SINGULAR HOMOLOGY GROUPS AND HOMOTOPY GROUPS OF FINITE TOPOLOGICAL SPACES

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1. Introduction. Finite topological spaces have more interesting topological properties than one might suspect at first. Without thinking about it very long, one might guess that the singular homology groups and homotopy groups of finite spaces vanish in dimension greater than zero. (One might jump to the conclusion that continuous maps of simplexes and spheres into a finite space must be constant.) However, we shall show (see Theorem 1) that exactly the same singular homology groups and homotopy groups occur for finite spaces as occur for finite simplicial complexes.

A map $f : X \to Y$ is a *weak homotopy equivalence* if the induced maps

$$f_* : \pi_i(X, x) \to \pi_i(Y, fx) \tag{1.1}$$

are isomorphisms for all x in X and all $i \geq 0$. (Of course in dimension 0, "isomorphism" is understood to mean simply "1–1 correspondence," since $\pi_0(X, x)$, the set of path components of X, is not in general endowed with a group structure.) It is a well-known theorem of J.H.C. Whitehead (see [4; 167]) that every weak homotopy equivalence induces isomorphisms on singular homology groups (hence also on singular cohomology rings.) Note that the general case is reduced to the case where X and Y are path connected by the assumption that (1.1) is a 1–1 correspondence for $i = 0$.

THEOREM 1. (i) *For each finite topological space X there exist a finite simplicial complex K and a weak homotopy equivalence $f : |K| \to X$.* (ii) *For each finite simplicial complex K there exist a finite topological space X and a weak homotopy equivalence $f : |K| \to X$.*

This theorem is a consequence of the stronger and more detailed Theorems 2, 3, and 4 stated in the next section.

The main idea for the correspondences $X \to K$ and $K \to X$ in the above theorem is contained in the paper [1] of P.S. Alexandroff. Finite spaces are special cases of what Alexandroff [1] called "discrete" spaces, but which we prefer to call *A-spaces* (since "discrete" commonly means now that every subset is open).

Definition. An *A-space* is a topological space in which the intersection of every collection of open subsets is open. A *T_0A-space* is an A-space satisfying the *T_0 separation axiom*: for each pair of distinct points, there exists an open set containing one but not the other.

All our theorems work just as well for A-spaces. A class of spaces between the

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class of finite spaces and the class of A-spaces is the class of *locally finite* spaces—those spaces in which every point has a finite neighborhood. An example of a compact T_0A-space which is not locally finite is the positive integers with basis consisting of the sets $U_n = \{m : m \geq n\}$, all $n \geq 1$.

The author is grateful to the referee for suggesting an alternate approach to the proofs of the theorems. This approach consists of applying the work of A. Dold and R. Thom in §2 of [2] on quasi-fibrations. The referee's suggestion resulted in stronger theorems with shorter proofs.

Some other aspects of the homotopy theory of finite spaces have been studied by R. E. Stong in a paper [8] to appear.

Part of the following results were announced in [5]. *Added in proof.* An application of this work to "ordinary" spaces is given in [7].

2. Statements of main theorems. In the following, if K is any simplicial complex, the underlying polyhedron $|K|$ will always be given the weak topology (see [3; 75]). We do not assume complexes to be locally finite. All maps of spaces are continuous.

THEOREM 2. *There exists a correspondence that assigns to each T_0A-space X a simplicial complex $\mathcal{K}(X)$, whose vertices are the points of X, and a weak homotopy equivalence $f_X : |\mathcal{K}(X)| \to X$. Each map $\varphi : X \to Y$ of T_0A-spaces is also a simplicial map $\mathcal{K}(X) \to \mathcal{K}(Y)$, and $\varphi f_X = f_Y |\varphi|$.*

The construction $X \to \mathcal{K}(X)$, which is described in §5 below, is due to Alexandroff [1; 506]. He calls $\mathcal{K}(X)$ the *barycentric subdivision* of the T_0A-space X. (The reason will be apparent.) For a special kind of locally finite T_0 spaces X it is shown in [1; 515] that the simplicial homology groups of the complex $\mathcal{K}(X)$ are isomorphic to the Kolmogoroff homology groups of X. No map $f_X : |\mathcal{K}(X)| \to X$ is discussed in [1]. The observation in Theorem 2 that every continuous map $X \to Y$ is a simplicial map $\mathcal{K}(X) \to \mathcal{K}(Y)$ is in [1; 508].

