MFE Math Foundations
Assignment 4 Solutions

1. (a) As discussed in class, if we can show that the postulated price function $P(S,t)$ satisfies the Black-Scholes PDE,

$$P_t + rP_S S + \frac{\sigma^2}{2} P_{SS} S^2 - rP = 0,$$

and also the boundary conditions

$$P(S,T) = \Phi(S),$$

where $\Phi(S)$ is the option's payoff at time T (where $\Phi(S) = S^n$ in this case) given that the price of the underlying asset is $S = S(T)$, then option pricing theory implies that $P(S(t),t)$, $0 \le t < T$, is the price of the option at time t, for the price of the underlying at time t, $S(t)$.

We calculate the partial derivatives of the postulated solution,

$$P(S(t),t) = e^{\alpha(T-t)} S(t)^n, \qquad \alpha = (n-1)r + n(n-1)\frac{\sigma^2}{2},$$

obtaining

$$\begin{aligned} P_t &= -\alpha P, \\ P_S &= n\frac{P}{S}, \\ P_{SS} &= n(n-1)\frac{P}{S^2}. \end{aligned}$$

Plugging this into the Black Scholes PDE yields

$$\begin{aligned} P_t + rSP_S + \frac{\sigma^2}{2} S^2 P_{SS} - rP &= (-\alpha P) + rS\left(n\frac{P}{S}\right) + \frac{\sigma^2}{2} S^2 \left(n(n-1)\frac{P}{S^2}\right) - rP, \\ &= P\left(-\alpha + nr + n(n-1)\frac{\sigma^2}{2} - r\right) \\ &= 0. \end{aligned}$$

Finally, it follows immediately that $P(S,T) = S^n = \Phi(S)$, so all the conditions for $P(S(t),t)$ to be the price function of the option are satisfied.

2. (a) *PDE*: Let $x = \ln(S)$ and $s = T - t$. Then

$$
\begin{aligned}
V_s &= \left(r - \frac{\sigma^2}{2}\right) V_x + \frac{\sigma^2}{2} V_{xx} - rV, \qquad (1)\\
V|_{x=\ln(50)} &= 0, \quad 0 \le s < T\\
V|_{x=\ln(100)} &= 0, \quad 0 \le s < T\\
V|_{s=0} &= (50 - e^x)(100 - e^x) \stackrel{\text{def}}{=} \Phi(x), \quad \ln(50) < x < \ln(100).
\end{aligned}
$$

(b) *Program*: See Matlab program for convection equation in class slides, for a comparable example.

(c) *Solution*: The approximate solution is shown with $N = 200$, corresponding to $\Delta x = \frac{1}{200}$. The grid contains $N + 1$ points in the x-direction, $n = 0, 1, \ldots, N$, but since the boundary values are fixed, $V_0^m = V_N^m = 0$, the approximation occurs for points $n = 1, \ldots, N - 1$.

