On the Revival of a Modified and Diluted Version of Hilbert's Consistency Program (Extended Abstract)

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Summary of Results: This article is a continuation of a series of results [52]–[62] that the author has published about generalizations and boundary-case exceptions to the Second Incompleteness Theorem. It will show how to revive a type of diluted version of Hilbert's Consistency Program.

Our results will obviously fall short of being full-scale evasions of the Second Incompleteness Theorem (since the robustness of Gödel's result is unquestionable). They will, be much stronger than our earlier evasions of Gödel's result, however. They will show that a **STRICTLY FINITE SIZED** axiom system can simultaneously:

1. recognize its own consistency under a Hilbert-style method of deduction,
2. prove "simulated" analogs of all of Peano Arithmetic's Π_1 like theorems
3. and recognize the existence of the full set of integers 0,1,2,3....

An earlier "ISINF" formalism in our year-2006 article [59] could be trivially modified to acheiving these tasks — in a context where its proofs were typically longer than the number of atoms residing in the universe. The remarkable aspect of our new "IS.Terse" system is that it will use normal-sized proofs that can fit into the memory space of a human brain or computer in a perfectly pragmatic manner.

By this we mean that there will never be more than a small Polynomial increase in length between the size of an initial proof of a Π_1 theorem of Peano Arithmetic and its "Π_1^E" counterpart under our self-justifying and finitized "IS.Terse" system.

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1 Introduction

It is well known that Gödel established two historic results in his centennial 1931 paper [18]. The First Incompleteness Theorem thus showed that there exists no decision procedure for identifying the true statements of Arithmetic. The Second Incompleteness Theorem demonstrated that conventional axiom systems cannot recognize their own consistency. Moreover, Hilbert-Bernays developed in 1939 a historic generalization of the Second Incompleteness Theorem, centering around their foundational Derivability Conditions [20, 21, 30].

It is fascinating that Hilbert, as the co-author of one of the most historic generalizations of the Second Incompleteness Theorem, chose to have the motto of his consistency program engraved on his tombstone:

> * "Wir mussen wissen. Wir werden wissen" (English translation: "We must know; We will know")

Also, the mystery enshrouding the Second Incompleteness Theorem is further amplified by the fact that Gödel was explicitly uncertain about the generality of the Second Incompleteness Theorem in his initial 1931 seminal paper [18] about this subject. His centennial paper thus included the following quite poignant caveat:

> ** "Theorem XI (the Second Incompleteness Theorem) represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and *there might conceivably be finite proofs* which cannot be stated in ... "

Thus, there have always been doubts, since the time of its inception, about whether or not the Second Incompleteness Theorem might allow for the existence of at least boundary-case exceptions to its formalism.

The author of this article has published a series of this paper during the last nineteen years [52]–[62] that have sought to both generalize the Second Incompleteness Theorem and explore its permissible boundary-case exceptions. There can be no doubt that both Gödel's original 1931 variant of the Second Incompleteness Theorem and the Hilbert-Bernays 1939 generalization of it [20, 21, 30] shall eternally mathematically overshadow any boundary-case exceptions to it.

But yet there is the vexing philosophical question as to how do human beings overcome the mysteries in the Tree of Knowledge and maintain enough *instinctive faith (?)* in their own consistency to motivate themselves to cogitate? In other words, how strong may a boundary-case exception to the Second Incompleteness Theorem become before it encounters the inevitable barriers that cannot be surpassed ?

The current paper is intended to be a sequel to our prior articles [52]-[62]. It will show that for any r.e. axiomatization of arithmetic α , it is possible to devise a **STRICTLY FINITE SIZED** axiom system, called IS.Terse(α) that can simultaneously:

1. recognize its own consistency under a Hilbert-style method of deduction,
2. prove "simulated" analogs of all of α's Π_1 like theorems
3. and recognize the existence of the full infinite set of integers 0,1,2,3.... via the employment of a finite number of starting axioms and constants.

Our earlier year-2006 paper [59] had used the term **Infinite Far Reach** to describe the above property (3). Its "ISINF" formalism came close to performing tasks 1-3 — in a context where we called its proofs "unnatural" because they were typically longer than the number of atoms residing in the universe. The remarkable aspect of our new "IS.Terse" system is that it will avoid this deleterious exponential growth by assuring that IS.Terse(α) can simulate α's Π_1^E knowledge with no more than a polynomial increase in proof length.

The main significance or our results will be explained in the next chapter. It is because a generalization of the Second Incompleteness Theorem, arriving from the combined work of essentially Pudlák, Solovay, Nelson and Wilkie-Paris [32, 36, 42, 51], had demonstrated that arithmetic axiom systems, recognizing merely successor as a total function, are unable to prove a theorem affirming their own Hilbert consistency.

We will overcome such difficulties by employing an Ajtai-like Pigeon Hole formalism [4], that had been privately communicated to us about 10-11 years ago by Pavel Pudlák [38], as a potential replacement for the successor operation, A summary of Pudlák's private communications and his open question can be found in Section 6 of our article [59].

The theme of our current Year-2012 research will essentially be that the hopes, raised by Hilbert and Gödel in the statement $*$ and $**$, can be partially vindicated via the hybridizing of [52]–[62]'s *"I am consistent"* axiom statements with Pudlák's suggested deployment [38] of Ajtai's Pigeon Hole [4] mathematics. Our new results, about self-justification, will thus exceed the prior results of [52]–[62] by pertaining to Self Justifying axiom systems that include *a robust* Modus Ponens rule.

2 Overall Formalism

Let β denote an axiom system, and d denote a deduction method. The ordered pair (β, d) will be called **Self Justifying** when:

i one of β's theorems states that the deduction method d, applied to the system β, will produce a consistent set of theorems, and

ii the axiom system β is in fact consistent.

For any (β, d), it is easy to construct a second axiom system $\beta^d \supseteq \beta$ that satisfies Part-i of this definition. For instance, β^d could consist of all of β's axioms plus the following added sentence, that we call **SelfRef**(β, d)

- There is no proof (using d's deduction method) of $0 = 1$ from the *union* of the axiom system β with *this* sentence "SelfRef(β, d)" (looking at itself).

Kleene [24] discussed how to encode approximate analogs of SelfRef(β, d)'s self-referential statement. Each of Kleene, Rogers and Jeroslow [24, 39, 22] noted β^d may, however, be inconsistent (despite SelfRef(β, d)'s assertion), thus causing it to violate Part-ii of self-justification's definition.

This problem arises in settings more general than Gödel's paradigm, where β was an extension of Peano Arithmetic. There are many settings where the Second Incompleteness Theorem does generalize [1, 2, 3, 9, 10, 11, 13, 16, 18, 20, 21, 26, 28, 35, 36, 37, 41, 42, 44, 46, 48, 50, 51, 55, 59, 56, 60]. Each such result formalizes a paradigm where self-justification is infeasible, due to a diagonalization issue. Most logicians have hesitated to thus employ a SelfRef(β, d) axiom because β+SelfRef(β, d) is usually inconsistent. [1].

Our research explored special circumstances [54, 57, 58, 59] where it is feasible to construct self-justifying formalisms. These paradigms involved weakening the properties a system can prove about addition and/or multiplication (to avoid the preceding difficulties). To be more precise, let $Add(x, y, z)$ and $Mult(x, y, z)$ denote two 3-way predicates indicating x, y and z satisfy $x + y = z$ and $x * y = z$. A logic will be said to **recognize** successor, addition and multiplication as **Total Functions** iff it includes 1-3 as axioms.

$$\forall x \; \exists z \quad Add(x, 1, z) \tag{1}$$

$$\forall x \; \forall y \; \exists z \quad Add(x, y, z) \tag{2}$$

$$\forall x \; \forall y \; \exists z \quad Mult(x, y, z) \tag{3}$$

[1] Typical ordered pairs (β, d) will have the property that the broader axiom system $\beta^d = \beta +$ SelfRef(β, d) will be inconsistent, even when β is consistent. This is because a standard Gödel-like self-referencing construction will typically produce a proof of $0 = 1$ from β^d, irregardless of whether or not β is consistent.

We will say a logic system α is **Type-M** iff it contains each of (1) – (3) as axioms, **Type-A** iff it contains only (1) and (2) as axioms, and **Type-S** iff it contains only (1) as an axiom. A system is called **Type-NS** iff it *does not* contain any of these axioms.

Our investigations [52]–[60] began by observing that some Type-A systems can recognize their consistency under semantic tableaux deduction, and several Type-NS systems can recognize their Hilbert consistency. We communicated these results to Robert Solovay during a series of telephone conversations during April of 1994. Shortly after Solovay learned about [52]'s evasion of the Second Incompleteness Effect, he privately communicated to us that he knew [42] how to combine the work of Pudlák, Nelson and Wilkie-Paris [32, 36, 51] to establish the following hybridized theorem:

Theorem 2.1 *(Solovay's 1994 Generalization [42] of a 1985 theorem of Pudlák [36] using some of Nelson and Wilkie-Paris [32, 51]'s methods)* : Let α denote any axiom system which contains Equation (1)'s Type-S statement and which thus assures the successor operation x' always satisfies

$$\forall x \ \ \forall y \quad [\ x' \neq 0 \quad \text{AND} \quad (\ x' = y' \ \Leftrightarrow \ x = y \ ) \ ] \tag{4}$$

Then α will be unable to recognize its own Hilbert consistency, whenever it treats addition and multiplication as 3-way relations satisfying their usual identity, associative, commutative and distributive properties (encoded in the obvious manner as Π_1 styled statements via the Add(x,y,z) and Mult(x,y,z) predicates).

Solovay never published any precise proof of Theorem 2.1's hybridizing of the work of Pudlák, Nelson and Wilkie-Paris [36, 32, 51], which he privately communicated [42] to us. A reader can find generalizations of the Second Incompleteness Theorem that are closely related to Theorem 2.1 in papers by Pudlák, Buss-Ignjatovic, Švejdar and Willard [11, 36, 44, 56], as well as in the Appendix A of [54].

Harvey Friedman [15] also deserves credit for developing a result related to Theorem 2.1, as early as 1979. Thus if $\text{Cons}_\alpha(n)$ denotes a "finistic consistency statement" which declares that there is no proof of 0=1 from the axiom system α *whose length is less than* n. then Friedman [15, 16] observed that for some constant $\epsilon > 0$, most consistent systems α require a bit-length at least as large as $O(\ n^\epsilon \ )$ to prove $\text{Cons}_\alpha(n)$. A partially analogous result was later reconstructed by Pudlák [35]. Also, [37] provides an excellent survey of the prior literature (up to the mid-1990's), including a review of Friedman's early 1979 results in [14, 15].