THEOREM 3. *There exists a correspondence that assigns to each simplicial complex K a locally finite T_0 space $\mathfrak{X}(K)$ whose points are the barycenters of simplexes of K, and a weak homotopy equivalence $f_K : |K| \to \mathfrak{X}(K)$. Furthermore, to each simplicial map $\psi : K \to L$ is assigned a map $\psi' : \mathfrak{X}(K) \to \mathfrak{X}(L)$ such that $\psi' f_K \simeq f_L |\psi|$.*

It will readily be seen from the construction below that $\mathcal{K}$ and $\mathfrak{X}$ are covariant functors. Theorem 2 will be used in proving Theorem 3. In fact, the functors $\mathcal{K}$ and $\mathfrak{X}$ are related by the equation $K' = \mathcal{K}(\mathfrak{X}(K))$, where K' is the first barycentric subdivision of the complex K.

Observe that part (ii) of Theorem 1 follows from Theorem 3; for if the complex K is finite, $\mathfrak{X}(K)$ is finite. And part (i) of Theorem 1 follows from Theorem 2, *except* for the fact that the finite space X in Theorem 1 is not assumed to be T_0. This is taken care of by the following theorem.

THEOREM 4. *There exists a correspondence that assigns to each A-space X a quotient space $\hat{X}$ of X with the following properties.* (i) *The quotient map $\nu_X : X \rightarrow \hat{X}$ is a homotopy equivalence.* (ii) *$\hat{X}$ is a T_0A-space.* (iii) *For each map $\varphi : X \rightarrow Y$ of A-spaces, there exists a unique map $\hat{\varphi} : \hat{X} \rightarrow \hat{Y}$ such that $\nu_Y \varphi = \hat{\varphi} \nu_X$.*

If X is any A-space, and $\mu : \hat{X} \rightarrow X$ is a homotopy inverse for ν_X , then the map $\mu f_X : |\mathcal{K}(\hat{X})| \rightarrow X$ is a weak homotopy equivalence by Theorems 2 and 4. By the same theorems, if X is finite, then the complex $\mathcal{K}(\hat{X})$ is finite. Thus we have seen that Theorem 1 follows from Theorems 2, 3, and 4.

For each space X, let $S(X)$ be the suspension of X; and for each map $\varphi : X \rightarrow Y$, let $S(\varphi) : S(X) \rightarrow S(Y)$ be the suspension of φ. If $\mathfrak{A}$ denotes the category of all spaces and maps, then $S : \mathfrak{A} \rightarrow \mathfrak{A}$ is a covariant functor. The next theorem asserts the existence of an analog $\mathcal{S}$ of S, which we call *non-Hausdorff suspension.*

THEOREM 5. *There exist a covariant functor $\mathcal{S} : \mathfrak{A} \rightarrow \mathfrak{A}$ and a natural transformation $g : S \rightarrow \mathcal{S}$ with the following properties.* (i) *The class of finite* (*locally finite*) $(A-)(T_0)$ *spaces is transformed into itself by $\mathcal{S}$.* (ii) *For each space X, the map $g_X : S(X) \rightarrow \mathcal{S}(X)$ is a weak homotopy equivalence.*

The maps f_X , f_K , and g_X of Theorems 2, 3, and 5 will be proved to be weak homotopy equivalences by using the theorem stated next. This theorem follows from a modification of the proof of Satz 2.2 of Dold and Thom [2]. The necessary modification will be pointed out in the next section. An open cover $\mathfrak{U}$ of a space B will be called *basis-like* if whenever $x \in U \cap V$ and U, $V \in \mathfrak{U}$, there exists a $W \in \mathfrak{U}$ such that $x \in W \subset U \cap V$. Notice that this condition in fact means precisely that $\mathfrak{U}$ is a basis for a topology on B smaller than the given one.