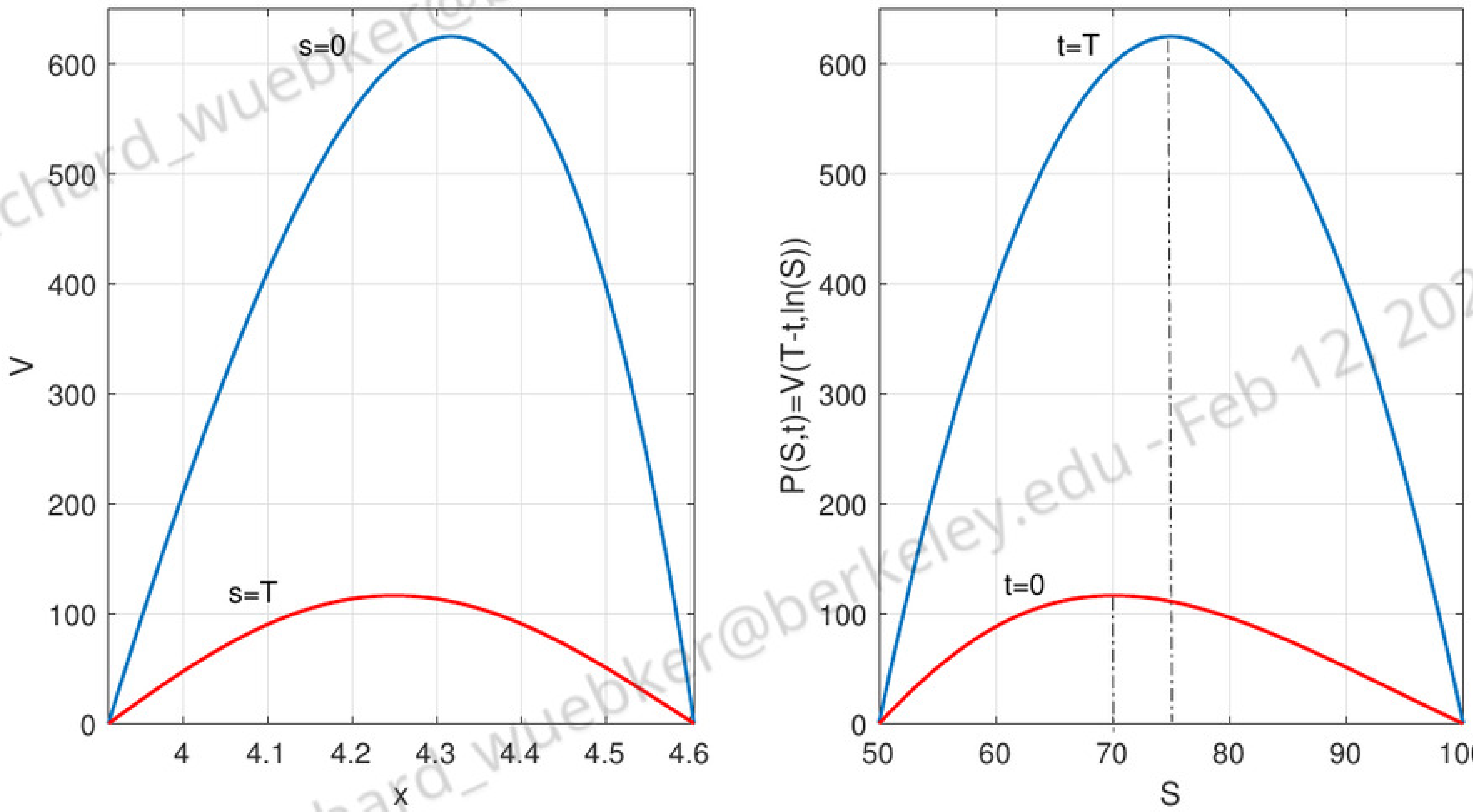

Figure 1: $V(x, 0)$, $V(x, 1)$, $P(S, 0)$, $P(S, 1)$.

We see from the figure that the maximum of the red line in the right panel occurs around $S = 70$, which is below the point of maximal payout which occur at $S = 75$, where $\Phi(\ln(S)) = (S - 50)(100 - S)$ is the highest. The reason is that the stock price is expected to increase between 0 and T. So if $S_0 = 75$, it is likely that $S_T > 75$, leading to a lower terminal payoff.

(d) *Stability*: By varying k, one finds that the approximate solution explodes for k somewhere between 6 and 6.5. For example, the maximum norm of the approximate solution at $T = 1$ is shown for $N = 200$ points and $N = 400$ points in Figure 2 for $k = 6, 6.01, 6.02, \ldots, 6.27$, plotted in logarithmic scale on the y-axis. As seen in the figure, the seems to be close to constant up until $k = 6.25$, and then explode for $k = 6.26$ and $k = 6.27$.

Indeed, although outside of the scope of this course, one can show that the theoretical stability bound is exactly $k_{\max} = 1/\sigma^2 = 1/0.4^2 = 6.25$ for this problem. This is shown by using the

