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A McCord Functor for Alexandroff Categories

Master thesis

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Abstract

We shall generalize the construction of McCord's weak homotopy equivalence between the realization of the nerve of a finite poset and the original poset to Alexandroff categories and the induced toposes between them. For this, we shall construct a functor, called the McCord functor, which will give us the basis for a geometric morphism. Finally, we attempt to find sufficient conditions for flatness. The resulting geometric morphism will provide an equivalence of categories on the level of locally constant finite objects.

What I cannot create,
I do not understand.

RICHARD FEYNMAN

Contents

Chapter 1

Introduction

1.1 Outline

Chapter 1 of this thesis gives a light introduction to the subject ending with question 1.4.2. After that, the general flow of this thesis is divided into two parts. The first part, chapters 2 and 3, deal with topos theory. It sets up some basic definitions, theorems and propositions. There is basically no original work in these two chapters, except at the end of chapter 3, where we prove proposition 3.2.2. This result will be used in chapter 6.

The second part of this thesis consists of the theory of simplicial sets. This is devoted to chapter 4. Half of this chapter is preliminary theory and should be well-known. Section 4.4 is all original work, where we set up a framework to work with open sets in a geometric realization. This it to be used in chapter 5. Chapter 5 is devoted to the construction of the McCord functor. This is all original work, so the proofs become somewhat detailed here. Chapter 6 proves the equivalence of categories on the level of locally constant finite objects. This, too, is all original work. So again the proofs are detailed. Chapter 7 ends with an outlook on what problems might be solved still. The reader knowledgeable in both topos theory and simplicial sets may thus skip to section 4.4 and read on. In summary, up to section 4.4 is preliminary text (with the exclusion of proposition 3.2.2), while section 4.4 and on is original work.

The bibliography and index pages may be found at the end of the paper. They do not appear in the contents because they would not fit.

1.2 Prerequisites and Notation

As for the prerequisites, it is assumed that the reader has a firm grasp on category theory. I contemplated whether to include all the relevant category theoretic definitions, but I refrained from doing so because the preliminary text is already substantial. Here are some notions which should be familiar. Category. Small category. Balanced category. Functor. Left exact functor. Right exact functor. Exact functor. Isomorphism-reflecting functor. Adjoint pair of functors. Monoid. Monomorphism. Epimorphism. Exponentials. Cartesian closed categories. (Finite) (co)limits. Moreover, I assume that the reader has seen (pre)sheaves, and is familiar with the classic construction of turning a continuous map $f : X \to Y$ of spaces into an adjoint pair of functors $f_* \dashv f^*$.

The set $\mathbb{N}$ is the set of natural numbers including zero. The set $\mathbb{Z}$ is the set of all integers. $\mathbb{R}$ denotes the real numbers. If something is in boldface, for instance, $\mathbf{C}$, then this is always a category. If something is in scriptface, for instance, $\mathscr{E}$, then this is always a topos. Sheaves and presheaves will be denoted by F, G, P, Q, etcetera (so *not* by $\mathcal{F}$, $\mathcal{G}$). Natural transformations are always greek small letters, for instance $\alpha, \eta, \varepsilon$. Every diagram that is drawn in this thesis is commutative, or will be proven to be commutative. If F and G are functors, the symbol $F \dashv G$ means that F is the left adjoint of G, or equivalently, G is the right adjoint of F. **Set** is the category of sets. **set** is the category of finite sets. **Top** is the category of topological spaces.

1.3 Acknowledgements

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1.4 Motivation and Problem Statement

Let X be a finite connected T_0-space. Then we have a lattice of open subsets $\mathcal{O}(X)$. We may form the category of sheaves on this lattice. Let us denote it by $\mathrm{Sh}(\mathcal{O}(X))$. From a different viewpoint, X can be regarded as a partially ordered set. Every partially ordered set is a category. So we may also form the presheaf category $\mathbf{Set}^{X^{op}}$. That is, $\mathbf{Set}^{X^{op}}$ is the category whose objects are functors $X^{op} \to \mathbf{Set}$ and whose morphisms are natural transformations between them. Observe that every point x of the space X has a minimal open neighborhood. Let us denote it by $U_x \subset X$. This thesis began with the following observation.

Theorem 1.4.1. *Let X be a finite T_0-space. Then there is an equivalence of categories*

$$\mathrm{Sh}(\mathcal{O}(X)) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbf{Set}^{X^{op}}$$

where the functor φ sends a sheaf F to the presheaf defined by

$$\varphi(F)(x) = F(U_x), \qquad x \in X,$$

while the quasi-inverse functor ψ sends a presheaf P to the sheaf defined on the basis of opens $\{U_x : x \in X\}$ by

$$\psi(P)(U_x) = P(x).$$

Proof. The functors are each other's quasi-inverse once we show that φ and ψ are well-defined. Note that $x \leq y \iff U_x \subseteq U_y$, so that φ is well-defined, and for every presheaf P on X the presheaf $\psi(P)$ is a sheaf on the basis $\{U_x : x \in X\}$. So after taking a projective limit it corresponds to a sheaf on all of $\mathcal{O}(X)$. $\square$

Meanwhile there is the notion of the nerve of a category, to be defined in definition 4.3.1. In the case of the poset X, the nerve $N(X)$ is simply the simplicial set defined by the n-chains of elements $(x_0 \leq \ldots \leq x_n)$, with $x_i \in X$. So there are n less-than-or-equal-to signs together with $n+1$ elements $x_i \in X$ in such an n-chain. One should view such an n-chain as being an actual n-simplex. To get the i'th face of an n-chain, simply forget the element x_i in the chain to get an $(n-1)$-chain. Similarly, one can turn an n-chain into a so-called degenerate $(n+1)$-chain by repeating the i'th element x_i. Every simplicial set has a so-called geometric realization, whereby we turn a simplicial set into a topological space. Let us denote that space by $|NX|$. Concretely, elements of $|NX|$ are equivalence classes of pairs $[\sigma, t]$ where $\sigma = (x_1, \ldots, x_n)$ is an n-simplex of NX and t is a point on the standard geometric n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$. Two representatives (σ, t) and (τ, s) are equivalent if and only if "one is a face of the other", that is, if one can find a sequence of "taking faces" which realizes σ as a face of τ, and the point t is geometrically reduced alongside the face maps to the point s. All this will be made precise in chapter 4. In any case, notice that every such n-chain is totally ordered with a minimal element. If $t \in \Delta^n$, let us define the *support* of t to be the set of indices

$$\operatorname{supp}(t) := \{i : t_i > 0\}.$$

Then, [Bar11, Theorem 1.4.6] tells us that there is a continuous map

$$\mu : |NX| \to X, \qquad [(x_0 \leq \ldots \leq x_n), t] \mapsto x_{\min \operatorname{supp}(t)},$$

called the *McCord map*, which moreover induces a weak homotopy equivalence

$$\widetilde{\mu} : \pi_n(|NX|) \xrightarrow{\sim} \pi_n(X), \qquad n \geq 0.$$

Combining this with theorem 1.4.1, we see that we have an adjoint pair of functors

$$\operatorname{Sh}(\mathcal{O}|NX|) \underset{\mu^* \circ \psi}{\overset{\mu_* \circ \varphi}{\rightleftarrows}} \mathbf{Set}^{X^{op}} \tag{1.1}$$

where the right adjoint $\mu_* \circ \varphi$ sends a sheaf F on $|NX|$ to the presheaf defined by

$$\left(\mu^{-1} \circ \varphi\right)(F)(x) = F\left(\mu^{-1} U_x\right), \qquad x \in X.$$

It turns out that categories such as in eq. (1.1) may be regarded as "spaces", and with that notion comes the notion of a "fundamental group". Such fundamental groups are not merely groups, but are in fact profinite groups carrying topological structure. By general category yoga, we may dedude then, using [Len08, Theorem 1.15], that the functor above induces an isomorphism of profinite groups

$$\widehat{\pi}_1\left(|NX|, |p|\right) \cong \pi_1\left(\mathbf{Set}^{X^{op}}, p\right).$$

Here, the notion of a "point" p will be defined in definition 2.2.3.

The small category X is part of a (big) category called the category of all small categories (with functors between them as morphisms), denoted by **Cat**. Moreover, it turns out that such an adjoint pair is so common that a theory around it was developed by Grothendieck and co. Let us call such an adjoint pair for which the left adjoint is left exact a *geometric morphism*. The precise definition is given in definition 2.2.1.

Question 1.4.2. For which (connected) categories $\mathbf{C} \in \mathbf{Cat}$ does there exist a geometric morphism

$$\operatorname{Sh}(\mathcal{O}|N\mathbf{C}|) \rightleftarrows \mathbf{Set}^{\mathbf{C}^{op}}$$

which induces an isomorphism of profinite groups

$$\widehat{\pi}_1(|N\mathbf{C}|, |p|) \cong \pi_1\left(\mathbf{Set}^{\mathbf{C}^{op}}, p\right)$$

and does all this even make sense?

The notion of connectedness is defined in definition 3.1.3. What a fundamental group is is treated in chapter 3. Because geometric morphisms are so common, we'll denote them from now on with a *single arrow* $\rightarrow$ instead of two arrows $\leftrightarrows$, but keep in mind that an adjoint pair is always meant.

This thesis will give a partial answer to question 1.4.2. As it turns out, for *every small category* there is at least a functor, dubbed the *McCord functor* which looks like

$$\mu : \mathbf{C} \rightarrow \mathbf{LH}/|N\mathbf{C}|.$$

Here, **LH** is the subcategory of **Top** consisting of all topological spaces together with *local homeomorphisms* between them as morphisms (instead of just continuous maps). To make this μ work, we needed to introduce a certain class of categories called *Alexandroff* categories which carry the right properties in order to turn μ into an equivalence of categories on the level of locally constant finite objects. The precise meaning of an Alexandroff category is given in definition 5.3.5. What locally constant finite objects are is explained in definition 3.1.4. We end here with the statement of the main result of the thesis, to be proved in corollary 6.2.4.

Theorem 1.4.3. *Let* $\mathbf{C}$ *be a small connected Alexandroff category. Then there exists a geometric morphism*

$$\tau(\mu) : \mathrm{Sh}(\mathcal{O}|N\mathbf{C}|) \rightarrow \mathbf{Set}^{\mathbf{C}^{op}}$$

which induces an isomorphism of profinite groups

$$\widehat{\pi}_1(|N\mathbf{C}|, |A|) \cong \pi_1\left(\mathbf{Set}^{\mathbf{C}^{op}}, A\right)$$

for every object $A \in \mathbf{C}$.

Chapter 2

Topos Theory Done Quick

This preliminary section treats some basic concepts of topos theory. The main ingredients that we will need are the definition of a topos, the concept of a geometric morphism, the concept of points on a topos and the "generalized $\otimes$- Hom adjunction". We will follow mostly Chapter 2, 3, 4 and 7 of [LM91]. The book [Joh77] is also a good reference. This section is very terse and contains a lot of information. The reader is invited to consult the references.

2.1 Basic Definitions

We made an attempt to give the definition of a topos in its most elementary form, but it turned out that for the purposes of this thesis, it was not necessary to know the definition in its utmost generality. Because of this, we shall give two definitions; they are both instances of a more general concept of a topos. The interested reader is invited to read [Joh77] or [LM91].

Definition 2.1.1. Let $\mathbf{C} \in \mathbf{Cat}$ be a small category. Recall that $\mathbf{C}^{op}$ denotes the opposite category. We call the category of set-valued functors

$$[\mathbf{C}^{op}, \mathbf{Set}] = \mathbf{Set}^{\mathbf{C}^{op}}$$

a *presheaf topos*.

Definition 2.1.2. Let $\mathbf{C}$ be a category. The contravariant *Yoneda functor* is defined as the functor which sends objects to representables. That is,

$$\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}, \qquad A \mapsto \mathrm{Hom}_{\mathbf{C}}(-, A).$$

Recall that if $X \in \mathbf{Top}$ and if $F : \mathcal{O}(X)^{op} \to \mathbf{Set}$ is a presheaf, then F is called a *sheaf* if every matching family has a unique amalgamation.

Definition 2.1.3. Let $X \in \mathbf{Top}$ be a topological space. The category of all sheaves
$$\mathrm{Sh}(\mathcal{O}(X))$$
is called a *Grothendieck topos.*

Definition 2.1.4. Let G be a topological group. The category G-**Set** is the category of all sets with a continuous right G-action, together with equivariant maps respecting the continuous G action between them. We call this a *topos of G-sets.*

In the special case that G is discrete, it is in fact a presheaf topos: consider G as a one-object category, call the object $\bullet$. For each $g \in G$, consider it as a morphism $g : \bullet \to \bullet$. Then G is a groupoid. Take the presheaves on this groupoid. This is the same thing as G-**Set**.

Definition 2.1.5. By a *topos* we mean either a presheaf topos, a Grothendieck topos or a topos of G-sets, for some topological group G. A topos will always be denoted by $\mathscr{E}$ or $\mathscr{F}$.

Fact 2.1.6 (A topos is like the category of sets)**.** Let $\mathscr{E}$ be a topos. Then $\mathscr{E}$ has all finite limits and all finite colimits, so in particular an initial object 0 and a terminal object 1. Every monic arrow is an equalizer. Every arrow both monic and epi is an isomorphism. $\mathscr{E}$ has exponentials. Every arrow f factors as $f = m \circ e$, with e epi and m mono. For any object $B \in \mathscr{E}$, the slice category $\mathscr{E}/B$ is also a topos. Any arrow $k : A \to 0$ is an isomorphism. Any arrow $0 \to B$ is monic. Any object $C \in \mathscr{E}$ has a monomorphism to an injective object. All of these facts can be found as theorems, lemma's and propositions in [LM91, Chapter 3].

We shall see during the thesis that $\mathscr{E}$ is also like a "space", with "points".

Definition 2.1.7. A *subobject* of $B \in \mathscr{E}$ is an isomorphism class of monomorphisms $m : S \rightarrowtail B$. Two subobjects lie in the same isomorphism class if and only if there is an isomorphism between them making the obvious triangle commute. The set of subobjects is denoted by $\mathrm{Sub}_{\mathscr{E}}(B)$.

Definition 2.1.8. Let $\mathbf{C}$ be a category and take an object $C \in \mathbf{C}$. By a *sieve* we mean a subfunctor $S \subseteq \mathbf{y}(C)$.

Among all objects in a topos is one which requires special attention. We shall construct it here using our definition of a topos, but we stress that in the elementary definition of a topos, it is *part of the definition.*

Construction 2.1.9 (Ω for presheaf toposes)**.** Suppose that $\mathscr{E}$ is a presheaf topos. Define a presheaf Ω as follows. For every object $C \in \mathbf{C}$, we set
$$\Omega(C) = \{\text{sieves on } C\}.$$

If $f : D \to C$ is a morphism in $\mathbf{C}$, the restriction map $\Omega(f)$ is given by pulling back the sieve. Define a natural transformation $\mathrm{true} : 1 \to \Omega$ which on the component C works by sending the unique element of 1 to the maximal sieve $\mathbf{y}(C)$.

Construction 2.1.10 (Ω for sheaf toposes)**.** Suppose that $\mathscr{E}$ is a sheaf topos on a topological space X. Define a presheaf Ω on $\mathcal{O}(X)$ by

$$\Omega(U) = \{W \mid W \subset U,\ W \text{ open in } X\}, \qquad U \in \mathcal{O}(X).$$

If $V \subset U$ are opens of X, the restriction map is given by intersecting with V. The so-constructed presheaf Ω is a sheaf by [LM91, Theorem II.9.2]. Define a natural transformation $\mathrm{true} : 1 \to \Omega$ which on the component $U \in \mathcal{O}(X)$ works by sending the unique element of 1 to the element $U \in \Omega(U)$.

Construction 2.1.11 (Ω for G-sets)**.** Suppose that $\mathscr{E}$ is equivalent to G-$\mathbf{Set}$ for some topological group G. Take $\Omega = 1 \sqcup 1$ with trivial G-action and choose a monomorphism $\mathrm{true} : 1 \to \Omega$.

Proposition 2.1.12. *For each object $A \in \mathscr{E}$ there is a natural isomorphism*

$$\mathrm{Sub}_{\mathscr{E}}(A) \cong \mathrm{Hom}_{\mathscr{E}}(A, \Omega)$$

Proof. Given a monomorphism $m : S \rightarrowtail A$, construct a classifying map. For the other way, take the pullback with the map $\mathrm{true} : 1 \to \Omega$. $\square$

Thus, Ω may be regarded as the representable object for subobjects. For this reason, we call it the *subobject classifier*.

Before we move on to the next section, there is one construction which we'll use too.

Definition 2.1.13. Let $P \in \mathbf{Set}^{\mathbf{C}^{op}}$ be a presheaf. The *category of elements*, denoted

$$\int_{\mathbf{C}} P$$

is the category whose objects are pairs (C, c) with $C \in \mathbf{C}$ and $c \in P(C)$. An arrow $f : (C, c) \to (C', c')$ is an arrow $f : C \to C'$ in $\mathbf{C}$ such that $c = P(f)(c')$, or in a more compact notation, such that $c = c' \cdot f$.