It should be mentioned that Willard had published several results about generalizations of the Second Incompleteness Theorem that closely complement our published boundary-case exceptions to it. Thus, [53, 55, 56, 57, 60] examined the Second Incompleteness Theorem in a semantic tableaux context, by generalizing pioneering theorems of Adamowicz-Zbierski and Wilkie-Paris's about $I\Sigma_0$ $[1, 3, 51]$. Also, our paper [61] was stimulated by a fascinating observation by L. A. Kołodziejczyk [25, 26], about the difference in lengths between semantic tableaux and Herbrandized proofs. It used Kołodziejczyk observations to show that some self-justifying Herbrandized systems [61] are surprisingly compatible with a full-fledged multiplication *function* symbol. We point out in [63] that its research into the "Translational Reflection Principle" nice complements reflection principles studied by Beklemishev, Kreisel-Takeuti and Verbrugge-Visser [6, 7, 8, 28, 46, 50].

A reader can find a more detailed summary of the prior literature in [57, 59, 62]'s literature survey chapters. The current article will be largely self-contained and not technically need an examination of the prior literature. This is because *all that is needed* for appreciating our new results will be the formal statement of Theorem 2.1. (Its proof is not even necessary to understand.) The two key points that should be remembered, when examining the remainder of this article, are that:

1. Theorem 2.1 represents essentially the Second Incompleteness Theorem's analog of a lighthouse. It thus warns investigators about which passages are too treacherous to traverse, and

2. Our main boundary-case exception to the Second Incompleteness Effect, in Theorem 4.17, will serve as a beacon, employing golden data stemming from Theorem 2.1's lighthouse, to

enable the scientific community to rekindle a moderated and diluted version of Hilbert's consistency program.

In combination, these two perspectives will lead to a new interpretation of the caveats $*$ and $**$, that Hilbert and Gödel had pronounced about the Second Incompleteness Theorem (e.g. see Section 1).

Our results will be much more pristine in this paper than most of our prior research because they will concern Hilbert-style deduction (rather than a lesser technology). They will also be happily compatible with a **STRICT FINITE LIMIT** on the size of the deployed self-justifying axiom systems.

3 Notation and Basic Concepts

A function F will be called **Non-Growth** iff $F(a_1, a_2, ...a_j) \leq Maximum(a_1, a_2, ...a_j)$ for all $a_1, a_2, ...a_j$. Six examples of non-growth functions are:

1. *Integer Subtraction* where " $x - y$ " is defined to equal zero in *the special case* where $x \leq y$,
2. *Integer Division* where " $x \div y$ " equals x when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise,
3. $Root(x, y)$ which equals $\lceil x^{1/y} \rceil$ when $y \geq 1$, and it equals x when $y = 0$.
4. $Maximum(x, y)$,
5. $Logarithm(x) = \lfloor \text{Log}_2(x) \rfloor$ when $x \geq 2$, and zero otherwise.
6. $Count(x, j) =$ the number of "1" bits among x's rightmost j bits.

These operations were called **Grounding Functions** in [54, 56, 57, 62]. Our starting language L^0 will also contain two atomic relations of " $=$ " and " $\leq$ " and three built in constants symbols c_0 , c_1 and c_2 , representing the values of 0, 1 and 2. These constant symbols allow us to formalize the additional functions of

a. Pred(x) $=$ $x-1$ (Thus, Item 1's definition for Subtraction implies Pred(0)= 0.)

b. Half(x) $=$ $\frac{x}{2}$ (in the context of Item 2's definition for Division).

c. $\text{Pred}^n(x)$ defined to be n iterations of the Predecessor operation

d. $\text{Half}^n(x)$ defined to be n iterations of the halving operation.

One awkward aspect of this notation is that it provides no guarantee that integers larger than 2 will exist without the presence of some further methodology for producing larger integers.

One method for solving this problem was presented in [59]. It employed an infinite number of further constant symbols. The latter's ISCE(A) system was compatible with self-justification, but such an infinite number of constant symbols clearly trespassed on Hilbert's goal of using a finite-sized formalism.

The "ISINF" formalism of [59] offered an alternate method for resolving this difficulty. It required the use of *only three* constant symbols. It could prove analogs of all Peano Arithmetic's Π_1 theorems, but almost all its proofs unfortunately had lengths longer than the number of atoms in the universe. We need a new definition to explain why ISINF is of interest despite its obvious eye-squinting limitations:

Definition 3.1 *An axiom system* α *will be said [59] to have* **Infinite Far Reach** *iff for each natural number* n *, the system* α *can prove a theorem of the form*

$$\exists \ x \quad Pred^n(x) \ = \ 1 \tag{5}$$

The ISINF framework in [59] was a self-justifying system with Infinite Far Reach. The opening paragraph of [59]'s Section 6, was quite frank about ISINF's limitations. It used the word "unnatural" to describe the ISINF system. Such a deliberately self-deprecating term was appropriate because proofs of trivial theorems under ISINF were typically much longer than the number of atoms in the universe.

The reason ISINF was worthy of mention, despite this obvious limitation, is that ISINF's formalism, demonstrated there existed a manner to evade the Theorem 2.1's generalization of the Second Incompleteness Theorem, under Section 2's notion of a Type-NS formalism. In other words, ISINF's minuscule evasion of the Second Incompleteness Theorem was significant because it showed Theorem 2.1 did have loopholes (albeit the most *very, very, very tiny* eye-squinting types of ones).

Pudlák appreciated the nature of the challenge we faced. (This is because he was one of the main architects of Theorem 2.1's formalism, and we had shared several private communications during approximately 1995-2005.) In one of those communications, Pudlák suggested [38], that we consider using an Ajtai-like Extender function to refine our analysis [2] .

Ajtai's research [4] had demonstrated that if one merely started with the initial integer 0 as a starting object, then one could essentially construct an infinite set of further integers, with a primitive that we will call an Ajtai-style **Extender Function**. It will be denoted as $\lambda(x)$. It will jump through the set of integers in an almost *haphazard manner* because the *sole equations* defining $\lambda(x)$ will be (6) and (7):

$$\forall \ x \quad \lambda(x) \neq 0 \tag{6}$$

$$\forall \ x \ \ \forall \ y \quad x \neq y \ \Rightarrow \ \lambda(x) \neq \lambda(y) \tag{7}$$

Thus for each integer $i > 0$, it is plausible that $\lambda(x) = i$ (because $\lambda(x)$ is accompanied by *no other defining equations* besides Equations (6) and (7)). There are an uncountable number distinct functional operations λ that can satisfy Equations (6) and (7), by an easy application of Cantor's diagonalization theory.

[2]We had received a similar suggestion that we consult Ajtai's research [4] from an anonymous NSF referee before receiving Pudlák crucial private communication. This anonymous referee's comments were only one sentence long and difficult to decipher. (They were quite different from Pudlák emailed private communications [38], which were almost 1 page long.) The anonymous referee's 1-sentence suggestion obviously displayed significant insight. It did not influence my research because my grant proposal was ultimately rejected, and I could not decipher the anonymous referee's terse 1-sentence comment until I later received Pudlák's email [38].

Thus, Pudlák's insightful email asked the question as to what extent an Ajtai-style Jump-Function could be used to refine our prior work about self justification ? It took us roughly 11-12 years to solve this problem because the uncountable number of entities that satisfy λ's defining equations would throw several awkward (and very confusing) monkey wrenches into our prior proof-methodologies.

Pudlák's suggested research topic was useful because Atjai's Extender function λ can feasibly grow at a slower rate than successor (as the footnote [3] documents). Thus after almost one dozen years of research into this topic, we discovered how one can apply Equations (6) and (7) to streamline [59]'s "ISINF" self-justifying system, so as to improve aspects of the efficiency of its self-justifying results.

Pudlák's suggested revision of [59]'s "ISINF" formalism, thus, turned out to be unquestionably useful. It still, however, fell short of achieving our main objectives. This is because its proofs were sharply shorter than ISINF's counterparts, but **still amazingly longer** than the number of atoms in the universe, for most simple Π_1 and Σ_1 style theorems (due to the bulky nature of a $\lambda(x)$ function).

The next part of this section will explain how we can achieve the basic aspirations, expressed by Hilbert and Gödel in Section 1's statements of $*$ and $**$, by replacing the Ajtai-style Extender Construct $\lambda(x)$ with a more complicated **Super-Extender** construct, called $\zeta(x)$.

3.1 "Up-Walking" under Super-Extender Theory

We will often avoid use the abbreviated symbols of "0", "1" and "2" to informally designate the built-in "constant names" of " c_0 ", " c_1 " and " c_2 ". Throughout this paper, the number " 1 " will be considered to be a power of 2 since $2^0 = 1$. Also the symbol of "Log(x)" will denote our "Grounding Logarithm" function (formally defined by Item 5 in Section 3's opening

[3]The operation $\lambda(x)$ will grow at a slower rate than Successor if it equals $x+1$ for all standard numbers x and if $\lambda(x) = x - 1$ when x is a non-standard integer. This seemingly minute detail causes Theorem 2.1's generalization of the Second Incompleteness Theorem to collapse for some non-standard interpretations of λ.

paragraph).

Definition 3.2 *The symbol Power(x) will denote that x is power of 2 . It is formally expressed by the statement (8) under our Grounding language notation:*

$$x = 1 \quad \vee \quad Log(x) \neq Log(x) - 1 \tag{8}$$

The symbol $\zeta(x)$ will essentially be the analog of the $\lambda(x)$ function that walks among the powers of 2 in a manner analogous to $\lambda(x)$'s haphazard walk through conventional integers. It will thus satisfy the axiomatic constraints below (which are $\zeta(x)$'s analog of the more modest constraints given in (6) and (7)). The most important difference between these two constructs is that axiom (9) requires that $\zeta(x)$ maps power of 2 onto powers of 2.

$$\forall\ x \quad \text{Power}(x) \quad \Rightarrow \quad \text{Power}(\ \zeta(x)\ ) \tag{9}$$

$$\forall\ x \quad \zeta(x) \neq 1 \tag{10}$$

$$\forall\ x\ \ \forall\ y \quad x \neq y \ \Rightarrow\ \zeta(x) \neq \zeta(y) \tag{11}$$

It needs to be emphasized that the sentences (9) – (11) will be the *only vehicle* our self-justifying axioms will have available to construct integers ≥ 3. They will be henceforth called the Up-Walking axioms.

There will, thus, be no other means for defining the operation $\zeta(x)$, besides (9) – (11)'s "Up-Walking" axioms. The operation $\zeta(x)$ is, thereby, similar to $\lambda(x)$, insofar as there are an *uncountably* infinite different number of eligible representations of the function " ζ ".

This latter fact is significant. It will mean that our efforts to rekindle a modified (and admittedly diluted) form of Hilbert's consistency program will ultimately amount to flirting with the maze of diagonalization paradoxes, which arise in what Hilbert called "Cantor's Paradise".

3.2 The Implications of (9)–(11)'s Up-Walking axioms for $\zeta(x)$

This section will prove baby lemmas about $\zeta(x)$ function that will explain why it is more useful than $\lambda(x)$.

