Piston Theory—A New Aerodynamic Tool for the Aeroelastician†

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SUMMARY

Representative applications are described which illustrate the extent to which simplifications in the solutions of high-speed unsteady aeroelastic problems can be achieved through the use of certain aerodynamic techniques known collectively as "piston theory." Based on a physical model originally proposed by Hayes and Lighthill, piston theory for airfoils and finite wings has been systematically developed by Landahl, utilizing expansions in powers of the thickness ratio δ and the inverse of the flight Mach Number M. When contributions of orders δ/M^3 and δ^2/M^2 are negligible, the theory predicts a point-function relationship between the local pressure on the surface of a wing and the normal component of fluid velocity produced by the wing's motion. The computation of generalized forces in aeroelastic equations, such as the flutter determinant, is then always reduced to elementary integrations of the assumed modes of motion.

Essentially closed-form solutions are given for the bending-torsion and control-surface flutter properties of typical section airfoils at high Mach Numbers. These agree well with results of more exact theories wherever comparisons can be fairly made. Moreover, they demonstrate the increasingly important influence of thickness and profile shape as M grows larger, a discovery that would be almost impossible using other available aerodynamic tools. The complexity of more practical flutter analyses—e.g., on three-dimensional wings and panels—is shown to be substantially reduced by piston theory. An iterative procedure is outlined, by which improved flutter eigenvalues can be found through the successive introduction of higher-order terms in δ and $1/M$.

Other applications to unsteady supersonic problems are reviewed, including gust response and rapid maneuvers of elastic aircraft. Steady-state aeroelastic calculations are also discussed, but for them piston theory amounts only to a slight modification of Ackeret's formulas.

Suggestions are made regarding future research based on the new aerodynamic method, with particular emphasis on areas where computational labor can be reduced with a minimum loss of precision. It is pointed out that a Mach Number zone exists where thermal effects are appreciable but nonlinear viscous interactions may be neglected, and that in this zone piston theory is the logical way of estimating air loads when analyzing aerodynamic-thermo-elastic interaction problems.

PRINCIPAL SYMBOLS

a	= speed of sound in air
b	= reference semichord of wing or half-length of panel
GJ	= torsional stiffness of uniform cantilever wing
k	= $(\omega b/U)$ reduced frequency of simple harmonic motion
l	= span of uniform cantilever wing
$L_1, \ldots, L_6$, $M_1, \ldots M_6$	= dimensionless coefficients defining lift and moment on an oscillating airfoil
m	= mass of panel or two-dimensional airfoil per unit distance perpendicular to the flow
M	= (U/a_∞) free-stream Mach Number
M_α	= pitching moment per unit span, about the axis $X = 2bx_0$, on a two-dimensional airfoil (positive nose up)
M_β	= hinge moment per unit span on a two-dimensional flap (positive nose up)
p	= ambient pressure in air
P	= lift per unit span on a two-dimensional airfoil (positive downward)
t	= physical time
U	= free-stream velocity or flight speed
w	= (δw_0) component of fluid velocity in z-direction or normal to wing surface
X, Y, Z	= $(2bx, 2by, 2bz)$ rectangular Cartesian coordinates, with X directed parallel to the free stream
Z_0	= Z-coordinate of upper wing surface, divided by δ
γ	= specific-heat ratio of air ($\gamma \cong 1.4$)
δ	= denotes the larger of the wing thickness ratio or the ratio of amplitude of unsteady motion to wing chord
ρ	= ambient density of air
ω	= circular frequency of simple harmonic motion

Properties of a Two-Dimensional "Typical Section" of a Wing‡

A_w, A_F	= cross-sectional areas of airfoil and flap, respectively
h	= bending displacement of axis of rotation (positive downward)
I_α	= mass moment of inertia about axis of rotation per unit span
M_w, M_F	= static moments of areas of airfoil and flap, respectively, about their leading edges
r_α	= $(\sqrt{I_\alpha/mb^2})$ dimensionless radius of gyration about axis of rotation
r_β	= dimensionless flap radius of gyration about hingeline
x_0	= dimensionless distance from leading edge to axis of rotation
x_1	= dimensionless distance from leading edge to flap hingeline
x_α	= dimensionless static unbalance about axis of rotation (positive for c.g. aft)
x_β	= dimensionless static unbalance of flap about its hingeline (positive for c.g. aft)
α	= torsional displacement about axis of rotation (positive nose up)
β	= angular displacement of flap relative to airfoil chordline (positive nose up)
μ	= $(m/4\rho_\infty b^2)$ relative density of typical section
$\tau(x)$	= dimensionless thickness distribution

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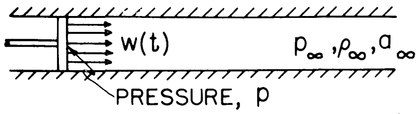
‡ See also Figs. 2 and 3.



SMALL-DISTURBANCE RELATION:

$$p - p_\infty = \rho_\infty a_\infty w(t)$$

SECOND-ORDER RELATION:

$$p - p_\infty = \rho_\infty a_\infty^2 \left\{ \left(\frac{w}{a_\infty}\right) + \frac{\gamma+1}{4}\left(\frac{w}{a_\infty}\right)^2 \right\}$$

LIGHTHILL'S THIRD-ORDER RELATION:

$$p - p_\infty = \rho_\infty a_\infty^2 \left\{ \left(\frac{w}{a_\infty}\right) + \frac{\gamma+1}{4}\left(\frac{w}{a_\infty}\right)^2 + \frac{\gamma+1}{12}\left(\frac{w}{a_\infty}\right)^3 \right\}$$

FIG. 1. Formulas for the pressure on a piston in a one-dimensional channel.

$\omega_h, \omega_\alpha, \omega_\beta$ = uncoupled natural frequencies of oscillation of spring-restrained typical section in bending, torsion, and flap rotation, respectively

INTRODUCTION

THE TERM "piston theory," as used in this paper, refers to any method for calculating the aerodynamic loads on aircraft in which the local pressure generated by the body's motion is related to the local normal component of fluid velocity in the same way that these quantities are related at the face of a piston moving in a one-dimensional channel. The circumstances under which such an analogy is valid are now well understood and are outlined below. In general, piston theory may be employed for large flight Mach Numbers or high reduced frequencies of unsteady motion, whenever the surface involved is nearly plane and not inclined too sharply to the direction of the free stream. The foregoing shape conditions are fulfilled by all but the immediate tip (and possibly leading-edge) regions of supersonic wings. In most aeronautical applications the normal component of fluid velocity is the given quantity and the surface pressure is the unknown to be determined, so that a point-function relationship between the two is a great convenience. These observations, coupled with the fact that arbitrary small deformations and arbitrarily time-dependent unsteady motions can be taken into account, recommend piston theory to the aeroelastician as a powerful tool for analyzing high-speed problems.