There is an evident projection functor

$$\pi_P : \int_{\mathbf{C}} P \to \mathbf{C}, \qquad (C, c) \mapsto C.$$

Thus, when composed with the Yoneda functor we have a functor back to the original presheaf topos that we started with, that is:

$$\mathbf{y} \circ \pi_P : \int_{\mathbf{C}} P \to \mathbf{Set}^{\mathbf{C}^{op}}, \qquad (C, c) \mapsto \mathrm{Hom}_{\mathbf{C}}(-, C).$$

Proposition 2.1.14. *We have a natural isomorphism*

$$P \cong \operatorname{colim}\left(\mathbf{y} \circ \pi_P\right).$$

In other words, every presheaf is a colimit of representables.

Proof. This is [LM91, Corollary I.3]. □

2.2 The Morphisms Between Toposes

Let $f : X \to Y$ be a continuous map of topological spaces. If $F \in \mathrm{Sh}(\mathcal{O}(X))$, we can turn it into a sheaf on Y using f by defining

$$f_*(F)(V) = F(f^{-1}V), \qquad V \in \mathcal{O}(Y).$$

On the other hand, given a sheaf $G \in \mathrm{Sh}(\mathcal{O}(Y))$, we can turn it into a sheaf on X using f by defininig

$$f^*(G)(U) = \operatorname*{colim}_{V \supset f(U)} G(V), \qquad U \in \mathcal{O}(X).$$

If f is an open map, we have the benefit of seeing that the colimit degenerates into the simpler definition $G(f(U))$. One can show that f^* is left adjoint to f_*, that is $f^* \dashv f_*$. Moreover, one can show that f^* is left exact (so preserves finite limits). This motivates the following definition.

Definition 2.2.1. Let $\mathscr{F}$ and $\mathscr{E}$ be toposes. A *geometric morphism*

$$f : \mathscr{F} \to \mathscr{E}$$

is an adjoint pair of functors $f^* \dashv f_*$ where f_* is a functor $f_* : \mathscr{F} \to \mathscr{E}$, f^* is a functor $f^* : \mathscr{E} \to \mathscr{F}$ and f^* is left exact. The functor f^* is called the *inverse image part* and the functor f_* is called the *direct image part*.

So we have a category **Topos** whose objects are toposes and whose morphisms are geometric morphisms. We shall be needing the fact that every Grothendieck topos always comes equipped with a unique geometric morphism to **Set**.

Proposition 2.2.2. *Let $\mathscr{E}$ be a Grothendieck topos. Then there exists a unique geometric morphism $\gamma : \mathscr{E} \to \mathbf{Set}$, called the* structure morphism. *Its direct image part is given by the global sections functor*

$$\gamma_*(F) = \Gamma(F) = \operatorname{Hom}_{\mathscr{E}}(1, F),$$

and its inverse image part is given by the constant sheaf functor

$$\gamma^*(S) = \Delta(S) = \coprod_{s \in S} 1.$$

Proof. The fact that γ exists is a matter of checking that $\gamma^* \dashv \gamma_*$ and checking that γ^* preserves finite limits. For uniqueness, suppose that $f : \mathscr{E} \to \mathbf{Set}$ is a geometric morphism. Then for any set S we have $S \cong \mathbf{Set}(1, S)$ and this gives

$$f_*E \cong \mathbf{Set}(1, f_*E) \cong \mathscr{E}(f^*1, E) \cong \mathscr{E}(1, E) \cong \Gamma E$$

for any object $E \in \mathscr{E}$. So we get a natural isomorphism $f_* \cong \gamma_*$. $\square$

A point of a topological space $X \in \mathbf{Top}$ is an element $x \in X$. Equivalently, it is a map $x : 1 \to X$. This gives us a geometric morphism $x : \mathrm{Sh}(\mathcal{O}(1)) = \mathbf{Set} \to \mathrm{Sh}(\mathcal{O}(X))$ with direct image part x_* a constant sheaf and inverse image part x^* a skyscraper sheaf.

Definition 2.2.3. Let $\mathscr{E}$ be a topos. A *point* of $\mathscr{E}$ is a geometric morphism $p : \mathbf{Set} \to \mathscr{E}$.

2.3 Tensor Products

Let $P : \mathbf{C}^{op} \to \mathbf{Set}$ be a presheaf and $A : \mathbf{C} \to \mathbf{Set}$ a set-valued functor.

Definition 2.3.1. The *tensor product* $P \otimes_{\mathbf{C}} A$ of P and A is defined to be the coequalizer of

$$\textstyle\coprod_{C,C'} P(C) \times \mathrm{Hom}(C', C) \times A(C') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_C P(C) \times A(C) \xrightarrow{\phi} P \otimes_{\mathbf{C}} A, \tag{2.1}$$

where for elements $p \in P(C)$, $u : C' \to C$ and $a' \in A(C')$ the maps θ and τ are given by

$$\theta(p, u, a') = (p \cdot u, a'), \qquad \tau(p, u, a') = (p, u \cdot a').$$

The elements of $P \otimes_{\mathbf{C}} A$ are all of the form $\phi(p, a)$. Let us introduce a useful notation for this set. Write

$$\phi(p, a) = p \otimes a, \qquad p \in P(C),\ a \in A(C).$$

By definition of θ and τ, we then see that

$$p \cdot u \otimes a' = p \otimes u \cdot a', \qquad p \in P(C),\ u : C' \to C,\ a' \in A(C'). \tag{2.2}$$

So in other words, the set $P \otimes_{\mathbf{C}} A$ is the quotient of the set

$$\coprod_C P(C) \times A(C)$$

by the equivalence relation generated by eq. (2.2).

Theorem 2.3.2. *Let $P : \mathbf{C}^{op} \to \mathbf{Set}$ be a presheaf and $A : \mathbf{C} \to \mathbf{Set}$ a set-valued functor. Then the functor*

$$R_A : \mathbf{Set} \to \mathbf{Set}^{\mathbf{C}^{op}}$$

defined for each set E and each object $C \in \mathbf{C}$ by

$$R_A(E)(C) = \mathrm{Hom}_{\mathbf{Set}}(A(C), E)$$

has a left adjoint L_A defined for each presheaf P as the equalizer $P \otimes_{\mathbf{C}} A$ of eq. (2.1).

Proof. See [LM91, Theorem VII.2.1] for the details. □

Definition 2.3.3. A set-valued functor $A : \mathbf{C} \to \mathbf{Set}$ is said to be *flat* if the induced tensor product functor $- \otimes_{\mathbf{C}} A$ is left exact.

Theorem 2.3.4. *Points of the presheaf topos $\mathbf{Set}^{\mathbf{C}^{op}}$ correspond to flat functors $A : \mathbf{C} \to \mathbf{Set}$.*

Proof. The proof can be found in [LM91, Theorem VII.5.2]. When a functor $A : \mathbf{C} \to \mathbf{Set}$ is flat, we get from theorem 2.3.2 an adjunction $- \otimes_{\mathbf{C}} A \dashv \underline{\mathrm{Hom}}_{\mathbf{C}}(A, -)$, and this constitutes a geometric morphism. To go from a point $p : \mathbf{Set} \to \mathbf{Set}^{\mathbf{C}^{op}}$ to a flat functor $A : \mathbf{C} \to \mathbf{Set}$, use the Yoneda embedding $\mathbf{C} \xrightarrow{\mathbf{y}} \mathbf{Set}^{\mathbf{C}^{op}} \xrightarrow{p^*} \mathbf{Set}$, and define A in this way. That is, take $A = p^* \circ \mathbf{y}$. □

Construction 2.3.5 (How to find points)**.** It is not immediately clear how to determine the points of a topos. For a presheaf topos $\mathscr{E} = \mathbf{Set}^{\mathbf{C}^{op}}$, we can use the following construct. Take an object $A \in \mathbf{C}$. Then we have a covariant representable functor $\mathrm{Hom}(A, -) : \mathbf{C} \to \mathbf{Set}$. This functor is flat, because if P is a presheaf on $\mathbf{C}$, then one can show that $P \otimes \mathrm{Hom}(A, -) \cong P(A)$. This preserves products and equalizers, so is left exact. The right adjoint is then given by the presheaf

$$\underline{\mathrm{Hom}}_{\mathbf{C}}(\mathrm{Hom}(A, -), -)$$

defined for each set S by

$$\underline{\mathrm{Hom}}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(A, -), S)(B) = \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathbf{C}}(A, B), S), \qquad B \in \mathbf{C}.$$

For a set-valued functor $A : \mathbf{C} \to \mathbf{Set}$, replace now the category $\mathbf{Set}$ with an arbitrary cocomplete category $\mathscr{E}$. The cocompleteness of $\mathscr{E}$ provides the same definition of $\otimes_{\mathbf{C}}$ as a coequalizer.

Definition 2.3.6. A functor $A : \mathbf{C} \to \mathscr{E}$ is said to be *flat* if the corresponding tensor product functor $- \otimes_{\mathbf{C}} A : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathscr{E}$ is left exact.

Theorem 2.3.7. *The adjunction of theorem 2.3.2 holds for cocomplete $\mathscr{E}$.*

Proof. See [LM91, Theorem VII.2.1bis]. □

Theorem 2.3.8. *Let $\mathscr{E}$ be a topos with small colimits and let $\mathbf{C}$ be any category. Geometric morphisms $\mathscr{E} \to \mathbf{Set}^{\mathbf{C}^{op}}$ correspond to flat functors $\mathbf{C} \to \mathscr{E}$.*

Proof. See [LM91, Theorem VII.7.2]. □

Theorem 2.3.9. *Let $\mathscr{E}$ be a topos with small colimits and let $\mathbf{C}$ be a category. Then a functor $A : \mathbf{C} \to \mathscr{E}$ is flat if and only if it is filtering.*

Proof. See [LM91, Theorem VII.9.1]. □

2.4 Etale Spaces are Sheaves

One problem with the generality of sites is that it is not easy to define the stalk of a sheaf, a classical notion. Here we quickly describe a useful equivalence of categories that we'll use throughout. Recall that if X is a space, then $\mathcal{O}(X)$ denotes the lattice of opens of X, regarded as category with its canonical Grothendieck topology.

Definition 2.4.1. Let X be a topological space and take a presheaf $P \in \mathbf{Set}^{\mathbf{C}^{op}}$ and take a point $x \in X$. The *stalk at x* is defined to be the colimit

$$P_x := \operatorname*{colim}_{x \in U} P(U).$$

Elements of the stalk are called *germs*.

Definition 2.4.2. Let X be a topological space. The category $\mathbf{LH}/X$ is defined as follows. Its objects are local homeomorphisms $f : Y \to X$. also called *etale maps*. The space Y is regarded as an *etale space* over X. An arrow from $(f : Y \to X)$ to $(g : Z \to X)$ is a continuous map $h : Y \to Z$ making the obvious triangle commute.

It's not hard to prove that h is in fact automatically an etale map too.

Theorem 2.4.3. *There is an equivalence of categories*

$$\mathbf{LH}/X \cong \mathrm{Sh}(\mathcal{O}(X))$$

which sends an etale space $p : Y \to X$ to the sheaf of sections $\Gamma(p)$ defined on opens U of X by

$$\Gamma(p)(U) = \{s : U \to Y \mid p \circ s = \mathrm{id}_U\}$$

and which sends a sheaf F on $\mathcal{O}(X)$ to the etale space

$$\pi : \bigsqcup_{x \in X} F_x \to X$$

where π is the etale map sending a germ to the point at which the colimit was taken. The etale space $\bigsqcup F_x$ is topologized in a suitable way.

Proof. It's a matter of checking the definitions. For the details, see [LM91, Corollary II.6.3]. $\square$

Hence we see that $\mathbf{LH}/X$ is a topos.

Chapter 3

The Fundamental Group

This preliminary section treats the concept of a fundamental group for a topos. $\mathscr{E}$ always denotes a topos. Here we follow mostly Chapter 8 of [Joh77], although much of the theory can also be found in [Gro71, Exposé V, Sections 4, 5, and 6] or [Len08, Chapter 3] or even [Sza10]. The stacks project tag [Sta16, 0BMQ] (clickable link if you read this on a computer) is also a good resource. We shall define the notion of a Galois category, then prove that a certain full subcategory of every topos is a Galois category, and with this define the fundamental group of a topos. At the end of this section we prove an original result regarding locally constant finite presheaves.

3.1 Galois Categories

Definition 3.1.1. Let $\mathscr{E}$ be a topos. Then $\mathscr{E}$ is called a *Boolean* topos if there is an isomorphism $\Omega \cong 1 + 1$ for its subobject classifier Ω.

For instance, **Set** is Boolean. Most Grothendieck toposes are not Boolean.

Definition 3.1.2. By a *Galois category* we mean a pair $(\mathscr{G}, F)$ where $\mathscr{G}$ is a small Boolean topos and $F : \mathscr{G} \to \mathbf{set}$ is an exact, isomorphism-reflecting functor to the category of finite sets, called the *fundamental functor* or *fiber functor*.

By a proposition yet to state and prove, there is up to equivalence only one such fiber functor F for any given $\mathscr{G}$, so we usually denote a Galois category by just $\mathscr{G}$.

Definition 3.1.3. Let $\mathscr{E}$ be a Grothendieck topos and recall the structure morphism $\gamma : \mathscr{E} \to \mathbf{Set}$ from proposition 2.2.2. Then $\mathscr{E}$ is called *connected* if γ^* is full and faithful.

If X is a topological space, then the topos of sheaves is connected in the sense of definition 3.1.3 if and only if X is connected as a topological space.

If $\mathscr{E}$ is a presheaf topos on $\mathbf{C}$, then $\mathscr{E}$ is connected in the sense of definition 3.1.3 if and only if the underlying non-directed graph of $\mathbf{C}$ where the vertices are the objects of $\mathbf{C}$ and the edges are the morphisms of $\mathbf{C}$ is connected as a graph.

Definition 3.1.4. Suppose that $\mathscr{E}$ is a connected Grothendieck topos. An object $X \in \mathscr{E}$ is called *locally constant finite*, or *l.c.f.*, if there exists an object $U \in \mathscr{E}$ whose unique morphism $U \to 1$ is epi, a finite set $n \in \mathbf{set}$ and an isomorphism $\varphi : X \times U \to \Delta(n) \times U$ in the slice topos $\mathscr{E}/U$. The full subcategory of all locally constant finite objects of $\mathscr{E}$ is denoted by $\mathscr{E}_{\mathrm{lcf}}$.

Spelled out, this means that we require a commutative diagram

$$\begin{tikzcd}
X \times U \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_U"'] & & \Delta(n) \times U \arrow[dl, "\mathrm{pr}_U"] \\
& U &
\end{tikzcd}$$

Proposition 3.1.5. *Let $\mathscr{E}$ be a connected Grothendieck topos. Then the following is true.*

1. *$\mathscr{E}_{\mathrm{lcf}}$ is a small Boolean topos.*

2. *If $f : \mathscr{F} \to \mathscr{E}$ is a geometric morphism, then f^* restricts to a functor $\mathscr{E}_{\mathrm{lcf}} \to \mathscr{F}_{\mathrm{lcf}}$ which preserves finite limits, exponentials and the subobject classifier.*

3. *If $p : \mathbf{Set} \to \mathscr{E}$ is a point, then the functor $\mathscr{E}_{\mathrm{lcf}} \to \mathbf{Set}_{\mathrm{lcf}} = \mathbf{set}$ induced by p^* reflects isomorphisms.*

Proof. [Joh77, Proposition 8.42]. The proof uses the Mitchell-Bénabou language of a topos. This is a fascinating topic, but we won't go into it here. □

An *atom* $A \in \mathscr{E}$ is an object which is not the initial object 0 but has no non-trivial subobjects. That is, $\# \operatorname{Sub}_{\mathscr{E}}(A) = 2$.

Lemma 3.1.6. *Let $(\mathscr{G}, F)$ be a Galois category. Then*

1. *For every $X \in \mathscr{G}$, there are atoms $A_1, \ldots, A_n$ such that $X = A_1 + \ldots + A_n$, and this decomposition is unique up to reordering.*

2. *If $A \in \mathscr{G}$ is an atom and $f : A \to A$ is an endomorphism, then f is an automorphism.*

Proof. The proof is in [Joh77, Lemma 8.44]. Here we give a sketch.

(1). Let $X \in \mathscr{G}$ be given. If $X \cong 0$, then it is a coproduct of zero atoms, so assume $X \ncong 0$. The fiber functor F preserves 0 and reflects iso's, so $F(X) \neq \emptyset$. If X is not an atom we can decompose it into a non-trivial coproduct $X \cong X_1 + X_2$, so we find $F(X) \cong F(X_1) + F(X_2)$. Since $F(X)$ is a finite set this process ends eventually.

