Let $\mathbf{C} \in \mathbf{Cat}$. Recall the nerve and geometric realization. Write $X_{\mathbf{C}} = |N\mathbf{C}|$.

**Question.** For which (connected) categories $\mathbf{C} \in \mathbf{Cat}$ does there exist a geometric morphism

$$\mathrm{Sh}(X_{\mathbf{C}}) \to \mathbf{Set}^{\mathbf{C}^{op}}$$

which induces an isomorphism of profinite groups

$$\widehat{\pi}_1(|N\mathbf{C}|, |p|) \cong \pi_1\left(\mathbf{Set}^{\mathbf{C}^{op}}, p\right) ?$$

**Definition.** Let $\mathbf{C}$ be a category, $N\mathbf{C}$ the simplicial nerve and $X_{\mathbf{C}} = |N\mathbf{C}|$. Let $\mathbf{C}/A$ be the slice category over $A$. Write $D_f$ for the domain of a morphism $f$. We define the *McCord space* of $A$ to be the topological space

$$\mu(A) := \left( \bigsqcup_{f \in \mathbf{C}/A} D_f^* \right) / \sim .$$

Elements of the coproduct $\bigsqcup D_f^*$ may be denoted as tuples $(f, p)$ where $f : D_f \to A$ is an object of the slice category $\mathbf{C}/A$ and $p \in D_f^* \subset X_{\mathbf{C}}$. Let $\rhd$ be the binary relation defined by $(f,p) \rhd (g,q) \iff p = q$ in $X_{\mathbf{C}}$ and there exists a morphism $h : f \to g$ in $\mathbf{C}/A$ and there exists an $n$-simplex $\sigma \in \mathrm{star}(h)$ such that $p \in \mathrm{int}(\sigma)$. This relation is reflexive, but in general neither symmetric nor transitive. Let $\sim$ be the smallest equivalence relation generated by $\rhd$.

Define a map of sets

$$e_A : \mu(A) \to X_{\mathbf{C}}, \qquad [f, p] \mapsto p.$$

**Definition.** Let $f : A \to B$ be a morphism in $\mathbf{C}$. Then we have a functor $\mathbf{C}/f : \mathbf{C}/A \to \mathbf{C}/B$ given by sending an object $g \in \mathbf{C}/A$ to the composition $f \circ g$. Define a map $\mu(f) : \mu(A) \to \mu(B)$ as sending an equivalence class $[g, p] \in \mu(A)$ to the equivalence class $[f \circ g, p]$.

**Corollary.** *$\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$ is a functor.*

**Definition.** Let $p \in X_{\mathbf{C}}$. We define the *support* of $p$ to be the full subcategory of $\mathbf{C}$ given by

$$\mathbf{C}(p) := \begin{pmatrix} A \in \mathbf{C} : p \in A^* \\ + \\ \text{morphisms from } \mathbf{C} \end{pmatrix}.$$

**Theorem.** *Let $\mathbf{C}$ be a finite $T_0$-space, or equivalently a finite poset. For each $x \in \mathbf{C}$, denote its minimal open set around $x$ by $U_x$. Then there is a natural homeomorphism*

$$\mu(x) \cong \mu_{\mathbf{C}}^{-1}(U_x).$$

*Proof.* I claim that the etale map $e_x : \mu(x) \to X_{\mathbf{C}}$ has a section on $\mu_{\mathbf{C}}^{-1}(U_x) \subset X_{\mathbf{C}}$. Take a point $p \in \mu_{\mathbf{C}}^{-1}(U_x)$. Write $M_p = \min \mathbf{C}(p)$. Then $M_p \in U_x$, so $M_p \leq x$. Write $u : M_p \to x$ for the unique morphism. We now have $[u, p] \in \mu(x)$. Define