The authors' main purpose is to focus attention on the remarkable simplicity and utility of the method by presenting a selection of new applications and comparisons with previously published results which were originally obtained at the expense of much more laborious computation. The number of different examples which can be given in a single paper is naturally limited, but it is hoped that readers engaged either in the field of aeroelasticity or elsewhere will be encouraged to seek other uses of piston theory on problems of current interest.

Fig. 1 depicts a piston moving with velocity $w(t)$ in the end of a channel containing perfect gas, whose undisturbed pressure, density, and speed of sound are p_∞, ρ_∞, and a_∞. Provided that the piston generates only simple waves and produces no entropy changes, the exact expression for the instantaneous pressure $p(t)$ on its face is

$$p/p_\infty = \{1 + [(\gamma - 1)/2](w/a_\infty)\}^{2\gamma/(\gamma-1)} \quad (1)$$

Depending on the magnitude of the ratio w/a_∞, Eq. (1) may be approximated by the linear relation,

$$p - p_\infty = \rho_\infty a_\infty w \quad (2)$$

by its second-order binomial expansion,

$$p - p_\infty = \rho_\infty a_\infty^2\{(w/a_\infty) + [(\gamma + 1)/4](w/a_\infty)^2\} \quad (3)$$

or by its third-order expansion,

$$p - p_\infty = \rho_\infty a_\infty^2\{(w/a_\infty) + [(\gamma + 1)/4](w/a_\infty)^2 + [(\gamma + 1)/12](w/a_\infty)^3\} \quad (4)$$

Eq. (3) resembles Busemann's quadratic formula for steady supersonic flow past an airfoil. Reasoning from a suggestion of Hayes,[1] Lighthill[2] pointed out that Eq. (4) can be used with excellent accuracy even under nonisentropic conditions to calculate the pressure on an airfoil in steady or unsteady motion whenever the flight Mach Number has such an order of magnitude that $M^2 \gg 1$. An additional limitation is that the product $M\delta$ cannot be too large, where for practical purposes δ represents the larger of the thickness ratio of the airfoil or the ratio of maximum amplitude of unsteady motion to airfoil chord length.

It is of historical interest that a theory equivalent to applying the linear relation Eq. (2) to simple harmonic oscillations of a flat plate with hinged flap in two-dimensional flow was given by Collar[3] in 1944. Collar's work consists of an extension by physical reasoning of Ackeret's well-known formula for steady supersonic flight. It was provided a mathematical foundation by Temple and Jahn,[4] who reduced the same results from a series expansion of the exact solution when $M \rightarrow \infty$, independent of the frequency of oscillation. (The condition of large Mach Number also implies that the factor $\sqrt{M^2 - 1}$ can be replaced by M, which brings Collar's formulation into exact agreement with linear piston theory.) In 1946 Jordan[5] presented the high Mach Number expansion for two-dimensional sinusoidal motion and noted the point-function relation between p and w which is the characteristic feature of piston theory.

More recently Hjelte[6] and Landahl, Mollø-Christensen, and Ashley[7] have shown that linearized piston theory can be used for arbitrary small motions of thin two-dimensional airfoils whenever any one of the conditions $M^2 \gg 1$, $kM^2 \gg 1$ or $k^2M^2 \gg 1$ is met, k being an appropriate measure of the "unsteadiness" of the flow. The last of these three conditions suggests that certain subsonic flows may be covered, but comparison with more exact solutions indicates that satisfactory accuracy will be attained only in very high-frequency problems such as local skin vibrations.

Landahl's systematic presentation[8] of the circumstances under which linear and second-order piston theory can be deduced from the general theory of unsteady potential flow brings together the various lines of approach outlined above and furnishes a rigorous estimate of the magnitude of errors to be expected when Eq. (2) or Eq. (3) is adapted to practical problems. He treats an upper wing surface close to the X-Y-plane in a uniform stream of velocity U parallel to the X-direction. The instantaneous surface position is given by

$$Z = \delta Z_0(X, Y, t) \tag{5}$$

the parameter δ being factored out so that Z_0 becomes a quantity of order unity when made dimensionless by division with a typical length—i.e., the midspan chord $2b$. The normal fluid velocity produced by the wing's motion is

$$\delta w_0 = [(\partial/\partial t) + U(\partial/\partial X)]\delta Z_0 \tag{6}$$

By means of an iterative procedure Landahl derives a general solution for the pressure coefficient

$$C_p = (p - p_\infty)/[(1/2)\rho_\infty U^2] \tag{7}$$

on the surface. When this is expanded in powers of δ and $1/M^2$, a series develops in the following form:

$$C_p = (\delta/M)C_{p_1}{}^{(1)} + (\delta/M^3)C_{p_2}{}^{(1)} + (\delta/M^5)C_{p_3}{}^{(1)} + O(\delta/M^7) + \delta^2 C_{p_1}{}^{(2)} + (\delta^2/M^2)C_{p_2}{}^{(2)} + O(\delta^2/M^4) \tag{8}$$

The various coefficients $C_{p_n}{}^{(1)}$ and $C_{p_n}{}^{(2)}$ are of order unity whenever the solution is uniformly convergent, so that the δ and M factors indicate the magnitude of the various terms. Those of importance to the present discussion are listed below.

$$C_{p_1}{}^{(1)} = 2(w_0/U) \tag{9a}$$

$$C_{p_1}{}^{(2)} = [(\gamma + 1)/2](w_0/U)^2 \tag{9b}$$

$$C_{p_2}{}^{(1)} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\int_{x_l}^{x}(x - \xi) \times \frac{w_0[\xi, y, (Ut/2b) - x + \xi]}{U}\,d\xi \tag{9c}$$

All space coordinates in Eq. (9c) are divided by the typical length $2b$; $x_l(y)$ gives the projection of the wing leading edge on the x-y-plane.

The δ/M and δ^2 terms in Eqs. (8) and (9) are obviously identical to the linear and second-order portions of Eq. (3). Hence, in general, the piston approximation contains absolute errors no greater than the orders of δ/M^3 and δ^2/M^2. There is also a general limitation that neither of the products δM or $k\delta M$ can be too large, which means physically that the surface can nowhere be inclined too sharply to the direction of U and that normal velocities produced by the unsteady motion must be limited in comparison with the speed of sound. Because the mathematical solution is not uniformly convergent in the neighborhood of a cut-off wing tip, as might be expected from the failure of the one-dimensional flow analogy when fluid can spill laterally around the tip, the error estimates are incorrect there. This does not forbid the use of piston theory to account for other three-dimensional effects, however, and the region where inaccuracies may occur is clearly bounded by the inboard Mach line from the tip leading edge. Finally, the third-order term in Eq. (4) cannot be predicted by Landahl's analysis, which is based on a flow assumed free of strong shocks and entropy changes. The justification for including third-order contributions, when they are appreciable, is regarded as excellent but nevertheless empirical.

