

Richard Wong

# Honors Multivariable Calculus

Math 32A/BH, UCLA

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## Preface

These lecture notes are written for Math 32A/BH, the honors multivariable calculus sequence at UCLA. Multivariable calculus is the mathematical language that allows us to describe the geometry of the physical world around us.

Math 32AH covers differential multivariable calculus, where you will learn how to mathematically describe the motion of planets in orbit, the behavior of electromagnetic forces, and the path of steepest ascent through the hills of Los Angeles. Math 32BH covers integral multivariable calculus, where you will learn how to calculate area, volume, arclength and surface area of various shapes, as well as how to calculate the amount of electricity flowing along a curve or through a surface.

The honors sequence differs from Math 32A/B in that it explores the topics of multivariable calculus with more mathematical rigor. In particular, in this course, you will explore not just the geometric meaning behind these definitions and theorems, but you will also learn how to use these definitions to prove that these theorems are true. I also plan to help you develop the reasoning and questioning skills needed to explore these mathematical ideas. Some questions you might see in 32AH include the following:

What changes, and what stays the same when we move from single variable calculus to multivariable calculus? What does it mean to take a derivative of a multivariable function?

In 32BH, we will shift our focus from proving theorems, to understanding complex mathematical definitions and concepts. While we will see the classical theorems in multivariable integral calculus, we will push the boundaries and study theorems in full generality.

What is the difference between an integral and an anti-derivative?

What is the relationship between derivatives and integrals in single variable calculus?

Finally, Math 32A/BH builds the foundation for more advanced topics in mathematics, such as linear algebra, real analysis, complex analysis, and differential geometry. This course is recommended for students interested in learning about advanced mathematics.

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These lecture notes are intended to supplement the course lecture slides, and they will provide a mix of both computational and proof-based questions. They draw on material from the following texts:

- *Calculus and Analysis in Euclidean Space*, by Shurman.
- *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach*, by Hubbard and Hubbard.
- *Multivariable Calculus*, by Rogawski and Adams.

You might find it convenient to obtain copies of these books for reference.

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Please let me know if you notice any typos or errors, or if you have any comments or suggestions!

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## Chapter 1

### Linear Algebra on $\mathbb{R}^n$

"Mathematics is the art of reducing any problem to linear algebra" - William Stein

Linear algebra is one of the foundational concepts in mathematics, and is essential to properly understanding multivariable calculus.

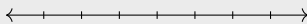
We will begin by studying certain properties of  $\mathbb{R}^n$ , and we will see how  $\mathbb{R}^n$  and its properties can be generalized to vector spaces. We will then study how vector spaces can be compared to each other, through linear maps.

#### 1.1 An introduction to $\mathbb{R}^n$

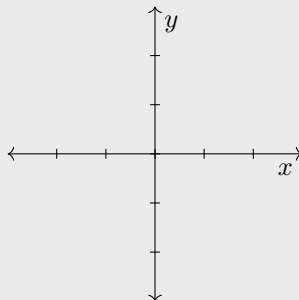
**Definition 1.1.1.** Given a positive integer  $n$ , ***n-dimensional Euclidean space*** (denoted  $\mathbb{R}^n$ ) is the set of all ordered  $n$ -tuples of real numbers. That is,

$$\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

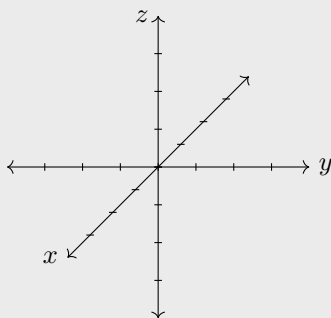
We are familiar with  $\mathbb{R}^1$  (often described as the real line  $\mathbb{R}$ ), which we can sketch as the real number line:



as well as  $\mathbb{R}^2$  (often described as the Euclidean plane), which we can sketch with the  $x$  and  $y$  axes.



$\mathbb{R}^3$  is often described as Euclidean space, and has 3 axes (denoted the  $x$ ,  $y$ , and  $z$  axes).



We can study and understand  $n$ -dimensional Euclidean space in many different ways by imposing different *structures* on the set  $\mathbb{R}^n$ . For example, we might wish to consider  $\mathbb{R}^n$  geometrically (more precisely, as a topological space).

Thinking geometrically, an element  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  can be thought of as describing *coordinates* that tell us precisely where  $\mathbf{x}$  is located inside of  $\mathbb{R}^n$ . In this framework, we will call elements of  $\mathbb{R}^n$  points.

**Definition 1.1.2.** A **point in  $\mathbb{R}^n$**  (often denoted  $\mathbf{x} \in \mathbb{R}^n$ ) is an element of  $\mathbb{R}^n$ , considered as a topological space. That is,

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

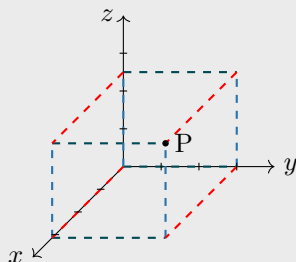
We say that  $x_1$  is the **first coordinate**,  $x_2$  is the **second coordinate**, and so on, until  $x_n$  (which is the  $n$ -th coordinate).

The point  $O = (0, 0, \dots, 0) \in \mathbb{R}^n$  is called the **origin** (in  $\mathbb{R}^n$ ).

We can also consider the point  $P = (\pi, 3, 2.5) \in \mathbb{R}^3$ .

The first coordinate tells us that along the  $x$ -axis, we are  $\pi$  units in the positive direction from the origin; the second coordinate tells us we are 3 units in the positive direction along the  $y$ -axis, and the third coordinate tells us we are 2.5 units in the positive direction along the  $z$ -axis.

We can visualize  $P$  by drawing a box such that one corner is at the origin, the edges are aligned with the axes in  $\mathbb{R}^3$ , and the dimensions of the box correspond with the coordinates.  $P$  is then the corner of the box that is furthest from the origin.



On the other hand, we are not required to think of  $\mathbb{R}^n$  in this way. Indeed, the structures that we impose on  $\mathbb{R}^n$  should reflect how we want to think.

For example, we can instead think of an element  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$  not as describing coordinates, but rather, as describing **displacement**. In other words, given a starting point  $P$  and an ending point

$Q$ , we can measure the *change* in each coordinate. In this framework, we will call elements of  $\mathbb{R}^n$  vectors:

**Definition 1.1.3.** A **vector** in  $\mathbb{R}^n$  (often denoted  $\mathbf{v} \in \mathbb{R}^n$ ) is an element of  $\mathbb{R}^n$ , considered as a vector space. That is,

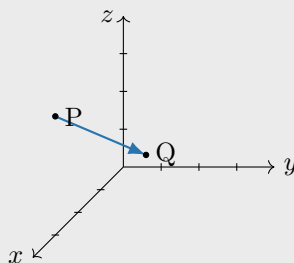
$$\mathbf{v} = \langle v_1, \dots, v_n \rangle$$

We say that  $v_1$  is the **first component**,  $v_2$  is the **second component**, and so on, until  $v_n$  (which is the  $n$ -th component).

Suppose that you started at the point  $P = (3, 0, \pi) \in \mathbb{R}^3$ , and ended up at the point  $Q = (4, 3, e) \in \mathbb{R}^3$ . Then your total displacement along the  $x$ -axis is  $4 - 3 = 1$  unit;  $3 - 0 = 3$  units along the  $y$ -axis; and  $e - \pi$  units along the  $z$ -axis. Then your total displacement can be described as a vector

$$\mathbf{PQ} = \langle 1, 3, e - \pi \rangle \in \mathbb{R}^3$$

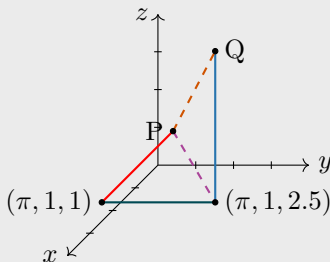
We can visualize the vector  $\mathbf{PQ}$  as an arrow in  $\mathbb{R}^3$ , starting at  $P$  and ending at  $Q$ :



We say that the terminal point  $Q$  is the **head** of the vector  $\mathbf{PQ}$ , and that the initial point  $P$  is the **tail** of the vector  $\mathbf{PQ}$ .

Given a vector  $\mathbf{PQ}$ , we can recover information about the relationship between the points  $P$  and  $Q$ . For example, we can calculate the distance between two points  $P$  and  $Q$  in  $\mathbb{R}^3$  by iterating the Pythagorean theorem twice:

The distance between  $P = (1, 1, 1)$  and  $Q = (\pi, 3, 2.5)$  is  $\sqrt{(\pi - 1)^2 + (3 - 1)^2 + (2.5 - 1)^2}$



Observe that the points  $P$ ,  $(\pi, 1, 1)$ , and  $(\pi, 1, 2.5)$  form a right triangle, whose hypotenuse has length  $\sqrt{(\pi - 1)^2 + (2.5 - 1)^2}$ . Similarly, the points  $P$ ,  $(\pi, 1, 2.5)$ , and  $Q$  also form a right triangle, where the length of the hypotenuse is precisely the distance between  $P$  and  $Q$ .

Observe that the distance between  $P = (1, 1, 1)$  and  $Q = (\pi, 3, 2.5)$  is precisely the square root of

the sum of the squares of the components of the vector  $\mathbf{PQ}$ . That is, if we write the components of  $\mathbf{PQ} = \langle \pi - 1, 2, 1.5 \rangle = \langle v_1, v_2, v_3 \rangle$ , then the distance between  $P$  and  $Q$  is

$$\sqrt{v_1^2 + v_2^2 + v_3^2}$$

This observation generalizes to  $\mathbb{R}^n$ , as we would need to iterate the Pythagorean theorem  $n - 1$  times. If we consider two points  $P = (x_1, \dots, x_n)$  and  $Q = (y_1, \dots, y_n)$ , and write the components of the vector  $\mathbf{PQ} = \langle x_1 - y_1, \dots, x_n - y_n \rangle = \langle v_1, \dots, v_n \rangle$ , we again find that the distance between  $P$  and  $Q$  is

$$\sqrt{v_1^2 + \dots + v_n^2}$$

Thus, we should give this quantity a name:

**Definition 1.1.4.** Given a vector  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ , its **magnitude** is given by

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^n v_i^2}$$

**Proposition 1.1.5.** The distance between two points  $P = (x_1, \dots, x_n)$  and  $Q = (y_1, \dots, y_n)$  is precisely  $\|\mathbf{PQ}\|$ .

The vector  $\mathbf{PQ} = \langle 1, 3, e - \pi \rangle \in \mathbb{R}^3$  has magnitude

$$\|\mathbf{PQ}\| = \sqrt{1^2 + 3^2 + (e - \pi)^2}$$

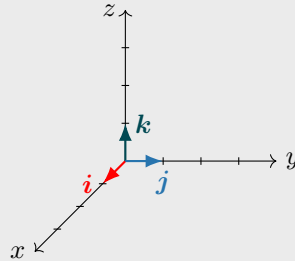
**Definition 1.1.6.** A **unit vector** in  $\mathbb{R}^n$  is a vector  $\mathbf{u} \in \mathbb{R}^n$  such that  $\|\mathbf{u}\| = 1$ .

In  $\mathbb{R}^3$ , there are special unit vectors, called the **standard basis vectors** of  $\mathbb{R}^3$  (see example 1.2):

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$



**Proposition 1.1.7.** Given a vector  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$  with magnitude  $\|\mathbf{v}\| = \sqrt{\sum_{i=1}^n v_i^2}$ , then the vector

$$\mathbf{e}_v := \left\langle \frac{v_1}{\|\mathbf{v}\|}, \dots, \frac{v_n}{\|\mathbf{v}\|} \right\rangle$$

is a unit vector.



The vector  $\mathbf{e}_{PQ} = \left\langle \frac{1}{\sqrt{(e-\pi)^2+10}}, \frac{3}{\sqrt{(e-\pi)^2+10}}, \frac{e-\pi}{\sqrt{(e-\pi)^2+10}} \right\rangle \in \mathbb{R}^3$  is a unit vector.

What are the differences between the points perspective and the vectors perspective?

Though the two perspectives might seem similar, there are important differences:

For example, from the points perspective, we should think of elements in  $\mathbb{R}^n$  as being analogous to *locations* on a map. It **doesn't make sense** to add the location of Los Angeles, California (say, coordinates with latitude  $34^\circ$  N, longitude  $118^\circ$  W) to the location of Austin, Texas (say, coordinates  $30^\circ$  N,  $98^\circ$  W).

On the other hand, from the vectors perspective, we should think of elements in  $\mathbb{R}^n$  as being analogous to *displacements*. In this case, let us consider two displacements: The displacement from Los Angeles to Austin means travelling  $4^\circ$  south, and travelling  $20^\circ$  east. Similarly, the displacement from Austin to New York City (coordinates  $41^\circ$  N,  $74^\circ$  W) means travelling  $11^\circ$  north, and travelling  $24^\circ$  east.

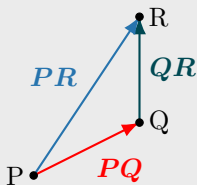
It **does make sense** to add these two displacements together, which entails travelling a total of  $7^\circ$  north, and  $44^\circ$  east. Observe that the result of this addition is the **total displacement** from Los Angeles to New York City (travelling  $7^\circ$  north, and  $44^\circ$  east).

We can formulate this in terms of vectors as follows:

**Definition 1.1.8** (Vector addition). *Given two vectors  $\mathbf{u} = \langle u_1, \dots, u_n \rangle$ ,  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ , we can add two vectors together to form the sum  $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$ . In terms of components,*

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, \dots, u_n + v_n \rangle$$

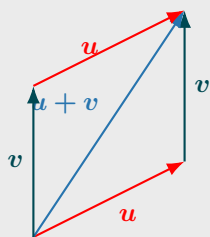
There is a nice geometric picture that accompanies vector addition: Given two vectors  $\mathbf{PQ}$  and  $\mathbf{QR}$ , then the sum of these two vectors is the vector  $\mathbf{PR}$  (that is, the vector with tail  $P$ , and head  $R$ )



That is, we add vectors "tip to tail".

**Proposition 1.1.9.** *Vector addition is commutative. That is, the vectors  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{v} + \mathbf{u}$  are the same vector.*

This follows immediately from the definition in terms of components. However, there is a nice geometric picture that accompanies this property:



That is, if we move our vectors freely, the  $\mathbf{u} + \mathbf{v}$  is the diagonal of the parallelogram spanned by  $\mathbf{u}$  and  $\mathbf{v}$ .

Let us be more precise about moving vectors freely: we can combine the points perspective with the vector perspective. Using our previous analogies, it makes sense to add the *displacement* from Los Angeles to Austin to the *location* New York City.

We should interpret this as saying to start at the coordinates  $41^\circ$  N,  $74^\circ$  W, and to then travel  $4^\circ$  south, and  $20^\circ$  east. The end result is a **location** with coordinates  $37^\circ$  N,  $54^\circ$  W (somewhere in the middle of the Atlantic Ocean!).

We can formulate this in terms of vectors as follows:

**Definition 1.1.10.** In terms of coordinates, given a point  $P = (x_1, \dots, x_n) \in \mathbb{R}^n$ , and a vector  $\mathbf{v} = \langle v_1, \dots, v_n \rangle \in \mathbb{R}^n$ , then we can define the **point**  $P + \mathbf{v}$  to be the point with coordinates

$$P + \mathbf{v} = (x_1 + v_1, \dots, x_n + v_n)$$

Informally, the point  $P + \mathbf{v}$  is the tail of the vector that starts at  $P$ , and travels along the length of  $\mathbf{v}$ . We can also make this mathematically precise through vector addition:

**Definition 1.1.11.** A **position vector** in  $\mathbb{R}^n$  is a vector whose tail (or initial point) is the origin.

If the head of the position vector is the point  $P$ , then we often denote the position vector as the vector  $\mathbf{OP}$ .

**Proposition 1.1.12.** Given a point  $P \in \mathbb{R}^n$ , and a vector  $\mathbf{v} \in \mathbb{R}^n$ , the point  $P + \mathbf{v}$  has the same coordinates as the head (or terminal point) of the vector  $\mathbf{OP} + \mathbf{v}$ .

We are now able to describe various vectors at different locations in  $\mathbb{R}^n$ . We have seen cases where two vectors indeed start and end at the same point, but in definition 1.1.10, we now see that it is possible for "the same vector" to have different starting points. This leads us to the following question:

When are two vectors the same?

We first must define what we mean for two vectors to be the same:

**Definition 1.1.13.** A vector  $\mathbf{w} = \mathbf{AB}$  is said to be **equivalent** to a vector  $\mathbf{v} = \mathbf{PQ}$  if the terminal point  $B$  is the same as the point  $A + \mathbf{v}$

We sometimes also say that  $\mathbf{w}$  is a **translation** of  $\mathbf{v}$ .

Suppose that we have a vector  $\mathbf{v} = PQ \in \mathbb{R}^n$ , and suppose that the components of  $\mathbf{v}$  are  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ .

Consider the point  $P = (v_1, \dots, v_n)$ . Then  $\mathbf{v}$  is equivalent to the position vector  $\mathbf{OP}$ , which starts at the origin  $O$ , and ends at the point  $P$ .

The example above is the key idea of the proof of the following proposition:

**Proposition 1.1.14.** *Consider the points  $A, B, P, Q \in \mathbb{R}^n$ . The vectors  $\mathbf{AB}$  and  $\mathbf{PQ}$  are equivalent if and only if they have the same components.*

We now turn to a second way to distinguish vectors:

**Definition 1.1.15.** A **scalar** (often denoted  $\lambda \in \mathbb{R}$ ) is an element of the field of real numbers,  $\mathbb{R}$ .

**Definition 1.1.16.** Given a vector  $\mathbf{v} = \langle v_1, \dots, v_n \rangle \in \mathbb{R}^n$  and a scalar  $\lambda \in \mathbb{R}$ , the **scalar multiple**  $\lambda \mathbf{v}$  is the vector

$$\lambda \mathbf{v} = \langle \lambda v_1, \dots, \lambda v_n \rangle$$

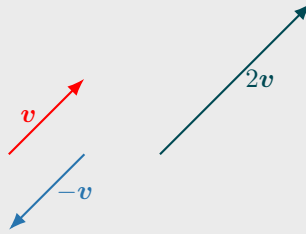
That is, we multiply  $\lambda$  with each component.

We can interpret scalar multiples geometrically through the following calculation:

**Proposition 1.1.17.** *The magnitude of a scalar multiple  $\lambda \mathbf{v}$  is the same as the magnitude of  $\mathbf{v}$  times the absolute value of  $\lambda$ . That is,*

$$\|\lambda \mathbf{v}\| = |\lambda|(\|\mathbf{v}\|)$$

This tells us that scalar multiplication changes the magnitude of a vector (that is, it *scales* it by a factor of  $|\lambda|$ ).



**Definition 1.1.18.** Two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  are said to be **parallel** if they are scalar multiples of each other. That is, there exists a scalar  $\lambda \in \mathbb{R}$  such that

$$\mathbf{v} = \lambda \mathbf{w}$$

Consider the points  $P = (3, -2, 1)$  and  $Q = (2, -1, 1)$  in  $\mathbb{R}^3$ , and consider the vectors  $\mathbf{u} = \langle -1, 1, 0 \rangle$ ,  $\mathbf{v} = \langle -2, 2, 0 \rangle$ ,  $\mathbf{w} = \langle 1, -1, 0 \rangle$ ,  $\mathbf{a} = \langle 1, -1, 1 \rangle$ .

- The vector  $\mathbf{PQ}$  is equivalent to  $\mathbf{u}$ .
- The vector  $\mathbf{PQ}$  is parallel to  $\mathbf{v}$ , with  $\lambda = 2$ .

- The vector  $PQ$  is parallel to  $w$ , with  $\lambda = -1$ .
- The vector  $PQ$  is **not** parallel to  $a$ . That is, there is no  $\lambda$  such that  $PQ = \lambda a$ .  
If such a  $\lambda$ , existed, that would require that the equations  $-1 = \lambda 1$  and  $0 = \lambda 1$  to both be true. But that would be a contradiction, since  $\lambda$  cannot be both  $-1$  and  $0$ .

Parallel vectors are so named because they give rise to parallel lines. See definition 1.1.24 in the exercises to learn how we describe lines in  $\mathbb{R}^n$ , and exercise 1.12.

Observe that in the example above, we saw that two vectors can be parallel, yet point in different directions. We can make this idea precise:

**Definition 1.1.19.** Given a non-zero vector  $v \in \mathbb{R}^n$ , its **direction vector**  $e_v$  is the unit vector such that

$$\|v\|e_v = v$$

Note that we used the word "the" in the definition above - in order for this definition to make sense, we should make sure to prove that the direction vectors is *unique*.

**Proposition 1.1.20.** Given a non-zero vector  $v \in \mathbb{R}^n$ , its **direction vector**  $e_v$  is unique.

The general strategy to prove that something is unique is as follows:

[Explain more](#)

*Proof.* Let  $v \in \mathbb{R}^n$  be a non-zero vector. Suppose that  $u_1$  and  $u_2$  are both direction vectors for  $v \in \mathbb{R}^n$ . We wish to show that  $u_1 = u_2$ .

Since  $u_i$  is a direction vector, we know that  $\|v\|u_i = v$ . Thus,

$$\|v\|u_1 = \|v\|u_2$$

Since  $v$  is non-zero, then  $\frac{1}{\|v\|}$  is a scalar. Thus we can scalar multiply equation (1) by  $\frac{1}{\|v\|}$  to see that

$$u_1 = u_2$$

as desired. □

**Proposition 1.1.21.** A formula for the direction vector is as follows:

$$e_v = \frac{1}{\|v\|}v$$

**Definition 1.1.22.** Suppose that the vector  $w$  is parallel to  $v$ , with  $v = \lambda w$ . If  $\lambda > 0$ , then we say  $w$  has the **same direction** as  $v$ . If  $\lambda < 0$ , then  $w$  has the **opposite direction** as  $v$ .

Thus, we can now algebraically explain the difference between the vectors  $PQ$  and  $QP$ . They are vectors that point in the opposite direction of each other.

**Proposition 1.1.23.** Two vectors are equivalent if and only if they have the same magnitude and the same direction vector.

In other words, if two vectors have different magnitudes, then they must be different vectors. Similarly, if two vectors have different direction vectors, then they must be different vectors.

To summarize, we have learned the following:

A vector in  $\mathbb{R}^n$  is determined equivalently by:

a. an initial point and a terminal point.

$$\mathbf{v} = \mathbf{PQ}$$

b. its components.

$$\mathbf{v} = \langle x_1, \dots, x_n \rangle$$

c. a direction and magnitude.

$$\mathbf{v} = \|\mathbf{v}\| \mathbf{e}_v$$

### Exercises

**1.1** Explain the relationship between points and vectors.

More concretely, given a point in  $\mathbb{R}^n$ , is there a natural way to obtain a vector in  $\mathbb{R}^n$ ? Conversely, given a vector in  $\mathbb{R}^n$ , is there a natural way to obtain a point in  $\mathbb{R}^n$ ?

**1.2** Sketch and find the components of the unit position vector in  $\mathbb{R}^2$  that makes an angle of 30 degrees counterclockwise from the positive  $x$ -axis.

**1.3** Sketch and find the components of the unit position vector in  $\mathbb{R}^2$  that makes an angle of 30 degrees counterclockwise from the negative  $x$ -axis.

a. Is this vector equivalent to the vector in exercise 1.2?

b. Is this vector parallel to the vector in exercise 1.2?

**1.4** Find the components of the unit position vector in  $\mathbb{R}^2$  that makes an angle of  $\theta$  degrees counterclockwise from the negative  $x$ -axis.

**1.5** Are these vectors parallel to the vector  $\mathbf{v} = \langle 2, -1, 1 \rangle$ ? Are they equivalent?

a.  $\mathbf{PQ}$ , where  $P = (1, 1, 0)$ , and  $Q = (3, 0, 1)$ .

b.  $\mathbf{QP}$ , where  $P = (1, 1, 0)$ , and  $Q = (3, 0, 1)$ .

c.  $\mathbf{AB}$ , where  $A = (1, 1, 0)$ , and  $B = (5, -1, 2)$ .

d.  $\mathbf{PQ} + 2\mathbf{AB}$ , where  $\mathbf{PQ}$  and  $\mathbf{AB}$  are as above.

e.  $\mathbf{PQ} + 2\langle 1, 0, 0 \rangle$ , where  $\mathbf{PQ}$  is as above.

**1.6** Let  $\mathbf{u} = \langle 1, 1, 0 \rangle$ , and let  $\mathbf{v} = \langle 0, 0, 1 \rangle$ .

a. Compute  $\mathbf{u} - \mathbf{v}$ .

b. Sketch  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{u} - \mathbf{v}$ .

c. In general, what is the relationship between  $\mathbf{u} - \mathbf{v}$  and  $\mathbf{v} - \mathbf{u}$ ?

**1.7** Given four points,  $A$ ,  $B$ ,  $C$ , and  $D$  in  $\mathbb{R}^3$ , we obtain a quadrilateral  $ABCD$ . If  $ABCD$  is a parallelogram, and the coordinates of the corners are  $A = (1, 0, 1)$ ,  $B = (3, 3, 2)$ , and  $C = (2, 4, 5)$ , then what are the coordinates of  $D$ ?

(**Hint:** Recall that  $ABCD$  is given by a line from  $A$  to  $B$  to  $C$  to  $D$ , and then back to  $A$ . Use vectors to solve this problem!).

**1.8** Consider the parallelogram  $ABCD$ . Prove that the diagonals  $BC$  and  $AD$  intersect each other at their midpoints.

(**Hint:** Describe the diagonals as vectors).

Lines in  $\mathbb{R}^n$ 

In the following exercises, we will describe lines in  $\mathbb{R}^n$ .

**Definition 1.1.24.** The line  $\mathcal{L}$  in  $\mathbb{R}^n$ , passing through the point  $P = (x_1, \dots, x_n)$ , in the direction of the vector  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ , can be described by the vector-valued function  $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  defined by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

where  $\mathbf{r}_0$  is the vector  $\mathbf{r}_0 = \mathbf{OP} = \langle x_1, \dots, x_n \rangle$ .

We call  $\mathbf{r}(t)$  the **vector parametrization of  $\mathcal{L}$** .

Observe that by definition, the vector  $\mathbf{r}(t)$  is a position vector for all  $t$ . The tips of these position vectors are precisely the points on the line  $\mathcal{L}$ .

**1.9** Recall that we can describe a line in  $\mathbb{R}^2$  as the set of points  $L = \{(x, y) \mid y = mx + b\}$ . Find a vector parametrization of the line  $L$ .

- What is the relationship between the constant  $b$  and the vector  $\mathbf{r}_0$ ?
- What is the relationship between the slope  $m$  and the direction vector  $\mathbf{v}$ ?

**1.10** Find the vector parametrization of the line that passes through the points  $P = (1, 0, 2)$  and  $Q = (2, 5, -1)$ .

**1.11** Given a point  $P$  and a direction vector  $\mathbf{v}$ , is the vector parametrization  $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$  of a line  $\mathcal{L}$  unique?

- Is there a vector parametrization of  $\mathcal{L}$  using the same direction vector  $\mathbf{v}$  but a different starting point  $Q$ ? If so, find the coordinates of a possible  $Q$ .
- Is there a vector parametrization of  $\mathcal{L}$  using the same starting point  $P$ , but a different direction vector,  $\mathbf{w}$ ? If so, find the components of a possible  $\mathbf{w}$ .
- Is there a vector parametrization of  $\mathcal{L}$  using a different direction vector  $\mathbf{w}$  and a different starting point  $Q$ ? If so, find a possible  $Q$  and  $\mathbf{w}$ .

**1.12** Suppose that  $\mathbf{r}_1(t) = \mathbf{r}_0 + t\mathbf{v}$  and  $\mathbf{r}_2(s) = \mathbf{r}_0 + s\mathbf{w}$  are two parametrizations of the same line. What is the relationship between  $\mathbf{v}$  and  $\mathbf{w}$ ?

**1.13** When are  $\mathbf{r}_1(t) = \mathbf{OP} + t\mathbf{v}$  and  $\mathbf{r}_2(s) = \mathbf{OQ} + s\mathbf{v}$  are two parametrizations of the same line?

**Definition 1.1.25.** We say that two lines  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(s)$  **intersect** if there is a point  $P$  lying on both lines.

**1.14** Determine whether the following two lines intersect. If they do, find the common point of intersection.

$$\mathbf{r}_1(t) = \langle 3, 2, 6 \rangle + t\langle 1, 0, -1 \rangle$$

$$\mathbf{r}_2(s) = \langle 4, 12, 24 \rangle + s\langle -1, 10, 20 \rangle$$

(**Hint:** If a point lies on both lines, then that point must satisfy both equations).

**1.15** Determine whether the following two lines intersect. If they do, find the common point of intersection.

$$\mathbf{r}_1(t) = \langle 3, 2, 6 \rangle + t\langle 1, 0, -1 \rangle$$

$$\mathbf{r}_2(s) = \langle 4, 5, 1 \rangle + s\langle -1, 3, 3 \rangle$$

**1.16** Suppose that you have two lines with parametrizations  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(s)$ , respectively. What is the difference between solving the vector equation  $\mathbf{r}_1(t) = \mathbf{r}_2(s)$  versus solving the vector equation  $\mathbf{r}_1(t) = \mathbf{r}_2(t)$ ?

**Definition 1.1.26.** We say that two distinct lines are **skew**, if they do not intersect each other, and they are not parallel to each other.

**1.17** Write down parametrizations of two distinct lines that are skew to each other.

## 1.2 Vector spaces

How can we generalize and abstract these properties of  $\mathbb{R}^n$ ?

In these lecture notes, we will only study vector spaces whose field of scalars is  $\mathbb{R}$  - that is, we are only considering *real* vector spaces<sup>1</sup>. So in these notes we will omit the adjective real and simply refer to them as vector spaces.

**Definition 1.2.1.** A **vector space**  $V$  is a set with a distinguished element  $\mathbf{0} \in V$ , together with two operations,  $+: V \times V \rightarrow V$ , and  $\cdot: \mathbb{R} \times V \rightarrow V$  satisfying the following axioms for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , and for all  $\lambda, \alpha \in \mathbb{R}$ .

(i)  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$  (Additive associativity)

(ii)  $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$  (Additive identity)

(iii) For all  $\mathbf{v} \in V$ , there exists  $\mathbf{w} \in V$  such that  $\mathbf{v} + \mathbf{w} = \mathbf{0}$  (Additive inverse)

(iv)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  (Additive commutativity)

(v)  $\lambda(\alpha\mathbf{v}) = (\lambda\alpha)\mathbf{v}$  (Scalar associativity)

(vi)  $1\mathbf{v} = \mathbf{v}$  (Scalar identity)

(vii)  $(\lambda + \alpha)\mathbf{u} = \lambda\mathbf{u} + \alpha\mathbf{u}$  (Distribution over scalar addition)

(viii)  $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$  (Distribution over vector addition)

The elements of  $V$  are called **vectors**, and elements of  $\mathbb{R}$  are called **scalars**<sup>a</sup>.

<sup>a</sup>see footnote 1

When faced with an abstract definition like this, it's especially important to first find a few concrete examples to keep in mind. Luckily, from our discussion in the previous section, we have in fact checked most of these axioms:

**Theorem 1.2.2.**  $\mathbb{R}^n$  is a vector space.

$\mathbb{R}^n$  is the prototypical example of a vector space<sup>2</sup>. What this means is that whenever you are trying to determine whether or not something is true about vector spaces, you should try to see if it

<sup>1</sup>That is, scalars must be elements of the field  $\mathbb{R}$  (as in definition 1.1.15). In other classes, such as linear algebra, you will see that we can generalize by replacing the field  $\mathbb{R}$  with any other field  $\mathbb{F}$ , and a lot of the theory of vector spaces and linear algebra still work over an arbitrary field  $\mathbb{F}$ .

<sup>2</sup>In fact, in exercise 1.51, you can show that any **finite dimensional** vector space is isomorphic to  $\mathbb{R}^n$ .

is true for  $\mathbb{R}^n$ . Then, you should try to translate the proof in the  $\mathbb{R}^n$  case to the general case, purely using the axioms and properties of abstract vector spaces.

However, you should also be careful not to trick yourself into thinking that every vector space behaves like  $\mathbb{R}^n$ ! Below is an example of a vector space that is quite different from  $\mathbb{R}^n$ .

**Theorem 1.2.3.** *The set  $\mathcal{F}(\mathbb{R})$  of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a vector space with pointwise addition and pointwise scalar multiplication.*

$$(f + g)(x) := f(x) + g(x) \quad (\lambda f)(x) := \lambda(f(x))$$

*Proof.* You will verify  $\mathcal{F}(\mathbb{R})$  satisfies the axioms in exercise 1.29. □

With these examples in mind, we can now look back at the axioms of a vector space (definition 1.2.1) and see *why* these axioms are important.

Axioms (i) through (iv) describe how to *add* vectors using the operation  $+: V \times V \rightarrow V$ . From previous experiences with addition, we should require that  $+$  should be (i) associative and (iv) commutative. Moreover, we also require that the zero vector  $\mathbf{0} \in V$  should behave the way that  $0 \in \mathbb{R}$  does, with regards to (ii) adding zero, as well as (iii) the existence of additive inverses (e.g. negative numbers in  $\mathbb{R}$ ).

Axioms (v) through (vii) describe how to *scalar multiply* vectors using the operation  $\cdot : \mathbb{R} \times V \rightarrow V$ . The intuition for these axioms comes from our knowledge of multiplication for real numbers, which is (v) associative and (vii) distributes over addition of real numbers. Moreover, the element  $1 \in \mathbb{R}$  is the *multiplicative identity* - that is, for any element  $a \in \mathbb{R}$ , we have that  $1(a) = a$ . This property of  $1 \in \mathbb{R}$  is mirrored in axiom (vi).

**Warning:** It is important to note, however, that scalar multiplication is **not** multiplication of vectors. Indeed, scalar multiplication is a function  $\mathbb{R} \times V \rightarrow V$ . That is, its input is precisely one scalar and one vector, and whose output is a vector. You will compare this to multiplication of real numbers in exercise 1.18.

In Section 1.4, we will discuss various vector operations. The dot product has as input two vectors, and outputs a scalar, whereas the cross product has input two vectors in  $\mathbb{R}^3$ , and whose output is a vector in  $\mathbb{R}^3$ .

## Vector subspaces

Let us now turn to other examples of vector spaces.

Consider the set

$$W = \{(x, 0, 0) \in \mathbb{R}^3\}$$

The first observation to make is that  $W$  is a subset of  $\mathbb{R}^3$  - that is, every element of  $W$  is also contained inside of  $\mathbb{R}^3$ .

Now, if we take any two elements in  $(x_1, 0, 0), (x_2, 0, 0) \in W$ , we can then add them together as elements of the vector space  $\mathbb{R}^3$  to get the vector  $(x_1 + x_2, 0, 0) \in \mathbb{R}^3$ . Observe that the sum  $(x_1 + x_2, 0, 0)$  is still an element of  $W$  - thus the vector addition in  $\mathbb{R}^3$  naturally defines a vector addition on  $W$ .

Furthermore, if we take an element  $(x_1, 0, 0) \in W$  and a scalar  $\lambda \in \mathbb{R}$ , we can again treat  $(x_1, 0, 0)$  as an element in  $\mathbb{R}^3$ , and form the scalar multiple  $\lambda((x_1, 0, 0)) = (\lambda x_1, 0, 0)$ . Once again, the scalar multiple  $\lambda((x_1, 0, 0))$  is still an element of  $W$  - thus the scalar multiplication in  $\mathbb{R}^3$  naturally defines a scalar multiplication on  $W$ .



Thus, we have shown that if we define vector addition and scalar multiplication on  $W$  as coming from  $\mathbb{R}^3$ , it then follows that  $W$  satisfies the axioms a vector space.

We can now give a name to this phenomenon of a subset of a vector space turning out to be a vector space:

**Definition 1.2.4.** A nonempty subset  $W$  of a vector space  $V$ , is a **vector subspace** of  $V$  if for all  $v, w \in W$  and for all  $\lambda \in \mathbb{R}$ ,

(i)  $v + w \in W$  (closure under addition)

(ii)  $\lambda(v) \in W$  (closure under scalar multiplication)

Again, to keep ourselves from thinking only in terms of  $\mathbb{R}^n$ , here is another vector subspace example:

**Theorem 1.2.5.** The set of  $\mathcal{C}(\mathbb{R})$  of continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a vector space with pointwise addition and pointwise scalar multiplication.

$$(f + g)(x) := f(x) + g(x) \quad (\lambda f)(x) := \lambda(f(x))$$

**Theorem 1.2.6.**  $\mathcal{C}(\mathbb{R})$  is a subspace of  $\mathcal{F}(\mathbb{R})$ .

*Proof.* Since the constant function 0 defined by sending  $x \mapsto 0$  is continuous, the set  $\mathcal{C}(\mathbb{R})$  is non-empty.

From single variable calculus, recall that if  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions, then their sum  $(f + g)(x) := f(x) + g(x)$  is also continuous. Similarly, recall if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, then the scalar multiple  $(\lambda f)(x) := \lambda(f(x))$  is also continuous.<sup>3</sup>

Thus,  $\mathcal{C}(\mathbb{R})$  is a subspace of  $\mathcal{F}(\mathbb{R})$ . □

**Warning:** It is not true that every subset of a vector space is a vector space. For a sophisticated example, see exercise 1.26. For a more immediate example, consider the following:

The set of two elements  $A = \{(1, 0), (0, 1)\}$  is a subset of  $\mathbb{R}^2$ , but is not a vector subspace of  $\mathbb{R}^2$ , as it fails to be closed under both addition and scalar multiplication.

In fact, we can generalize this example:

**Proposition 1.2.7.** If  $A$  is a finite set of elements in a vector space contains a non-zero vector, then  $A$  cannot be a vector subspace<sup>a</sup>.

<sup>a</sup>Remember, we are considering only real vector spaces in these notes.

One proof, using vector addition, is as follows:

*Proof.* Suppose that  $A$  be a finite set of elements in a vector space with cardinality  $|A| = n$ , and suppose that  $\mathbf{a} \in A$  is a non-zero vector. Then consider the sums  $\sum_{i=1}^j \mathbf{a}$  for  $1 \leq j \leq n + 1$ . Since  $\mathbf{a}$  is non-zero, we have described  $n + 1$  different vectors, and so at least one of the sums  $\sum_{i=1}^j \mathbf{a}$  cannot be contained in  $A$ . Thus  $A$  cannot be a vector subspace. □

A variation on this proof can be phrased using scalar multiplication:

<sup>3</sup>This is proven in theorem 2.4.5. In fact, the set of continuous functions from  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  is also a subspace of all functions from  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ !

*Proof.* Let  $W$  be a vector subspace of  $V$ , and let  $\mathbf{w} \in W$  be a non-zero vector. Then since vector subspaces are closed under scalar multiplication,  $W$  contains  $\lambda\mathbf{w}$  for all  $\lambda \in \mathbb{R}$ . Moreover, since  $\mathbf{w}$  is non-zero, we have shown that  $W$  must have at least as many elements as  $\mathbb{R}$ . In other words, we have described an (uncountably) infinite number of elements in  $W$ , and so any finite set of vectors in a vector space containing a non-zero vector cannot be a vector subspace.  $\square$

To phrase this in another way, we have proved that a vector subspace with finitely many elements must be the **zero** vector space<sup>4</sup>. However, one might wonder if we can fix this example in some way:

Given a (finite) set of elements  $A$  in a vector space  $V$ , can we produce a vector subspace?  
Even better, can we produce the **smallest** vector space containing  $A$  (that is, without including any extraneous vectors)?

We see from our observations above to construct the smallest vector space containing  $A$ , we must first include all scalar multiples of elements in  $A$ . Once we have done so, we then need to include any possible sum of scalar multiples. This is called a **linear combination**:

**Definition 1.2.8.** Let  $V$  be a vector space. A **linear combination** of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  is a vector of the form

$$\lambda_1\mathbf{v}_1 + \dots + \lambda_k\mathbf{v}_k$$

where the  $\lambda_i$  are scalars.

#### PICTURE

Thus, we now have a candidate set for the smallest vector space containing  $A$ : the set of all linear combinations of vectors in  $A$ .

**Definition 1.2.9.** Let  $V$  be a vector space, and let  $A$  be a subset of vectors in  $V$ . The set of all linear combinations of vectors in  $A$  is called the **span** of  $A$ , and is written

$$\text{span}(A)$$

We first must prove that  $\text{span}(A)$  is a vector subspace:

**Theorem 1.2.10.** Let  $V$  be a vector space, and let  $A = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a set of vectors in  $V$ . Then  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a vector subspace of  $V$ .

*Proof.* Since  $\mathbf{v}_i \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ , we see that this subset is non-empty.

We now prove that  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is closed under addition. Let  $\mathbf{w}_1, \mathbf{w}_2 \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ . Then by definition,

$$\mathbf{w}_1 = \lambda_1\mathbf{v}_1 + \dots + \lambda_k\mathbf{v}_k \quad \text{and} \quad \mathbf{w}_2 = \alpha_1\mathbf{v}_1 + \dots + \alpha_k\mathbf{v}_k$$

Then by the vector space axioms,

$$\mathbf{w}_1 + \mathbf{w}_2 = (\lambda_1 + \alpha_1)\mathbf{v}_1 + \dots + (\lambda_k + \alpha_k)\mathbf{v}_k,$$

Hence  $\mathbf{w}_1 + \mathbf{w}_2 \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ .

We now prove it is closed under scalar multiplication. Suppose  $\lambda$  is an arbitrary scalar. Then by the vector space axioms,

$$\lambda\mathbf{w}_1 = (\lambda\lambda_1)\mathbf{v}_1 + \dots + (\lambda\lambda_k)\mathbf{v}_k,$$

Hence  $\lambda\mathbf{w}_1 \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ .  $\square$

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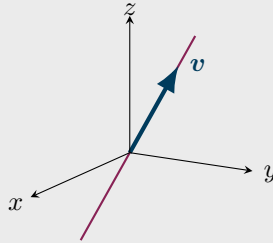
<sup>4</sup>see exercise 1.23.

