A Note About Taylor Series

Taylor Series are used in many problems in astrophysics where a "first order approximation" is appropriate instead of a full solution.

Taylor Series

If a function, *f(x)*, is *continuously differentiable* (i.e., the first, second, third, etc. derivatives exist), then you can express *f(x)* as an infinite sum of polynomials and the function's derivatives:

$$f(x) = f(a) + (x-a)\left.\frac{df}{dx}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=a} + \cdots$$

This is a Taylor Series, where a is a constant, usually some initial or "zero" value of *f(x)* of interest, e.g., *f(x)* at $x=0$. For most functions the function and the Taylor series are equal near $x=a$. When $a=0$, you will sometimes see this called a Maclaurin series.

In the limit where x is small, because small^2 is smaller and small^3 is smaller still, we can approximate *f(x)* by dropping all but the first two terms in the Taylor Series expansion:

$$f(x) \approx f(a) + (x-a)\left.\frac{df}{dx}\right|_{x=a}$$

This gives us a "first-order approximation" of *f(x)*.

Worked Example

You are given a function

$$f(x) = c(1+x)^4$$

Where c is a constant. In the limit $x \ll 1$, we can approximate *f(x)* as a first-order Taylor Series about $a=0$ as follows:

$$f(x) \approx f(a) + (x-a)\left.\frac{df}{dx}\right|_{x=a}$$

Set $a=0$ and substitute in *f(x)* into this equation:

$$\begin{aligned} f(x) &\approx c(1+0)^4 + (x-0)\frac{d}{dx}[c(1+x)^4]_{x=0} \\ &\approx c(1)^4 + x[4c(1+x)^3]_{x=0} \\ &\approx c + x[4c(1+0)^3] \end{aligned}$$

Evaluating all the bits and re-arranging algebraically into a cleaner form:

$$f(x) \approx c(1+4x)$$

gives us a 1st order approximation of *f(x)* valid when $x \ll 1$.

Care must be taken that the condition $x \ll 1$ is indeed satisfied before using the first-order approximation. How small is small enough? One way to assess this is to numerically evaluate the full and 1st order versions and see where the difference is smaller than ~1% (that means you could use the 1st order form and make only a 1% systematic error). If your precision requirements are greater, a smaller difference (0.1% etc.) should be used as appropriate. Try this with the equation above for c=1. At what value of x does the 1st order form become different by 1%?