(2) Let $f : A \to A$ be an endomorphism of an atom A. Since $A \ncong 0$, so is the image of f. So the image must be A. So f is epi. So $F(f)$ is an epi from $F(A)$ to $F(A)$, hence a bijection. Since F reflects iso's, f is an automorphism. □

Proposition 3.1.7. *Let $(\mathscr{G}, F)$ be a Galois category. Then F is pro-presentable. This means that there is a filtered inverse system $\{A_i : i \in \mathbf{I}\}$ where $A_i \in \mathscr{G}$ and a natural isomorphism*

$$F(X) \cong \lim_{i \in \mathbf{I}} \operatorname{Hom}(A_i, X)$$

Proof. This is [Joh77, Proposition 8.45]. We'll give a proof sketch here. Let $\mathbf{I}$ be the category whose objects are pairs (A, a) where $A \in \mathscr{G}$ is an atom and $a \in F(A)$ an element. A morphism $f : (A, a) \to (B, b)$ in $\mathbf{I}$ is a morphism $f : A \to B$ such that $F(f)(a) = b$. Then I claim that $\mathbf{I}$ is a poset. Indeed, if $f, g : A \rightrightarrows B$ are two morphisms such that $F(f)(a) = F(g)(a)$, then their equalizer is non-zero so must be all of A. Next I claim that $\mathbf{I}^{op}$ is filtered. So let $(A, a), (B, b) \in \mathbf{I}$. Then they are preceded by $(C, (a, b))$ where $(a, b) \in F(A \times B) \cong F(A) \times F(B)$, and C is the component in the decomposition (from lemma 3.1.6) of $A \times B$ that contains (a, b).

Now if $(A, a) \in \mathbf{I}$, then we can interpret a as a natural transformation $\operatorname{Hom}_{\mathscr{G}}(A, -) \to F$ where a morphism $f \in \operatorname{Hom}_{\mathscr{G}}(A, X)$ is sent to $F(f)(a) \in F(X)$. By construction of $\mathbf{I}$ these natural transformations combine into one natural transformation

$$\eta : G = \lim_{(A,a) \in \mathbf{I}} \operatorname{Hom}_{\mathscr{G}}(A, -) \to F.$$

η is epi by the first part of lemma 3.1.6. I claim that η is also mono. For suppose that $x, y \in G(X)$ are such that $\eta_X(x) = \eta_X(y)$. Then both x, y can be represented as morphism $x, y : A \rightrightarrows X$ such that $F(x)(a) = F(y)(a)$. Then it follows that their equalizer is non-zero, so their equalizer is all of A, from which we conclude that $x = y$ in $\operatorname{Hom}_{\mathscr{G}}(A, X)$ and hence in $G(X)$. □

Let $(A, a) \in \mathbf{I}$. Then we have a natural map

$$\operatorname{Aut}_{\mathscr{G}}(A) = \operatorname{Hom}_{\mathscr{G}}(A, A) \to F(A), \qquad f \mapsto F(f)(a). \tag{3.1}$$

This map is mono, because the maps in $\mathbf{I}^{op}$ are epi.

Definition 3.1.8. We say that (A, a) is a *Galois* or *normal* object if the above map in eq. (3.1) is also epi.

Proposition 3.1.9. *For any $X \in \mathscr{G}$ there is a Galois object (A, a) such that the natural map $\mathrm{Hom}_{\mathscr{G}}(A, X) \to F(X)$ is a bijection. So in particular the Galois objects form a cofinal subcategory of $\mathbf{I}$.*

Proof. This is [Joh77, Proposition 8.46]. We'll give a proof sketch. It's possible to find $(B, b) \in \mathbf{I}$ such that every $x \in F(X)$ comes from a morphism $\overline{x} : B \to X$ because $F(X)$ is finite and $\mathbf{I}^{op}$ is filtered. Let A be the image of the map

$$B \to \prod_{x \in F(X)} X$$

in $\mathscr{G}$ whose component at x is the morphism $\overline{x}$. Then A is a quotient of B. So A is an atom. So if a is the image of b in $F(A)$ the pair (A, a) is an element of $\mathbf{I}$ so that we have an isomorphism $\mathrm{Hom}_{\mathscr{G}}(A, X) \xrightarrow{\sim} F(X)$. Prove that the constructed pair (A, a) is Galois. □

If G is a topological group, recall that the category of all continuous G-sets, denoted G-**Set**, is a topos. It's not hard to show that

$$(G\text{-}\mathbf{Set})_{\mathrm{lcf}} = G\text{-}\mathbf{set}.$$

Theorem 3.1.10 (Grothendieck)**.** *Let $\mathscr{G}$ be a small category and $F : \mathscr{G} \to$ **set** a functor. Then the following are equivalent.*

1. *$(\mathscr{G}, F)$ is a Galois category.*
2. *There exists a topological group G and an equivalence of categories $\mathscr{G} \cong G$-**set** which identifies F with the forgetful functor.*

Moreover, if we demand that G be profinite, then it is determined up to isomorphism by $(\mathscr{G}, F)$.

Proof. This is [Joh77, Theorem 8.47]. We'll give a proof sketch.

(2) $\implies$ (1) is easy; G-**set** is a Boolean topos and the functor which forgets the G-action preserves finite limits, exponentials and the subobject classifier.

(1) $\implies$ (2). Let $\mathbf{N} \subset \mathbf{I}$ be the full subcategory of Galois objects. Let $f : (A, a) \to (B, b)$ be a morphism in $\mathbf{N}$. Then we have bijections $F(A) \cong \mathrm{Aut}_{\mathscr{G}}(A)$ and $F(B) \cong \mathrm{Aut}_{\mathscr{G}}(B)$, so we can view the map $F(f) : F(A) \to F(B)$ as a map $\varphi : \mathrm{Aut}(A) \to \mathrm{Aut}(B)$. Show that φ is a group homomorphism. For every $X \in \mathscr{G}$, put a continuous G-action on $F(X)$ by *choosing* a Galois object (A, a). This defines a factorization of F through the inclusion functor G-**set** $\to$ **set**. Construct a quasi-inverse functor for F. The last statement of the theorem follows from [Len08, Exercise 3.11]. □

Definition 3.1.11. Let $(\mathscr{G}, F)$ be a Galois category. The topological group G from theorem 3.1.10 is called the *fundamental group* of the Galois category.

Corollary 3.1.12. *Let $(\mathscr{G}, F)$ be a Galois category and $F' : \mathscr{G} \to$ **set** be another functor such that $(\mathscr{G}, F')$ is also a Galois category. Then there is a natural isomorphism $F \cong F'$.*

Proof. This is [Joh77, Corollary 8.48]. We'll give a sketch here. By theorem 3.1.10 we may assume that $\mathscr{G} = G$-**set** for some profinite group G, and that F is the forgetful functor forgetting the G-action. Now F' preserves coproducts so it suffices to prove the natural isomorphism on atoms by lemma 3.1.6. Let $\mathbf{I}'$ be the poset consisting of objects (A, a') where $A \in \mathscr{G}$ is an atom and $a' \in F'(A)$. Let X be the limit

$$X := \lim_{(A,a') \in \mathbf{I}'} F(A)$$

in **Set**. It is straightforward to show that X is not the empty set (apply Zorn's lemma, use that every $F(A)$ is finite and non-empty, that $\mathbf{I}'^{op}$ is filtered and that the transition maps in the inverse system are epis). *Choose* an element $x \in X$. Let $x(a')$ be the image of x in $F(A)$ corresponding to the factor (A, a') in the limit. The assigment $a' \mapsto x(a')$ defines a function $\varphi : F'(A) \to F(A)$ which is natural in A. We'll show that φ is mono. For suppose that $x(a') = x(a'')$. Then find $(B, b') \in \mathbf{I}'$ and morphisms $f, g : B \to A$ in $\mathscr{G}$ such that $F'(f)(b') = a'$ and $F'(g)(b') = a''$. Then $F(f)$ and $F(g)$ agree on $x(b')$ in $F(B)$, so their equalizer is non-empty, hence $f = g$. Finish by proving that φ is a natural isomorphism on atoms, hence on all objects by lemma 3.1.6. □

The term "natural isomorphism" for $F \cong F'$ is somewhat misleading in this case, because it is not canonical. It depends on the choice of x.

Finally, we are ready to define the fundamental group of a Grothendieck topos $\mathscr{E}$.

3.2 The Fundamental Group of a Topos

Definition 3.2.1. Let $\mathscr{E}$ be a connected Grothendieck topos with a point $p :$ **Set** $\to \mathscr{E}$. The *fundamental group* $\pi_1(\mathscr{E}, p)$ is defined to be the fundamental group of the Galois category $(\mathscr{E}_{\mathrm{lcf}}, p^*)$.

Proposition 3.2.2. *Let* $\mathbf{C}$ *be a category and* $P \in \mathbf{Set}^{\mathbf{C}^{op}}$. *If the topos* $\mathbf{Set}^{\mathbf{C}^{op}}$ *is connected, then the following are equivalent.*

1. *P is locally constant finite.*

2. *Every restriction map of P is a bijection.*

Proof. (1) $\implies$ (2). There exists a presheaf U with $U \to 1$ epi, a set $S \in$ **Set** and an isomorphism $\varphi : P \times U \to \Delta S \times U$ in $\mathbf{Set}^{\mathbf{C}^{op}}/U$. Let

$A \in \mathbf{C}$. From the commutativity of the diagram

$$\begin{tikzcd} PA \times UA \arrow[rr, "\varphi_A"] \arrow[dr, "\mathrm{pr}_{UA}"'] & & S \times UA \arrow[dl, "\mathrm{pr}_{UA}"] \\ & UA & \end{tikzcd}$$

we may conclude that φ_A is of the form

$$\varphi_A = (\psi_A, \mathrm{pr}_{UA}),$$

where

$$\psi_A : PA \times UA \to S$$

is a map of sets. So for every morphism $f : A \to B$ in $\mathbf{C}$ we have a commutative diagram

$$\begin{tikzcd} PB \times UB \arrow[r, "(\psi_B, \mathrm{pr}_{UB})"] \arrow[d, "Pf \times Uf"'] & S \times UB \arrow[d, "\mathrm{id}_S \times Uf"] \\ PA \times UA \arrow[r, "(\psi_A, \mathrm{pr}_{UA})"'] & S \times UA \end{tikzcd}$$

where the horizontal arrows are bijections. We'll show that Pf is a bijection. First the injectivity part. Suppose that $x, y \in PB$ are such that $x \cdot f = y \cdot f$. Take an element $z \in UB$. This is possible, because $U \to 1$ is epi. Then we see that

$$\begin{tikzcd} (x, z) \arrow[r, mapsto] \arrow[d, mapsto] & (\psi_B(x, z), z) \arrow[d, mapsto] \\ (x \cdot f, z \cdot f) \arrow[r, mapsto] & (\psi_A(x \cdot f, z \cdot f), z \cdot f) = (\psi_B(x, z), z \cdot f) \end{tikzcd}$$

Therefore,

$$\psi_B(x, z) = \psi_A(x \cdot f, z \cdot f) = \psi_A(y \cdot f, z \cdot f) = \psi_B(y, z).$$

So $\varphi_B(x, z) = \varphi_B(y, z)$. Since φ_B is a bijection, we conclude that $x = y$. Injectivity is proven. Let us now prove surjectivity. So let $y \in PA$ be an arbitrary element. Take an element $z \in UB$. Then $(y, z \cdot f) \in PA \times UA$. Thus $(\psi_A(y, z \cdot f), z \cdot f) \in S \times UA$. So $(\psi_A(y, z \cdot f), z) \in S \times UB$. This element corresponds uniquely to an element $(x, z') \in PB \times UB$ via φ_B^{-1}. Since the diagram commutes, we must have $y = x \cdot f$ and $z \cdot f = z' \cdot f$. Thus Pf is surjective. We conclude that Pf is a bijection.

(2) $\implies$ (1). Suppose P is a presheaf with the property that all of its restriction maps are bijections. Take any object $A_0 \in \mathbf{C}$. Let

$$S = \coprod_{a \in PA_0} 1.$$

Observe that for all objects $A \in \mathbf{C}$ we now have a bijection of sets $PA \cong S$ because the topos is assumed to be connected. Define the presheaf U as follows. For every object $A \in \mathbf{C}$, we set

$$UA = \mathrm{Iso}(S, PA).$$

Given a morphism $f : A \to B$ in $\mathbf{C}$, the restriction map $Uf : UB \to UA$ is defined by

$$\mathrm{Iso}(S, PB) \to \mathrm{Iso}(S, PA), \qquad g \mapsto (Pf) \circ g.$$

This is well-defined, because Pf is a bijection for each morphism f by assumption. The restriction maps are compatible with composition of morphisms because P is a presheaf. The unique morphism $U \to 1$ is epi since the set $\mathrm{Iso}(S, PA)$ is non-empty for all $A \in \mathbf{C}$. Define the natural transformation $\varphi : P \times U \to \Delta S \times U$ as follows. Given an object $A \in \mathbf{C}$, the component φ_A is defined as

$$\varphi_A : PA \times \mathrm{Iso}(S, PA) \to S \times \mathrm{Iso}(S, PA), \qquad (a, g) \mapsto \left(g^{-1}(a), g\right).$$

The inverse of φ_A is then given by $\varphi_A^{-1}(s, g) = (g(s), g)$. Clearly, φ_A respects the projection onto UA for every $A \in \mathbf{C}$. It remains to check that for every $f : A \to B$ in $\mathbf{C}$ the usual diagram commutes. So let $f : A \to B$ be any morphism in $\mathbf{C}$. Let $(a, g) \in PB \times \mathrm{Iso}(S, PB)$. Then on the one hand,

$$((\mathrm{id}_S \times Uf) \circ \varphi_B)(a, g) = (\mathrm{id}_S \times Uf)\left(g^{-1}(a), g\right) = \left(g^{-1}(a), (Pf) \circ g\right).$$

On the other hand,

$$(\varphi_A \circ (Pf \times Uf))(a, g) = \varphi_A((Pf)(a), (Pf) \circ g) = \left(g^{-1}(a), (Pf) \circ g\right).$$

Hence φ is a natural isomorphism. $\square$

3.3 Worked Example: The Topos of Graphs

In this section we explore the topos of graphs. We'll compute its fundamental group. Let $\mathbf{C}$ be the category given by

$$
\begin{tikzcd}
V \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & E
\end{tikzcd}
$$

and let $\mathscr{E} = \mathbf{Set}^{\mathbf{C}^{op}}$. The objects of $\mathscr{E}$ may be viewed as graphs. The topos $\mathscr{E}$ furthermore has a structure morphism to $\mathbf{Set}$, namely

$$\gamma : \mathscr{E} \to \mathbf{Set}, \qquad \gamma_*(X) = \mathrm{Hom}_{\mathscr{E}}(1, X),\ \gamma^*(S) = \Delta S.$$

The topos $\mathscr{E}$ is connected. The next thing to do is to find a point of $\mathscr{E}$. Applying construction 2.3.5 with the object E of $\mathscr{E}$, we obtain a geometric morphism

$$E : \mathbf{Set} \to \mathscr{E}$$

whose left exact left adjoint E^* is given by sending a graph $X \in \mathscr{E}$ to the set of its edges $X(E)$, and whose right adjoint E_* is given by sending a set S to the "underline Hom", that is, for any set $S \in \mathbf{Set}$ we find

$$\underline{\mathrm{Hom}}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(E,-), S)(V) = \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathbf{C}}(E, V), S) = \mathrm{Hom}(\emptyset, S) = 1,$$
$$\underline{\mathrm{Hom}}_{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(E,-), S)(E) = \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathbf{C}}(E, E), S) = \mathrm{Hom}(1, S) = S.$$

Note that the subobject classifier Ω, like for all presheaf categories, is given by

$$
\begin{aligned}
\Omega(V) &= \{\text{sieves on } V\} = \{\emptyset, \mathbf{y}V\},\\
\Omega(E) &= \{\text{sieves on } E\} = \{\emptyset, \{s\}, \{t\}, \{s,t\}, \mathbf{y}E\},\\
\Omega(s) &= \text{ pullback the sieve by } s,\\
\Omega(t) &= \text{ pullback the sieve by } t.
\end{aligned}
$$

So in a picture, the graph Ω looks like Figure 3.1.

The subobject classifier is thus given by

$$\mathrm{true} : 1 \to \Omega, \qquad v \mapsto \mathbf{y}V,\ e \mapsto \mathbf{y}E.$$

Now let $\mathscr{E}_{\mathrm{lcf}}$ be the Boolean topos of locally constant objects of $\mathscr{E}$. Since $\mathscr{E}_{\mathrm{lcf}}$ is Boolean, its subobject classifier is given by

$$\Omega_{\mathrm{lcf}} = 1 + 1$$

generated by the morphisms

$$\mathrm{true} : 1 \to \Omega, \qquad \mathrm{false} = \neg \circ \mathrm{true} : 1 \to \Omega.$$

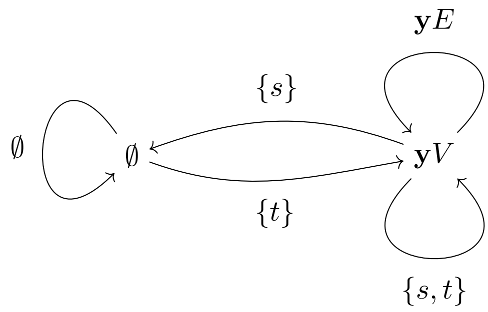


Figure 3.1: The subobject classifier of $\mathscr{E}$

Figure 3.2: The subobject classifier of $\mathscr{E}_{\mathrm{lcf}}$

So the graph Ω_{lcf} looks like Figure 3.2.

Let C_n be a cyclic graph with n vertices and n edges, $n > 0$. Then C_n is locally constant finite; take U to be the connected graph with n vertices and $n-1$ edges. The product $C_n \times U$ is then a disjoint union of n copies of U, so that we have an isomorphism $C_n \times U \cong \Delta(n) \times U$, where $\Delta(n)$ are n copies of the graph with 1 vertex and 1 edge. Moreover, this isomorphism respects the projection down to U, so that C_n is locally constant finite as per definition 3.1.4.