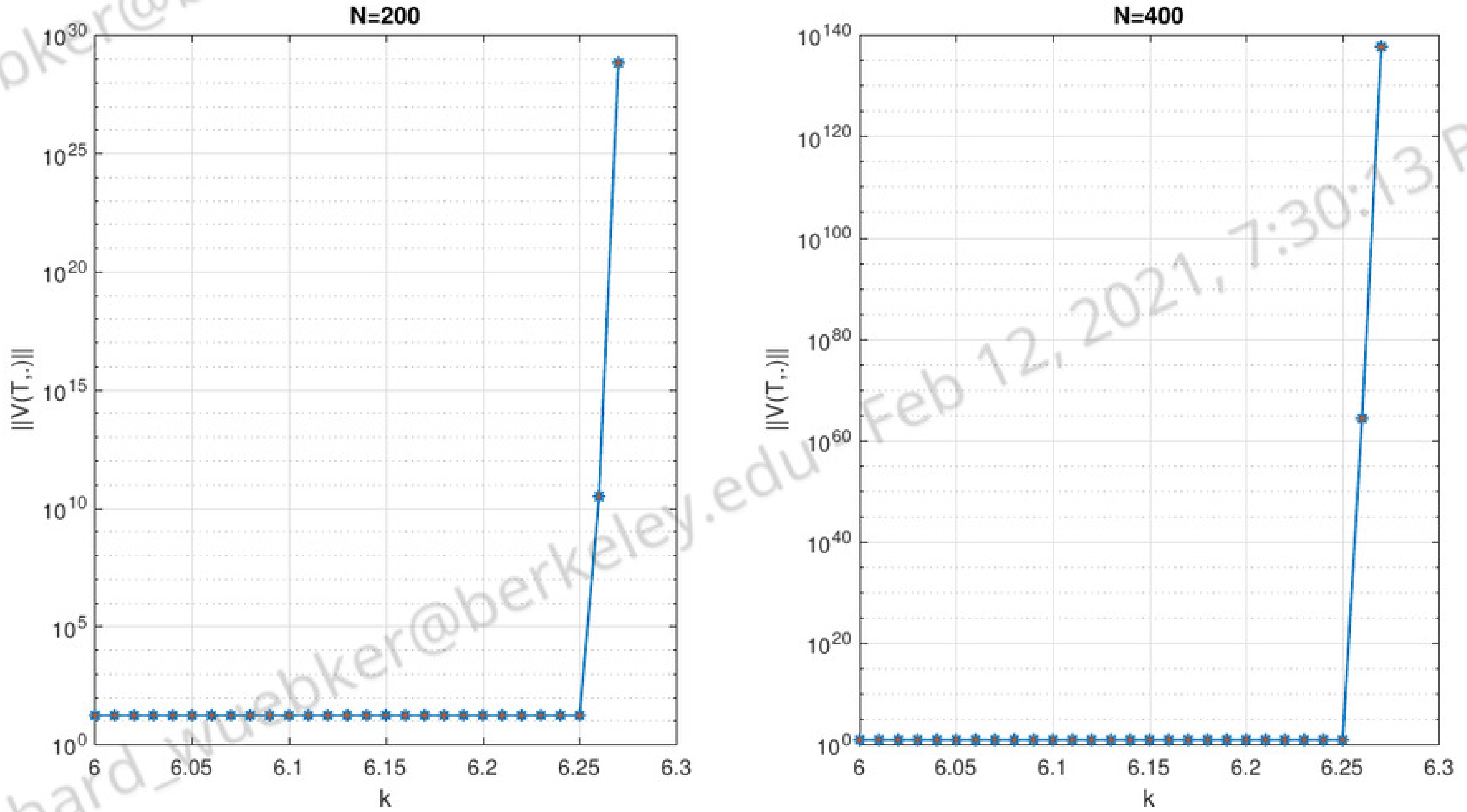

Figure 2: *Norm of solution,* $\|V(T,\cdot)\|_\infty$*, as function of* k*, for* $N = 200$ *(left panel) and* $N = 400$ *(right panel).*

following argument: Denote $a = r - \frac{\sigma^2}{2}$ and $b = \frac{\sigma^2}{2}$. Using the finite difference scheme

$$D_{+s}V = aD_{0x}V + bD_{+x}D_{-x}V - rV \tag{2}$$

on the grid function $V = \{V_n^m\}_{m,n}$ gives

$$\frac{V_n^{m+1} - V_n^m}{\Delta s} = a\frac{V_{n+1}^m - V_{n-1}^m}{2\Delta x} + b\frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{\Delta x^2} - rV_n^m,$$

where $m = 0, 1, \ldots, M-1$, $n = 1, 2, \ldots, N-1$. Let $k = \Delta s/\Delta x^2$. Rearranging both sides of the above equation gives the following form of the law of motion

$$V_n^{m+1} = \left[1 - \left(r + \frac{2b}{\Delta x^2}\right)\Delta s\right]V_n^m + \left(\frac{b}{\Delta x^2} - \frac{a}{2\Delta x}\right)\Delta s V_{n-1}^m + \left(\frac{a}{2\Delta x} + \frac{b}{\Delta x^2}\right)\Delta s V_{n+1}^m.$$