$$s_x : \mu_{\mathbf{C}}^{-1}(U_x) \to \mu(x)$$

by sending the point $p$ to $[u, p]$. The definition of $s_x$ is unambiguous, because there is only one choice for $u$. Clearly we have $e_x \circ s_x = \mathrm{id}_{\mu_{\mathbf{C}}^{-1}(U_x)}$. So $s_x$ is a section. We shall now prove that $s_x \circ e_x = \mathrm{id}_{\mu(x)}$. Take $[g, p] \in \mu(x)$ and suppose that $(s_x \circ e_x)([g, p]) = [u, p]$. We want to show that $[g, p] = [u, p]$. Note that $g : D_g \to x$ is unique. Moreover, $D_g \in \mathbf{C}(p)$, so $M_p \leq D_g$. Let $h : M_p \to D_g$ be the unique morphism. Again by uniqueness, $h \circ g = u$. We now have a morphism $h : g \to u$ in $\mathbf{C}/x$, and $(g, h) \in \mathrm{star}(h)$ with $p \in \mathrm{int}(g, h)$. So $[g, p] = [u, g]$. □

**Definition.** We say $\mathbf{C}$ is an *Alexandroff* category if for all $p \in X_{\mathbf{C}}$

1. the support $\mathbf{C}(p)$ is totally ordered with a (unique) minimal element $M_p$, and
2. for every $A \in \mathbf{C}$ and for every $[f, p] \in e_A^{-1}(p)$ there exists a unique morphism $m : M_p \to D_f$ with the property that there is an $n$-simplex $\sigma \in \mathrm{star}(m)$ with $p \in \mathrm{int}(\sigma)$.

**Definition.** We say that $\mathbf{C}$ is *well-fibered* if for all $p \in X_{\mathbf{C}}$ there exists an object $B \in \mathbf{C}$ such that $p^* \circ \mu \cong \mathrm{Hom}_{\mathbf{C}}(B, -)$.

**Lemma.** *If $\mathbf{C}$ is Alexandroff, then it is well-fibered.*

*Proof.* Let $p \in X_{\mathbf{C}}$. The claim is that $p^* \circ \mu \cong \mathrm{Hom}_{\mathbf{C}}(M_p, -)$, where $M_p$ is defined as in **??** . In other words, we need to find a natural isomorphism $\alpha : p^* \circ \mu \to \mathrm{Hom}_{\mathbf{C}}(M_p, -)$. To that end, define $\beta : \mathrm{Hom}_{\mathbf{C}}(M_p, -) \to p^* \circ \mu$ as follows. For each component $A \in \mathbf{C}$, we set

$$\beta_A : \mathrm{Hom}_{\mathbf{C}}(M_p, A) \to e_A^{-1}(p), \qquad h \mapsto [h, p].$$

Then naturality of $\beta$ is clear. The natural transformation $\beta$ will be the inverse for the natural transformation $\alpha$. For the natural transformation $\alpha$, define it as follows.

Take $[g, p] \in e_A^{-1}(p)$. Then $g : D_g \to A$ and $p \in D_g^*$. So $D_g \in \mathbf{C}(p)$. Since $\mathbf{C}$ is Alexandroff, there exists a unique morphism $m : M_p \to D_g$ with the property that there is some $\sigma \in \mathrm{star}(m)$ such that $p \in \mathrm{int}(\sigma)$. For each component $A \in \mathbf{C}$, we set

$$\alpha_A : e_A^{-1}(p) \to \mathrm{Hom}_{\mathbf{C}}(M_p, A), \qquad [g, p] \mapsto g \circ m.$$

Because this $m$ is unique, $\alpha_A$ is well-defined. Observe now that

$$(\beta_A \circ \alpha_A)[g, p] = [g \circ m, p].$$

But $m$ has the property that we are also given a simplex $\sigma \in \mathrm{star}(m)$ such that $p \in \mathrm{int}(\sigma)$. That means that $(g \circ m, p) \triangleright (g, p)$, so $[g \circ m, p] = [g, p]$. In the other direction we find

$$(\alpha_A \circ \beta_A)(h) = h,$$

so we conclude that $\alpha$ and $\beta$ are each other's inverse transformations. $\square$