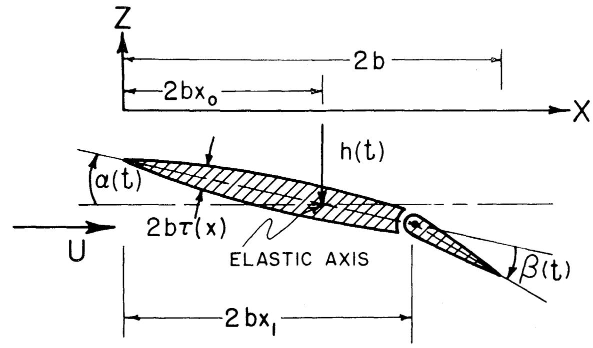


FIG. 2. Geometric properties of an airfoil section with flap.

Two-Dimensional Flutter Problems

The aerodynamic derivatives used for predicting the flutter characteristics of a two-dimensional airfoil or "typical section" are calculated by considering sinusoidal motion in the three degrees of freedom h (vertical translation), α (pitching), and β (flap rotation) illustrated in Fig. 2. The axis of pitch and the hingeline of the aerodynamically unbalanced flap are located at distances $2bx_0$ and $2bx_1$, respectively, behind the leading edge. Up to terms of second order, this motion gives rise to normal components of fluid velocity

$$\frac{w}{U} = \begin{cases} \mp\ [(\dot h/U) + \alpha + (2b/U)\dot\alpha(x - x_0)] + (1/2)(d\tau/dx), & x < x_1 \\ \mp\ [(\dot h/U) + \alpha + (2b/U)\dot\alpha(x - x_0) + \beta + (2b/U)\dot\beta(x - x_1)] + (1/2)(d\tau/dx), & x \geq x_1 \end{cases} \tag{10}$$

Here the negative sign refers to the upper surface and the positive to the lower, since w must always be taken in the outward direction from the surface as used in piston formulas like Eqs. (2)–(4).

Although arbitrary time dependence can be handled without additional difficulty, only the case of simple harmonic motion is treated here, because it permits the introduction of the familiar notation of Garrick and Rubinow.[9] For example, the lift and pitching moment about $x = x_0$ per unit span, due to

$$h = \bar{h}e^{i\omega t} \tag{11a}$$

$$\alpha = \bar{\alpha}e^{i\omega t} \tag{11b}$$

$$\beta = \bar{\beta}e^{i\omega t} \tag{11c}$$

can be written in general

$$P = -4\rho_\infty b U^2 k^2 \left[\frac{h}{b}(L_1 + iL_2) + \alpha(L_3 + iL_4) + \beta(L_5 + iL_6)\right] \quad (12)$$

$$M_\alpha = -4\rho_\infty b^2 U^2 k^2 \left[\frac{h}{b}(M_1 + iM_2) + \alpha(M_3 + iM_4) + \beta(M_5 + iM_6)\right] \quad (13)$$

where

$$L_3 = L_3' - 2x_0 L_1 \quad (14a)$$

$$L_4 = L_4' - 2x_0 L_2 \quad (14b)$$

$$M_1 = M_1' - 2x_0 L_1 \quad (14c)$$

$$M_2 = M_2' - 2x_0 L_2 \quad (14d)$$

$$M_3 = M_3' - 2x_0[(M_1' + L_3') - 2x_0 L_1] \quad (14e)$$

$$M_4 = M_4' - 2x_0[(M_2' + L_4') - 2x_0 L_2], \text{ etc.} \ldots \quad (14f)$$

Note that k now represents the reduced frequency $\omega b/U$ of the oscillation.

When second-order piston theory, according to Eq. (3), is employed to approximate these dimensionless coefficients, the following relatively simple expressions are found:

$$L_1 = 0 \quad (15a)$$

$$L_2 = 1/kM \quad (15b)$$

$$L_3' = 1/k^2 M \quad (15c)$$

$$L_4' = (1/kM) - [(\gamma + 1)/k](A_w/8b^2) \quad (15d)$$

$$L_5 = [(1 - x_1)/k^2 M] - [(\gamma + 1)/k^2][\tau(x_1)/4] \quad (15e)$$

$$L_6 = [(1 - x_1)^2/kM] - [(\gamma + 1)/k](A_F/8b^2) \quad (15f)$$

$$M_1' = 0 \quad (15g)$$

$$M_2' = (1/kM) - [(\gamma + 1)/k](A_w/8b^2) \quad (15h)$$

$$M_3' = (1/k^2 M) - [(\gamma + 1)/k^2](A_w/8b^2) \quad (15i)$$

$$M_4' = (4/3kM) - [(\gamma + 1)/k](M_w/4b^3) \quad (15j)$$

$$M_5 = \{[(1 - x_1^2) - 2x_0(1 - x_1)]/k^2 M\} - [(\gamma + 1)/k^2]\{(x_1 - x_0)[\tau(x_1)/2] + (A_F/8b^2)\} \quad (15k)$$

$$M_6 = \frac{(4/3)(1 - x_1^3) - 2(x_1 + x_0)(1 - x_1^2) + 4x_0 x_1(1 - x_1)}{kM} - \frac{(\gamma + 1)}{k}\left[(x_1 - x_0)\frac{A_F}{4b^2} + \frac{M_F}{4b^3}\right] \quad (15l)$$

In each of Eqs. (15) the first term is the linear part, which arises even in the absence of thickness; each contains the Mach Number as a factor M^{-1}, in agreement with Eq. (8). The thickness terms, when not identically zero, are linearly dependent on the thickness distribution function $\tau(x)$. Most of them involve simple geometric properties of the profile, such as A_w and M_w, which are the area and first moment of the area about the leading edge. Similarly the flap derivatives are dependent on flap cross-sectional area A_F and first moment M_F about the hingeline. Certain of these geometrical interpretations have been pointed out by Miles[10] in the discussion of second-order effects in his comprehensive monograph on unsteady flow.