**Proposition 1.2.11.** Fix a vector  $\mathbf{v} \in V$ . Then

$$\text{span}\{\mathbf{v}\} = \{\lambda\mathbf{v} \mid \lambda \in \mathbb{R}\}$$

*Proof.* By definition,  $\{\lambda\mathbf{v} \mid \lambda \in \mathbb{R}\} \subseteq \text{span}\{\mathbf{v}\}$ . On the other hand, since there is only one element in the set  $\{\mathbf{v}\}$ , the sum of any elements in  $\{\mathbf{v}\}$  is again a scalar multiple of  $\mathbf{v}$ . Thus we have that  $\{\lambda\mathbf{v} \mid \lambda \in \mathbb{R}\} \supseteq \text{span}\{\mathbf{v}\}$ .  $\square$

For example, if  $\mathbf{v}$  is a non-zero element in  $\mathbb{R}^n$ , then by definition,  $\text{span}\{\mathbf{v}\}$  is a line passing through the origin.



Let  $A = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$  be a subset of  $\mathbb{R}^2$ . Then  $\text{span}(A) = \mathbb{R}^2$ .

*Proof.* By definition,  $\text{span}(A) \subseteq \mathbb{R}^2$ . On the other hand, pick an arbitrary element  $\langle x, y \rangle$  in  $\mathbb{R}^2$ . We see from the axioms that  $x\langle 1, 0 \rangle + y\langle 0, 1 \rangle = \langle x, y \rangle$ . Thus  $\text{span}(A) \supseteq \mathbb{R}^2$ .  $\square$

We can now prove that  $\text{span}(A)$  is the **smallest** vector subspace containing  $A$ .

**Theorem 1.2.12.** Let  $V$  be a vector space, and let  $A = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a set of vectors in  $V$ . Suppose  $W$  is a vector subspace containing  $A$ . Then  $\text{span}(A) \subset W$ . That is,  $W$  contains  $\text{span}(A)$ .

*Proof.* We wish to show that for any  $\mathbf{v} \in \text{span}(A)$ , then  $\mathbf{v} \in W$ .

Let  $\mathbf{v}$  be a vector in  $\text{span}(A)$ . By definition, we can write  $\mathbf{v} = \sum \lambda_i \mathbf{v}_i$ .

By our hypothesis,  $W$  is a vector subspace that contains  $\mathbf{v}_i$ . Thus, for all  $1 \leq i \leq n$ , we have that  $\lambda_i \mathbf{v}_i \in W$  since  $W$  is closed under scalar multiplication. Furthermore, since  $W$  is closed under vector addition, then  $\sum \lambda_i \mathbf{v}_i \in W$ .

Thus, we have shown that  $\mathbf{v} = \sum \lambda_i \mathbf{v}_i \in W$ , as desired.  $\square$

## Bases

We have figured out how to construct a vector subspace  $\text{span}(A)$  from a given subset  $A$ . One natural question to ask is the converse:

Given a vector subspace  $W$ , can we construct a subset  $B$  such that  $W = \text{span}(B)$ ? Moreover, is it possible to find a smallest such subset  $B$ ?

Let us first consider an example:

Consider the vector subspace

$$W = \{\langle x, y, 0 \rangle \in \mathbb{R}^3\}$$

We can write  $W$  as the span of several different subsets:

- (i)  $W = \text{span}\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle\}$
- (ii)  $W = \text{span}\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 2, 0, 0 \rangle\}$
- (iii)  $W = \text{span}\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 1, 1, 0 \rangle\}$
- (iv)  $W = \text{span}\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 0 \rangle\}$
- (v)  $W = \text{span}\{\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle\}$
- (vi)  $W = \text{span}\{\langle 1, 1, 0 \rangle, \langle 1, 2, 0 \rangle\}$

From this collection of examples, we can make a few observations. Firstly, observe that the sets in examples (ii) to (iv) all contain the set in example (i), yet they all span the subspace  $W$ . What this means is that in terms of linear combinations, examples (ii) to (iv) contain extraneous information:

- example (ii) contains two vectors that are scalar multiples of each other;
- example (iii) contains a vector that is a linear combination of the other two vectors;
- example (iv) also contains a vector that is a linear combination of the other two vectors (see exercise 1.35).

We can make this idea of “extraneous information” rigorous with the following definition:

**Definition 1.2.13.** A set of vectors  $A \subset V$  is said to be **linearly dependent** if for every nonempty finite subset of vectors  $\{v_1, \dots, v_k\} \subset A$ , there exist scalars  $a_i$ , not all zero, such that

$$a_1 v_1 + \dots + a_k v_k = 0$$

Otherwise, the set  $A$  is said to be **linearly independent**.

This condition is precisely the same as our linear combination criterion as discussed above.

**Proposition 1.2.14.** A set of vectors  $A \subset V$  is linearly dependent if and only if one of the vectors is a linear combination of the others.

In some sense, the notion of linear independence gives us a partial answer to finding a “smallest subset” that spans  $W$ :

**Definition 1.2.15.** An ordered set of vectors  $\mathcal{B} \subset V$  is said to be a **basis** of  $V$  if

- $\text{span}(\mathcal{B}) = V$ , and
- $\mathcal{B}$  is linearly independent.

That is, a **basis** is a linearly independent spanning set.

Consider the vector subspace

$$W = \{\langle x, y, 0 \rangle \in \mathbb{R}^3\}$$

The following subsets are bases of  $W$ :

- (i)  $\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle\}$
- (ii)  $\{\langle 0, 2, 0 \rangle, \langle 1, 0, 0 \rangle\}$

(iii)  $\{\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle\}$

(iv)  $\{\langle 1, 1, 0 \rangle, \langle 1, 2, 0 \rangle\}$

Observe that examples (i) and (ii) are different bases, because the vectors are listed in a different order!

Examples (iii) and (iv) demonstrate the reason that we used quotes in talking about the "smallest subset". There is no meaningful relationship between examples (i), (iii), and (iv) - none of these subsets contain the others. This shows us that the set of bases of a subspace does not have a natural well-ordering - that is, there is no good notion of "smallest" when talking about bases.

In other words, there is no reason to prefer any of the bases over the other for an arbitrary vector space.

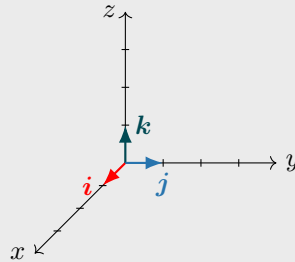
However, when specifically talking about the vector space  $\mathbb{R}^n$ , there **is** a preferred basis, called the standard basis:

**Definition 1.2.16.** The **standard basis** of  $\mathbb{R}^n$  is the collection of vectors

$$\{\mathbf{e}_i \mid i = 1, \dots, n\},$$

where  $\mathbf{e}_i$  denotes the vector with a 1 in the  $i$ -th coordinate and 0 elsewhere.

For example, the standard basis of  $\mathbb{R}^3$  is the ordered set  $\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$ .



Why is having a basis for a vector space useful?

There are several reasons that bases are useful!

**Theorem 1.2.17.** Given a basis  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  of  $V$ , any vector  $\mathbf{v} \in V$  can be expressed **uniquely** as a linear combination  $\mathbf{v} = \sum c_i \mathbf{v}_i$ .

*Proof.* Let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis of  $V$ .

Since  $\mathcal{B}$  is a basis,  $\text{span}(\mathcal{B}) = V$ , so every vector  $\mathbf{v} \in V$  can be expressed as some linear combination  $\mathbf{v} = \sum c_i \mathbf{v}_i$ .

We now turn towards uniqueness. We wish to show that if we can write  $\mathbf{v}$  in two different ways, such as  $\mathbf{v} = \sum a_i \mathbf{v}_i$  and  $\mathbf{v} = \sum c_i \mathbf{v}_i$ , then  $a_i = c_i$  for all  $1 \leq i \leq k$ .

Suppose we can write  $\mathbf{v} = \sum a_i \mathbf{v}_i$  and  $\mathbf{v} = \sum c_i \mathbf{v}_i$ . Then, observe that we can write

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = \sum a_i \mathbf{v}_i - \sum c_i \mathbf{v}_i$$

Thus, we can write

$$\mathbf{0} = \sum (a_i - c_i) \mathbf{v}_i$$

However, since  $\mathcal{B}$  is a basis,  $\mathcal{B}$  is a linearly independent set. Hence  $a_i - c_i = 0$  for all  $1 \leq i \leq k$ , and so

$$a_i = c_i$$

for all  $1 \leq i \leq k$ , as desired. □

Since we can write  $\mathbf{v}$  uniquely in terms of basis vectors, this allows us to make the following definition:

**Definition 1.2.18.** Let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis of  $V$ . The **coordinates of a vector**  $\mathbf{v} = \sum c_i \mathbf{v}_i$  is defined as the ordered set

$$[\mathbf{v}]_{\mathcal{B}} := (c_1, \dots, c_n)$$

This is a generalization of the notion of coordinates in  $\mathbb{R}^n$ !

Consider the vector  $\mathbf{v} = \langle x, y, z \rangle \in \mathbb{R}^3$ , and consider the standard basis in  $\mathbb{R}^3$ . Then we can write  $\mathbf{v}$  as

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

The coordinates of  $\mathbf{v}$  in the standard basis is the ordered set  $(x, y, z)$ .

Consider the vector subspace  $W = \{(x, y, 0) \in \mathbb{R}^3\}$  with basis  $\{\langle 1, 1, 0 \rangle, \langle 1, 2, 0 \rangle\}$ .

Observe that  $W$  is a two dimensional vector space<sup>a</sup>, so the coordinates of an arbitrary vector  $\mathbf{v} = \langle x, y, 0 \rangle \in W$  will be an ordered set of two elements, say  $(a, b)$ . To do so, we must solve the linear system of equations

$$a\langle 1, 1, 0 \rangle + b\langle 1, 2, 0 \rangle = \langle x, y, 0 \rangle$$

So we have two equations,  $a + b = x$ , and  $a + 2b = y$ , thus subtracting the first from the second, we see that  $b = y - x$ . This in turn tells us that  $a + (y - x) = x$ , so  $a = 2x - y$ .

Thus the coordinates for  $\mathbf{v} = \langle x, y, 0 \rangle \in W$  is the ordered set  $(2x - y, y - x)$ . That is,

$$(2x - y)\langle 1, 1, 0 \rangle + (y - x)\langle 1, 2, 0 \rangle = \langle x, y, 0 \rangle$$

---

<sup>a</sup>We define the notion of dimension in definition 1.2.21

The concept of a basis also helps us distinguish vector spaces.

**Theorem 1.2.19.** If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for  $V$ , then any other basis has cardinality  $n$ .

**Lemma 1.2.20.** Suppose that the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for a vector space  $V$ . Then any subset of  $V$  that contains more than  $n$  vectors is linearly dependent.

exposition

**Definition 1.2.21.** If  $V$  has a basis of  $n$  elements, then the **dimension** of  $V$  is  $n$ .

Since  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is a basis for  $\mathbb{R}^n$ , the dimension of  $\mathbb{R}^n$  is  $n$ .

Consider the vector subspace  $W = \{(x, y, 0) \in \mathbb{R}^3\}$ , which has basis  $\{\langle 1, 1, 0 \rangle, \langle 1, 2, 0 \rangle\}$ .  
Then  $W$  is a two dimensional subspace of  $\mathbb{R}^3$ .

Theorem 1.2.19 tells us that the dimension of a vector space  $V$  is an invariant of vector spaces. That is, if we have two isomorphic vector space  $V \cong W$ , and we know that  $\dim(V) = n$ , then it must follow that  $\dim(W) = n$ . You will prove this in exercise 1.50.

**Corollary 1.2.22.** *Let  $V$  be a vector space of dimension  $n$ , and  $W$  is a vector space of dimension  $m$ . Then if  $n \neq m$ , then  $V$  and  $W$  cannot be the same vector space.*

In fact, an even stronger statement is true (which you will prove in exercise 1.51:

**Theorem 1.2.23.** *If  $V$  is a vector space of dimension  $n$ , then  $V \cong \mathbb{R}^n$*

Thus, we completely understand all of the finite dimensional vector spaces.

**Proposition 1.2.24.** *Let  $V$  be a vector space of dimension  $n$ . Then any set of  $n$  linearly independent vectors  $\{x_1, \dots, x_n\} \subset V$  forms a basis for  $V$ .*

**Proposition 1.2.25.** *Let  $V$  be a vector space of dimension  $n$ . Then any set of  $n$  vectors  $\{y_1, \dots, y_k\} \subset V$  that spans  $V$  also forms a basis for  $V$ .*

### Exercises

**1.18** How is scalar multiplication a generalization of multiplication in  $\mathbb{R}$ ?

(**Hint:** Describe scalar multiplication for the vector space  $V = \mathbb{R}$ ).

**1.19** Let  $u = \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$  and  $v = \frac{1}{\sqrt{2}}\langle 1, -1 \rangle$  be vectors in  $\mathbb{R}^2$ . Write  $w = \langle 20, 4 \rangle \in \mathbb{R}^2$  as a linear combination of  $u$  and  $v$ .

**1.20** Consider the set of complex numbers

$$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$$

a. Show that  $\mathbb{C}$  is a vector space with the usual addition and scalar multiplication.

b. Find a basis for  $\mathbb{C}$  as a vector space.

**1.21** Prove that for any given vector space  $V$ , its distinguished element  $\mathbf{0}$  must be unique.

**1.22** Given a vector space  $V$ , prove that for any vector  $v \in V$ , the scalar multiple of  $0v$  is the zero vector. That is,

$$0v = \mathbf{0}$$

**1.23** Given a vector space  $V$ , prove that the set  $\{\mathbf{0}\}$  is a vector subspace of  $V$ .

**1.24** What is the span of the set  $\{\mathbf{0}\} \in \mathbb{R}^n$ ?

**1.25** For a fixed vector  $\langle a_1, \dots, a_n \rangle \in \mathbb{R}^n$ , is the set  $W = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i^n a_i x_i = 0\}$  always a subspace of  $\mathbb{R}^n$ ?

**1.26** For a fixed vector  $\langle a_1, \dots, a_n \rangle \in \mathbb{R}^n$ , is the set  $W = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i^n a_i x_i = 1\}$  always a subspace of  $\mathbb{R}^n$ ?

**1.27** For a fixed vector  $\langle a_1, \dots, a_n \rangle \in \mathbb{R}^n$ , is the set

$$W = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i^n a_i (x_i)^2 = 0\}$$

always a subspace of  $\mathbb{R}^n$ ?

**1.28** Suppose that  $U_1$  and  $U_2$  are subspaces of a vector space  $V$ . Prove that their intersection,

$$U_1 \cap U_2 := \{v \in V \mid v \in U_1 \text{ and } v \in U_2\}$$

is also a subspace of  $V$ .

**1.29** Prove that the set  $\mathcal{F}(\mathbb{R})$  of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a vector space with pointwise addition and pointwise scalar multiplication.

$$(f + g)(x) := f(x) + g(x) \quad (\lambda f)(x) := \lambda(f(x))$$

**1.30** Prove that the set  $\mathcal{C}(\mathbb{R})$  of *continuous* functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a vector subspace of  $\mathcal{F}(\mathbb{R})$ . You can use any facts about continuous function that you know from single variable calculus.

**1.31** Is the set  $\{(1, 0, 1), (2, 1, 0)\}$  a basis of  $\mathbb{R}^3$ ?

**1.32** Is the set  $\{(1, 0, 1), (2, 1, 0), (1, 1, -1)\}$  a basis of  $\mathbb{R}^3$ ?

**1.33** Is the set  $\{(1, 0, 1), (2, 1, 0), (1, 1, 1)\}$  a basis of  $\mathbb{R}^3$ ?

**1.34** Is the set  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$  a basis of  $\mathbb{R}^3$ ?

**1.35** Prove that any subset of a vector space that contains the zero vector is linearly dependent.

**1.36** Show that the collection  $P_n$  of all polynomials (with real coefficients) of degree less than or equal to  $n$  is a **subspace** of the vector space  $\mathcal{F}(\mathbb{R})$ , and that it has  $\{1, x, x^2, \dots, x^n\}$  as a basis.

**1.37** Given a basis  $\mathcal{B} = \{v_1, \dots, v_k\}$  of  $V$ , prove that the linear combination  $v = \sum c_i v_i$  is unique.

### 1.3 Maps of vector spaces

Now that we have a good grasp on what vector spaces are, we can now study how they relate to each other. In other words, we would like to study certain kinds of maps of vector spaces. For example, in this section, we will see what it means for two vector spaces to be the same!

Let us first consider two examples of maps:

Fix a vector  $\mathbf{a} = \langle a, b, c \rangle$ , and consider the map  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that sends a vector  $\mathbf{v} = \langle x, y, z \rangle$  to the vector  $\mathbf{v} + \mathbf{a} = \langle x + a, y + b, z + c \rangle$ .

Observe that the map  $f$  satisfies the following properties:

- (i) For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , we have that  $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ .
- (ii) For any scalar  $\lambda \in \mathbb{R}$ , and for any vector  $\mathbf{v} \in V$ , we have that  $f(\lambda \mathbf{u}) = \lambda f(\mathbf{u})$ .

Consider the function  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that sends a vector  $\langle x, y, z \rangle$  to the vector  $\langle x^2, y^2, z^2 \rangle$ .

Observe that the map  $g$  satisfies neither of the two properties

- (i) For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , we have that  $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ .
- (ii) For any scalar  $\lambda \in \mathbb{R}$ , and for any vector  $\mathbf{v} \in \mathbb{R}^3$ , we have that  $f(\lambda \mathbf{u}) = \lambda f(\mathbf{u})$ .

See the exercises (1.38) for examples of maps that preserve one property, but not the other.

Why are properties (i) and (ii) important?

Observe that if a map  $f$  satisfies properties (i) and (ii), then it preserves the vector space structure on  $\mathbb{R}^n$ . That is, it preserves vector space addition and scalar multiplication.



**Definition 1.3.1.** A map of vector spaces  $T : V \rightarrow W$  is called a **linear map** (or sometimes a **linear transformation**) if

$$T\left(\sum_i^k \alpha_i x_i\right) = \sum_i^k \alpha_i T(x_i)$$

for all  $k \in \mathbb{N}$ , for all  $\alpha_i \in \mathbb{R}$ , and all vectors  $x_i \in V$ .

This is a good definition to use for proving theorems, but it is difficult to check that a map of vector spaces satisfies this property. However, there is an equivalent definition of linearity, which allows us to check only two conditions:

**Proposition 1.3.2.** A map of vector spaces  $f : V \rightarrow W$  is a linear map if and only if

- (i) For any two vectors  $\mathbf{u}, \mathbf{v} \in V$ , we have that  $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ .
- (ii) For any scalar  $\lambda \in \mathbb{R}$ , and for any vector  $\mathbf{v} \in V$ , we have that  $f(\lambda \mathbf{u}) = \lambda f(\mathbf{u})$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the map that sends  $x$  to  $ax$ . Then  $f$  is linear.

*Proof.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the map that sends  $x$  to  $ax$ .

We first show that for all  $x, y \in \mathbb{R}$ ,  $f(x + y) = f(x) + f(y)$ .

By definition,  $f(x + y) = a(x + y)$ . However, by distributivity of scalars, we see that

$$f(x + y) = a(x + y) = ax + ay = f(x) + f(y)$$

as desired.

We now show that for all  $\lambda, x \in \mathbb{R}$ ,  $f(\lambda x) = \lambda f(x)$ .

By definition,  $f(\lambda x) = a(\lambda x)$ . But by commutativity of scalars, we see that

$$f(\lambda x) = a(\lambda x) = \lambda(ax) = \lambda f(x)$$

as desired. □

In later sections we will see how to generalize the proposition above (theorem 1.3.18).

The map  $i : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by  $i(x, y) = (x, y, 0)$  is linear.

The map  $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $p(x, y, z) = (x, y)$  is linear.

We can prove a lot of properties about linear maps of vector spaces simply using definition 1.3.1. Just as when we studied vector spaces, whenever you are trying to determine whether or not something is true about linear maps of vector spaces, you should try to see if it is true for linear maps in  $\mathbb{R}^n$ . Then, you should try to translate the proof to the general case, purely using the definition of a linear map.

The reason that we rely on definition 1.3.1 to prove things about linear maps is because it saves us time from having to prove a property for each specific linear map that we are interested. By using the axioms, we can prove things to be true for *all* linear maps.

**Proposition 1.3.3.** *Let  $T : V \rightarrow W$  be a linear map. Then  $T$  sends the 0 object in  $V$  to the 0 object in  $W$ . That is,*

$$T(\mathbf{0}_V) = \mathbf{0}_W$$

*Proof.* Suppose that  $T : V \rightarrow W$  is a linear map. We wish to calculate  $T(\mathbf{0}_V)$ .

Observe that  $\mathbf{0}_V = \mathbf{v} - \mathbf{v}$ . Thus,

$$T(\mathbf{0}_V) = T(\mathbf{v} - \mathbf{v})$$

by linearity, we see that

$$T(\mathbf{0}_V) = T(\mathbf{v}) - T(\mathbf{v}) = \mathbf{0}_W$$

as desired.  $\square$

**Proposition 1.3.4.** *Let  $T_1, T_2 : V \rightarrow W$  be two linear maps. Then the sum of two linear maps  $T_1 + T_2$  is a linear map.*

**Proposition 1.3.5.** *Let  $T : V \rightarrow W$  be a linear map. Then for any scalar  $\lambda \in \mathbb{R}$ , the map  $\lambda T$  is a linear map.*

**Proposition 1.3.6.** *Given two linear maps  $S : V \rightarrow W$  and  $T : U \rightarrow V$ , prove that the composite  $S \circ T : U \rightarrow W$  is also linear.*

#### Properties of linear maps

Let us now investigate some properties of linear maps. The properties are important not just in linear algebra, but in set theory, abstract algebra and topology as well.

**Definition 1.3.7.** *Let  $f : V \rightarrow W$  be a linear map. We say that  $f$  is **injective** or **one-to-one** (or sometimes,  $f$  is an **injection**) if the following holds:*

$$\text{For all } \mathbf{v}_1, \mathbf{v}_2 \in V, \text{ if } f(\mathbf{v}_1) = f(\mathbf{v}_2), \text{ then } \mathbf{v}_1 = \mathbf{v}_2.$$

That is, a map  $f$  is injective if any element in the codomain of  $f$  is the image of **at most** one element in its domain.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the map that sends  $x$  to  $ax$ . Show that  $f$  is injective.

Show that the map  $i : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by  $i(x, y) = (x, y, 0)$  is injective.

Equivalently, the contrapositive statement of injectivity states that  $f$  is **injective** (**one-to-one**) if the following holds:

$$\text{For all } \mathbf{v}_1, \mathbf{v}_2 \in V, \text{ if } \mathbf{v}_1 \neq \mathbf{v}_2, \text{ then } f(\mathbf{v}_1) \neq f(\mathbf{v}_2).$$

**Definition 1.3.8.** *Let  $f : V \rightarrow W$  be a linear map. We say that  $f$  is **surjective** or **onto** (or sometimes,  $f$  is a **surjection**) if the following holds:*

$$\text{For all } \mathbf{w} \in W, \text{ there exists a } \mathbf{v} \in V \text{ such that } f(\mathbf{v}) = \mathbf{w}.$$

That is, any element in the codomain of  $f$  is the image of **at least** one element in its domain.

Show that the map  $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $p(x, y, z) = (x, y)$  is surjective.

**Definition 1.3.9.** Let  $f : V \rightarrow W$  be a linear map. We say that  $f$  is **bijective** (or sometimes,  $f$  is a **bijection**) if  $f$  is both injective and surjective.

That is, any element in the codomain of  $f$  is the image of **exactly** one element in its domain.

We will study bijective linear maps

### Characterising linear maps

Can we describe characterize linear maps  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ?

Let us first start with the simplest case, where  $n = m = 1$ .

**Proposition 1.3.10.** Let  $T : \mathbb{R} \rightarrow \mathbb{R}$  be a linear map. Observe that if  $T(1) = a$ , then  $T(x) = ax$ .

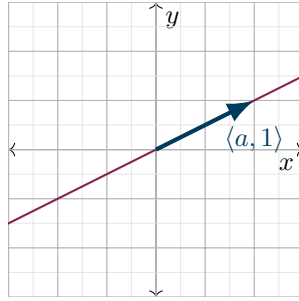
Thus we have shown that linear maps from  $\mathbb{R} \rightarrow \mathbb{R}$  must take the form of multiplication by  $a$ . From proposition 1.3, we saw that the converse holds as well. Thus, we have classified all linear maps from  $\mathbb{R} \rightarrow \mathbb{R}$ !

**Theorem 1.3.11.** A map  $T : \mathbb{R} \rightarrow \mathbb{R}$  is a linear map if and only if

$$T(x) = ax$$

for some  $a \in \mathbb{R}$ .

In other words, the graph of a linear map  $\mathbb{R} \rightarrow \mathbb{R}$  can be thought of as a line in  $\mathbb{R}^2$ , passing through the origin.



**Remark 1.3.12. Warning.** Note that if  $b \neq 0$ , the graph of  $f(x) = ax + b$  is a straight line, but  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **not linear**.

We proved in proposition 1.3.3 that linear maps must send  $o$  to  $o$ . However,  $f(0) = b \neq 0$ .

How can we generalize this result?

Observe that the set  $\{1\}$  is a basis for the vector space  $\mathbb{R}$ . Proposition 1.3.10 tells us that a linear map  $T : \mathbb{R} \rightarrow \mathbb{R}$  is determined completely by where  $T$  sends the basis vector <sup>5</sup>.

Recall also that given a basis  $\mathcal{B} = \{v_1, \dots, v_k\}$  of a vector space  $V$ , we can uniquely describe any vector  $v \in V$  as a linear combination of basis vectors (Theorem 1.2.17). That is,

$$v = \sum c_i v_i$$

<sup>5</sup>That is,  $a = f(1)$

In other words, in terms of its coordinates.

Let us consider the vector space  $\mathbb{R}^3$ , with the standard basis. Then, as we saw before, we can write the vector  $\mathbf{v} = \langle x, y, z \rangle \in \mathbb{R}^3$  in terms of the standard basis:

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

Let us consider a linear map  $T : \mathbb{R}^3 \rightarrow W$ , where  $W$  is an arbitrary vector space. Observe that using linearity, we have that

$$\begin{aligned} T(\mathbf{v}) &= T(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) \\ &= xT(\mathbf{e}_1) + yT(\mathbf{e}_2) + zT(\mathbf{e}_3) \end{aligned}$$

Thus, if we know the values of  $T(\mathbf{e}_1)$ ,  $T(\mathbf{e}_2)$  and  $T(\mathbf{e}_3)$  (e.g. how  $T$  acts on the basis vectors), then we can use the above formula to calculate  $T(\mathbf{v})$  for any vector  $\mathbf{v} \in \mathbb{R}^3$ .

This idea generalizes to the following theorem:

**Theorem 1.3.13.** *Let  $V$  be a vector space with basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , and let  $W$  be an arbitrary vector space. A linear map  $T : V \rightarrow W$  is determined by what it does on basis vectors.*

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear mapping such that  $T\langle 1, 0, 0 \rangle = \langle 1, 0, 1 \rangle$ ,  $T\langle 0, 1, 0 \rangle = \langle 3, 0, 2 \rangle$ , and  $T\langle 0, 0, 1 \rangle = \langle 4, 2, 1 \rangle$ . Then we can compute  $T\langle 3, 3, 2 \rangle$  as follows:

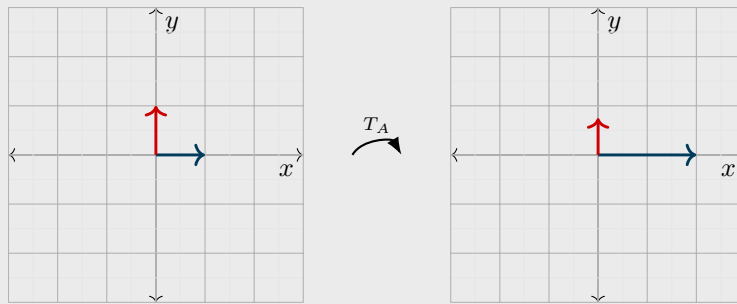
$$\begin{aligned} T\langle 3, 3, 2 \rangle &= 3T(\mathbf{e}_1) + 3T(\mathbf{e}_2) + 2T(\mathbf{e}_3) \\ &= 3\langle 1, 0, 1 \rangle + 3\langle 3, 0, 2 \rangle + 2\langle 4, 2, 1 \rangle \\ &= \langle 20, 4, 11 \rangle \end{aligned}$$

How can we geometrically think about linear maps  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ ?

In other words, what do graphs of linear maps from  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  look like?

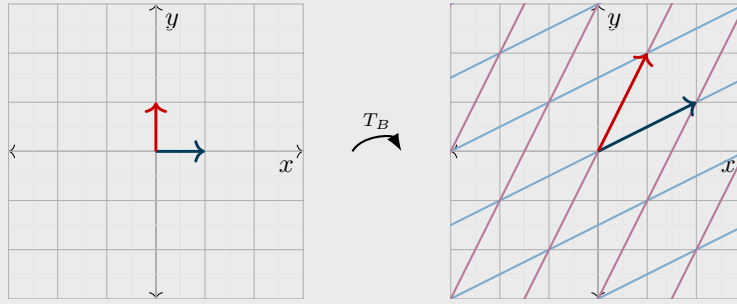
By theorem 1.3.13, we know that a linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  is determined by what it does on basis vectors. Let us consider a few linear maps  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

Consider the linear map  $T_A$  sends the vector  $\mathbf{e}_1 = \langle 1, 0 \rangle$  to the vector  $T_A(\mathbf{e}_1) = \langle 2, 0 \rangle$ , and the vector  $\mathbf{e}_2 = \langle 0, 1 \rangle$  to the vector  $T_A(\mathbf{e}_2) = \langle 0, \frac{3}{4} \rangle$ . Geometrically, the map looks like this:



Not every linear map is as simple, however!

Let us consider the linear map  $T_B$  that sends the vector  $\mathbf{e}_1$  to the vector  $T_B(\mathbf{e}_1) = \langle 2, 1 \rangle$ , and the vector  $\mathbf{e}_2$  to the vector  $T_B(\mathbf{e}_2) = \langle 1, 2 \rangle$ . Geometrically, the map looks like this:



We observe that the linear map  $T_B$  sends the unit square spanned by  $\mathbf{e}_1$  and  $\mathbf{e}_2$  to the parallelogram spanned by  $T_B(\mathbf{e}_1)$  and  $T_B(\mathbf{e}_2)$ . By linearity,  $T_B$  then sends the grid lines to a grid of congruent parallelograms.

In general,  $T_B$  is a good mental image for the geometric interpretation of a linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . This is why linear maps are sometimes called linear transformations - as they transform the vectors in the domain into (potentially different) vectors in the range.

### Matrices

Is there a more efficient way to describe how a linear transformation acts on a basis?

Let us first set up some machinery:

**Definition 1.3.14.** An  $m \times n$  (*real*) **matrix**  $A$  is an array of elements  $a_{i,j} \in \mathbb{R}$  with  $m$  rows and  $n$  columns:

$$A_{m,n} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

The set of  $m \times n$  (*real*) matrices is denoted  $M_{m \times n}(\mathbb{R})$ .

**Theorem 1.3.15.** The set  $M_{m \times n}(\mathbb{R})$ , with entry-wise addition and scalar multiplication, is a vector space.

$$A_{m,1} = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}$$

Observe that we can write vectors in  $\mathbb{R}^m$  as  $m \times 1$  matrices.

An  $m \times n$  matrix can act on a vector in  $\mathbb{R}^m$  in the following way:

**Definition 1.3.16** (Matrix by vector multiplication). Let  $A \in M_{m \times n}(\mathbb{R})$ , and let  $\mathbf{x} \in \mathbb{R}^n$ . The product  $A\mathbf{x} \in \mathbb{R}^m$  is defined the vector whose  $i$ -th entry is sum  $\sum_j a_{i,j}x_j$ . That is,

$$A\mathbf{x} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n \end{bmatrix}$$

Compute  $A\mathbf{x}$ , where  $\mathbf{x} = \langle 7, 8, 9 \rangle$ , and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

**Proposition 1.3.17.** Observe that the matrix  $A \in M_{m \times n}(\mathbb{R})$  determines a map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

Thus  $m \times n$  matrices are not the same as  $n \times m$  matrices, unless  $m = n$ . If  $m = n$ , we call  $A \in M_{n \times n}(\mathbb{R})$  a **square matrix**.

**Theorem 1.3.18.** Let  $A \in M_{m \times n}(\mathbb{R})$ . The map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined by  $T(\mathbf{x}) = A\mathbf{x}$  is linear.

Now, we will show the converse:

**Definition 1.3.19.** Given a basis  $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  for  $\mathbb{R}^n$ , the **standard matrix**  $A$  of a linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is given by

$$[A] = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix} \in M_{m \times n}(\mathbb{R})$$

Thus, we have precisely characterized the linear maps from  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  - they correspond exactly to  $n \times m$  matrices!

**Theorem 1.3.20.** A linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  determined exactly by a matrix  $A$  such that

$$T(\mathbf{x}) = A\mathbf{x}$$

We call  $A$  the **standard matrix** of  $T$ .

$$A_{1,n} = [a_{1,1} \quad a_{1,2} \quad \cdots \quad a_{1,n}]$$

Observe that the coordinate vector in  $\mathbb{R}^n$  is an  $1 \times n$  matrix.

**Definition 1.3.21.** Given a matrix  $A = [a_{i,j}] \in M_{n \times n}(\mathbb{R})$ , the **transpose matrix** is the matrix  $A^\top$  defined by

$$A^\top = [b_{i,j}] \in M_{n \times n}(\mathbb{R})$$

where  $b_{i,j} := a_{j,i}$ .

say more

We previously saw in proposition 1.3.6 that the composition of linear maps is again linear. What is the standard matrix of the composite?

Let  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$  be linear maps, with standard matrices  $A \in M_{m \times n}(\mathbb{R})$  and  $B \in M_{n \times p}(\mathbb{R})$ .

Then the composite  $S \circ T$  is a map from  $\mathbb{R}^p \rightarrow \mathbb{R}^m$ , and should have a standard matrix  $C \in M_{m \times p}(\mathbb{R})$

**Definition 1.3.22** (Matrix multiplication). *Let  $A \in M_{m \times n}(\mathbb{R})$  and  $B \in M_{n \times p}(\mathbb{R})$  be two matrices. Denote the columns of  $B$  by the vectors  $\mathbf{b}_1, \dots, \mathbf{b}_p$ . Then the product of  $A$  and  $B$  is the matrix*

$$AB = \begin{bmatrix} | & | & & | \\ [A]\mathbf{b}_1 & [A]\mathbf{b}_2 & \cdots & [A]\mathbf{b}_p \\ | & | & & | \end{bmatrix} \in M_{m \times p}(\mathbb{R})$$

In terms of entries, if  $A = [a_{i,j}]$  and  $B = [b_{i,j}]$ , then

$$AB = \left[ \sum_k a_{i,k} b_{k,j} \right]$$

Find the matrix  $AB$ , where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

**Theorem 1.3.23** (Properties of matrix multiplication). a.  $A(BC) = (AB)C$  for  $A \in M_{m \times n}(\mathbb{R})$ ,  $B \in M_{n \times p}(\mathbb{R})$ ,  $C \in M_{p \times q}(\mathbb{R})$  (associativity).

b.  $A(B + C) = AB + AC$  for  $A \in M_{m \times n}(\mathbb{R})$ ,  $B, C \in M_{n \times p}(\mathbb{R})$  (distribution).

c.  $(A + B)C = AC + BC$  for  $A, B \in M_{m \times n}(\mathbb{R})$ ,  $C \in M_{n \times p}(\mathbb{R})$  (distribution).

d.  $\lambda(AB) = (\lambda A)B = A(\lambda B)$  (scalars).

**Warning:** Matrix multiplication is **not** commutative. In fact, commutativity is meaningless unless  $A$  and  $B$  are both  $n \times n$  matrices.

**Definition 1.3.24.** The  $n \times n$  **identity matrix**  $I_n$  is the standard matrix of the identity transformation  $\text{id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , which sends a vector  $\mathbf{v}$  to itself.

The  $n \times n$  identity matrix has the form

$$I_n = [\delta_{ij}]$$

That is, the entry in the  $i$ -th row and  $j$ -th column is described by the **Kronecker delta** function, which is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The  $3 \times 3$  identity matrix is of the form

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Proposition 1.3.25.** *The identity matrix  $I_n \in M_{n \times n}(\mathbb{R})$  satisfies the property that for all  $A \in M_{m \times n}(\mathbb{R})$ ,*

$$I_n A = A = A I_n$$

**Proposition 1.3.26.** *The identity matrix  $I_n \in M_{n \times n}(\mathbb{R})$  corresponds to the linear transformation  $Id_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , determined by  $Id_n(\mathbf{v}) = \mathbf{v}$ .*

Invertible linear maps

**Definition 1.3.27.** *A matrix  $A \in M_{n \times n}(\mathbb{R})$  is **invertible** if there exists a matrix  $B \in M_{n \times n}(\mathbb{R})$  such that  $AB = BA = I_n$ .*

*$B$  is called the **inverse** of  $A$ .*

Show that the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

has inverse

$$B = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

The matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is **not** invertible. (**Hint:** use the fact that  $A^2 = 0$  to obtain a contradiction)

left inverse, right inverse

What does invertibility look like for linear transformations?

**Definition 1.3.28.** *A linear transformation  $T : V \rightarrow W$  is **invertible** if there exists a linear transformation  $S : W \rightarrow V$  such that  $S \circ T = id_V$  and  $T \circ S = id_W$ .*

**Proposition 1.3.29.** *If  $T : V \rightarrow W$  is linear and invertible, then  $T^{-1} : W \rightarrow V$  is also linear and invertible.*

**Proposition 1.3.30.** *A linear map  $T : V \rightarrow W$  is invertible if and only if it is bijective*<sup>6</sup>.

We will frequently interchange the two equivalent notions of invertibility and bijectivity. However, oftentimes when proving things we would like to either emphasize that  $T : V \rightarrow W$  has an inverse, or that  $T : V \rightarrow W$  is either surjective or injective.

<sup>6</sup>See definition 1.3.9



However, we are also interested in studying the consequences of invertibility (equivalently, bijectivity) for a linear map  $T : V \rightarrow W$ . Thus, we will introduce a third term for when we are agnostic about which properties of  $T$  we would like to use.

**Definition 1.3.31.** A linear transformation  $T : V \rightarrow W$  is an **isomorphism of vector spaces** if  $T$  satisfies any of the following equivalent conditions:

- a.  $T$  is invertible
- b.  $T$  is bijective

If  $T : V \rightarrow W$  is an isomorphism, we say that  $V$  and  $W$  are **isomorphic vector spaces**.

The map  $[-]_{\mathcal{S}} : \mathbb{R}^n \rightarrow M_{n \times 1}$  that sends a point to its coordinate vector<sup>a</sup> (with respect to the standard basis) is an isomorphism.

<sup>a</sup>See definition 1.2.18.

The vector space  $M_{m \times n}(\mathbb{R})$  is isomorphic to the vector space  $\mathbb{R}^{mn}$ .

Note that an isomorphism of vector spaces preserves precisely the vector space structure, but does not preserve any further properties.

How do we determine when a linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is an isomorphism? Equivalently, when is a matrix  $A \in M_{m \times n}(\mathbb{R})$  invertible?

From our discussion about dimension earlier, the dimension<sup>7</sup> of a vector space  $V$  is an invariant of  $V$ . For example, the dimension of  $\mathbb{R}^n$  is  $n$ <sup>8</sup>.

Thus we restrict our attention to linear maps  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , or equivalently, square matrices  $A \in M_{n \times n}(\mathbb{R})$ . For a square matrix, we can develop the theory of **determinants**:

**Definition 1.3.32.** The **determinant** of a  $2 \times 2$  matrix is denoted and defined as follows:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

**Definition 1.3.33.** The **determinant** of a  $3 \times 3$  matrix is denoted and defined as follows:

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

<sup>7</sup>That is, the cardinality of any basis of  $V$ . See definition 1.2.21

<sup>8</sup>see example 1.2

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 5 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

To see what the determinant measures geometrically, see section 1.4.

How can we generalize the determinant to  $n \times n$  matrices?

**Theorem 1.3.34** (Expansion along the  $i$ -th row). *Let  $A$  be an  $n \times n$  matrix, and let  $A_{ij}$  denote the  $(n-1) \times (n-1)$  matrix obtained by deleting row  $i$  and column  $j$  from  $A$ .*

$$\det(A) = (-1)^{i+1}a_{i,1}\det(A_{i1}) + \cdots + (-1)^{i+n}a_{i,n}\det(A_{in})$$

This theorem tells us that given an  $n \times n$  matrix  $A$ , we can expand along **any row**, and reduce the problem to computing determinants of  $(n-1) \times (n-1)$ . We can iterate this algorithm, and reduce computing the determinant of an  $n \times n$  matrix to a problem of computing determinants of  $2 \times 2$  matrices, which we know how to do.

Where does this formula come from? See the exercises (1.3) to see how one can define and construct the determinant abstractly. This leads to a proof of the explicit formula in theorem 1.3.34.

We have the following properties of the determinant

**Proposition 1.3.35.** *Given a matrix  $A$ , swapping any two rows or any two columns changes the sign of the determinant.*

**Remark 1.3.36.** *Let  $A, B \in M_{n \times n}(\mathbb{R})$ . In general,  $\det(A+B) \neq \det(A) + \det(B)$ .*

**Theorem 1.3.37.** *Let  $A, B \in M_{n \times n}(\mathbb{R})$ . Then  $\det(AB) = \det(A)\det(B)$ .*

Recall the definition of the transpose matrix (definition 1.3.21): If  $A = [a_{i,j}]$ , the transpose matrix is the matrix  $A^\top = [a_{j,i}]$ .

**Theorem 1.3.38.** *Given a matrix  $A = [a_{i,j}] \in M_{n \times n}(\mathbb{R})$ , we have that*

$$\det(A^\top) = \det(A)$$

**Theorem 1.3.39.** *A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with standard matrix  $A$  is an isomorphism if and only if  $\det(A) \neq 0$ .*

Thus, the determinant allows us to detect isomorphisms.

**Corollary 1.3.40.** *A set  $\mathcal{B} = \{v_1, \dots, v_n\}$  is a basis of  $\mathbb{R}^n$  if and only if*

$$\det \begin{bmatrix} | & | & | \\ v_1 & \cdots & v_n \\ | & | & | \end{bmatrix} \neq 0$$

Bucketlist of injective, surjective

## Exercises

**1.38** Consider the map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(x, y) = \begin{cases} x, & \text{if } y = 0 \\ y, & \text{if } y \neq 0 \end{cases}$$

Show that  $f$  preserves scalar multiplication, but not vector addition.