**Lemma.** *If $\mathbf{C}$ is well-fibered, then $\mu : \mathbf{C} \to \mathbf{LH}/X_{\mathbf{C}}$ is flat.*

*Proof.* By **??**, it suffices to prove that for every $p \in X_{\mathbf{C}}$ the functor $p^* \circ \mu : \mathbf{C} \to \mathbf{Set}$ is flat. This is the same thing as proving that the categoy of elements $\int_{\mathbf{C}} (p^* \circ \mu)$ is filtered, by **??**. Since $\mathbf{C}$ is well-fibered, there exists some object $B \in \mathbf{C}$ such that $p^* \circ \mu \cong \mathrm{Hom}_{\mathbf{C}}(B, -)$. Therefore,

$$\int_{\mathbf{C}} (p^* \circ \mu) \cong \int_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}}(B, -) \cong B \backslash \mathbf{C}.$$

Now the over-category $B\backslash\mathbf{C}$ is always filtered, because $\mathrm{id}_B : B \to B$ is an initial object. $\square$

So we see that when $\mathbf{C}$ is Alexandroff, $\mu$ is flat.

**Proposition.** *There exists a geometric morphism*

$$\tau(\mu) : \mathbf{LH}/X_{\mathbf{C}} \to \mathbf{Set}^{\mathbf{C}^{op}}$$

*for which the left-exact left adjoint $\tau(\mu)^*$ is given by sending a presheaf $P$ on $\mathbf{C}$ to the tensor product $P \otimes_{\mathbf{C}} \mu$, and for which the right adjoint $\tau(\mu)_*$ sends an etale space $e : E \to X_{\mathbf{C}}$ to the presheaf $\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)$ defined for every object $A \in \mathbf{C}$ by*

$$\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)(A) = \mathrm{Hom}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu(A), E).$$

*Proof.* Follows directly from the theory in [**?**, Chapter VII, Paragraph 7]. In particular, in [**?**, Theorem VII.7.2], take $\mathscr{E} = \mathbf{LH}/X_{\mathbf{C}}$. Alternatively, we spoke of the bijection between flat functors and geometric morphisms in **??**. $\square$

We shall be needing the following proposition.

**Proposition.** *Let $p : Y \to X$ be a (not necessarily finite) covering map, where $Y$ is a topological space and $X$ is a locally connected space. Let $f, g : Z \to Y$ be two continuous maps satisfying $p \circ f = p \circ g$, where $Z$ is a connected topological space. If there is a point $z \in Z$ with $f(z) = g(z)$, then $f = g$.*

*Proof.* This is [**?**, Proposition 2.2.2]. We'll give a sketch of the proof here. Let $U = \{w \in Z : f(w) = g(w)\}$. Then prove that $U$ is both open and closed in $Z$. Conclude that $U$ must be all of $Z$ by connectedness. $\square$

The following proposition is central.

**Proposition.** *Let $\pi_E : E \to X_{\mathbf{C}}$ be a finite covering map of degree $d > 0$ and let $A$ be an object of $\mathbf{C}$. Then we have a natural bijection of sets*

$$\alpha_{A,E} : (\mathbf{LH}/X_{\mathbf{C}})\,(\mu A, E) \to \pi_E^{-1}(|A|), \qquad \varphi \mapsto \varphi[\mathrm{id}_A, |A|].$$