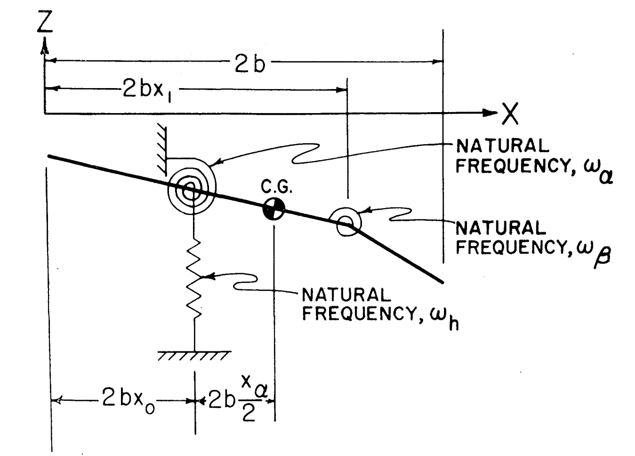


FIG. 3. Elastic and inertial properties of a two-dimensional typical section.

It is of considerable importance to the flutter analyst that, although Eqs. (15) are derived from a nonlinear theory, symmetries between the upper and lower surface motions cause the lift and moments to relate linearly to the oscillations producing them. At the higher Mach Numbers, aerodynamic loads computed from Eqs. (12)–(15) are known to agree closely with calculations by much more complicated second-order theories, such as that of van Dyke.[11] One instructive comparison can be made in the case of single degree of freedom rotational instability of the whole airfoil or the flap: both theories predict the impossibility of such an instability and show the same influence of thickness on the damping moment associated with the rotation. Taking airfoil pitching as an example, stability is dependent on the sign of the coefficient M_4, being positive when M_4 is positive. From Eqs. (15), the piston theory value is

$$M_4 = (1/kM)[(4/3) - 4x_0 + 4x_0^2] - [(\gamma + 1)/k](M_{wx_0}/4b^3) \quad (16)$$

where M_{wx_0} denotes the first moment of the cross-sectional area about $x = x_0$. Eq. (16) is positive for all values of x_0 and all airfoils thin enough to fall within the scope of the theory. If the section has fore-and-aft symmetry (double-wedge, circular-arc, etc.), thickness is seen to make an unstable contribution to M_4 when the axis of rotation is ahead of midchord.

For flutter computations involving more than one degree of freedom, the airfoil is assumed to have the elastic and inertia properties shown in Fig. 3. The flutter equations are not reproduced here, since their derivation and solution have been extensively discussed in the literature—e.g., Garrick and Rubinow.[9] The significant point to be made is that, when the aerodynamic coefficients have forms as simple as Eqs. (15), implicit or explicit algebraic expressions can be worked out for the critical flutter speed U_F and frequency ω_F as functions of M and the other parameters defining the system. Thus, the trial-and-error process which was formerly characteristic even of two-dimensional flutter analyses is eliminated. Flutter prediction becomes more elementary at high speeds than any other similar calculation—incompressible flow included—in the aeroelastician's experience. The influences of thickness, profile shape and initial angle of attack* can be introduced, for the first time, in a rigorous and routine fashion.

As a first illustration, Fig. 4 shows exact and approximate curves of dimensionless bending-torsion ($h - \alpha$) flutter speed $U_F/b\omega_\alpha$ vs. M for a typical section with a low frequency ratio, elastic axis at mid-chord and center of gravity at 60 per cent chord. The notation is that of Garrick and Rubinow[9] and is reviewed in the list of symbols. Fig. 4 is taken from a previous paper by Heller and the present authors,[12] in which the application of piston theory to two-dimensional bending-torsion flutter is extensively elaborated. A conclusion stated there, to which no exceptions have yet been found, is that the simplified approach yields $U_F/b\omega_\alpha$ values within 10 per cent of their exact counterparts for $M \geqslant 2.5$. Incidentally, third-order terms from Eq. (4) were carried through the original calculations but have since been discovered to have negligible influence within the ranges of variables treated.

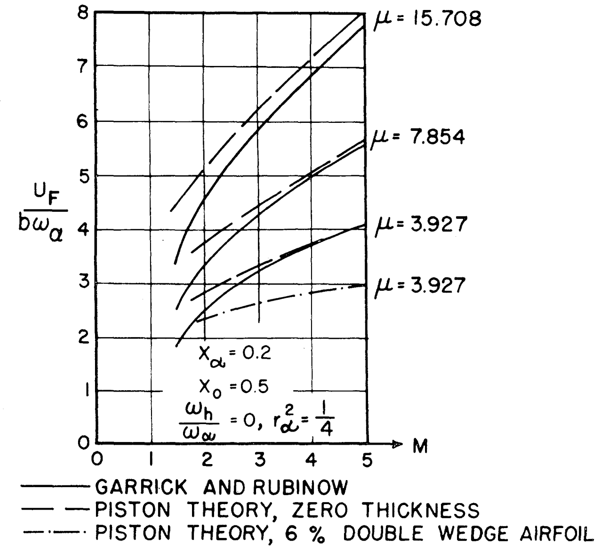


FIG. 4. Bending-torsion flutter of typical section.

All the piston theory curves on Fig. 4, together with similar curves for any other combinations of the bending-torsion parameters, are calculable from the single closed-form solution

$$\frac{U_F}{b\omega_\alpha} = \frac{\mu M}{\sqrt{\chi}}\sqrt{\frac{x_\alpha^2 - [(\omega_h/\omega_\alpha)^2\chi - 1]r_\alpha^2(\chi - 1)}{\mu M\{(1 - 2x_0 + \bar{A})[(\omega_h/\omega_\alpha)^2\chi - 1] + \chi_\alpha\} + (1 - 2x_0 + \bar{A})^2 - [(4/3) - 4x_0 + 4x_0^2 + \bar{B}]}} \quad (17)$$

where

$$\chi = \frac{r_\alpha^2 - 2x_\alpha(1 - 2x_0 + \bar{A}) + [(4/3) - 4x_0 + 4x_0^2 + \bar{B}]}{r_\alpha^2 + (\omega_h/\omega_\alpha)^2[(4/3) - 4x_0 + 4x_0^2 + \bar{B}]} \quad (18a)$$

$$\bar{A} = -M(\gamma + 1)(A_w/8b^2) \quad (18b)$$

$$\bar{B} = -M(\gamma + 1)(M_{wx_0}/4b^3) \quad (18c)$$

Thickness enters only through the constants $\bar{A}$ and $\bar{B}$; its influence may be strongly destabilizing, however, as suggested by the data for a 6 per cent double-wedge airfoil given in Fig. 4. Other interesting implications of Eq. (17) and various reduced forms thereof have been discussed previously.[12]

Figs. 5 and 6 compare piston theory results with exact calculations by Woolston and Huckel[13] for two-dimensional bending-aileron ($h - \beta$) flutter of an airfoil with zero thickness and a 20 per cent chord unbalanced flap. In each case the ordinate, abscissa, and section parameters are chosen to correspond with those given by reference 13. Accordingly, Fig. 5 presents the dimensionless bending stiffness $b\omega_h/a_\infty$ required to produce neutral stability, plotted vs. static unbalance parameter x_β for a system with unrestrained aileron ($\omega_\beta = 0$) and the other parameters stated. Fig. 6 presents the same ordinate, plotted vs. frequency ratio ω_β/ω_h, for two values of static unbalance.