THEOREM 6. *Suppose p is a map of a space E into a space B for which there exists a basis-like open cover $\mathfrak{U}$ of B satisfying the following condition: For each $U \in \mathfrak{U}$, the restriction $p|p^{-1}(U) : p^{-1}(U) \rightarrow U$ is a weak homotopy equivalence. Then p itself is a weak homotopy equivalence.*

3. Proof of Theorem 6.

Before showing what is necessary for the proof, let us indicate why Theorem 6 "almost" follows directly from the *statement* of Satz 2.2 of [2]. Recall that a map p of a space E onto a space B is called a *quasi-fibration* if the induced map

$$p_* : \pi_i(E, p^{-1}(x), y) \rightarrow \pi_i(B, x)$$

is an isomorphism for each $x \in B$, $y \in p^{-1}(x)$, and each $i \geq 0$. From the exactness of the homotopy sequence of $(E, p^{-1}(x), y)$ (and a separate argument in dimension 0), it is easy to see that *an onto map $p : E \rightarrow B$ with homotopically trivial fibers is a quasi-fibration if and only if it is a weak homotopy equivalence.* Hence if in Theorem 6 p were also assumed to be onto with homotopically trivial fibers, then the conclusion would follow directly from the statement of Satz 2.2 in [2].

The main idea of adapting the proof of Satz 2.2 on pages 246-251 of [2] to a proof of Theorem 6 is simply to assign a different meaning to the phrase "distinguished subset (ausgezeichnete Teilmenge) of B." Let us say that a subset U of B is *distinguished* provided that $p|p^{-1}(U) : p^{-1}(U) \to U$ is a weak homotopy equivalence. Then we replace Hilfssatz 2.4 by the following lemma, whose proof is quite similar.

LEMMA 1. *Let $p : E \to B$ be a map and let U be a distinguished subset of B. Then p is a weak homotopy equivalence if and only if the induced map*

$$p_* : \pi_i(E, p^{-1}(U), y) \to \pi_i(B, U, py)$$

is an isomorphism for all $i \geq 0$ and all y in $p^{-1}(U)$.

Hilfssatz 2.6 is kept as it is. The main thing to observe is that Satz 2.7 remains valid when "distinguished" is given the meaning specified above. For by Lemma 1, Hilfssatz 2.6 is still applicable. Also, the proof of Hilfssatz 2.8 depends only on the fact that the open cover $\mathfrak{U}$ is basis-like. Just as in §2.9, one applies Satz 2.7 to show that $p_* : \pi_i(E, p^{-1}(U), y) \to \pi_i(B, U, py)$ is an isomorphism whenever $U \in \mathfrak{U}$, $y \in p^{-1}(U)$, and $i \geq 0$. (One does not have to assume p is onto to show that p_* is onto for $i = 0$. This follows from the facts that $\mathfrak{U}$ covers B and that the maps $\pi_0(p^{-1}(V)) \to \pi_0(V)$, $V \in \mathfrak{U}$, are onto.) Then one applies Lemma 1 and the fact that $\mathfrak{U}$ covers B to obtain the final result that p is a weak homotopy equivalence.

4. The equivalence of transitive, reflexive relations and A-space structures. The material in this section is essentially contained in [1]; because of the notation and terminology involved we summarize it. Also, we have found it better in our context to replace Alexandroff's use of closed sets by the use of open sets.

If $M \subset X$, where X is an A-space, the *open hull* $U(M)$ of M is the intersection of all open subsets of X containing M. We write U_x for the open hull $U(\{x\})$ of a point x in X. Obviously the system $(U_x : x \in X)$ is a basis for X, which refines every basis.

Now we define a relation $\leq$ on X by saying $x \leq y$ if $x \in U_y$ (equivalently, $U_x \subset U_y$). We write $x < y$ if $x \leq y$ but $x \neq y$. Clearly, $\leq$ is transitive and reflexive. It is easy to see that a map $f : X \to Y$ of A-spaces is continuous if and only if it is *order-preserving* ($x \leq y$ implies $f(x) \leq f(y)$). An A-space is a T_0 space if and only if the transitive, reflexive relation $\leq$ is also antisymmetric—in other words, if and only if $\leq$ is a *partial order*.