**1.39** Prove that a map of vector spaces  $f : V \rightarrow W$  is a linear map if and only if for all vectors  $\mathbf{v}, \mathbf{w} \in V$ , and for any scalar  $\lambda \in \mathbb{R}$ , we have that

$$f(\lambda \mathbf{u} + \mathbf{v}) = \lambda f(\mathbf{u}) + f(\mathbf{v})$$

**1.40** Given two linear maps  $S : V \rightarrow W$  and  $T : U \rightarrow V$ , prove that the composite  $S \circ T : U \rightarrow W$  is also linear.

**1.41** Let  $T : V \rightarrow W$  be a linear map. Is the set

$$\text{Ker}(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

a subspace of  $V$ ?

**1.42** Let  $T : V \rightarrow W$  be a linear map. Is the set

$$\text{Im}(T) := \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$$

a subspace of  $W$ ?

**1.43** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear mapping such that  $T\langle 0, 1, 1 \rangle = \langle 1, 0, 1 \rangle$ ,  $T\langle 1, 0, 1 \rangle = \langle 3, 0, 2 \rangle$ , and  $T\langle 1, 1, 0 \rangle = \langle 4, 2, 1 \rangle$ .

Compute  $T\langle 3, 3, 2 \rangle$ .

**1.44** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear mapping such that  $T\langle 0, 1, 1 \rangle = 1$ ,  $T\langle 1, 0, 1 \rangle = 2$ , and  $T\langle 1, 1, 0 \rangle = 3$ . Find the standard matrix of  $T$ .

**1.45** A subset  $A \subset \mathbb{R}^n$  is said to be a **convex subset** of  $\mathbb{R}^n$  if it contains the line segment joining any two points of  $A$ . That is, for all  $\mathbf{a}, \mathbf{b} \in A$ , and for all  $t \in [0, 1]$ , then  $\mathbf{a} + t(\mathbf{b} - \mathbf{a}) \in A$ .

Let  $U \subset \mathbb{R}^n$  be a convex subset of  $\mathbb{R}^n$ . Show that if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear, then  $T(U)$  is a convex subset.

**1.46** Show that the rotation matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

has inverse

$$S_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

**1.47** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Prove that  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

**1.48** Show that the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is **not** invertible. (**Hint:** use the fact that  $A^2 = 0$  to obtain a contradiction)

**1.49** Let  $V$  and  $W$  be vector spaces such that  $\dim(V) = n = \dim(W)$ . Is a linear map  $T : V \rightarrow W$  necessarily an isomorphism?

**1.50** Show that an isomorphism of vector spaces must preserve dimension. That is, an isomorphism sends a basis of  $n$  elements to a basis of  $n$  elements.

**1.51** Prove that if  $V$  is a vector space of dimension  $n$ , then  $V \cong \mathbb{R}^n$ .

(**Hint:** Linear maps are determined by their actions on basis vectors).

**1.52** Show that if  $A$  is invertible, then  $\det(A) \neq 0$ .

**1.53** Prove that if a map  $T : V \rightarrow W$  is an isomorphism, then  $T$  is surjective.

**1.54** Prove that if a map  $T : V \rightarrow W$  is an isomorphism, then  $T$  is injective.

**1.55** Suppose that  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a linear map such that  $S(e_1) = S(e_2)$ . Is  $S$  an isomorphism?

**1.56** If a map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is both surjective and injective (bijective), what is the standard matrix of  $T^{-1}$ ?

**1.57** Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 5 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

Is the linear map associated to  $A$  invertible?

**1.58** Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 5 & 4 \\ 4 & -1 & 0 \end{bmatrix}$$

Is the linear map associated to  $A$  invertible?

**1.59** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear mapping such that  $T\langle 0, 1, 1 \rangle = \langle 1, 0, 1 \rangle$ ,  $T\langle 1, 0, 1 \rangle = \langle 3, 0, 2 \rangle$ , and  $T\langle 1, 1, 0 \rangle = \langle 4, 2, 1 \rangle$ . Is  $T$  invertible?

The determinant, abstractly

In this set of problems, we will see an abstract way to define the determinant map.

First, we need to define a property of maps of vector spaces:

**Definition 1.3.41.** A map  $M : V_1 \times \cdots \times V_k \rightarrow W$  is said to be **multilinear** if it is linear in each component. That is, for all  $1 \leq i \leq k$ , and for any fixed  $v_\ell \in V_\ell, \ell \neq i$ , the map  $V_i \rightarrow W$  given by

$$v \mapsto M(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$$

is linear.

If  $k = 2$ , then we say  $M$  is **bilinear**, instead of multilinear.

**1.60** Show that the map  $\cdot : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\langle u_1, \dots, u_n \rangle \cdot \langle v_1, \dots, v_n \rangle = \sum u_i v_i$$

is bilinear.

(This map is called the **dot product**, which we will discuss in the next section).

**1.61** Show that if we think of a matrix  $A = [a_{i,j}] \in M_{m \times n}(\mathbb{R})$  as an array of  $n$  column vectors, then the determinant map for  $2 \times 2$  and  $3 \times 3$  matrices is multilinear in the columns of  $A$ .

**Definition 1.3.42.** A multilinear map  $M$  is said to be **alternating** or **skew-symmetric** if

$$M(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_k) = -M(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k)$$

for all  $1 \leq i < j \leq k$  and for any  $\mathbf{v}_\ell \in V_\ell$ .

**1.62** Show that the determinant map for  $2 \times 2$  and  $3 \times 3$  matrices is alternating.

**1.63** Show that the dot product is **not** alternating. It is instead symmetric (commutative).

**Definition 1.3.43** (The determinant). There is exists a **unique** map,  $\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the following:

- a.  $\det$  is multilinear
- b.  $\det$  is alternating (skew-symmetric)
- c.  $\det(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$ .

This is an abstract way to determine the determinant of an  $n \times n$  matrix.

## 1.4 Operations on vector spaces

So far, we have discussed vector spaces and linear maps of vector spaces. Now we will turn to studying certain operations on  $\mathbb{R}^n$  (and  $\mathbb{R}^3$ ), which will tell us about the geometry of  $\mathbb{R}^n$  (and  $\mathbb{R}^3$ )

### Dot product

**Definition 1.4.1.** Given two vectors  $\mathbf{u} = \langle u_1, \dots, u_n \rangle$  and  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$  in  $\mathbb{R}^n$ , their (**dot product**) is the scalar defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum u_i v_i$$

This defines a function  $\cdot : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ .

**Remark 1.4.2.** Observe that we can identify  $\mathbf{u} \cdot \mathbf{v}$  as the matrix product  $\mathbf{u}^T \mathbf{v}$ .

**Theorem 1.4.3** (Axioms of the dot product). a.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$  (commutativity).

b.  $\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \lambda(\mathbf{v})$  (scalars).

c.  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$  (distribution).

d.  $\mathbf{v} \cdot \mathbf{v} \geq 0$ , with equality only when  $\mathbf{v} = \mathbf{0}$ . (positive definite).

**Proposition 1.4.4.** If  $\mathbf{v} \cdot \mathbf{w} = 0$  for all  $\mathbf{v} \in V$ , then  $\mathbf{w} = \mathbf{0}$ .

**Corollary 1.4.5.** If  $\mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w}$  for all  $\mathbf{w} \in V$ , then  $\mathbf{v} = \mathbf{u}$ .

What is the geometric meaning behind the dot product in  $\mathbb{R}^n$ ?

**Proposition 1.4.6.** Observe that for all  $\mathbf{v} \in \mathbb{R}^n$ , we have that  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ .

We see that the dot product allows us to talk about the notion of distance in  $\mathbb{R}^n$ . Thus the dot product has a geometric interpretation!

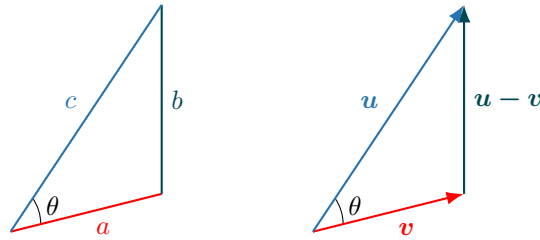
**Proposition 1.4.7.** *Let  $\theta$  be the angle between two non-zero position vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$ . Then*

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

*Proof.* This proposition follows from the Law of cosines: Given a triangle with side lengths  $a, b, c$ , where  $\theta$  is the angle between the sides of length  $a$  and  $b$ , we have that

$$c^2 = a^2 + b^2 - 2ab \cos(\theta)$$

Given two non-zero vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ , we can apply the Law of cosines to the triangle **spanned** by  $\mathbf{u}$  and  $\mathbf{v}$  pictured below:



The law of cosines tells us that for this triangle,

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$$

However, on the other hand, the properties of dot products tell us that

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

Given these two ways of evaluating  $\|\mathbf{u} - \mathbf{v}\|^2$ , we thus conclude that

$$\|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta) = \mathbf{u} \cdot \mathbf{v}$$

as desired. □

Thus, we can use the dot product in  $\mathbb{R}^n$  to determine whether the angle between the vectors  $\mathbf{u}$  and  $\mathbf{v}$  is acute, obtuse, or if  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal (that is, if  $\theta = \frac{\pi}{2}$ ).

**Proposition 1.4.8.** *The angle  $\theta$  between two non-zero position vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  is:*

- Acute ( $0 < \theta < \frac{\pi}{2}$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} > 0$ ,
- Orthogonal ( $\theta = \frac{\pi}{2}$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ ,
- Obtuse ( $\frac{\pi}{2} < \theta < \pi$ ) if and only if  $\mathbf{u} \cdot \mathbf{v} < 0$ .

The standard basis vectors  $\{\mathbf{e}_i\}$  in  $\mathbb{R}^n$  are all orthogonal to each other.

**Definition 1.4.9.** A subset of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$  is said to be **orthogonal** if  $\mathbf{v}_i \cdot \mathbf{v}_j = 0$  for all  $i \neq j$ .

Furthermore, if  $\|\mathbf{v}_i\| = 1$  for all  $1 \leq i \leq k$ , we say that the subset  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$  is **orthonormal**.

**Proposition 1.4.10.** Equivalently, a subset  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$  is orthonormal if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$$

where  $\delta_{ij}$  is the **Kronecker delta**, which is the symbol

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Orthonormal bases (such as the standard basis) are nice, because it turns out that we can compute the coordinate vectors for an orthonormal basis easily:

**Proposition 1.4.11.** Recall that if  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$  is an orthonormal basis of  $V$ , then for any  $\mathbf{v} \in V$ ,

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{b}_1)\mathbf{b}_1 + \dots + (\mathbf{v} \cdot \mathbf{b}_n)\mathbf{b}_n$$

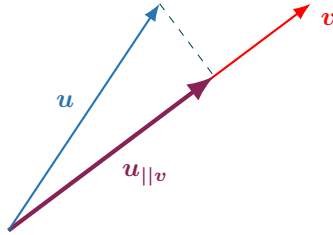
That is, the coordinates of a vector in an orthonormal basis  $\{\mathbf{b}_i\}$  of  $\mathbb{R}^n$  are exactly given by the dot products  $\mathbf{v} \cdot \mathbf{b}_i$ .

**Definition 1.4.12.** Assume  $\mathbf{v} \neq 0$ . The **projection of  $\mathbf{u}$  along  $\mathbf{v}$**  is the vector

$$\mathbf{u}_{||\mathbf{v}} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \right) \mathbf{e}_v$$

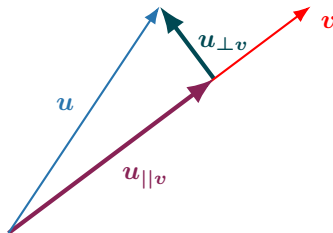
This is sometimes denoted  $\text{proj}_{\mathbf{v}} \mathbf{u}$ .

The scalar  $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$  is called the **scalar component of  $\mathbf{u}$  along  $\mathbf{v}$** .



**Definition 1.4.13.** The **orthogonal projection of  $\mathbf{u}$  along  $\mathbf{v}$**  is the vector

$$\mathbf{u}_{\perp \mathbf{v}} = \mathbf{u} - \mathbf{u}_{||\mathbf{v}}$$



**Theorem 1.4.14** (Further properties of the dot product). *The following properties of the dot product tell us about the geometry for  $\mathbb{R}^n$ :*

- a. *Cauchy-Schwarz inequality:*  $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$
- b. *Triangle inequality:*  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

#### Geometric interpretation

Observe that the dot product on  $\mathbb{R}^n$  is a function  $\cdot : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose we fix a vector  $\mathbf{n} \in \mathbb{R}^n$ . We can then consider the function

$$-\cdot \mathbf{n} : \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}$$

Given a vector  $\mathbf{n} \in \mathbb{R}^n$ , can we describe the set of vectors  $\mathbf{v} \in \mathbb{R}^n$  such that  $\mathbf{v} \cdot \mathbf{n} = 0$ ?

We know that  $\mathbf{0}$  is always in this set. We also know from proposition 1.4.7 that this is equivalent to describing all of the non-zero vectors that are *orthogonal* to  $\mathbf{n}$ .

**Proposition 1.4.15.** *For a fixed vector  $\mathbf{n} \in \mathbb{R}^n$ , the set*

$$\{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{n} = 0\}$$

*is a subspace of  $\mathbb{R}^n$ .*

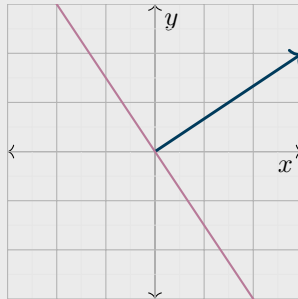
Let us examine a few examples.

Let us fix a vector  $\mathbf{n} = \langle a, b \rangle \in \mathbb{R}^2$ . Then we can write the subspace  $\{\mathbf{v} \in \mathbb{R}^2 \mid \mathbf{v} \cdot \mathbf{n} = 0\}$  as

$$\{\mathbf{v} = \langle x, y \rangle \in \mathbb{R}^2 \mid ax + by = 0\}$$

In other words if  $b \neq 0$ , this subspace is precisely the line  $y = -\frac{a}{b}x$ . If  $b = 0$ , then this subspace is the line  $x = 0$ .

Let us take for example  $\mathbf{n} = \langle 3, 2 \rangle \in \mathbb{R}^2$  in blue. Then the subspace is the line in red below, passing through the origin, in the direction of the vector  $\langle -2, 3 \rangle$ .



Thus, we see that the subspace of vectors in  $\mathbb{R}^2$  that are orthogonal to a vector  $\mathbf{n} = \langle a, b \rangle$  is a line passing through the origin, in the direction of the vector  $\langle -b, a \rangle$ .

In other words,

$$\{\mathbf{v} = \langle x, y \rangle \in \mathbb{R}^2 \mid ax + by = 0\} = \text{span}\{\langle -b, a \rangle\}$$

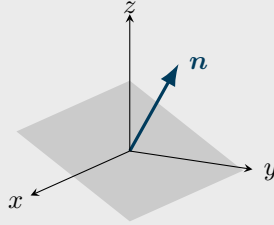
A vector parametrization of this line is  $\mathbf{r}(t) = t\mathbf{v}$ .



Let us fix a vector  $\mathbf{n} = \langle a, b, c \rangle \in \mathbb{R}^3$ . Then we can write the subspace  $\{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} \cdot \mathbf{n} = 0\}$  as

$$\{\mathbf{v} = \langle x, y, z \rangle \in \mathbb{R}^3 \mid ax + by + cz = 0\}$$

Let us take for example  $\mathbf{n} = \langle 3, 2, 1 \rangle \in \mathbb{R}^3$  in blue.



We again observe that the following two subspaces are the same!

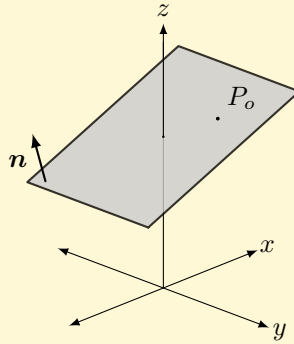
$$\{\mathbf{v} = \langle x, y, z \rangle \in \mathbb{R}^3 \mid 3x + 2y + 1z = 0\} = \text{span}\{\langle 1, 0, -3 \rangle, \langle 0, 1, -2 \rangle\}$$

That is, the subspace of vectors in  $\mathbb{R}^3$  orthogonal to  $\mathbf{n} = \langle a, b, c \rangle$  is a plane in  $\mathbb{R}^3$  passing through the origin.

In order to define a plane  $P$  in  $\mathbb{R}^3$  (not necessarily passing through the origin), we simply need to describe a point on the plane  $P$ , and a normal vector  $\mathbf{n} \in \mathbb{R}^3$ .

**Definition 1.4.16.** A plane  $P$  in  $\mathbb{R}^3$  is determined by a point  $P_0 = (x_0, y_0, z_0)$  and a **normal vector**  $\mathbf{n} = \langle a, b, c \rangle$ .

The points in the plane  $P$  consist of all the vectors that originate from  $P_0$  and are orthogonal to  $\mathbf{n}$ .



**Theorem 1.4.17.** The plane  $P$  in  $\mathbb{R}^3$  determined by a point  $P_0 = (x_0, y_0, z_0)$  and a **normal vector**  $\mathbf{n} = \langle a, b, c \rangle$  is described by the equations:

Vector form:	$\mathbf{n} \cdot \langle x, y, z \rangle$	$= d$
Scalar forms:	$a(x - x_0) + b(y - y_0) + c(z - z_0)$	$= 0$
	$ax + by + cz$	$= d$

Where we set  $d = ax_0 + by_0 + cz_0$ .

**Proposition 1.4.18.** *Given a vector  $\mathbf{n} \in \mathbb{R}^n$ , the set*

$$\{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{n} = 0\}$$

*is a vector subspace of  $\mathbb{R}^n$  of dimension  $n - 1$ .*

From the previous proposition, we can now generalize our examples from  $\mathbb{R}^2$  and  $\mathbb{R}^3$ :

**Definition 1.4.19.** *Let  $\mathbf{n} \in V$ , with  $\mathbf{n} \neq \mathbf{0}$ . The **hyperplane**  $W$  normal to  $\mathbf{n}$  (passing through the origin) is the subspace*

$$W = \{\mathbf{v} \in V \mid \mathbf{n} \cdot \mathbf{v} = 0\}$$

*We say that  $\mathbf{n}$  is a normal vector of  $W$ .*

Observe that we can also use vector addition to describe arbitrary planes in  $\mathbb{R}^n$  - that is, in order to define a plane  $P$  in  $\mathbb{R}^n$  (not necessarily passing through the origin), we simply need to describe a point on the plane  $P$ , and two non-parallel vectors. That is, we are describing planes as two-dimensional objects in the following way:

**Definition 1.4.20.** *The plane  $\mathcal{P}$  through the point  $P = (x_1, \dots, x_n)$  and determined by two non-parallel vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ , can be described by the vector function  $\mathbf{r}(s, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^n$  defined by*

$$\mathbf{r}(s, t) = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$$

*where  $\mathbf{r}_0$  is the vector  $\mathbf{r}_0 = \mathbf{OP} = \langle x_1, \dots, x_n \rangle$ .*

*We call  $\mathbf{r}(s, t)$  the **parametrization** of  $\mathcal{P}$ .*

That is, the plane is the set of all linear combinations of  $\mathbf{u}$  and  $\mathbf{v}$  that originate from  $P$ .

**Remark 1.4.21.** *In other words, we are describing a plane as the image of a linear map, whereas we can also describe the plane as the kernel of a linear map.*

How can we reconcile these two notions?

This leads us to the notion of the cross product of vectors in  $\mathbb{R}^3$ .

### Cross product

We saw previously that two non-zero vectors in  $\mathbb{R}^3$  span a plane  $P$  in  $\mathbb{R}^3$ . Is there a way to determine the normal vector  $\mathbf{n}$  associated to this plane?

In other words, given two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , can we construct a vector  $\mathbf{w} \in \mathbb{R}^3$  that is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ ?

The answer is yes (at least, in  $\mathbb{R}^3$ ). We will define the cross product of two vectors in  $\mathbb{R}^3$  in a few different ways (see definition 1.4.23, definition 1.4.24, and theorem 1.4.26). Each of these ways has its strengths.

We will first define the cross product in terms of direction and magnitude.

It turns out that in order to describe the direction of the cross product, we will need to make an arbitrary choice. This choice is called the right hand rule:

**Definition 1.4.22.** *We say that an ordered list of three orthogonal vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  in  $\mathbb{R}^3$  satisfy the **Right Hand Rule** if the following is true.*

- a. Point the fingers of your right hand in the direction of  $\mathbf{u}$
- b. Curl your fingers towards  $\mathbf{v}$  (maybe rotating your hand!)
- c. If your thumb points in the direction of  $\mathbf{w}$ , then  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  satisfy the right hand rule.

**Note that the order matters!** If the ordered list  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  satisfies the right hand rule, then the ordered list  $\mathbf{v}$ ,  $\mathbf{u}$  and  $\mathbf{w}$  does not satisfy the right hand rule.

For example, consider the standard basis vectors of  $\mathbb{R}^3$ :  $\mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\mathbf{j} = \langle 0, 1, 0 \rangle$ , and  $\mathbf{k} = \langle 0, 0, 1 \rangle$ .

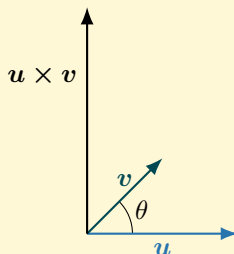
We see that the ordered list  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  does satisfy the right hand rule, while the ordered list  $\mathbf{i}$ ,  $\mathbf{k}$ , and  $\mathbf{j}$  doesn't.

We can now define the cross product in terms of direction and magnitude.

**Definition 1.4.23** (Magnitude and Direction). Given two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , with an angle  $\theta$  between them, their **cross product** is the unique vector  $\mathbf{u} \times \mathbf{v} \in \mathbb{R}^3$  with the following properties:

- (i)  $\mathbf{u} \times \mathbf{v}$  is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ .
- (ii)  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{u} \times \mathbf{v}$  satisfies the Right Hand Rule.
- (iii)  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$

Geometrically, we can draw the cross product as follows:



Note that together, (i) and (ii) specify a unique direction for the cross product - in  $\mathbb{R}^3$ , the set of vectors orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$  spans a one-dimensional subspace (a.k.a. a line). Thus, to specify one out of the two directions in this line, we must make an arbitrary choice via the right hand rule. Specifying a magnitude via (3) thus gives us a unique vector. We will discuss later the choice of magnitude later in 1.4.28.

What is the cross product  $\mathbf{i} \times \mathbf{j}$ ?

What is the cross product  $\mathbf{j} \times \mathbf{k}$ ?

How can we compute the cross product algebraically? In other words, what are the components of the cross product?

We now give a second, algebraic definition of the cross product. This will lead to a formula for the components of the cross product (theorem 1.4.26).

**Definition 1.4.24** (Algebraic characterization). *Given two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , their **cross product** is the unique vector  $\mathbf{u} \times \mathbf{v} \in \mathbb{R}^3$  defined by the property:*

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \quad \text{for all } \mathbf{w} \in \mathbb{R}^3$$

This is an interesting perspective on defining a vector in  $\mathbb{R}^3$ : it is determined by how it relates to all other vectors in  $\mathbb{R}^3$ . This kind of idea shows up a lot in higher level mathematics!

First of all, using this algebraic perspective, we can prove the following properties of the cross product:

**Theorem 1.4.25.** *Given two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ ,*

- a.  $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$  (anti-commutativity).
- b.  $\mathbf{u} \times \mathbf{v}$  is orthogonal to  $\mathbf{u}$  and  $\mathbf{v}$ .
- c. The cross product  $\times : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is bilinear.
- d.  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are parallel.

Moreover, we can use the algebraic characterization of the cross product to find the components of the cross product. Recall from proposition 1.4.11, if we consider the standard basis for  $\mathbb{R}^3$ ,  $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , we have that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (\mathbf{u} \times \mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{u} \times \mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{u} \times \mathbf{v} \cdot \mathbf{e}_3)\mathbf{e}_3 \\ &= \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{e}_1 \end{bmatrix} \mathbf{e}_1 + \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{e}_2 \end{bmatrix} \mathbf{e}_2 + \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{e}_3 \end{bmatrix} \mathbf{e}_3 \\ &= \left\langle \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{e}_1 \end{bmatrix}, \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{e}_2 \end{bmatrix}, \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{e}_3 \end{bmatrix} \right\rangle \end{aligned}$$

Thus, given the components of two vectors two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , we can compute the components of the cross product  $\mathbf{u} \times \mathbf{v}$ .

**Theorem 1.4.26.** *Given two vectors  $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$  and  $\mathbf{v} = \langle x_2, y_2, z_2 \rangle$ , their **cross product**  $\mathbf{u} \times \mathbf{v}$  can be described as:*

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left\langle \det \begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}, -\det \begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix}, \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \right\rangle \\ &= \langle y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1 \rangle \end{aligned}$$

We saw previously that two non-zero, non-parallel vectors in  $\mathbb{R}^3$  (or equivalently, 3 non-collinear points) span a plane  $P$  in  $\mathbb{R}^3$ , and we can now determine the normal vector  $\mathbf{n}$  associated to this plane:

Find the equation for the plane  $P$  determined by the points  $P = (1, 0, -1)$ ,  $Q = (2, 2, 1)$ ,  $R = (4, 1, 2)$ .

Observe that  $\mathbf{PQ} \times \mathbf{PR}$  is orthogonal to the plane, and we can use any of  $P$ ,  $Q$ , or  $R$  to determine the plane.

How are these two definitions of the cross product related to each other?

In other words, (1) what does the right hand rule have to do with determinants? (2) why did we choose the magnitude of  $\mathbf{u} \times \mathbf{v}$  to be  $\|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$ ?

More generally, what is the geometry of the determinant?

**Proposition 1.4.27.** *The right hand rule corresponds to having a positive determinant.*

**explain** (relaxed right-hand rule, adding scalar multiples does not change right-handedness - turn into diagonal matrix).

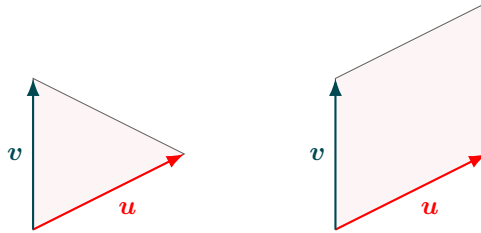
The standard basis  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  satisfies the right hand rule, and  $\det \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$  has positive determinant.

We now turn to question 2: Why did we choose the magnitude of  $\mathbf{u} \times \mathbf{v}$  to be  $\|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$ ?

**Proposition 1.4.28.** *Given two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , with an angle  $\theta$  between them, then*

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$$

First, observe that if two non-parallel vectors (say,  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ ) start at the same point, then they uniquely determine a triangle, and a parallelogram.



**Theorem 1.4.29.** *Let  $P$  be the parallelogram spanned by  $\mathbf{u}$  and  $\mathbf{v}$ , and  $T$  be the triangle spanned by  $\mathbf{u}$  and  $\mathbf{v}$ . Then*

$$\text{area}(P) = \|\mathbf{u} \times \mathbf{v}\| \quad \text{and} \quad \text{area}(T) = \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|$$

Observe that by geometry, the area of  $P$  is twice the area of the triangle  $T$ . Furthermore, if the angle between  $\mathbf{u}$  and  $\mathbf{v}$  is  $\theta$ , we can calculate the area of the parallelogram as base times height, e.g.,

$$(\|\mathbf{u}\|)(\|\mathbf{v}\| \sin(\theta)) = \|\mathbf{u} \times \mathbf{v}\|$$

However, here's another way to think about this parallelogram: Without loss of generality, we can rotate our vectors so that they lie in the  $xy$ -plane. (Equivalently, we can choose a basis  $\mathcal{B}$  so that the coordinates of  $\mathbf{u} = \langle a, b, 0 \rangle$ , and that  $\mathbf{v} = \langle c, d, 0 \rangle$ ).

**Proof without words:**  
**A  $2 \times 2$  determinant is the area of a parallelogram**

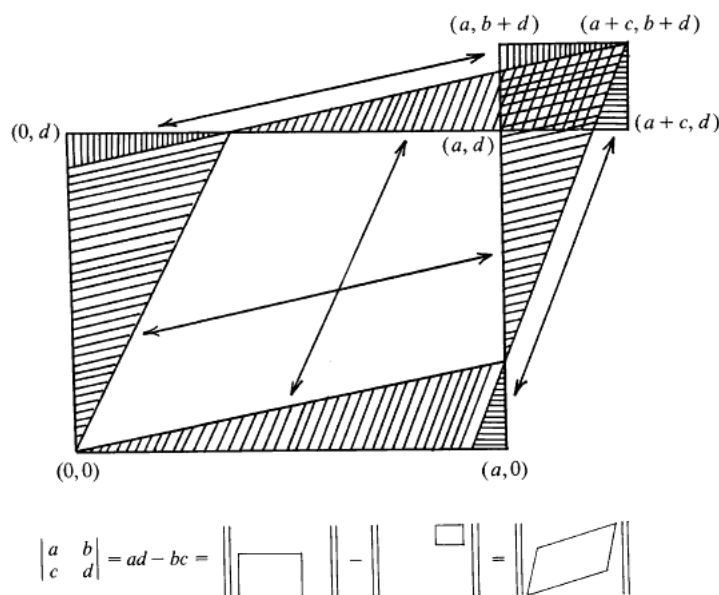


Figure 1.1: A Proof without Words, by Solomon W. Golomb ([Mathematics Magazine](#), March 1985)

Observe that the figure above shows that

$$\left| \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = \text{area}(P)$$

That is, the absolute value of a  $2 \times 2$  determinant equals the area of the parallelograms spanned by the rows.

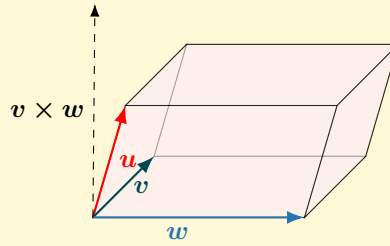
In other words, the determinant of a  $2 \times 2$  matrix  $A$  measures the area of the parallelogram spanned by  $A\mathbf{e}_1 = \langle a, c \rangle$  and  $A\mathbf{e}_2 = \langle b, d \rangle$ . That is, it measures the area of the image of the unit square under the linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(\mathbf{v}) = A\mathbf{v}$ .

PICTURE

Determinants as volume

How does this generalize to linear transformations in higher dimensions?

**Definition 1.4.30.** Given three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $\mathbb{R}^3$ , they span a **parallelepiped**. That is, a 3-dimensional solid, such that each face is a parallelogram.



We can again use geometry to compute the volume of this parallelepiped:

**Theorem 1.4.31.** Let  $D$  be the parallelepiped spanned by  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ , and let  $\theta$  be the angle between  $\mathbf{u}$  and  $\mathbf{v} \times \mathbf{w}$ . Then

$$\text{volume}(D) = \|\mathbf{v} \times \mathbf{w}\| \|\mathbf{u}\| |\cos(\theta)|$$

Observe that we can write this formula in terms of a dot product.

**Definition 1.4.32.** The scalar  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$  is called the **scalar triple product**.

Let  $D$  be the parallelepiped spanned by  $\mathbf{u} = \langle 1, 3, 2 \rangle$ ,  $\mathbf{v} = \langle 6, 5, 4 \rangle$ , and  $\mathbf{w} = \langle 2, 2, 2 \rangle$ . Compute the volume of  $D$ .

This computation should seem familiar to you: see example 1.3

**Theorem 1.4.33.** Let  $D$  be the parallelepiped spanned by  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ ,  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ , and  $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ . Then

$$\text{volume}(D) = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left| \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \right|$$

**Definition 1.4.34.** We say that three vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  are **co-planar** if they all lie in the same plane.

Show that if  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$  are co-planar, then  $\text{volume}(D) = 0$ . What is the geometric interpretation of this fact?

Just as we saw in the  $2 \times 2$  case, we see again that the determinant of a  $3 \times 3$  matrix  $A$  measures the volume of the parallelepiped spanned by  $A\mathbf{e}_1$ ,  $A\mathbf{e}_2$ , and  $A\mathbf{e}_3$ . That is, it measures the volume of the image of the unit cube under the linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by  $T(\mathbf{v}) = A\mathbf{v}$ .

This generalizes to  $n \times n$  matrices and their associated linear transformations  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Let us write  $\mathbf{v}_i = T(\mathbf{e}_i)$ .

**Theorem 1.4.35.** Let  $D \subset \mathbb{R}^n$  be the  $n$ -dimensional parallelepiped spanned by  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Then the volume of  $D$  is given by

$$\text{volume}(D) = \det \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & | & | \end{bmatrix}$$

That is, the determinant of the standard matrix of  $T$  measures the  $n$ -dimensional volume of the  $n$ -dimensional parallelepiped spanned by the image of the unit  $n$ -cube. In other words, the absolute

value of  $\det A$  is the factor by which  $T$  magnifies volume; and  $\det A$  is zero if and only if  $T$  is not invertible.

"The determinant is astonishing." — Jerry Shurman, Calculus and Analysis in Euclidean Space.

### Inner products

How can we generalize the dot product in  $\mathbb{R}^n$  to an arbitrary vector space  $V$ ?

For  $\mathbb{R}^n$ , we saw that the dot product satisfies the property  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ . Thus, our generalization should allow us to define a notion of distance in an abstract vector space.

**Definition 1.4.36.** Given two vectors  $\mathbf{u}, \mathbf{v} \in V$ , an **inner product** on  $V$  is a map  $\langle -, - \rangle : V \times V \rightarrow \mathbb{R}$  that is:

- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$  (symmetric).
- $\langle -, - \rangle$  is bilinear.
- $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ , with equality only when  $\mathbf{v} = \mathbf{0}$ . (positive definite).

**Definition 1.4.37.** Given an inner product  $\langle -, - \rangle$  on  $V$ , the **norm** (or **modulus**) of a vector  $\mathbf{v} \in V$  is defined as

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Given two vectors  $\mathbf{u} = \langle u_1, \dots, u_n \rangle$  and  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$  in  $\mathbb{R}^n$ , their **dot product** is an inner product.

$$\mathbf{u} \cdot \mathbf{v} = \sum u_i v_i$$

**Theorem 1.4.38** (Properties of an inner product). Let  $V$  be a vector space with inner product  $\langle -, - \rangle$ , and let  $\mathbf{u}, \mathbf{v} \in V$ . Then

- Cauchy-Schwarz inequality:  $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$
- Triangle inequality:  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

What is the geometric interpretation of the inner product for an arbitrary vector space?

**Definition 1.4.39.** A subset  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$  is said to be **orthogonal** if  $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$  for all  $i \neq j$ .

Furthermore, if  $\|\mathbf{v}_i\| = 1$  for all  $1 \leq i \leq k$ , we say that the subset  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$  is **orthonormal**.

**Proposition 1.4.40.** Equivalently, a set subset  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq V$  is orthonormal if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$$



where  $\delta_{ij}$  is the **Kronecker delta**, which is the symbol

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

### Exercises

- 1.64** Find all values of  $b$  such that the vectors  $\mathbf{u} = \langle b, 3, 2 \rangle$  and  $\mathbf{v} = \langle 1, b, 1 \rangle$  are orthogonal.
- 1.65** Given that  $\|\mathbf{u}\| = 1$ ,  $\|\mathbf{v}\| = 3$ , and  $\mathbf{u} \cdot \mathbf{v} = 2$ , evaluate the expression  $2\mathbf{u} \cdot (3\mathbf{u} - \mathbf{v})$ .
- 1.66** Given that  $\|\mathbf{u}\| = 2$ ,  $\|\mathbf{v}\| = 3$ , and the angle between  $\mathbf{u}$  and  $\mathbf{v}$  is  $120^\circ$ , calculate  $\|3\mathbf{u} - \mathbf{v}\|^2$ .
- 1.67** Consider the points  $P = (0, 8, 3)$ ,  $Q = (1, 3, 0)$ , and  $R = (1, 5, -2)$ . Does the triangle  $PQR$  have (A) an obtuse angle (and two acute angles); (B) a right angle (and two acute angles); or (C) three acute angles?
- 1.68** Consider the points  $P = (0, 8, 3)$ ,  $Q = (1, 3, 0)$ , and  $R = (1, 5, -2)$ , find the projection of  $\mathbf{PQ}$  in the direction of  $\mathbf{PR}$ .
- 1.69** Find the projection of  $\mathbf{u} = \langle 5, 7, -4 \rangle$  along  $\mathbf{v} = \langle 0, 0, 1 \rangle$ .
- 1.70** Find the projection of  $\mathbf{u} = \langle a, a, b \rangle$  along  $\mathbf{v} = \mathbf{i} - \mathbf{j}$ .
- 1.71** Given  $\mathbf{u} = \langle 4, -1, 2 \rangle$ , and  $\mathbf{v} = \langle 1, 0, 1 \rangle$
- Find the vectors  $\mathbf{u}_{\parallel \mathbf{v}}$  and  $\mathbf{u}_{\perp \mathbf{v}}$
  - Find the vectors  $\mathbf{v}_{\parallel \mathbf{u}}$  and  $\mathbf{v}_{\perp \mathbf{u}}$
- 1.72** Write the equation of the plane in  $\mathbb{R}^3$  with normal vector  $\mathbf{n} = \langle -1, 2, 1 \rangle$  passing through the point  $(4, 1, 5)$ .
- 1.73** Find the equation of the plane in  $\mathbb{R}^3$  passing through the points  $P = (5, 1, 1)$ ,  $Q = (1, 1, 2)$ ,  $R = (2, 1, 1)$ .
- 1.74** Find the equation of the plane in  $\mathbb{R}^3$  that contains the lines

$$\mathbf{r}_1(t) = \langle 2 + t, 2 + 3t, 3 + t \rangle$$

$$\mathbf{r}_2(s) = \langle 5 + s, 15 + 4s, 10 + 2s \rangle$$

- 1.75** Consider the plane  $P$  in  $\mathbb{R}^3$  given by the equation  $3x + 5y - 2z = 29$ . Find the equation of the plane  $R$  that is parallel to  $P$  and passes through the point  $(3, -1, 1)$ .

**Definition 1.4.41.** Let  $A$  be a nonempty subset of an inner product space  $V$ . The **orthogonal complement** of  $A$  is the subspace

$$A^\perp := \{\mathbf{v} \in V \mid \langle \mathbf{a}, \mathbf{v} \rangle = 0 \text{ for every } \mathbf{a} \in A\}$$

- 1.76** Prove that  $A^\perp$  is a vector subspace.  
 Let  $A = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  be  $k$  linearly independent vectors in a vector space  $V$  of dimension  $n$ . Prove that  $A^\perp$  is a subspace of dimension  $n - k$ .  
 Describe  $A^\perp$  as the intersection of  $k$  hyperplanes.
- 1.77** Calculate the cross product of  $\mathbf{u} = \langle 1, 1, 0 \rangle$  and  $\mathbf{v} = \langle 0, 1, 1 \rangle$

**1.78** Consider the points  $P = (0, 8, 3)$ ,  $Q = (1, 3, 0)$ , and  $R = (1, 5, -2)$ , find the area of the parallelogram spanned by  $\mathbf{PQ}$  and  $\mathbf{PR}$ .

**1.79** Find the area of the parallelogram determined by the vectors  $\langle a, 0, 0 \rangle$  and  $\langle 0, b, c \rangle$

**1.80** Consider the points  $P = (0, 8, 3)$ ,  $Q = (1, 3, 0)$ , and  $R = (1, 5, -2)$ , find the area of the triangle spanned by  $\mathbf{PQ}$  and  $\mathbf{PR}$ .

**1.81** Consider the points  $P = (-1, 8, 3)$ ,  $Q = (2, 3, 0)$ ,  $R = (2, 5, -2)$ , and  $S = (3, 5, 2)$ . Find the volume of the parallelepiped that has vertices  $P$ ,  $Q$ ,  $R$ , and  $S$ .

**1.82** Find the volume of the parallelepiped in  $\mathbb{R}^3$  spanned by the vectors  $\mathbf{u} = \langle 1, 0, 4 \rangle$ ,  $\mathbf{v} = \langle 1, 3, 1 \rangle$ ,  $\mathbf{w} = \langle -4, 2, 6 \rangle$ .

**1.83** Let  $P$  be the parallelepiped contained in the first octant (that is, all points in  $P$  satisfy  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ ) determined by the vertices  $A = (2, 4, 1)$ ,  $B = (5, 2, 5)$ ,  $C = (1, 2, 4)$ , and  $D = (1, 1, 1)$ . Find the point  $X$  in the parallelepiped  $P$  that is **furthest** from the origin. (**Hint:** Draw a picture).

**1.84** Do the points  $P = (0, 1, 1)$ ,  $Q = (1, 1, 2)$ ,  $R = (3, 3, 3)$  determine a plane or a line? Find the equation of either the plane or the line.

**1.85** Let  $P$  be the plane given by the equation  $-3x - 4y + 2z = -10$ . The plane  $P$  intersects the plane  $z = 0$  at some **acute** angle  $\theta$ . Determine the value of  $\cos(\theta)$ .

**1.86** The intersection of the plane  $-3x - 4y + 2z = -10$  and the plane  $z = 0$  is a line  $L$ . Find a vector parametrization of the line  $L$ .

**1.87** The line  $\mathbf{r}(t) = \langle 2, 2, 8 \rangle + t\langle 0, 1, 2 \rangle$  intersects the sphere  $x^2 + (y + 3)^2 + (z + 2)^2 = 9$  in two points. Find these two points.

## Chapter 2

### Analysis on $\mathbb{R}^n$

Having studied the vector space structure of  $\mathbb{R}^n$  and the geometric behavior of linear maps, we will now turn our focus to studying **non-linear** functions from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . In particular, this will lead us to study the geometric and topological properties of  $\mathbb{R}^n$ .

In section 2.1, we will first explore functions  $\mathbb{R} \rightarrow \mathbb{R}^n$  (**vector-valued functions**), as well as functions  $\mathbb{R}^n \rightarrow \mathbb{R}$  (**multivariable functions**). These turn out to be enough to capture the behavior of **general multivariable functions**  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ .

We will then study the notions of limits (section 2.2) and continuity (section 2.4) for these classes of functions, and see how it differs from limits and continuity in the single-variable case.

### 2.1 Multivariable functions

#### Vector-valued functions

We will begin by studying vector-valued functions. That is, maps of the form  $f : \mathbb{R} \rightarrow \mathbb{R}^n$ . Note that the domain of  $f$  is  $\mathbb{R}$ , so that the input of this function is a scalar. On the other hand, the codomain of  $f$  is  $\mathbb{R}^n$ , which means that the output is a point (position vector), hence the name vector-valued function.