I claim that disjoint unions of the C_n are the only type of objects in $\mathscr{E}_{\mathrm{lcf}}$. Indeed, by proposition 3.2.2, the restriction maps of an object $X \in \mathscr{E}_{\mathrm{lcf}}$ must be constant. This just translates to the fact that there are as many vertices as there are edges.

Moreover, each C_n is an atom in $\mathscr{E}_{\mathrm{lcf}}$. Indeed, from the definition of Ω_{lcf} we find that removing an edge means that we are forced to remove the vertices that the edge was attached to, too (this need not happen in $\mathscr{E}$). Thus C_n has no subobjects other than the empty graph.

Next, we claim that every C_n is also a Galois object. Indeed, there is a $\mathbb{Z}/n\mathbb{Z}$ action on the edges of C_n; simply cyclic shift the edges. Clearly then $\mathrm{Aut}(C_n) \cong C_n(E)$. Thus we may conclude that

$$\pi_1(\mathscr{E}, E) = \lim_{\leftarrow n} \mathrm{Aut}(C_n) \cong \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}.$$

It is to be noted that if we chose a different point of $\mathscr{E}$, say the object V, using construction 2.3.5, we would end up with the same argumentation; there's a $\mathbb{Z}/n\mathbb{Z}$ action on the vertices, and it can only be a $\mathbb{Z}/n\mathbb{Z}$ action because the source and target vertices of the edges of C_n must match up.

Chapter 4

Simplicial Sets

In this preliminary section we touch upon some concepts regarding simplicial complexes. In some way they are just a special case of a presheaf topos. On the other hand, they have some intrinsic properties not found in other presheaf toposes. Most of this section is from [GJ09] and [JT08]. The aim of this section is to define the notion of a star, the nerve of a category and the geometric realization.

4.1 Basic Definitions

Definition 4.1.1. Let $n \geq 0$ be a natural number. By the boldface $\mathbf{n}$ we mean the *category* consisting of precisely $n+1$ objects, denoted by $0, 1, 2, \ldots, n-1, n$, and whose morphisms are precisely

$$0 \to 1 \to \cdots \to (n-1) \to n.$$

so that $\mathbf{n}$ may be regarded as a totally ordered set.

Definition 4.1.2. By $\mathbf{\Delta}$ we mean the category whose objects are the $\mathbf{n}$'s and whose morphisms are functors $\theta : \mathbf{n} \to \mathbf{m}$.

If $\theta : \mathbf{n} \to \mathbf{m}$ is a morphism in $\mathbf{\Delta}$, we can also view it as an order-preserving function on the totally ordered sets $\mathbf{n}$ and $\mathbf{m}$. It is fruitful to confuse the two viewpoints sometimes. Among the morphisms in $\mathbf{\Delta}$ are two special types, namely the coface and codegeneracy maps.

Definition 4.1.3. Let $\mathbf{n} \in \mathbf{\Delta}$ and let $j \in \mathbf{n}$. The j'th *coface* map is the morphism

$$d^j : \mathbf{n-1} \to \mathbf{n}$$

defined by

$$d^j\left(0 \to \cdots \to (n-1)\right) = 0 \to \cdots \to (j-1) \to (j+1) \to \cdots \to (n+1),$$

i.e., "skip j", and the j'th *codegeneracy* map is the morphism

$$s^j : \mathbf{n+1} \to \mathbf{n}$$

defined by

$$s^j\,(0 \to \cdots \to (n+1)) = 0 \to \cdots \to (j-1) \to j \to j \to (j+1) \to \cdots \to n,$$

i.e., "repeat j".

If $\theta : \mathbf{n} \to \mathbf{m}$ is an order-preserving map, one can decompose it uniquely into a composite $\theta = \alpha \circ \beta$ where $\beta : \mathbf{n} \to \mathbf{p}$ is surjective and $\alpha : \mathbf{p} \to \mathbf{m}$ is injective. Then β may be decomposed into a bunch of codegeneracy maps and α may be decomposed into a bunch of coface maps. So the morphisms in $\mathbf{\Delta}$ are generated by the coface and codegeneracy maps. Moreover, we have

Lemma 4.1.4 (The Cosimplicial Identities)**.** *Let $n \geq 2$. Consider the diagram*

$$\begin{tikzcd} \mathbf{n} \arrow[r, shift left, "s^j"] \arrow[d, shift left, "s^i"] & \mathbf{n-1} \arrow[l, shift left, "d^j"] \arrow[d, shift left, "s^i"] \\ \mathbf{n-1} \arrow[u, shift left, "d^i"] \arrow[r, shift left, "s^{j-1}"] & \mathbf{n-2} \arrow[l, shift left, "d^{j-1}"] \arrow[u, shift left, "d^i"] \end{tikzcd}$$

in $\mathbf{\Delta}$. Then

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \forall\, i < j \\ s^{j-1} s^i &= s^i s^j && \forall\, i < j \\ s^j d^i &= d^i s^{j-1} && \forall\, i < j \\ s^i d^i &= \mathrm{id} && \\ s^i d^j &= d^{j-1} s^i && \forall\, i < j-1 \\ s^i d^{i+1} &= \mathrm{id} && i = j-1 \end{aligned}$$

Moreover, the square of s's is an absolute pushout, and the square of d's is an absolute pullback.

Proof. The proof is easy. It's just a lot of bookkeeping. □

Recall that a (co)limit is termed absolute when it is preserved by *any* functor. Since every morphism $\theta : \mathbf{n} \to \mathbf{m}$ factors uniquely into a composite of coface and codegeneracy maps, we conclude from the previous lemma

Theorem 4.1.5. *The category $\mathbf{\Delta}$ has absolute pushouts of surjections and absolute non-empty intersections of injections.*

Definition 4.1.6. The category of *simplicial sets* $\mathbf{S}$ is defined to be $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$.

In practice, a simplicial set $X \in \mathbf{S}$ is determined when for each natural number n a set X_n is given together with face maps $d_i : X_{n+1} \to X_n$ and degeneracy maps $s_i : X_{n-1} \to X_n$.

Definition 4.1.7. The *standard n-simplex* is the simplicial set $\mathrm{Hom}_{\Delta}(-, \mathbf{n}) = \mathbf{\Delta}(-, \mathbf{n}) = \mathbf{yn}$.

By a standard application of Yoneda's lemma we immediately obtain

Lemma 4.1.8. *Let $X \in \mathbf{S}$ be a simplicial set. Then for each natural number $n \geq 0$ we have*

$$X_n \cong \mathrm{Hom}_{\mathbf{S}}\left(\mathbf{\Delta}(-, \mathbf{n}), X\right).$$

Definition 4.1.9. An n-simplex $\sigma \in X_n$ is called *degenerate* if there is a surjection $f : \mathbf{n} \to \mathbf{m}$ with $m < n$ and an m-simplex $\tau \in X_m$ such that $\sigma = X(f)(\tau)$. A simplex is called *non-degenerate* if it is not degenerate.

A very useful lemma which we shall use over and over again is the following.

Lemma 4.1.10 (Eilenberg-Zilber)**.** *For each n-simplex $\sigma \in X_n$ there exists a unique order-preserving surjection $f : \mathbf{n} \to \mathbf{m}$ and a unique non-degenerate m-simplex $\tau \in X_m$ such that $\sigma = X(f)(\tau)$.*

Proof. We follow [JT08, Proposition 1.2.2]. If σ is already non-degenerate, take $m = n$ and $f = \mathrm{id}$. If σ is degenerate, then there exists some surjection $f_1 : \mathbf{n} \to \mathbf{m}_1$ with $m_1 < n$ and an m_1-simplex σ_1 such that $\sigma = X(f_1)(\sigma_1)$. If σ_1 is non-degenerate, we are done. Otherwise, continue in this way until we reach a σ_j that is non-degenerate. This process ends eventually since with each iteration, $m_i < m_{i-1}$, and vertices are always non-degenerate. So the existence of such a pair (f, τ) with $\sigma = X(f)(\tau)$ is established. Suppose then that (f', τ') is another pair with $\sigma = X(f')(\tau')$. Consider the pushout

$$\begin{tikzcd}
\mathbf{n} \arrow[r, "f"] \arrow[d, "f'"'] & \mathbf{m} \arrow[d, "\pi"] \\
\mathbf{m'} \arrow[r, "\pi'"'] & \mathbf{p}
\end{tikzcd}$$

in $\mathbf{\Delta}$. We can apply the Yoneda functor to get a commutative diagram

$$\begin{tikzcd}
\mathbf{\Delta}(-, \mathbf{n}) \arrow[r, "\mathbf{\Delta}(-,f)"] \arrow[d, "\mathbf{\Delta}(-,f')"'] & \mathbf{\Delta}(-, \mathbf{m}) \arrow[d, "\mathbf{\Delta}(-,\pi)"] \\
\mathbf{\Delta}(-, \mathbf{m'}) \arrow[r, "\mathbf{\Delta}(-,\pi')"'] & \mathbf{\Delta}(-, \mathbf{p})
\end{tikzcd}$$

in $\mathbf{S}$. This diagram is still a pushout by theorem 4.1.5. While the n-simplex σ is an element of the set X_n, it may equivalently be regarded as a functor

$\sigma : \mathbf{\Delta}(-,\mathbf{n}) \to S$ describing how the simplex is laid into the simplicial set. Similarly for τ and τ'. Hence we have a commutative diagram

$$\begin{tikzcd}
\mathbf{\Delta}(-,\mathbf{n}) \arrow[r, "\mathbf{\Delta}(-,f)"] \arrow[d, "\mathbf{\Delta}(-,f')"'] & \mathbf{\Delta}(-,\mathbf{m}) \arrow[d, "\mathbf{\Delta}(-,\pi)"] \arrow[ddr, bend left, "\tau"] & \\
\mathbf{\Delta}(-,\mathbf{m}') \arrow[r, "\mathbf{\Delta}(-,\pi')"'] \arrow[drr, bend right, "\tau'"'] & \mathbf{\Delta}(-,\mathbf{p}) \arrow[dr, dotted, "\exists! \rho"] & \\
 & & S
\end{tikzcd}$$

Since $\tau \circ \mathbf{\Delta}(-,f) = \sigma = \tau' \circ \mathbf{\Delta}(-,f')$, there exists a unique mediating ρ such that $\mathbf{\Delta}(-,\pi) \circ \rho = \tau$ and $\mathbf{\Delta}(-,\pi') \circ \rho = \tau'$ as indicated by the dotted arrow. We can regard ρ equivalently as a p-simplex in the set X_p, while π and π' are order-preserving surjections. So we see that $\tau = X(\pi)(\rho)$ and $\tau' = X(\pi')(\rho)$. But τ and τ' are both non-degenerate. So we must have $\pi = \pi' = \mathrm{id}$, $m = p = m'$, and hence $\tau = \tau'$. $\square$

Recall that every presheaf is a colimit of representable presheaves by proposition 2.1.14. The good news is that there are not many different representable presheaves in $\mathbf{S}$. So we conclude

Lemma 4.1.11. *If $X \in \mathbf{S}$ is a simplicial set, there is a canonical isomorphism*

$$X \cong \operatorname{colim}\left(\int_{\mathbf{\Delta}} X \xrightarrow{\pi_X} \mathbf{\Delta} \xrightarrow{\mathbf{y}} \mathbf{Set}^{\mathbf{\Delta}^{op}}\right)$$

The above lemma tells you "how the standard simplices comprise X", and is very combinatorial in nature. It's often convenient to view lemma 4.1.11 as a different (but equivalent, of course) colimit. Namely, define $\mathbf{\Delta} \downarrow X$ to be the category whose objects are maps $\sigma : \mathbf{\Delta}(-,\mathbf{n}) \to X$ (or just simplices of X). An arrow in $\mathbf{\Delta} \downarrow X$ is a commutative diagram of simplicial maps

$$\begin{tikzcd}
\mathbf{\Delta}(-,\mathbf{n}) \arrow[dd, "\theta"'] \arrow[dr, "\sigma"] & \\
 & X \\
\mathbf{\Delta}(-,\mathbf{m}) \arrow[ur, "\tau"'] &
\end{tikzcd}$$

It is not hard to prove that the colimit in lemma 4.1.11 is the same thing as

Lemma 4.1.12.

$$X \cong \operatorname*{colim}_{\substack{\mathbf{\Delta}(-,\mathbf{n}) \to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \mathbf{\Delta}(-,\mathbf{n}).$$

4.2 Geometric Realization

Definition 4.2.1. The *standard geometric n-simplex* is the topological space

$$\Delta^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \text{ for all } i \right\}$$

endowed with the subspace topology from the metric space $\mathbb{R}^{n+1}$.

There is a functor $|-| : \mathbf{\Delta} \to \mathbf{Top}$ defined by sending $\mathbf{n}$ to the standard geometric n-simplex in $\mathbb{R}^{n+1}$. It's easy to visualize what $|-|$ should do with the coface and codegeneracy maps. As with all such constructions, once we have a way to turn representable objects into an object of interest in a category, we have a way to turn every presheaf into an object of interest in a category. For a more precise meaning of the previous sentence, we have the following definition.

Definition 4.2.2. The *geometric realization* of X is defined to be the colimit of spaces

$$|X| := \operatorname*{colim}_{\substack{\mathbf{\Delta}(-,\mathbf{n}) \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} \Delta^n \in \mathbf{Top}$$

Elements of $|X|$ are equivalence classes of pairs $[\sigma, t]$, where $\sigma \in X_n$ is an n-simplex and $t \in \Delta^n$. The equivalence relation is defined by the rule

$$(X(\theta)(\sigma), t) \sim (\sigma, |\theta|(t)), \qquad \theta : \mathbf{n} \to \mathbf{m}.$$

In fact, this is reminiscent of the tensor product! Unfortunately, **Top** is not a topos. Moreover, $|X|$ is a space, not just a set (as it is for the tensor product). So we cannot formally define the geometric realization in this way. A map $f : X \to Y$ of simplicial sets (so, a natural transformation between the presheaves) determines a functor

$$\mathbf{\Delta} \downarrow f : \mathbf{\Delta} \downarrow X \to \mathbf{\Delta} \downarrow Y, \qquad \sigma \mapsto f \circ \sigma.$$

So we get a continuous map

$$|f| : |X| \to |Y|, \qquad [\sigma, t] \mapsto [f \circ \sigma, t].$$

Hence $|-| : \mathbf{\Delta} \to \mathbf{Top}$ extends to a functor $|-| : \mathbf{S} \to \mathbf{Top}$. There's also a functor going the other way. Observe that for any given topological space $S \in \mathbf{Top}$ we have a simplicial set $\mathrm{Sing}(S)$ defined on objects by $\mathrm{Sing}(S)_n := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, S)$. It's an exercise to determine what Sing should do with continuous maps, and what the face and degeneracy maps are for $\mathrm{Sing}(S)$.

Theorem 4.2.3. *The functor $|-|$ is the left-adjoint of* Sing.

Proof. There are natural bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Top}}\left(|X|, S\right) &= \mathrm{Hom}_{\mathbf{Top}}\left(\operatorname*{colim}_{\substack{\mathbf{\Delta}(-,\mathbf{n})\to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \Delta^n, S\right) \\
&\cong \lim_{\substack{\mathbf{\Delta}(-,\mathbf{n})\to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \mathrm{Hom}_{\mathbf{Top}}\left(\Delta^n, S\right) \\
&\cong \lim_{\substack{\mathbf{\Delta}(-,\mathbf{n})\to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \mathrm{Hom}_{\mathbf{S}}\left(\mathbf{\Delta}(-,\mathbf{n}), \mathrm{Sing}(S)\right) \\
&\cong \mathrm{Hom}_{\mathbf{S}}\left(X, \mathrm{Sing}(S)\right).
\end{aligned}$$

□

4.3 The Nerve of a Category

From the way we defined the category $\mathbf{\Delta}$ we have an obvious inclusion functor

$$\mathbf{\Delta} \to \mathbf{Cat}, \qquad \mathbf{n} \mapsto \mathbf{n}.$$

So we have a way to change representable objects into objects of a category that we are interested in. We can do the whole construction again, but this time in **Cat** instead of **Top**. It's possible to set up an adjoint pair of functors between **S** and **Cat** just like for the adjoint pair $|-| \dashv \mathrm{Sing}$. We will not be needing the left adjoint. What we will be needing from this adjunction, though, is the right adjoint. We define formally

Definition 4.3.1. Let $\mathbf{C} \in \mathbf{Cat}$ be a small category. We define the *nerve* of $\mathbf{C}$ to be the simplicial set $N\mathbf{C}$ defined for each natural number $n \geq 0$ as the set

$$(N\mathbf{C})_n := \mathrm{Hom}_{\mathbf{Cat}}\left(\mathbf{n}, \mathbf{C}\right).$$

Spelled out in detail, this means that the 0-simplices are the objects of $\mathbf{C}$ (which is indeed a set). If $n > 0$, then an n-simplex of $(N\mathbf{C})_n$ is an n-tuple

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

of composable morphisms $f_i : A_{i-1} \to A_i$. We'll denote such simplices more compactly by $(f_1, \ldots, f_n)$. The face maps

$$d_j : (N\mathbf{C})_n \to (N\mathbf{C})_{n-1}$$

compose two consecutive morphisms at the j'th position to get an $(n-1)$-tuple of composable morphisms when $0 < j < n$, and they discard the outer morphism when $j = 0$ or n. The degeneracy maps

$$s_j : (N\mathbf{C})_n \to (N\mathbf{C})_{n+1}$$

insert an identity morphism at the j'th position to get a degenerate $(n+1)$-tuple.

