Math 32B, Calculus of Several Variables
Lecture 26

- TA/Instructor Surveys posted on my.ucla.edu due Saturday at 8am

- LA Survey link on Canvas. due Saturday at 8am

Final exam review Thursday 6/8 4-5pm on Zoom

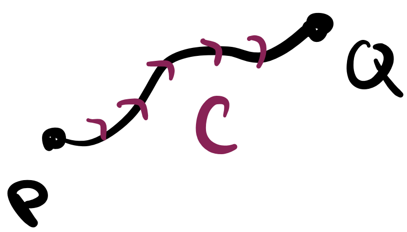
- End of Quarter reflection assignment due before final exam.

Richard Wong

Spring 2023, UCLA

Slides can be found on Canvas.

① State Fundamental Thm of Conservative vector fields.



If $F = \nabla f$ (Scalar potential function)

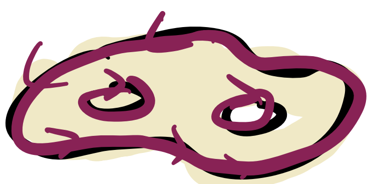
$$\int_C F \cdot dr = f(Q) - f(P)$$

"P & Q are the bdry of C"

Green's Thm If $D \subseteq \mathbb{R}^2$ $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$\oint_{\partial D} F \cdot dr = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

bdry orientation

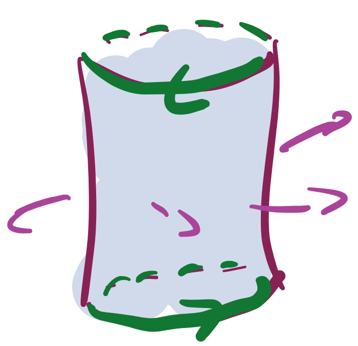


Stoke's Thm

S surface in $\mathbb{R}^3$
∂S is a union of closed curves

$F : \mathbb{R}^3 \to \mathbb{R}^3$
$A : \mathbb{R}^3 \to \mathbb{R}^3$

$F = \text{curl} A$ (the vector potential of F is A)



$$\oint_{\partial S} A \cdot dr = \iint_S \text{curl} A \cdot dS$$

bdry orientation.

Divergence Thm

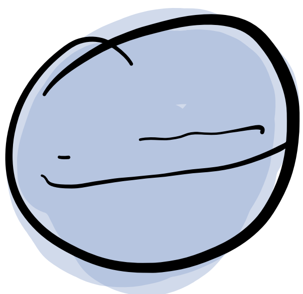
$F : \mathbb{R}^3 \to \mathbb{R}^3$

W is a solid region in $\mathbb{R}^3$

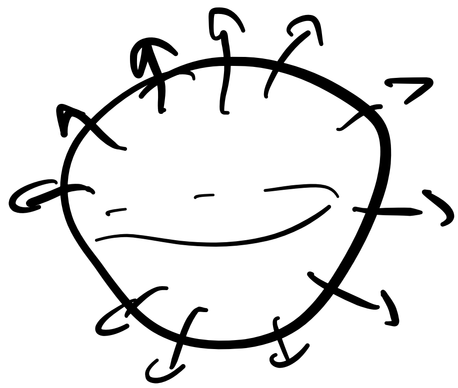
∂W surface that encloses W.

$$\iint_{\partial W} F \cdot dS = \iiint_{W} \operatorname{div} F \, dV$$

how orientation



W is solid region in $\mathbb{R}^3$



∂W is surface w/ normal vectors pointing away from W.

How do we keep track of all of these theorems?

Scalar function · vector field · vector field · Scalar function

$f \longmapsto \nabla f$ · $F \longmapsto \text{curl} F$ · $F \longrightarrow \text{div} F$

$$\{f : \mathbb{R}^3 \to \mathbb{R}\} \xrightarrow{\nabla} \{\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3\} \xrightarrow{\text{curl}} \{\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3\} \xrightarrow{\text{div}} \{f : \mathbb{R}^3 \to \mathbb{R}\}$$

$\text{curl}(\nabla f) = \vec{0}$ · $\text{div}(\text{curl}(F)) = 0$

evaluate f · calculate $\int_C F \cdot dr$ · Calculate $\iint_S F \cdot dS$ · Calculate $\iiint_W f \, dV$

How do we keep track of all of these theorems?