Although the Mach Number is only 2—the highest value included in the exact calculations[13]—the agreement with piston theory seems quite satisfactory. As shown by a few representative points, changes in the critical condition caused by replacing the flat plate with a 6 per cent double-wedge profile are small in this situation. No explicit formulas, comparable to Eqs. (17)–(18), can be written for the critical conditions of bending-aileron, torsion-aileron, or bending-torsion-aileron flutter. From the practical standpoint, however, the implicit formulas that must be solved demand little more computational effort than the bending-torsion case.

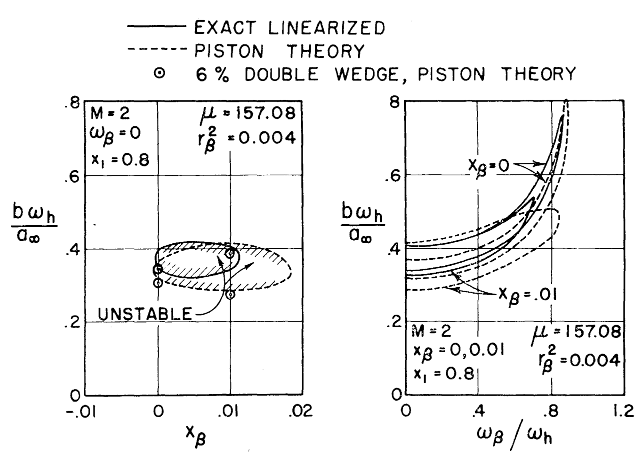


FIG. 5 (*left*). Bending-aileron flutter at $M = 2$ of typical section with aileron free. FIG. 6 (*right*). Bending-aileron flutter at $M = 2$ of typical section for various x_β and ω_β/ω_h.

* Camber and initial angle are accounted for only by Eq. (4).

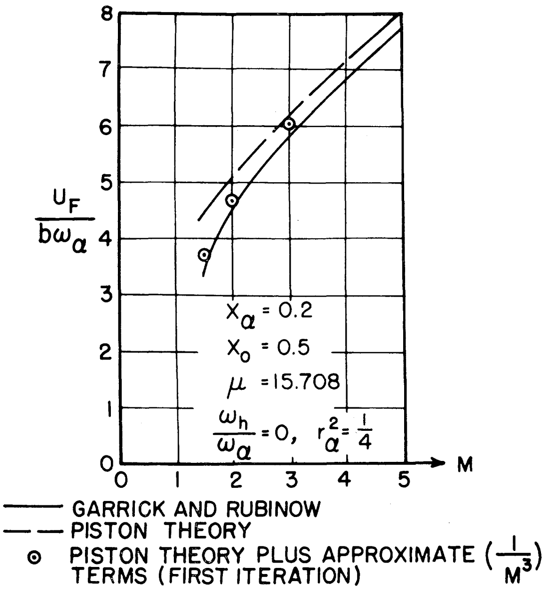


FIG. 7. Bending-torsion flutter of typical section.

In connection with certain of the foregoing conclusions and also with flutter analyses on finite wings in the Mach Number range between 2 and 3, it must be pointed out that a theoretical inconsistency may arise from including thickness effects. Examination of Eq. (8) reveals that, if the δ^2 term is retained while the δ/M^3 term is omitted, it is automatically implied that

$$\delta \gg 1/M^3 \tag{19}$$

This inequality is not fulfilled, for example, by a 6 per cent thick airfoil at $M = 2$. A preliminary study of the influence of the δ/M^3 term on flutter eigenvalues has demonstrated that its omission often produces compensating errors in the elements of the flutter determinant. This would be a fortunate conclusion, if generally valid, because this term is not of the piston theory type.

Following a suggestion by Landahl, it is possible to introduce the higher terms of Eq. (8) into an iterative flutter calculation. For a given M, a first estimate of flutter reduced frequency k_F is taken from piston theory. Corrections based on this k_F are then inserted into the flutter determinant, and a second approximation is computed with relative ease. It appears that a single iteration will normally be sufficient for most purposes, and this involves no especial difficulty. The δ/M^3 corrections to the aerodynamic coefficients for bending-torsion flutter are as follows:

$$\Delta L_1 = \sin 2k/2kM^3 \tag{20a}$$

$$\Delta L_2 = \cos 2k/2kM^3 \tag{20b}$$

$$\Delta L_3' = \sin 2k/4k^3M^3 \tag{20c}$$

$$\Delta L_4' = -(\sin^2 k/2k^3M^3) \tag{20d}$$

$$\Delta M_1' = (\sin 2k/kM^3) + (\cos 2k/2k^2M^3) - (\sin 2k/4k^3M^3) \tag{20e}$$

$$\Delta M_2' = (\cos 2k/kM^3) - (\sin 2k/2k^2M^3) + (\sin^2 k/2k^3M^3) \tag{20f}$$

$$\Delta M_3' = (\sin 2k/2k^3M^3) - (\sin^2 k/2k^4M^3) \tag{20g}$$

$$\Delta M_4' = (\cos 2k/2k^3M^3) - (\sin 2k/4k^4M^3) \tag{20h}$$

Fig. 7 illustrates the improvement in predicted bending-torsion flutter speed that is attained when piston theory is modified by a first iteration in the manner just described. The directions and magnitudes of the changes in $U_F/b\omega_\alpha$ thus found can be expected to turn out about the same, whether or not thickness effects are also introduced into the calculation.

Three-Dimensional Flutter Problems

When adapted to flutter calculations for finite lifting surfaces, piston theory is automatically "strip theory," in the sense that the pressure at any point, as well as the loading along any chordwise section, is independent of the motions of other portions of the system. As discussed above, the loads are predicted with the least accuracy near a cut-off tip, but this same criticism can be leveled at any of the other strip theories which have been used so extensively in the past. If the flutter problem is set up in terms of assumed modes and frequencies of vibration, the principal novelty is that the generalized forces acting on the various modes can usually be written down in simple algebraic forms for arbitrary Mach Number and reference reduced frequency. Certain unwieldy trial-and-error procedures common to more conventional methods are thus avoided or minimized, as with the typical section in the foregoing discussion.