4.4 Star Neighborhoods

From this point on, the thesis consists of mostly original work, so the proofs become more detailed.

Lemma 4.4.1. *Let $S \in \mathbf{S}$ be a simplicial complex and $T \subset S$ a subcomplex. Then $|T| \subset |S|$ is a closed set.*

Proof. Let $\sigma \in S_n$ be an n-simplex of S. Let I be the set of all possible faces of σ which are contained in T. Note that I is a finite set. Therefore $\bigcup_{\tau \in I} |\tau|$ is closed in $|\sigma|$. $\square$

Definition 4.4.2. Let $S \in \mathbf{S}$ be a simplicial set, and an n-simplex $\sigma \in S_n$. The *star* of σ, denoted $\mathrm{star}(\sigma)$, is the set of all simplices in S which contain σ as an eventual face.

This means that $\sigma \in S_n$ is an eventual face of some $\tau \in S_m$ with $m > n$ if and only if there exist natural numbers $i_1, i_2, \ldots, i_{m-n}$ such that $\sigma = (d_{i_{m-n}} \circ \cdots \circ d_{i_2} \circ d_{i_1})(\tau)$. If $m = n$, then σ is a face of σ, so $\sigma \in \mathrm{star}(\sigma)$. If $m < n$, then $S_m \cap \mathrm{star}(\sigma) = \emptyset$. Put in a more functorial way, identify σ and τ as functors $\sigma : \mathbf{\Delta}(-, \mathbf{n}) \to S$ and $\tau : \mathbf{\Delta}(-, \mathbf{m}) \to S$. Then σ is an eventual face of τ if and only if there exists an injective order-preserving map $\theta : \mathbf{n} \to \mathbf{m}$ such that

$$\begin{tikzcd}
\mathbf{\Delta}(-, \mathbf{n}) \arrow[d, "\mathbf{\Delta}(-,\theta)"'] \arrow[dr, "\sigma"] & \\
\mathbf{\Delta}(-, \mathbf{m}) \arrow[r, "\tau"'] & S
\end{tikzcd} \tag{4.1}$$

is a commutative diagram of simplicial sets.

Definition 4.4.3. Let $S \in \mathbf{S}$ and take an n-simplex $\sigma \in S_n$. The simplicial set $S - \sigma$ is defined as follows. For any $n \in \mathbb{N}$, we set

$$(S - \sigma)_n := \{\tau \in S_n : \tau \notin \mathrm{star}(\sigma)\}.$$

The face and degeneracy maps for $S - \sigma$ are the same as from S, and by definition of the star they are well-defined.

It is obvious that $S - \sigma \subset S$. As a consequence, $|S - \sigma|$ is a closed subspace inside $|S|$ by lemma 4.4.1.

Definition 4.4.4. Let $S \in \mathbf{S}$ be a simplicial set and $\sigma \in S_n$ an n-simplex. We define the *realization of the star* $\sigma^* \subset |S|$ as the open subspace $|S| - |S - \sigma|$.

Observe that if $v \in S_0$ is a 0-simplex (a vertex), then the only point $p \in v^*$ with the property that $p = |w|$ for some $w \in S_0$ is v.

Definition 4.4.5. Let $S \in \mathbf{S}$ be a simplicial set and take an n-simplex $\sigma \in S_n$. Identify the realization of σ in $|S|$ with its continuous map $|\sigma| : \Delta^n \to |S|$. We define the *interior* of σ as the image of $\{(t_0, \dots, t_n) \in \Delta^n : t_i > 0,\ i = 0, \dots, n\}$.

Observe that this definition works fine even for degenerate simplices, and note also that the interior of a 0-simplex is a point in $|S|$. Note also that the interior of a simplex is not necessarily open in $|S|$. However, we do have

Lemma 4.4.6. *Let $\sigma \in S_n$ be an n-simplex of a simplicial set $S \in \mathbf{S}$. Then $\sigma^* = \bigcup_{\tau \in \mathrm{star}(\sigma)} \mathrm{int}(\tau)$.*

Proof. The inclusion $\bigcup_{\tau \in \mathrm{star}(\sigma)} \mathrm{int}(\tau) \subseteq \sigma^*$ is trivial. Let us prove the other inclusion.

Let $p \in \sigma^*$. Then $p \notin |S - \sigma|$. This means that p is not in the (closed) image of any m-simplex $|\tau| : \Delta^m \to |S|$ for which σ is *not* an eventual face. But $p \in |S|$, so what remains is that p is in the image of an m-simplex $|\tau| : \Delta^m \to |S|$ for which σ is an eventual face. By lemma 4.1.10, we may assume τ to be non-degenerate. Let $(t_0, \dots, t_m) \in \Delta^m$ be the barycentric coordinates such that $|\tau|(t_0, \dots, t_m) = p$. If $m = 0$ then we are done, so assume $m > 0$. If $t_i = 0$ for some i, we may replace τ by its face $d_i\tau$ and replace the barycentric coordinates by $(t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_m)$. After such a replacement, we still have $d_i\tau \in \mathrm{star}(\sigma)$. Continue in this way until we reach a k-simplex τ', with $k \le m$, with barycentric coordinates $(t_0, \dots, t_k)$ with $t_i > 0$ for all $i = 0, \dots, k$.

□

Lemma 4.4.7. *Let $n > 0$ and take an n-simplex $\sigma \in S_n$ in a simplicial set $S \in \mathbf{S}$. Then*

$$\sigma^* \subset \bigcap_{i=0}^{n} (d_i\sigma)^*.$$

Proof. If $\tau \in \mathrm{star}(\sigma)$, then τ is eventually a face of σ, so it's clearly also eventually a face of $d_i\sigma$ for $i = 0, \dots, n$. Hence $\tau \in \bigcap_{i=0}^{n} \mathrm{star}(d_i\sigma)$. Therefore $\mathrm{int}(\tau) \subset (d_i\sigma)^*$ for every $i = 0, \dots, n$. □

Lemma 4.4.8. *Let $S \in \mathbf{S}$. For every n-simplex $\sigma \in S_n$, the realization of the star $\sigma^* \subset |S|$ is a connected subset.*

Proof. Suppose that we can write $\sigma^* = U \cup V$, where U and V are two disjoint non-empty open subsets of σ^*. Let $\tau \in \mathrm{star}(\sigma)$. Then $|\tau|$ is a compact connected metric space, so $\mathrm{int}(\tau)$ is a connected metric space. but $\mathrm{int}(\tau) = (U \cap \mathrm{int}(\tau)) \cup (V \cap \mathrm{int}(\tau))$, so either $\mathrm{int}(\tau) \subset U$ or $\mathrm{int}(\tau) \subset V$. Let I be the set of all $\tau \in \mathrm{star}(\sigma)$ for which $\mathrm{int}(\tau) \subset U$ and let J be the set of all $\tau \in \mathrm{star}(\sigma)$ for which $\mathrm{int}(\tau) \subset V$. Both I and J are non-empty. By lemma 4.4.6, we find a decomposition

$$\sigma^* = \bigcup_{\tau \in \mathrm{star}(\sigma)} \mathrm{int}(\tau) = \bigcup_{\tau \in I} \mathrm{int}(\tau) \cup \bigcup_{\tau \in J} \mathrm{int}(\tau).$$

Without loss of generality, $\sigma \in I$. If $\tau \in \mathrm{star}(\sigma)$, with τ an m-simplex, $n \leq m$, then there exists a commutative diagram as in eq. (4.1). Thus, we have a commutative diagram of topological spaces and continuous maps

$$\begin{tikzcd}
\Delta^n \arrow[d, "|\mathbf{\Delta}(-,\theta)|"'] \arrow[dr, "|\sigma|"] & \\
\Delta^m \arrow[r, "|\tau|"'] & |S|
\end{tikzcd}$$

By connectedness of Δ^m and Δ^n, we have $\tau \in I$. But this holds for arbitrary τ, so J is empty. A contradiction. □

Lemma 4.4.9. *The realizations of the stars of vertices form an open cover of $|S|$. That is, $|S| = \bigcup_{v \in S_0} v^*$.*

Proof. By lemma 4.4.6,

$$\bigcup_{v \in S_0} v^* = \bigcup_{v \in S_0} \bigcup_{\tau \in \mathrm{star}(v)} \mathrm{int}(\tau).$$

The right-hand-side of this equation exhausts all possible simplices of S. So it suffices to prove that every point $p \in |S|$ is contained in the interior of some n-simplex σ. Let $p \in |S|$. Then $p \in |\sigma|$ for some n-simplex $\sigma \in S_n$. By lemma 4.1.10, there exists a unique non-degenerate m-simplex $\tau \in S_m$ and a surjective order-preserving map $f : \mathbf{n} \to \mathbf{m}$, with $m \leq n$, such that $(Sf)(\tau) = \sigma$. Thus $p \in |\tau|$. If $p \in \mathrm{int}(\tau)$ then we are done. Otherwise, $p \in |d_i\tau|$ for some i. Continue in this way. Eventually this process stops. □

Chapter 5

Construction of the McCord Functor

Let $\mathbf{C} \in \mathbf{Cat}$ be a small category and write $X_{\mathbf{C}} := |N\mathbf{C}|$. The goal is to find a functor $\mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$, where $\mathbf{LH}$ denotes the category of topological spaces together with local homeomorphisms (etale maps) as morphisms. Recall from definition 2.4.2 and theorem 2.4.3 that $\mathbf{LH}/X_{\mathbf{C}}$ is a Grothendieck topos, and it is connected if and only if $X_{\mathbf{C}}$ is connected, which is the case if and only if $\mathbf{C}$ is connected (viewed as a graph).

We shall construct a functor $\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$ which we'll call the *McCord functor*. It is this functor, together with an appropriate class of small categories, that will give us an isomorphism on the level of fundamental groups.

5.1 What It Does on Objects

Recall that if $\sigma \in (N\mathbf{C})_n$ is an n-simplex, the realization of the star (viz. definition 4.4.4) is an open subset in $X_{\mathbf{C}}$. In particular, if $A \in \mathbf{C}$ is an object, then $A \in (N\mathbf{C})_0$, so we have an open subset $A^* \subset X_{\mathbf{C}}$ "centered around" $|A| \in X_{\mathbf{C}}$.

Definition 5.1.1. Let $\mathbf{C}$ be a category, $N\mathbf{C}$ the simplicial nerve and $X_{\mathbf{C}} = |N\mathbf{C}|$. Let $\mathbf{C}/A$ be the slice category over A. Write D_f for the domain of a morphism f. We define the *McCord space* of A to be the topological space

$$\mu(A) := \left(\coprod_{f \in \mathbf{C}/A} D_f^* \right) / \sim .$$

Elements of the coproduct $\coprod D_f^*$ may be denoted as tuples (f, p) where $f : D_f \to A$ is an object of the slice category $\mathbf{C}/A$ and $p \in D_f^* \subset X_{\mathbf{C}}$. Let $\triangleright$ be the binary relation defined by $(f, p) \triangleright (g, q) \iff p = q$ in $X_{\mathbf{C}}$ and there exists a morphism $h : f \to g$ in $\mathbf{C}/A$ and there exists an n-simplex $\sigma \in \text{star}(h)$ such that $p \in \text{int}(\sigma)$. This relation is reflexive, but in general neither symmetric nor transitive. Let $\sim$ be the smallest equivalence relation generated by $\rhd$.

Denote the quotient map by $q_A : \bigsqcup D_f^* \to \mu(A)$. The topology on $\mu(A)$ can be described as follows. A subset $U \subset \mu(A)$ is open in $\mu(A)$ if and only if for all objects $g \in \mathbf{C}/A$ the set $D_g^* \cap q_A^{-1}(U)$ is open in $X_{\mathbf{C}}$.

Lemma 5.1.2. *For every $A \in \mathbf{C}$, the space $\mu(A)$ is connected.*

Proof. Let $f \in \mathbf{C}/A$ be a morphism $f : D_f \to A$. Then we have a morphism $f : f \to \text{id}_A$ in $\mathbf{C}/A$ and a 1-simplex $f \in \text{star}(f)$. Therefore, every constituent in the coproduct $\bigsqcup_{f \in \mathbf{C}/A} D_f^*$ is glued with the terminal object $\text{id}_A : A \to A$ of $\mathbf{C}/A$ along the interior of at least one 1-simplex. □

The next thing to do is to construct an etale map $\mu(A) \to X_{\mathbf{C}}$. In order to do that, as an intermediate step we shall prove that the quotient map $q_A : \bigsqcup_{f \in \mathbf{C}/A} D_f^* \to \mu(A)$ is etale.

Lemma 5.1.3 (The Gluing-Compatible-Opens Lemma)**.** *Let $\{U_i\}_{i \in I}$ be a collection of open subsets of a topological space X, and for each $i, j \in I$, let V_{ij} be an open subset of $U_i \cap U_j$. If $(i, x), (j, y) \in \bigsqcup_{i \in I} U_i$, define $(i, x) \rhd (j, y)$ if and only if $x = y$ in X and $x \in V_{ij}$. Let $\sim$ be the smallest equivalence relation generated by $\rhd$. Let $Y = (\bigsqcup_{i \in I} U_i) \,/ \sim$ and denote the quotient map by $q : \bigsqcup U_i \to Y$. Then q is a surjective etale map.*

Proof. Without loss of generality we may assume that $V_{ij} = V_{ji}$ and $V_{ii} = U_i$ for all $i, j \in I$. The fact that q is surjective and continuous follows from the fact that it is a quotient map. First we'll prove that q is an open map. Let $j \in I$ and let $W_j \subset U_j$ be an open set in U_j. We need to prove that $q^{-1}(q(W_j))$ is again open. Thus we need to prove that for every $i \in I$ the set $U_i \cap q^{-1}(q(W_j))$ is open in U_i. Unraveling the definition of $\sim$, we have

$$U_i \cap q^{-1}(q(W_j)) = \{x \in U_i : q(x, i) \in q(W_j)\}$$

$$= W_j \cap \bigcup_{l=2}^{\infty} \bigcup_{\substack{(k_1, \ldots, k_l) \in I^l \\ k_1 = i,\, k_l = j}} \left(V_{k_1 k_2} \cap V_{k_2 k_3} \cap \ldots \cap V_{k_{l-1} k_l} \right).$$

Since each $V_{k_x k_y}$ is open, the finite intersections $V_{k_1 k_2} \cap \ldots \cap V_{k_{l-1} k_l}$ are open. Thus $U_i \cap q(q(W_j))$ is open. Now we'll show that q is etale. So let $x \in U_i$. The claim is that q restricts to a homeomorphism $U_i \cong q(U_i)$. Define a map of sets $s_i : q(U_i) \to U_i$ by sending an element $[y, i] \in q(U_i)$ to the element $y \in U_i$. Then s_i is independent of the representative of the equivalence class, for suppose that $[y, i] = [y', i']$ in Y. Then $y = y'$ in X, so $[y', i'] = [y, i']$. Thus s_i defines a section of $q|_{U_i}$ as sets. But since q is an open map, s_i is continuous. Thus q is etale. □

Lemma 5.1.4. *Suppose that*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

is a diagram where X, Y and Z are topological spaces, f surjective etale, h etale, and g is a map of sets (not necessarily continuous). If the diagram commutes then g is etale (and hence continuous).

Proof. Let $y \in Y$. Take $x \in f^{-1}(y)$. Then there exist open neighborhoods $x \in U \subset X$ and $x \in V \subset X$ such that $U \cong f(U)$ via $f|_U$ and $V \cong h(V)$ via $h|_V$. Let $W = f(U \cap V)$. Then $y \in W$, and W is open because $f|_U$ is a homeomorphism. Also $h|_{U\cap V} = g|_W \circ f|_{U\cap V}$. So $g|_W = h|_{U\cap V} \circ (f|_{U\cap V})^{-1}$. $\square$

Define a map of sets

$$e_A : \mu(A) \to X_{\mathbf{C}}, \qquad [f, p] \mapsto p.$$

Proposition 5.1.5. *The map e_A is an etale map.*

Proof. We first apply lemma 5.1.3. The index set I is to be the objects of the slice category $\mathbf{C}/A$. For the collection of opens, take $\{D_f^*\}_{f\in\mathbf{C}/A}$. For the opens in the intersections, take the realizations of the 1-simplices (morphisms) $h : D_f \to D_g$ in the slice category $\mathbf{C}/A$. By lemma 4.4.7 we have $h^* \subset D_f^* \cap D_g^*$. It follows that $q_A : \bigsqcup_{f\in\mathbf{C}/A} D_f^* \to \mu(A)$ is a surjective etale map.