Fundamental thm of conservative v.fields

Stoke's Thm

divergence theorem

$f \longmapsto \nabla f$

$\mathbf{F} \longmapsto \operatorname{curl} \mathbf{F}$

$\mathbf{F} \longmapsto \operatorname{div} \mathbf{F}$

$$\{f : \mathbb{R}^3 \to \mathbb{R}\} \xrightarrow{\nabla} \{\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3\} \xrightarrow{curl} \{\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3\} \xrightarrow{div} \{f : \mathbb{R}^3 \to \mathbb{R}\}$$

$\operatorname{curl}(\nabla f) = \vec{0}$

$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$

evaluate f

calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$

calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$

calculate $\iiint_W f \, dV$

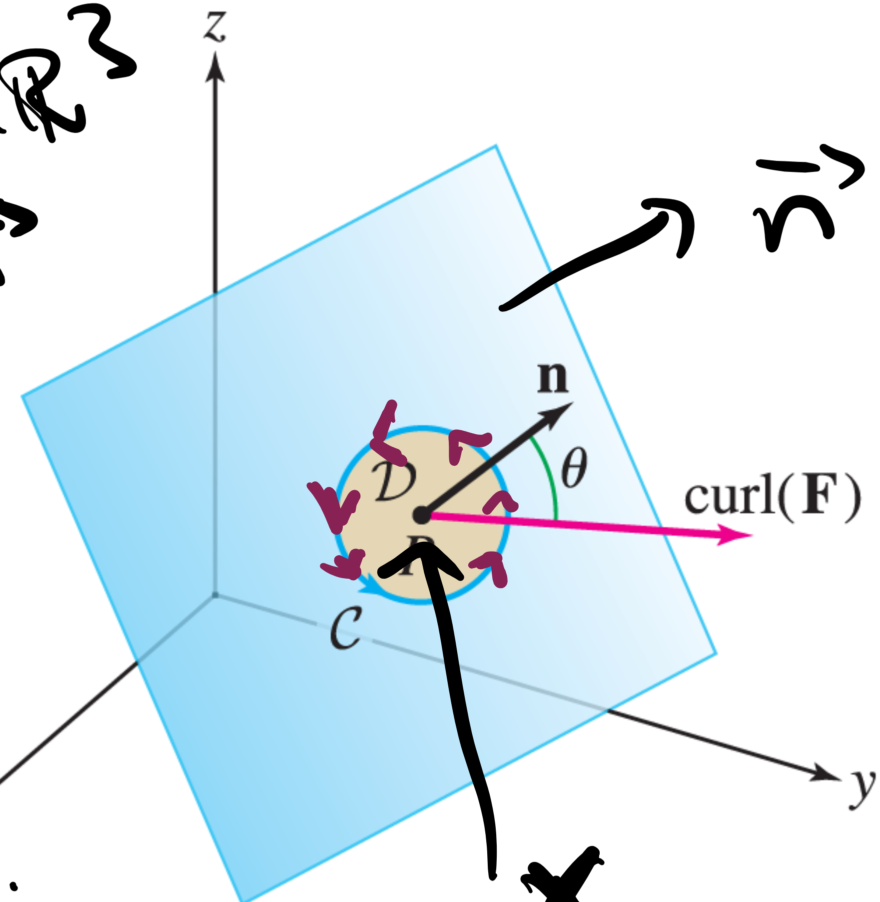
If $\mathbf{F} = \text{curl}\,\mathbf{A}$, S is a surface in $\mathbb{R}^3$

then $\displaystyle\oint_{\partial S} \mathbf{A} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{A} \cdot d\mathbf{S}$

Question

How does Stoke's theorem relate to the curl of a vector field in $\mathbb{R}^3$*?*

- D is a surface in $\mathbb{R}^3$ oriented by $\vec{n}$
- ∂D is C oriented w/ bdry orientation.