The process is reversible. Let $(X, \leq)$ be a set with a transitive, reflexive relation. For each x in X, let $U_x = \{y \in X : y \leq x\}$. The system $(U_x : x \in X)$ forms a basis for a topology on X that makes X into an A-space, in which U_x is the open hull of x.

In the following sections we will freely use this equivalence between A-spaces and sets with a transitive, reflexive relation.

5. Proof of Theorem 2.

DEFINITION OF $\mathcal{K}(X)$. Let X be a T_0A-space. The vertices of the complex $\mathcal{K}(X)$ are the points of X. The simplexes of $\mathcal{K}(X)$ are the finite, totally ordered subsets of X.

Example. Let $X = \{a, b, c, d\}$ be the four-point space with basis $\{\{a\}, \{c\}, \{a, b, c\}, \{a, d, c\}\}$. Then $|\mathcal{K}(X)|$ is a simple closed curve.

Lemma 2. If Y is a subspace of the T_0A-space X, then $\mathcal{K}(Y)$ is a full subcomplex of $\mathcal{K}(X)$.

Proof. If $y \in Y$, clearly the open hull of y in Y equals the intersection of Y with the open hull of y in X. Thus the relation $\leq$ in Y is the restriction of the relation $\leq$ in X. From this the lemma follows.

DEFINITION OF THE MAP $f_X : |\mathcal{K}(X)| \to X$. If $u \in |\mathcal{K}(X)|$ (where X is a T_0A-space), then u is contained in a unique open simplex $(x_0, x_1, \cdots, x_r)$ where $x_0 < x_1 < \cdots < x_r$ in X. We let $f_X(u) = x_0$.

LEMMA 3. *If Y is an open subset of the T_0A-space X, then $(f_X)^{-1}(Y)$ is the regular neighborhood of $\mathcal{K}(Y)$ in $\mathcal{K}(X)$. Hence by Lemma 2, $(f_X)^{-1}(Y)$ deformation retracts onto $|\mathcal{K}(Y)|$.*

Proof. In other words, we wish to establish the equation

$$(f_X)^{-1}(Y) = \bigcup\{\text{star } (y) : y \in Y\}. \tag{5.1}$$

Here star (y) is the union of all open simplexes of $\mathcal{K}(X)$ having y as a vertex.

Suppose $u \in (f_X)^{-1}(Y)$. Then $u \in \sigma = (x_0, x_1, \cdots, x_r)$, where $x_0 < x_1 < \cdots < x_r$ in X and $f_X(u) = x_0 \in Y$. Thus $\sigma \subset$ star (x_0) so that u belongs to the right-hand side of (5.1). Conversely, suppose $\sigma \subset$ star (y), where $y \in Y$. If $\sigma = (x_0, x_1, \cdots, x_r)$ where $x_0 < x_1 < \cdots < x_r$, then some x_i equals y. Thus $x_0 \leq y$, so that $x_0 \in U_y$. Since Y is open, $U_y \subset Y$, so that $f_X(\sigma) = x_0 \in Y$. This completes the proof.

LEMMA 4. *f_X is continuous.*

Proof. Regular neighborhoods are open.

LEMMA 5. *Let Y be a space containing a point ω such that the only open subset of Y containing ω is Y itself. Then Y is contractible.*

Proof. Define a homotopy $F: Y \times I \to Y$ by letting $F(y, t) = y$ if $0 \leq t < 1$ and $F(y, 1) = \omega$ for all y in Y. We need only show that F is continuous. For this, let G be any open subset of Y. *Case* 1: $\omega \in G$. Then by assumption, $G = Y$, so that $F^{-1}(G) = Y \times I$ is open. *Case* 2: $\omega \notin G$. Then it is easy to see that $F^{-1}(G) = G \times [0, 1)$, which is open.

LEMMA 6. *If X is an A-space and $x \in X$, then U_x is contractible. Thus X has a basis of contractible, open sets; in particular, X is locally contractible.*

Proof. Take $Y = U_x$ and $\omega = x$ in Lemma 5.