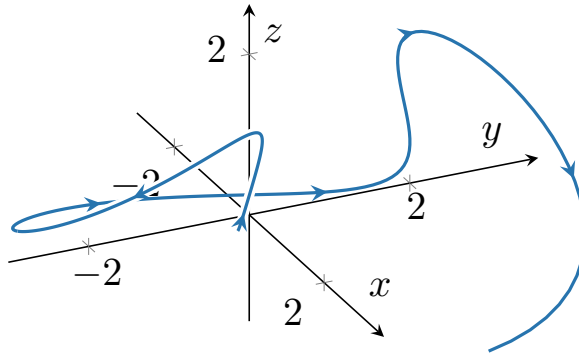
**Definition 2.1.1.** A **vector-valued function** is a function  $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ , given by

$$\mathbf{r}(t) = \langle x_1(t), x_2(t), \dots, x_n(t) \rangle = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + \dots + x_n(t)\mathbf{e}_n$$

where all of the  $x_i(t) : \mathbb{R} \rightarrow \mathbb{R}$  are single-variable functions.

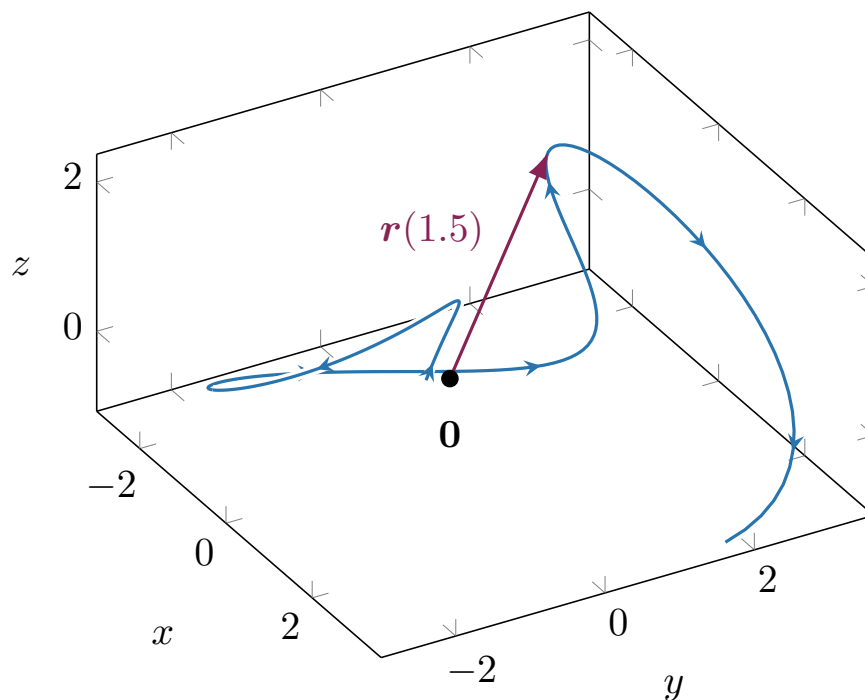
$\mathbf{r}(t)$  is also called a **parametrization** (with **parameter**  $t$ ).

In other words, vector-valued functions are precisely described by their component functions  $x_i(t)$ . We will see that this will cause vector-valued functions  $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  to behave extremely similarly to scalar functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . In some sense, a vector-valued function  $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  is simply a collection of  $n$  single-variable scalar functions.



How should we geometrically interpret vector-valued functions?

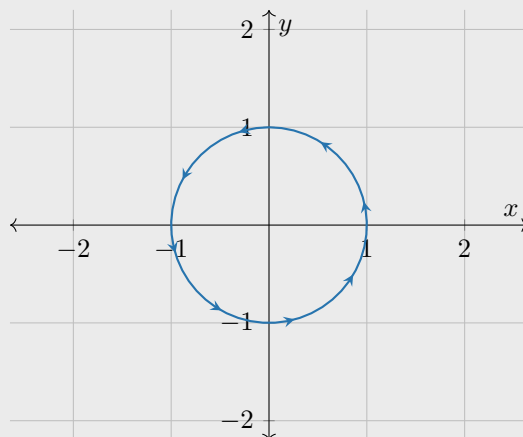
We should think of a vector-valued function  $\mathbf{r}(t)$  as describing the motion of a particle in  $\mathbb{R}^n$ . For example, let us think of the units of  $t$  as begin seconds. Then we can interpret the vector  $\mathbf{r}(0)$  as describing the position of a particle at time  $t = 0$ . We can then think of the vector  $\mathbf{r}(1)$  as describing the position of the particle 1 second after  $t = 0$ ; and similarly, the vector  $\mathbf{r}(-1)$  describes the position of the particle 1 second **before**  $t = 0$ .



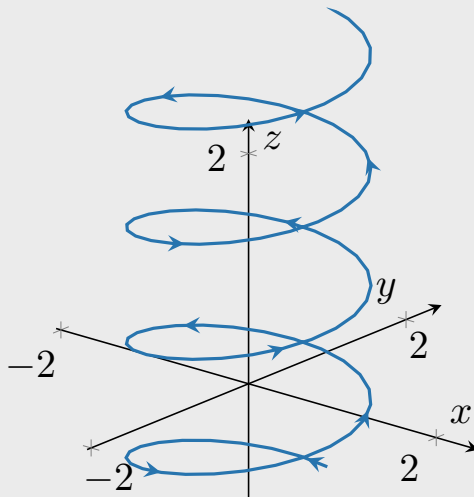
Let us turn to some examples:

A scalar function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a vector-valued function for  $n = 1$ .

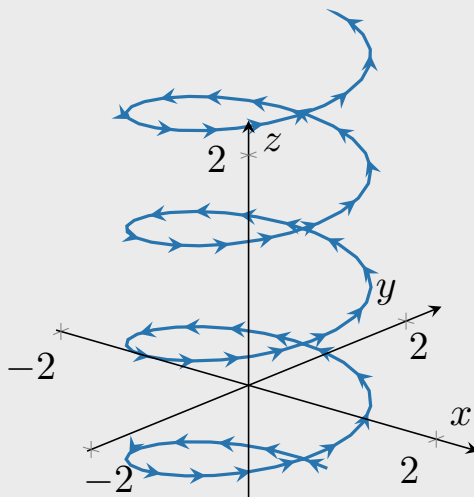
Sketch the vector-valued function  $\mathbf{c}(t) = \langle -\sin(t), \cos(t) \rangle$  in  $\mathbb{R}^2$ .



Sketch the vector-valued function  $\mathbf{h}(t) = \langle -\sin(t), \cos(t), t \rangle$  in  $\mathbb{R}^3$ .



Sketch the vector-valued function  $\mathbf{r}(s) = \langle -\sin(5s), \cos(5s), 5s \rangle$  in  $\mathbb{R}^3$ .



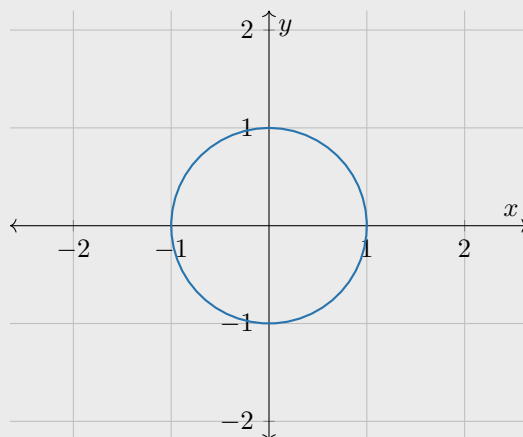
How do the two examples  $\mathbf{h}(t)$  and  $\mathbf{r}(s)$  differ from each other?

**Definition 2.1.2.** Given a parametrization  $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ , we say that its **parametric curve** is the set of points

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{r}(t)\} \subset \mathbb{R}^n$$

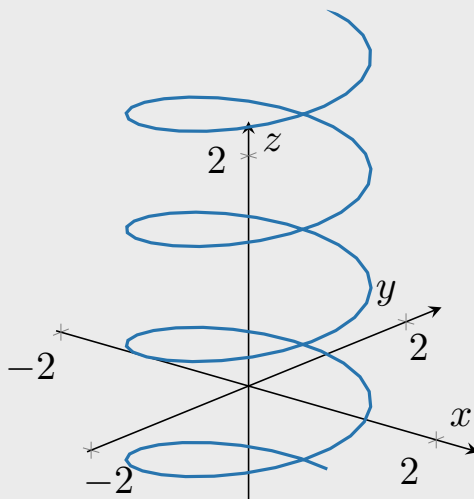
That is, we can geometrically interpret the parametric curve as being the path traced out by the parametrization  $\mathbf{r}(t)$ .

The parametric curve of  $\mathbf{c}(t) = \langle -\sin(t), \cos(t) \rangle$  is the circle of radius 1, centered at the origin. That is, it is the set of points  $\{(x, y) \mid x^2 + y^2 = 1\}$



**Remark 2.1.3.** The two examples  $\mathbf{h}(t)$  and  $\mathbf{r}(s)$  show us that there are multiple possible parametrizations of the same curve.

The parametric curve of  $\mathbf{h}(t) = \langle -\sin(t), \cos(t), t \rangle$  and  $\mathbf{r}(s) = \langle -\sin(5s), \cos(5s), 5s \rangle$  is called a helix.



In other words, the parametrization describes how fast and in what direction you walk the path, whereas the parametric curve is the path itself.

How can we sketch parametrizations and parametric curves?

Plotting points is something that a computer can do pretty easily<sup>1</sup>; however, plotting points by hand is often a very inefficient way for humans to study parametric curves. We will discuss two tools that can help us more easily sketch and understand vector-valued functions in  $\mathbb{R}^3$ .

<sup>1</sup>in fact, this is how all the graphics are plotted in these lecture notes!

### Projections

Given a vector-valued function  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ , if we isolate and study the behavior of any two components, then we can gain information about the parametrization.

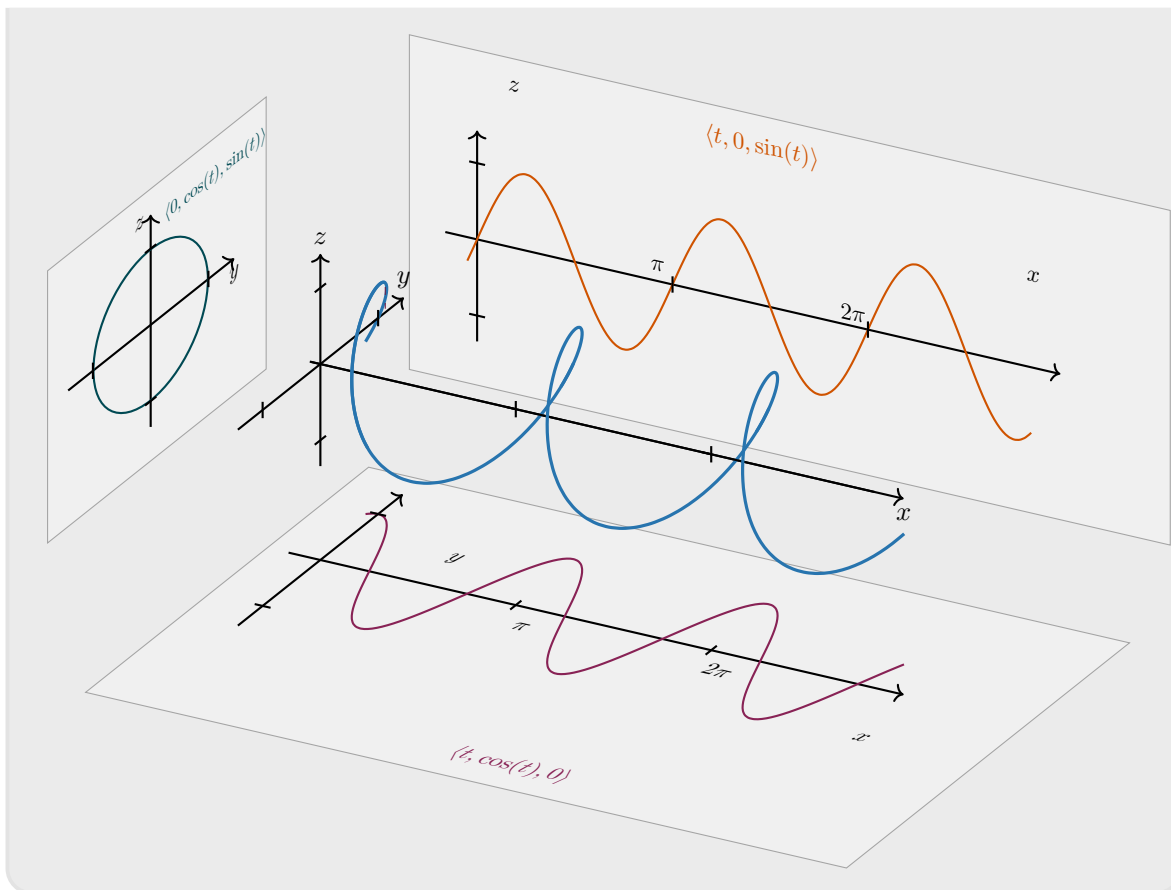
**Definition 2.1.4.** *Given a vector-valued function  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ , we can study the following **projections**:*

- The projection onto the  $xy$ -plane, which corresponds to  $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$
- The projection onto the  $xz$ -plane, which corresponds to  $\mathbf{r}(t) = \langle x(t), 0, z(t) \rangle$
- The projection onto the  $yz$ -plane, which corresponds to  $\mathbf{r}(t) = \langle 0, y(t), z(t) \rangle$

That is, we can think of these as shadows of the vector-valued function  $\mathbf{r}(t)$ .

Consider the helix parametrization  $\mathbf{h}(t) = \langle t, \cos(t), \sin(t) \rangle$  in  $\mathbb{R}^3$ .

- The projection onto the  $xy$ -plane corresponds to  $\mathbf{r}(t) = \langle t, \cos(t), 0 \rangle$ . This corresponds to a cosine curve in the  $z = 0$  plane, depicted on the bottom, in red.
- The projection onto the  $xz$ -plane corresponds to  $\mathbf{r}(t) = \langle t, 0, \sin(t) \rangle$ . This corresponds to a sine curve in the  $y = 0$  plane, depicted on the right in orange.
- The projection onto the  $yz$ -plane corresponds to  $\mathbf{r}(t) = \langle 0, \cos(t), \sin(t) \rangle$ . This describes counterclockwise motion around the unit circle centered at the origin, in the  $yz$ -plane, depicted on the left in green.



### Quadric surfaces in $\mathbb{R}^3$

Another tool we can use to graph and sketch vector-valued functions in  $\mathbb{R}^3$  are called quadric surfaces: These are certain well-studied surfaces in  $\mathbb{R}^3$ , that have been completely classified.

**Definition 2.1.5.** A *quadric surface* in  $\mathbb{R}^3$  is defined as the set of points

$$\{(x, y, z) \mid Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + ax + by + cz + d = 0\}$$

A plane of the form  $ax + by + cz + d = 0$  is a quadric surface.

Thus, studying these quadric surfaces is a generalization of our discussion of planes. We are now allowing for non-linear behavior!

[say more](#)

Let us investigate a few non-linear examples:

Let us consider the quadric surface  $C$  given by

$$C := \{(x, y, z) \mid x^2 + y^2 = 1\}$$

Observe that if we restrict our attention to the points that lie in both  $C$  and in the plane  $z = 0$ , we see that these points can be described precisely as a unit circle centered at the origin, in the  $z = 0$  plane.



$$\{(x, y, 0) \mid x^2 + y^2 = 1\}$$

Furthermore, this same picture holds if we instead consider any plane of the form  $z = k$ ! We again see a unit circle centered at the origin, but now in the  $z = k$  plane.

$$\{(x, y, k) \mid x^2 + y^2 = 1\}$$

Thus, the surface  $C$  consists of a collection (one for each  $k \in \mathbb{R}$ ) of unit circles centered at the origin, stacked on top of each other. In other words,  $C$  is an infinite cylinder of radius 1.

### define cylinders

We can use this technique in general to study the solutions to an arbitrary equation in  $\mathbb{R}^3$  [ref later](#)

**Definition 2.1.6.** The *trace in the plane*  $P$  of a subset  $S \subset \mathbb{R}^3$  is the intersection of  $S$  with  $P$ . That is,

$$S \cap P = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \in S \text{ and } \mathbf{x} \in P\}$$

We typically study  $x$ -traces,  $y$ -traces, and  $z$ -traces - that is, traces in the planes  $x = k$ ,  $y = k$ , and  $z = k$  respectively.

We can think of traces as being slices of the graph. [say more](#)

**Theorem 2.1.7.** Up to rotation and translation, there are 3 types of quadratic cylinders:

- Parabolic cylinders, which have standard form

$$y = \left(\frac{x}{a}\right)^2$$

- Elliptic cylinders, which have standard form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

- Hyperbolic cylinders, which have standard form

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

**Theorem 2.1.8.** Up to rotation and translation, there are 8 types of quadric surfaces:

- Planes, which have standard form

$$ax + by + cz = d$$

- Quadratic cylinders (described in theorem 2.1.7).
- Elliptic paraboloids, which have standard form

$$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

- Hyperbolic paraboloids, which have standard form

$$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

- *Ellipsoids, which have standard form*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

- *Elliptic cones, which have standard form*

$$\left(\frac{z}{c}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

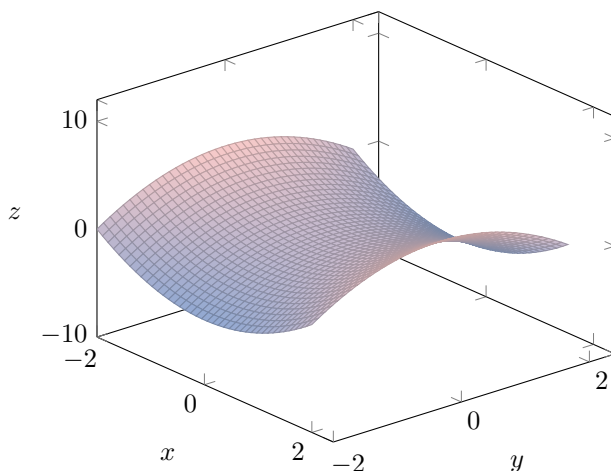
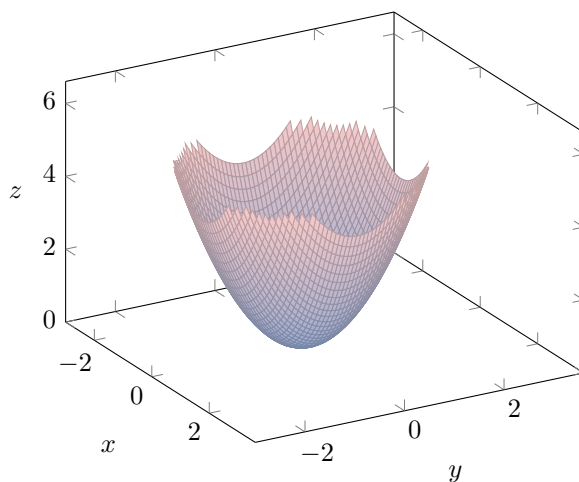
- *Hyperboloids of one sheet, which have standard form*

$$\left(\frac{z}{c}\right)^2 + 1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$$

- *Hyperboloids of two sheets, which have standard form*

$$\left(\frac{z}{c}\right)^2 - 1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

That is, given any quadric surface, corresponding to a quadratic equation  $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + ax + by + cz + d = 0$ , one can rotate it and translate it so that the quadratic equation can be written in the standard form above.



The unit sphere of radius 1 in  $\mathbb{R}^3$  can be described as the set of vectors  $\{\mathbf{v} \in \mathbb{R}^3 \mid \|\mathbf{v}\| = 1\}$ . Equivalently, the unit sphere in  $\mathbb{R}^3$  is the set of points

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

Thus, the unit sphere of radius 1 is an example of an ellipsoid!

Sketch and describe the surface

$$x = \left(\frac{y-1}{4}\right)^2 + \left(\frac{z}{3}\right)^2$$

Describe and sketch the the quadric surface  $S$  given by

$$\frac{9(x-1)^2}{4} + 4z^2 = 9y^2 + 36$$

How can we use this to sketch vector-valued functions?

Given a vector-valued function  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ , if we can find a relationship between the components of the coordinate vectors, then we know that  $\mathbf{r}(t)$  must lie on the quadric surface.

Consider the helix parametrization  $\mathbf{h}(t) = \langle -\sin(t), \cos(t), t \rangle$  in  $\mathbb{R}^3$ .

Observe that the components of  $\mathbf{h}(t)$  satisfy the equation  $x(t)^2 + y(t)^2 = 1$ . This tells us indeed that the helix lies on the cylinder  $x^2 + y^2 = 1$ .

[draw picture](#)

## Multivariable functions

We now turn to studying functions of the form  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ :

**Definition 2.1.9.** A **multivariable function** is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . We sometimes say that  $f$  is a **function of  $n$  variables**.

A scalar function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function of 1 variable.

The distance between the origin and a point  $P = (x_1, \dots, x_n)$  is a multivariable function  $d : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$d(x_1, \dots, x_n) = \sqrt{\sum_{i=1}^n x_i^2}$$

**Definition 2.1.10.** Consider a subset  $D \subset \mathbb{R}^n$ . The **indicator function**  $1_D$  is the piecewise

function defined by

$$1_D(\mathbf{x}) := \begin{cases} 1 & \text{if } \mathbf{x} \in D \\ 0 & \text{if } \mathbf{x} \notin D \end{cases}$$

Just like with single-variable functions, we might be interested in functions that are defined only on a subset  $D \subseteq \mathbb{R}^n$ . Nevertheless, the following construction shows us that it is sufficient to study the theory of multivariable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Let  $D \subset \mathbb{R}^n$ . Then a function  $f : D \rightarrow \mathbb{R}$  can be extended to a function of  $n$  variables  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  in the following way:

$$\tilde{f}(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in D \\ 0 & \text{if } \mathbf{x} \notin D \end{cases}$$

That is, to every point  $\mathbf{x} = (x_1, \dots, x_n)$  in the domain  $D$ , we assign the real number  $f(\mathbf{x}) \in \mathbb{R}$ . To every other point  $\mathbf{x} \notin D$ , we assign the value 0. In other words,  $\tilde{f}(\mathbf{x}) = f(\mathbf{x})1_D(\mathbf{x})$ .

How should we geometrically interpret multivariable functions?

Recall that given a single variable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we can draw its graph as the set of points

$$\{(x, f(x))\} \subseteq \mathbb{R}^2$$

We can generalize this to graph multivariable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ :

**Definition 2.1.11.** Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , its **graph** is the following subset of  $\mathbb{R}^{n+1}$ :

$$\Gamma_f := \{(x_1, \dots, x_n, f(x_1, \dots, x_n))\} \subset \mathbb{R}^{n+1}$$

In other words, the graph is given by the equation

$$x_{n+1} = f(x_1, \dots, x_n)$$

in  $\mathbb{R}^{n+1}$ .

We can easily sketch and visualize graphs of functions with 2 variables:

**Definition 2.1.12.** The **graph** of a function of two variables  $f(x, y)$  is the surface in  $\mathbb{R}^3$  consisting of the set of points  $(x, y, z)$  that are solutions to the equation

$$z = f(x, y)$$

How can we sketch the graph of a function of two variables?

To sketch graphs in  $\mathbb{R}^3$ , we can again use traces!

**Definition 2.1.13.** The **trace in the plane**  $P$  of a graph  $\Gamma \subset \mathbb{R}^3$  is the intersection of  $\Gamma$  with  $P$ . That is,

$$\Gamma \cap P = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \in \Gamma \text{ and } \mathbf{x} \in P\}$$

We typically study *x*-traces, *y*-traces, and *z*-traces - that is, traces in the planes  $x = k$ ,  $y = k$ , and  $z = k$  respectively.

We can think of traces as being slices of the graph. [say more](#)

**Definition 2.1.14.** The *level curves* (isoclines, contour map) of a function of two variables  $f(x, y)$  are the *z*-traces of the graph  $z = f(x, y)$ .

[picture](#)

Sketch the graph of  $f(x, y) = x^2 + y^2$ .

Sketch the graph of  $f(x, y) = \sqrt{9 - x^2 - y^2}$ .

### Implicit functions

From the previous section, we observed that some surfaces, such as the sphere in  $\mathbb{R}^3$ , are not graphs of multivariable functions. That is, the sphere cannot be described as  $z = f(x, y)$  for  $f(x, y)$  a function of 2 variables. (As we will see later, they are graphs of multivariable functions **locally**).

Nevertheless, we can still describe them using multivariable functions implicitly!

**Definition 2.1.15.** Given a multivariable function  $G(x_1, \dots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$ , its **vanishing locus** is the set of points

$$\{(x_1, \dots, x_n) \mid G(x_1, \dots, x_n) = 0\}$$

The unit sphere in  $\mathbb{R}^3$  is the vanishing locus of the function

$$G(x, y, z) = x^2 + y^2 + z^2 - 1$$

All quadric surfaces are the vanishing loci of the general quadratic equation

$$Q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + ax + by + cz + d$$

We can also describe graphs of functions as vanishing loci!

The graph of a multivariable function  $x_{n+1} = f(x_1, \dots, x_n)$  is the vanishing locus of the function

$$G(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_n) - x_{n+1}$$

Note that the vanishing locus of  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  is a subset of  $\mathbb{R}^n$ , whereas the graph of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a subset of  $\mathbb{R}^{n+1}$ ! Again, this is analogous to the difference between the kernel of a linear map, versus the image of a linear map.

### General Multivariable functions

We now turn to the general case:

**Definition 2.1.16.** A *general multivariable function* is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , defined by

$$f(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$$

That is, a general multivariable function is a vector-valued function, whose components are multivariable functions!

A vector-valued function is a general multivariable function  $f : \mathbb{R} \rightarrow \mathbb{R}^m$ .

A multivariable function is a general multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

### Exercises

**2.1** Consider the vector-valued function  $\mathbf{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$ , which parametrizes a curve called a **cycloid**. Sketch  $\mathbf{r}(t)$  from 0 to  $2\pi$ .

**2.2** Sketch and describe the vector-valued function  $\mathbf{r}(t) = \langle e^{-t} \sin(t), e^{-t} \cos(t), e^{-t} \rangle$ .

a. How many times does  $\mathbf{r}(t)$  intersect a sphere of radius 1?

**2.3** Sketch and describe the vector-valued function  $\mathbf{r}(t) = \langle t \sin(t), t \cos(t), t \rangle$ .

**2.4** Find a parametrization of the intersection of the cylinder  $x^2 + y^2 = 1$  and the plane  $x + y + z = 1$ . Sketch and describe the parametric curve.

**2.5** Find a parametrization of the intersection of the cylinder  $x^2 + y^2 = 1$  and the parabolic cylinder  $z = 4x^2$ .

Sketch and describe the parametric curve.

**2.6** Vistani's curve is the intersection of the surfaces  $x^2 + y^2 = z^2$  and  $y = z^2$ . Find a parametrization of Vistani's curve.

**2.7** Show that Vistani's curve lies on the unit sphere of radius 1, centered at  $(0, 1, 0)$ .

**2.8** Show that the vector-valued function  $\mathbf{r}(t) : \langle t^2 - 1, t - 2t^2, 4 - 6t \rangle$  lies in a plane. What is the equation of the plane?

We say that two lines  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(s)$  **intersect** if there is a point  $P$  lying on both curves. We say that  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(s)$  **collide** if there exists some  $t_0$  such that  $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$ .

**2.9** What is the difference between a point of intersection and a point of collision?

**2.10** Consider the vector-valued functions

$$\mathbf{r}_1(t) = \langle t^2 - 3, -1, t^2 \rangle \quad \mathbf{r}_2(s) = \langle 2s, 2s + 1, 2s + 3 \rangle$$

a. Do  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(s)$  **collide**? If so, find the point(s) of collision.

b. Find all values of  $t$  and  $s$  where the curves of  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(s)$  **intersect**.

**2.11** Sketch traces of the surface  $x^2 + (\frac{y}{4})^2 + z^2 = 1$ .

**2.12** Classify and sketch the quadric surface  $S$  given by

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 5z^2 - 1$$

**2.13** Classify and sketch the quadric surface  $S$  given by

$$9y^2 + 4(z - 1)^2 = \frac{9x^2}{4} + 36$$

**2.14** Sketch traces of the function  $f(x, y) = \frac{y}{x^2}$ .

**2.15** Sketch traces of the function  $f(x, y) = \frac{1}{x^2 + y^2 + 1}$ .

**2.16** Sketch traces of the function  $f(x, y) = \cos(x - y)$ .

**2.17** Consider the following equations:

$$\begin{aligned} \text{(i)} \quad & \left(\frac{y}{2}\right)^2 = \left(\frac{x}{4}\right)^2 + \frac{8}{9} \\ \text{(ii)} \quad & \left(\frac{y}{2}\right)^2 = \left(\frac{x}{4}\right)^2 + 1 \\ \text{(iii)} \quad & \left(\frac{y}{2}\right)^2 + \left(\frac{z-1}{3}\right)^2 = 1 \end{aligned}$$

- If a quadric surface  $S$  has (i) and (ii) as traces, then can we determine  $S$ ?
- If a quadric surface  $S$  has (i), (ii), and (iii) as traces, then can we determine  $S$ ?
- If (i) is a trace of a quadric surface  $9y^2 + 4(z - 1)^2 = \frac{9x^2}{4} + 36$ , what plane could the trace be in?
- If (iii) is a trace of a quadric surface  $9y^2 + 4(z - 1)^2 + 9(x - 1)^2 = 36$ , what plane could the trace be in?

**2.18** Sketch the level curves of the multivariable function  $f(x, y) = |x| + |y|$

## 2.2 Limits of sequences

Now that we have a good understanding of multivariable functions, let us turn to rigorously studying limits and continuity in multivariable calculus. We will first study limits of sequences (of **vectors**), and then turn to limits of functions (both **vector-valued** and **multivariable** functions).

First, recall the notion of a limit of sequences of real numbers:

**Definition 2.2.1.** Let  $\{a_k\}$  be a sequence of real numbers. We say that the sequence **converges** if the following holds:

For all  $\varepsilon > 0$ , there exists  $M$  such that for all  $m > M$ ,

$$|a_m - L| < \varepsilon$$

We say  $L$  is the **limit** of the sequence  $\{a_k\}$ . If no such  $L$  exists, we say  $\{a_k\}$  **diverges**.

In other words, the limit of a sequence  $\{a_k\}$  is  $L$  if for every small value  $\varepsilon$ , we can find an  $M$  such that  $a_m$  is  $\varepsilon$ -close to  $L$  for  $m > M$ .

Consider the sequence  $\{\frac{1}{k^3+1}\}$ . Then  $\lim_{k \rightarrow \infty} \frac{1}{k^3+1} = 0$ .

*Proof.* Let  $\varepsilon > 0$ . We wish to find  $M$  such that for  $m > M$ ,  $|a_m - 0| = a_m < \varepsilon$ .

Let us choose  $M > (\frac{1}{\varepsilon} - 1)^{\frac{1}{3}}$ . Then if  $m > M$ , we have that

$$m^3 > M^3 > \left(\frac{1}{\varepsilon} - 1\right)$$

Thus,

$$m^3 + 1 > \frac{1}{\varepsilon}$$

We can then deduce that for  $m > M$ ,

$$\varepsilon > \frac{1}{m^3 + 1} = a_m$$

That is, we have shown that  $\lim_{k \rightarrow \infty} \frac{1}{k^3 + 1} = 0$ .

□

### Why do we need to use $\varepsilon$ when talking about limits?

Ideally, what we would like to say is that the sequence  $\{a_k\}$  will eventually reach its limit,  $L$ . However, if we consider the sequence

$$\left\{ \frac{1}{k^3 + 1} \right\}$$

We can observe that  $\lim_{k \rightarrow \infty} \frac{1}{k^3 + 1} = 0$ , but  $\frac{1}{k^3 + 1} > 0$ .

Thus, since the sequence  $\{a_k\}$  might not ever attain its limit  $L$ , we need a different way to describe the limit:

**Proposition 2.2.2.** *Let  $x \in \mathbb{R}_{\geq 0}$ . Then the following are equivalent:*

- a. *For all  $\varepsilon > 0$ ,  $x < \varepsilon$*
- b.  *$x = 0$ .*

As a consequence, to show that  $a = b$ , it is equivalent to show that there is no number between  $a$  and  $b$ !

Consider the sequence  $\{\sum_{i=1}^k \frac{9}{10^i}\} = \{0.9, 0.99, 0.999, \dots\}$ . Then  $\lim_{k \rightarrow \infty} a_k = 1$ .  
In other words, the notion of limits explains why  $0.999 = 1$ .

Observe that in our definition, we use the term "**the** limit", which should imply that the limit of a sequence of real numbers is unique. We verify that this is true:

**Proposition 2.2.3** (Limits are unique). *Suppose  $\{a_k\}$  is a sequence of real numbers that converges to both  $a$  and  $b$ . Then  $a = b$ .*

*Proof.* By proposition 2.2.2, it is equivalent to show that for all  $\varepsilon > 0$ ,  $|a - b| < \varepsilon$ . Thus, let  $\varepsilon > 0$ .

Since  $\lim_{k \rightarrow \infty} a_k = a$ , we know that there exists  $M_a$  such that for all  $m > M_a$ ,  $|a_m - a| < \frac{\varepsilon}{2}$ . Similarly, since  $\lim_{k \rightarrow \infty} a_k = b$ , we know that there exists  $M_b$  such that for all  $m > M_b$ ,  $|a_m - b| < \frac{\varepsilon}{2}$ . Set  $M = \max\{M_a, M_b\}$ .

Observe that  $|a - b| = |a - a_m + a_m - b|$ , and we can apply the triangle inequality to observe that for all  $m > M$ ,

$$\begin{aligned} |a - b| &= |a - a_m + a_m - b| \\ &\leq |a - a_m| + |a_m - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, we have proven that for all  $\varepsilon > 0$ ,  $|a - b| < \varepsilon$ . Therefore,  $a = b$ , as desired.

□



We can now rigorously prove the limit properties that we've seen in single variable calculus.

**Theorem 2.2.4** (Properties of Limits in  $\mathbb{R}$ ). *Let  $\{a_k\}$  and  $\{b_k\}$  be sequences in  $\mathbb{R}$  that converge to  $a$  and  $b$ , respectively. Then*

- a.  $\lim_{k \rightarrow \infty} (a_k + b_k) = a + b$
- b.  $\lim_{k \rightarrow \infty} (ca_k) = ca$  for all  $c \in \mathbb{R}$ .
- c.  $\lim_{k \rightarrow \infty} (a_k b_k) = ab$ .
- d. If  $\{b_k\}, b \neq 0$ , then  $\lim_{k \rightarrow \infty} \frac{1}{b_k} = \frac{1}{b}$ .
- e. If  $\{b_k\}, b \neq 0$ , then  $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{a}{b}$ .

**Theorem 2.2.5** (Squeeze theorem for sequences). *Let  $\{a_k\}$ ,  $\{b_k\}$ , and  $\{c_k\}$  be sequences in  $\mathbb{R}$  such that there exists an  $M$  such that  $a_m \leq b_m \leq c_m$  for all  $m > M$ .*

*Then if  $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} c_k = L$ , then  $\lim_{k \rightarrow \infty} b_k = L$ .*

**Corollary 2.2.6.** *If  $\lim_{k \rightarrow \infty} |a_k| = 0$ , then  $\lim_{k \rightarrow \infty} a_k = 0$ .*

Limits of sequences of vectors

How do we generalize the notion of sequences in  $\mathbb{R}$  to sequences of vectors in  $\mathbb{R}^k$ ?

Recall that the idea of a limit of a sequence of real numbers was the following:  $L$  is the limit of a sequence  $\{a_n\}$  if for every small value  $\varepsilon$ , we can find an  $N$  such that  $a_n$  is  $\varepsilon$ -close to  $L$  for  $n > N$ .

The same idea should hold true for vectors in  $\mathbb{R}^n$  - we say that two points  $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^n$  are close to each other if the distance  $\|\mathbf{P} - \mathbf{Q}\|$  between them is small!

**Definition 2.2.7.** *Let  $\{\mathbf{a}_n\}$  be a sequence of vectors in  $\mathbb{R}^k$ . We say that the sequence  $\{\mathbf{a}_n\}$  **converges** to the vector  $\mathbf{L} \in \mathbb{R}^k$  if:*

*For all  $\varepsilon > 0$ , there exists  $M$  such that for all  $m > M$ ,*

$$\|\mathbf{a}_m - \mathbf{L}\| < \varepsilon$$

*We say  $\mathbf{L}$  is the limit of the sequence  $\{\mathbf{a}_n\}$ . If no such  $\mathbf{L}$  exists, we say that  $\{\mathbf{a}_n\}$  **diverges**.*

Another way to state the definition of limits is through topology:

**Definition 2.2.8.** *Let  $\mathbf{P} \in \mathbb{R}^n$ . The **open ball of radius  $\varepsilon$  around  $\mathbf{P}$** , denoted  $B_\varepsilon(\mathbf{P})$ , is the set of points*

$$B_\varepsilon(\mathbf{P}) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{P}\| < \varepsilon\}$$

That is,  $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{L}$  if for every  $\varepsilon > 0$ , there is some  $M$  such that  $\mathbf{a}_m \in B_\varepsilon(\mathbf{L})$ .

[picture](#)

Once again, we should confirm that the terminology "**the** limit" is correct. The key idea of the proof is the same as in proposition 2.2.3.

**Proposition 2.2.9** (Limits are unique). *Suppose  $\{\mathbf{a}_n\}$  converges to both  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$ . Then  $\mathbf{a} = \mathbf{b}$ .*

*Proof.* By proposition 2.2.2, it is equivalent to show that for all  $\varepsilon > 0$ ,  $\|\mathbf{a} - \mathbf{b}\| < \varepsilon$ . Thus, let  $\varepsilon > 0$ .

Since  $\lim_{k \rightarrow \infty} \mathbf{a}_k = \mathbf{a}$ , we know that there exists  $M_a$  such that for all  $m > M_a$ ,  $\|\mathbf{a}_m - \mathbf{a}\| < \frac{\varepsilon}{2}$ . Similarly, since  $\lim_{k \rightarrow \infty} \mathbf{a}_k = \mathbf{b}$ , we know that there exists  $M_b$  such that for all  $m > M_b$ ,  $\|\mathbf{a}_m - \mathbf{b}\| < \frac{\varepsilon}{2}$ . Set  $M = \max\{M_a, M_b\}$ .

Observe that  $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} - \mathbf{a}_m + \mathbf{a}_m - \mathbf{b}\|$ , and we can apply the triangle inequality to observe that for all  $m > M$ ,

$$\begin{aligned} \|\mathbf{a} - \mathbf{b}\| &= \|\mathbf{a} - \mathbf{a}_m + \mathbf{a}_m - \mathbf{b}\| \\ &\leq \|\mathbf{a} - \mathbf{a}_m\| + \|\mathbf{a}_m - \mathbf{b}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, we have proven that for all  $\varepsilon > 0$ ,  $\|\mathbf{a} - \mathbf{b}\| < \varepsilon$ . Therefore,  $\mathbf{a} = \mathbf{b}$ , as desired.  $\square$

**Theorem 2.2.10** (Limits in  $\mathbb{R}^m$  are determined componentwise). *Let  $\{\mathbf{a}_n\}$  be a sequence in  $\mathbb{R}^m$ , where  $\mathbf{a}_n = \langle a_{n,1}, a_{n,2}, \dots, a_{n,m} \rangle$ .*

*Then  $\{\mathbf{a}_n\}$  converges to  $\mathbf{a} = \langle a_1, \dots, a_m \rangle$  if and only if  $\{a_{n,i}\}$  converges to  $a_i$  for all  $1 \leq i \leq m$ .*

*Proof.* The key idea is the following inequalities  $|a_{n,i} - a_i| \leq \|\mathbf{a}_{n,i} - \mathbf{a}\| \leq \sum_{i=1}^m |a_{n,i} - a_i|$ .  $\square$

**Corollary 2.2.11.** *Let  $\{\mathbf{a}_n\}$  and  $\{\mathbf{b}_n\}$  be sequences in  $\mathbb{R}$  that converge to  $\mathbf{a}$  and  $\mathbf{b}$ , respectively. Then*

- a.  $\lim_{n \rightarrow \infty} (\mathbf{a}_n + \mathbf{b}_n) = \mathbf{a} + \mathbf{b}$
- b.  $\lim_{n \rightarrow \infty} (c\mathbf{a}_n) = c\mathbf{a}$  for all  $c \in \mathbb{R}$ .

### Divergent sequences

Do limits of sequences always exist?

There are two main types of sequences that diverge:

The sequence  $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$  in  $\mathbb{R}$  diverges.

The sequence  $\{n\} = \{1, 2, 3, 4, \dots\}$  in  $\mathbb{R}$  diverges.

In this section, we will investigate the behavior of these sequences, and quantify why they don't converge.

**Definition 2.2.12.** *Let  $\{\mathbf{a}_n\}$  be a sequence of vectors in  $\mathbb{R}^k$ . A **subsequence** of  $\{\mathbf{a}_n\}$  is a sequence  $\{\mathbf{b}_i\}$ , where*

$$\mathbf{b}_i = \mathbf{a}_{n_i}$$

*such that  $n_1 < n_2 < \dots < n_i < \dots$ .*

**Theorem 2.2.13.** *Let  $\{\mathbf{a}_n\}$  be a sequence of vectors in  $\mathbb{R}^k$  such that  $\{\mathbf{a}_n\}$  has two different subsequences  $\{\mathbf{a}_{n_i}\}$  and  $\{\mathbf{a}_{n_j}\}$  that converge to two different limits.*

*Then  $\{\mathbf{a}_n\}$  diverges.*

The sequence  $\{(-1)^n\}$  has the subsequences

$$\{(-1)^{2n}\} = \{1, 1, 1, \dots\}$$

and

$$\{(-1)^{2n+1}\} = \{-1, -1, -1, \dots\}$$

which converge to 1 and  $-1$ , respectively.

Hence  $\{(-1)^n\}$  diverges by theorem 2.2.13.

Can also prove it fails to follow definition.