In our discussion, the term **E-Grounding Language** will be defined to be a language identical to the Grounding language L^0, defined in Section 3's opening paragraph, except that it will employ the further functional operation of $\zeta(x)$ as its only allowed growth operation. This language will be henceforth denoted as L^E. Also, our discussion will employ the following notation:

1. The symbol $\zeta^n(x)$ will denote a primitive that applies n iterations of the ζ function to an initial input of x. Thus, it is the operation of $\zeta(\ \zeta(\ ...\zeta(\ x\ ))).)$.

2. The symbol a_n denotes the quantity of $\zeta^n(1)$.

3. The symbol b_n denotes the quantity of $\text{Max}(a_1, a_2, ...a_n)$. (This quantity can be formally denoted using L^E's language machinery because Section 3's opening paragraph indicated that Maximum was one of our allowed function primitives.)

Lemma 3.3 *For the sake of simplicity, let us presume that the function " ζ " always maps a standard integer x onto another standard integer $\zeta(x)$. (We will later see how this assumption is useful.) Then for each non-negative integer j, it is possible to encode a term in the E-Grounding language,* called E_J, *that uses no more than $O(\ j^2\ )$ symbols to represent the quantity of 2^j.* (Also, the symbol E_j will use more than j^2 symbols for representing the quantity of 2^j, by a trivial argument. Thus our $O(j^2)$ upper bound on E_j's length will be accompanied by a matching $\Omega(j^2)$ lower bound.)

Proof: The justification of Lemma 3.3 is easy. Its main results (the $O(\ j^2\ )$ upper bound) shall have its proof divided into two cases:

The Case where $j\ =\ 0$ or 1. Easy because then E_j can be formally represented by the symbols c_1 or c_2 that represent the quantities of "1" and "2".

The Case where $j \geq 2$. The application of (9)–(11)'s Up-Walking axioms to Item 3's object

b_j implies:

$$b_j \geq 2^{j+1} \tag{12}$$

Equation (12) and our definitions for the Division and the Halving operations then implies that we can write E_j as a term using $O(\ j^2\ )$ symbols via (13)'s formalism.

$$E_j \quad =_{\text{def}} \quad \{\ b_j \ \div\ \text{Half}^j(b_j)\ \} \tag{13}$$

Again, we emphasize that our $O(\ j^2\ )$ upper bound on E_j's length is accompanied by a matching $\Omega(\ j^2\ )$ lower bound (because E_j employs more than j^2 symbols for its encoding in all but a finite number of cases.) □

Lemma 3.4 *For an arbitrary standard number* n *, it is possible to write a term in the E-Grounding language,* called T_n *, that uses no more than* $O(\ 1{+}Log^3 n\ )$ *symbols to represent the quantity of* n. (For $n > 2^5$, it will require at least $Log^2(n)$ symbols to represent the quantity of n .)

Proof: Lemma 3.4 is a fairly easy consequence of Lemma 3.3. Its proof is also divided into two cases:

The Case where $n \leq 2$. Easy because we again use one of the three built-in constant symbols c_0 , c_1 and c_2 to represent one of the three quantities of 0, 1 and 2.

The Case where $n \geq 3$. Let j denote the least integer such that $2^j \geq n$. Then one of the objects used by our encoding of the term T_n will be term E_n (defined under Lemma 3.3's formalism). The remainder of the term T_n for formalizing n's value is obtained by subtracting from E_n the required subset among the objects E_0 , E_1 , E_2 ... E_{n-1} needed to calculate n's value. (Such an encoding for T_n clearly involves no more than $O(\ 1{+}\text{Log}^3 n\ )$ symbols). □

The proof of the baby results in Lemmas 3.3 and 3.4 were, of course, quite straightforward. The importance of these results, however, should not be underestimated. No analogs of Lemma

3.4's $O(1+\text{Log}^3 n)$ symbol-length are feasible when $\lambda(x)$'s alternate formalism under Equations (6) and (7) is employed. (This is because these defining axioms do not employ an analog of Equation (9)'s constraint.)

This issue is crucial because an integer n would essentially require in excess of n symbols for encoding it, if we used $\lambda(x)$'s formalism to encode it. The latter would mean that a proof employing L symbols, whose Gödel encoding would correspond to a number $n > 2^L$ would require more symbols than there are atoms in universe when merely $L > 200$. In contrast, Lemma 3.4's PolyLog constraint will cause its formalization of a Length-L proof to have a much more compact encoding using no more than Polynomial(L) symbols.

The Lemmas 3.3 and 3.4 will thus be central to our formalisms in this article because we will wish our main IS.Terse(α) self-justifying systems to **BOTH** simulate the base axiom system α's Π_1 style knowledge **AND TO HAVE** its simulations of α's knowledge involve no more than a polynomial expansion of α's initial proof lengths.

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It will thus launch a very lengthy train of theorems that will provide at least a partial (and certainly a quite diluted) positive answer to a variant of Hillbert's fascinating Second Open Question [4] .

The remainder of this paper will follow Hilbert's advice about following a maze of diagonalization theorems into what Hilbert had called "Cantor's Paradise" — in order to explore certain offshoots and side-issues that are related to Hillbert's Second Open Question.

Thus while "applied" mathematicians have traditionally considered uncountable sets to possess questionable significance, we will use the fact that there are an *uncountably* infinite

[4] This is intuitively because Lemma 3.4 will allow one to encode an arbitrary integer n , with a term T_n whose PolyLog(n) length is sharply shorter than the exponentially longer objects that the $\lambda(x)$ function would produce. It is unnecessary for us to provide any proofs, in this footnote, because its sole purpose is to explain our motivation for developing Lemma 3.4's formalism.

number of different eligible ζ−style functions, all satisfying (9)–(11)'s Up-Walking axioms, as a winning point to help our train of theorems engage in a dizzying Cantor-like diagonalization journey to further explore added details about the precise meaning of certain mutated versions of Hilbert's Second Open Question.

3.3 The Properties of Π_n^E and Σ_n^E Formulae

The analogs of classic meta-arithmetic Π_n and Σ_n formulae [20] in our language L^E will be called Π_n^E and Σ_n^E expressions. This section will define these expressions and also introduce some related useful notation:

Definition 3.5 An **E-Unit** will be defined to be an object that is either a variable, one of our three built in constants of c_0, c_1 or c_2, or one of Lemmas 3.3's and 3.4's terms of E_j and T_n. Also, an **E-Term** will be either an E-Unit or a functional object, built out of E-Units, variable names and the *six non-growth* Grounding functions (that were defined in Section 3). This definition is subtle. The footnote [5] explains exactly why E-terms are defined in the exactly manner that they are.

The phrase **E-Bounded Quantifier** will refer to expressions of the form "$\exists\ v \leq t$" or "$\forall\ v \leq t$" where t is an E-term. A formula will be called Δ_0^E when all its quantifiers are so bounded and all its unbounded terms are also **FORMALLY** E-terms.

Our formal definition of Π_n^E, and Σ_n^E formulae will rely upon the above notation in a straight-forward manner. It is given below:

Definition 3.6 Every Δ_0^E formula will be automatically considered to be also a Π_0^E and Σ_0^E formula. For $n \geq 1$, our " Π_n^E " and " Σ_n^E " formulae will be defined by the the following quite conventional rules:

[5] The definition of an E-term prevents it from using ζ's function symbol in any part of this term, except when enclosed within one of the designated subterms of " E_j " and " T_n ". (This seemingly minor caveat is a non-trivial matter. It will be essential to establish our main results.)

1. A formula will be called Π_n^E iff it can be written in the canonical form of

   $\forall v_1 \, \forall v_2 \, ... \, \forall v_k \quad \Phi(v_1, v_2, ..v_k),$ where Φ is Σ_{n-1}^E.

2. Likewise, a formula will be called Σ_n^E iff it can be written in the canonical form of

   $\exists v_1 \, \exists v_2 \, ... \, \exists v_k \quad \Phi(v_1, v_2, ..v_k),$ where Φ is Π_{n-1}^E.

Our main goal will be to study self-justifying axiom systems that can prove all Π_1^E theorems of any arbitrary initial consistent r.e. base axiom system (including the possibility that this base be Peano Arithmetic). Our core theorems, in the next several sections, will also employ the following lemma:

Lemma 3.7 *The language L^E is able to encode two Δ_0^E formulae, called Add(x, y, z) and Mult(x, y, z), that formalize the two 3-way relations of $x + y = z$ and $x * y = z$.*

Proof. Easy because these two relations are encoded by the following two Δ_0^E formulae:

$$z - y = x \quad \wedge \quad z \geq x \tag{14}$$

$$[\ (x = 0 \vee y = 0) \Rightarrow z = 0\ ] \ \wedge \ [\ (x \neq 0 \wedge y \neq 0\ ) \ \Rightarrow \ (\ \frac{z}{x} = y \ \wedge \ \frac{z-1}{x} < y \ \ )\ ] \tag{15}$$

Remark 3.8 The proofs of Lemmas 3.3, 3.4 and 3.7 were all, of course, all quite straightforward. These results are significant, however, because they collectively represent the first three steps of our method for rekindling a type of modified version of Hilbert's consistency program.

4 Summary of Main Mathematical Perspective

The symbol **NN** will denote the set of non-negative integers in the standard model of the natural numbers, throughout this paper. We already noted that there will be an uncountably infinite number of different eligible ζ function-objects that will satisfy the requirements of (9)–(11)'s Up-Walking axioms. Such a function will be called **E-Regular** iff it maps standard numbers always onto standard numbers (as is formally defined below).

$$\forall \ x \in \text{NN} \quad \Rightarrow \quad \zeta(x) \in \text{NN} \tag{16}$$

Unless otherwise specified, we will always presume that ζ is E-regular.

It also should be remarked that Equation (16) is a set-theoretic declaration that lies outside the prior section's language L^E. (This because it uses the " $\in$ " symbol.) It is thus intended to convey intuitions to the reader, that lie outside the formal domains of strictly "arithmetic" axiom systems, such as the IS.Terse and IS.Pure formalisms (defined later).

Definition 4.1 Let t denote one of Definition 3.5's E-terms. It will be called **E-Simple** iff it contains no variables as inputs. (Thus, its sole atomic inputs are either one of the three built-in constants of c_0, c_1 and c_2 or one of Lemmas 3.3's and 3.4's terms of E_j and T_n.)

In addition to discussing E-Simple terms, we will also refer to **G-Simple Terms**. The latter will be defined to be any term built out of our three built in constants of c_0, c_1 and c_2, connected in an arbitrary manner by the six non-growth functions in our language L^E and by its seventh functional object of " ζ ".