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D is a surface w/ bdry

$\vec{n}$

∂D = circle

Interpreting the curl in $\mathbb{R}^3$

Corollary

Suppose $\boldsymbol{F}$ *is a vector field in* $\mathbb{R}^3$*, and consider a plane through* $X \in \mathbb{R}^3$ *with unit normal vector* $\boldsymbol{n}$*. Let* C *be a small circle of radius* ε *in the plane, centered at* X*, which encloses a disk* D *in the plane.*

Stokes thm: $\oint_{\partial D} F \cdot dr = \iint_D \operatorname{curl} F \cdot dS.$

Interpreting the curl in $\mathbb{R}^3$

Corollary

Suppose $\boldsymbol{F}$ is a vector field in $\mathbb{R}^3$, and consider a plane through $X \in \mathbb{R}^3$ with unit normal vector $\boldsymbol{n}$. Let C be a small circle of radius ε in the plane, centered at P, which encloses a disk D in the plane. Then

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n} \, dS \approx (\operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}) \text{ area } D$$

vector line integral

surface flux integral.

if we make D very small

Interpreting the curl in $\mathbb{R}^3$

Corollary

Suppose $\boldsymbol{F}$ *is a vector field in* $\mathbb{R}^3$*, and consider a plane through* $X \in \mathbb{R}^3$ *with unit normal vector* $\boldsymbol{n}$*. Let* C *be a small circle of radius* ε *in the plane, centered at* P*, which encloses a disk* D *in the plane. Then*

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n}\, dS \approx (\operatorname{curl}(\boldsymbol{F})(P) \cdot \boldsymbol{n}) \operatorname{area}(D)$$

approximate surface integral

$$\operatorname{curl} F(p) \cdot \vec{n} \approx \frac{\oint_{\partial D} F \cdot dr}{\text{Area } D}$$

Interpreting the curl in $\mathbb{R}^3$

Corollary

Suppose $\boldsymbol{F}$ *is a vector field in* $\mathbb{R}^3$*, and consider a plane through* $X \in \mathbb{R}^3$ *with unit normal vector* $\boldsymbol{n}$*. Let* C *be a small circle of radius* ε *in the plane, centered at* P*, which encloses a disk* D *in the plane. Then*

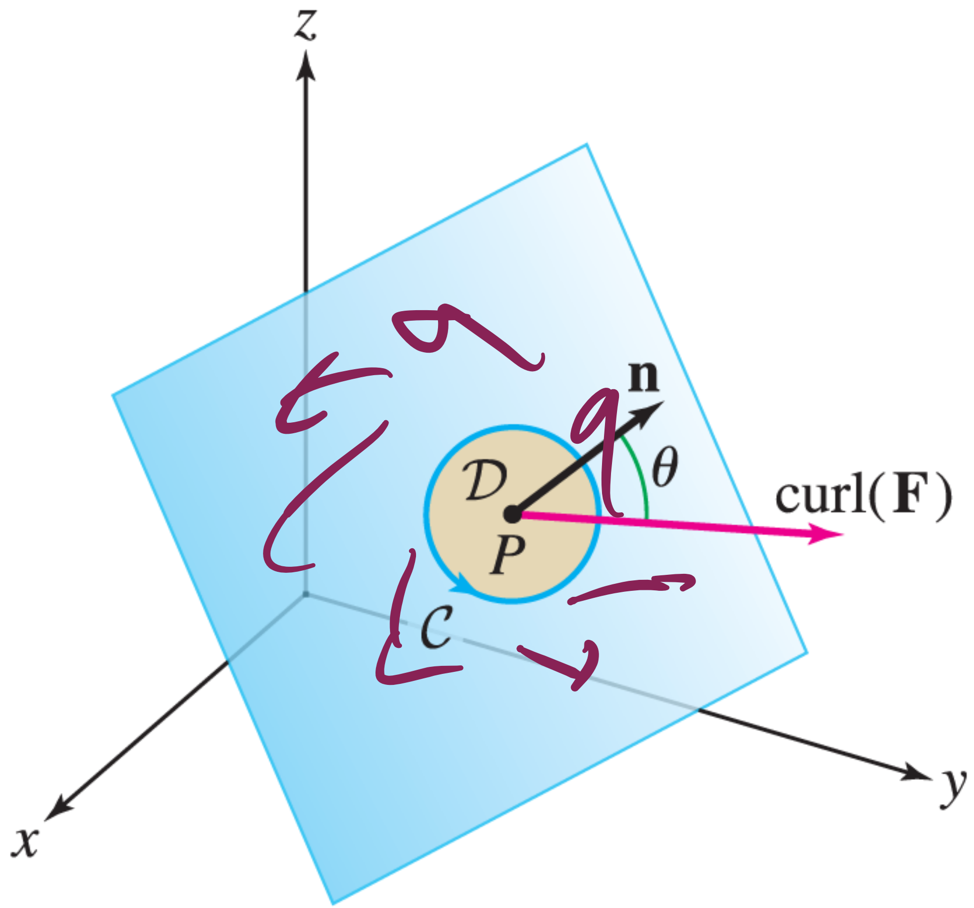
$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n}\, dS \approx (\operatorname{curl}(\boldsymbol{F})(P) \cdot \boldsymbol{n})\operatorname{area}(D)$$