**Theorem 2.2.14** (Convergent sequences are bounded). *Let  $\{a_n\}$  be a sequence in  $\mathbb{R}^k$  such that  $\lim_{n \rightarrow \infty} a_n = L$ .*

*Then the sequence  $\{a_n\}$  is **bounded**. That is, there exists a number  $A \in \mathbb{R}$  such that  $\|a_n\| \leq A$  for all  $n$ .*

*Proof.* We would like to say that  $A$  should be the largest value out of all the  $\|a_n\|$ . However, since the set  $\{\|a_n\|\}$  is an *infinite* set, there does not necessarily exist a maximum.<sup>2</sup>

However, we can take the maximum of a finite set! Since  $\lim_{n \rightarrow \infty} a_n = L$ , if we choose  $\varepsilon = 1$ , then by definition, there exists  $M$  such that for all  $m > M$ ,

$$\|a_m - L\| < 1$$

By the triangle inequality, this implies that

$$\|a_m\| < \|L\| + 1$$

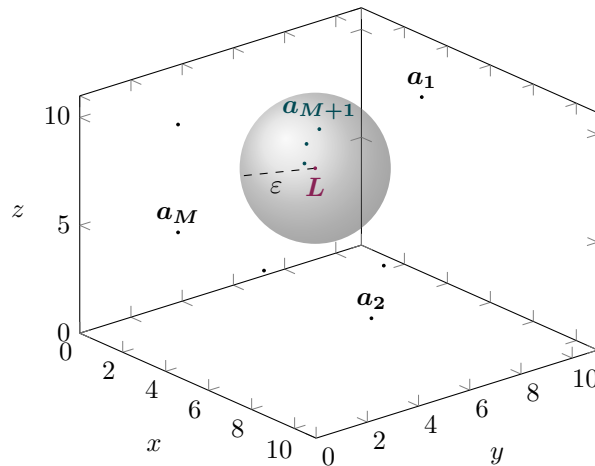
for all  $m > M$ .

Observe that the set  $\{\|a_1\|, \dots, \|a_M\|, \|L\| + 1\}$  is finite, and we can then choose

$$A = \max\{\|a_1\|, \dots, \|a_M\|, \|L\| + 1\}$$

By construction,  $\|a_n\| \leq A$  for all  $n$ , as desired. □

The essence of the proof can be summed up in the following picture:



<sup>2</sup>The sequence  $\{1 - \frac{1}{n}\}$  has no maximal element, for example. Nevertheless, it is still bounded!

All vectors  $\mathbf{a}_m$  for  $m > M$  satisfy the property that  $\|\mathbf{a}_m\| \leq \|\mathbf{L}\| + \varepsilon$ . that is, those vectors are contained inside the small ball. Outside of this ball, there are finitely many vectors. Thus, if we take  $A = \max\{\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_M\|, \|\mathbf{L}\| + \varepsilon\}$ , then we have bounded all the vectors in the sequence.

**Corollary 2.2.15.** *If  $\{\mathbf{a}_n\}$  be a sequence in  $\mathbb{R}^k$  that is unbounded, then  $\{\mathbf{a}_n\}$  diverges.*

The sequence  $\{n\}$  in  $\mathbb{R}$  is unbounded, hence it diverges.

It is not necessary that every term be unbounded - it's enough to find an unbounded subsequence!

**Theorem 2.2.16.** *Let  $\{\mathbf{a}_n\}$  be a sequence of vectors in  $\mathbb{R}^k$ . If  $\{\mathbf{a}_n\}$  has a subsequence  $\{\mathbf{a}_{n_i}\}$  that diverges, then  $\{\mathbf{a}_n\}$  diverges.*

### Exercises

**2.19** Let  $\{\mathbf{a}_n\}$  and  $\{\mathbf{b}_n\}$  be sequences in  $\mathbb{R}^n$  that converge to  $\mathbf{a}$  and  $\mathbf{b}$ , respectively. Show that

a.  $\lim_{n \rightarrow \infty} (\mathbf{a}_n \cdot \mathbf{b}_n) = \mathbf{a} \cdot \mathbf{b}$

b.  $\lim_{n \rightarrow \infty} (\|\mathbf{a}_n\|) = \|\mathbf{a}\|$

**2.20** Let  $\{\mathbf{a}_n\}$  be a sequence in  $\mathbb{R}^n$  that converges to  $\mathbf{a}$ . Show that

a.  $\lim_{n \rightarrow \infty} (\|\mathbf{a}_n\|) = \|\mathbf{a}\|$

**2.21** Why don't we have product and quotient rules for limits of two sequences  $\{\mathbf{a}_n\}$ ,  $\{\mathbf{b}_n\}$  in  $\mathbb{R}^n$ ?

**2.22** Conjecture and prove product and quotient rules for a sequence  $\{\mathbf{a}_n\}$  in  $\mathbb{R}^n$  and a sequence  $\{b_n\}$  in  $\mathbb{R}$ .

**2.23** Does the limit of the sequence  $\{\frac{(-1)^n}{n}\}$  exist? Prove your answer.

**2.24** Does the limit of the sequence  $\{\frac{n}{n+1}\}$  exist? Prove your answer.

**2.25** Let  $\{\mathbf{a}_n\}$  be a sequence of real numbers. The sequence of averages is the sequence  $\{b_n\}$  defined by

$$b_n = \frac{a_1 + \dots + a_n}{n}$$

Prove that if  $\lim_{n \rightarrow \infty} a_n = 0$ , then the sequence of averages  $\{b_n\}$  satisfies  $\lim_{n \rightarrow \infty} b_n = 0$ .

**Definition 2.2.17.** *Let  $S$  be a non-empty subset of  $\mathbb{R}$ . A real number  $x$  is called an **upper bound** of  $S$  if  $x \geq s$  for all  $s \in S$ .*

**Definition 2.2.18.** *Let  $S$  be a non-empty subset of  $\mathbb{R}$ . A real number  $x$  is called the **least upper bound** (alternatively, **supremum**) of  $S$  if*

a.  $x$  is an upper bound for  $S$

b. If  $y$  is an upper bound for  $S$ , then  $x \leq y$

We often write the least upper bound as  $\sup(S)$

It turns out that if  $S$  is a non-empty subset of  $\mathbb{R}$ , then if  $S$  has an upper bound, then  $S$  has a least upper bound. (see theorem 4.1.18)

**Definition 2.2.19.** Let  $S$  be a non-empty subset of  $\mathbb{R}$ . A real number  $x$  is called a **lower bound** of  $S$  if  $x \leq s$  for all  $s \in S$ .

**Definition 2.2.20.** Let  $S$  be a non-empty subset of  $\mathbb{R}$ . A real number  $x$  is called the **greatest lower bound** (alternatively, **infimum**) of  $S$  if

- a.  $x$  is a lower bound for  $S$
- b. If  $y$  is a lower bound for  $S$ , then  $x \geq y$

We often write the greatest lower bound as  $\inf(S)$

It turns out that if  $S$  is a non-empty subset of  $\mathbb{R}$ , then if  $S$  has a lower bound, then  $S$  has a greatest lower bound.

**Definition 2.2.21.** A sequence of real numbers  $\{a_n\}$  is said to be **increasing** (resp. **decreasing**) if  $a_n \leq a_{n+1}$  (resp.  $a_n \geq a_{n+1}$ ) for all  $n$ .

**2.26** Prove that if a sequence of real numbers  $\{a_n\}$  is increasing and bounded above, then  $\lim_{n \rightarrow \infty} a_n = \sup(\{a_n\})$

**2.27** Prove that if a sequence of real numbers  $\{a_n\}$  is decreasing and bounded below, then  $\lim_{n \rightarrow \infty} a_n = \inf(\{a_n\})$

**2.28** Prove the following theorem:

Let  $\{a_n\}$  be an increasing or a decreasing sequence. Then  $\{a_n\}$  converges if and only if  $\{a_n\}$  is bounded.

## 2.3 Limits of functions

Now let us turn to limits of functions! Let us first recall the notion of limits of single-variable functions:

**Definition 2.3.1.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has the **limit**  $b$  at  $a$  if

For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in \mathbb{R}$ ,

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - b| < \varepsilon$$

We write  $\lim_{x \rightarrow a} f(x) = b$ .

Since  $\mathbb{R}$  is a well-ordered field (in other words, since  $\mathbb{R}$  is a complete metric space), we can turn this into a statement about left limits and right limits.

left and right picture

**Definition 2.3.2** (Limit of a general multivariable function). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  has the **limit**  $b$  at  $a$  if

For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in \mathbb{R}^n$ ,

$$0 < \|x - a\| < \delta \quad \text{implies} \quad \|f(x) - b\| < \varepsilon$$

We write  $\lim_{x \rightarrow a} f(x) = b$ .

**Proposition 2.3.3** (Limits of functions are unique). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and suppose  $\lim_{x \rightarrow a} f(x) = b_1$  and  $\lim_{x \rightarrow a} f(x) = b_2$ . Then  $b_1 = b_2$ .*

We will study this for the two different classes of functions (vector-valued functions  $\mathbb{R} \rightarrow \mathbb{R}^n$  and multivariable functions  $\mathbb{R}^n \rightarrow \mathbb{R}$ ) we've seen. This turns out to be enough, as we see in section 2.3.

#### Limits of vector valued functions

**Definition 2.3.4** (Limit of a vector-valued function). *A function  $f : \mathbb{R} \rightarrow \mathbb{R}^m$  has the **limit**  $b$  at  $a$  if*

*For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in \mathbb{R}$ ,*

$$0 < |x - a| < \delta \quad \text{implies} \quad \|f(x) - b\| < \varepsilon$$

**Theorem 2.3.5** (Limits of vector-valued functions are determined componentwise). *For any vector valued function  $\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle$ ,*

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \langle x_1, \dots, x_n \rangle$$

*if and only if for all  $1 \leq i \leq n$ ,*

$$\lim_{t \rightarrow a} x_i(t) = x_i$$

*Proof.* The proof is essentially the same as the proof of theorem 2.2.10:

The key idea is again the following inequalities

$$|x_i(t) - x_i| \leq \|\mathbf{r}(t) - \langle x_1, \dots, x_n \rangle\| \leq \sum_{i=1}^m |x_i(t) - x_i|$$

□

picture

**Theorem 2.3.6.** *Let  $\mathbf{r}, \mathbf{s} : \mathbb{R} \rightarrow \mathbb{R}^m$  be vector-valued functions. Suppose that  $\lim_{t \rightarrow a} \mathbf{r}(t)$  and  $\lim_{t \rightarrow a} \mathbf{s}(t)$  exist. Then*

a. **Sum Law:**

$$\lim_{t \rightarrow a} (\mathbf{r}(t) + \mathbf{s}(t)) = \lim_{t \rightarrow a} \mathbf{r}(t) + \lim_{t \rightarrow a} \mathbf{s}(t)$$

b. **Scalar Multiple Law:**

$$\lim_{t \rightarrow a} \lambda(\mathbf{r}(t)) = \lambda \left( \lim_{t \rightarrow a} \mathbf{r}(t) \right)$$

#### Limits of multivariable functions

**Definition 2.3.7** (Limit of a multivariable function). *A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has the **limit**  $b$  at  $a$  if*

*For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $\mathbf{x} \in \mathbb{R}^n$ ,*

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \quad \text{implies} \quad |f(\mathbf{x}) - b| < \varepsilon$$

**Theorem 2.3.8.** Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  be functions of  $n$  variables. Suppose that  $\lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x})$  and  $\lim_{\mathbf{x} \rightarrow \mathbf{P}} g(\mathbf{x})$  exist. Then

a. **Sum Law:**

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} (f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x}) + \lim_{\mathbf{x} \rightarrow \mathbf{P}} g(\mathbf{x})$$

b. **Scalar Multiple Law:**

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} \lambda(f(\mathbf{x})) = \lambda \left( \lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x}) \right)$$

c. **Product Law:**

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} (f(\mathbf{x})g(\mathbf{x})) = \left( \lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x}) \right) \left( \lim_{\mathbf{x} \rightarrow \mathbf{P}} g(\mathbf{x}) \right)$$

d. **Quotient Law:** If  $\lim_{\mathbf{x} \rightarrow \mathbf{P}} g(\mathbf{x}) \neq 0$ ,

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x})}{\lim_{\mathbf{x} \rightarrow \mathbf{P}} g(\mathbf{x})}$$

This theorem allows us to reduce to studying limits of single-variable functions.

Let  $f(x, y) = x$ . Then

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = a$$

*Proof.* Let  $\varepsilon > 0$ . We wish to show that there exists  $\delta > 0$  such that if  $0 < \|(x, y) - (a, b)\| < \delta$ , then  $|x - a| < \varepsilon$ .

Observe that

$$|x - a| = \sqrt{(x - a)^2} \leq \sqrt{(x - a)^2 + (y - b)^2} = \|(x, y) - (a, b)\|$$

Therefore, choosing  $\delta = \varepsilon$ , we see that if  $\|(x, y) - (a, b)\| < \varepsilon$ , then  $|x - a| < \varepsilon$ . □

Let  $f(x, y) = \frac{x \sin(y)}{e^x + 1}$ . Then

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \frac{a \sin(b)}{e^a + 1}$$

Observe that theorem 2.3.8 allows us to evaluate the limits, except when we possibly divide by zero.

When do limits of multivariable functions **not** exist?

**Definition 2.3.9.** Let  $X \subset \mathbb{R}^n$ . We say that a point  $\mathbf{p} \in \mathbb{R}^n$  is a **limit point of  $X$**  if there is a sequence  $\{\mathbf{a}_n\}$  contained inside  $X$  such that  $\{\mathbf{a}_n\}$  converges to  $\mathbf{p}$ .

**Theorem 2.3.10.** Let  $X \subset \mathbb{R}^n$ ,  $f : X \rightarrow \mathbb{R}^m$  a function, and  $\mathbf{a}$  a limit point of  $X$ . Then the following statements are equivalent:

a.  $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{b}$

b. For every sequence  $\{\mathbf{a}_n\}$  converging to  $\mathbf{a}$  (with  $\mathbf{a}_n \neq \mathbf{a}$ ), the sequence  $\{f(\mathbf{a}_n)\}$  converges to  $\mathbf{b}$ .

In other words, in order for a limit of a multivariable function to exist, it must yield the same value along all possible approaches.

Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2+y^2}$  does not exist.

Consider the paths  $\mathbf{r}_1(t) = \langle 0, t \rangle$ , and  $\mathbf{r}_2(t) = \langle t, t \rangle$ .

What other tools can we use to prove limits exist?

It's often difficult to use epsilon-delta arguments to prove that limits exist (especially limits involving non-polynomial functions like  $\sin(x)$ ,  $\cos(x)$ ,  $e^x$ , etc.). Thus, we would like other ways to prove that limits exist.

In these notes, we will discuss two tools that one can use:

**Theorem 2.3.11** (Squeeze Theorem). *Let  $f(\mathbf{x})$ ,  $g(\mathbf{x})$ , and  $h(\mathbf{x})$  be functions of  $n$  variables such that*

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x}) = L = \lim_{\mathbf{x} \rightarrow \mathbf{P}} h(\mathbf{x})$$

*If there exists  $\delta > 0$  such that for all  $\mathbf{x} \in B_\delta(\mathbf{P}) - \{\mathbf{P}\}$ , we have that*

$$f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x})$$

*Then  $\lim_{\mathbf{x} \rightarrow \mathbf{P}} g(\mathbf{x}) = L$ .*

Consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} x \sin\left(\frac{1}{x^2+y^2}\right)$$

Observe that for all  $\theta$ ,  $|\sin(\theta)| \leq 1$ . Therefore, it follows that  $|x \sin(\frac{1}{x^2+y^2})| \leq |x|$ . Thus,

$$-|x| \leq x \sin\left(\frac{1}{x^2+y^2}\right) \leq |x|$$

Since we know that  $\lim_{(x,y) \rightarrow (0,0)} -|x| = \lim_{(x,y) \rightarrow (0,0)} |x| = 0$ , by the squeeze theorem it follows that

$$\lim_{(x,y) \rightarrow (0,0)} x \sin\left(\frac{1}{x^2+y^2}\right) = 0$$

We can also use polar coordinates to compute limits of functions in two variables. The key idea is the following proposition:

**Proposition 2.3.12.** *If we set  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ , then  $(x, y)$  approaches  $(0, 0)$  if and only if  $r$  approaches 0.*

So if we have a limit of a function of two-variables  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ , we can often use polar coordinates to **try** to reduce our limit to a single-variable limit, by considering the function

$$g(r, \theta) = f(r \cos(\theta), r \sin(\theta))$$

and studying the limit

$$\lim_{r \rightarrow 0} g(r, \theta)$$

Consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$$



Upon converting to polar coordinates, we obtain

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} &= \lim_{r \rightarrow 0} \frac{(r \cos(\theta))^2 r \sin(\theta)}{(r \cos(\theta))^2 + (r \sin(\theta))^2} \\ &= \lim_{r \rightarrow 0} \frac{r^3 \cos^2(\theta) \sin(\theta)}{r^2} \\ &= \lim_{r \rightarrow 0} r \cos^2(\theta) \sin(\theta)\end{aligned}$$

Observe that since both  $|\cos(\theta)| \leq 1$  and  $|\sin(\theta)| \leq 1$ , we can conclude that  $|r \cos^2(\theta) \sin(\theta)| \leq |r|$ . Thus,

$$\lim_{(x,y) \rightarrow (0,0)} \left| \frac{x^2 y}{x^2 + y^2} \right| \leq \lim_{r \rightarrow 0} |r| = 0$$

So by the squeeze theorem, we can conclude that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0$$

However, converting to polar coordinates will not always be successful, as you will see in the exercises below [REFERENCE](#). A precise statement about limits in polar coordinates is as follows:

**Theorem 2.3.13** (Limits using polar coordinates). *Let  $f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function of two variables, which we can express in polar coordinates as  $g(r, \theta) := f(r \cos(\theta), r \sin(\theta))$ . Then*

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L$$

*if and only if there exists  $\delta > 0$  and a function  $h : \mathbb{R} \rightarrow \mathbb{R}$  such that*

- *If  $0 < r < \delta$ , then  $|g(r, \theta) - L| \leq h(r)$  for all  $\theta$ , AND*
- $\lim_{r \rightarrow 0} h(r) = 0$

**Corollary 2.3.14.** *If  $\lim_{r \rightarrow 0} g(r, \theta)$  depends on  $\theta$ , then the value of the limit will differ for different straight line paths. Thus,  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.*

### Limits of general multivariable functions

Now that we have studied limits of both [vector-valued functions](#)  $\mathbb{R} \rightarrow \mathbb{R}^n$  and [multivariable functions](#)  $\mathbb{R}^n \rightarrow \mathbb{R}$ , this turns out to capture the behavior of limits of general multivariable functions  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ !

The key idea is the following theorem:

**Theorem 2.3.15** (Limits of general multivariable functions are determined componentwise). *For any general multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , defined by*

$$f(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$$

*Then*

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x}) = \langle b_1, \dots, b_m \rangle$$

*if and only if for all  $1 \leq i \leq m$ ,*

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} f_i(\mathbf{x}) = b_i$$

*Proof.* The proof idea is exactly the same as theorem [2.3.5](#). □

Thus, it is indeed enough to study limits of multivariable functions  $\mathbb{R}^n \rightarrow \mathbb{R}$ .

## Exercises

**2.29** Show that  $\lim_{(x,y) \rightarrow (0,0)} x \sin\left(\frac{1}{x^2+y^2}\right) = 0$ .

**2.30** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{x^2+y^2}$  does not exist.

**2.31** Use the squeeze theorem to compute  $\lim_{(x,y) \rightarrow (0,0)} \frac{5xy^2}{x^2+y^2}$ .

**2.32** Show that the following limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + 3y^4}$$

**2.33**  $\lim_{(x,y) \rightarrow (1,2)} x^2 + y$  exist? Prove or disprove your claim.

**2.34** Does  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$  exist? Prove or disprove your claim.

**2.35** Does  $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$  exist? Prove or disprove your claim.

**2.36** Does  $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xyz}{x^3+y^3+z^3}$  exist? Prove or disprove your claim.

**2.37** Consider the function  $f(x, y) = \begin{cases} \cos(x) & \text{if } x^2 + y^2 = \frac{1}{n} \text{ for some } n = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$

Compute  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  or show that it does not exist.

**2.38** Consider the function

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \neq (0, 0) \text{ with } x + y \text{ rational} \\ 0 & \text{if } (x, y) \neq (0, 0) \text{ with } x + y \text{ irrational} \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

Compute  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  or show that it doesn't exist.

**2.39** We can use polar coordinates to compute limits of functions in two variables approaching the point  $(0, 0)$ . How can we modify the polar coordinate method to compute limits of functions in two variables approaching the point  $(a, b)$ ?

**2.40** Use polar coordinates to show that the limit  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2+y^2}$  does not exist.

**2.41** Convert the limit  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2}$  to polar coordinates, and show that for all  $\theta$ ,  $\lim_{r \rightarrow 0} g(r, \theta) = 0$ .

**2.42** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2}$  doesn't exist by finding two paths  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(t)$ , passing through the origin, such that the limit of  $f(x, y)$  along these two paths have different values.

**2.43** Why do your answers in exercises 2.41 and 2.42 not contradict each other? How do they relate to theorem 2.3.13?

## 2.4 Continuity

Let us now turn to continuity. Let us first recall the notion of continuity for a single-variable function:

**Definition 2.4.1.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $a \in \mathbb{R}$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

That is, the limit of the function at  $a$  is precisely the value of the function at  $a$ .

**picture**

The idea for general multivariable functions is identical:

**Definition 2.4.2.** A general multivariable function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is **continuous** at a point  $\mathbf{P} \in \mathbb{R}^m$  if

$$\lim_{\mathbf{x} \rightarrow \mathbf{P}} f(\mathbf{x}) = f(\mathbf{P})$$

$f$  is said to be continuous on a subset  $X \subset \mathbb{R}^m$  if it is continuous for every  $\mathbf{P} \in X$ .

That is, the limit of the function at  $\mathbf{P}$  is precisely the value of the function at  $\mathbf{P}$ .

Thus, questions about continuity are really questions about limits: To show that a function is continuous, you need to show that the limit exists, and then check that the limit matches the value of the function.

Thus, all of our theorems about limits immediately yield theorems about continuity.

**Theorem 2.4.3.** Suppose  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$  are general multivariable functions that are **continuous** at a point  $\mathbf{P}$ . Then

- a. **Sum Law:**  $f + g$  is continuous at  $\mathbf{P}$ .
- b. **Scalar Multiple Law:**  $\lambda f$  is continuous at  $\mathbf{P}$ .

Vector-valued functions

**Proposition 2.4.4.** For any vector valued function

$$\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle$$

$\mathbf{r}(t)$  is continuous at  $P$  if and only if for all  $1 \leq i \leq n$ ,  $x_i(t)$  is continuous at  $P$ .

Multivariable functions

**Theorem 2.4.5** (Properties of continuity of multivariable functions). Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  be functions of  $n$  variables. Suppose that  $f(\mathbf{x})$  and  $g(\mathbf{x})$  are continuous at  $\mathbf{P}$ .

- a. **Product Law:**  $f(\mathbf{x})g(\mathbf{x})$  is continuous at  $\mathbf{P}$ .
- b. **Quotient Law:** If  $g(\mathbf{P}) \neq 0$ ,  $\frac{f(\mathbf{x})}{g(\mathbf{x})}$  is continuous at  $\mathbf{P}$ .

The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given below is continuous for all  $(x, y) \in \mathbb{R}^2$ .

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x, y) \neq \mathbf{0} \\ 0 & \text{if } (x, y) = \mathbf{0} \end{cases}$$

*Proof.* By the quotient law,  $f$  is continuous for all  $(x, y) \neq \mathbf{0}$ . We thus only need to show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2} = 0$

We will show this directly, but we could have also used the squeeze theorem, or the polar coordinates method.

Let  $\varepsilon > 0$ . We need to find  $\delta > 0$  such that if  $0 < \|(x, y)\| < \delta$ , then

$$\left| \frac{xy^2}{x^2+y^2} - 0 \right| < \varepsilon$$

Observe that since  $x^2 > 0$ , then

$$\left| \frac{xy^2}{x^2+y^2} \right| \leq \left| \frac{xy^2}{y^2} \right| = |x|$$

Thus, if we choose  $\delta = \varepsilon$ , then we know that  $|x| \leq ||\langle x, y \rangle|| < \varepsilon$ . Hence  $|\frac{xy^2}{x^2+y^2}| \leq |\frac{xy^2}{y^2}| = |x| < \varepsilon$ , as desired.  $\square$

**Theorem 2.4.6** (The composition of continuous functions is continuous.). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and let  $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$  be multivariable functions such that  $f$  is continuous at  $\mathbf{P} \in \mathbb{R}^n$ , and  $g$  is continuous at  $f(\mathbf{P}) \in \mathbb{R}^m$ .*

*Then  $g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is continuous at  $\mathbf{P} \in \mathbb{R}^n$ .*

*Proof.* **do proof**  $\square$

**Theorem 2.4.7.**  $\det : M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} \rightarrow \mathbb{R}$  is continuous.

**Corollary 2.4.8.** The cross product  $\times : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is continuous.

**Theorem 2.4.9.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then  $T$  is continuous (in fact,  $T$  is uniformly continuous).

**Definition 2.4.10.** A multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is **uniformly continuous** if

For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $\mathbf{a}, \mathbf{b} \in X$ ,

$$0 < ||\mathbf{a} - \mathbf{b}|| < \delta \quad \text{implies} \quad ||f(\mathbf{a}) - f(\mathbf{b})|| < \varepsilon$$

#### Exercises

**2.44** Consider the function  $f(x, y) = \begin{cases} y^2 \sin(\frac{1}{(x-4)^2+y^2}) & \text{if } (x, y) \neq (4, 0) \\ c & \text{otherwise} \end{cases}$

Either show that there is a  $c$  that will make  $f$  continuous at  $(4, 0)$ , or show that  $f$  will not be continuous for any choice of  $c$ .

**2.45** Consider the function  $f(x, y) = \begin{cases} \frac{x^2 + \sin(y)}{2x^2 + y + e^{xy} - 1} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{otherwise} \end{cases}$

Is  $f$  continuous at  $(0, 0)$ ?

**2.46** Consider the function  $f(x, y) = \begin{cases} \frac{x^2 + \sin(xy) + y^2}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ c & \text{otherwise} \end{cases}$

Either show that there is a  $c$  that will make  $f$  continuous, or show that  $f$  will not be continuous for any choice of  $c$ .

**2.47** Consider the function  $f(x, y) = \begin{cases} \frac{y^5}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{otherwise} \end{cases}$

Is  $f$  continuous at  $(0, 0)$ ?

**2.48** Consider the function  $g(x, y) = \begin{cases} x^2 \cos(\frac{1}{x^2+y^2}) & \text{if } (x, y) \neq (0, 0) \\ c & \text{otherwise} \end{cases}$

Either show that there is a  $c$  that will make  $g$  continuous, or show that  $g$  will not be continuous for any choice of  $c$ .

**2.49** Consider the function  $f(x, y) = \begin{cases} \frac{x^2}{x^2+|y|^3} & \text{if } (x, y) \neq (0, 0) \\ c & \text{otherwise} \end{cases}$

Either show that there is a  $c$  that will make  $f$  continuous at  $(0, 0)$ , or show that  $f$  will not be continuous for any choice of  $c$ .

**2.50** Consider the function  $f(x, y) = \begin{cases} x \sin(\frac{1}{x^2+y^2}) & \text{if } (x, y) \neq (0, 0) \\ c & \text{otherwise} \end{cases}$

Either show that there is a  $c$  that will make  $f$  continuous at  $(0, 0)$ , or show that  $f$  will not be continuous for any choice of  $c$ .

**2.51** Consider the function  $g(x, y) = \begin{cases} \frac{x^4+2x^2y^2+y^4}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ c & \text{otherwise} \end{cases}$

Either show that there is a  $c$  that will make  $g$  continuous at  $(0, 0)$ , or show that  $g$  will not be continuous for any choice of  $c$ .

**Definition 2.4.11.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to be **Lipschitz** if there exists a constant  $K \geq 0$  such that for all  $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ ,

$$\|f(\mathbf{x}_1) - f(\mathbf{x}_2)\| \leq K\|\mathbf{x}_1 - \mathbf{x}_2\|$$

**2.52** Prove that if a function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is Lipschitz, then  $f$  is continuous.

## Chapter 3

### The Multivariable Derivative

Now that we have understood the notions of limits and continuity, we can turn to defining the notion of the multivariable derivative.

Let us first recall the notion of the derivative of a single-variable function:

**Definition 3.0.1.** A function  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$  is **differentiable** at a point  $x_0 \in D$  if

- There exists  $\delta$  such that  $B_\delta(x_0) \subseteq D$
- The following limit exists:

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = L$$

If  $f$  is differentiable at  $x_0$ , we say that  $L$  is **the derivative of  $f$  at  $x_0$** , and we write

$$f'(x_0) := L$$

(sometimes written  $\frac{df}{dx}(x_0) = L$ )

In other words, the derivative depends precisely on the local behavior of  $f(x)$  near  $x_0$ . That is, we look precisely at the behavior of  $f(x)$  in a small neighborhood  $B_\varepsilon(x_0)$ .

**interior point**

The notion of the derivative in single-variable calculus has many interpretations:

- The derivative  $f'(a)$  tells us the slope of the tangent line to the graph  $y = f(x)$  at the point  $x = a$ .
- The derivative  $f'(a)$  tells us the instantaneous rate of change of  $f(x)$  at the point  $x = a$ .
- The derivative  $\frac{d}{dx}$  is an operator, whose input is a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and whose output is a function  $f' : \mathbb{R} \rightarrow \mathbb{R}$ .
- The derivative is the linear approximation of  $f(x)$  in a small neighborhood  $B_\varepsilon(x_0)$ .

We will see how all of these various notions generalize to the case of multivariable functions.

### 3.1 The Multivariable Derivative

It takes some thought to come up with the correct definition of the derivative of a multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  at a point  $\mathbf{x}_0$ !

For instance, one might naïvely define the multivariable derivative as the following limit:

$$f'(\mathbf{x}_0) \text{ “=” } \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)}{\|\mathbf{h}\|}$$

However, this definition **fails** for multiple reasons:

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function  $f(x, y) = x$ , and consider the point  $\mathbf{x}_0 = (0, 0)$ .

We see that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - f(0, 0)}{\|\langle x, y \rangle\|}$$

**does not exist** by considering the following sequences:

- Along the sequence  $\mathbf{x}_n = \langle \frac{1}{n}, 0 \rangle$ , we have that

$$\lim_{n \rightarrow \infty} \frac{f(\frac{1}{n}, 0)}{\|\langle \frac{1}{n}, 0 \rangle\|} = 1$$

- Along the sequence  $\mathbf{y}_n = \langle \frac{1}{n}, 0 \rangle$ , we have that

$$\lim_{n \rightarrow \infty} \frac{f(\frac{-1}{n}, 0)}{\|\langle \frac{-1}{n}, 0 \rangle\|} = -1$$

- Along the sequence  $\mathbf{z}_n = \langle 0, \frac{1}{n} \rangle$ , we have that

$$\lim_{n \rightarrow \infty} \frac{f(0, \frac{1}{n})}{\|\langle 0, \frac{1}{n} \rangle\|} = 0$$

Moreover, this definition also fails because it does not correctly generalize the notion of the single variable derivative:

Specializing the naïve definition to  $n = 1$ , we obtain  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{|x|}$ .

However, if we consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = |x|$ . We know from single-variable calculus that this function is not differentiable at  $x = 0$ .

Nevertheless, we see that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{|x|} = 1$$

**Remark 3.1.1.** One can correct for the issue in 3.1 by instead considering

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)|}{\|\mathbf{h}\|}$$

Nevertheless, the issue in 3.1 continues to exist for the  $f(x, y) = x$  at  $(0, 0)$  by considering the sequences  $\mathbf{x}_n = \langle \frac{1}{n}, 0 \rangle$  and  $\mathbf{z}_n = \langle 0, \frac{1}{n} \rangle$ .

Thus, we must make a paradigm shift in how we think of the multivariable derivative:

How should we interpret the multivariable derivative?

The derivative of a single variable function  $f(x)$  at a point  $x_0$  is the **approximation** of  $f(x)$  by a linear transformation at  $x_0$ .

**Definition 3.1.2.** A single variable function is **differentiable at**  $x_0$  if there exists a linear transformation  $T : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\lim_{h \rightarrow 0} \left| \frac{f(x_0 + h) - f(x_0) - T(h)}{h} \right| = 0$$

Observe that by our characterization of linear transformations  $T : \mathbb{R} \rightarrow \mathbb{R}$ , correspond to scalars  $m \in \mathbb{R}$  via the correspondence

$$T(x) = mx$$

We define the derivative of  $f$  at  $x_0$  to be  $m$ . That is,

$$f'(x_0) := m$$

We can generalize this definition as follows:

**Definition 3.1.3.** A multivariable function  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is **differentiable** at an interior point  $\mathbf{x}_0$  of  $A$  if there exists a linear transformation  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

The **derivative** of  $f$  at  $x_0$  is the linear transformation

$$Df(\mathbf{x}_0) := T$$

By our characterization of linear transformations,  $Df(\mathbf{x}_0) : \mathbb{R}^m \rightarrow \mathbb{R}^n$  corresponds to a matrix  $[Df(\mathbf{x}_0)] \in M_{n \times m}(\mathbb{R})$

**Remark 3.1.4.** The derivative of a function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  at a point  $x_0$  is the **approximation** of  $f$  by a **linear transformation**  $Df(\mathbf{x}_0)$  at  $x_0$ .

**Proposition 3.1.5** (Uniqueness). Suppose  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is differentiable at  $\mathbf{x}_0$ .

Then the derivative  $Df(\mathbf{x}_0) : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is unique.

The constant map  $c : \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined by  $c(\mathbf{x}) = \mathbf{a}$  is differentiable everywhere, and  $Dc(\mathbf{x}_0) = 0$ .

A linear map  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is differentiable everywhere, and  $DT(\mathbf{x}_0) = T$ .

**Proposition 3.1.6.** Suppose that  $f, g : A \rightarrow \mathbb{R}^n$  are differentiable at  $x_0 \in A^\circ$ . Then  $f + g$  and  $\lambda f$  are differentiable at  $x_0 \in A^\circ$ .

**Proposition 3.1.7** (Differentiability implies continuity). Suppose  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is differentiable at  $\mathbf{x}_0$ .

Then  $f$  is continuous at  $\mathbf{x}_0$ .

**Theorem 3.1.8** (The Chain rule). Suppose that  $g : \mathbb{R}^k \rightarrow \mathbb{R}^m$  is differentiable at  $x_0 \in \mathbb{R}^k$ , and  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is differentiable at  $g(\mathbf{x}_0) \in \mathbb{R}^m$ .

Then  $f \circ g : \mathbb{R}^k \rightarrow \mathbb{R}^n$  is differentiable at  $x_0 \in \mathbb{R}^k$ , and

$$D(f \circ g)(\mathbf{x}_0) = D(f)(g(\mathbf{x}_0)) \circ Dg(\mathbf{x}_0)$$



The derivative of a vector valued function

**Theorem 3.1.9.** Let  $\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle$  be a vector-valued function.

The **derivative** of  $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  at an interior point  $\mathbf{t}_0$  is the linear transformation  $T : \mathbb{R} \rightarrow \mathbb{R}^n$  given by

$$T = \begin{bmatrix} x'_1(t_0) \\ \vdots \\ x'_n(t_0) \end{bmatrix}$$

We sometimes write this derivative as  $\mathbf{r}'(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ .

**Theorem 3.1.10.** A vector-valued function  $\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle$  is differentiable if and only if the functions  $x_i(t)$  are differentiable.

$$\mathbf{r}'(t) = \langle x'_1(t), \dots, x'_n(t) \rangle$$

We can think of the derivative  $\mathbf{r}'(t)$  as a tangent vector to the parametric curve of  $\mathbf{r}(t)$ .

**Definition 3.1.11.** The **tangent line** at  $\mathbf{r}(t_0)$  is the line determined by the direction vector  $\mathbf{r}'(t_0)$  and the point  $\mathbf{r}(t_0)$ . That is, the line can be parametrized as

$$\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

picture

**Theorem 3.1.12** (Differentiation Rules). Assume that  $\mathbf{r}(t)$  and  $\mathbf{s}(t)$  are differentiable vector-valued functions  $\mathbb{R} \rightarrow \mathbb{R}^n$ .

a.  $\frac{d}{dt}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$

b. **Scalar product rule.** For any differentiable scalar-valued function  $f(t)$ ,

$$\frac{d}{dt}(f(t)\mathbf{r}(t)) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$$

c. **Chain rule.** For any differentiable scalar-valued function  $f(t)$ ,

$$\frac{d}{dt}(\mathbf{r}(f(t))) = \mathbf{r}'(f(t))f'(t)$$

**Theorem 3.1.13** (Dot product rule). Assume that  $\mathbf{r}(t)$  and  $\mathbf{s}(t)$  are differentiable vector-valued functions  $\mathbb{R} \rightarrow \mathbb{R}^n$ . Then

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$$

**Theorem 3.1.14** (Cross product rule). Assume that  $\mathbf{r}(t)$  and  $\mathbf{s}(t)$  are differentiable vector-valued functions  $\mathbb{R} \rightarrow \mathbb{R}^3$ . Then

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$$

## The derivative of multivariable functions

How should we think of the derivative of a multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ?

Given a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , we can graph it as a surface  $x_{n+1} = g(x_1, \dots, x_n)$  in  $\mathbb{R}^{n+1}$ , which has points

$$(x_1, \dots, x_n, g(x_1, \dots, x_n))$$

If  $T$  is the derivative of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , then  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  is a linear map, which corresponds to a  $n \times 1$  matrix.

$$T(x_1, \dots, x_n) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_i^n a_i x_i$$

**Proposition 3.1.15.** *The graph of  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  is the hyperplane  $x_{n+1} = \sum_i^n a_i x_i$ .*

**Definition 3.1.16.** *The  $i$ -th partial derivative  $D_i f$  of a multivariable function  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}$  for  $\mathbf{x}_0 \in A^\circ$  is defined as the limit*

$$D_i f(\mathbf{x}_0) = \lim_{t \rightarrow 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)}{t}, \quad i = 1, \dots, n$$

*if it exists.*

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) = xe^{xy}$ . Then the partial derivatives of  $f$  are

$$\frac{\partial f}{\partial x} = ye^{xy} + e^{xy} \quad \frac{\partial f}{\partial y} = x^2 e^{xy}$$

**Proposition 3.1.17.** *Given a graph of  $z = f(x_1, \dots, x_n)$ , the tangent (hyperplane) in  $\mathbb{R}^{n+1}$  is spanned by vectors of the form  $\mathbf{e}_i + \frac{\partial f}{\partial x_i} \mathbf{e}_{n+1}$*

picture

**Proposition 3.1.18.** *We can take the normal vector of the tangent hyperplane in  $\mathbb{R}^{n+1}$  to the graph of a multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  to be*

$$\mathbf{n} = \sum_i^n \left( \frac{\partial f}{\partial x_i} \mathbf{e}_i \right) - \mathbf{e}_{n+1} = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, -1 \right\rangle$$

**Theorem 3.1.19.** *Let  $\mathbf{a} = \langle a_1, \dots, a_n \rangle \in \mathbb{R}^n$ . The equation of the tangent hyperplane in  $\mathbb{R}^{n+1}$  to the graph of a multivariable function  $f(\mathbf{a}) : \mathbb{R}^n \rightarrow \mathbb{R}$  at a point*

$$\bar{\mathbf{a}} = (a_1, \dots, a_n, f(\mathbf{a}))$$

*is given by*

$$\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, -1 \right\rangle \cdot (\mathbf{x} - \bar{\mathbf{a}}) = 0$$

*Equivalently,*

$$x_{n+1} = f(\mathbf{a}) + [D_1 f(\mathbf{a}) \quad D_2 f(\mathbf{a}) \quad \cdots \quad D_n f(\mathbf{a})] (\mathbf{x} - \mathbf{a})$$

We can use the multivariable derivative to approximate multivariable functions!

**Theorem 3.1.20** (Linear approximation). *If  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}$  is differentiable at a point  $\mathbf{a} = (a_1, \dots, a_n)$ , and  $\mathbf{x} = (x_1, \dots, x_n)$  is close to  $\mathbf{a}$ , then*

$$\begin{aligned} f(\mathbf{x}) &\approx f(\mathbf{a}) + [D_1 f(\mathbf{a}) \quad D_2 f(\mathbf{a}) \quad \cdots \quad D_n f(\mathbf{a})] (\mathbf{x} - \mathbf{a}) \\ &= f(\mathbf{a}) + \sum_i^n \left( \frac{\partial f}{\partial x_i}(\mathbf{a}) \right) (x_i - a_i) \end{aligned}$$

Equivalently, one can say that the change in  $f$  near  $\mathbf{a}$  can be approximated by the partial derivatives and the change in  $x_i$ .

$$\Delta f \approx \sum_i^n \left( \frac{\partial f}{\partial x_i}(\mathbf{a}) \right) \Delta x_i$$

The multivariable derivative in coordinates

How do we calculate the linear transformation  $Df(\mathbf{x}_0)$  for general multivariable functions  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ ?

**Definition 3.1.21.** *Let  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a multivariable function defined by  $f_i : A \subset \mathbb{R}^m \rightarrow \mathbb{R}$ :*

$$f(\mathbf{x}) = \begin{bmatrix} f^1(\mathbf{x}) \\ \vdots \\ f^n(\mathbf{x}) \end{bmatrix}$$

*The **Jacobian matrix** of  $f$  at  $\mathbf{x}_0$  is*

$$[J_f(\mathbf{x}_0)] = \begin{bmatrix} D_1 f^1(\mathbf{x}_0) & D_2 f^1(\mathbf{x}_0) & \cdots & D_m f^1(\mathbf{x}_0) \\ D_1 f^2(\mathbf{x}_0) & D_2 f^2(\mathbf{x}_0) & \cdots & D_m f^2(\mathbf{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f^n(\mathbf{x}_0) & D_2 f^n(\mathbf{x}_0) & \cdots & D_m f^n(\mathbf{x}_0) \end{bmatrix}$$

*if the partial derivatives exist.*

**Theorem 3.1.22** (The derivative in coordinates). *Let  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a multivariable function. If  $f$  is **differentiable at  $\mathbf{x}_0$** , then all the partial derivatives  $D_i f^j(\mathbf{x}_0)$  exist, and the standard matrix of  $Df(\mathbf{x}_0)$  is  $[J_f(\mathbf{x}_0)]$ . That is,*

$$Df(\mathbf{x}_0)(\mathbf{h}) = [J_f(\mathbf{x}_0)]\mathbf{h}$$

Consider the transformation from polar coordinates to rectangular coordinates, which is the function  $f(r, \theta) : (0, \infty) \times [0, 2\pi) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$f(r, \theta) = \langle r \cos(\theta), r \sin(\theta) \rangle$$

Then the Jacobian of  $f$  is the matrix

$$[J_f(r, \theta)] = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix}$$

**Remark 3.1.23.** The converse does not hold - there exists functions  $f$  such that all the partial derivatives exist at some point  $\mathbf{x}_0$ , but  $f$  is not differentiable at  $\mathbf{x}_0$ .