*Proof.* By **??**,

$$e_A^{-1}(|A|) = \{[\mathrm{id}_A, |A|]\} \subset \mu(A).$$

Write

$$\pi_E^{-1}(|A|) = \{x_1, \ldots, x_d\} \subset E.$$

Now take a morphism $\varphi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu(A), E)$. Then

$$\varphi[\mathrm{id}_A, |A|] \in \{x_1, \ldots, x_d\}.$$

I claim that these $d$ choices for $\varphi[\mathrm{id}_A, |A|]$ completely determine $\varphi$. So let $\psi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu(A), E)$ be another morphism and suppose that

$$\varphi[\mathrm{id}_A, |A|] = x_1 = \psi[\mathrm{id}_A, |A|].$$

We will apply **??** . Take $Y = E$, $X = X_{\mathbf{C}}$, $Z = \mu(A)$, $p = \pi_E$, $f = \varphi$, $g = \psi$ and $z = [\mathrm{id}_A, |A|]$ in **??** . Then $X_{\mathbf{C}}$ is a locally connected space, because it is a CW-complex by [**?**, Proposition I.2.3]. Moreover, $\mu(A)$ is connected by **??**. Finally,

$$p \circ f = \pi_E \circ \varphi = e_A = \pi_E \circ \varphi = p \circ g.$$

This proves that

$$\#\,(\mathbf{LH}/X_{\mathbf{C}})\,(\mu A, E) \leq d.$$

Let us now prove that the map $\alpha_{A,E}$ is surjective. Thus, given $x \in \pi_E^{-1}(|A|)$ we want to show that there exists some $\varphi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu\, A, E)$ such that $\varphi[\mathrm{id}_A, |A|] = x$. We shall actually construct such a $\varphi$. First, observe that $\pi_E$ is a Serre fibration. Then apply **??** to see that any two lifts of some $|\sigma| : \Delta^n \to X_{\mathbf{C}}$ are unique. For each $f \in \mathbf{C}/A$ (and so in particular for $\mathrm{id}_A$) we have a commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook, "x"] \arrow[d, hook, "|d_0|"'] & E \arrow[d, "\pi_E"] \\
\Delta^1 \arrow[r, "|f|"'] \arrow[ur, dashed, "\exists!\widetilde{f}"] & X_{\mathbf{C}}
\end{tikzcd}$$

and a unique diagonal filler $\widetilde{f} : \Delta^1 \to E$ as indicated by the dotted arrow in the diagram. Thus we have a collection of lifted paths $\widetilde{f} : \Delta^n \to E$ all ending up at the point $x \in E$ and starting at some arbitrary point in $E$. Let us call the starting point $\widetilde{f}(0)$.

Now let $[f, p] \in \mu(A)$ be an arbitrary point. We are going to define what $\varphi[f, p]$ is. We have $f : D_f \to A$ and $p \in D_f^*$, so $p \in \mathrm{int}(\sigma)$ for some $n$-simplex $\sigma \in \mathrm{star}(D_f)$. We may assume that $\sigma$ is non-degenerate by **??**. If $n = 0$, then $p = |D_f|$. In that case, we define

$$\varphi[f, p] := \widetilde{f}(0).$$

Suppose now that $n > 0$. Let $\theta : \mathbf{0} \to \mathbf{n}$ be an injective order-preserving map as in **??** such that $D_f = \sigma \circ \mathbf{\Delta}(-, \theta)$. Then we have a commutative diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, hook, "\widetilde{f}(0)"] \arrow[d, hook, "|\mathbf{\Delta}(-,\theta)|"'] & E \arrow[d, "\pi_E"] \\ \Delta^n \arrow[r, "|\sigma|"'] \arrow[ur, dashed, "\exists! \widetilde{\sigma}"] & X_{\mathbf{C}} \end{tikzcd}$$

and a unique diagonal filler $\widetilde{\sigma} : \Delta^n \to E$. Let $t \in \Delta^n$ be the unique coordinates such that $p = |\sigma|(t)$. We define

$$\varphi[f, p] := \widetilde{\sigma}(t).$$

We must prove that this definition is independent of the chosen representative of the equivalence relation in $\mu(A)$. So suppose that $(f, p) \rhd (g, p)$. Then $p \in \mathrm{int}(\sigma_f)$ and $p \in \mathrm{int}(\sigma_g)$ for some $\sigma_f$ having $D_f$ as a vertex and some $\sigma_g$ having $D_g$ as a vertex. Suppose that we have uniquely lifted $\sigma_f$ and $\sigma_g$ to maps

$$\Delta^n \xrightarrow{\widetilde{\sigma_f}} E, \qquad \Delta^m \xrightarrow{\widetilde{\sigma_g}} E.$$