We have used the letter "G" as a helpful mnemonic because "G-Simple Terms" are a generalization of the "E-Simple" concept. This is because an E-Simple term is allowed to employ the functional object of " ζ " only within the confines of one of Lemma 3.4's terms of T_n. In contrast, G-Simple Terms can employ the symbol " ζ " *anywhere.*

Definition 4.2 A G-Simple Term will be said to be **E-Fixed** iff it produces the same integer-output for all functional objects " ζ " that are E-regular. (This construct will be crucial in this paper because Lemma 4.3 will show all E-Simple terms are a E-Fixed, *but the same is not true* for G-Simple terms.

Lemma 4.3 *Every E-Simple term is E-Fixed, but many G-Simple terms* **FAIL TO** *be E-Fixed.*

Proof. We have set up the notation in our discussion so that Lemma 4.3 is a straightforward consequence of our prior notation. The E-Fixed property of all E-Simple terms is thus an easy consequence of Lemmas 3.3 and 3.4, combined with the definition of an E-Simple term.

In contrast, the G-Simple objects of $\zeta(1)$, $\zeta(2)$, $\zeta(4)$, $\zeta(8)$ ... are allowed to represent any set of distinct powers of 2 (all greater than 1), under various choices of different eligible ζ functions. This makes it it obvious that many G-Simple terms *are certainly not* E-Fixed. □

Remark 4.4 We close this section by reminding the reader that Π_1^E sentences are a significant topic because Lemma 3.7 implies that they can formalize the 3-way relations of addition and multiplication as Δ_0^E formulae. (Thus , Lemma 4.3's unusual notational mechanism is *certainly germane* to classic arithmetic, since every conventional integer n can be formalized by an E-Fixed term of " T_n " and addition and multiplication can receive Δ_0^E encoding via Lemma 3.7's machinery.)

4.1 Definition of the "IS.Pure" Axiom System

Most of the self-justifying axiom systems in [52, 54, 57, 59] had employed names that began with the letters "IS". This 2-letter sequence was an acronym for "Introspective Semantics".

The main two axiomatic frameworks, defined in this article, will be called "IS.Pure" and "IS.Terse". They will possess differing advantages.

Given any initial axiom system α , our IS.Pure(α) formalism will duplicate α's Π_1^E knowledge in a stronger sense. In contrast, IS.Terse(α) will be desirable in Hilbert-like settings, where a self-justifying system is sought to have a *strictly defined* finite cardinality. Thus, our "IS.Terse" formalism will essentially formalize a trade-off where α's Π_1^E knowledge is diluted from a "pure" to a "simulated" state, so that the number of deployed proper axioms is strictly finite.

Both the IS.Pure(α) and IS.Terse(α) axiom systems will be comprised of three axiom-groups, called the Group-1, Group-2 and Group-3 schemas. This section will define the three axiom groups of the IS.Pure(α) formalism. The next section will then outline how one may incrementally modify this formalism to define IS.Terse(α).

We will start our description of IS.Pure(α) by defining its "Group-1" axioms. The following notation will be helpful during our discourse.

Definition 4.5 A Δ_0^E formula will be said to be **Purely Closed** iff it contains no free variables, and it also excludes all appearances of the ζ function symbol outside its appearance in one of Definition 4.2's "E-unit" terms of T_n and E_j .

Example 4.6 The formula (17) is clearly Purely-Closed, since it contains no use of the ζ function symbol beyond the internal confines of its " E_5 " term.

$$\forall\, x \ \leq \ E_5 \ \ \exists\, y \ \leq \ x-1 \qquad y = \frac{x}{2} \tag{17}$$

In contrast, (18) fails to be Purely-Closed because its term " $\zeta(E_5)$ " term uses " ζ " as an external object.

$$\forall\, x \ \leq \ \zeta(E_5) \ \ \exists\, y \ \leq \ x-1 \qquad y = \frac{x}{2} \tag{18}$$

Likewise, (19) fails to be Purely-Closed because it contains a free variable of v .

$$\forall\, x \ \leq \ v \ \ \exists\, y \ \leq \ x-1 \qquad y = \frac{x}{2} \tag{19}$$

Remark 4.7 At an intuitive level, the set of Purely-Closed Δ_0^E formula represents the minimum knowledge that one would want every arithmetic system, using L^E's language, to display. Thus, it is imperative that the " ζ " function symbol appear within the internal confines of the objects of T_n and E_j because our formalism would otherwise be unable to formally represent integers $\geq\ 3$. On the other hand, *it is unnecessary* to use the " ζ " function symbol anywhere else within a Δ_0^E formula besides these places, since Lemma 3.7 indicated that addition and multiplication have Δ_0^E encodings in L^E's language. This means that it is possible for a straightforward and trivial primitive recursive algorithm to translate all of classic arithmetic's (conventionally closed) Δ_0 formula into equivalent "Purely-Closed" Δ_0^E encodings. The set of Purely-Closed Δ_0^E, thus, captures a very non-trivial amount of arithmetic's traditional information.

Lemma 4.8 *There exists in L^E's language a consistent axiom system, called G_1, that can prove every Purely-Closed Δ_0^E formula that holds true under NN's Standard model, where* G_1 **consists of merely** *the union of (9)–(11)'s "Up-Walking" axioms with* **a strictly finite** *number of Π_1^E axiom sentences.*

The key intuition justifying Lemma 4.8 is that its analogs *are trivially valid* under languages that employ more traditional variants of Δ_0-like formula, whenever *at least one of* the usual growth functions of say successor, addition or multiplication are present. The proof of Lemma 4.8 is slightly more complicated than its traditional analogs only because it does not contain any of these traditional growth function. (It relies, instead, upon " ζ " as the only available growth function).

The central point is that this added complication is only a minor further wrinkle because the combined force of Lemmas 3.3, 3.4 and 3.7 can provide our axiom system G_1 with an adequately sufficient knowledge for it to appreciate the meanings of its E_j and T_n terms and the significance of the Δ_0^E encodings for $\mathrm{Add}(x,y,z)$ and $\mathrm{Mult}(x,y,z)$. As a result of this information, it is possible to construct a *finite-sized* axiom system G_1 that satisfies Lemma 4.8's claim.

A formal proof of Lemma 4.8 is provided in the Appendix C. of the unabridged version of this paper [63]. We recommend that the reader omit examining this appendix during one's first pass through this material. This is because the intuition behind Lemma 4.8 has been explained by the preceding two paragraphs, and the added details are a distraction from more important issues.

We are now ready to define the first of the self-justifying axiom systems that will be examined in this article:

Definition 4.9 Let α be any axiom system that uses all the symbols in L^E's language. Then IS.Pure(α) will denote a 3-part axiom system that can prove all α's Π_1^E theorems and recognize its own Hilbert consistency. Its three sub-schemes, called Group-1, Group-2 and Group-3, are defined below:

1. IS.Pure(α)'s "Group-1" scheme will denote any axiom system, of finite cardinality, that satisfies Lemma 4.8's invariant. This system, often denoted as G_1, must be able to prove every Purely-Closed Δ_0^E formula that holds true under NN's Standard model. (There are infinitely many different possible G_1 systems that can be chosen that satisfies this criteria. Any one may be used to define IS.Pure(α)'s Group-1 scheme.) Thus, the efficiency of our formalism may be affected by the particular " G_1 " system used, but the correctness of our basic results is unaffected by which G_1 we choose. (For example a consequence of Lemma 3.4's upper bound is that a properly chosen Group-1 axiom systems can verify in O(PolyLog(n)) time that the object " T_n " corresponds to the integer of n).

2. Let $\lceil \Phi \rceil$ denote Φ's Gödel number, and $\text{HilbPrf}_\alpha(x,q)$ denote a Δ_0^E formula indicating q is a Hilbert-styled proof from axiom system α of the theorem x. Then for each Π_1^E sentence Φ, IS.Pure(α)'s "Group-2" scheme will contain one axiom of the form:

$$\forall q \quad \{ \text{HilbPrf}_\alpha (\lceil \Phi \rceil , q) \ \Rightarrow \ \Phi \ \} \tag{20}$$

(From a pragmatic perspective, (20)'s *full infinite* set of Group-2 sentences will obviously be awkward. It will, however, be useful from a theoretical perspective by showing that our IS.Pure(α) formalism can prove all the Π_1^E theorems of α in a pristine "purist" sense.)

3. The Group-3 axiom of IS.Pure(α) will consist of a single Π_1^E sentence that essentially corresponds to the following statement:

   $\oplus$ "There is no Hilbert-style proof of 0=1 from the union of the Group-0, 1 and 2 axioms with **THIS SENTENCE** (referring to itself)".

We have already illustrated in several papers [52, 54, 57, 59, 61, 62] how similar self-referential Π_1 like constructions can construct analogs of the sentence $\oplus$ above. In essence, the Π_1^E encoding of $\oplus$ rests on constructing a special Δ_0^E formula called " $\text{HilbPrf}_{\text{IS.Pure}(\alpha)}(x, y)$ " such that Equation (21) (below) can be roughly thought of as being semantically equivalent to the sentence $\oplus$.

$$\forall p \quad \neg \, \text{HilbPrf}_{\text{IS.Pure}(\alpha)}(\lceil 0 = 1 \rceil, p) \tag{21}$$

Theorem 4.10 . *There exists a Δ_0^E encoding for the $HilbPrf_{\text{IS.Pure}(\alpha)}(x, y)$ predicate which will cause (21)'s statement to be semantically equivalent $\oplus$'s Group-3 statement.*

Several results, similar to Theorem 4.10, have been displayed in [52, 54, 57, 59, 61, 62]. We have also summarized the machinery, needed to prove Theorem 4.10, in Appendixes A and B of the unabridged version of this paper [63]. (The first appendix provides a quite routine description of our Godel-like method for encoding proofs. The Appendix B then illustrates how the prior methodologies of [52, 54, 57, 59, 61, 62] can endow " $\text{HilbPrf}_{\text{IS.Pure}(\alpha)}(x, y)$ " with a Δ_0^E encoding in this context.)

We recommend that a reader omit examining both these appendixes during one's first pass through this paper (since they repeat methods that were previously vented in [52, 54, 57, 59, 61, 62]). One further definition is needed before we can turn to our first main theorem:

Definition 4.11 *Let I denote a function that maps an initial axiom system α onto an axiom system $I(\alpha)$. This mapping operation will be called* **Consistency Preserving** *iff $I(\alpha)$ is guaranteed to be consistent whenever the union of α with the Group-1 axiom system G_1 (defined by Lemma 4.8) is consistent.*

One of our goals will be to use Definition 4.11 in a context where I is an operation that maps an initial axiom system α onto a self-justifying formalism, similar to say the IS.Pure(α). In this case, Theorem 4.12's invariant will hold:

Theorem 4.12 *The IS.Pure(α) formalism is Consistency Preserving. (Thus, IS.Pure(α) is guaranteed to be consistent whenever $\alpha \cup G_1$ is consistent.)*

Remark 4.13 The reason Theorem 4.12 is of interest is that "Consistency Preserving" transformations are usually difficult to construct. This is because most initial base axiom systems β become inconsistent when an analog of $\oplus$'s "I am consistent" statement is added to them. The intuitive reason Theorem 4.12 shall avoid this difficulty is that it will *lie just below the threshold* where Theorem 2.1's variation of the Second Incompleteness Theorem becomes active. The formal justification of Theorem 4.12 will appear at the end of Section 6.1 of the unabridged version of this paper [63], under its Corollary 6.1.