Now we show that e_A is etale. Denote the inclusion maps by $i_{A,f} : D_f^* \to X_{\mathbf{C}}$. Then we get a natural map

$$i_A = \bigsqcup_{f\in\mathbf{C}/A} i_{A,f} : \bigsqcup_{f\in\mathbf{C}/A} D_f^* \to X_{\mathbf{C}}.$$

Observe that we have a commutative diagram

$$\begin{tikzcd} \bigsqcup_{f\in\mathbf{C}/A} D_f^* \arrow[rr, "q_A"] \arrow[dr, "i_A"'] & & \mu(A) \arrow[dl, "e_A"] \\ & X_{\mathbf{C}} & \end{tikzcd}$$

in which i_A is etale and q_A is surjective etale. Thus e_A is etale by lemma 5.1.4. $\square$

5.2 What it Does on Morphisms

Definition 5.2.1. Let $f : A \to B$ be a morphism in $\mathbf{C}$. Then we have a functor $\mathbf{C}/f : \mathbf{C}/A \to \mathbf{C}/B$ given by sending an object $g \in \mathbf{C}/A$ to the composition $f \circ g$. Define a map $\mu(f) : \mu(A) \to \mu(B)$ as sending an equivalence class $[g, p] \in \mu(A)$ to the equivalence class $[f \circ g, p]$.

Lemma 5.2.2. *The definition of $\mu(f)$ is independent of the equivalence relation on $\mu(A)$.*

Proof. Let $[g, p], [h, p] \in \mu(A)$ and suppose without loss of generality that $(f, p) \rhd (g, p)$. This means that there exists a morphism $k : g \to h$ in $\mathbf{C}/A$ and an n-simplex $\sigma \in \mathrm{star}(k)$ such that $p \in \mathrm{int}(k)$. But $D_g = D_{f \circ g}$, $D_h = D_{f \circ h}$, and $D_f = A$. So we have a commutative diagram

$$\begin{tikzcd}
D_g = D_{f\circ g} \arrow[rr, "k"] \arrow[dr, "g"] \arrow[ddr, bend right, "f\circ g"'] & & D_{f\circ h} = D_h \arrow[dl, "h"'] \arrow[ddl, bend left, "f\circ h"] \\
& A = D_f \arrow[d, "f"] & \\
& B &
\end{tikzcd} \tag{5.1}$$

in $\mathbf{C}$. Therefore k is a morphism in $\mathbf{C}/B$ too. So $[f \circ g, p] = [f \circ h, p]$ in $\mu(B)$. □

Lemma 5.2.3. *The map $\mu(f) : \mu(A) \to \mu(B)$ is continuous.*

Proof. Denote the quotient maps by $q_A : \bigsqcup_{f \in \mathbf{C}/A} D_f^* \to \mu(A)$ and $q_B : \bigsqcup_{f \in \mathbf{C}/B} D_f^* \to \mu(B)$, respectively. Let $U \subset \mu(B)$ be an open set. We want to show that $\mu(f)^{-1}(U)$ is an open set in $\mu(A)$. Now $\mu(f)^{-1}(U)$ is open in $\mu(A)$ if and only if for all $g \in \mathbf{C}/A$ the set $D_g^* \cap (\mu(f) \circ q_A)^{-1}(U)$ is open in $X_{\mathbf{C}}$. I claim that for all $g \in \mathbf{C}/A$, the equality

$$D_g^* \cap (\mu(f) \circ q_A)^{-1}(U) = D_{f \circ g}^* \cap q_B^{-1}(U)$$

holds in $X_{\mathbf{C}}$. Indeed, we have

$$\begin{aligned}
p \in D_g^* \cap (\mu(f) \circ q_A)^{-1}(U) &\iff p \in D_g^* \text{ and } (\mu(f) \circ q_A)(g, p) \in U \\
&\iff p \in D_g^* \text{ and } [f \circ g, p] \in U \\
&\iff p \in D_{f \circ g}^* \text{ and } [f \circ g, p] \in U \\
&\iff p \in D_{f \circ g}^* \text{ and } q_B(f \circ g, p) \in U \\
&\iff p \in D_{f \circ g}^* \cap q_B^{-1}(U).
\end{aligned}$$

Since U is open in $\mu(B)$, we know that for all $h \in \mathbf{C}/B$ the set $D_h^* \cap q_B^{-1}(U)$ is open in $X_{\mathbf{C}}$, hence the claim follows. □

We summarize our work.

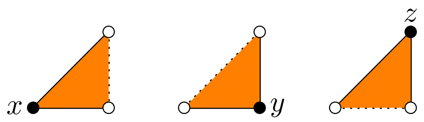
Corollary 5.2.4. *$\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$ is a functor.*

We work out a few examples to show the functor μ in action.

Example 5.2.5. Take $\mathbf{C} = \mathbf{3} = \left(x \xrightarrow{f} y \xrightarrow{g} z \right)$. Then $X_{\mathbf{C}}$ is a triangle with vertices $|x|$, $|y|$ and $|z|$. The McCord spaces are

$$\begin{aligned} \mu(x) &= x^*_{\mathrm{id}}, \\ \mu(y) &= x^*_f \sqcup y^*_{id} / \sim, \\ \mu(z) &= x^*_{g\circ f} \sqcup y^*_g \sqcup z^*_{\mathrm{id}} / \sim . \end{aligned}$$

Observe that $\mu(x) = \mu_{\mathbf{C}}^{-1}(U_x)$, the inverse image of the McCord map. Pictured below are the open sets x^*, y^* and z^* of $X_{\mathbf{C}}$.



The stars x^* and y^* share the open interior of the 1-simplex f, so they glue on that piece. Moreover, they share the open interior of the 2-simplex (f, g), so the "interiors" are glued together too. What we are left with is the whole space minus the point z. So, $\mu(y) \cong X_{\mathbf{C}} - \{z\}$. Notice that $\mu(y) = \mu_{\mathbf{C}}^{-1}(U_y)$. For $\mu(z)$, we get the whole space $X_{\mathbf{C}}$. Again, $\mu(z) = \mu_{\mathbf{C}}^{-1}(U_z)$.

Example 5.2.6 (The Finite Circle)**.** Take $\mathbf{C}$ to be the following poset: $a \leq c, d$, $b \leq c, d$. Then $\mathbf{C}$ may also be regarded as a finite T_0-space. The minimal opens are $U_a = \{a, c, d\}$, $U_b = \{b, c, d\}$, $U_c = \{c\}$ and $U_d = \{d\}$. The star sieve construction agrees with the inverse images of the McCord map.

Example 5.2.7 (The Graph Category)**.** Now it gets interesting. Take $\mathbf{C}$ to be the graph category $x \rightrightarrows y$, with $f, g : x \to y$. The realization $X_{\mathbf{C}}$ is a circle with two distinguished vertices x and y. We have

$$\begin{aligned} \mu(x) &= x^*_{\mathrm{id}}, \\ \mu(y) &= x^*_f \sqcup x^*_g \sqcup y^*_{\mathrm{id}} / \sim . \end{aligned}$$

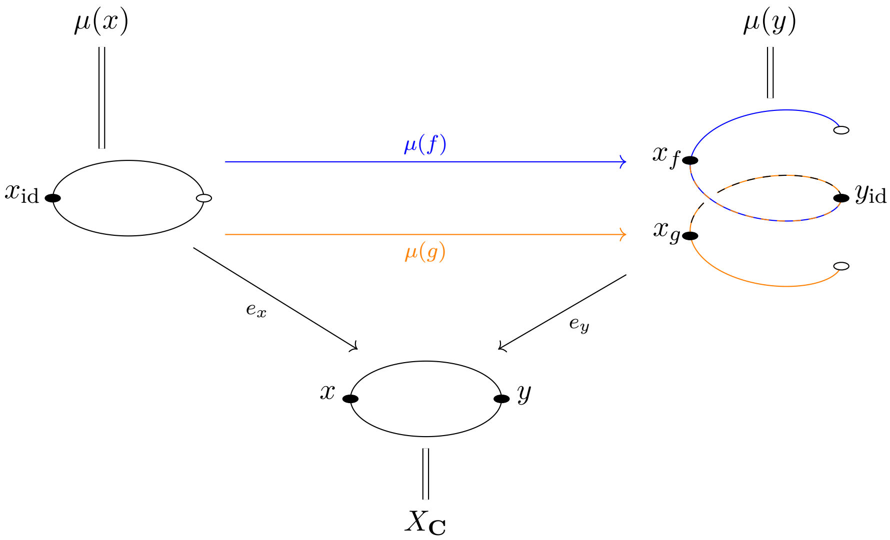
Let's compute

$$\begin{aligned} \mathrm{star}(x) &= \{x, f, g\}, \\ \mathrm{star}(y) &= \{y, f, g\}, \\ \mathrm{star}(f) &= \{f\}, \\ \mathrm{star}(g) &= \{g\}. \end{aligned}$$

Note that we leave out the degenerate simplices. The reader may convince himself that these are also taken care of during gluing. All possible morphisms in $\mathbf{C}/y$ are

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "f"'] & y \arrow[d, "\mathrm{id}"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "g"] \arrow[dr, "g"'] & y \arrow[d, "\mathrm{id}"] \\ & y \end{tikzcd}$$

Here, the open interior of the 1-simplex f is shared by the first copy x^*_f and y^*_{id}, so they glue on their top halves. The open interior of the 1-simplex g is shared by the second copy x^*_g and y^*_{id}, so they glue on their lower halves. What we are left with is a spiral that goes around y once.



Example 5.2.8 (The Coequalizer Category)**.** Take $\mathbf{C}$ to be the coequalizer category $x \rightrightarrows y \to z$. Denote the morphisms by $f, g : x \to y$, $h : y \to z$, $hf = hg = k$. The realization $X_{\mathbf{C}}$ looks like a pancake, with vertex x on the left border, vertex z in the middle and y on the right border. We have

$$\begin{aligned} \mu(x) &= x^*_{\mathrm{id}}, \\ \mu(y) &= x^*_f \sqcup x^*_g \sqcup y^*_{\mathrm{id}} / \sim, \\ \mu(z) &= x^*_k \sqcup y^*_h \sqcup z^*_{\mathrm{id}} / \sim . \end{aligned}$$

Let's compute

$$\begin{aligned}
\operatorname{star}(x) &= \{x, f, g, k, (h, f), (h, g)\},\\
\operatorname{star}(y) &= \{y, f, g, h, (h, f), (h, g)\},\\
\operatorname{star}(z) &= \{z, h, k, (h, f), (h, g)\},\\
\operatorname{star}(x) \cap \operatorname{star}(y) &= \{f, g, (h, f), (h, g)\},\\
\operatorname{star}(x) \cap \operatorname{star}(z) &= \{k, (h, f), (h, g)\},\\
\operatorname{star}(y) \cap \operatorname{star}(z) &= \{h, (h, f), (h, g)\}.
\end{aligned}$$

Note that we leave out the degenerate simplices. The reader may convince himself that these are also taken care of during gluing. The open set x^* is the whole space $X_{\mathbf{C}}$ minus the closed line segment from y to z. That is minus the closed 1-simplex h. The open set y^* is the whole space minus the closed line segment from x to y. That is minus the closed 1-simplex k. The open set z^* is the interior of $X_{\mathbf{C}}$. All possible morphisms in $\mathbf{C}/z$ are

$$\begin{tikzcd} z \arrow[r, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & z \arrow[d, "\mathrm{id}"] \\ & z \end{tikzcd} \quad \begin{tikzcd} y \arrow[r, "h"] \arrow[dr, "h"'] & z \arrow[d, "\mathrm{id}"] \\ & z \end{tikzcd} \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "k"'] & y \arrow[d, "h"] \\ & z \end{tikzcd} \quad \begin{tikzcd} x \arrow[r, "g"] \arrow[dr, "k"'] & y \arrow[d, "h"] \\ & z \end{tikzcd}$$

$\mu(x)$ The McCord space $\mu(x)$ is the open set x^*, the "pacman face".

$\mu(y)$ The interior of the 1-simplex f is shared by the first copy x^*_f and y^*_{id}, so they glue there. The interior of the 2-simplex (f, h) is shared by the first copy x^*_f and y^*_{id}, so they glue there. The interior of the 1-simplex g is shared by the second copy x^*_g and y^*_{id}, so they glue there. The interior of the 2-simplex (g, h) is shared by the second copy x^*_g and y^*_{id}, so they glue there. What we are left with is a spiral staircase making one roundabout.

$\mu(z)$ The 2 copies of the 2-simplex (f, h) sitting inside $\mu(y)$ are shared by the same 2-simplex (f, h) sitting inside z^*, so they glue. The same holds for the 2-simplex (g, h). Moreover the simplices k and h glue too. So we are left with the whole space $X_{\mathbf{C}}$.

Example 5.2.9 (The Equalizer Category)**.** Take $\mathbf{C}$ to be the equalizer category $x \to y \rightrightarrows z$. Denote $h : x \to y$ and $f, g : y \to z$ with $fh = gh = k$. The realization $X_{\mathbf{C}}$ is again a pancake, but this time y is at the left border, x is in the middle and z is at the right border. You can also view $X_{\mathbf{C}}$ as a coffee-filter, where the coffee runs down to the vertex x. We have

$$\begin{aligned}
\mu(x) &= x^*,\\
\mu(y) &= x^* \sqcup y^* / \sim,\\
\mu(z) &= x^* \sqcup y^*_f \sqcup y^*_g \sqcup z^* / \sim .
\end{aligned}$$

The open set x^* is the whole space $X_{\mathbf{C}}$ minus the border, so minus the closed 1-simplex f and minus the closed 1-simplex g. The open sets y^* and z^* are pacman faces.

$\mu(x)$ This is the interior x^* of $X_{\mathbf{C}}$.

$\mu(y)$ The interior of the 2-simplex (h, f) is shared by both x^* and y^*, so they glue there. The interior of the 2-simplex (h, g) is shared by both x^* and y^*, so they glue there. The interior of the 1-simplex h is shared by x^* and y^*, so they glue there. What we are left with is the whole space $X_{\mathbf{C}}$ minus the point z.

$\mu(z)$ We go over the relevant morphisms in $\mathbf{C}/z$.

$k : x \to z$ We have $\mathrm{star}(x) \cap \mathrm{star}(z) = \{k, (h, f), (h, g)\}$. All three of those simplices have k as a face. So x^* and z^* glue on the interiors of those three simplices.

$f : y \to z$ We have $\mathrm{star}(y) \cap \mathrm{star}(z) = \{f, g, (h, f), (h, g)\}$. The morphism f is an eventual face of f and (h, f), so y_f^* and z^* glue on the interiors of $|f|$ and $|(h, f)|$.

$g : y \to z$ We have $\mathrm{star}(y) \cap \mathrm{star}(z) = \{f, g, (h, f), (h, g)\}$. The morphism g is an eventual face of g and (h, g). So y_g^* and z^* glue on the interiors of $|g|$ and $|(h, g)|$.

What $\mu(z)$ looks like is hard to describe. It is sort of an oreo cookie with two infinitely close boundaries which twist at one end.

Example 5.2.10 (The 2-Element Monoid Category)**.** Let M be the monoid $\{1, f\}$ with $f^2 = f$, regarded as a category. The set of n-simplices of the nerve NM are $(NM)_n = M^n$. In particular, NM has just one vertex, call it $\bullet$. The realization X_M is something which I cannot describe easily. It may be vaguely described as a “fat” point. We see that

$$\mathrm{star}(\bullet) = \bigsqcup_{n \geq 0} (NM)_n \,.$$

In the slice category $M/\bullet$, all possible morphisms are

$$\begin{tikzcd} \bullet \arrow[r, "1"] \arrow[dr, "1"'] & \bullet \arrow[d, "1"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "f"'] & \bullet \arrow[d, "1"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "f"'] & \bullet \arrow[d, "f"] \\ & \bullet \end{tikzcd}$$

Now the construction of $\mu(\bullet)$ goes as follows. We have the object $\bullet \xrightarrow{1} \bullet$ in $M/\bullet$ and the object $\bullet \xrightarrow{f} \bullet$ in $M/\bullet$. The realization $\bullet_1^*$ is X_M. The realization $\bullet_f^*$ is also X_M. What parts do we glue? The only valid morphism of the possible three above is $f : \bullet_f \to \bullet_1$. In other words, every simplex $\sigma \in \mathrm{star}(\bullet)$ which contains f is glued between the two copies of X_M. What remains are the "completely degenerate" simplices $\sigma = (1, 1, \ldots, 1)$. But their realizations are just a point, so $\mu(\bullet)$ is X_M with a "double vertex", akin to the well-known real line with double origin (which is indeed etale over the real line).

5.3 Flatness Conditions

To understand μ better, we shall study its fibers (or stalks).

Definition 5.3.1. Let $p \in X_{\mathbf{C}}$. We define the *support* of p to be the full subcategory of $\mathbf{C}$ given by

$$\mathbf{C}(p) := \begin{pmatrix} A \in \mathbf{C} : p \in A^* \\ + \\ \text{morphisms from } \mathbf{C} \end{pmatrix}.$$

Note that $\mathbf{C}(p)$ is not the empty category by lemma 4.4.9. Put a relation on $\mathbf{C}(p)$ by declaring

$$A \leq B \iff \mathrm{Hom}_{\mathbf{C}}(A, B) \neq \emptyset, \qquad A, B \in \mathbf{C}(p).$$

Then $\leq$ is a preorder, but usually not a partial order.