COROLLARY. *Every A-space is locally pathwise connected, hence is pathwise connected if and only if it is connected.*

LEMMA 7. *If X is a T_0A-space and $x \in X$, then $(f_X)^{-1}(U_x)$ is a contractible (open) subset of the polyhedron $|\mathcal{K}(X)|$.*

Proof. By Lemma 3, it suffices to show that $|\mathcal{K}(U_x)|$ is contractible. Let $V_x = U_x - x$. It suffices to show that $\mathcal{K}(U_x) = \text{cone}\,(\mathcal{K}(V_x), x)$. Every simplex of $\mathcal{K}(V_x)$ is of the form $\{x_0, \cdots, x_r\}$ where $x_0 < \cdots < x_r < x$. Hence $\{x_0, \cdots, x_r, x\}$ is a simplex of $\mathcal{K}(U_x)$. And it is clear that every simplex of $\mathcal{K}(U_x) - \mathcal{K}(V_x)$ is of the form $\{x_0, \cdots, x_r, x\}$ where $x_0 < \cdots < x_r$ in V_x.

Now we can apply Theorem 6 to show that the map $f_X : |\mathcal{K}(X)| \to X$ is a weak homotopy equivalence. For the basis-like open cover of X we take the basis $(U_x : x \in X)$. By Lemmas 6 and 7, each set U_x is a distinguished subset of B. Thus Theorem 6 applies.

Remark. An alternate method of proof is to show that the fibers of the map f_X are contractible. Then by Lemmas 6 and 7, f_X is a quasi-fibration over each U_x, so that one may apply Satz 2.2 of [2] directly to obtain that f_X is a quasi-fibration, hence a weak homotopy equivalence. However, we have chosen to state and use Theorem 6 since this theorem seems to be of interest in its own right. Also, the use of Theorem 6 gives a stronger and simpler version of Lemma 13 in §8. *Added in proof.* Theorem 6 is used in [7].

It remains only to establish the second sentence in the statement of Theorem 2. Suppose $\varphi : X \to Y$ is a map of T_0A-spaces. Since φ is order preserving, φ maps simplexes of $\mathcal{K}(X)$ onto simplexes of $\mathcal{K}(Y)$; therefore $\varphi : \mathcal{K}(X) \to \mathcal{K}(Y)$ is simplicial. Suppose $u \in (x_0, \cdots, x_r) \subset |\mathcal{K}(X)|$, where $x_0 < \cdots < x_r$. Now, $|\varphi|(u)$ belongs to the simplex $(\varphi(x_0), \cdots, \varphi(x_r))$, where $\varphi(x_0) \leq \cdots \leq \varphi(x_r)$. (There may be repetitions.) Hence $f_Y(|\varphi|(u)) = \varphi(x_0) = \varphi f_X(u)$. This completes the proof of Theorem 2.

COROLLARY 1. *If X is a T_0A-space and $Y \subset X$, then f_X induces an isomorphism of the singular homology exact sequence of the pair $(|\mathcal{K}(X)|, |\mathcal{K}(Y)|)$ onto that of the pair (X, Y).*

Proof. It is clear that $f_Y : |\mathcal{K}(Y)| \to Y$ is simply the restriction of f_X to $|\mathcal{K}(Y)|$. Thus the result follows from Theorem 2, the Whitehead Theorem, and the "five lemma."

COROLLARY 2. *If X is a T_0A-space, then for each point x in X the local singular homology groups $H_*(|\mathcal{K}(X)|, |\mathcal{K}(X)| - x)$ and $H_*(X, X - x)$ are isomorphic.*

Proof. From Corollary 1, $H_*(|\mathcal{K}(X)|, |\mathcal{K}(X - x)|) \approx H_*(X, X - x)$. However, clearly $|\mathcal{K}(X)| - x = |\mathcal{K}(X - x)| \cup ((\text{star}\ x) - x)$, so that $|\mathcal{K}(X - x)|$ is a deformation retract of $|\mathcal{K}(X)| - x$.