4.2 The Rekindling of Hilbert's Consistency Program

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The attached $1\frac{1}{2}$ page addendum section to this article will describe a much more mature version of Theorem 4.12's IS.Pure(α)'s axiom system, called IS.Terse(α).

The evident difficulty with the prior section's IS.Pure(α) formalism is that this system essentially *rides piggy-back* upon the Π_1^E knowledge of the base axiom system α, via IS.Pure(α)'s bulky Group-2 axiom schema. Most logicians would agree that this difficulty would be resolved if one could compress IS.Pure(α)'s infinite-sized Group-2 axiom schema into one single finite-sized sentence, as shall be done by the IS.Terse(α) formalism, described in the attached 2-page addendum.

This is because the finite cardinality of IS.Terse(α) demonstrates finite-sized self-justifying systems can recognize their own consistency and prove a non-trivial subset of Peano Arithmetic's Π_1^E theorems. It should thus rekindle the hopes that Gödel and Hilbert expressed in their statements $*$ and $**$ (by showing that at least a diluted fragment of what Hilbert sought to accomplish in his consistency program is feasible).

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by the next section's IS.Terse(α) formalism.

and the subsequent paragraph deleted.

Addendum Summarizing the Properties of IS.Terse(α)

4.2.1 Notation for Defining IS.Terse(α)

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The IS.Terse formalism, unlike IS.Pure, will be an axiom systems of strictly finite cardinality (in response to Hilbert's Second Open Question). The penalty for this *finite-sized* "memory tightness" is that the IS.Terse framework will compress its memory space *so tightly* that its Group-2 scheme will duplicate α's Π_1^E knowledge in *only* a "simulated" sense. (It will not duplicate this knowledge in Section 4.1's "purist" sense.)

The following notation will explain the type of simulated knowledge our IS.Pure(α) axiom system will be capable of representing:

1. Check(t) will denote a Δ_0^E formula that checks to see whether t is the Gödel number of a Π_1^E sentence.

2. Test(t, x) will denote any Δ_0^E formula where (22)'s invariant is true under NN's Standard model for every Π_1^E sentence Φ simultaneously. There are infinitely many different Δ_0^E formulae that can serve as Test(t, x) predicates satisfying this condition. (Example 4.15 will illustrate one such example.)

$$\Phi \quad \longleftrightarrow \quad \forall\, x \;\; \text{Test}(\,\lceil\, \Phi \,\rceil\,,\; x\,) \tag{22}$$

The expression (23) will be called the axiom system α's **Global Simulation Sentence**. Its Test(t, x) clause essentially will allow IS.Terse(α) to simulate the Π_1^E knowledge of α .

$$\forall\, t \;\; \forall\, q \;\; \forall\, x \;\; \{ \;\; [\;\; \text{HilbPrf}_{\,\alpha}\,(\,t\,,\;q\,)\;\;) \;\; \wedge \;\; \text{Check}(t) \;\;] \quad \longrightarrow \quad \text{Test}(t,x) \;\; \} \tag{23}$$

The beauty of (23)'s sentence is that it is a finitized object that is capable of providing *almost as much information* as IS.Pure(α)'s infinite-sized Group-2 axiom schema. This is because statement (22) holds true under the standard model of NN, whenever Φ represents a Π_1^E sentence.

Thus, it will turn out that each time the base axiom system α proves a Π_1^E sentence Φ, our self-justifying and *finite-sized* IS.Terse(α) styled axiom systems can prove a corresponding equivalent statement of:

$$\forall\, x \;\; \text{Test}(\ \ulcorner \Phi \urcorner\ ,\ x\ ) \tag{24}$$

Definition 4.14 Let τ denote any function that maps a Π_1^E sentence Φ onto a Π_1^E statement, similar to (24), such that these two statements are logically equivalent under the standard model (in the sense that Equation (22) is true for all Π_1^E inputs of Φ). Such a mapping will be called a Π_1^E **Isomorphism.** (There are infinitely many different available Π_1^E isomorphisms, with different levels of memory, computing-time and logical-syntax efficiencies.) For this article's present purposes, we will need to identify merely one such Π_1^E Isomorphism to revive interest in Hilbert's consistency program. That example is given below.

Example 4.15 Let BootProof(s,x) denote a Δ_0^E formula specifying that s is a Σ_1^E sentence and that x is a Hilbert-styled proof of the statement s *using only* Lemma 4.8's " G_1 " axiom system. We will also employ the following notation:

1. NegPrf(t,x) will denote a Δ_0^E formula specifying that t is a Π_1^E sentence and that BootProof(s,x) is true when s is the Σ_1^E sentence representing t's negation.

2. Test$_0(t,x)$ will denote the following Δ_0^E formula:

$$\text{Test}_0(t,x) \quad =_{\text{def}} \quad \neg\ \text{NegPrf}(t,x) \tag{25}$$

Then for each Π_1^E sentence Φ, it is easy to verify [6] that statement (26) holds true under NN's

[6] This is because the fact that Lemma 4.8's G_1 axiom system can prove all true Purely-Closed Δ_0^E sentences implies that it can also prove every Σ_1^E sentence that is true under NN's standard model. Hence, the "BootProof" deduction methods is Σ_1^E complete (and also obviously contradiction-free). This is exactly what what we needed to establish to show the statement (26) holds true under NN's Standard model.

Standard model.

$$\Phi \quad \longleftrightarrow \quad \forall\, x \;\; \mathrm{Test}_0^E(\; \ulcorner \Phi \urcorner \;,\; x \;) \tag{26}$$

The latter implies that (27) is a global simulation sentence for the axiom system α .

$$\forall\, t \;\; \forall\, q \;\; \forall\, x \;\; \{ \;\; [\;\; \mathrm{HilbPrf}_{\,\alpha} \;(\; t \;,\; q \;) \;\;) \;\; \wedge \;\; \mathrm{Check}(t) \;\;] \quad \longrightarrow \quad \mathrm{Test}_0(t,x) \quad \} \tag{27}$$

We emphasize that there are an infinite number of different examples of $\mathrm{Test}_i(t,x)$ predicates that generate global simulation sentences, and (27) illustrates only one such example.

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4.2.2 Definition of IS.Terse(α) and Main Theorem

We are now ready to define our main axiom family and state the main theorem it will satisfy.

Definition 4.16 Let us presume that the superscript τ denotes some operational object that meets Definition 4.14's Π_1^E Isomorphism mapping requirement. (Thus, τ maps every Π_1^E sentence Φ onto a variant of (24)'s formula that is equivalent to it under NN's Standard Model.) Then the axiom system IS.Terse$^\tau(\alpha)$ (which is often informally designated as "IS.Terse(α)" when τ's name is not needed) is defined as follows:

1. The Group-1 axioms for IS.Terse$^\tau(\alpha)$ are identical to those used by IS.Pure [7] .

2. The one and only Group-2 axiom of IS.Terse$^\tau(\alpha)$ will be (28)'s Π_1^E sentence, in a context where " τ " formally specifies which among an infinite number of allowed possibilities is the code for " $\mathrm{Test}(t,x)$ ".

$$\forall\, t \;\; \forall\, q \;\; \forall\, x \;\; \{ \;\; [\;\; \mathrm{HilbPrf}_{\,\alpha} \;(\; t \;,\; q \;) \;\;) \;\; \wedge \;\; \mathrm{Check}(t) \;\;] \quad \longrightarrow \quad \mathrm{Test}(t,x) \quad \} \tag{28}$$

As we have already noted, IS.Terse's Group-2 scheme is different from that of IS.Pure because it consists of one Π_1^E sentence (rather than an infinite collection of such sentences).

[7] Thus, they will provide an ability to prove every Purely-Closed Δ_0^E sentences that holds true under NN's standard model.

3. The *"I am consistent"* statement in the Group-3 scheme of IS.Terse will be identical to IS.Pure's counterpart, except that its notion of " I " (or "me") has a different meaning because the two formalisms invoke different forms of Group-2 axioms [8]. Its formal definition is given below:

$$\forall\, p \quad \neg \,\text{HilbPrf}_{\,\text{IS.Terse}(\alpha)}\,(\, \ulcorner 0 = 1 \urcorner\,,\; p\,) \tag{29}$$

Theorem 4.17 (A very much diluted but *not entirely negligible* type of at least partially positive answers to the goals sought by Hilbert's Consistency Program): *For any Π_1^E Isomorphism τ, the IS.Terse$^\tau$ mapping operation satisfies Definition 4.11's requirements for Consistency Preservation.* (Thus, IS.Terse$^\tau(\alpha)$ is guaranteed to be consistent whenever α is consistent.)

Theorem 4.17's proof is provided in Section 6.1 of the unabridged version of this paper [63]. It essentially is a partial (but not full) positive reply [9] to Hilbert's Second Open Question.

This extended abstract has been written in a style so that its text is verbatim identical to Sections 1-4 of [63]'s unabridged version of this paper. The curious reader, thus, now has an opportunity to turn to Sections 5–6 of [63] for seeing the proofs of the claims made in this Extended Abstract.

The Section 6.2 of [63] will explain why there are likely to be a variety of pragmatic applications of IS.Pure(α) and IS.Terse(α) in the distant future. This is because there are many complicated renditions of these formalisms (some of which should support a pragmatically small polynomial increase between the length of α's proof of a Π_1^E theorem and its analogs under the Group-2 schemes.)

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[8] This distinction is significant because IS.Terse's Group-2 axiom is more compressed, while IS.Terse's Group-2 axiom chooses to traverse an alternate road along the path of trade-offs, by providing sufficient structure to garner information from α in a "purist" rather than simulated sense.

[9] Theorem 4.17 is a more direct reply to Hilbert's Second Open Question than Theorem 4.12 because only IS.Terse(α) assuredly contains no more than a **happily finite** number of axiom sentences

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PROBABLY DELETE above 2 paragraphs in journal paper.

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There is obviously no space to prove Theorem 4.12 within the confines of this extended abstract. This topic will be addressed in the longer unabridged version of this paper [63] (see especially Corollary ??? in the latter document).

The two obvious drawbacks of IS.Pure(α) are that:

1. It employs an infinite number of proper axioms (because its Group-2 scheme consists of an infinite number of axioms).
2. It is obviously less than ideal for IS.Pure(α) to learn about its consistency only via a use of $\oplus$'s "I am consistent" axiomatic statement.

It turns out that both these challenges are addressed and solved in a completely satisfactory manner in the unabbreviated version [63] of this paper.