Lemma 5.3.2. *Let $p \in X_{\mathbf{C}}$. Then $p \in |\sigma|$ for a unique non-degenerate n-simplex $\sigma \in (N\mathbf{C})_n$. The objects of the category $\mathbf{C}(p)$ consist of the vertices of this σ.*

Proof. If $A \in \mathbf{C}$ is a vertex of σ, then $\sigma \in \mathrm{star}(A)$. So $p \in A^*$. On the other hand, suppose that $A \in \mathbf{C}(p)$. We want to show that A is a vertex of σ. We know that $p \in A^*$. So $p \in \mathrm{int}(\tau)$ for some $\tau \in \mathrm{star}(A)$ by lemma 4.4.6. So $p \in |\tau|$. Let σ' be the unique non-degenerate simplex for τ. Then $|\tau| = |\sigma'|$. Then A is a vertex of σ'. By uniqueness, $\sigma = \sigma'$. $\square$

Observe that if $p = |A|$ for some object $A \in \mathbf{C}$, then $\mathbf{C}(p)$ consists of a single object.

Lemma 5.3.3. *If $\mathbf{C}$ is a finite poset, then $\mathbf{C}(p)$ is totally ordered with a minimal element for all $p \in X_{\mathbf{C}}$.*

Proof. Take $p \in X_{\mathbf{C}}$. Let σ be the unique non-degenerate simplex such that $p \in |\sigma|$. By lemma 5.3.2, the objects of the category $\mathbf{C}(p)$ are the vertices of σ. The vertices of an n-simplex in $(N\mathbf{C})_n$ are totally ordered, because $\mathbf{C}$ is a poset. Since $n < \infty$, there is a minimal element. $\square$

The results of example 5.2.5 and example 5.2.6 generalize to a theorem. Recall that for a finite T_0-space $\mathbf{C}$, the McCord map $\mu_{\mathbf{C}}$ is defined to be the continuous map

$$\mu_{\mathbf{C}} : X_{\mathbf{C}} \to \mathbf{C}, \qquad p \mapsto \min \mathbf{C}(p).$$

Theorem 5.3.4. *Let $\mathbf{C}$ be a finite T_0-space, or equivalently a finite poset. For each $x \in \mathbf{C}$, denote its minimal open set around x by U_x. Then there is a natural homeomorphism*

$$\mu(x) \cong \mu_{\mathbf{C}}^{-1}(U_x).$$

Proof. I claim that the etale map $e_x : \mu(x) \to X_{\mathbf{C}}$ has a section on $\mu_{\mathbf{C}}^{-1}(U_x) \subset X_{\mathbf{C}}$. Take a point $p \in \mu_{\mathbf{C}}^{-1}(U_x)$. Write $M_p = \min \mathbf{C}(p)$. Then $M_p \in U_x$, so $M_p \leq x$. Write $u : M_p \to x$ for the unique morphism. We now have $[u,p] \in \mu(x)$. Define

$$s_x : \mu_{\mathbf{C}}^{-1}(U_x) \to \mu(x)$$

by sending the point p to $[u,p]$. The definition of s_x is unambiguous, because there is only one choice for u. Clearly we have $e_x \circ s_x = \mathrm{id}_{\mu_{\mathbf{C}}^{-1}(U_x)}$. So s_x is a section. We shall now prove that $s_x \circ e_x = \mathrm{id}_{\mu(x)}$. Take $[g,p] \in \mu(x)$ and suppose that $(s_x \circ e_x)([g,p]) = [u,p]$. We want to show that $[g,p] = [u,p]$. Note that $g : D_g \to x$ is unique. Moreover, $D_g \in \mathbf{C}(p)$, so $M_p \leq D_g$. Let $h : M_p \to D_g$ be the unique morphism. Again by uniqueness, $h \circ g = u$. We now have a morphism $h : g \to u$ in $\mathbf{C}/x$, and $(g,h) \in \mathrm{star}(h)$ with $p \in \mathrm{int}(g,h)$. So $[g,p] = [u,g]$. ☐

Thus, we have found a generalization of the McCord map.

Definition 5.3.5. We say $\mathbf{C}$ is an *Alexandroff* category if for all $p \in X_{\mathbf{C}}$

1. the support $\mathbf{C}(p)$ is totally ordered with a (unique) minimal element M_p, and
2. for every $A \in \mathbf{C}$ and for every $[f,p] \in e_A^{-1}(p)$ there exists a unique morphism $m : M_p \to D_f$ with the property that there is an n-simplex $\sigma \in \mathrm{star}(m)$ with $p \in \mathrm{int}(\sigma)$.

Remark 5.3.6. I chose the name "Alexandroff category" because a topological space is termed an "Alexandroff space" when every point has a minimal open neighborhood with respect to inclusion. The second property is a technical one but it is required for lemma 5.3.10.

Example 5.3.7. If $\mathbf{C}$ is a finite poset, then $\mathbf{C}$ is Alexandroff. The graph category from example 5.2.7 is Alexandroff. The (co)equalizer category of example 5.2.8 and example 5.2.9 is Alexandroff. The category $x \leftrightarrows y$ where the two arrows are each other's inverse, is not an Alexandroff category. On the other hand, its skeleton is the trivial category, so is Alexandroff. Hence being Alexandroff is an "evil" notion.

Lemma 5.3.8. *If* $\mathbf{C}$ *is Alexandroff, then for all objects* $A \in \mathbf{C}$ *we have* $\mathrm{Hom}_{\mathbf{C}}(A, A) = \{\mathrm{id}_A\}$.

Proof. Take $A \in \mathbf{C}$ and set $p = |A| \in X_{\mathbf{C}}$. Take an element $[f, p] \in e_A^{-1}(p)$ in the fiber. Then $f : D_f \to A$ and $p \in D_f^*$. By lemma 5.3.2, $D_f = M_p = A$. Then since $\mathbf{C}$ is Alexandroff, f is unique, so we must have $f = \mathrm{id}_A$. $\square$

From lemma 5.3.8 we see that being Alexandroff and a monoid at the same time implies that $\mathbf{C}$ is the trivial category with one object and one morphism. It also shows that being Alexandroff greatly restricts the category.

Definition 5.3.9. We say that $\mathbf{C}$ is *well-fibered* if for all $p \in X_{\mathbf{C}}$ there exists an object $B \in \mathbf{C}$ such that $p^* \circ \mu \cong \mathrm{Hom}_{\mathbf{C}}(B, -)$.

Lemma 5.3.10. *If* $\mathbf{C}$ *is Alexandroff, then it is well-fibered.*

Proof. Let $p \in X_{\mathbf{C}}$. The claim is that $p^* \circ \mu \cong \mathrm{Hom}_{\mathbf{C}}(M_p, -)$, where M_p is defined as in definition 5.3.5. In other words, we need to find a natural isomorphism $\alpha : p^* \circ \mu \to \mathrm{Hom}_{\mathbf{C}}(M_p, -)$. To that end, define $\beta : \mathrm{Hom}_{\mathbf{C}}(M_p, -) \to p^* \circ \mu$ as follows. For each component $A \in \mathbf{C}$, we set

$$\beta_A : \mathrm{Hom}_{\mathbf{C}}(M_p, A) \to e_A^{-1}(p), \qquad h \mapsto [h, p].$$

Then naturality of β is clear. The natural transformation β will be the inverse for the natural transformation α. For the natural transformation α, define it as follows.

Take $[g, p] \in e_A^{-1}(p)$. Then $g : D_g \to A$ and $p \in D_g^*$. So $D_g \in \mathbf{C}(p)$. Since $\mathbf{C}$ is Alexandroff, there exists a unique morphism $m : M_p \to D_g$ with the property that there is some $\sigma \in \mathrm{star}(m)$ such that $p \in \mathrm{int}(\sigma)$. For each component $A \in \mathbf{C}$, we set

$$\alpha_A : e_A^{-1}(p) \to \mathrm{Hom}_{\mathbf{C}}(M_p, A), \qquad [g, p] \mapsto g \circ m.$$

Because this m is unique, α_A is well-defined. Observe now that

$$(\beta_A \circ \alpha_A)[g, p] = [g \circ m, p].$$

But m has the property that we are also given a simplex $\sigma \in \mathrm{star}(m)$ such that $p \in \mathrm{int}(\sigma)$. That means that $(g \circ m, p) \rhd (g, p)$, so $[g \circ m, p] = [g, p]$. In the other direction we find

$$(\alpha_A \circ \beta_A)(h) = h,$$

so we conclude that α and β are each other's inverse transformations. $\square$

Lemma 5.3.11. *The topos* $\mathbf{LH}/X_{\mathbf{C}}$ *has enough points.*

Proof. $X_{\mathbf{C}}$ is a geometric realization of a simplicial set, so $X_{\mathbf{C}}$ is a CW-complex by [GJ09, Proposition I.2.3]. In particular it is a compactly generated Hausdorff space, and Hausdorff spaces always have enough points. $\square$

Recall from definition 2.3.3 that μ is flat when by definition the induced functor $- \otimes_{\mathbf{C}} \mu$ is left exact.

Lemma 5.3.12. *If $\mathbf{C}$ is well-fibered, then $\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$ is flat.*

Proof. By lemma 5.3.11, it suffices to prove that for every $p \in X_{\mathbf{C}}$ the functor $p^* \circ \mu : \mathbf{C} \to \mathbf{Set}$ is flat. This is the same thing as proving that the category of elements $\int_{\mathbf{C}} (p^* \circ \mu)$ is filtered, by theorem 2.3.9. Since $\mathbf{C}$ is well-fibered, there exists some object $B \in \mathbf{C}$ such that $p^* \circ \mu \cong \mathrm{Hom}_{\mathbf{C}}(B, -)$. Therefore,

$$\int_{\mathbf{C}} (p^* \circ \mu) \cong \int_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(B, -) \cong B \backslash \mathbf{C}.$$

Now the over-category $B \backslash \mathbf{C}$ is always filtered, because $\mathrm{id}_B : B \to B$ is an initial object. $\square$

So we see that when $\mathbf{C}$ is Alexandroff, μ is flat.

Lemma 5.3.13. *Assume that $\mathbf{C}$ is a monoid (so consists of a single object). Let $p \in X_{\mathbf{C}}$. Denote the unique object of $\mathbf{C}$ by $\bullet$. Let $e_\bullet : \mu(\bullet) \to X_{\mathbf{C}}$ be the etale space of $\bullet$.*

1. *If $p = |\bullet|$, then $e_\bullet^{-1}(|\bullet|) \cong \mathrm{Hom}_{\mathbf{C}}(\bullet, \bullet)$.*
2. *If $p \neq |\bullet|$, then $e_\bullet^{-1}(p)$ is a one-element set.*

Proof. Follows from the definition of $\mu(\bullet)$. $\square$

We'll end this section with a conjecture.

Conjecture 5.3.14. *If $\mathbf{C}$ is a monoid, then $\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$ is flat.*

The evidence lies in the fact that $X_{\mathbf{C}}$ may be regarded as the classifying space of the monoid $\mathbf{C}$, so its fundamental group is the groupification of $\mathbf{C}$ (for instance, see [Len11]). Moreover, the fundamental group of the presheaf topos on $\mathbf{C}$ is (the profinite completion of) the groupification of $\mathbf{C}$. So in this sense, the McCord functor μ should detect that.

Chapter 6

The Equivalence

In this section, we assume that $\mathbf{C}$ is an Alexandroff category (viz. definition 5.3.5). Recall (viz. definition 5.1.1) that we constructed a flat (viz. lemma 5.3.12, lemma 5.3.10) functor

$$\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}.$$

We shall make some preparations first by proving some useful propositions, and then finish by proving that we can restrict to an equivalence of categories on the level of locally constant finite objects.

6.1 Preparation

Proposition 6.1.1. *There exists a geometric morphism*

$$\tau(\mu) : \mathbf{LH}/X_{\mathbf{C}} \to \mathbf{Set}^{\mathbf{C}^{op}}$$

for which the left-exact left adjoint $\tau(\mu)^*$ is given by sending a presheaf P on $\mathbf{C}$ to the tensor product $P \otimes_{\mathbf{C}} \mu$, and for which the right adjoint $\tau(\mu)_*$ sends an etale space $e : E \to X_{\mathbf{C}}$ to the presheaf $\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)$ defined for every object $A \in \mathbf{C}$ by

$$\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)(A) = \mathrm{Hom}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu(A), E).$$

Proof. Follows directly from the theory in [LM91, Chapter VII, Paragraph 7]. In particular, in [LM91, Theorem VII.7.2], take $\mathscr{E} = \mathbf{LH}/X_{\mathbf{C}}$. Alternatively, we spoke of the bijection between flat functors and geometric morphisms in theorem 2.3.8. $\square$

We shall be needing the following proposition.

Proposition 6.1.2. *Let $p : Y \to X$ be a (not necessarily finite) covering map, where Y is a topological space and X is a locally connected space. Let*

$f, g : Z \to Y$ be two continuous maps satisfying $p \circ f = p \circ g$, where Z is a connected topological space. If there is a point $z \in Z$ with $f(z) = g(z)$, then $f = g$.

Proof. This is [Sza10, Proposition 2.2.2]. We'll give a sketch of the proof here. Let $U = \{w \in Z : f(w) = g(w)\}$. Then prove that U is both open and closed in Z. Conclude that U must be all of Z by connectedness. □

The following proposition is central to this section.

Proposition 6.1.3. *Let $\pi_E : E \to X_{\mathbf{C}}$ be a finite covering map of degree $d > 0$ and let A be an object of $\mathbf{C}$. Then we have a natural bijection of sets*

$$\alpha_{A,E} : (\mathbf{LH}/X_{\mathbf{C}})\,(\mu A, E) \to \pi_E^{-1}(|A|), \qquad \varphi \mapsto \varphi[\mathrm{id}_A, |A|].$$

Proof. By lemma 5.3.8,

$$e_A^{-1}(|A|) = \{[\mathrm{id}_A, |A|]\} \subset \mu(A).$$

Write

$$\pi_E^{-1}(|A|) = \{x_1, \ldots, x_d\} \subset E.$$

Now take a morphism $\varphi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu(A), E)$. Then

$$\varphi[\mathrm{id}_A, |A|] \in \{x_1, \ldots, x_d\}.$$

I claim that these d choices for $\varphi[\mathrm{id}_A, |A|]$ completely determine φ. So let $\psi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu(A), E)$ be another morphism and suppose that

$$\varphi[\mathrm{id}_A, |A|] = x_1 = \psi[\mathrm{id}_A, |A|].$$

We will apply proposition 6.1.2. Take $Y = E$, $X = X_{\mathbf{C}}$, $Z = \mu(A)$, $p = \pi_E$, $f = \varphi$, $g = \psi$ and $z = [\mathrm{id}_A, |A|]$ in proposition 6.1.2. Then $X_{\mathbf{C}}$ is a locally connected space, because it is a CW-complex by [GJ09, Proposition I.2.3]. Moreover, $\mu(A)$ is connected by lemma 5.1.2. Finally,

$$p \circ f = \pi_E \circ \varphi = e_A = \pi_E \circ \varphi = p \circ g.$$

This proves that

$$\#\,(\mathbf{LH}/X_{\mathbf{C}})\,(\mu A, E) \leq d.$$

Let us now prove that the map $\alpha_{A,E}$ is surjective. Thus, given $x \in \pi_E^{-1}(|A|)$ we want to show that there exists some $\varphi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu\, A, E)$ such that $\varphi[\mathrm{id}_A, |A|] = x$. We shall actually construct such a φ. First, observe that π_E is a Serre fibration. Then apply proposition 6.1.2 to see that any two lifts of some $|\sigma| : \Delta^n \to X_{\mathbf{C}}$ are unique. For each $f \in \mathbf{C}/A$ (and so in particular for id_A) we have a commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook, "x"] \arrow[d, hook, "|d_0|"'] & E \arrow[d, "\pi_E"] \\
\Delta^1 \arrow[r, "|f|"'] \arrow[ur, dashed, "\exists! \widetilde{f}"] & X_{\mathbf{C}}
\end{tikzcd}$$

and a unique diagonal filler $\widetilde{f} : \Delta^1 \to E$ as indicated by the dotted arrow in the diagram. Thus we have a collection of lifted paths $\widetilde{f} : \Delta^n \to E$ all ending up at the point $x \in E$ and starting at some arbitrary point in E. Let us call the starting point $\widetilde{f}(0)$.