The first example presented in this section is selected because unclassified experimental data are available and because it offers startling evidence of the importance of thickness in supersonic flutter. The case is one of

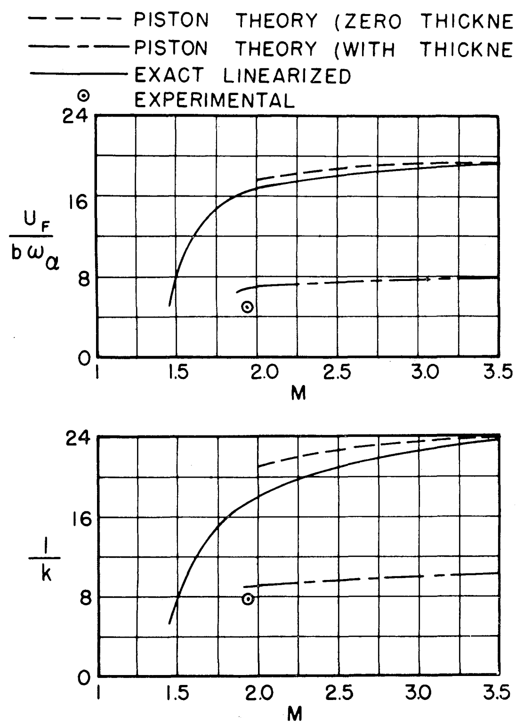


FIG. 8. Bending-torsion flutter of supersonic wing—comparison of experiment with various theories.

bending-torsion flutter of a straight-tapered wing of moderate aspect ratio,* whose other properties are carefully chosen to be representative of lifting surfaces on aircraft designed for appreciable supersonic speeds. The actual properties cannot be revealed for reasons of security.

Fig. 8 presents theoretical curves of the dimensionless flutter characteristics vs. Mach Number. These were obtained from the standard Rayleigh-Ritz equations, including one uncoupled mode of cantilever bending and one of cantilever torsional vibration. The solid curve results from using the two-dimensional linearized aerodynamic theory of Garrick and Rubinow;[9] at higher Mach Numbers it is seen to merge smoothly with the curve based on zero-thickness piston theory, as would be expected. The dot-dashed curve differs only in that the influences of profile shape and thickness have been introduced through Eqs. (15). There is little doubt about which calculation agrees best with the measured data, even though M is less than 2. The linearized prediction is dangerously unconservative, despite the fact that previous experience with lower Mach Numbers suggests that neglecting tip effects should lead to an error in the opposite direction. Introducing terms of the order δ/M^3, as outlined at the end of the preceding section, might be likely to displace the dot-dashed $U_F/b\omega_\alpha$ curve downward by roughly the distance between the two zero-thickness curves, thus giving an even better comparison with experiment.

Quite a different sort of application of piston theory to three-dimensional wings concerns the possibility of finding exact solutions for the damping and frequency of the aeroelastic modes, when the inertia and elastic properties are uniform along the span. A laborious calculation of this type was once made, using incompressible strip techniques, by Goland and Luke.[14] By contrast, the aerodynamic derivatives for arbitrary motion according to piston theory are such that little more difficulty is encountered than when analyzing modes of free vibration in a vacuum.

One simple example which serves to illustrate the method is uncoupled pure torsion in a supersonic stream of a wing having constant chord $2b$, constant torsional stiffness GJ, and constant running moment of inertia I_α about the axis of twist. The differential equation governing the spanwise distribution of twist $\theta(Y, t)$ is

$$I_\alpha(\partial^2\theta/\partial t^2) = GJ(\partial^2\theta/\partial Y^2) - (8\rho_\infty U^2 b^2/M) \times \left\{(2b/U)(\partial\theta/\partial t)[(1/3) - x_0 + x_0^2] + \theta[(1/2) - x_0]\right\} \quad (21)$$

In Eq. (21) the aerodynamic moment acting about the axis $2bx_0$ behind the leading edge was found directly from Eq. (2), with

$$w = \mp[U\theta + 2b(\partial\theta/\partial t)(x - x_0)] \quad (22)$$

* This wing was flutter tested at M.I.T. as a portion of experimental research conducted for Dynamics Branch, Aircraft Laboratory, Wright Air Development Center, under Contract No. AF33(038)-22955.

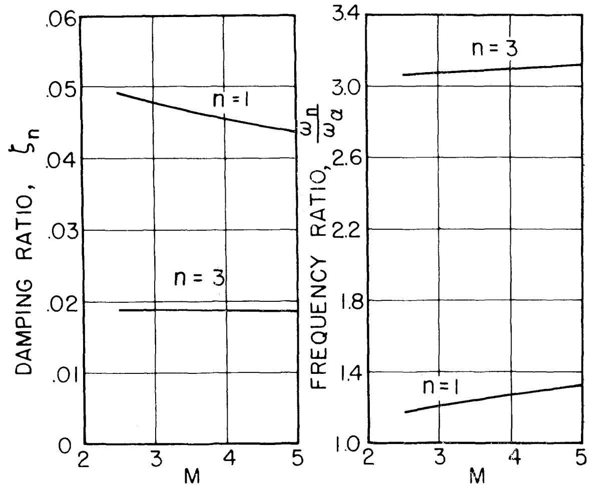


FIG. 9. Frequency and damping ratios of two torsional aeroelastic modes of uniform wing, $\mu r_\alpha^2 = 12$, $x_0 = 0.443$, $k_1 = 0.25/M$.

Thickness effects are omitted, and the minus and plus signs in Eq. (22) refer to the upper and lower wing surfaces, respectively.

Eq. (21) is linear with constant coefficients and is obviously solved by products of exponential functions of Y and t. The boundary conditions of zero twist at the root and zero torque at the tip force the Y-functions to be simple sine curves, as in the case of free vibration. These conditions also yield a characteristic equation, from which can be computed the frequency and the ratio of actual damping to critical damping for the mode of each order. For the nth mode, the damping ratio and the frequency ratio prove to be

$$\zeta_n = \frac{(1/2\mu r_\alpha^2)[(4/3) - 4x_0 + 4x_0^2]}{\sqrt{n^2(Mk_1)^2 + [(1 - 2x_0)/\mu r_\alpha^2]M}} \quad (23a)$$

$$\omega_n/\omega_\alpha = (1/Mk_1)\sqrt{n^2(Mk_1)^2 + [(1 - 2x_0)/\mu r_\alpha^2]M} \times \sqrt{1 - \zeta_n^2}; \quad n = 1, 3, 5, \ldots \quad (23b)$$

where k_1 is the reduced frequency $\omega_\alpha b/U$ corresponding to the fundamental mode of vibration in vacuum—i.e.,

$$\omega_\alpha = (\pi/2l)\sqrt{GJ/I_\alpha} \quad (23c)$$

All other quantities are defined in the list of symbols. ζ_n is always positive, confirming the previously mentioned stability of pure rotational motions at high Mach Numbers, but, curiously, it turns out to be inversely proportional to n for the modes of larger order. Fig. 9 shows curves of damping ratio and ratio of damped natural frequency in air to the fundamental frequency ω_α, both plotted vs. M for the two lowest orders $n = 1$ and 3. Numerical properties of the particular wing are listed on the Figure.