6. Proof of Theorem 3. Let K be a simplicial complex with first barycentric subdivision K'. For each simplex σ of K, let $b(\sigma)$ be the barycenter of σ. Let

$\mathfrak{X}(K) = \{b(\sigma): \sigma \in K\}$ = set of all vertices of K'. Now $\mathfrak{X}(K)$ has a partial order defined by $b(\sigma) \leq b(\sigma')$ if $\sigma \subset \sigma'$. Thus $\mathfrak{X}(K)$ becomes a T_0A-space. Clearly $\mathfrak{K}(\mathfrak{X}(K)) = K'$. For each point $b(\sigma)$ of $\mathfrak{X}(K)$, $U_{b(\sigma)}$ is simply $\mathfrak{X}(\bar{\sigma})$, where $\bar{\sigma}$ is the subcomplex of K consisting of all faces of σ. Hence $\mathfrak{X}(K)$ is locally finite.

Let the map $f_K : |K| \rightarrow \mathfrak{X}(K)$ be simply the map $f_{\mathfrak{X}(K)}$ of Theorem 2, which indeed maps $|\mathfrak{K}(\mathfrak{X}(K))| = |K'| = |K|$ onto $\mathfrak{X}(K)$. Thus we get immediately from Theorem 2 that f_K is a weak homotopy equivalence.

Let us now establish the correspondence $\psi \rightarrow \psi'$ of Theorem 3. Let $\psi : K \rightarrow L$ be a simplicial map. Define a simplicial map $\psi' : K' \rightarrow L'$ by $\psi'(b(\sigma)) = b(\psi\sigma)$. Clearly the maps $|\psi|, |\psi'| : |K| \rightarrow |L|$ are homotopic. Now ψ' maps the points of $\mathfrak{X}(K)$ into the points of $\mathfrak{X}(L)$, hence is also a map $\mathfrak{X}(K) \rightarrow \mathfrak{X}(L)$, which is order-preserving, therefore continuous. By the commutativity relation in Theorem 2, $\psi' f_K = f_L \, |\psi'| \simeq f_L \, |\psi|$. This completes the proof of Theorem 3.

7. Proof of Theorem 4. Let X be an A-space. We define an equivalence relation $\sim$ on X by saying $x \sim y$ if $U_x = U_y$ (equivalently, if $x \leq y$ and $y \leq x$). Let $\hat{X}$ be the quotient space $X/\sim$, and let $\nu = \nu_X : X \rightarrow \hat{X}$ be the quotient map. It is easy to see that *any* quotient space of an A-space is an A-space. The A-space structure on $\hat{X}$ defines a transitive, reflexive relation on $\hat{X}$. The next lemma shows that this relation is the same as the one induced by ν from the relation on X.

LEMMA 8. *If $x, y \in X$, then $\nu(x) \leq \nu(y)$ if and only if $x \leq y$.*

Proof. First we show that for each x in X,

$$\nu(U_x) = U_{\nu(x)} \, . \tag{7.1}$$

Observe that $\nu^{-1}\nu(U_x) = U_x$. (For if $z \in \nu^{-1}\nu(U_x)$, then $\nu(z) = \nu(w)$ for some w in U_x; thus $z \in U_z = U_w \subset U_x$.) Since ν is a quotient map, $\nu(U_x)$ is therefore a neighborhood of $\nu(x)$. Thus $\nu(U_x) \subset U_{\nu(x)}$. The reverse inclusion follows from the continuity of ν.

Now if $\nu(x) \leq \nu(y)$, by (7.1) $\nu(x) \in \nu(U_y)$. Thus $\nu(x) = \nu(z)$, where $z \in U_y$, so that $x \leq z \leq y$. Conversely, if $x \leq y$, by continuity of ν (or by (7.1), $\nu(x) \leq \nu(y)$.

From this lemma, part (ii) of Theorem 4 follows, since from the definition of ν it is now obvious that the relation $\leq$ on $\hat{X}$ is antisymmetric. Notice that (7.1) also shows that $\hat{X}$ is locally finite whenever X is. The next lemma is part (i) of the theorem.