The next two pages of this Extended Abstract, entitled "Supplement", provides a short 2-page executive summary about how an adequate response to these two challenges can be mustered. We strongly encourage the referees to briefly scan these two pages of supplementary material because they describe the most surprising results that we have obtained during the last 20 years of our research.

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5 Mathematical Machineries for Starting Our Proofs

The author of this article shares with many other academics a strong dislike for excessive and unnecessary notation. It turns out, however, that all the tedious notational winding turns in this article will be absolutely imperative for our purposes.

The is because the somehow-never-published Theorem 2.1, due to the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris, formalized a substantial barrier that self-justifying axiom systems can never surpass.

Thus, the only hope in developing a self-justifying system, that employs Hilbert deduction and has a substantial amount of arithmetic knowledge, falls upon studying what Section 2 called Type-NS formalisms.

The discussion in this section will introduce two intermediate machineries, called the Linear Finitization Theorem for exploring these issues

The discussion in this chapter will never explicitly reference Theorems 4.12 and 4.17 because our Linear Finitization Theorem and the Interp(J) framework are intended to have other applications to logic, as well.

The use of these two mathematical machineries for helping to prove Theorems 4.12 and 4.17 will not be discussed in this article until Section ???.

5.1 The Linear Finitization Theorem

The Linear Finitization Theorem, explored in the present chapter, can be applied to any of the classic textbook definitions of Hilbert Deduction. For instance, we initially developped this theorem as a serendepitous and fully accidental event, while preparing a lecture presentation that was related to Section 2.4 of Enderton's introductory logic textbook [12]. We later decided that it would be better to present our Linear Finitization Theorem, using Mendelson's definition [30] of Hilbert Deduction, because it will lead to a more succinct presentation.

Mendelson's definition of a Hilbert-style deductive mechanism appears on Page 62 of [30]. Its formalization of a Hilbert-style deductive system will consist of two rules of inference (Modus Ponens and Generalization) and the following five schemes of Logical axioms:

A1 $B \Rightarrow (C \Rightarrow B)$

A2 $[B \Rightarrow (C \Rightarrow D] \Rightarrow [(B \Rightarrow C) \Rightarrow (B \Rightarrow D)]$

A3 $(\neg C \Rightarrow \neg B) \Rightarrow [(\neg C \Rightarrow B) \Rightarrow C]$

A4 $(\forall x\, B(x)) \Rightarrow B(t)$ when t is substituionable for x in B.

A5 $[\forall x\, (B \Rightarrow C)] \Rightarrow (B \Rightarrow (\forall x\, C)$ if B contains no free occurrences of x

Throughout our discussion in this section, we will call the formulae in sentences (30) and (31) the **Single-Universally** and **Double-Universally** versions of quantified formulae.

$$\forall x \quad \psi(x) \tag{30}$$

$$\forall x\ \forall y \quad \psi(x,y) \tag{31}$$

Also, we will employ the following notation:

a. For a fixed language L and for (30)'s single-universally quantified expression, the symbol $\text{Extend}_1^L(\psi)$ will denote all the formulae of the form $\psi(t_1)$, $\psi(t_2)$, $\psi(t_3)$ ... where t_1, t_2, t_3 .... is a list of all the terms that appear in the language L.

b. Likewise for a fixed language L and for (31)'s double-universally quantified expression, the symbol $\text{Extend}_2^L(\psi)$ will denote the set of all the formulae of the form $\psi(s,t)$, where s and t are any allowed terms in the language L.

All of our major results in the next two sections of this article will employ Theorem 5.1 and/or its corollary.

Theorem 5.1 *Suppose* p *is a Hilbert-styled proof of the theorem* Φ *from the axiom system* $\Gamma + \forall x\, \psi(x)$. *Then there exists a Hilbert-styled proof of the theorem* Φ *from the axiom system* $\Gamma \cup \text{Extend}_1^L(\psi)$, *called* q , **whose length is no more than three times** *the length of the proof* p.

Proof. Most conventional axiom systems contain a built-in constant called "0" and are compatible with the assumption "0=0". Our theorem does not technically require Γ to have these properties. It is, however, easier to summarize Theorem 5.1's proof, if we assume Γ contains the constant zero and the axiom "$0 = 0$".

Let p^* be a sequence of formulae, identical to p's proof sequence, except that it replaces each phrase of the form

$$* \qquad \text{“} \forall\, v\, \psi(v) \;\Rightarrow\; \psi(t_i) \text{”}$$

with the line

$$** \qquad \text{“} 0 = 0 \;\Rightarrow\; \psi(t_i) \text{”}$$

This replacement rule will be applied in both the cases where the i-th line of p's initial proof consists of solely the statement $*$, as well as in the alternate case where it contains $*$ as a subformula (contained within a longer statement).

It is easy to see that every line of the resulting new proof p^* can be justified as being valid, EXCEPT FOR the case where such a line has $**$ appearing as an isolated formula (lying by itself in the $i-$th line without any other accompanying information). Such a line can be easily justified, ***in a modified proof,*** where we arrange for each isolated appearance of $**$ to be preceded by the following two lines:

i. $\psi(t_i)$

ii. $\psi(t_i) \;\Rightarrow (\; 0 = 0 \;\Rightarrow\; \psi(t_i) \;)$

Such a modified version of p^* will be called q. It will satisfy Theorem 5.1's claim of being a valid proof of Φ from the axiom system $\Gamma \cup \text{Extend}_1^L(\psi)$ because

1. Each instance of a Type-i-line will be an axiom of $\text{Extend}_1^L(\psi)$,
2. Each of the inserted Type-ii-lines is an instance of an "A1" logical axiom, and
3. Each line of type $**$ that follows after the newly-inserted Lines of (i) and (ii) is justified via an application of Modus Ponens.

Moreover, the proof q will have a length no more than three times greater than p's length because each of the preceding paragraph's repairs involve adding two lines to q's proof for justifying one defective line in p^*'s proof. $\square$.

Our main use of Theorem 5.1 in the next section will consist of applications of its corollary.

Corollary 5.2 *Let Γ be any set of axioms that includes the built-in constant " 0 " and which includes the axiom " $0 = 0$ ". Let Λ and Λ^* have the following two properties*

1. Λ *is a set of a finite number of single and double-universally quantified formulae.*

2. Λ^* *consists of the union of all the* $Extend_1^L(\psi)$ *and* $Extend_2^L(\psi)$ *sets associated with* Λ *'s axioms.*

Then if P *is a proof of a theorem* Φ *from* $\Gamma \cup \Lambda$, *there will exist a second proof* Q *of this theorem from* $\Gamma \cup \Lambda^*$, *whose length exceeds* P*'s length by no more than a factor of 3.*

Proof Easy because one can apply essentially repeated iterations of Theorem 5.1's procedure to process each element in Λ's set of starting axioms until Q is produced. If the proof P contained n lines and $m \leq n$ instances of Type-A4 axioms, then the resulting proof Q can be trivially demonstrated to consist of no more than $n + 2m$ lines. (This is because our procedure will insert no more than m pairs of Type-i and Type-ii lines when it creates Q.) Hence, Q's length will exceed P's length by no more than a factor of 3. □

It is well-known that there are many results in Mathematics that may initially appear to have trivial proofs, but which are essential towards solving a greater open question. Theorem 5.1 and its corollary are results of this type. It was, thus, immediately after I derived Theorem 5.1 and its corollary I realized that my 20-year old research project, that began with the article [52], would soon reach its culmination.

5.2 The Finitized Model(H) and Its Interp(H) Framework

The next section will prove Theorems 4.12 and 4.17 by using a finitized Model(H) and its Interp(H) framework, as helpful intermediate steps. These constructs, as well as their analogs where $H = \infty$, will be defined in this section.

For any $H \geq 1$, the symbol **Model(H)** will denote the set of integers $\leq 2^H$. Each of our six Grounding functions of Subtraction, Integer-Division, Maximum ... can be accomodated within this finitized model because they are non-growth functions (that never produce outputs greater than their inputs). Also, our three built in constants of c_0, c_1 and c_2 can be incorporated within this model because they represent quanities $\leq$ 2, which can obviously fit into Model(H)'s range of allowed integer values whenever $H \geq 1$.

One difficulty with the Model(H) is that if $H < \infty$ then it cannot accomodate a $\zeta(x)$ function because the Up-Walking axioms (from statements (9)-(11)) imply $\zeta(x)$ will have an infinite range. Our next definition will explain how we will address this challenge and turn it to our advantage, during the next chapter's proofs of Theorems 4.12 and 4.17.

Definition 5.3 *The symbol* $\Im_H(x)$ *will denote* **a crude and imperfect** *approximation of* $\zeta(x)$*'s behavior defined by the following three invariants:*

A. $\Im_H(x) = 2x$ *when* x *is a power of 2 less than* 2^H ,

B. $\Im_H(x) = 1$ *when* $x = 2^H$, *and*

C. $\Im_H(x) = x$ *in all other cases*

Remark 5.4 At the intuitive level, each of the functions $\Im_1$, $\Im_2$, $\Im_3$ ... *fall short* of being " ζ " class functions because Item B causes them to fail to satisfy the constraint-structure of (9)–(11)'s Up-Walking axioms. However, their infinite limit, denoted as $\Im_\infty$, falls into the class of permissible " ζ " functions. This is because it has the following "mutated" version of Item A replace Items A and B.

A* $\Im_\infty(x) = 2x$ when x *is any power* of 2.

Definition 5.5 The primitive $\Im_\infty$ will henceforth be called the **Slow-ζ Function.** At an intuitive level, $\Im_\infty$ can be roughly thought of as the slowest feasible growing function, lying in the class of " ζ " styled functions. (This is because it essentially satisfies the requirements of (9)–(11)'s Up-Walking axioms with the minimal possible growth.)

Definition 5.6 Throughout this article, the symbol **Assumption(H)** will denote the *temporarily held assumption* that the symbol " $\zeta(x)$ " satisfies the following equality:

$$\zeta(x) \; =_{\text{def}} \; \Im_H(x) \tag{32}$$

Comment: The Definition 5.6 used the phrase of *"temporarily held assumption"* to characterize (32) because such a definition of " $\zeta(x)$ " is incompatible with the requirements of (9)–(11)'s Up-Walking axioms, for every $H < \infty$. (This is because Items A-C will cause $\zeta(x)$ to range over a finite range of values, while (9)–(11)'s Up-Walking axioms require it to have infinite range.) In essence, the creative tension between $\Im_H(x)$'s unaccpetible behavior when $H < \infty$ and its premissible behaviour under Defintion 5.5's "Slow-ζ" function, will be the central issue used to prove Theorems 4.12 and 4.17.