Let $t_f \in \Delta^n$ and $t_g \in \Delta^m$ be the unique coordinates such that

$$|\sigma_f|(t_f) = p = |\sigma_g|(t_g).$$

By the definition of $\rhd$, There exists a morphism $h : f \to g$ in $\mathbf{C}/A$ and a $k$-simplex $\tau \in \mathrm{star}(h)$ such that $p \in \mathrm{int}(\tau)$. Consider first the 2-simplex $\beta$ given by

$$\beta = (h, g) : \mathbf{\Delta}(-, \mathbf{2}) \to N\mathbf{C}.$$

Let $\widetilde{\beta} : \Delta^2 \to E$ be the unique lift of $|\beta| : \Delta^2 \to X_{\mathbf{C}}$. By [**?**, Exercise A.1], the face of a lift is the lift of a face, so we see that two of the faces of $\widetilde{\beta}$ upstairs in $E$ are the lifts $\widetilde{f}$ and $\widetilde{g}$. Denote by $\widetilde{h}$ the third lift of $|h| : \Delta^1 \to X_{\mathbf{C}}$.

We may assume by **??** that $\tau$, $\sigma_f$ and $\sigma_g$ are non-degenerate. This implies that $\tau = \sigma_f = \sigma_g$, and $n = m = k$, and $t_f = t_g$. Let $\theta' : \mathbf{1} \to \mathbf{m}$ be an injective order-preserving map as in **??** such that $h = \tau \circ \mathbf{\Delta}(-, \theta')$. Then we have a commutative diagram

$$\begin{tikzcd} \Delta^1 \arrow[r, "\widetilde{h}"] \arrow[d, hook, "|\mathbf{\Delta}(-,\theta')|"'] & E \arrow[d, "\pi_E"] \\ \Delta^m \arrow[r, "|\tau|"'] \arrow[ur, dashed, "\exists! \widetilde{\tau}"] & X_{\mathbf{C}} \end{tikzcd}$$

By uniqueness of the lifts, $\widetilde{\sigma_f} = \widetilde{\sigma_g} = \widetilde{\tau}$.

Continuity of $\varphi$ follows from the fact that all the liftings $\widetilde{\sigma} : \Delta^n \to E$ from the continuous maps $|\sigma| : \Delta^n \to X_{\mathbf{C}}$ are continuous. Moreover, the realization $X_{\mathbf{C}}$ is defined as the colimit

$$X_{\mathbf{C}} = \operatorname*{colim}_{\substack{\mathbf{\Delta}(-,\mathbf{n}) \to N\mathbf{C} \\ \text{in } \mathbf{\Delta}\downarrow N\mathbf{C}}} \Delta^n$$

So the gluing data comes from $X_{\mathbf{C}}$.

□

**Theorem.** *If $\pi : E \to X_{\mathbf{C}}$ is a finite covering map, then $\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)$ is a locally constant finite presheaf on $\mathbf{C}$. Conversely, if $P$ is a locally constant finite presheaf on $\mathbf{C}$, then $P \otimes_{\mathbf{C}} \mu$ has the structure of a finite covering space over $X_{\mathbf{C}}$.*

*Proof.* The claim that $P \otimes_{\mathbf{C}} \mu$ is a finite covering map whenever $P \in \left(\mathbf{Set}^{\mathbf{C}^{op}}\right)_{\mathrm{lcf}}$ is covered in **??**. (Use **??** there for the translation between sheaves and etale spaces). Explicitly, the finite covering map is given by

$$\pi : P \otimes_{\mathbf{C}} \mu \to X_{\mathbf{C}}, \qquad x \otimes [f, p] \mapsto p.$$

The degree of $\pi$ is the number of elements in $P(A)$, for any $A \in \mathbf{C}$. This is well-defined by **??** and the assumption that $X_{\mathbf{C}}$ is connected.