LEMMA 9. *$\nu : X \rightarrow \hat{X}$ is a homotopy equivalence.*

Proof. Let $\mu : \hat{X} \rightarrow X$ be *any* right inverse for the onto map ν; that is, $\nu\mu = 1_{\hat{X}}$. By Lemma 8, μ is order preserving, hence continuous. We need to show that the map $\pi = \mu\nu : X \rightarrow X$ is homotopic to 1_X. Since $\nu\pi(x) = \nu\mu\nu(x) = \nu(x)$, we see that for each x in X,

$$U_{\pi(x)} = U_x \, . \tag{7.2}$$

Now define $F : X \times I \to X$ by $F(x, t) = x$ if $t < 1$, and $F(x, 1) = \pi(x)$. To show that F is continuous, let $(x, s) \in X \times I$. Now $U_x \times I$ is a neighborhood of (x, s), which is shown as follows to be mapped by F into $U_{F(x,s)}$. By (7.2) $U_{F(x,s)} = U_x$. Now take any $(y, t) \in U_x \times I$. If $t < 1$, then $F(y, t) = y \in U_x$. If $t = 1$, $F(y, t) = \pi(y) \in U_{\pi(y)} =$ (by (7.2)) $U_y \subset U_x$. This completes the proof.

To establish part (iii), suppose $\varphi : X \to Y$ is a map of A-spaces. Since φ is order preserving, φ maps equivalent points under ν_X into equivalent points under ν_Y. Hence there is a unique function $\hat{\varphi} : \hat{X} \to \hat{Y}$ such that $\hat{\varphi}\nu_X = \nu_Y\varphi$. Since ν_X is a quotient map, $\hat{\varphi}$ is continuous (alternately, from Lemma 8, $\hat{\varphi}$ is order preserving, hence continuous.)

8. Non-Hausdorff suspension. Proof of Theorem 5. If X is a topological space, the *non-Hausdorff cone of* X, denoted by $\mathcal{C}(X)$, is defined as follows. Take a point ω_X not in X. Then $\mathcal{C}(X) = X \cup \{\omega_X\}$, the topology for $\mathcal{C}(X)$ consisting of all open subsets of X, as well as the whole space $\mathcal{C}(X)$. The *non-Hausdorff suspension*, denoted by $\mathcal{S}(X)$, is defined as follows. Take two points ω_X, ω_X' not in X. Then $\mathcal{S}(X) = X \cup \{\omega_X, \omega_X'\}$, the topology consisting of all open subsets of X, as well as $X \cup \{\omega_X\}$, $X \cup \{\omega_X'\}$, and $\mathcal{S}(X)$. Note that $\mathcal{S}(X)$ is the union of two copies of $\mathcal{C}(X)$, whose intersection is X. If $\varphi : X \to Y$, we extend φ to maps $\mathcal{C}(\varphi) : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{S}(\varphi) : \mathcal{S}(X) \to \mathcal{S}(Y)$ by letting $\mathcal{C}(\varphi)(\omega_X) = \omega_Y$ and $\mathcal{S}(\varphi)(\omega_X) = \omega_Y$, $\mathcal{S}(\varphi)(\omega_X') = \omega_Y'$. Continuity of these maps is obvious. Clearly $\mathcal{C}$ and $\mathcal{S}$ are covariant functors. In the following we write simply ω for ω_X, etc.

LEMMA 10. *Both $\mathcal{C}$ and $\mathcal{S}$ preserve each of the following properties: that a space be finite, locally finite, T_0, or be an A-space.*

Proof. Straightforward.

LEMMA 11. *For each space X, $\mathcal{C}(X)$ is contractible.*

Proof. This follows from Lemma 5.

We shall prove a stronger result than that stated in Theorem 5: we shall produce a natural transformation g^n from the n-fold suspension S^n to the n-fold non-Hausdorff suspension $\mathcal{S}^n$ such that for each space X, $g_X^n : S^n(X) \to \mathcal{S}^n(X)$ is a weak homotopy equivalence. First we make a definition.