Panel Flutter

A problem which has caused analytical difficulties out of all proportion to its apparent complexity is the flutter of a flat skin panel exposed on one side to a supersonic air stream. Since most of the trouble stems from inability to construct exact solutions based on available

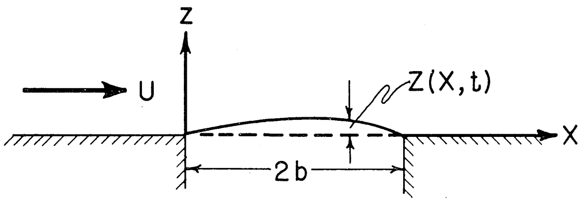


FIG. 10. Two-dimensional panel adjacent to a supersonic air stream.

aerodynamic theories, it is reasonable to expect that piston theory can yield valuable information within the ranges of M and k for which it is valid. This is definitely the case, as will be illustrated here by studying the elementary example of the unbuckled two-dimensional panel.

The panel, pictured in Fig. 10, is embedded in an otherwise rigid, plane surface parallel to the X-axis. The two-dimensional displacement $Z(X, t)$ occurs between supports at $X = 0$ and $X = 2b$. m represents the mass per unit distance perpendicular to the direction of the stream U; EI is the *effective* bending stiffness; and there is a uniform initial tension T per unit span which lies somewhere between the yield strength and the (negative) value which would cause buckling. The differential equation of motion, including air loads by linear piston theory, reads

$$EI(\partial^4 Z/\partial X^4) - T(\partial^2 Z/\partial X^2) + (\rho_\infty U^2/M)(\partial Z/\partial X) + (m/2b)(\partial^2 Z/\partial t^2) + (\rho_\infty U/M)(\partial Z/\partial t) = 0 \quad (24)$$

Pressure variations on the back of the panel are omitted from Eq. (24) because they can usually be expected to increase the stability by removing energy from vibrations. If the panel is simply supported, Z and $\partial^2 Z/\partial X^2$ must vanish at $X = 0$ and $2b$.

The procedure for determining the stability of solutions of Eq. (24) is to assume that $Z(X, t)$ is of the form $\{\bar{Z}(X)e^{i\Omega t}\}$, and to try to find whether, for some combinations of the physical parameter EI, T, etc., it is possible for Ω to have a zero or negative imaginary part. If such is not the case, one is led to the conclusion that all small motions are damped. It will now be proved that for the membrane problem, in which the fourth derivative term of Eq. (24) is assumed to be negligible, Ω can have only positive imaginary part, implying stability.

The equation governing the complex dimensionless amplitude function

$$\bar{z}(x) = (1/2b)\bar{Z}(X/2b)$$

of the membrane can be written in symbolic form

$$[(d^2/dx^2) + a(d/dx) + (c + id)]\bar{z} = 0 \quad (25)$$

where the constants a, c, and d are real numbers. The boundary conditions are

$$\bar{z}(0) = z(1) = 0 \quad (26)$$

If $\bar{z}$ is separated into real and imaginary parts

$$\bar{z}(x) = z_r(x) + iz_i(x) \quad (27)$$

an elimination of z_i leads to the following real differential equation and boundary conditions for z_r:

$$[(d^2/dx^2) + a(d/dx) + c]^2 z_r + d^2 z_r = 0 \qquad d \neq 0 \quad (28)$$

$$\left.\begin{aligned} z_r &= 0 \\ [(d^2/dx^2) + a(d/dx)]z_r &= 0 \end{aligned}\right\} \quad \text{for} \quad x = 0, 1 \quad (29)$$

If the same procedure is used to eliminate z_r, it develops that the differential equation and boundary conditions for z_i are identical with Eqs. (28) and (29). However, it can be shown that Eqs. (28) and (29) have no eigenvalues if $d \neq 0$. If $d = 0$, one obtains a very simple formula for the imaginary part of Ω:

$$\Omega_i = \rho_\infty a_\infty b/m$$

This shows that the membrane panel is always stable, the degree of stability depending on the ratio of the air density to the running mass of the plate. A few calculations of damping ratios have been compared with numerical results from the exact solution of Goland and Luke[15] at $M = \sqrt{8}$; satisfactory agreement is found in all cases.

The foregoing proof holds true only in the case of a membrane.* When the fourth-derivative bending term is included one can demonstrate that, even though the separated differential equations for z_r and z_i are identical with the same boundary conditions, d does not necessarily have to be zero, and the resulting equations will have *two* linearly independent eigensolutions for a given set of eigenvalues (c, $d \neq 0$). In this case, the sign of Ω_i cannot be determined readily.

Recently doubts have been cast on the validity of the Rayleigh-Ritz method as applied to supersonic panel flutter. The relationship between Rayleigh-Ritz and exact solutions is easy to work out at Mach Numbers where piston theory applies. As an illustration, a Rayleigh-Ritz calculation was made on a panel with membrane stiffness only, having $(mM/\rho_\infty b^2) = 40$ (this single parameter fixes the dimensionless stability properties). According to the exact analysis, the panel is absolutely stable, having $(\Omega_i b/U) = 0.025$. However, the assumed-mode procedure yields the dimensionless flutter frequencies and speeds, based on the fundamental frequencies ω_1 of free vibration, which are tabulated below:

Number of Normal Modes Used in the Analysis of a Membrane	$\left(\dfrac{\omega_F}{\omega_1}\right)$	$\left(\dfrac{U_F}{b\omega_1}\right)$
2	1.58	4.81
3	2.48	4.82
4	3.43	4.84

It is seen that taking a few modes results in a finite flutter speed, inconsistent with the exact solution. The

* The authors are indebted to J. M. Hedgepeth of NACA and Professor Budiansky of Harvard University for pointing out this fact. The proof given in the original preprint for the plate-type panel is incorrect. (For further discussion, cf. Readers' Forum note by Hedgepeth.[21])

values of $U_F/b\omega_1$ might even seem to be converging toward a real limit, except that reference to the sequence of ω_F/ω_1's and careful examination of the flutter determinant for higher numbers of modes shows that some sort of divergence toward an infinite or imaginary U_F ultimately will take place. Unfortunately, no such careful attention to convergence is observed in routine flutter calculations, so this difficulty could be completely overlooked in the present case. It is interesting to note that if only odd or only even modes are considered, the Rayleigh-Ritz solution will always indicate a damped motion as in the exact solution.