Thus,

$$(\operatorname{curl}(\boldsymbol{F})(P) \cdot \boldsymbol{n}) \approx \frac{1}{\operatorname{area}(D)} \oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r}$$

measures how much of F flows along C

Therefore, the ***circulation*** *of* $\boldsymbol{F}$ *in a given plane* X *depends on the angle between* $\operatorname{curl}(\boldsymbol{F})$ *and* $\boldsymbol{n}$*.*



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curl F(p) tells us how much F swirls around p

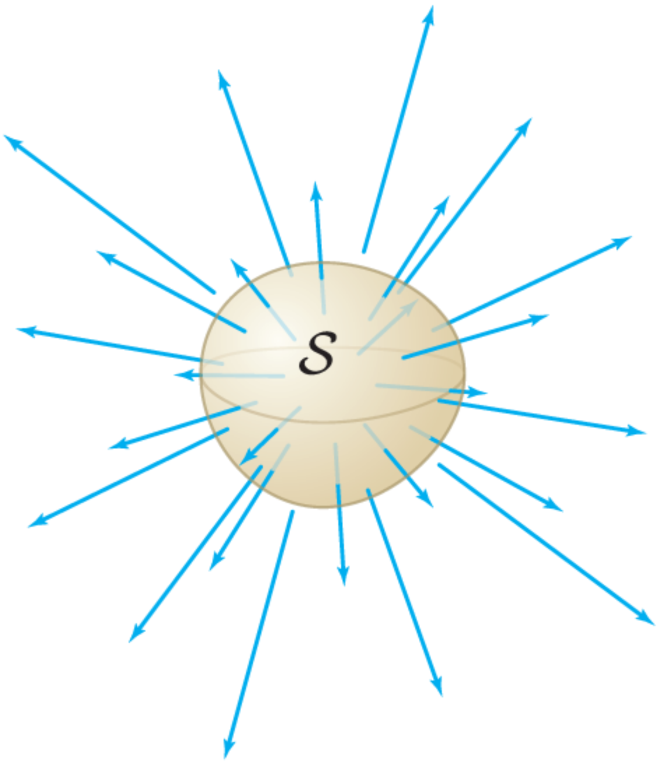
C

$\oint_{\partial D} F \cdot dr$

Saturday Morning Breakfast Cereal, 2/24/2014

Question

How does the divergence theorem relate to the divergence of a vector field in $\mathbb{R}^3$?



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Interpreting divergence in $\mathbb{R}^3$

Corollary

Recall that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ can be intepreted as the <u>flow rate</u> across S. Suppose $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, and consider a small sphere S around the point $P \in \mathbb{R}^3$ with outward-pointing normal. Then

divergence thm

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div}(\mathbf{F})\, dV \approx (\operatorname{div} \mathbf{F}) \iiint_W 1\, dV$$

$$= (\operatorname{div} \mathbf{F})(\text{volume } W)$$

$P \in \mathbb{R}^3$

S sphere centered around P.