For each map $\varphi : X \to X'$ we define a map $T(\varphi) : S(X) \to \mathcal{S}(X')$. Suppose $X \neq \phi$, and let $\nu : X \times [-1, 1] \to S(X)$ be the quotient map, identifying $X \times \{1\}$ and $X \times \{-1\}$ to points v and v', respectively. If $(x, t) \in X \times (-1, 1)$, let $T(\varphi)(\nu(x, t)) = \varphi(x)$. Let $T(\varphi)(v) = \omega$, $T(\varphi)(v') = \omega'$. In case $X = \phi$, $S(X)$ is a pair of points $\{v, v'\}$, which again we map to $\{\omega, \omega'\}$. It is easy to see that $T(\varphi)$ is continuous. The following lemma is straightforward to check.

LEMMA 12. *Commutativity of the first diagram implies commutativity of the second diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ \downarrow \psi & & \downarrow \psi' \\ Y & \xrightarrow{\varphi'} & Y' \end{array} \qquad \begin{array}{ccc} S(X) & \xrightarrow{T(\varphi)} & \mathcal{S}(X') \\ \downarrow S(\psi) & & \downarrow \mathcal{S}(\psi') \\ S(Y) & \xrightarrow{T(\varphi')} & \mathcal{S}(Y') \end{array}$$

LEMMA 13. *If* $\varphi : X \to X'$ *is a weak homotopy equivalence, then so is* $T(\varphi) : S(X) \to \mathcal{S}(X')$.

Proof. We apply Theorem 6. For the basis-like open cover of $\mathcal{S}(X')$ we take simply $\{X' \cup \{\omega\}, X' \cup \{\omega'\}, X'\}$. By Lemma 11, $X' \cup \{\omega\}$ is contractible; and $(T(\varphi))^{-1}(X' \cup \{\omega\}) = \nu(X \times (-1, 1])$, which is contractible. Hence $X' \cup \{\omega\}$ is distinguished. The treatment of $X' \cup \{\omega'\}$ is symmetric. On $(T(\varphi))^{-1}(X')$, $T(\varphi)$ is essentially the composition of the projection $X \times (-1, 1) \to X$ and the weak homotopy equivalence $\varphi : X \to X'$, hence is a weak homotopy equivalence.

Now we define the natural transformation $g^n : S^n \to \mathcal{S}^n$ $(n \geq 1)$. For each space X, let $g_X^n : S^n(X) \to \mathcal{S}^n(X)$ be simply $T^n(1_X)$, the n-fold iterate of T applied to the identity function on X. Naturality of g^n says that for any map $\psi : X \to Y$, the diagram

$$\begin{array}{ccc} S^n(X) & \xrightarrow{g_X^n} & \mathcal{S}^n(X) \\ \downarrow S^n(\psi) & & \downarrow \mathcal{S}^n(\psi) \\ S^n(Y) & \xrightarrow{g_Y^n} & \mathcal{S}^n(Y) \end{array}$$

commutes. This follows by applying Lemma 12 inductively, beginning with the trivially commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ \downarrow \psi & & \downarrow \psi. \\ Y & \xrightarrow{1} & Y \end{array}$$

Finally, applying Lemma 13 inductively, beginning with the identity map $X \to X$, we see that g_X^n is a weak homotopy equivalence $S^n(X) \to \mathcal{S}^n(X)$. This completes the proof of Theorem 5.

As an example, consider the $(2n + 2)$-point space

$$\Sigma^n = \mathcal{S}^n(S^0),$$

where S^0 is the 0-sphere (pair of points). The preceding results give a weak homotopy equivalence of the ordinary n-sphere $S^n(S^0)$ onto Σ^n. (This may also be seen by Theorem 2.) Not only does Σ^n have all the homotopy groups and singular homology groups of an n-sphere, it even has the same local singular homology groups (it is a "homology n-manifold".) This may be seen from Corollary 2 at the end of §5. In fact, it may be seen directly by induction using the Mayer-Vietoris sequence that the complement of each point in Σ^n is acyclic.

The 6-point space Σ^2 "would be homeomorphic to a 2-sphere if it were only Hausdorff." More precisely, consider the following conditions on a topological space X : (1) *The complement of each point in X is acyclic* (*in singular homology*); (2) $H_2(X) \neq 0$. We have seen that the T_0 space Σ^2 satisfies these two conditions. However, simply by adding the extra condition (3) *X is Hausdorff*, one can conclude that X is homeomorphic to the 2-sphere. (See [5].)

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