We shall prove the other direction, which is the remarkable one. So we want to show that given a finite covering map $\pi_E : E \to X_{\mathbf{C}}$ and given a morphism $f : A \to B$ in $\mathbf{C}$, the map

$$(\mathbf{LH}/X_{\mathbf{C}})(\mu\, B, E) \to (\mathbf{LH}/X_{\mathbf{C}})(\mu\, A, E), \qquad \varphi \mapsto \varphi \circ \mu(f) \tag{1}$$

is a bijection. By **??** , it suffices to prove that the map in eq. (1) is injective. So take two morphisms $\varphi, \psi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu\, B, E)$ and suppose that $\varphi \circ \mu(f) = \psi \circ \mu(f)$. We want to prove that $\varphi = \psi$. In **??** , take $Y = E$, $X = X_{\mathbf{C}}$, $Z = \mu(B)$, $p = \pi_E$, $f = \varphi$, $g = \psi$. As in the proof of **??** , all conditions of **??** are satisfied, except that we need to supply a point $[h, p] \in \mu(B)$ such that $\varphi[h, p] = \psi[h, p]$. But we know that

$$\forall\ [g, p] \in \mu(A) : \varphi[f \circ g, p] = \psi[f \circ g, p].$$

Now $\mu(A)$ is non-empty, because $[\mathrm{id}_A, |A|] \in \mu(A)$. Therefore

$$\varphi[f, |A|] = \psi[f, |A|]$$

and we are done. □

**Proposition.** *Let $E \in \mathbf{FinCov}/X_{\mathbf{C}}$. Then the counit at the component $E$*

$$\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mu \to E$$

*of the adjunction $- \otimes_{\mathbf{C}} \mu \dashv \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, -)$ is an isomorphism.*

*Proof.* The counit is given by the continuous map over the base space $X_{\mathbf{C}}$

$$\varepsilon_E : \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mu \to E, \qquad \varphi \otimes [f,p] \mapsto \varphi([f,p]),$$

where $\varphi \in (\mathbf{LH}/X_{\mathbf{C}})(\mu(A), E)$ for some $A \in \mathbf{C}$ and $[f,p] \in \mu(A)$. Like for sheaves, it suffices to prove that $\varepsilon_E$ is an isomorphism on the level of stalks, i.e. fibers of the finite covering maps. First of all, it suffices to look at points $p$ of the form $p = |A|$ for some object $A \in \mathbf{C}$, for recall (viz. **??** , **??** ) that $\mu$ is well-fibered, so that $p^* \circ \mu \cong \mathrm{Hom}(M_p, -)$. This gives

$$\begin{aligned} p^* \circ \left(\underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mu\right) &\cong \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} (p^* \circ \mu) \\ &\cong \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E) \otimes_{\mathbf{C}} \mathrm{Hom}(M_p, -) \\ &\cong \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, E)(M_p) \\ &= \mathrm{Hom}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu(M_p), E) \\ &\cong \pi_E^{-1}(|M_p|) \end{aligned}$$

Now if we follow the isomorphisms, the composition is precisely the counit. ☐

A similar thing occurs with the unit.