Interpreting divergence in $\mathbb{R}^3$

Corollary

Recall that $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ can be intepreted as the flow rate across S. Suppose $\boldsymbol{F}$ is a vector field in $\mathbb{R}^3$, and consider a small sphere S around the point $P \in \mathbb{R}^3$ with outward-pointing normal. Then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_W div(\boldsymbol{F})\ dV \approx div(\boldsymbol{F})(P)\ \mathrm{vol}(W)$$

Interpreting divergence in $\mathbb{R}^3$

Corollary

Recall that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ can be intepreted as the <u>flow rate</u> across S. Suppose $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, and consider a small sphere S around the point $P \in \mathbb{R}^3$ with outward-pointing normal. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div}(\mathbf{F})\, dV \approx \operatorname{div}(\mathbf{F})(P)\, \operatorname{vol}(W)$$

Flux of F through S

Thus,

$$\underline{\operatorname{div}(\mathbf{F})(P)} \approx \frac{1}{\operatorname{vol}(W)} \iint_S \mathbf{F} \cdot d\mathbf{S}$$

*Therefore, the **divergence** of $\mathbf{F}$ at a point P can be interpreted as the outward flux of $\mathbf{F}$ near P.*

$\iint_S F \cdot dS > 0$

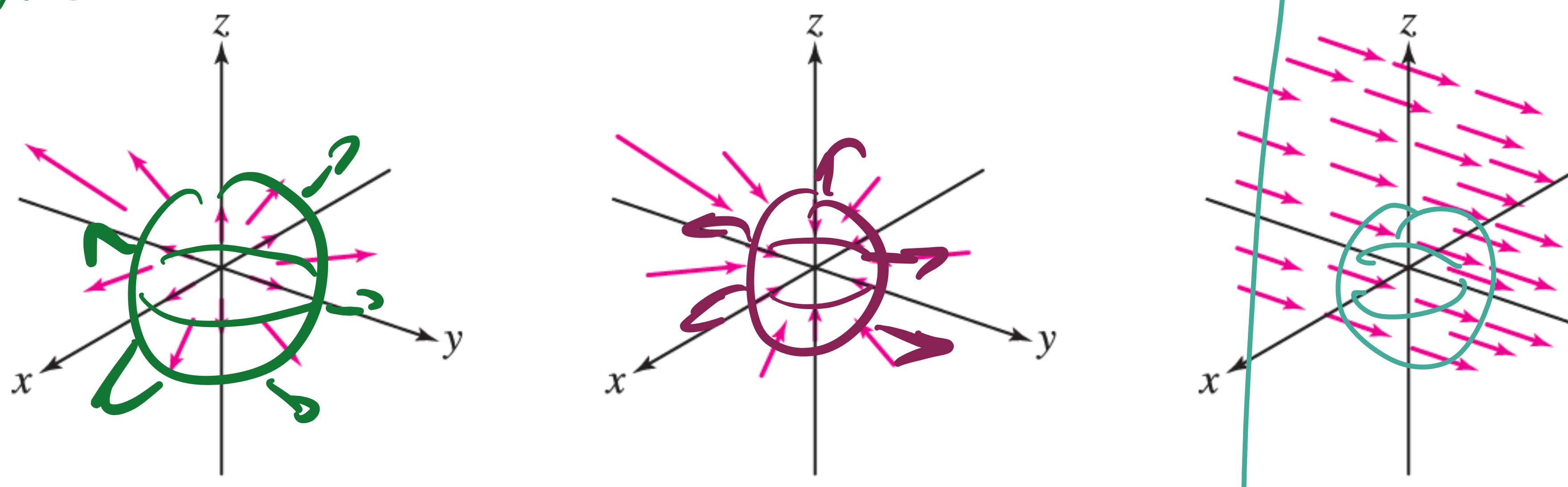
hence $\operatorname{div} F(p) > 0$

$\iint_S F \cdot dS < 0$

hence $\operatorname{div}(F)(p) < 0$

$\iint_S F \cdot dS = 0$

$\operatorname{div} F(p) = 0$



(A) (B) (C)

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P being a source of F

P being a sink of F.

The amount flowing out of S is equal to the amount flowing