Consider the function  $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$

Then the partial derivatives of  $f$  exist at  $(0, 0)$ , but  $f$  is not continuous at  $(0, 0)$ !

picture

Consider the function  $f(x, y) = \begin{cases} \frac{2xy(x+y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$

Then the partial derivatives exist at  $(0, 0)$ , but  $f$  is not differentiable at  $(0, 0)$ . You will prove this in the exercises below.

picture

However, if we strengthen the hypotheses, then we do have the following result:

**Theorem 3.1.24.** Let  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a multivariable function. If all the partial derivatives  $D_i f^j(\mathbf{x}_0)$  exist and are continuous in some open ball  $B_\varepsilon(\mathbf{x}_0)$ , then  $f$  is differentiable at  $\mathbf{x}_0$ .

The chain rule

Let us now revisit the previous theorems, now that we can compute general multivariable derivatives!

**Theorem 3.1.25** (The Chain rule). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and let  $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$  be multivariable functions such that  $f$  is differentiable at  $\mathbf{x}_0 \in \mathbb{R}^n$ , and  $g$  is differentiable at  $f(\mathbf{x}_0) \in \mathbb{R}^m$ .

Then  $g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is differentiable at  $\mathbf{x}_0 \in \mathbb{R}^n$ , and

$$D(g \circ f)(\mathbf{x}_0) = Dg(f(\mathbf{x}_0)) \circ Df(\mathbf{x}_0)$$

We can prove this using the definition of derivative.

However, since we know that the derivative can be computed in terms of the Jacobian, we equivalently have

**Theorem 3.1.26** (The Chain rule in coordinates). Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $\mathbf{x}_0 \in \mathbb{R}^n$ , and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$  is differentiable at  $f(\mathbf{x}_0) \in \mathbb{R}^m$ . Then

$$[J_{g \circ f}(\mathbf{x}_0)] = [J_g(f(\mathbf{x}_0))] [J_f(\mathbf{x}_0)]$$

We can spell this out in terms of coordinates:

Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $\mathbf{x}_0 \in \mathbb{R}^n$ , and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$  is differentiable at  $f(\mathbf{x}_0) \in \mathbb{R}^m$ . Then

$$\begin{bmatrix} D_1(g \circ f)^1(\mathbf{x}_0) & \cdots & D_n(g \circ f)^1(\mathbf{x}_0) \\ \vdots & \vdots & \vdots \\ D_1(g \circ f)^k(\mathbf{x}_0) & \cdots & D_n(g \circ f)^k(\mathbf{x}_0) \end{bmatrix} = \begin{bmatrix} D_1 g^1(f(\mathbf{x}_0)) & \cdots & D_m g^1(f(\mathbf{x}_0)) \\ \vdots & \vdots & \vdots \\ D_1 g^k(f(\mathbf{x}_0)) & \cdots & D_m g^k(f(\mathbf{x}_0)) \end{bmatrix} \begin{bmatrix} D_1 f^1(\mathbf{x}_0) & \cdots & D_n f^1(\mathbf{x}_0) \\ \vdots & \vdots & \vdots \\ D_1 f^m(\mathbf{x}_0) & \cdots & D_n f^m(\mathbf{x}_0) \end{bmatrix}$$

How can we compute  $D(g \circ f)(\mathbf{x}_0)$ ? Equivalently, what are the entries of the matrix  $[J_{g \circ f}(\mathbf{x}_0)]$ ?

To answer this question, let us look at the chain rule for a special case: composites of the form  $\mathbb{R}^n \rightarrow \mathbb{R}^m \rightarrow \mathbb{R}$

Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $\mathbf{x}_0 \in \mathbb{R}^n$ , and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is differentiable at  $f(\mathbf{x}_0) \in \mathbb{R}^m$ . Then

$$\begin{aligned} [D_1(g \circ f)(\mathbf{x}_0) \quad \cdots \quad D_n(g \circ f)(\mathbf{x}_0)] &= \\ [D_1 g(f(\mathbf{x}_0)) \quad \cdots \quad D_m g(f(\mathbf{x}_0))] &\begin{bmatrix} D_1 f^1(\mathbf{x}_0) & \cdots & D_n f^1(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ D_1 f^m(\mathbf{x}_0) & \cdots & D_n f^m(\mathbf{x}_0) \end{bmatrix} \end{aligned}$$

Therefore,

$$\mathcal{D}_i(g \circ f) = \sum_k D_i f^k D_k g$$

**Corollary 3.1.27.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $\mathbf{x}_0 \in \mathbb{R}^n$ , and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is differentiable at  $f(\mathbf{x}_0) \in \mathbb{R}^m$ . Then

$$[J_{g \circ f}(\mathbf{x}_0)] = [D_i(g \circ f)^j] = \left[ \sum_k D_i f^k D_k g^j \right]$$

Let  $f(x, y, z) = xy + z$ . Calculate  $\frac{\partial f}{\partial s}$ , where  $x = s^2$ ,  $y = st$ ,  $z = t^2$ .

Let us now turn to another specific case of the chain rule:

**Proposition 3.1.28** (Chain rule for paths). Let  $f(x_1, \dots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable function, and let  $\mathbf{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle : \mathbb{R} \rightarrow \mathbb{R}^n$  be a vector-valued function. Then  $f(\mathbf{r}(t)) : \mathbb{R} \rightarrow \mathbb{R}$  is a single variable function, and

$$\begin{aligned} \frac{d}{dt} f(\mathbf{r}(t_0)) &= \left[ \frac{\partial f}{\partial x_1}(\mathbf{r}(t_0)) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{r}(t_0)) \right] \begin{bmatrix} x'_1(t_0) \\ \vdots \\ x'_n(t_0) \end{bmatrix} \\ &= \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i}(\mathbf{r}(t_0)) \right) x'_i(t_0) \end{aligned}$$

This measures the rate of change of  $f$  along the path  $\mathbf{r}(t)$ .

**PICTURE**

Consider the linear path through  $\mathbf{x}_0 \in \mathbb{R}^n$  in the direction of a vector  $\mathbf{v}$ , say

$$\mathbf{r}(t) = \mathbf{x}_0 + t\mathbf{v}$$

Observe that the chain rule depends on the magnitude of  $\mathbf{v}$

Let us make a definition that is independent of  $\|\mathbf{V}\|$

**Definition 3.1.29.** If  $\mathbf{u} = \langle u_1, \dots, u_n \rangle$  is a unit vector in  $\mathbb{R}^n$ , then the **directional derivative** in the direction of  $\mathbf{u}$  at the point  $\mathbf{x}_0 \in \mathbb{R}^n$  is defined as

$$D_{\mathbf{u}}f(\mathbf{x}_0) = u_1 \frac{\partial f}{\partial x_1}(\mathbf{x}_0) + \dots + u_n \frac{\partial f}{\partial x_n}(\mathbf{x}_0)$$

The directional derivative measures the rate of change of  $f$  in the direction of  $\mathbf{u}$ . That is,

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \frac{d}{dt}f(\mathbf{r}(t))$$

for  $\mathbf{r}(t) = \mathbf{x}_0 + t\mathbf{u}$ .

PICTURE

The gradient

**Definition 3.1.30.** If  $f(x_1, \dots, x_n)$  is a function of  $n$  variables, then the **gradient** of  $f$  is the vector-valued function

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

That is,

$$\nabla f = [Df(\mathbf{x}_0)]^\top$$

where  $[A]^\top$  indicates the transpose matrix (see definition 1.3.21).

**Proposition 3.1.31.** The chain rule for paths can be rewritten as

$$\frac{d}{dt}f(\mathbf{r}(t_0)) = \nabla f(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0)$$

**Corollary 3.1.32.** If  $\mathbf{u} \in \mathbb{R}^n$  is a unit vector, then the directional derivative in the direction of  $\mathbf{u}$  at the point  $\mathbf{P} \in \mathbb{R}^n$  can be computed as

$$D_{\mathbf{u}}f(\mathbf{P}) = \nabla f(\mathbf{P}) \cdot \mathbf{u}$$

This helps us geometrically interpret the gradient:

**Proposition 3.1.33.** The directional derivative  $D_{\mathbf{u}}f$  is **maximized** when  $\theta = 0$ , so when  $\mathbf{u} = \mathbf{e}_{\nabla f}$ . The maximum value of  $D_{\mathbf{u}}f$  is  $\|\nabla f\|$ .

The directional derivative  $D_{\mathbf{u}}f$  is **minimized** when  $\theta = \pi$ , so when  $\mathbf{u} = -\mathbf{e}_{\nabla f}$ . The minimum value of  $D_{\mathbf{u}}f$  is  $-\|\nabla f\|$ .

Thinking of  $z$  as the height of  $z = f(x, y)$ , the gradient  $\nabla f$  points in the direction of **steepest ascent**.

The opposite of the gradient,  $-\nabla f$ , points in the direction of **steepest descent**.

Furthermore, if  $\mathbf{r}(t)$  parameterizes a level curve  $f(x_1, \dots, x_n) = k$ , recall that means that for all  $t$ ,

$$f(\mathbf{r}(t)) = k$$

**Proposition 3.1.34.** The gradient  $\nabla f$  is **orthogonal** to the (tangent lines of) the level curves.

## PICTURE

**Corollary 3.1.35.** Let  $f(x_1, \dots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable function at a point  $\mathbf{a} = (a_1, \dots, a_n)$ . Moreover, suppose that  $f(a_1, \dots, a_n) = k$ .

Then  $\nabla f = \mathbf{0}$ , or  $\nabla f$  is **orthogonal** to the surface

$$f(x_1, \dots, x_n) = k$$

**Corollary 3.1.36.** The **tangent hyperplane** to the surface  $f(x_1, \dots, x_n) = k$  in  $\mathbb{R}^n$  at the point  $\mathbf{a} = (a_1, \dots, a_n)$  is given by

$$\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$$

How does this relate to previous notions of tangent hyperplane?

**Theorem 3.1.37.** The graph of a multivariable function  $g(x_1, \dots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$  can be described as

$$x_{n+1} = g(x_1, \dots, x_n)$$

in  $\mathbb{R}^{n+1}$ . The equation of the tangent hyperplane in  $\mathbb{R}^{n+1}$  at a point

$$\bar{\mathbf{a}} = (a_1, \dots, a_n, g(a_1, \dots, a_n))$$

is given by

$$\left\langle \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}, -1 \right\rangle \cdot (\bar{\mathbf{x}} - \bar{\mathbf{a}}) = 0$$

Observe that if we set  $f(x_1, \dots, x_{n+1}) = g(x_1, \dots, x_n) - x_{n+1}$ , then

$$\nabla f = \left\langle \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}, -1 \right\rangle$$

## Exercises

**3.1** Find a function  $f(x) : \mathbb{R} \rightarrow \mathbb{R}$  that shows that defining the derivative of a multivariable function as

$$f'(\mathbf{x}_0) = \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)}{\|\mathbf{h}\|}$$

does not generalize the single variable derivative.

**3.2** Consider the function  $f(x, y) = x$ . Show that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)}{\|\mathbf{h}\|}$$

does not exist at  $(0, 0)$ . (Hence, this is also not a good definition of the multivariable derivative).

**3.3** Use linear approximation to approximate  $\frac{8.01}{\sqrt{(1.99)(2.01)}}$ .

**3.4** Use linear approximation to approximate  $(2.92)^2 \sqrt{4.08}$ .

**3.5** Consider the function  $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$  Use the limit definition of the partial derivative to compute  $f_x(0, 0)$  and  $f_y(0, 0)$ .

**3.6** Consider the function  $f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$

Use the limit definition of the partial derivative to compute  $f_x(0, 0)$  and  $f_y(0, 0)$ .

**3.7** Let  $f(u, v) = \tan(uv^3)$ . Find the partial derivatives  $f_u(u, v)$  and  $f_v(u, v)$ .

**3.8** Let  $f(x, y, z) = xy + z$ . Calculate  $\frac{\partial f}{\partial t}$ , where  $x = s^2$ ,  $y = st$ ,  $z = t^2$ .

**3.9** Let  $f(x, y) = \tan(xy^2)$ , and consider the path  $\mathbf{r}(t) = \langle t, e^t \rangle$ . Compute  $\frac{d}{dt}f(\mathbf{r}(t))$  at the point  $(2, e^2)$ .

**3.10** Calculate the gradient of  $g(x, y, z) = x \ln(y + z)$ .

**3.11** Find the maximum rate of change of  $f(x, y) = e^{xy-y^2}$  at the point  $(1, 1)$ .

**3.12** Let  $f(x, y) = e^{xy-y^2}$ . Compute the directional derivative in the direction of  $\mathbf{u} = \langle 5, 12 \rangle$  at the point  $P = (1, 1)$ .

**3.13** Find the directional derivative of  $g(x, y, z) = xy + z^2$  at the point  $P = (3, 2, -1)$ , in the direction pointing to the origin.

**3.14** Find the directional derivative of  $g(x, y, z) = x \ln(y+z)$  in the direction of the vector  $\mathbf{v} = \langle 2, -1, 1 \rangle$  at the point  $P = (2, e, e)$ .

**3.15** Find an equation of the plane tangent to the graph of  $f(x, y) = xy^3 + x^2$  at the point  $(2, -2, -12)$

## 3.2 Optimizing Multivariable functions

Recall that in single-variable calculus, the derivative is also useful in optimising single-variable functions. That is, finding local maxima and minima, as well as global maxima and minima.

**Definition 3.2.1.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a **local maximum** at  $P \in \mathbb{R}$  if there exists some  $r > 0$  such that

$$f(P) \geq f(Q)$$

for all points  $Q \in B_r(P)$ .

**Definition 3.2.2.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a **local minimum** at  $P \in \mathbb{R}$  if there exists some  $r > 0$  such that

$$f(P) \leq f(Q)$$

for all points  $Q \in B_r(P)$ .

**Definition 3.2.3.** A function  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$  has a **global maximum on  $D$**  at  $P \in \mathbb{R}$  if

$$f(P) \geq f(Q)$$

for all points  $Q \in D$ .

**Definition 3.2.4.** A function  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$  has a **global minimum on  $D$**  at  $P \in \mathbb{R}$  if

$$f(P) \leq f(Q)$$



for all points  $Q \in D$ .

How can we find and classify local maxima and minima?

**Definition 3.2.5.** A point  $P \in \mathbb{R}$  in the domain of  $f(x)$  is a **critical point** if

- a.  $f'(P) = 0$ , OR
- b.  $f'(P)$  does not exist.

**Theorem 3.2.6** (The single-variable first derivative test). If  $f(x)$  has a local maximum or minimum at  $P$ , then  $P$  is a critical point of  $f(x)$ .

We can classify the critical points of  $f(x)$  using the single-variable second derivative test (theorem 3.2.18).

We can also find global maxima and minima of  $f(x)$  on an interval  $D = [a, b]$ :

**Theorem 3.2.7.** The global maxima and minima of  $f(x)$  on  $[a, b]$  either occur at the critical points of  $f$  in  $[a, b]$ , or on the boundary of  $[a, b]$ .

To classify the global maxima and minima, we simply evaluate the function at these critical points. In the following sections, we will see how to generalize these ideas to multivariable calculus:

#### Local optimization

First, let us define the notion of a local extremum for a multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ :

**Definition 3.2.8.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has a **local maximum** at  $P \in \mathbb{R}^n$  if there exists some  $r > 0$  such that

$$f(P) \geq f(Q)$$

for all points  $Q \in B_r(P)$ .

**Definition 3.2.9.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has a **local minimum** at  $P \in \mathbb{R}^n$  if there exists some  $r > 0$  such that

$$f(P) \leq f(Q)$$

for all points  $Q \in B_r(P)$ .

**Note:** local extrema only make sense for functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

**Definition 3.2.10.** A function  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  has a **global maximum** at  $x_0 \in A$  if

$$f(x_0) \geq f(x)$$

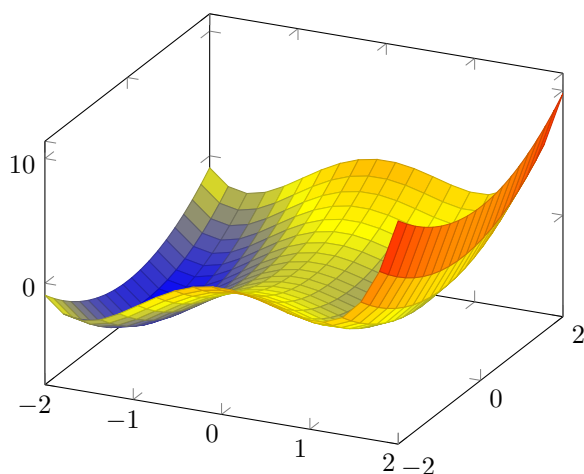
for all  $x \in A$ .

**Definition 3.2.11.** A function  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  has a **global minimum** at  $\mathbf{x}_0 \in A$  if

$$f(\mathbf{x}_0) \leq f(\mathbf{x})$$

for all  $\mathbf{x} \in A$ .

**Remark 3.2.12.** Observe that **local** maxima/minima are not necessarily **global** maxima/minima.



What is the multivariable analogue of the first derivative test?

**Remark 3.2.13.** Observe that if  $\mathbf{P} \in \mathbb{R}^n$  is a local maxima or minima of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , and if  $f$  is differentiable at  $\mathbf{P}$ , then  $Df(\mathbf{P})$  is the zero linear transformation (equivalently,  $\nabla f = \mathbf{0}$ ).

That is, the tangent plane at  $\mathbf{P}$  is parallel to the plane  $x_{n+1} = 0$ .

**Definition 3.2.14.** A point  $\mathbf{P} \in \mathbb{R}^n$  is said to be a **critical point** of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  if either

- $Df(\mathbf{P}) = \mathbf{0}$ , OR
- $Df(\mathbf{P})$  does not exist.

**Proposition 3.2.15.** Equivalently,  $\mathbf{P}$  is a critical point of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  if

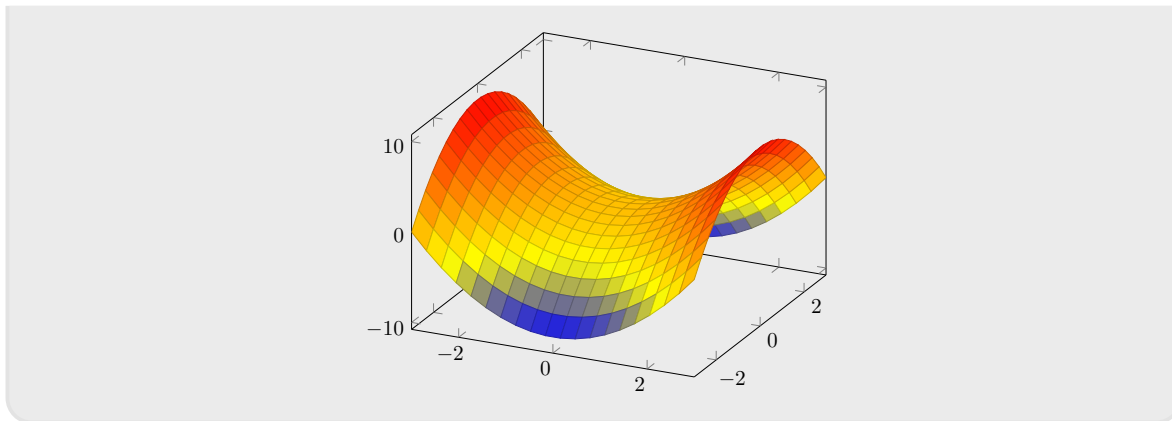
- $\nabla f(\mathbf{P}) = \mathbf{0}$ , OR
- $\nabla f(\mathbf{P})$  does not exist.

**Theorem 3.2.16** (The multivariable first derivative test). If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has a local maximum or minimum at  $\mathbf{P}$ , then  $\mathbf{P}$  is a critical point of  $f$ .

**Remark 3.2.17.** The converse of the first derivative test does not hold:

If  $\mathbf{P}$  is a critical point of  $f$ , then  $\mathbf{P}$  is not necessarily a local maximum or minimum.

Consider the function  $f(x, y) = x^2 - y^2$ . Observe that  $(0, 0)$  is a critical point, but it is neither a local maximum or a minimum.



### Classifying local extrema

How can we algebraically classify the behavior of critical points?

Recall that in single-variable calculus, we can do so using the **second derivative test**:

**Theorem 3.2.18** (The single-variable second derivative test). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a single variable function.*

- a. *If  $f'(P) = 0$  and  $f''(P) > 0$ , then there is a **local minimum** at  $x = P$ .*
- b. *If  $f'(P) = 0$  and  $f''(P) < 0$ , then there is a **local maximum** at  $x = P$ .*
- c. *If  $f'(P) = 0$  and  $f''(P) = 0$ , then the test is **inconclusive**.*

*If  $f''(P)$  does not exist, then the test is **inconclusive**.*

What is the multivariable analogue of the second derivative test?

First, we need to figure out what we mean by the second derivative of a multivariable function. Recall from example 3.1 that the derivative of a linear transformation  $T$  is  $T$  itself. That is,

$$DT(\mathbf{x}_0) = T$$

Thus, we cannot naïvely define the second derivative of  $f$  as  $D(Df(\mathbf{x}_0))$ .

**Remark 3.2.19.** *We have defined the derivative  $Df(\mathbf{x}_0)$  to be the linear approximation of  $f$  at a point  $\mathbf{x}_0 \in \mathbb{R}^n$ .*

*This is analogous to finding the slope of the tangent line of a single variable function at a point  $x \in \mathbb{R}$ .*

Therefore, we wish to find the derivative of  $Df$ , not of  $Df(\mathbf{x}_0)$ , where

$$Df : \mathbb{R}^n \rightarrow M_{1 \times n}(\mathbb{R})$$

However, observe that we have only defined derivatives of functions  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . We have not defined the derivative of a function that outputs matrices! However, since we know that

$$M_{1 \times n}(\mathbb{R}) \cong \mathbb{R}^n$$

via the transpose map  $(-)^T$

$$M \mapsto M^T$$

We can then consider the map

$$(Df)^\top : \mathbb{R}^n \cong M_{1 \times n}(\mathbb{R}) \rightarrow \mathbb{R}^n$$

which sends a vector

$$\mathbf{x}_0 \mapsto \begin{bmatrix} D_1 f(\mathbf{x}_0) \\ \vdots \\ D_n f(\mathbf{x}_0) \end{bmatrix}$$

Observe that

$$(Df)^\top(\mathbf{x}_0) = \nabla f(\mathbf{x}_0)$$

Thus, we can make the following definition:

**Definition 3.2.20.** A multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is **twice-differentiable** if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are both differentiable.

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is twice-differentiable, then the **second derivative of  $f$  at  $\mathbf{x}_0$**  is defined as

$$D^2 f(\mathbf{x}_0) := D(\nabla f)(\mathbf{x}_0)$$

**Definition 3.2.21.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a twice-differentiable function. Then the standard matrix of  $D^2 f(\mathbf{x}_0)$  is called the **Hessian matrix**,

$$[H_f(\mathbf{x}_0)]$$

**Proposition 3.2.22.** The **Hessian matrix** of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $\mathbf{x}_0$  is

$$[H_f(\mathbf{x}_0)] = \begin{bmatrix} D_1 D_1 f(\mathbf{x}_0) & D_2 D_1 f(\mathbf{x}_0) & \cdots & D_n D_1 f(\mathbf{x}_0) \\ D_1 D_2 f(\mathbf{x}_0) & D_2 D_2 f(\mathbf{x}_0) & \cdots & D_n D_2 f(\mathbf{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 D_n f(\mathbf{x}_0) & D_2 D_n f(\mathbf{x}_0) & \cdots & D_n D_n f(\mathbf{x}_0) \end{bmatrix}$$

In other words, the Hessian matrix is the  $n \times n$  matrix of all second-order partial derivatives of  $f$ ,

$$f_{x_j x_i} = D_i D_j f = \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_j} \right)$$

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) = xe^{xy}$ . Compute the partial derivatives, the Jacobian matrix,  $\nabla f$ , and the Hessian matrix of  $f$  at  $(1, 1)$ .

**Definition 3.2.23.** A function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be of class  $C^2$  (writing  $f \in C^2(U)$ ) if all second-order partial derivatives exist and are continuous on  $U$ .

**Definition 3.2.24.** A function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be of class  $C^k$  (writing  $f \in C^k(U)$ ) if all  $k$ th-order partial derivatives exist and are continuous on  $U$ .

**Definition 3.2.25.** A function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be of class  $C^\infty$  (writing  $f \in C^\infty(U)$ ) if all partial derivatives exist and are continuous on  $U$ .

**Theorem 3.2.26** (Clairaut's theorem). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose that  $D_i f$ ,  $D_j f$ , and  $D_i D_j f$  exist and are continuous on an open disk  $D \subset \mathbb{R}^n$ . Then  $D_j D_i f$  exists on  $D$ , and moreover

$$D_i D_j = D_j D_i \text{ on the disk } D$$

Clairaut's theorem is more commonly stated (and easily proved) as the following corollary:

**Corollary 3.2.27.** Let  $f \in C^2(U)$ . Then

$$D_i D_j = D_j D_i \text{ on the region } U$$

We can use Clairaut's theorem to compute the partial derivative  $g_{zzwx}$  for

$$g(x, y, w, z) = x^2 w^2 z^2 + \sin\left(\frac{xy}{z^2}\right)$$

Note that as long as  $z \neq 0$ , we have that all partial derivatives of  $g$  exist. Thus,  $g_{zzwx} = g_{wxzz}$  as long as  $z \neq 0$ .

Observe that  $g_w = 2x^2 w z^2$ . Thus,  $g_{zzwx} = g_{wxzz} = 8wx$ , as long as  $z \neq 0$

**Corollary 3.2.28.** If  $f \in C^2(U)$ , then

$$[H_f(\mathbf{x}_0)]^\top = [H_f(\mathbf{x}_0)]$$

With the second derivative defined, we can now state the two-variable second derivative test:

**Theorem 3.2.29** (The two-variable second derivative test). Let  $\mathbf{x}_0 \in U$  be a critical point of  $f(x, y) : U \rightarrow \mathbb{R}$ , and suppose that  $f \in C^2(U)$ . Let us write  $D = \det[H_f(\mathbf{x}_0)]$ .

- If  $D > 0$  and  $f_{xx}(\mathbf{x}_0) > 0$ , then there is a **local minimum** at  $\mathbf{x}_0$ .
- If  $D > 0$  and  $f_{xx}(\mathbf{x}_0) < 0$ , then there is a **local maximum** at  $\mathbf{x}_0$ .
- If  $D < 0$ , then  $f$  has a **saddle point** at  $\mathbf{x}_0$ .
- If  $D = 0$ , then the test is **inconclusive**.

If  $D$  does not exist, then the test is **inconclusive**.

**Remark 3.2.30.** At a critical point  $P = (a, b)$ , we can apply the single-variable second derivative test to

- the trace  $f(a, y) = g(y)$  in the plane  $x = a$
- the trace  $f(x, b) = h(x)$  in the plane  $y = b$

For the former, observe that

$$f_{yy}(a, b) = g''(b) \text{ is } \begin{cases} > 0 & \text{if } f(a, y) = g(y) \text{ has a local min at } b \\ < 0 & \text{if } f(a, y) = g(y) \text{ has a local max at } b \end{cases}$$

For the latter, observe that

$$f_{xx}(a, b) = h''(a) \text{ is } \begin{cases} > 0 & \text{if } f(x, b) = h(x) \text{ has a local min at } a \\ < 0 & \text{if } f(x, b) = h(x) \text{ has a local max at } a \end{cases}$$

How does this generalize to multiple variables?

**Theorem 3.2.31** (Second order Taylor approximation). *If  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at a point  $\mathbf{a} \in \mathbb{R}^n$ , and  $\mathbf{x}$  is close to  $\mathbf{a}$ , then*

$$f(\mathbf{x}) \approx f(\mathbf{a}) + [Df(\mathbf{a})](\mathbf{x} - \mathbf{a}) + \frac{1}{2} ([H_f(\mathbf{a})](\mathbf{x} - \mathbf{a})) \cdot (\mathbf{x} - \mathbf{a})$$

**Definition 3.2.32.** *Let  $A \in M_{n \times n}(\mathbb{R})$  be a symmetric matrix (that is,  $A^\top = A$ ). Then  $A$  is said to be*

- a. **positive definite**, if for all  $\mathbf{x} \neq \mathbf{0}$ , then  $(A\mathbf{x}) \cdot \mathbf{x} > 0$ ,
- b. **negative definite** if for all  $\mathbf{x} \neq \mathbf{0}$ , then  $(A\mathbf{x}) \cdot \mathbf{x} < 0$ ,
- c. **indefinite** otherwise.

**Theorem 3.2.33** (The multivariable second derivative test). *Let  $\mathbf{x}_0 \in U \subset \mathbb{R}^n$  be a critical point of a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ .*

*Suppose that  $f \in C^2(U)$ , and that the Hessian  $[H_f(\mathbf{x}_0)]$  is invertible.*

- a. *If  $[H_f(\mathbf{x}_0)]$  is positive definite, then there is a **local minimum** at  $\mathbf{x}_0$ .*
- b. *If  $[H_f(\mathbf{x}_0)]$  is negative definite, then there is a **local maximum** at  $\mathbf{x}_0$ .*
- c. *If  $[H_f(\mathbf{x}_0)]$  is indefinite, then  $f$  has a **saddle point** at  $\mathbf{x}_0$ .*

### Global Optimization

We saw from the previous section that the gradient and the second derivative test allow us to find and classify **local** maxima and minima.

(On a given domain), how can we find **global** maxima and minima?

In single-variable calculus, we have the following theorem:

**Theorem 3.2.34** (Single-variable global minima and maxima). *If  $f(x)$  is continuous on a closed and bounded interval  $[a, b]$ , then  $f$  has a global maximum and a minimum on  $[a, b]$ .*

*That is, there exists an  $M \in [a, b]$  and an  $m \in [a, b]$  such that  $f(m) \leq f(x) \leq f(M)$  for all  $x \in [a, b]$ .*

**Theorem 3.2.35.** *The global maxima and minima of  $f(x)$  on  $[a, b]$  either occur at the critical points of  $f$  in  $[a, b]$ , or on the boundary of  $[a, b]$ .*

**Remark 3.2.36.** *Observe that if  $A \subset \mathbb{R}$  is not closed, or not bounded, then global maxima/minima are not guaranteed to exist.*

The interval  $A = [0, 1)$  is bounded, but not closed.

The function  $f(x) = x$  has no global maximum in  $A$ , but has a global minimum at  $x = 0$ .

The function  $f(x) = -x$  has a global maximum at  $x = 0$ , but no global minimum in  $A$ .

The interval  $A = \{x \mid x \geq 0\}$  is closed, but not bounded.

The function  $f(x) = x$  has no global maximum in  $A$ , but has a global minimum at  $x = 0$ .

The function  $f(x) = -x$  has a global maximum at  $x = 0$ , but no global minimum in  $A$ .

The interval  $A = \{x \mid x > 0\}$  is neither closed nor bounded.

The function  $f(x) = x$  has no global maximum or minimum in  $A$ .

The function  $f(x) = \sin(x)$  has (infinitely many) global maxima and minima in  $A$ .

Thus, we have a few questions to consider:

How do we generalize the notion of a bounded and closed interval  $[a, b] \subset \mathbb{R}$  to a subset  $A \subset \mathbb{R}^n$ ?

Can we find the possible points where global maxima/minima occur?

To answer the first question, we will need to define the notions of bounded and closed sets in  $\mathbb{R}^n$ .

**Definition 3.2.37.** A subset  $D \subset \mathbb{R}^n$  is **bounded** if there exists some  $r > 0$  such that  $D \subset B_r(\mathbf{0})$ .

**Definition 3.2.38.** A point  $\mathbf{x}_0 \in \mathbb{R}^n$  is a **boundary point** of  $D \subset \mathbb{R}^n$  if: for all  $\varepsilon > 0$ ,

- a.  $B_\varepsilon(\mathbf{x}_0) \cap D$  is non-empty, **and**
- b.  $B_\varepsilon(\mathbf{x}_0) \cap D^c$  is non-empty.

**Definition 3.2.39.** A subset  $D \subset \mathbb{R}^n$  is **closed** if it contains all of its boundary points.

Consider the set

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$$

The boundary of  $D$  is the set

$$\{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\} \cup \{(0, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

$D$  is closed, but not bounded.

Consider the set

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 < (x - a)^2 + (y - b)^2 \leq 3\}$$

is bounded, but not closed.

The boundary of  $D$  is the set

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 = 3\}$$

$D$  is bounded, but not closed.

If a region  $D$  is closed and bounded, then we can guarantee the existence of global maxima and minima for a continuous function on  $D$ :

**Theorem 3.2.40** (Multivariable global minima and maxima). *If  $D$  is a closed and bounded subset of  $\mathbb{R}^n$ , and  $f$  is a continuous function on  $D$ , then  $f$  has a global maximum and a global minimum in  $D$ .*

*That is, there exists an  $\mathbf{M} \in D$  and an  $\mathbf{m} \in D$  such that  $f(\mathbf{m}) \leq f(\mathbf{x}) \leq f(\mathbf{M})$  for all  $\mathbf{x} \in D$ .*

**Theorem 3.2.41.** *The global maxima and minima of  $f$  on  $D$  either occur at the critical points of  $f$  in  $D$ , or on the boundary of  $D$ .*

We know how to use the gradient and the second derivative test allow us to find and classify **local** maxima and minima. The question we turn to now is how to find maxima and minima on the boundary of a region  $D$ :

### Constrained optimization

Given a region  $D$ , how can we find the maxima and minima on  $D$ ?

**Remark 3.2.42.** *The boundary of  $D \subset \mathbb{R}^2$  can be thought of as:*

- *the parametric curve corresponding to  $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ .*
- *The  $z = 0$  trace of a graph  $z = g(x, y)$*

To find local maxima and minima of  $f(x, y)$  on the boundary of  $D$ , we are trying to find the local maxima and minima of  $f(x, y)$  subject to the condition  $g(x, y) = 0$ . In other words, we are trying to optimize  $f(x, y)$  given the constraint  $g(x, y) = 0$ . We can do so using the method of Lagrange multipliers:

**Theorem 3.2.43** (The method of Lagrange Multipliers). *Assume that  $f(x, y)$  and  $g(x, y)$  are differentiable functions. If*

- a.  *$f(x, y)$  has a local maximum or minimum subject to the constraint  $g(x, y) = 0$  at a point  $(a, b)$ ,*  
**AND**

- b. *if  $\nabla g(a, b) \neq 0$*

*then there is a scalar  $\lambda$  such that*

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

**Corollary 3.2.44.** *We can use the **Lagrange equations***

$$f_x(a, b) = \lambda g_x(a, b) \quad \text{and} \quad f_y(a, b) = \lambda g_y(a, b)$$

*to determine the critical points on the boundary.*

**Definition 3.2.45.** *Points  $P = (a, b)$  that satisfy the Lagrange equations for the optimization problem of  $f(x, y)$  with constraint  $g(x, y) = 0$  are called **critical points for optimization with constraint**.*

**Remark 3.2.46.** *A constrained critical point is not necessarily a local max or a local min.*



Maximize the area of a rectangular fence subject to the constraint that the perimeter is equal to 20.

Maximize  $f(x, y) = xy$  subject to the constraint  $4x^2 + 9y^2 = 32$ .

### Classifying constrained extrema

How can we tell if the Lagrange multiplier method give us a local max? a local min?

We need an analogue of the second-derivative test for constrained optimization:

**Definition 3.2.47.** Assume that  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  are differentiable functions. Suppose that we are trying to optimize  $f(x, y)$  subject to the constraint  $g(x, y) = 0$ .

Then the **Lagrange function** is a function  $\Lambda : \mathbb{R}^3 \rightarrow \mathbb{R}$  defined as

$$\Lambda(x, y, \lambda) := f(x, y) + \lambda g(x, y)$$

**Definition 3.2.48.** The **bordered Hessian matrix**  $[H_\Lambda]$  is the Hessian matrix of the Lagrange function  $\Lambda(x, y, \lambda) = f(x, y) + \lambda g(x, y)$ :

$$\begin{aligned} [H_\Lambda] &= \begin{bmatrix} D_x D_x \Lambda & D_y D_x \Lambda & D_\lambda D_x \Lambda \\ D_x D_y \Lambda & D_y D_y \Lambda & D_\lambda D_y \Lambda \\ D_x D_\lambda \Lambda & D_y D_\lambda \Lambda & D_\lambda D_\lambda \Lambda \end{bmatrix} \\ &= \begin{bmatrix} f_{xx} + \lambda g_{xx} & f_{xy} + \lambda g_{xy} & g_x \\ f_{yx} + \lambda g_{yx} & f_{yy} + \lambda g_{yy} & g_y \\ g_x & g_y & 0 \end{bmatrix} \\ &= \begin{bmatrix} H_{\Lambda(x,y)} & [J_{g(x,y)}]^\top \\ J_{g(x,y)} & 0 \end{bmatrix} \end{aligned}$$

**Theorem 3.2.49.** Assume that  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  are differentiable functions. Suppose that we are trying to optimize  $f(x, y)$  subject to the constraint  $g(x, y) = 0$ .

Let  $\mathbf{x}_0$  be a constrained critical point, with associated Lagrange multiplier  $\lambda_0$ . Let us write  $|\overline{H}| = \det[H_\Lambda(\mathbf{x}_0, \lambda_0)]$ .

- If  $|\overline{H}| > 0$  then there is a **local maximum** at  $\mathbf{x}_0$ .
- If  $|\overline{H}| < 0$ , then  $f$  has a **local minimum** at  $\mathbf{x}_0$ .
- If  $|\overline{H}| = 0$ , then the test is **inconclusive**.

### Exercises

**3.16** Consider the function

$$f(x, y, z, w) = \frac{3x^2 + e^y z + x}{3y^2 + 2e^{w^2}}$$

Compute the fourth order partial derivative  $f_{wyzx}$ .

**3.17** Consider the function

$$f(x, y) = x^3 - xy + y^3$$

Find the critical points of  $f$ , and use the second derivative test to classify the critical points of  $f$ .

**3.18** Consider the function

$$f(x, y) = x^2 + y^3 - 2x - 6y$$

Find the critical points of  $f$ , and use the second derivative test to classify the critical points of  $f$ .

**3.19** Consider the function

$$f(x, y) = \ln(x) + 2\ln(y) - x - 4y$$

Find the critical points of  $f$ , and use the second derivative test to classify the critical points of  $f$ .

**3.20** Is the set

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x + y \leq 1\}$$

closed? bounded? What is its boundary?

**3.21** Is the set  $A = \{(x, y) \mid -1 < x \leq 1, -1 \leq y < 1\}$  closed? bounded? What is its boundary?

**3.22** Find the critical points of  $f(x, y) = x^4 + y^4$  subject to the constraint  $x^2 + y^2 = 1$ .

**3.23** Determine the coordinates of the global maxima and minima of  $f(x, y) = x^4 + y^4$  on the disk  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ .

**3.24** Find the critical points of  $f(x, y) = -x^2 + 2y^2 + 6x$  subject to the constraint  $x^2 + y^2 = 1$ .

**3.25** Determine the coordinates of the global maxima and minima of  $f(x, y) = -x^2 + 2y^2 + 6x$  on the disk  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ .

**3.26** Let  $f(x, y) = (x^3 - 3x) + (y^3 - 3y)$ . Find the constrained critical points of  $f(x, y)$  on the boundary of the region  $S = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$

**3.27** Let  $f(x, y) = (x^3 - 3x) + (y^3 - 3y)$ . Find the global maxima and minima of  $f(x, y)$  on the region  $S = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$

**3.28** Find a continuous function that does not have a global maximum on the domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x + y \leq 1\}$$

**3.29** Find the critical points of  $f(x, y) = 4x^2 + 9y^2$  subject to the constraint  $xy = 4$ .

**3.30** Find the point  $(a, b)$  on the line  $4x + 9y = 12$  that is closest to the origin.

**3.31** Find the point  $\mathbf{x}_0 = (x_0, y_0, z_0)$  on the plane  $ax + by + cz = d$  that is closest to the origin. What is the distance from  $\mathbf{x}_0 = (x_0, y_0, z_0)$  to the origin?

## Chapter 4

### Integration on $\mathbb{R}^n$

What is the definition of the integral in single variable calculus?

#### 4.1 Defining the integral

What is the definition of the integral in single variable calculus?

PICTURE

- a. Can we define the integral such that we can deal with both  $\int_{-\infty}^{\infty} f(x) dx$  and  $\int_a^b f(x) dx$ ?
- b. Can we define the integral in a way that easily generalizes to multivariable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ?
- c. What kinds of functions can we integrate?

In these lecture notes, we'll be dealing with proper/definite integrals. That is, we will want our integrals to have finite values.<sup>1</sup>

That is, we'll be working with functions that are **bounded** with **bounded support**.

**Definition 4.1.1.** A subset  $D \subset \mathbb{R}^n$  is **bounded** if there exists some  $r > 0$  such that  $D \subset B_r(\mathbf{0})$ .

recall

**Definition 4.1.2.** A function  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is **bounded** if its image

$$\{f(\mathbf{x}) \mid \mathbf{x} \in A\}$$

is a bounded subset of  $\mathbb{R}$ .

recall

**Definition 4.1.3.** A point  $\mathbf{x} \in \mathbb{R}^n$  is a **boundary point** of  $D \subset \mathbb{R}^n$  if: for all  $\varepsilon > 0$ ,

- a.  $B_\varepsilon(\mathbf{x}) \cap D$  is non-empty, **and**
- b.  $B_\varepsilon(\mathbf{x}) \cap D^c$  is non-empty.

**Definition 4.1.4.** The **closure** of a set  $D \subset \mathbb{R}^n$  is the union of  $D$  and the boundary of  $D$ .

---

<sup>1</sup>It takes some care to define non-proper integrals in multivariable calculus!