We will now need to introduce two more definitions for formalizing the precise manner in which we shall apply model theory to prove Theorems 4.12 and 4.17. These constructs are defined below:

Definition 5.7 Let us recall that Section 2.2 of Mendelson's textbook used the symbol " Σ " to denote the set of all enumerations of the elements in the domain D for an arbitrary model. It also had $s \in \Sigma$ denote one such enumeration. We will assume throughout this article that s denotes a sequence of "all zeroes" (i.e. as in " $0, 0, 0, ...$ "). The operator s will thus assign all *free* occurrences of variables " v_i ", lying in an arbitrary formula, an interpretational value of "0". This particular sequence of "all zeroes" will be called the **All-Null-Sequence.** It will *always be denoted* as s during our discourse.

Definition 5.8 Let us recall that the second paragraph of this section had defined Model(H) as a framework for modeling the set of integers $\leq 2^H$, where our six Grounding functions and the three built in constants of c_0, c_1 and c_2 are assigned their natutally intended meanings. Within this context, **Interp(H)** will be defined to be an extension of Model(H) that contains the following two additional caveats:

i. The function " s ", from Definition 5.7, will be the valuation function for interpreting the meaning of free variables under our Interp(H)'s model. (We remind the reader that Definition 5.7 specified that s would be the " All-Null-Sequence".)

ii. The Interp(H)'s semantics will treat the Equation (32) from Assumption(H) as formulating the definition of $\zeta(x)$ (e.g. it will presume $\zeta(x) = \Im_H(x)$.)

We close this section by emphasizing the fragile nature of the Interp(H)'s framework. This fragile nature arises because Item (ii)'s equality of " $\zeta(x) = \Im_H(x)$ " will be incompatible with (9)–(11)'s Up-Walking axioms whenever [10] $H < \infty$. (It will, thus, be compatible with the Up-Walking axioms *only under the extremal assumption* that $H = \infty$.)

Our proofs of Theorems 4.12 and 4.17, in this article, will take advantages of the creative tension that stems from Interp(H)'s fragile nature. The fact, that $\Im_H(x)$'s behavior is unaccpetible when $H < \infty$ and premissible only if $H = \infty$, will ultimately be the linchpin, that will provide the proofs of these two theorems.

[10]The incompatability arises because Assumption(H) presumes the presence of a finite model, where all integers have their sizes bounded by 2^H. This upper bouund is incompatible with (9)–(11)'s Up-Walking axioms because the latter implies the existence of an infinite range of available integers.

5.3 A Unifying Intermdeiate Machinery

This section will formalize a more advanced result, called Theorem 5.11, that will hybridize the two divergent sets of observations, that were made in the prior two sections. Our Theorem 5.11 will be intended to constitute the main intermediate step, used during the next chapter's proofs of Theorems 4.12 and 4.17.

It is helpful to begin our discussion by introducing some further notation that will be used throughout this article. The capitol letter symbols, such as P, Q and R, will denote the Gödel numbers of proofs.

A Gödel Encoding scheme will be called **Hand-Natural** iff Log(P) has a size that is proportional to the number of strokes that a pen uses when writing a proof. (The essentially natural "B-adic" encoding methods, described by Hájek-Pudlák [20], Wilkie-Paris [51] and Buss [10], have the nice philosophical quality of being Hand-Natural.) Also, Appendix A provides another example of a Hand-Natural encoding scheme. It will correspond to the particular Hand-Natural method that will be employed in this article.

Let N_P denote the number of lines in the proof p, and L_p denote the number of logical symbols appearing in this proof. A Gödel Encoding scheme will be called **Conventional** iff all its proofs P satisfy

$$\text{Log}_2(P) \quad > \quad N_P \; + \; L_P \tag{33}$$

All natural encoding methods, including all Hand-Natural ones, satisfy Equation (33)'s constraint. (We will therefore presume this property is present when examining the proof-strings, used by the axiom systems of α, IS.Pure(α) and IS.Terse(α), throughout this article).

Our discussion in the remainder of this article will mostly focus around structures that should will called "Almost Π_1^E Systems". This construct is defined below:

Definition 5.9 An axiom system β will be said to be an **Almost Π_1^E System** iff all but three of its axioms have Π_1^E encodings, and it also satisfies the following two constraints:

1. The axiom system β will include (9)–(11)'s three "Up-Walking" axioms. (These will be the only allowed axioms in β that are not required to have a Π_1^E encoding. Thus, an inspection of sentences (9)–(11) will reveal that these sentences do have a Π_1 like structure, but they *are not* " Π_1^E " (e.g. they do not satisfy Definition 3.6 exact requirements).

2. This second part of Definition 5.9 will presume that Δ_0^E sentences are degenerate forms of Π_1^E sentences (and that they are therefore allowed to appear in β's list of axioms). Within this context, the additional axioms of β may be *any set of* closed Π_1^E sentences, subject to the constraint that β must be rich enough to prove every one of Definition 4.5's "Purely-Closed" Δ_0^E formula that hold true under the Standard-NN model.

Clarification Upon the Meaning of Definition 5.9: The Items (1) and (2) do not require that the axiom system β be consistent. Thus, Item (2) merely requires that β be complete (e.g. able to prove all the Purely-Closed Δ_0^E formula that hold true under the Standard-NN model). There will be some sytsems β (examined later in this paper) where it will be initially unclear whether or not β is consistent.

Example 5.10 The Lemma 4.8 had proven the existence of an axiom system, called G_1, which will certainly satisfy Definition 5.9's requirements. Also, if S is any set of additional Π_1^E sentences, then $S \cup G_1$ will be another example of an "Almost Π_1^E System", irregardeless of whether or not $S \cup G_1$ is consistent.

Our next therom will capture the main formalism that will be used to prove Theorems 4.12 and 4.17 in the next section:

Theorem 5.11 *Suppose β is an Almost Π_1^E Formalism, all of whose axiom sentences* **hold true** *under the Standard-NN model. Let us also use the following notation:*

1. Γ will denote the set of Π_1^E axiom sentences that lie in β.

2. Λ will denote the three remaining Up-Walking axioms, from Equations (9)–(11), which also lie in β (but fall short of being Π_1^E sentences).

3. Λ^* will consist of the union of all the Extend$_1^L(\psi)$ and Extend$_2^L(\psi)$ sets associated with Λ's axioms.

4. The symbol P will denote a proof of some theorem Φ from the axiom system " $\Gamma \cup \Lambda$ ". (Please note that Items (1) and (2) indicate " $\Gamma \cup \Lambda$ " is just another name for " β ").

5. The symbol H will denote the quantity $\text{Log}_2(P) - 1$.

Let us recall that Corollary 5.2's procedure is capable of transforming P's proof of Φ into an alternate proof, called "Q", of Φ, where the difference between these proofs is essentially that Q's proof uses " $\Gamma \cup \Lambda^*$ " (rather than " $\Gamma \cup \Lambda$ ") as its starting set of axioms. It will be automatically true, in this context, that every line of Q's generated proof, including its final theorem Φ, will be a formally true statement under Interp(H) finistic model (e.g see Definition 5.8).

Remark 5.12 (explaining the intuition behind Theorem 5.11): The reason Theorem 5.11 is of interest is that there exists no finistic model that can capture the contents of Up-Walking axioms, from Equations (9)–(11), as we have noted several times in this article. The Theorem 5.11 presents a type of pragmatic solution to this dilemma, by demonstrating that all the axiom-sentences, invoked in at least Q's proof, hold true in Interp(H)'s finistic model. (This is because Q's axiom come from a subset of the system " $\Gamma \cup \Lambda^*$ ", which has very different propeties than " $\Gamma \cup \Lambda$ ".)

Proof of Theorem 5.11 : Let M_Q denote the largest integer value, assigned to any term in the proof Q, under Definition 5.8's Interp(H)'s model. Then the footnote [11] shows how our notation trivially implies that Equation (34) must hold:

$$M_Q \leq \frac{P}{4} < 2^H \quad \text{(where the latter inequality follows from Theorem 5.11's Assumption 5)} \qquad (34)$$

Let us next recall that the axioms, used in the proof Q, will typically be a proper susbset of $\Gamma \cup \Lambda^*$'s set of allowed axioms. The Equation (34), in this context, immediately implies that all axiom-sentences, physically placed within the proof Q, will have sufficiently small values

[11] Let L_P again denote the number of logical symbols that can appear in the proof P symbols. Let V denote the value of the largest term of the form $\zeta(\ \zeta(\ \zeta(\ \ldots\ \zeta(\ 1))))$ that can be squeezed into any proof, possessing P's length. This term can be easily shown to contain no more than $\text{Log}_2(P) - 2$ iterations in of the " ζ " function. (This is because all proofs will require using at least two symbols besides the " ζ " symbols to formulate one well-defined line, and Equation (33)'s "Conventional" Godel Encoding Methodology assumes that each symbol, appearing in a proof, must require at least one bit to encode.) Also, the Definition 5.8's Interp(H) model implies every term that is coded in k symbol will be smaller than the $k-1$ iterated expression of $\zeta^{k-1}(1) = 2^{k-1}$ (that was encoded by V). These results, in combination, imply the validity of Equation (34). (Indeed, (34)'s inequality can be easily shown to be an excessively conservative estimate, under a more refined analysis).

must clearly be Π_1^E axiom systems of β, due to the nomenclature [13] that we have employed.

Moreover, the footnote [14] uses the soundness of Hilbert deduction to demonstrate that at least one of these sentences Ψ_i must also be false under Interp(H)'s model. The latter corroborates Corrolary 5.13's claim because this sentence Ψ_i has a Π_1^E format. □

6 Main Results

The main goal in this section will be to prove Theorems 4.12 and 4.17 and explore some of their generalizations. Both these theorems have nearly identical proofs and implications. We will focus most of our attention on proving Theorem 4.17 because its target subject comes closer to the essence [15] of Hilbert's Second Open Question.

The justification of Theorem 4.17 will consist of the proof-by-contradiction, given in Section 6.1. (At this section, we will also explain how an analogous argument also justifies Theorem 4.12.)

6.1 A Proof-by-Contradiction Justifying Theorem 4.17

Suppose for the sake of establishing a contradiction that α is consistent but IS.Terse(α) is inconsistent. Then a proof of $0 = 1$ from IS.Terse(α) must exist that has a minimal Gödel number, which we will henceforth call P. $***$

We will prove that P's existence is impossible, in this section, by constructing from P another proof of $0 = 1$, called R such that $R < P$.

[13] An inspection of the sentences (9)-(11) will indicate that each of their axiom sentences hold true under the Standard-NN model. Hence, some other axiom of β must be the delinquent axiom, violating the requirements of the Standard-NN model. Such formally delinquent axioms must have a Π_1^E structure because of manner manner in which "Almost Π_1^E " axiom systems were formalized (e.g. see Definition 5.9).

[14] The structure of this soundness proof is quite simple: Our previous proof of Theorem 5.11 had noted that the three Up-Walking axioms all held true in the model Interp(H). This forces the conclusion that *at least* one of the Π_1^E axioms of Ψ_1, Ψ_2, Ψ_3 ... must be false under Interp(H)'s model, since a proof Q cannot establish a theorem Φ, that is false in Interp(H)'s model, without one of its axiomatic assumptions also being false in this model.