The reader can imagine many other more complicated cases of high-speed panel flutter whose prediction will be expedited by the use of piston theory. Almost-plane plates with finite dimensions normal to the stream direction, and, in fact, with arbitrary shape, fall rigorously within its scope, since they do not have cut-off tips. Nonlinear aerodynamic effects due to initial bucklings are readily included and fit well into an analysis which also involves nonlinear structural properties. No further discussion is presented here on these aspects of the problem, because they are currently under study by Sechler and Fung at California Institute of Technology.

Other Aeroelastic Problems

Equations like (3) and (8) hold for motions with arbitrary time dependence, so that it is natural to consider employing them in unsteady aeroelastic calculations where the displacements are not sinusoidal. Typical cases of this sort are the response of aircraft to gusts or blasts, and high-speed loads due to rapid maneuvers. Certain steady-state analyses may also be simplified because of the point-function character of piston theory, or rendered more accurate through the introduction of thickness effects. A good example of the latter kind is Biot's section on nonlinearities in his study of the chordwise divergence of supersonic wings.[16] To the authors' knowledge, his is the first publication where Lighthill's theory is applied to a practical aeroelastic problem.

Steady-state supersonic calculations with two-dimensional aerodynamics are not facilitated by piston theory, because Ackeret's formula for the supersonic airfoil already exhibits the point-function relationship between pressure and normal velocity. A considerable saving is achieved, on the other hand, when three-dimensional air loads are needed but when some inaccuracy in the treatment of the immediate tip region can be overlooked. Weatherill's recent thesis[17] on the high-speed divergence and loss of aileron control of elastic wings confirms this point very well; his use of piston theory came as a brief afterthought following an elaborate analysis based on three-dimensional aerodynamic influence coefficients.

To demonstrate the utility of piston theory as a tool for dealing with unsteady transients, consider the gust response of a two-dimensional airfoil which can bend or

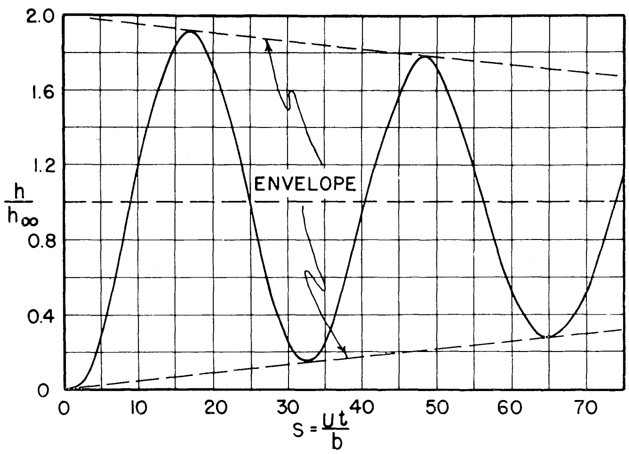


Fig. 11. Bending response of typical section to sharp-edged gust (properties given in text).

twist but undergoes no elastic chordwise deformation. Bisplinghoff, Ashley, and Halfman[18] discuss how all such problems can be solved by appropriate superposition of six indicial response functions, three lifts and three pitching moments. For example, the piston theory lift and moment following a step change $\dot{h}_0$ in the airfoil's vertical velocity at $t = 0$ assume the following constant values instantaneously:

$$P = -(4\rho_\infty bU\dot{h}_0/M) \tag{30}$$

$$M_\alpha = -(4\rho_\infty b^2U\dot{h}_0/M)(1 - 2x_0) \tag{31}$$

Thickness effects are omitted from Eqs. (30) and (31), but they can be introduced by means of a single multiplicative factor. Similarly, the lift and moment developed due to an encounter at $t = 0$ with a uniform sharp-edged gust of vertical velocity w_g are

$$P = \begin{cases} -(4\rho_\infty Ubw_g/M)(Ut/2b) & 0 \leq Ut/b \leq 2 \\ -(4\rho_\infty Ubw_g/M) & 2 \leq Ut/b \end{cases} \tag{32}$$

$$M_\alpha = \begin{cases} -(4\rho_\infty Ub^2w_g/M)[(1/4) \times (Ut/b)^2 - (Ut/b)x_0] & 0 \leq Ut/b \leq 2 \\ -(4\rho_\infty Ub^2w_g/M)\,[1 - 2x_0] & 2 \leq Ut/b \end{cases} \tag{33}$$

At Mach Numbers above 2, Eqs. (30)–(33) check very well with the more exact supersonic indicial functions given by Chang[19] and others. To illustrate the use of these functions, the bending response to a sharp-edged gust has been calculated for the airfoil shown in Fig. 2 with infinite torsional and control-surface stiffnesses ($\omega_\alpha = \omega_\beta = \infty$). The various constants entering the computation are $M = 3$, $\mu = 33^1/_3$, $\omega_h = 100$ rad./sec., $b = 3$ ft., $U \cong 3{,}000$ ft./sec. Fig. 11 presents this response in the form of the ratio of the instantaneous bending displacement $h(t)$ to its final steady-state value. The curve is actually just the transient of a second-order, linear system with constant coefficients. The slow rate of decay indicates that the aerodynamic damping at high Mach Numbers is quite small, so that a large number of cycles of vibration will go by before a significant amplitude reduction occurs.

It should be evident that computations like the above can be performed on three-dimensional wings with other elastic and rigid-body degrees of freedom included in the equations of motion. The only increased difficulties are organizational and numerical; the gain in simplicity relative to more exact aerodynamic theories is proportional to the size of the problem.

Concluding Remarks

As was convincingly demonstrated by Bisplinghoff in his 1955 Wright Brothers Lecture,[20] high-temperature structural problems, boundary-layer heat transfer, and a host of related effects are certain to complicate the work of the aeroelastician in the not-too-distant future. It is, therefore, a pleasure to be able to report the practical success of one new element in the picture which represents a significant reduction in labor as compared with corresponding theories appropriate to flight at slower speeds. A Mach Number zone can be defined, lying roughly between 2.5 and the lower limit of hypersonic flow (a boundary determined by the size of the product δM); it is characterized by the fact that thermal effects are usually quite important but that the distinction can still be made between a boundary layer and an external potential flow. Within this zone it seems clear that piston theory is the logical tool to use for estimating aerodynamic forces when analyzing a large majority of all aeroelastic and aerothermoelastic phenomena.

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