Now let $[f, p] \in \mu(A)$ be an arbitrary point. We are going to define what $\varphi[f, p]$ is. We have $f : D_f \to A$ and $p \in D_f^*$, so $p \in \mathrm{int}(\sigma)$ for some n-simplex $\sigma \in \mathrm{star}(D_f)$. We may assume that σ is non-degenerate by lemma 4.1.10. If $n = 0$, then $p = |D_f|$. In that case, we define

$$\varphi[f, p] := \widetilde{f}(0).$$

Suppose now that $n > 0$. Let $\theta : \mathbf{0} \to \mathbf{n}$ be an injective order-preserving map as in eq. (4.1) such that $D_f = \sigma \circ \mathbf{\Delta}(-, \theta)$. Then we have a commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook, "\widetilde{f}(0)"] \arrow[d, hook, "|\mathbf{\Delta}(-,\theta)|"'] & E \arrow[d, "\pi_E"] \\
\Delta^n \arrow[r, "|\sigma|"'] \arrow[ur, dashed, "\exists! \widetilde{\sigma}"] & X_{\mathbf{C}}
\end{tikzcd}$$

and a unique diagonal filler $\widetilde{\sigma} : \Delta^n \to E$. Let $t \in \Delta^n$ be the unique coordinates such that $p = |\sigma|(t)$. We define

$$\varphi[f, p] := \widetilde{\sigma}(t).$$

We must prove that this definition is independent of the chosen representative of the equivalence relation in $\mu(A)$. So suppose that $(f, p) \rhd (g, p)$. Then $p \in \mathrm{int}(\sigma_f)$ and $p \in \mathrm{int}(\sigma_g)$ for some σ_f having D_f as a vertex and some σ_g having D_g as a vertex. Suppose that we have uniquely lifted σ_f and σ_g to maps

$$\Delta^n \xrightarrow{\widetilde{\sigma_f}} E, \qquad \Delta^m \xrightarrow{\widetilde{\sigma_g}} E.$$

Let $t_f \in \Delta^n$ and $t_g \in \Delta^m$ be the unique coordinates such that

$$|\sigma_f|(t_f) = p = |\sigma_g|(t_g).$$

By the definition of $\rhd$, There exists a morphism $h : f \to g$ in $\mathbf{C}/A$ and a k-simplex $\tau \in \mathrm{star}(h)$ such that $p \in \mathrm{int}(\tau)$. Consider first the 2-simplex β given by

$$\beta = (h, g) : \mathbf{\Delta}(-, \mathbf{2}) \to N\mathbf{C}.$$

Let $\widetilde{\beta} : \Delta^2 \to E$ be the unique lift of $|\beta| : \Delta^2 \to X_{\mathbf{C}}$. By [Hat02, Exercise A.1], the face of a lift is the lift of a face, so we see that two of the faces of $\widetilde{\beta}$ upstairs in E are the lifts $\widetilde{f}$ and $\widetilde{g}$. Denote by $\widetilde{h}$ the third lift of $|h| : \Delta^1 \to X_{\mathbf{C}}$.

We may assume by lemma 4.1.10 that τ, σ_f and σ_g are non-degenerate. This implies that $\tau = \sigma_f = \sigma_g$, and $n = m = k$, and $t_f = t_g$. Let $\theta' : \mathbf{1} \to \mathbf{m}$ be an injective order-preserving map as in eq. (4.1) such that $h = \tau \circ \mathbf{\Delta}(-, \theta')$. Then we have a commutative diagram

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\widetilde{h}"] \arrow[d, hook, "|\mathbf{\Delta}(-,\theta')|"'] & E \arrow[d, "\pi_E"] \\
\Delta^m \arrow[r, "|\tau|"'] \arrow[ur, dashed, "\exists! \widetilde{\tau}"] & X_{\mathbf{C}}
\end{tikzcd}$$

By uniqueness of the lifts, $\widetilde{\sigma_f} = \widetilde{\sigma_g} = \widetilde{\tau}$.

Continuity of φ follows from the fact that all the liftings $\widetilde{\sigma} : \Delta^n \to E$ from the continuous maps $|\sigma| : \Delta^n \to X_{\mathbf{C}}$ are continuous. Moreover, the realization $X_{\mathbf{C}}$ is defined as the colimit

$$X_{\mathbf{C}} = \underset{\substack{\mathbf{\Delta}(-,\mathbf{n}) \to N\mathbf{C} \\ \text{in } \mathbf{\Delta} \downarrow N\mathbf{C}}}{\operatorname{colim}} \Delta^n$$

So the gluing data comes from $X_{\mathbf{C}}$.

$\square$

From definition 2.3.1 and eq. (2.2), we see that $P \otimes_{\mathbf{C}} \mu$ can be described as the space

$$\left(\bigsqcup_{A \in \mathbf{C}} P(A) \times \mu(A) \right) / \sim,$$

where elements of this space (wherein $P(A)$ carries the discrete topology for each $A \in \mathbf{C}$) are denoted by

$$x \otimes [f, p] \in P \otimes_{\mathbf{C}} \mu, \qquad A \in \mathbf{C},\ x \in P(A),\ [f, p] \in \mu(A)$$

under the rule

$$x \cdot g \otimes [f, p] = x \otimes [g \circ f, p], \qquad x \in P(A), g : A \to B, [f, p] \in \mu(A)$$

defined by the equivalence relation $\sim$. In particular, we note that if P is representable, say $P = \mathrm{Hom}_{\mathbf{C}}(-, A)$ with $A \in \mathbf{C}$, then

$$\mathrm{Hom}_{\mathbf{C}}(-, A) \otimes_{\mathbf{C}} \mu \cong \mu(A).$$

Theorem 6.1.4. *If $\pi : E \to X_{\mathbf{C}}$ is a finite covering map, then $\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)$ is a locally constant finite presheaf on $\mathbf{C}$. Conversely, if P is a locally constant finite presheaf on $\mathbf{C}$, then $P \otimes_{\mathbf{C}} \mu$ has the structure of a finite covering space over $X_{\mathbf{C}}$.*

Proof. The claim that $P \otimes_{\mathbf{C}} \mu$ is a finite covering map whenever $P \in \left(\mathbf{Set}^{\mathbf{C}^{op}}\right)_{\mathrm{lcf}}$ is covered in proposition 3.1.5. (Use theorem 2.4.3 there for the translation between sheaves and etale spaces). Explicitly, the finite covering map is given by

$$\pi : P \otimes_{\mathbf{C}} \mu \to X_{\mathbf{C}}, \qquad x \otimes [f, p] \mapsto p.$$

The degree of π is the number of elements in $P(A)$, for any $A \in \mathbf{C}$. This is well-defined by proposition 3.2.2 and the assumption that $X_{\mathbf{C}}$ is connected.

We shall prove the other direction, which is the remarkable one. So we want to show that given a finite covering map $\pi_E : E \to X_{\mathbf{C}}$ and given a morphism $f : A \to B$ in $\mathbf{C}$, the map

$$(\mathbf{LH}/X_{\mathbf{C}})(\mu\, B, E) \to (\mathbf{LH}/X_{\mathbf{C}})(\mu\, A, E), \qquad \varphi \mapsto \varphi \circ \mu(f) \tag{6.1}$$

is a bijection. By proposition 6.1.3, it suffices to prove that the map in eq. (6.1) is injective. So take two morphisms $\varphi, \psi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu\, B, E)$ and suppose that $\varphi \circ \mu(f) = \psi \circ \mu(f)$. We want to prove that $\varphi = \psi$. In proposition 6.1.2, take $Y = E$, $X = X_{\mathbf{C}}$, $Z = \mu(B)$, $p = \pi_E$, $f = \varphi$, $g = \psi$. As in the proof of proposition 6.1.3, all conditions of proposition 6.1.2 are satisfied, except that we need to supply a point $[h, p] \in \mu(B)$ such that $\varphi[h, p] = \psi[h, p]$. But we know that

$$\forall\ [g, p] \in \mu(A) : \varphi[f \circ g, p] = \psi[f \circ g, p].$$

Now $\mu(A)$ is non-empty, because $[\mathrm{id}_A, |A|] \in \mu(A)$. Therefore

$$\varphi[f, |A|] = \psi[f, |A|]$$

and we are done. □

Lemma 6.1.5. *Let P be a locally constant finite presheaf on $\mathbf{C}$ and take an object $A \in \mathbf{C}$. Then the fiber of the finite covering map $\pi : P \otimes_{\mathbf{C}} \mu \to X_{\mathbf{C}}$ above $|A| \in X_{\mathbf{C}}$ is given by*

$$\pi^{-1}(|A|) = \{x \otimes [\mathrm{id}_A, |A|] : x \in P(A)\}$$

and thus there is an isomorphism of presheaves $\pi^{-1}(|-|) \cong P$ on $\mathbf{C}$.

Proof. We have

$$\begin{aligned}
\pi^{-1}(|A|) &= \left\{x \otimes [f, p] : x \in P(B),\ f : D_f \to B,\ p = |A| \in D_f^*,\ B \in \mathbf{C}\right\} \\
&= \{x \otimes [f, |A|] : x \in P(B),\ f : A \to B,\ B \in \mathbf{C}\} \\
&= \{x \cdot f \otimes [\mathrm{id}_A, |A|] : x \in P(B),\ f : A \to B,\ B \in \mathbf{C}\} \\
&= \{P(f)(x) \otimes [\mathrm{id}_A, |A|] : x \in P(B),\ f \in \mathrm{Hom}(A, B),\ B \in \mathbf{C}\} \\
&= \{x \otimes [\mathrm{id}_A, |A|] : x \in P(A)\}.
\end{aligned}$$

In the second equality, we use the fact that D_f^* has $|D_f|$ as its only "vertex", so that $D_f = A$. In the third equality we use the tensor product rule, and in the last equality we use the fact that $P(f)$ is a bijection for all f by proposition 3.2.2 and lemma 5.3.8. $\square$

Corollary 6.1.6. *Let P be a locally constant finite presheaf on $\mathbf{C}$. For every $A \in \mathbf{C}$, we have a natural bijection of sets*

$$(\mathbf{LH}/X_{\mathbf{C}})\,(\mu(A), P \otimes_{\mathbf{C}} \mu) \cong P(A).$$

Proof. Combine the results of proposition 6.1.3, theorem 6.1.4 and lemma 6.1.5. $\square$

The category of finite covering spaces over $X_{\mathbf{C}}$ is denoted by $\mathbf{FinCov}/X_{\mathbf{C}}$. It is a full subcategory of $\mathbf{LH}/X_{\mathbf{C}}$. The equivalence of categories in theorem 2.4.3 restricts to an equivalence

$$\mathbf{FinCov}/X_{\mathbf{C}} \cong \mathrm{Sh}(\mathcal{O}(X_{\mathbf{C}}))_{\mathrm{lcf}}.$$

Corollary 6.1.7. *Let $\pi_E : E \to X_{\mathbf{C}}$ be a finite covering space. Consider the finite covering space $\pi : \underline{\mathrm{Hom}}(\mu, E) \otimes \mu \to X_{\mathbf{C}}$. Then for each object $A \in \mathbf{C}$, we have a natural bijection of fibers (i.e. stalks)*

$$\pi^{-1}(|A|) \cong \pi_E^{-1}(|A|).$$

Proof. By lemma 6.1.5, we have

$$\pi^{-1}(|A|) \cong \mathrm{Hom}\left(\mu(A), E\right).$$

And by proposition 6.1.3,

$$\mathrm{Hom}\left(\mu(A), E\right) \cong \pi_E^{-1}(|A|).$$

So the claim follows. $\square$

6.2 The Unit and Counit are Natural Isomorphisms

Proposition 6.2.1. *Let $E \in \mathbf{FinCov}/X_{\mathbf{C}}$. Then the counit at the component E*

$$\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mu \to E$$

of the adjunction $- \otimes_{\mathbf{C}} \mu \dashv \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, -)$ is an isomorphism.

Proof. The counit is given by the continuous map over the base space $X_{\mathbf{C}}$

$$\varepsilon_E : \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mu \to E, \qquad \varphi \otimes [f,p] \mapsto \varphi\left([f,p]\right),$$

where $\varphi \in (\mathbf{LH}/X_{\mathbf{C}})\,(\mu(A), E)$ for some $A \in \mathbf{C}$ and $[f,p] \in \mu(A)$. Like for sheaves, it suffices to prove that ε_E is an isomorphism on the level of stalks, i.e. fibers of the finite covering maps. First of all, it suffices to look at points p of the form $p = |A|$ for some object $A \in \mathbf{C}$, for recall (viz. definition 5.3.9, lemma 5.3.10) that μ is well-fibered, so that $p^* \circ \mu \cong \mathrm{Hom}(M_p, -)$. This gives

$$\begin{aligned} p^* \circ \left(\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mu\right) &\cong \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} (p^* \circ \mu) \\ &\cong \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mathrm{Hom}(M_p, -) \\ &\cong \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)(M_p) \\ &= \mathrm{Hom}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu(M_p), E) \\ &\cong \pi_E^{-1}(|M_p|) \end{aligned}$$

Now if we follow the isomorphisms, the composition is precisely the counit. □

A similar thing occurs with the unit.

Proposition 6.2.2. *Let P be a locally constant finite presheaf. Then the unit of the adjunction $- \otimes_{\mathbf{C}} \mu \dashv \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, -)$ at the component P is an isomorphism.*

Proof. The unit is a map of presheaves

$$\eta : P \to \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, P \otimes_{\mathbf{C}} \mu) \tag{6.2}$$

which for a given object $A \in \mathbf{C}$ is a map of sets

$$\eta_A : P(A) \to \mathrm{Hom}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu(A), P \otimes_{\mathbf{C}} \mu)$$

and, since $\mathbf{LH}/X_{\mathbf{C}}$ is cartesian closed (because it is a topos), this is the same thing as giving a map of sets

$$\eta_A^{\top} : P(A) \times \mu(A) \to P \otimes \mu$$

and this map is given by

$$\eta_A^{\top}(x, [f, p]) = x \otimes [f, p].$$

Now the isomorphism in corollary 6.1.6 is precisely the unit. □

Corollary 6.2.3. *The left and right adjoint of proposition 6.1.1 restrict to an equivalence of categories*

$$\mathbf{FinCov}/X_{\mathbf{C}} \cong \left(\mathbf{Set}^{\mathbf{C}^{op}}\right)_{\mathrm{lcf}}.$$

Proof. Apply proposition 6.2.2 and proposition 6.2.1. □

Corollary 6.2.4. *Let $A \in \mathbf{C}$ be an object. Then there is a natural isomorphism of profinite groups*

$$\widehat{\pi}_1 (X_{\mathbf{C}}, |A|) \cong \pi_1 \left(\mathbf{Set}^{\mathbf{C}^{op}}, A \right).$$

Proof. Interpret A as a geometric morphism (point)

$$A : \mathbf{Set} \to \mathbf{Set}^{\mathbf{C}^{op}}$$

where the inverse image part sends a presheaf P on $\mathbf{C}$ to $P(A)$, and the direct image part sends a set S to the "underline Hom" from construction 2.3.5. So A is a point of the topos $\mathbf{Set}^{\mathbf{C}^{op}}$. The category $\left(\mathbf{Set}^{\mathbf{C}^{op}}\right)_{\mathrm{lcf}}$ is a Galois category with fundamental functor given by the inverse image part of the point A. From corollary 6.2.3, we obtain

$$\pi_1 \left(\mathbf{Set}^{\mathbf{C}^{op}}, A \right) \cong \pi_1 (\mathrm{Sh}(X_{\mathbf{C}}), |A|).$$

Then from [Len08, Theorem 1.15, or 3.10], we obtain

$$\pi_1 (\mathrm{Sh}(X_{\mathbf{C}}), |A|) \cong \widehat{\pi}_1 (X_{\mathbf{C}}, |A|).$$

□

Example 6.2.5. Take $\mathbf{C}$ to be the graph category $x \rightrightarrows y$ with $f, g : x \to y$. Then $X_{\mathbf{C}}$ is a circle with fundamental group $\mathbb{Z}$, so corollary 6.2.4 tells us that

$$\pi_1 \left(\mathbf{Sets}^{\mathbf{C}^{op}}, x \right) = \widehat{\mathbb{Z}}.$$

Compare this with section 3.3.

Example 6.2.6. Take $\mathbf{C}$ to be the (co)equalizer category from example 5.2.9 or example 5.2.8. Both realizations $X_{\mathbf{C}}$ are disks, so we can immediately conclude that $\pi_1 \left(\mathbf{Set}^{\mathbf{C}^{op}}, x \right) = 0$.

Example 6.2.7. Take $\mathbf{C}$ to be the category given by $x \rightrightarrows y \leftleftarrows z$. Then the realization $X_{\mathbf{C}}$ is a figure-8. The fundamental group of the figure-8 can be computed using the Van Kampen theorem to find that $\pi_1 \left(\mathbf{Set}^{\mathbf{C}^{op}}, x \right) = \widehat{\mathbb{Z} * \mathbb{Z}}$.

Example 6.2.8. Take $\mathbf{C}$ to be any finite poset. Then the fundamental group of $\mathbf{Set}^{\mathbf{C}^{op}}$ is the profinite completion of the fundamental group of $\mathbf{C}$ viewed as a finite T_0-space by theorem 5.3.4.

Chapter 7

Outlook and Open Problems

We have partially answered question 1.4.2 with theorem 5.3.4 and theorem 1.4.1. Here we shall consider some problems that one might still want to solve.

First, there is the evidence from [Len11] that theorem 1.4.1 also holds for monoid categories **C**. In order to move forward in that direction, perhaps it is best if conjecture 5.3.14 is answered positively.

The definition of an Alexandroff category (definition 5.3.5) is rather technical, and it would be benificial to provide a theorem that gives sufficient conditions to detect an Alexandroff category.

During the development of the theorems, the suspicion arose that there might be some redundancy in definition 5.1.1. It would be a good approach to reconsider it and optimize away any redundant conditions.

The appendix in [Hat02] treats a systematic way to build a space given some opens from it and equivalence relations on them. Perhaps the construction of the McCord space can be more streamlined by using that.

Finally, I feel that there is some inherent degeneracy in definition 5.1.1. More precisely, we seem to only care about 1-morphisms and thus 1-simplices. The higher-order simplices are only there because the nerve functor creates them. Perhaps if one is willing to work with higher categories, more needs to be taken care of in definition 5.1.1. For instance, one could imagine that there be not only a 1-morphism, but also a 2-morphism such that ..., a 3-morphism such that ..., and so forth. The author hopes that in this way, higher homotopy groups of toposes will reveal themselves naturally.

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Index