**Proposition.** *Let $P$ be a locally constant finite presheaf. Then the unit of the adjunction $- \otimes_{\mathbf{C}} \mu \dashv \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, -)$ at the component $P$ is an isomorphism.*

*Proof.* The unit is a map of presheaves

$$\eta : P \to \underline{\mathrm{Hom}}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu, P \otimes_{\mathbf{C}} \mu) \tag{2}$$

which for a given object $A \in \mathbf{C}$ is a map of sets

$$\eta_A : P(A) \to \mathrm{Hom}_{\mathbf{LH}/X_{\mathbf{C}}}(\mu(A), P \otimes_{\mathbf{C}} \mu)$$

and, since $\mathbf{LH}/X_{\mathbf{C}}$ is cartesian closed (because it is a topos), this is the same thing as giving a map of sets

$$\eta_A^{\top} : P(A) \times \mu(A) \to P \otimes \mu$$

and this map is given by

$$\eta_A^{\top}(x, [f,p]) = x \otimes [f,p].$$

Now the isomorphism in **??** is precisely the unit. ☐

**Corollary.** *The left and right adjoint of* **??** *restrict to an equivalence of categories*

$$\mathbf{FinCov}/X_{\mathbf{C}} \cong \left(\mathbf{Set}^{\mathbf{C}^{op}}\right)_{\mathrm{lcf}}.$$

*Proof.* Apply **??** and **??** . □

**Corollary.** *Let $A \in \mathbf{C}$ be an object. Then there is a natural isomorphism of profinite groups*

$$\widehat{\pi}_1 (X_{\mathbf{C}}, |A|) \cong \pi_1 \left( \mathbf{Set}^{\mathbf{C}^{op}}, A \right).$$

*Proof.* Interpret $A$ as a geometric morphism (point)

$$A : \mathbf{Set} \to \mathbf{Set}^{\mathbf{C}^{op}}$$

where the inverse image part sends a presheaf $P$ on $\mathbf{C}$ to $P(A)$, and the direct image part sends a set $S$ to the "underline Hom" from **??**. So $A$ is a point of the topos $\mathbf{Set}^{\mathbf{C}^{op}}$. The category $\left(\mathbf{Set}^{\mathbf{C}^{op}}\right)_{\mathrm{lcf}}$ is a Galois category with fundamental functor given by the inverse image part of the point $A$. From **??** , we obtain

$$\pi_1 \left( \mathbf{Set}^{\mathbf{C}^{op}}, A \right) \cong \pi_1 (\mathrm{Sh}(X_{\mathbf{C}}), |A|).$$

Then from [**?**, Theorem 1.15, or 3.10], we obtain

$$\pi_1 (\mathrm{Sh}(X_{\mathbf{C}}), |A|) \cong \widehat{\pi}_1 (X_{\mathbf{C}}, |A|).$$

□

**Example.** Take $\mathbf{C}$ to be the graph category $x \rightrightarrows y$ with $f, g : x \to y$. Then $X_{\mathbf{C}}$ is a circle with fundamental group $\mathbb{Z}$, so **??** tells us that

$$\pi_1 \left( \mathbf{Sets}^{\mathbf{C}^{op}}, x \right) = \widehat{\mathbb{Z}}.$$

Compare this with section 3.3.

**Example.** Take $\mathbf{C}$ to be the (co)equalizer category from **??** or **??**. Both realizations $X_{\mathbf{C}}$ are disks, so we can immediately conclude that $\pi_1 \left( \mathbf{Set}^{\mathbf{C}^{op}}, x \right) = 0$.

**Example.** Take $\mathbf{C}$ to be the category given by $x \rightrightarrows y \leftleftarrows z$. Then the realization $X_{\mathbf{C}}$ is a figure-8. The fundamental group of the figure-8 can be computed using the Van Kampen theorem to find that $\pi_1 \left( \mathbf{Set}^{\mathbf{C}^{op}}, x \right) = \widehat{\mathbb{Z} * \mathbb{Z}}$.

**Example.** Take $\mathbf{C}$ to be any finite poset. Then the fundamental group of $\mathbf{Set}^{\mathbf{C}^{op}}$ is the profinite completion of the fundamental group of $\mathbf{C}$ viewed as a finite $T_0$-space by **??** .