay, R. M.: Private Telephone conversations during April of 1994 between Dan Willard and Robert M. Solovay. During those conversations Solovay described an unpublished generalization of one of Pudlák's theorems [28], using some methods of Nelson and Wilkie-Paris [23, 37]. (The Appendix A of [39] offers a 4-page summary of our interpretation of Solovay's remarks. Several other articles [10, 19, 23, 26, 28, 37] have also noted that Solovay often has chosen to privately communicate noteworthy insights that he has elected not to formally publish.)

[35] Švejdar, V.: An interpretation of Robinson arithmetic in its Grzegorczjk's weaker variant. *Fundamenta Informaticae* 81: 347-354 (2007).

[36] Visser, A.: Faith and falsity. *Annals Pure & Applied Logic* 131(1-3): 103–131 (2005).

[37] Wilkie, A. J., Paris, J. B.: On the scheme of induction for bounded arithmetic. *Annals Pure & Applied Logic* 35: 261-302 (1987).

[38] Willard, D. E.: Self-verifying axiom systems. In Proceedings of the Third Kurt Gödel Colloquium (with eds. Gottlob, G., Leitsch, A., Munduci, D.), *LNCS* vol. 713. pp. 325-336, Springer, Heidelberg (1993). https://doi.org/10.1007/BFb0022580.

[39] Willard, D. E.: Self-verifying systems, the incompleteness theorem and the tangibility reflection principle. *Journal of Symbolic Logic* 66(2): 536-596 (2001).

[40] Willard, D. E.: How to extend the semantic tableaux and cut-free versions of the second incompleteness theorem almost to Robinson's arithmetic Q. *Journal of Symbolic Logic* 67(1): 465–496 (2002).

[41] Willard, D. E.: An exploration of the partial respects in which an axiom system recognizing solely addition as a total function can verify its own consistency. *Journal of Symbolic Logic* 70(4): 1171-1209 (2005).

[42] Willard, D. E.: A generalization of the second incompleteness theorem and some exceptions to it. *Annals Pure & Applied Logic* 141(3): 472-496 (2006).

[43] Willard, D. E.: On the available partial respects in which an axiomatization for real valued arithmetic can recognize its consistency. *Journal of Symbolic Logic* 71(4): 1189-1199 (2006).

[44] Willard, D. E.: Passive induction and a solution to a Paris-Wilkie open question. *Annals Pure & Applied Logic* 146(2): 124-149 (2007).

[45] Willard, D. E.: Some specially formulated axiomizations for $I\Sigma_0$ manage to evade the Herbrandized version of the second incompleteness theorem, *Information and Computation* 207(10): 1078-1093 (2009).

[46] Willard, D. E.: On how the introducing of a new θ function symbol into arithmetic's formalism is germane to devising axiom systems that can appreciate fragments of their own Hilbert consistency. *Cornell Archives arXiv Report* 1612.08071 (2016).

[47] Willard, D. E.: About the Characterization of a Fine Line That Separates Generalizations and Boundary-Case Exceptions for the Second Incompleteness Theorem under Semantic Tableau Deduction", *Journal of Logic and Computation* 31(1):375-392 (2021).

[48] Yourgrau, P.: *A World Without Time: The Forgotten Legacy of Gödel and Einstein.* (See page 58 for the passages we have quoted.) Basic Books (2005).