That is, the closure is the set

$$\overline{D} = \{x \in \mathbb{R}^n \mid B_r(x) \cap D \neq \emptyset \text{ for all } r \in \mathbb{R}\}$$

**Proposition 4.1.5.** A set  $D$  is closed (*recall*) if and only if

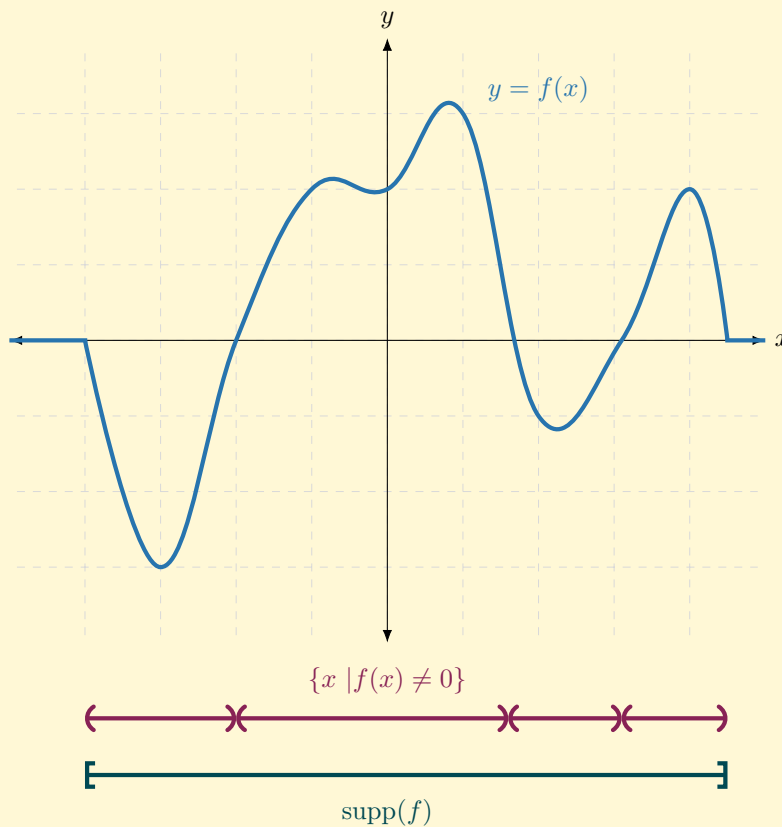
$$D = \overline{D}$$

**Definition 4.1.6.** The **set of non-zero values of a function**  $f : A \subset \mathbb{R}^n$  is the set

$$\{x \in A \mid f(x) \neq 0\}$$

**Definition 4.1.7.** The **support** of a function  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is the **closure** of the set of non-zero values of a function:

$$\text{supp}(f) := \overline{\{x \in A \mid f(x) \neq 0\}}$$



**Definition 4.1.8.** A function has **bounded support** if its support is bounded. Equivalently, there exists  $R > 0$  such that  $f(x) = 0$  for all  $\|x\| > R$ .

The function  $f(x) = \sin(x)$  does not have bounded support. In fact, the set of non-zero points of  $\sin(x)$  is the set

$$\{x \in \mathbb{R} \mid x \neq n\pi\}$$

This set is already unbounded, and when we take the closure,  $\text{supp}(f) = \mathbb{R}$ .

However, the function

$$g(x) = \begin{cases} \sin(x) & \text{if } -\pi \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

has bounded support. The set of non-zero points of  $g(x)$  is the set

$$(-\pi, 0) \cup (0, \pi)$$

When we take the closure,  $\text{supp}(f) = [-\pi, \pi]$ , which is bounded.

We can now define the notion of multivariable integral for bounded functions with bounded support.

### Integration over $\mathbb{R}^n$

The main idea of integrating a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is not so difficult:

- We chop up our region into pieces.
- We sample  $f$  on each of the pieces, and sum over the pieces.
- We take the limit of the sum as the size of the pieces shrinks.

However, **we will need to be careful**, as we will encounter some technical difficulties in making these ideas rigorous.

We will begin with chopping up the domain of  $f$  (e.g.  $\mathbb{R}^n$ ) into pieces. In technical terms, we will construct a **partition** of  $\mathbb{R}^n$ .

**Definition 4.1.9.** A **partition** of a set  $X$  is a collection of non-empty subsets  $X_\alpha \subset X$  such that every element of  $x \in X$  is in **exactly one**  $X_\alpha$ .

**Definition 4.1.10.** Given a vector  $\mathbf{k} = \langle k_1, \dots, k_n \rangle \in \mathbb{Z}^n \subset \mathbb{R}^n$  (that is,  $k_i \in \mathbb{Z}$  for all  $i$ ), we can define the **dyadic cube**  $C_{\mathbf{k},N}$  in  $\mathbb{R}^n$  as

$$C_{\mathbf{k},N} := \left\{ \mathbf{x} = \langle x_1, \dots, x_n \rangle \in \mathbb{R}^n \mid \frac{k_i}{2^N} \leq x_i < \frac{k_i + 1}{2^N} \right\}$$

What is the dyadic cube  $C_{1,1}$  in  $\mathbb{R}^1$ ?

What is the dyadic cube  $C_{1,2}$  in  $\mathbb{R}^1$ ?

**picture**

What is the dyadic cube  $C_{\langle 1,1 \rangle,1}$  in  $\mathbb{R}^2$ ?

**Proposition 4.1.11.** The volume of a dyadic cube  $C_{\mathbf{k},N}$  in  $\mathbb{R}^n$  is  $\frac{1}{2^{Nn}}$ .

**Proposition 4.1.12.** For a fixed  $N$ , the collection of all dyadic cubes

$$D_N(\mathbb{R}^n) := \{C_{\mathbf{k},N} \mid \text{for all } \mathbf{k} \in \mathbb{Z}^n\}$$

partitions  $\mathbb{R}^n$ . We call  $D_N(\mathbb{R}^n)$  the *N-th dyadic partition of  $\mathbb{R}^n$* .

The dyadic partition is a convenient choice of **partition** of  $\mathbb{R}^n$ . We have chosen this particular partition as it is systematic and easy to define.

There are many other ways to partition  $\mathbb{R}^n$ , and they each lead to an equivalent definition of the multivariable integral. For example, it turns out that the partitions do not need to be uniform (e.g. we could take rectangles with differing widths). Moreover, we could take partitions depending on the range of the function (this leads to the notion of the Lebesgue integral).

Given our dyadic cube partition, we now turn to sampling points from the partition:

lower left corner, midpoint, etc.

**Definition 4.1.13.** Given a choice of  $\mathbf{x}_{\mathbf{k},N} \in C_{\mathbf{k},N}$  for every cube, the *N-th Riemann sum* is defined as

$$R_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} f(\mathbf{x}_{\mathbf{k},N}) \text{vol}(C)$$

We have many choices for sampling points, and it is a priori unclear that these various sampling methods should produce the same value for our integral when we take the limit as  $N$  goes to  $\infty$ .

Thus, we will use Darboux sums to define the multivariable integral - this will guarantee that if the limit as  $N$  goes to  $\infty$  exists, then any sampling method will produce the same value<sup>2</sup>.

**Definition 4.1.14.** Let  $X \subset \mathbb{R}$ . A number  $M \in \mathbb{R}$  is an **upper bound of  $X$**  if for every  $x \in X$ , we have that  $x \leq M$ .

**Definition 4.1.15.** Let  $q$  be an upper bound of  $X$ . We say  $q$  is the **supremum of  $X$**  (or **least upper bound of  $X$** ) if for all upper bounds  $M$  of  $X$ , we have that  $q \leq M$ .

We write  $q := \sup(X)$ . If  $X$  is not bounded above, we write  $\sup(X) = \infty$ .

**Definition 4.1.16.** Let  $X \subset \mathbb{R}$ . A number  $m \in \mathbb{R}$  is an **lower bound of  $X$**  if for every  $x \in X$ , we have that  $m \leq x$ .

**Definition 4.1.17.** Let  $p$  be a lower bound of  $X$ . We say  $p$  is the **infimum of  $X$**  (or **greatest lower bound of  $X$** ) if for all lower bounds  $m$  of  $X$ , we have that  $m \leq p$ .

We write  $p := \inf(X)$ . If  $X$  is not bounded below, we write  $\inf(X) = -\infty$ .

Consider the set  $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ . What is  $\sup(X)$  and  $\inf(X)$ ?

We would often like to describe the largest and smallest element of a subset of  $\mathbb{R}$ . However, if we consider sets like  $X = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$  and  $Y = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ , we see immediately that there is no largest element in  $X$ , and that there is no smallest in  $Y$ . Thus, we need to use the notions of the supremum and infimum, instead.

**Theorem 4.1.18** (Completeness of  $\mathbb{R}$ ). Every nonempty subset  $X \subset \mathbb{R}$  has a supremum and infimum. Moreover,  $\sup(X)$  and  $\inf(X)$  are unique.

<sup>2</sup>See proposition 4.1.32 and exercise 4.10

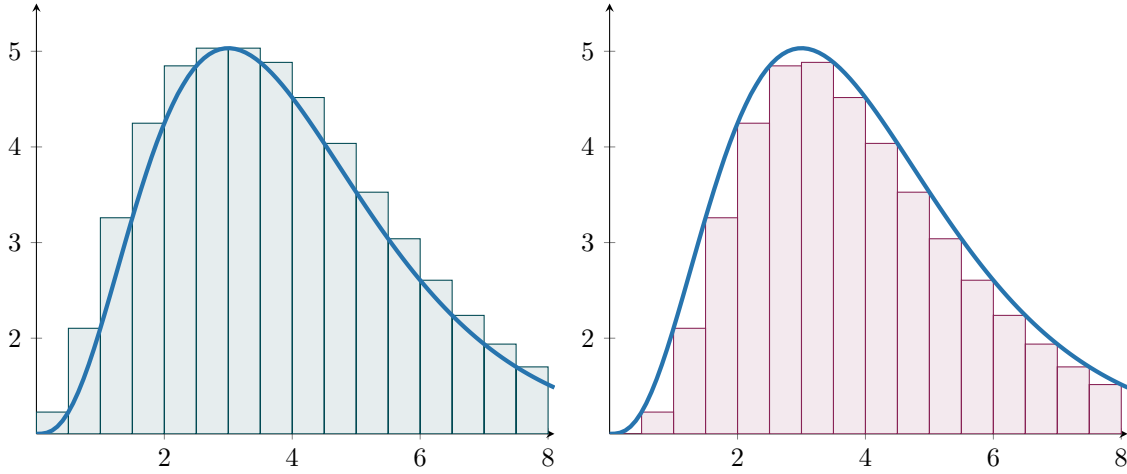
**Definition 4.1.19.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function, and  $D \subset \mathbb{R}^n$  an arbitrary subset. We will consider the following quantities:

$$M_D(f) := \sup(\{f(\mathbf{x}) \mid \mathbf{x} \in D\}) \quad m_D(f) := \inf(\{f(\mathbf{x}) \mid \mathbf{x} \in D\})$$

Consider the subset  $C_{1,2}$  in  $\mathbb{R}^1$ , and let  $f(x) = \frac{1}{x}$ .  
Calculate  $M_{C_{1,2}}(f)$  and  $m_{C_{1,2}}(f)$ .

**Definition 4.1.20.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function with bounded support. The *N-th upper Darboux sum* and *N-th lower Darboux sum*  $f$  are defined as

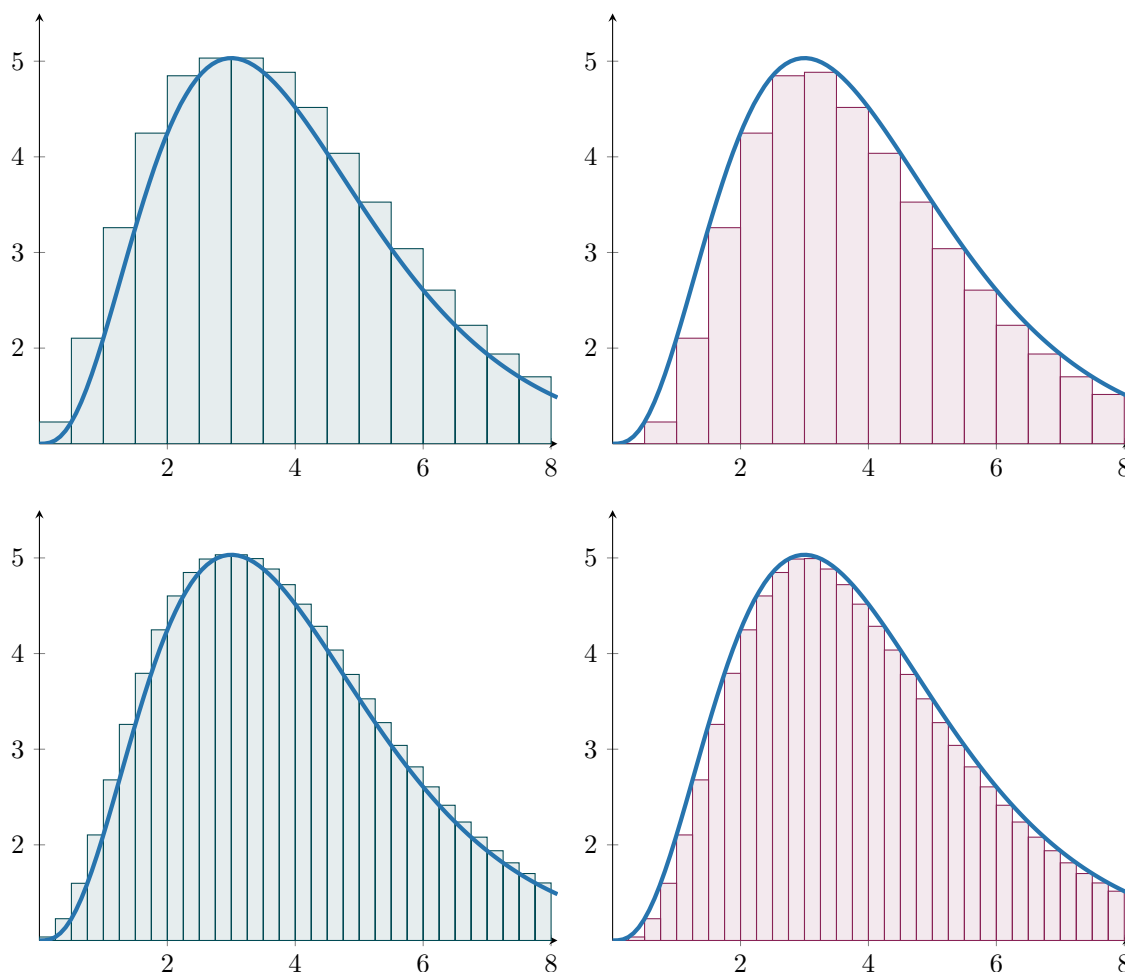
$$U_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} M_C(f) \text{vol}(C) \quad L_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} m_C(f) \text{vol}(C)$$



**Proposition 4.1.21.**

$$U_N(f) := \frac{1}{2^{Nn}} \sum_{C \in D_N(\mathbb{R}^n)} M_C(f) \quad L_N(f) := \frac{1}{2^{Nn}} \sum_{C \in D_N(\mathbb{R}^n)} m_C(f)$$

**Proposition 4.1.22.** As  $N$  increases,  $U_N(f)$  decreases and  $L_N(f)$  increases.



**Definition 4.1.23.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function with bounded support. The **upper Darboux integral** and **lower Darboux integral** of  $f$  are defined as

$$U(f) := \lim_{N \rightarrow \infty} U_N(f) \quad L(f) := \lim_{N \rightarrow \infty} L_N(f)$$

**Remark 4.1.24.** *limits exist*

**Definition 4.1.25.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function with bounded support. We say that  $f$  is **integrable** if  $U(f) = L(f)$ .

The **integral** of  $f$  is

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \, dV := U(f) = L(f)$$

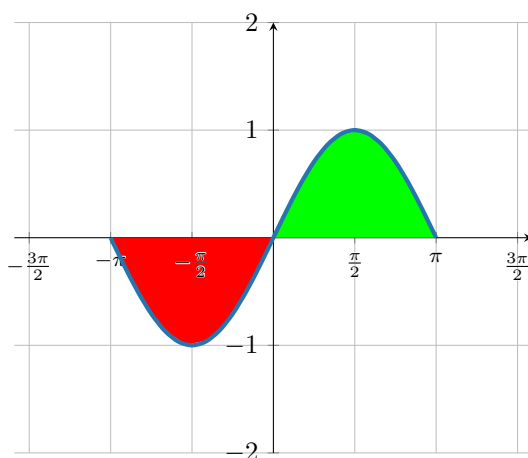
By proposition 2.2.2, this is equivalent to asking that for any  $\varepsilon > 0$ , there exists  $N$  such that  $|U_N(f) - L_N(f)| < \varepsilon$

**Corollary 4.1.26.** If  $f$  is integrable, we can use  $U_N(f)$  or  $L_N(f)$  to estimate  $\int_{\mathbb{R}^n} f(\mathbf{x}) \, dV$ .

How should we interpret our definition of the integral?



**Remark 4.1.27.** If  $f(\mathbf{x})$  is positive, then the integral is the  $n$ -dimensional volume under the graph of  $f(\mathbf{x})$ . Otherwise, we can think of the integral as being the **signed** volume under the graph of  $f(\mathbf{x})$ .



What functions are integrable?

**Theorem 4.1.28.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function that is bounded with bounded support, then  $f$  is integrable.

PICTURE

However, there are some functions that are “obviously” integrable, yet are not continuous:

Consider the unit square  $A = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ . We can define an indicator function to be the function  $1_A : \mathbb{R}^n \rightarrow \mathbb{R}$  is the function

$$1_A(\mathbf{x}) := \begin{cases} 1 & \text{if } \mathbf{x} \in A \\ 0 & \text{if } \mathbf{x} \notin A \end{cases}$$

We expect the integral of  $1_A(\mathbf{x})$  to give us the area of  $A$ . However, this function is not continuous.

In section 4.3, we will discuss criteria for a function  $f$  to be integrable. For now, whenever we say integrable, you should think of continuous functions that are bounded with bounded support.

**Remark 4.1.29.** Note that if the integral  $\int_{\mathbb{R}^n} f \, dV$  exists, then these two boundedness conditions ( $f$  is bounded and  $f$  has bounded support) guarantee that the integral is finite.

**Remark 4.1.30.** Using the Darboux formulation, it follows that if  $f$  is integrable, then any choice of sampling *will result in the same value for the integral*

**Definition 4.1.31.** Recall that given a choice of  $\mathbf{x}_{\mathbf{k},N} \in C_{\mathbf{k},N}$  for every cube, the  $N$ -th **Riemann sum** is defined as

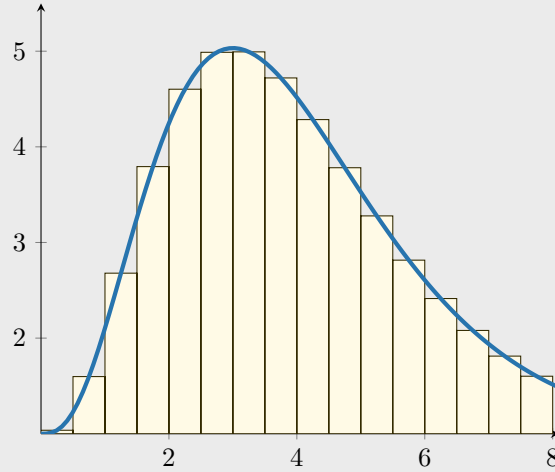
$$R_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} f(\mathbf{x}_{\mathbf{k},N}) \, \text{vol}(C)$$

This yields the **Riemann integral**

$$\lim_{N \rightarrow \infty} R_N(f) = \int_{\mathbb{R}^n} f(\mathbf{x}) \, dV$$

For example, we can choose the **midpoint rule** by choosing

$$\mathbf{x}_{k,N} = \left\langle \frac{2k_1 + 1}{2^{N+1}}, \dots, \frac{2k_n + 1}{2^{N+1}} \right\rangle$$



**Warning!** This only works if you know  $f$  is integrable. The Riemann sum may converge even if the function is not integrable. Consider the function:

$$f(x) := \begin{cases} 1 & \text{if } x \in [0, 1] \text{ and } x \notin \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 4.1.32.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable, then the integral is the limit of the Riemann sum.*

*Proof.* Observe that for all cubes  $C$ ,

$$M_C(f) \geq f(\mathbf{x}_{k,N}) \geq m_C(f)$$

Thus,

$$U_N(f) \geq R_N(f) \geq L_N(f)$$

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable, by the squeeze theorem,  $\lim_{N \rightarrow \infty} R_N(f) = \int_{\mathbb{R}^n} f(\mathbf{x}) \, dV$  □

**Integration over subsets of  $\mathbb{R}^n$**

Observe that we have defined the analogue of  $\int_{-\infty}^{\infty} f(x) \, dx$  in single variable calculus.

How can we define the analogue of  $\int_a^b f(x) \, dx$ ?

In other words, how do we integrate over sub-regions of  $\mathbb{R}^n$ ?

Suppose that  $B \subset \mathbb{R}^n$  is a region in  $\mathbb{R}^n$ . We have two potential ways to define the integral of  $f$  over  $B$  (denoted  $\int_B f \, dV$ ). We can either:

- a. incorporate  $B$  into the definition of the integral  $\int_B f \, dV$ . In other words, given any subset  $B$ , we would need to partition  $B$  into smaller subsets, and take a limit.
- b. or we can treat  $B$  as part of the **input** of the integral. That is, we can use the partition of  $\mathbb{R}^n$  that we constructed previously to yield a partition of  $B$ .

We will take the second perspective, and define the integral of  $f$  over  $B$  in terms of some integral  $\int_{\mathbb{R}^n} g \, dV$ . We can do so using **indicator functions**:

**Definition 4.1.33.** Let  $B \subset \mathbb{R}^n$  be a subset. The **indicator function**  $1_B : \mathbb{R}^n \rightarrow \mathbb{R}$  is the function

$$1_B(\mathbf{x}) := \begin{cases} 1 & \text{if } \mathbf{x} \in B \\ 0 & \text{if } \mathbf{x} \notin B \end{cases}$$

Observe that given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , then the function  $f(\mathbf{x})1_B(\mathbf{x})$  is the piecewise function

$$f(\mathbf{x})1_B(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in B \\ 0 & \text{if } \mathbf{x} \notin B \end{cases}$$

This construction restricts the support of  $f$  to the region  $B$ . Thus, to integrate a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  over  $B$ , it is equivalent to integrate the function  $f(\mathbf{x})1_B(\mathbf{x})$  instead.

**Definition 4.1.34.** Let  $B \subset \mathbb{R}^n$ , and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an integrable function. Then we can define the **integral of  $f$  over the region  $B$**  (denoted  $\int_B f(\mathbf{x}) \, dV$ ) as

$$\int_B f(\mathbf{x}) \, dV := \int_{\mathbb{R}^n} f(\mathbf{x})1_B(\mathbf{x}) \, dV$$

picture

**Definition 4.1.35.** Let  $A \subset \mathbb{R}^n$ . If  $1_A : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable, then the  **$n$ -dimensional volume of  $A$**  is

$$\text{vol}_n(A) := \int_{\mathbb{R}^n} 1_A \, dV$$

Let  $A = [0, 1] \times [0, 1]$  be the unit square in  $\mathbb{R}^2$ . Then the 2-dimensional volume of  $A$  (a.k.a the area of  $A$ ) is 1.

Recall definition 2.1.11, which defines the **graph of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$**  to be the following subset of  $\mathbb{R}^{n+1}$ :

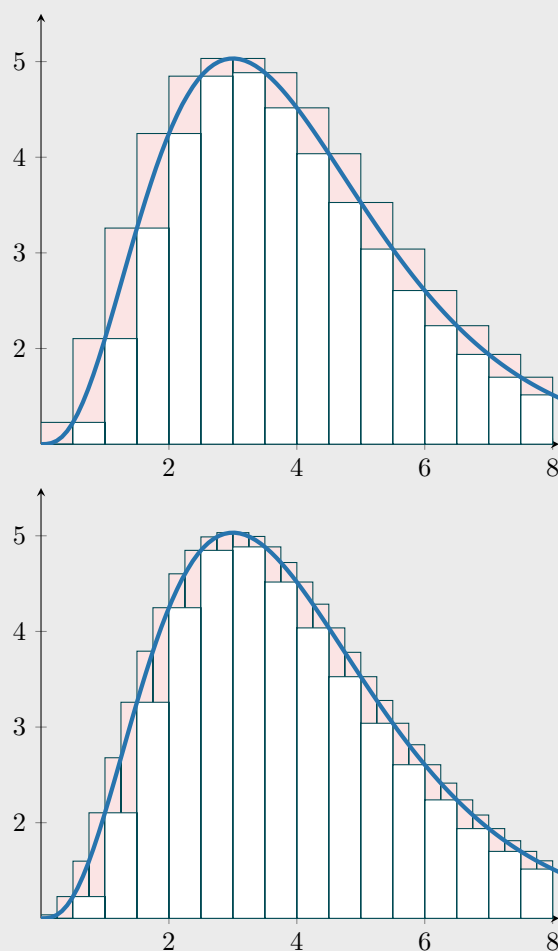
$$\Gamma_f := \{(x_1, \dots, x_n, f(x_1, \dots, x_n))\} \subset \mathbb{R}^{n+1}$$

In other words, the graph is given by the equation

$$x_{n+1} = f(x_1, \dots, x_n)$$

in  $\mathbb{R}^{n+1}$ .

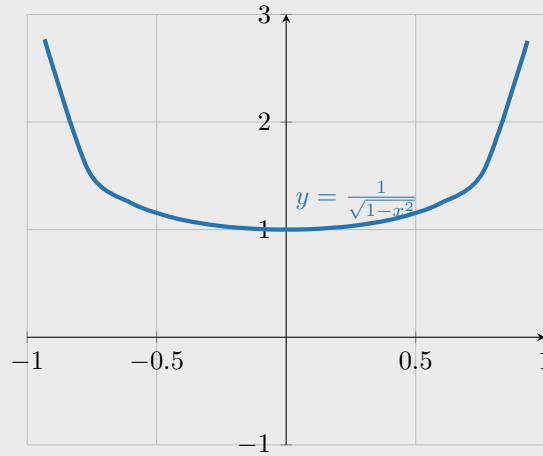
**Theorem 4.1.36.** If  $X \subset \mathbb{R}^n$  is a closed and bounded (compact) region, and  $f : X \rightarrow \mathbb{R}$  be a continuous function, then the  $(n+1)$ -dimensional volume of the graph  $\Gamma_f$  is 0.



Moreover, we can also use indicator functions to integrate functions that are not defined on all of  $\mathbb{R}^n$ !

$$f(x) = e^{-\frac{1}{\sqrt{1-x^2}}}$$

Consider the function  $f(x) = \frac{1}{\sqrt{1-x^2}}$ , which is only defined on  $(-1, 1)$ .



Recall that  $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = [\arcsin x]_{-1}^1 = \pi$ .

Thinking of the integral as the signed volume under the graph, we should expect

$$\int_{\mathbb{R}} \frac{1}{\sqrt{1-x^2}} dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \pi$$

We can make this precise using indicator functions!

**Definition 4.1.37.** Let  $A \subset \mathbb{R}^n$ , and let  $f : A \rightarrow \mathbb{R}$  be a function. We can extend  $f$  to a function  $\tilde{f}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  by defining

$$\tilde{f}(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in A \\ 0 & \text{if } \mathbf{x} \notin A \end{cases}$$

We will often use the following abusive notation when we want to indicate the domain  $A$ :

$$f(\mathbf{x})1_A(\mathbf{x}) := \tilde{f}(\mathbf{x})$$

**Definition 4.1.38.** A function  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is **integrable** if the function  $f(\mathbf{x})1_A(\mathbf{x})$  is integrable, and we can define the integral of  $f$  (denoted  $\int_A f(\mathbf{x}) dV$ ) as

$$\int_A f(\mathbf{x}) dV := \int_{\mathbb{R}^n} f(\mathbf{x})1_A(\mathbf{x}) dV$$

Let  $f : (-1, 1) \rightarrow \mathbb{R}$  be the function defined by  $f(x) = \frac{1}{\sqrt{1-x^2}}$ . We can extend it to the function  $\tilde{f}(x) : \mathbb{R} \rightarrow \mathbb{R}$  by defining

$$\tilde{f}(x) := \begin{cases} \frac{1}{\sqrt{1-x^2}} & \text{if } x \in (-1, 1) \\ 0 & \text{else} \end{cases}$$

We can combine these two notions as well to define integrals of integrable functions  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  over regions  $B \subset \mathbb{R}^n$ .

**Definition 4.1.39.** Let  $B \subset \mathbb{R}^n$ , and let  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be an integrable function. Then we can define the integral  $\int_B f(\mathbf{x}) \, dV$  as

$$\int_B f(\mathbf{x}) \, dV := \int_{\mathbb{R}^n} f(\mathbf{x}) 1_A(\mathbf{x}) 1_B(\mathbf{x}) \, dV$$

By construction,

$$\begin{aligned} \int_B f(\mathbf{x}) \, dV &= \int_B \tilde{f}(\mathbf{x}) \, dV \\ &= \int_{\mathbb{R}^n} \tilde{f}(\mathbf{x}) 1_B(\mathbf{x}) \, dV \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) 1_A(\mathbf{x}) 1_B(\mathbf{x}) \, dV \end{aligned}$$

### Properties of the integral

In this section, we will prove some properties about the multivariable integral. Recall that when you see integrability, you should think about continuous functions that are bounded with bounded support.

**Theorem 4.1.40.** Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  be two integrable functions. Then

a.  $f + g$  is also integrable, and

$$\int_{\mathbb{R}^n} f + g \, dV = \int_{\mathbb{R}^n} f \, dV + \int_{\mathbb{R}^n} g \, dV$$

b. If  $\lambda \in \mathbb{R}$ , then  $\lambda f$  is integrable, and

$$\int_{\mathbb{R}^n} \lambda f \, dV = \lambda \int_{\mathbb{R}^n} f \, dV$$

c. If  $f(\mathbf{x}) \leq g(\mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} f \, dV \leq \int_{\mathbb{R}^n} g \, dV$$

d.  $|f|(\mathbf{x}) := |f(\mathbf{x})|$  is integrable, and

$$\left| \int_{\mathbb{R}^n} f \, dV \right| \leq \int_{\mathbb{R}^n} |f| \, dV$$

Note that these properties follow immediately from the properties of limits.

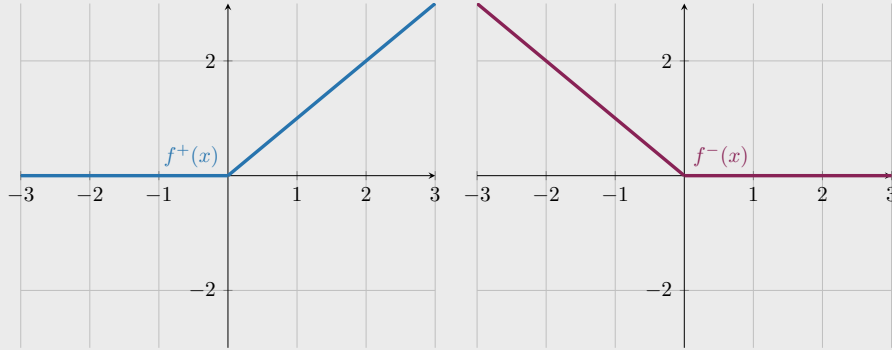
**Corollary 4.1.41.** We can reduce to studying integrals of non-negative functions.

**Definition 4.1.42.** Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define two auxillary **non-negative functions**,  $f^+$  and  $f^-$ .

$$f^+(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } f(\mathbf{x}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad f^-(\mathbf{x}) := \begin{cases} -f(\mathbf{x}) & \text{if } f(\mathbf{x}) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let  $f(x) = x$ . Then

$$f^+(x) := \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad f^-(x) := \begin{cases} -x & \text{if } x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



**Proposition 4.1.43.**  $f(x) = f^+(x) - f^-(x)$ .

**Proposition 4.1.44.** Suppose that  $f(\mathbf{x})$  is integrable on  $\mathbb{R}^n$ , and  $g(\mathbf{y})$  is integrable on  $\mathbb{R}^m$ . Then  $h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$  is integrable on  $\mathbb{R}^{n+m}$ , and

$$\int_{\mathbb{R}^{n+m}} h \, dV \, dW = \int_{\mathbb{R}^n} f \, dV \int_{\mathbb{R}^m} g \, dW$$

**Remark 4.1.45. Warning:** Note that  $\mathbf{x}$  and  $\mathbf{y}$  are necessarily different variables. Furthermore,

$$\int_{\mathbb{R}^n} f(\mathbf{x})g(\mathbf{x}) \, dV \neq \int_{\mathbb{R}^n} f(\mathbf{x}) \, dV \int_{\mathbb{R}^n} g(\mathbf{x}) \, dV$$

Consider the functions  $f(x) = x^2$ ,  $g(x) = x$ , and the region  $A = [0, 1] \subset \mathbb{R}$ . Observe that

$$\int_0^1 f(x)g(x) \, dx = \int_0^1 x^3 \, dx \neq \left( \int_0^1 x^2 \, dx \right) \left( \int_0^1 x \, dx \right)$$

The correct interpretation of the right hand side should be

$$\int_{A \times A} x^2 y \, dA$$

We will see how to calculate this in the next section.

### Exercises

**4.1** Come up with examples of

- a multivariable function that is not bounded.
- a multivariable function that does not have bounded support.
- A multivariable function that is neither bounded nor has bounded support.

**4.2** Come up with an example of a function  $f(x) : \mathbb{R} \rightarrow \mathbb{R}$  with bounded support, but  $\text{supp}(f)$  is not an interval  $[a, b]$ .

**4.3** Prove that if  $f$  and  $g$  are functions that are bounded with bounded support, then  $f + g$  and  $\lambda f$  are also functions that are bounded with bounded support.

**4.4** Explicitly describe and sketch the dyadic cube  $C_{\langle 1,1 \rangle,2}$  in  $\mathbb{R}^2$ .

**4.5** Explicitly describe and sketch the dyadic cube  $C_{\langle 1,1,1 \rangle,1}$  in  $\mathbb{R}^3$ .

**4.6** Consider the subset  $C_{\langle 1,1 \rangle,2}$  in  $\mathbb{R}^2$ , and let  $f(x, y) = x^2 + y^2$ . Calculate  $M_{C_{\langle 1,1 \rangle,2}}(f)$  and  $m_{C_{\langle 1,1 \rangle,2}}(f)$ .

**4.7** Consider the subset  $C_{\langle 1,1 \rangle,2}$  in  $\mathbb{R}^2$ , and let  $f(x, y) = \frac{1}{x^2 + y^2}$ . Calculate  $M_{C_{\langle 1,1 \rangle,2}}(f)$  and  $m_{C_{\langle 1,1 \rangle,2}}(f)$ .

**4.8** Consider the subset  $C_{0,2}$  in  $\mathbb{R}^1$ , and let  $f(x) = x^2$ . Calculate  $M_{C_{0,2}}(f)$  and  $m_{C_{0,2}}(f)$ .

**4.9** Consider the subset  $C_{0,2}$  in  $\mathbb{R}^1$ , and let  $f(x) = \frac{1}{x}$ . Calculate  $M_{C_{0,2}}(f)$  and  $m_{C_{0,2}}(f)$ .

**4.10** Show that if  $f(x) < g(x)$  for all  $x$ , then it is not necessarily the case that  $\sup(f(x)) < \sup(g(x))$  for all  $x$ . (**Hint:** consider the function  $f(x) := \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$ )

**4.11** Let  $A, B \subset \mathbb{R}$  be nonempty sets. Define the subset  $A + B$  by

$$A + B := \{z \in \mathbb{R} \mid \text{there exist } x \in X, y \in Y \text{ such that } z = x + y\}$$

a. Show that  $\sup(A + B) = \sup(A) + \sup(B)$ .

b. Show that  $\inf(A + B) = \inf(A) + \inf(B)$ .

**4.12** Let  $A, B \subset \mathbb{R}$  be nonempty sets. Define the subset  $A - B$  by

$$A - B := \{z \in \mathbb{R} \mid \text{there exist } x \in X, y \in Y \text{ such that } z = x - y\}$$

a. Show that  $\sup(A - B) = \sup(A) - \inf(B)$ .

b. Show that  $\inf(A - B) = \inf(A) - \sup(B)$ .

**4.13** Consider the set  $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ .

a. Do  $\max(A)$  or  $\min(A)$  exist?

b. Find  $\sup(A)$  and  $\inf(A)$ .

**4.14** Consider the set  $A = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$ .

a. Do  $\max(A)$  or  $\min(A)$  exist?

b. Find  $\sup(A)$  and  $\inf(A)$ .

**4.15** Prove that  $U_N(f) \geq L_N(f)$ .

**4.16** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function that is bounded with bounded support. Prove that  $U(f) := \lim_{N \rightarrow \infty} U_N(f)$  exists. (You will need to use exercise 2.26).

**reference problem**

**4.17** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function that is bounded with bounded support. Prove that  $L(f) := \lim_{N \rightarrow \infty} L_N(f)$  exists. (You will need to use exercise 2.27).



**4.18** Let  $\mathbf{R}_{\mathbf{k},N} \in C_{\mathbf{k},N}$  be an arbitrary sampling of the dyadic partition, and define

$$R_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} f(\mathbf{R}_{\mathbf{k},N}) \text{vol}(C)$$

Prove that if  $f$  is integrable, then

$$\lim_{N \rightarrow \infty} R_N(f) = \int_{\mathbb{R}^n} f \, dV$$

(**Hint:** use the squeeze theorem!)

**4.19** Give an example of an integrable function  $f$  such that

$$\left| \int_{\mathbb{R}^n} f \, dV \right| < \int_{\mathbb{R}^n} |f| \, dV$$

(that is, an example of strict inequality)

**4.20** Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable functions, and consider the function  $\sup(f, g)$  defined by

$$\sup(f, g)(\mathbf{x}) := \sup(f(\mathbf{x}), g(\mathbf{x}))$$

Show that  $\sup(f, g)$  is integrable. (**Hint:** Write  $\sup(f, g)$  in terms of  $f^+, f^-, g^+, g^-$ ).

**4.21** Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable functions, and consider the function  $\inf(f, g)$  defined by

$$\inf(f, g)(\mathbf{x}) := \inf(f(\mathbf{x}), g(\mathbf{x}))$$

Show that  $\inf(f, g)$  is integrable. (**Hint:** Write  $\inf(f, g)$  in terms of  $f^+, f^-, g^+, g^-$ ).

## 4.2 Calculating multivariable integrals

Given a region  $B \subset \mathbb{R}^n$  and an integrable function  $f$ , how can we compute  $\int_B f(\mathbf{x}) \, dV$ ?

**Theorem 4.2.1** (Fubini). *Let  $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function that is bounded with bounded support, and let  $(i_1, \dots, i_n)$  be a permutation of the set  $\{1, \dots, n\}$ . Then*

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \, dV = \int_{-\infty}^{\infty} \left( \cdots \left( \int_{-\infty}^{\infty} f(\mathbf{x}) \, dx_{i_1} \right) \cdots \right) dx_{i_n}$$

That is, we can compute an integral  $\int_{\mathbb{R}^n} f(\mathbf{x}) \, dV$  as an iterated integral, in any variable order!

Let  $R$  be the region  $[0, 1] \times [0, 1]$ . Compute the integral

$$\iint_R x e^{xy} \, dA$$

By Fubini's theorem, there are two iterated integrals we can use (namely,  $\int_0^1 \int_0^1 x e^{xy} \, dx \, dy$  or  $\int_0^1 \int_0^1 x e^{xy} \, dy \, dx$ ).

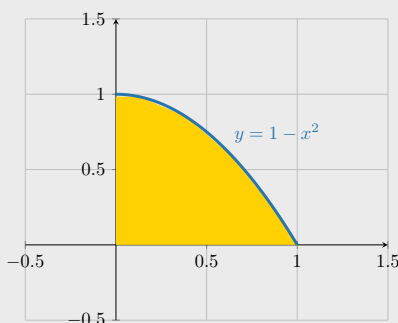
However, observe that the former requires integration by parts. However, we can compute

$$\begin{aligned}
 \iint_R x e^{xy} \, dA &= \int_0^1 \int_0^1 x e^{xy} \, dy \, dx \\
 &= \int_0^1 [e^{xy}]_{y=0}^{y=1} \, dx \\
 &= \int_0^1 e^x - 1 \, dx \\
 &= e - 2
 \end{aligned}$$

**Remark 4.2.2.** *Relationship to Clairaut's theorem* (see corollary 3.2.27).

Typically, the most difficult thing to do in multivariable integration is to visualize the region that you're integrating over.

Compute the volume of the solid region enclosed between the graph of  $f(x, y) = y$  and the  $xy$ -plane, over the region below:



Calculate the integral of  $f(x, y, z) = xyz$  over the region  $W$  in the first octant (that is,  $x, y, z \geq 0$ ) bounded by  $z = 4 - y^2$ ,  $z = 0$ ,  $y = 2x$ , and  $x = 0$ .

We can sketch this region as the region between  $z = 0$  and the parabolic cylinder ( $z = 4 - y^2$ ), sitting above the triangle cut out by  $x = 0$ ,  $y = 2x$ , and  $y = 2$ . We can thus write the region  $W$  as

$$\{(x, y, z) \mid 0 \leq z \leq 4 - y^2, 0 \leq x \leq \frac{y}{2}, 0 \leq y \leq 2\}$$

Thus, we can compute the integral as

$$\int_0^2 \int_0^{\frac{y}{2}} \int_0^{4-y^2} xyz \, dz \, dx \, dy = \frac{2}{3}$$

Alternatively, we can sketch this region as the region between  $x = 0$  and the plane  $x = \frac{y}{2}$ , sitting above the parabola cut out by  $z = 4 - y^2$ ,  $z = 0$ , and  $y = 0$ . We can thus write the region  $W$  as

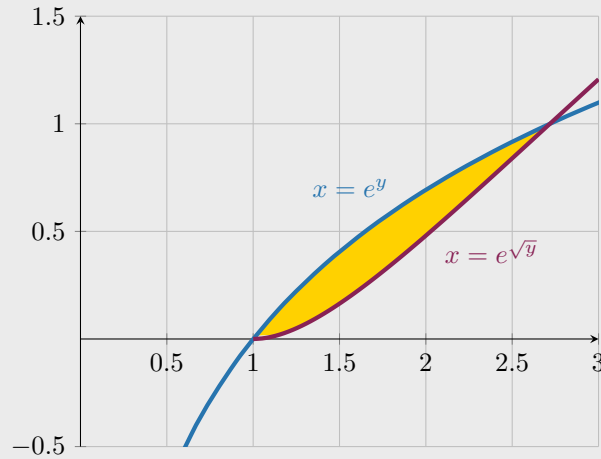
$$\{(x, y, z) \mid 0 \leq x \leq \frac{y}{2}, 0 \leq z \leq 4 - y^2, 0 \leq y \leq 2\}$$

Thus, we can also compute the integral as

$$\int_0^2 \int_0^{4-y^2} \int_0^{\frac{y}{2}} xyz \, dx \, dz \, dy = \frac{2}{3}$$

As Clairaut's theorem stipulates, we can choose the iterated integral in the order that we want.