[15] This is because the strict finitism of IS.terse's Group-2 formalism is analogous to Hilbert's finistic goals.

Our construction shall begin by taking the proof P and using it to construct a different proof Q of $0 = 1$ that removes all formal appearances of the axioms (9) – (11) from P's proof. This proof, called Q, will be constructed via the procedure that was outlined in Corollary 5.2. (Its construction is thus straightforward.)

This proof Q will actually be longer than the proof P. It will technically **ALSO NOT BE** a proof that derives from the **literally-specified** assumptions of the axiom system IS.Terse(α). Thus in order to construct our needed proof $R < P$, we will need to determine what are the exact properties that the proof Q will possess to facilitate this construction.

Section 5.3's Corolllary's 5.13 will provide the answer to this question. It will constitute the needed linchpin for proving Theorem 4.17.

Corolllary's 5.13's formalism, in particular, is applicable to the IS.Terse(α) axiom system because the latter is an Almost-E axiom. It implies that one of IS.Terse(α)'s axiom sentences must be false under **BOTH** the infinite-sized Standard-NN model **and also** under the finite-sized Interp(H) model, when H is defined by [16] Equation (35)'s equality:

$$H \;=\; \text{Log}(P) \;-\; 1 \tag{35}$$

This delinquent axiom of IS.Terse(α) must be its Group-3 "I am consistent" statement because all other possibilities are precluded by the Items (1) and (2), as is explained below:

1. Lemma 4.8 shows all IS.Terse(α)'s Group-1 axioms hold in the Standard-NN model. They, thus, cannot be the delinquent axiom (which Corollary 5.13 states will exist).

2. Likewise, IS.Terse(α)'s Group-2 axiom (formalized by statement (28)) will also hold true in Standard-NN model as a trivial consequence of $***$'s assumption that α is consistent. It also, thus, cannot be Corollary 5.13's delinquent axiom.

Thus, IS.Terse(α)'s Group-3 axiom must be the delinquent axiom, that is false under both the Standard-NN and Interp(H) models, because all the other possibilities have been precluded by Items (1) and (2).

[16] Our discussion in the current section has H defined by Equation (35) because this definition for H was used in Section 5.3 by both its Theorem 5.11 and Corollary 5.13 to make their formalisms operative.

The observations from the preceding two paragraphs are, essentially, the crux of our justification of Theorem 4.17, via a proof-by-contradiction. This is because Equation (35) immediately implies that the statement (29)'s *"I am consistent"* Group-3 axiom can be false in the Interp(H) model *only when* there exists [17] a proof $R < P$ that represents another proof of $0 = 1$.

The latter observation brings our proof to contradiction to its sought-after conclusion because it shows that $***$'s initial assumption that P represented the minimal proof of $0 = 1$ has been contradicted. □

Corollary 6.1 *The Theorem 4.12 (asserting the consistency-preservation of the "IS.Pure" formalism) can be easily corroborated by an analog of the preceding proof of IS.Terse's consistency-preservation property.*

Proof. The only significant difference between the IS.Pure and IS.Terse formalisms is that they use slightly diifferent types of Group-2 axioms. The proof of IS.Pure's consistency-preservation property is identical to the preceding analysis for IS.Terse because it is easy to see that Item 2, in the preceding proof, will also apply to IS.Pure's Group-2 scheme (e.g. the consistency of α easily implies that IS.Pure(α)'s Group-2 axioms hold true under true under the Standard-NN model). □

6.2 Extensions and Limitations of the IS.Pure and IS.Terse Formalisms

The proofs of Theorems 4.12 and 4.17 can be easily extended to construct self-justifying axiom systems that possess what [62] called a Level(∞) appreciation of their own consistency (encoded via the techniques of [62]'s Appendix A). These would amount to minor revisions of our formalisms, which are knowldegeable that there exists no proofs of an arbitrary sentence Φ and its negation from themselves.

The latter implies that the Theorems 6.12 and 6.13 from [62], discussing its "Translational Reflection Principle", can also be applied to minor modifications of our IS.Pure(α) and IS.Terse(α) formalisms that seek to hybridize our current results with those of [62]. These

[17] In other words, the existence of an $R < P$ that represents another proof of $0 = 1$ is an *immediate semantic consequence* of the the combination of the definition of (29)'s *"I am consistent"* Group-3 axiom statement and Equation (35)'s inequality.

reflection princibles are very interesting, but they should not be confussed with alternate reflection methodologies that Beklemishev, Kreisel-Takeuti and Verbrugge-Visser discuss in in [6, 7, 8, 28, 46, 50]. Such alternate reflection machines nicely complement our results by examining an essentially *entirely different topic.* This is because [6, 7, 8, 28, 46, 50]'s systems typically can recognize additon, multiplication and successor as total functions, while our reflection machines attain their broader compass only by restricting themselves to what Section 2 called "Type-NS" systems (that recognize none of these primitives as total functions)

Remark 6.2 Before closing this article, it should be mentioned that a trivial construction will further establish that both IS.Pure(α) and IS.Terse(α) can:

1. prove the existence of essentially any integer n in no more than O(PolyLog(n)) steps.
2. require no more than O(Polynomial(L)) steps to simulate any $O(L)$ length proof of a Π_1^E theorem Φ of α . (This means that they can confirm Φ's simulated and purist reality for the respective cases of IS.Terse(α) and IS.Pure(α), under proofs with some quite eminently reasonable degrees of "short-proof-length" efficiency.

Moreover for numerous integers $K > 2$, the Polylog and Polynomial efficiencies of Items (1) and (2) can be further enhanced if the formula "Power(x)" in Equation (9)'s axiom is changed to read that *" x is a power of K "* . Part of the reason we suspect that the formalisms of IS.Terse(α) and IS.Pure(α) are significant is that they will gain much added levels of efficiency, when one uses "Power$_K(x)$" declarations in (9)'s axiom, with an optimally chosen K.

Remark 6.3 The suggestion, formulated by Remark 6.2, *should not* be interpreted as implying that one can use arbitrarily large K in Remark 6.2's "Power$_K(x)$" predicates. This is because analogs of the Incompleteness Effect, from Theorem 6 in Section 6.2 of [57], will then become active. It (or more precisely its analog) will then imply that the insertion of an unusually large constant into a self-justifying formalism can cause it to collapse. Likewise an excessively large K in the Power$_K(x)$ predicate must be guarded against. (These incompleteness effects are essentially are essentially analogous to some of earlier incompleteness results of Friedman and Pudlák [15, 35] about how it is usually impossible for an $O(n)$ length proof to preclude the existence any $O(n^\epsilon)$ proof of $0 = 1$, where we are now revisiting this topic in the context of a "self-justifying" environmemt.)

Remark 6.4 Let "DoubleExp(x)" denote that x is an integer of the form 2^{2^n} . It can be formally proven that the IS.Terse(α) and IS.Pure(α) will lose their self-justification properties, when one replaces "Power(x)" with "DoubleExp(x)" in (9)'s axiom sentence. It is unnecessary for us to prove this fact here because its proof is straightforward analog [18] of the variant of the Second Incompleteness Theorem appearing in [59]'s Theorem 4. These properties of DoubleExp(x), combined with Remark 6.3's observations, imply that the IS.Terse and IS.Pure formalisms are useful *when not employed with excessive zeal.*

Remark 6.5 For a suitably small chosen constant C, the combination of Remarks 6.2–6.4 suggests the efficiency of our proof formalism will often be best managed if one replaces the Power(x) formula inside (9)'s axiom sentence with a more sophisticated Power$^C(x)$ formula, which states that x satisfies the following equality:

$$\exists\ n \geq 0 \qquad x\ =\ 2^{n^C} \tag{36}$$

Future researchers, investigating this topix, should note that the choice of C is critical because too large a constant C will invoke the force of the Second Incompleteness Theorem, while a needlessly small C here (or a needlessly small K in Remark 6.5) will introduce unfortunate levels of inefficiencies.

6.3 Epistemological Perspective

The combination of all our results, in this section, suggests that the predictions that Hilbert and Gödel made in the statements $*$ and $**$ (appearing in Section 1 of the current article) were at least partially realistic. The best way to reinforce this point is to examine a statement from the 1934 (first) edition of the Hilbert-Bernays textbook [21], which was brought to my attention by Karim Joeseph Maroud [?] at a panel discussion, chaired by Harvey Friedman [17], at a March 17-18 meeting of the AMS. This statement was translated into English by Solomon Feferman in his English translation of [21]. Thus, Hilbert spoke the following words in the preface of the 1934 version of [21]:

> $***$ "With respect to this goal (e.g. the successful completion of Hilbert's consistency program), *I (e.g. David Hilbert)* would like to emphasize the following

[18]The Theorem 4 of [59] formalized a variant of the Second Incompleteness Theorem that can be easily generalized whenever a doubly exponential growth rate is present.

view, which temporarily arose and which maintained that certain results of Gödel showed that my proof theory can't be carried out, *has been shown to be erroneous.* In fact, these results show only that one must explore the finitary standpoint in a sharper way for the farther reaching consistency proofs"

The above statement obviously engages in somewhat of a flavor of over-exaggeration when the 1934 edition of [21] speaks of a "temporary" interpretation of Gödel's centennial theorem that "has been shown to be erroneous". Indeed, if our Theorems 4.12 and 4.17 represent near-maximal forms of boundary-case evasions of the Second Incompleteness that are feasible (as we suspect that they do), then it will *always be ambiguous* whether Hilbert's goals were fully reasonable (as opposed to being only partially so). This is because one could argue that Hilbert's Second Open Question lies in the twilight zone, separating the fully realistic from the unrealistic. Such types of muddled matters occur often in philosophy, as well as in many related endeavors, because many foundational questions seek to address issues so challenging that the prevailing deficiencies in the human language initially encourages over-simplification.

In other words, a complex answer is often needed during an intellectual discourse — because there are often two sets of definitions, called say S_1 and S_2, where either set of definitions is reasonable in certain intended applications (and neither is fully ubiquitous). The goal of our investigation of "boundary-case" evasions of the Second Incompleteness Theorem was, thus, to investigate a middle-case circumstance where neither of the initial foundational perspectives, of say S_1 or S_2, is entirely correct. (This is because reality is often so complex that some initial approximations of some philosophical phenomena, called say S_1 and S_2, are good initial approximatons of an issue, that may be too complex to be fully explained by either S_1 or S_2.)

7 Conclusion

Our positive reply to at least a diluted version of Hilbert's Second Open Question is obviously significant. This is because it explains how human beings do possess a *partial level* of instinctive faith in their own consistency.

We do not consider these results to be full-scale evasions of the Second Incompleteness Theorem. This is because the various generalizations of the Gödel's centennial observation are obviously major results. Moreover, the Remarks 6.3 and 6.4 indicate that there exist two