Compute the integral of  $f(x, y) = \frac{1}{\ln(x)}$  over the domain  $D$  bounded by  $x = e^y$  and  $x = e^{\sqrt{y}}$ .



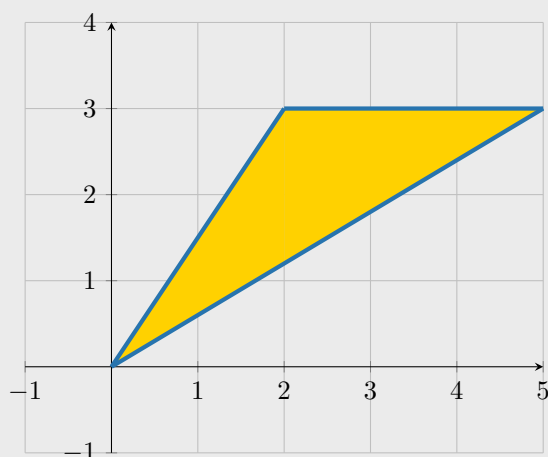
**Theorem 4.2.3** (Decomposition of Domains). *Let  $K$  be a compact (closed and bounded) subset in  $\mathbb{R}^n$  such that its boundary  $\partial D$  has volume zero. Furthermore, let  $K = K_1 \cup K_2$ , such that  $K_1$  and  $K_2$  are compact, and the intersection  $K_1 \cap K_2$  has volume  $o^3$ .*

*Let  $f : K \rightarrow \mathbb{R}$  be a continuous function. Then  $f$  is integrable over  $K_1$  and  $K_2$ , and*

$$\int_K f(\mathbf{x}) \, dA = \int_{K_1} f(\mathbf{x}) \, dA + \int_{K_2} f(\mathbf{x}) \, dA$$

Evaluate the double integral of  $f(x, y) = 1 - 2x$  over the region  $K$  as drawn below:

<sup>3</sup>Typically,  $K_1 \cap K_2$  will be an  $(n - 1)$ -dimensional subset (like the graph of a function  $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ , as in theorem 4.1.36)



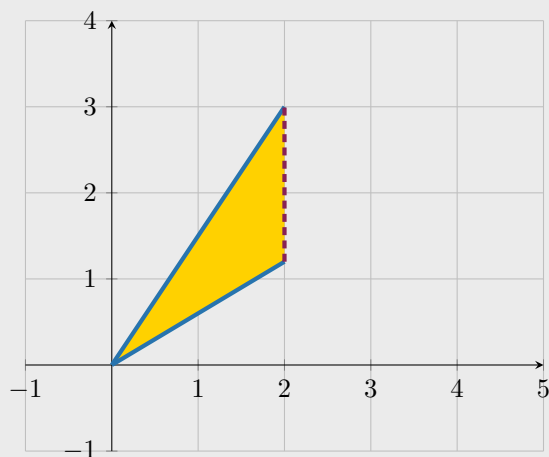
We can decompose  $K$  into two regions,  $K_1$  and  $K_2$  defined as

$$K_1 = \{(x, y) \mid 0 \leq x \leq 2, \frac{3}{5}x \leq y \leq \frac{3}{2}x\}$$

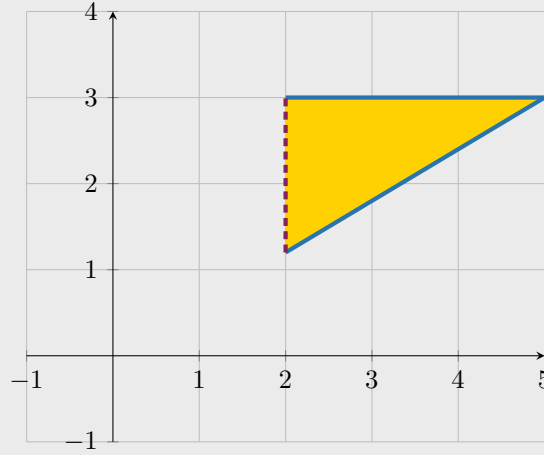
and

$$K_2 = \{(x, y) \mid 2 \leq x \leq 5, \frac{3}{5}x \leq y \leq 3\}$$

Note that  $K_1$  and  $K_2$  are both **vertically simple** (which simplifies calculating the integral).



The region  $K_1$

The region  $K_2$ 

Observe that  $K = K_1 \cup K_2$ , and the intersection  $K_1 \cap K_2$  is the curve  $\{(x, y) \mid \frac{6}{5} \leq y \leq 3, x = 2\}$ , which has volume 0. Therefore,

$$\begin{aligned}
 \iint_K 1 - 2x \, dA &= \iint_{K_1} 1 - 2x \, dA + \iint_{K_2} 1 - 2x \, dA \\
 &= \int_0^2 \int_{\frac{3}{5}x}^{\frac{3}{2}x} 1 - 2x \, dy \, dx + \int_2^5 \int_{\frac{3}{5}x}^3 1 - 2x \, dy \, dx \\
 &= \int_0^2 (1 - 2x) \left( \frac{3}{2}x - \frac{3}{5}x \right) dx + \int_2^5 (1 - 2x) \left( 3 - \frac{3}{5}x \right) dx \\
 &= -3 + -13.5 = -16.5
 \end{aligned}$$

**Definition 4.2.4.** A subset  $D \subset \mathbb{R}^2$  is **vertically simple** if it is the region between graphs of two continuous functions  $y = g_1(x)$  and  $y = g_2(x)$  over a fixed interval of  $x$ -values  $[a, b]$ .

The regions  $K_1 = \{(x, y) \mid 0 \leq x \leq 2, \frac{3}{5}x \leq y \leq \frac{3}{2}x\}$  and  $K_2 = \{(x, y) \mid 2 \leq x \leq 5, \frac{3}{5}x \leq y \leq 3\}$  are both **vertically simple**.

**Definition 4.2.5.** A subset  $D \subset \mathbb{R}^2$  is **horizontally simple** if it is the region between graphs of two continuous functions  $x = h_1(y)$  and  $x = h_2(y)$  over a fixed interval of  $y$ -values  $[c, d]$ .

The region  $K = \{(x, y) \mid \frac{2}{3}y \leq x \leq \frac{5}{3}y, 0 \leq y \leq 3\}$  is horizontally simple. **Warning:** Be careful about the inequalities!

We could therefore have instead computed

$$\begin{aligned}
 \iint_K 1 - 2x \, dA &= \int_0^3 \int_{\frac{2}{3}y}^{\frac{5}{3}y} 1 - 2x \, dx \, dy \\
 &= \int_0^3 \left[ x - x^2 \right]_{\frac{2}{3}y}^{\frac{5}{3}y} dy \\
 &= \int_0^3 \frac{5}{3}y - \left(\frac{5}{3}y\right)^2 - \frac{2}{3}y + \left(\frac{2}{3}y\right)^2 \, dy \\
 &= -16.5
 \end{aligned}$$

### Exercises

**4.22** Let  $R$  be the region  $[1, 2] \times [2, 4]$ . Compute the integral

$$\iint_R e^{3x-y} \, dA = \int_1^2 \int_2^4 e^{3x-y} \, dy \, dx$$

**4.23** Let  $R$  be the region  $[0, 1] \times [0, 1]$ . Compute the integral

$$\iint_R x e^{xy} \, dA$$

**4.24** Compute the integral  $\iint_R x^2 \tan(y) \, dA$  for  $R$  the rectangle  $[0, 2] \times [0, \frac{\pi}{3}]$

**4.25** Compute the integral  $\iint_R \frac{x}{1+xy} \, dA$  for  $R$  the rectangle  $[0, 1] \times [0, 1]$

**4.26** Find the volume of the solid region enclosed between the graph of  $f(x, y) = \frac{1}{\sqrt{x+y}}$  and the  $xy$ -plane, over the rectangle  $[0, 4] \times [0, 5]$ .

**4.27** Compute the integral of  $f(x, y) = \frac{1}{\ln(x)}$  over the domain  $D$  bounded by  $x = e^{4y}$  and  $x = e^{2\sqrt{y}}$ .

**4.28** Find the volume of the region bounded by  $y = 1 - x^2$ ,  $z = 1$ ,  $y = 0$ , and  $z + y = 2$ .

**4.29** The volume contained in a sphere of radius 2 in  $\mathbb{R}^3$  is  $\frac{32\pi}{3}$ . What is the value of the following integral?

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 \int_0^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx$$

**4.30** Integrate  $f(x, y, z) = x$  over the region  $W$  bounded above by  $z = 4 - x^2 - y^2$  and below by  $z = x^2 + 3y^2$  in the octant  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ .

**4.31** Find the volume between the intersection of the surfaces  $z = x^2 + y^2$  and  $z = 10 - x^2 - y^2$ .

**4.32** Compute  $\iiint_W x \, dV$ , where  $W$  is the region in the first octant below the planes  $z = 1 - x$  and  $z = 3 - x - y$ .

### 4.3 Integrability

In the previous sections, we have defined the notion of integrability, which we will restate below:

**Definition 4.3.1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function with bounded support. We say that  $f$  is **integrable** if  $U(f) = L(f)$ .

The **integral** of  $f$  is

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \, dV := U(f) = L(f)$$

**Proposition 4.3.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function with bounded support. Then the following are equivalent:

- a.  $f$  is integrable.
- b. For any  $\varepsilon > 0$ , there exists  $N$  such that for all  $n > N$ ,

$$U_n(f) - L_n(f) < \varepsilon$$

- c. For any  $\varepsilon > 0$ , there exists  $N$  such that,

$$U_N(f) - L_N(f) < \varepsilon$$

*Proof.* To see that (1) is equivalent to (2), we expand out the definitions. That is,  $f$  is integrable if and only if

$$\lim_{N \rightarrow \infty} (U_N(f) - L_N(f)) = 0$$

However, this is equivalent to (2) by proposition 2.2.2 and exercise 4.7 (which tells us that  $U_N(f) - L_N(f) \geq 0$ ).

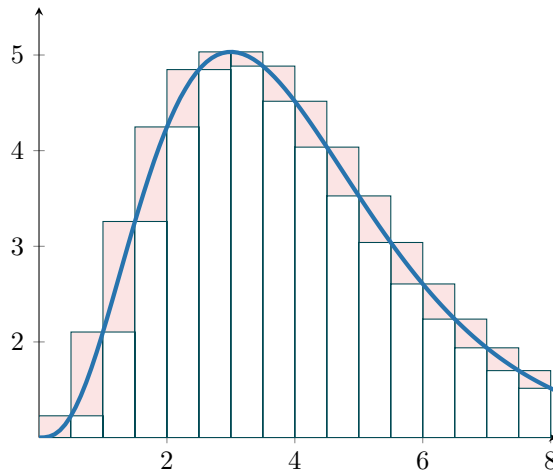
To see that (2) is equivalent to (3), observe from proposition 4.1.22 that  $U_N(f) - L_N(f) \leq U_{N+1}(f) - L_{N+1}(f)$   $\square$

The examples we've had so far are the following:

**Theorem 4.3.3.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function that is bounded with bounded support, then  $f$  is integrable.

**Definition 4.3.4.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function, and let  $A \subset \mathbb{R}^n$ . The **oscillation** of  $f$  over  $A$  is defined as

$$\text{osc}_A(f) := M_A(f) - m_A(f)$$



**Theorem 4.3.5.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable if and only if

- a.  $f$  is bounded with bounded support
- b. For all  $\varepsilon > 0$ , there exists  $N$  such that

$$\sum_{\{C \in D_N \mid \text{osc}_C(f) > \varepsilon\}} \text{vol}_n C < \varepsilon$$

*Proof.* fixthis □

**Proposition 4.3.6.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has bounded support, then there are finitely many cubes  $C_{\mathbf{k},N}$  where  $\text{osc}_{C_{\mathbf{k},N}}(f) \neq 0$ .

**Theorem 4.3.7.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function that is bounded with bounded support, then  $f$  is integrable.

*Proof.* Let  $\varepsilon > 0$ . For a fixed vector  $\mathbf{k}$ , we can always find  $M_{\mathbf{k}}$  such that for  $m > M_{\mathbf{k}}$ ,

$$\text{osc}_{C_{\mathbf{k},m}}(f) < \varepsilon$$

since  $f$  is continuous.

There are finitely many cubes to consider, so we can take  $M = \max\{M_{\mathbf{k}}\}$ . So

$$\sum_{C \in D_M(\mathbb{R}^n)} \text{osc}_C(f) \text{vol}(C) < \sum_{C \in D_M(\mathbb{R}^n)} \varepsilon \frac{1}{2^{Mn}}$$

□

That is, there are **no cubes** where  $\text{osc}_C(f) > \varepsilon$ .

**Proposition 4.3.8.** The function:  $g(x) := \begin{cases} 1 & \text{if } x \in [0, 1] \text{ and } x \notin \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$  is **not** integrable.

*Proof.* For all  $m$ ,  $\text{osc}_{C_{\mathbf{k},m}}(g) = 1$  for **every cube** contained in the region  $[0, 1]$ . Thus,

$$\sum_{C \subset [0,1]} \text{osc}_C(g) \text{vol}(C) = \sum_{C \subset [0,1]} \text{vol}(C) = 1$$

But

$$\sum_{C \subset [0,1]} \text{osc}_C(g) \text{vol}(C) < \sum_{C \in D_m(\mathbb{R})} \text{osc}_C(g) \text{vol}(C)$$

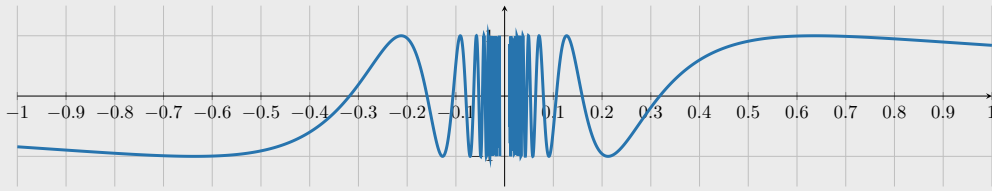
So  $f$  is **not integrable**. □

From these examples, it is tempting to conclude if you cannot bound the oscillation of a function  $f$ , then  $f$  is not integrable. However, this is not the case (see theorem 4.3.10).

**Proposition 4.3.9.** Consider the function:

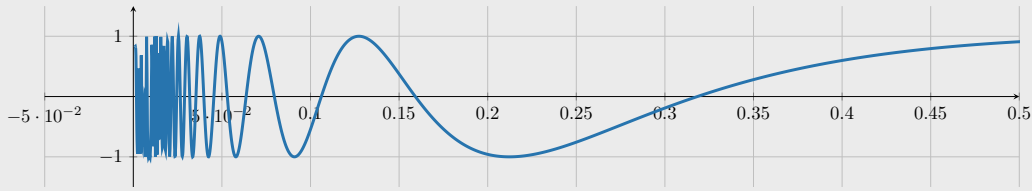
$$h(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$





Observe that as  $x$  approaches 0,  $h(x)$  oscillates between  $-1$  and  $1$ . However,  $h(x)$  **IS** integrable on any interval  $[a, b] \subset \mathbb{R}^n$ .

*Proof.* Observe that  $h(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$  is continuous everywhere except at  $x = 0$ .



Thus, if  $D = C_{0,N}$  or  $D = C_{-1,N}$ , then  $\text{osc}_D(f) = 2$ .

However, for all other  $C_{k,N}$ , we can choose  $M$  large enough so that  $\text{osc}_C(f) < \varepsilon$  for all  $C$ .

So

$$\begin{aligned} \sum_{C \in D_M(\mathbb{R}^n)} \text{osc}_C(f) \text{vol}(C) &= 4\text{vol}(C) + \sum_{\substack{C \in D_M(\mathbb{R}^n) \\ C \neq D}} \text{osc}_C(f) \text{vol}(C) \\ &= \frac{4}{2^{Mn}} + \sum_{\substack{C \in D_M(\mathbb{R}^n) \\ C \neq D}} \varepsilon \frac{1}{2^{Mn}} \end{aligned}$$

□

The correct statement is as follows: a bounded function with bounded support  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable if you can bound the volumes of the cubes where  $f$  oscillates too much.

**Theorem 4.3.10** (Integrability Criterion I). *A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable if and only if*

- $f$  is bounded with bounded support*
- For all  $\varepsilon > 0$ , there exists  $N$  such that*

$$\sum_{\{C \in D_N(\mathbb{R}^n) \mid \text{osc}_C(f) > \varepsilon\}} \text{vol}_n C < \varepsilon$$

This is a useful theorem to have, but it is difficult to use in practice. Here is another integrability theorem:

**Theorem 4.3.11** (Integrability Criterion II). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function with bounded support. If  $f$  is continuous except on a set of volume zero, then  $f$  is integrable.*

**Remark 4.3.12.** *This is **not** an if and only if statement! We will see an example of a function that is integrable, but is discontinuous on a set whose volume is not defined<sup>4</sup>.*

*We will refine this integrability theorem to an if and only if statement in theorem 4.3.19*

<sup>4</sup>see example 4.3

**Proposition 4.3.13.** *A bounded set  $X \subset \mathbb{R}^n$  has  $n$ -dimensional volume 0 if and only if for every  $\varepsilon > 0$ , there exists  $M$  such that*

$$\sum_{\substack{C \in D_M(\mathbb{R}^n) \\ C \cap X \neq \emptyset}} \text{vol}_n(C) < \varepsilon$$

If  $X \subset \mathbb{R}^n$  is a closed and bounded (compact) region, and  $f : X \rightarrow \mathbb{R}$  is a continuous function, then

$$\text{vol}_{n+1}(\Gamma_f) = 0$$

ref

The unit circle in  $\mathbb{R}^2$ ,  $S := \{(x, y) \mid x^2 + y^2 = 1\}$  has 2-dimensional volume 0.

Let  $D$  be the unit disk in  $\mathbb{R}^2$ . That is,  $D := \{(x, y) \mid x^2 + y^2 \leq 1\}$ .

Then the indicator function  $1_D(\mathbf{x}) : \mathbb{R}^2 \rightarrow \mathbb{R}$  is integrable by theorem 4.3.11, as it not continuous on a subset of 2-dimensional volume 0.

**Corollary 4.3.14.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable, and let  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function. If  $f = g$  except on a set of volume 0, then  $g$  is integrable, and*

$$\int_{\mathbb{R}^n} f \, dV = \int_{\mathbb{R}^n} g \, dV$$

Let us now turn to some pathological functions:

We first describe a set that does not have  $n$ -dimensional volume.

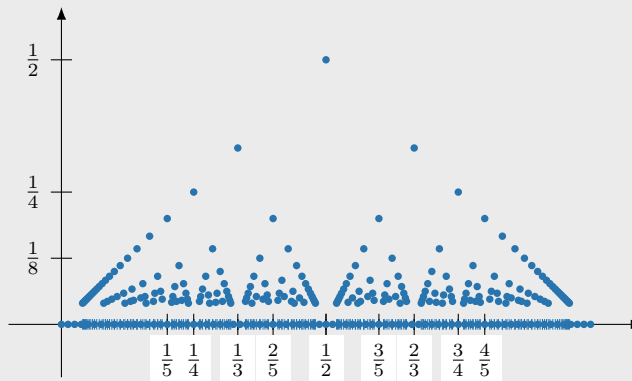
Consider the set  $A \subset \mathbb{R}$  defined by

$$A := \{x \in \mathbb{R} \mid x \in (0, 1) \text{ and } x \in \mathbb{Q}\}$$

$A$  does not have volume, since the indicator function  $1_A(x) : \mathbb{R} \rightarrow \mathbb{R}$  is not integrable (osc( $1_A$ ) = 1 on the interval  $(0, 1)$ ) example

Consider Thomae's function, which is defined as the function

$$g(x) := \begin{cases} \frac{1}{q} & \text{if } x \in (0, 1) \text{ and } x = \frac{p}{q} \text{ is rational, written in lowest terms, with } p \in \mathbb{Z}, q \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$



We see that  $f$  is discontinuous precisely on the set  $A := \{x \in \mathbb{R} \mid x \in (0, 1) \text{ and } x \in \mathbb{Q}\}$

However, by theorem 4.3.10, we see that we can bound the volumes of the cubes where  $f$  oscillates too much, and hence Thomae's function is integrable.

So let us now turn to generalizing theorem 4.3.11.

**Definition 4.3.15.** An open ball  $B \subset \mathbb{R}^n$  of radius  $\delta > 0$ , centered on  $\mathbf{x}$ , is the set

$$B = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{v}\| < \delta\}$$

**Definition 4.3.16.** A set  $X \subset \mathbb{R}^n$  has **measure zero** if for every  $\varepsilon > 0$ , there exists an infinite sequence of open balls  $B_i$  such that

$$X \subset \bigcup B_i \quad \text{and} \quad \sum \text{vol}_n(B_i) < \varepsilon$$

**Proposition 4.3.17.** A set of volume 0 has measure zero.

**Remark 4.3.18.** On the other hand, it is possible that  $X$  has measure zero, but  $\text{vol}(X)$  is undefined.

The set

$$A = \mathbb{Q} \cap [0, 1]$$

that is, the set of rational numbers in the interval  $[0, 1]$ , has undefined volume<sup>a</sup>. However, the set

$$A = \mathbb{Q} \cap [0, 1]$$

has measure zero!

We can enumerate the elements of  $A$  as

$$\{F_n\} := \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \dots\}$$

To each element in the sequence, we can associate  $B_n$  as the open ball of radius  $\frac{1}{2^{n+1}}$ , centered at  $F_n$ .

We can then see that

$$A \subset \bigcup B_n \quad \text{and} \quad \sum \text{vol}_n(B_n) < \varepsilon$$

---

<sup>a</sup>see example 4.3

**Theorem 4.3.19** (Integrability Criterion III). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable if and only if

- a.  $f$  is bounded with bounded support
- b.  $f$  is continuous except on a set of measure 0.

Thomae's function is bounded with bounded support, and is continuous except on a set of measure 0.

### Exercises

**4.33** Come up with (and prove) an example of a multivariable function that is not integrable.

**4.34** Come up with (and prove) an example of a function where  $|f|$  is integrable, but  $f$  is not integrable.

## 4.4 Change of variables

Consider the unit ball (that is, the solid region bounded by the unit sphere) in  $\mathbb{R}^3$ , denoted

$$B := \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

Observe that it is difficult to compute  $\int_B 1 \, dV$  as an iterated integral.

In this section, we will see that changing coordinate systems will make integration easier.

## Polar coordinates

Recall that we've studied vectors in  $\mathbb{R}^2$  primarily in terms of their components. However, we have seen that it is equivalent to describe a position vector in terms of its magnitude and direction. Polar coordinates is a way to describe vectors in  $\mathbb{R}^2$  using this perspective:

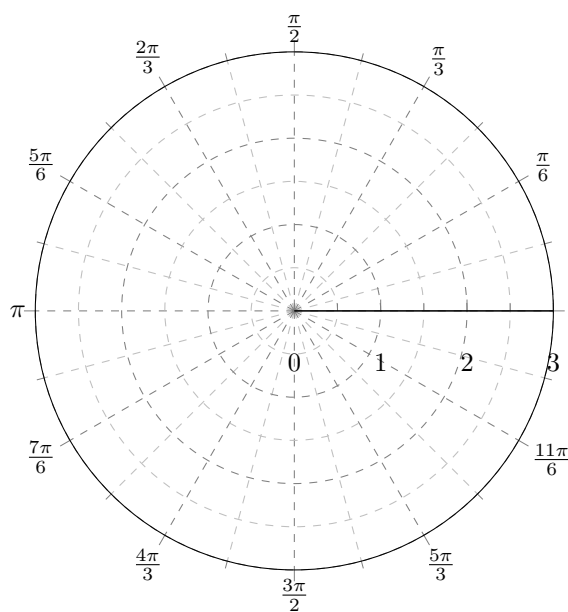
**Definition 4.4.1.** Given a point  $(x, y)$  in rectangular coordinates, we can convert it to a point  $(r, \theta)$  in polar coordinates by setting

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} \quad (\text{assuming } x \neq 0)$$

We can also convert from polar coordinates back to the standard rectangular coordinates:

**Definition 4.4.2.** Given a point  $(r, \theta)$  in polar coordinates, we can convert it to a point  $(x, y)$  in rectangular coordinates by setting

$$x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta)$$



The line  $y = mx$  corresponds to the polar equation

$$\theta = \tan^{-1}(m)$$

[explain more](#)

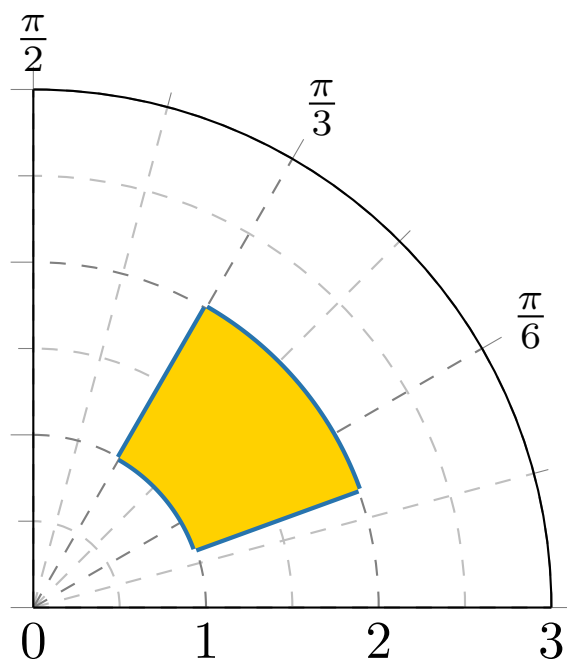
The disk of radius  $R$ , centered at the origin, can be described in polar coordinates as the set of points

$$\{(r, \theta) \mid r \leq R\}$$

Moreover, polar coordinates allows us to describe certain curves in  $\mathbb{R}^2$  much more easily than in rectangular coordinates, such as the three-petaled rose described by  $r = \cos(3\theta)$ .

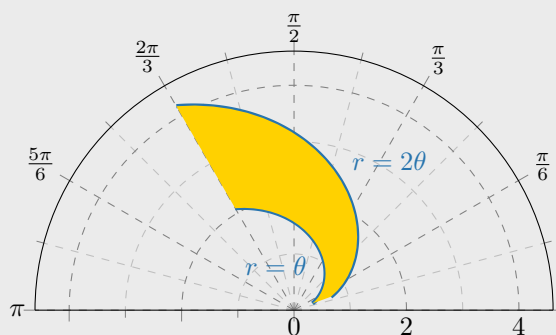
Polar coordinates are convenient when the domain of integration  $R$  is an angular sector, or a **polar rectangle**. That is, if we can describe  $R$  in the form:

$$R = \{(r, \theta) \mid \theta_1 \leq \theta \leq \theta_2, r_1 \leq r \leq r_2\}$$



**Definition 4.4.3.** A region  $R$  is called **radially simple** if it is the region between graphs of two continuous functions  $r_1(\theta)$  and  $r_2(\theta)$  over a fixed interval of  $\theta$ -values. That is,

$$R = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}$$

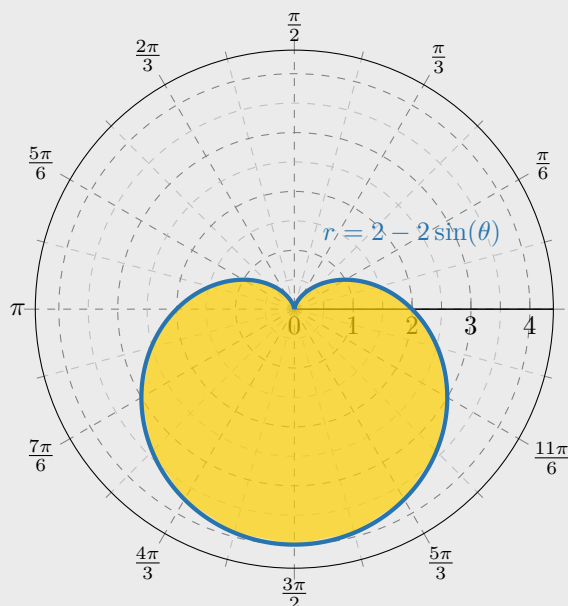


**Theorem 4.4.4** (Double integral in polar coordinates). *If  $f(x, y)$  is a continuous function on a radially simple domain  $R$ , then*

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta$$

**Remark 4.4.5.** *Observe that there is an extra  $r$  term in the integral - we will see why when we consider the general change of variables formula.*

Find the area of the region enclosed by the cardioid  $r = 2 - 2 \sin(\theta)$ .



### Cylindrical coordinates

We can extend polar coordinates in  $\mathbb{R}^2$  to cylindrical coordinates in  $\mathbb{R}^3$  in the following way:

**Definition 4.4.6.** *Given a point  $(x, y, z)$  in Euclidean coordinates, we can convert it to a point  $(r, \theta, z)$  in cylindrical coordinates by setting*

$$z = z \quad \text{and} \quad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} \quad (\text{assuming } x \neq 0)$$

We can also convert from cylindrical coordinates back to the standard Euclidean coordinates:

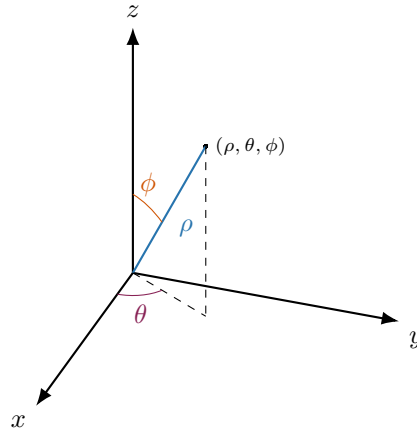
**Definition 4.4.7.** Given a point  $(r, \theta, z)$  in polar coordinates, we can convert it to a point  $(x, y, z)$  in rectangular coordinates by setting

$$x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta) \quad \text{and} \quad z = z$$

Let  $B$  denote the unit ball (that is, the solid region bounded by the unit sphere) in  $\mathbb{R}^3$ . Write  $\int_B 1 \, dV$  as an iterated integral in cylindrical coordinates.

### Spherical coordinates

Another coordinate system for  $\mathbb{R}^3$  can be defined in terms of spherical coordinates:



**Proposition 4.4.8.** Given a point  $(\rho, \theta, \phi)$  in spherical coordinates, we can convert it to the standard Euclidean coordinates by setting

$$x = \rho \sin(\phi) \cos(\theta) \quad y = \rho \sin(\phi) \sin(\theta) \quad z = \rho \cos(\phi)$$

**Proposition 4.4.9.** Given a point  $(x, y, z)$  in standard Euclidean coordinates, we can convert it to spherical coordinates by setting

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \tan(\theta) = \frac{y}{x} \quad \cos(\phi) = \frac{z}{\rho}$$

**Definition 4.4.10.** A solid region  $R \subset \mathbb{R}^3$  is called **centrally simple** if  $R$  is of the form

$$R = \{(\rho, \theta, \phi) \mid \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2, \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)\}$$

### PICTURE

**Theorem 4.4.11** (Triple integrals in spherical coordinates). Let  $f(x, y, z)$  be a continuous function on a centrally simple region  $R$ . Write

$$f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) = g(\theta, \phi, \rho)$$

Then  $\iiint_R f(x, y, z) \, dV$  equals

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho=\rho_1(\theta, \phi)}^{\rho=\rho_2(\theta, \phi)} g(\theta, \phi, \rho) \, \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

**Remark 4.4.12.** Observe that there is an extra  $\rho^2 \sin(\phi)$  term in the integral - we will see why when we consider the general change of variables formula.

Let  $B$  denote the unit ball (that is, the solid region bounded by the unit sphere) in  $\mathbb{R}^3$ . Write  $\int_B 1 \, dV$  as an iterated integral in spherical coordinates.

### General Change of Variables

Why do we need to add extra terms when changing coordinates?

### Linear change of variables

We will first study what happens for linear change of variable maps:

[references](#)

**Proposition 4.4.13.** A function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a **linear map** if for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  and for all  $\lambda \in \mathbb{R}$ ,

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ .
- $T(\lambda \mathbf{u}) = \lambda T(\mathbf{u})$ .

**Theorem 4.4.14.** A map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear if and only if there is a matrix  $A \in M_{m \times n}(\mathbb{R})$  such that

$$T(\mathbf{x}) = A\mathbf{x}$$

We call  $A$  the **standard matrix** of  $T$ .

**Definition 4.4.15.** The **standard matrix**  $A$  of a linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is given by

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix} \in M_{m \times n}(\mathbb{R})$$

Consider the linear map  $T(x, y) = (Ax + By, Cx + Dy)$ . What is the image of the unit square  $R = [0, 1] \times [0, 1]$  under  $G$ ?

**Proposition 4.4.16.** Let  $P$  be the parallelogram spanned by  $\mathbf{u} = \langle A, B \rangle$  and  $\mathbf{v} = \langle C, D \rangle$  in  $\mathbb{R}^2$ .

$$\left| \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right| = \text{area}(P)$$

That is, the absolute value of a  $2 \times 2$  determinant equals the area of the parallelograms spanned by the rows.

**Theorem 4.4.17** (Expansion along the  $i$ -th row). Let  $A$  be an  $n \times n$  matrix, and let  $A_{ij}$  denote the  $(n-1) \times (n-1)$  matrix obtained by deleting row  $i$  and column  $j$  from  $A$ .

$$\det(A) = (-1)^{i+1} a_{i,1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{i,n} \det(A_{in})$$



The **determinant** of a  $3 \times 3$  matrix is denoted and defined as follows:

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

**Theorem 4.4.18.** Let  $D$  be the parallelepiped spanned by  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $\mathbb{R}^3$ . Then

$$\text{volume}(D) = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left| \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \right|$$

**Theorem 4.4.19.** Let  $D$  be the parallelepiped spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in  $\mathbb{R}^n$ . Then

$$\text{vol}_n(D) = \left| \det \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} \right|$$

**Corollary 4.4.20.** Let  $T(\mathbf{x}) = A\mathbf{x}$  be a linear transformation from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . Then the volume of the image of the unit cube in  $\mathbb{R}^n$  is  $|\det A|$ .

**Theorem 4.4.21.** The  $n$ -dimensional volume of a region  $D \subset \mathbb{R}^n$  under a linear transformation  $T(\mathbf{x}) = A\mathbf{x}$  is given by

$$\text{vol}_n(T(D)) = |\det A| \text{vol}_n(D)$$

Nonlinear change of variables

What happens if  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a **non-linear** change of coordinates map?

If  $G$  is differentiable, we can approximate  $G$  with its **derivative** (aka its linear approximation).

**Definition 4.4.22.** Let  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a multivariable function defined by  $f_i : A \subset \mathbb{R}^m \rightarrow \mathbb{R}$ :

$$f(\mathbf{x}) = \begin{bmatrix} f^1(\mathbf{x}) \\ \vdots \\ f^n(\mathbf{x}) \end{bmatrix}$$

The **Jacobian matrix** of  $f$  at  $\mathbf{x}_0$  is

$$[J_f(\mathbf{x}_0)] = \begin{bmatrix} D_1 f^1(\mathbf{x}_0) & D_2 f^1(\mathbf{x}_0) & \cdots & D_m f^1(\mathbf{x}_0) \\ D_1 f^2(\mathbf{x}_0) & D_2 f^2(\mathbf{x}_0) & \cdots & D_m f^2(\mathbf{x}_0) \\ \vdots & \vdots & \vdots & \vdots \\ D_1 f^n(\mathbf{x}_0) & D_2 f^n(\mathbf{x}_0) & \cdots & D_m f^n(\mathbf{x}_0) \end{bmatrix}$$

if the partial derivatives exist.

**Theorem 4.4.23** (The derivative in coordinates). Let  $f : A \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a multivariable function. If  $f$  is **differentiable at  $\mathbf{x}_0$** , then all the partial derivatives  $D_i f^j(\mathbf{x}_0)$  exist, and the standard matrix of  $Df(\mathbf{x}_0)$  is  $[J_f(\mathbf{x}_0)]$ . That is,

$$Df(\mathbf{x}_0)(\mathbf{h}) = [J_f(\mathbf{x}_0)]\mathbf{h}$$

**Remark 4.4.24.** The derivative of a function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  at a point  $\mathbf{x}_0$  is the **approximation** of  $f$  by a **linear transformation** at  $\mathbf{x}_0$ .

Given a differentiable map  $G(u, v) = (x(u, v), y(u, v))$ , we can define its **Jacobian matrix**,  $[J_G]$ , as the matrix of partial derivatives:

$$[J_G] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

The determinant of the Jacobian matrix is denoted  $\text{Jac}(G)$ .

$$\det([J_G]) = \text{Jac}(G) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

**Proposition 4.4.25.** *Let  $D \subset \mathbb{R}^n$  be a region such that  $\text{vol}_n(D)$  is small, and let  $p \in D$ . Let  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be differentiable map. Then*

$$\text{vol}_n(G(D)) \approx |\det([J_G](p))| \text{vol}_n(D)$$

*That is, the  $n$ -dimensional volume of  $G(D)$  can be approximated by the  $n$ -dimensional volume of  $[J_G](p)(D)$ .*

When is  $\text{vol}_n(G(D)) = \text{vol}_n(D)$ ?

There are a few problems that can occur:

- $G$  might not be differentiable
- $\det([J_G](p)) = 0$ . That is,  $J_G$  is not invertible.
- $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is **not injective** on  $D$ . That is, there exist points  $\mathbf{x} \neq \mathbf{y} \in D$  such that  $G(\mathbf{x}) = G(\mathbf{y})$

Consider the linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$T(x, y) = (x + y, 2x + 2y)$$

Note that  $\det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 0$ .

Consider the polar coordinates map defined by

$$G(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

$G$  is **not injective** on the region

$$R = \{(r, \theta) \mid 0 \leq \theta \leq 3\pi, 0 \leq r \leq 2\}$$

**PICTURE**

Consider the polar coordinates map defined by

$$G(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

**$G$  IS injective** on the region

$$R = \{(r, \theta) \mid 0 < \theta < 2\pi, 0 < r < 2\}$$

**Theorem 4.4.26** (Change of variables for volume). *Let  $K \subset \mathbb{R}^n$  be a compact set such that  $\text{vol}_n(\partial K) = 0$ . Let  $U \subset \mathbb{R}^n$  be an open set containing  $K$ . Let*

$$G : U \rightarrow \mathbb{R}^n$$

*be a map such that*

- a.  $G$  is differentiable.
- b.  $G$  is injective on the interior of  $K$ .
- c.  $\det[J_G] \neq 0$  on the interior of  $K$ .

*Then*

$$\int_{G(K)} 1 \, dV = \int_K |\det[J_G]| \, dV$$

**Theorem 4.4.27** (Change of variables). *Let  $K \subset \mathbb{R}^n$  be a compact set such that  $\text{vol}_n(\partial K) = 0$ . Let  $U \subset \mathbb{R}^n$  be an open set containing  $K$ . Let*

$$G : U \rightarrow \mathbb{R}^n$$

*be a map such that*

- a.  $G$  is differentiable.
- b.  $G$  is injective on the interior of  $K$ .
- c.  $\det[J_G] \neq 0$  on the interior of  $K$ .

*Then if  $f : G(K) \rightarrow \mathbb{R}$  is a continuous function, then*

$$\int_{G(K)} f \, dV = \int_K (f \circ G) |\det[J_G]| \, dV$$

Consider the polar coordinates map defined by

$$G(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

**Corollary 4.4.28** (Double integral in polar coordinates). *If  $f(x, y)$  is a continuous function on a polar rectangle  $G(R)$ , then*

$$\iint_{G(R)} f(x, y) \, dA = \int_{\theta_1}^{\theta_2} \int_{r=r_1}^{r=r_2} f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta$$

Let  $D$  be the region in the first quadrant defined by

$$D = \{(x, y) \mid 1 \leq xy \leq 4, 1 \leq y/x \leq 4\}$$

Compute  $\iint_D (x^2 + y^2) \, dx \, dy$ .

**Remark 4.4.29.** Sometimes, it is easier to find a map going in the wrong direction,

$$F(x, y) = (u(x, y), v(x, y))$$

Then  $G = F^{-1}$ .

**Theorem 4.4.30.** If  $G = F^{-1}$ , and  $\det[J_F] \neq 0$ , then  $\det[J_G] = \frac{1}{\det[J_F]}$

#### Exercises

**4.35** Sketch a graph of and find the area enclosed by the polar equation  $r = \sin(2\theta)$ .

**4.36** Sketch a graph of and find the area enclosed by the polar equation  $r = 4 \cos(\theta)$ .

**4.37** Sketch a graph of and find the area enclosed by the polar equation  $r = 1 + \sin(\theta)$ .

**4.38** Sketch a graph of and find the area enclosed by the polar equation  $r = 2 \cos(\theta) - 1$ .

**4.39** The volume of a sphere of radius 2 is  $\frac{32\pi}{3}$ . What is the value of the following integral?

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\pi} \int_0^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

**4.40** Compute the integral

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^1 z \sqrt{x^2 + y^2} \, dz \, dx \, dy$$

**4.41** Let  $D$  be the region in the first quadrant defined by

$$D = \{(x, y) \mid 1 \leq xy \leq 4, 1 \leq y/x \leq 4\}$$

Compute  $\iint_D (x^2 + y^2) \, dx \, dy$ .

**4.42** Consider the parallelogram  $P$  with vertices  $A = (0, 0)$ ,  $B = (3, 1)$ ,  $C = (2, 3)$ ,  $D = (5, 4)$ .

a. Find a change of variables  $G(u, v) = (x(u, v), y(u, v))$  that transforms the unit square to  $P$ . Compute the determinant of the Jacobian of  $G$ .

b. Use your work from the previous problem to compute the integral

$$\iint_P e^{3x-y} \, dA$$

**4.43** Consider the change of variables  $G(u, v) = (uv^{-1}, uv)$  that transforms the square  $[1, 2] \times [1, 2]$  to the region  $D$  in the first quadrant bounded by  $1 \leq xy \leq 4$  and  $1 \leq \frac{y}{x} \leq 4$ . Use this to compute the integral

$$\iint_D x^2 + y^2 \, dA$$

## Chapter 5

### Integration on manifolds

## Chapter 6

### Divergence theorem

## Hints and Solutions

**1.1.** Position vectors.