Let

$$\begin{aligned}\alpha &= -\frac{a}{2\Delta x} + \frac{b}{\Delta x^2},\\ \beta &= \frac{a}{2\Delta x} + \frac{b}{\Delta x^2},\end{aligned}$$

and note that $\alpha > 0$ and $\beta > 0$ for small Δx, since $b = \frac{\sigma^2}{2} > 0$. We can now write the law of motion in the form

$$V_n^{m+1} = [1 - (r + \alpha + \beta)\Delta s]V_n^m + \alpha\Delta s V_{n-1}^m + \beta\Delta s V_{n+1}^m. \tag{3}$$

Moreover, we can view the evaluation of (3) as applying a linear operator, $\mathcal{F}$ on a one-dimensional grid function (in the x-dimension) that steps forward one period in time,

$$V^{m+1} = \mathcal{F}[V^m], \qquad \text{where } V^m = (V_1^m, \ldots, V_{N-1}^m) \in \mathbb{R}^{N-1}.$$

We wish to find a bound on the operator $\mathcal{F}$ in the maximum norm. Recall that

$$\|V^m\|_\infty \stackrel{\text{def}}{=} \max_{1\le n\le N-1} |V_n^m|,$$

and the maximum norm of $\mathcal{F}$ is

$$\sup_{V\neq \mathbf{0}} \frac{\|\mathcal{F}[V]\|_\infty}{\|V\|_\infty}.$$

It then follows from (3) that

$$|V_n^{m+1}| \le |V_n^m| \times |1-(r+\alpha+\beta)\,\Delta s| + \alpha\Delta s \times |V_{n-1}^m| + \beta\Delta s \times |V_{n+1}^m|,$$

in turn implying

$$\begin{aligned}\|\mathcal{F}[V^m]\|_\infty &\le \|V^m\|_\infty \times (|1-(r+\alpha+\beta)\,\Delta s| + \alpha\Delta s + \beta\Delta s) \\ &\stackrel{\text{def}}{=} K \times \|V^m\|_\infty,\end{aligned}$$

so $\|F\|_\infty \le K$. Since $V^m = \mathcal{F}^m[V^0]$, it therefore follows that $\|V^m\|_\infty \le K^m \left\|V^0\right\|_\infty$. So, as long as $K \le 1$, V^m does not blow up regardless of m.
To ensure $K \le 1$, we note that when $1-(r+\alpha+\beta)\,\Delta s \ge 0$,

$$\begin{aligned}K = |1-(r+\alpha+\beta)\,\Delta s| + \alpha\Delta s + \beta\Delta s &= 1-(r+\alpha+\beta)\,\Delta s + \alpha\Delta s + \beta\Delta s \\ &= 1 - r\Delta s \\ &\nearrow 1,\end{aligned}$$

as $\Delta s \searrow 0$.
But

$$1-(r+\alpha+\beta)\,\Delta s \ge 0$$

is equivalent to

$$\Delta s \le \frac{1}{\alpha+\beta+r} = \left(r + \frac{2b}{\Delta x^2}\right)^{-1} = \frac{\Delta x^2}{r(\Delta x)^2+\sigma^2}, \tag{4}$$

an inequality that holds for small enough positive Δx if and only the stability constraint

$$\frac{\Delta s}{\Delta x^2} < \frac{1}{\sigma^2} = 6.25 \tag{5}$$

is satisfied. It then follows that $\|V^m\|_\infty \le \left\|V^0\right\|_\infty \le \max_{x\in A}|\Phi(x)| = \|\Phi\|_\infty$, where $A = [\ln(50), \ln(100)]$, and the continuous maximum norm for Φ on the domain A is used. Putting it all together, we arrive at $|V_n^m|$ being bounded by $\|\Phi\|_\infty$ regardless of m and n, as long as (5) is satisfied.

(e) *Order of convergence*: We verify the order of convergence numerically but plotting the error, $\|e(T,\cdot)\|_\infty$ against the number of points N in logarithmic coordinates. The result is shown in Figure 3, indicating that the the method is of second order in N (of first order in M, since M is proportional to N^2). Specifically, define the grid function

$$\hat{V}_n^m = V(m\Delta s, n\Delta x),$$

where $V(m\Delta s, n\Delta x)$ is the exact solution to the PDE at $s = m\Delta s$, $x = n\Delta x$. The error then satisfies

$$\left|\hat{V}_n^{T/\Delta s} - V_n^{T/\Delta s}\right| \le C_T(\Delta x^2) = C'_T(N^{-2}),$$

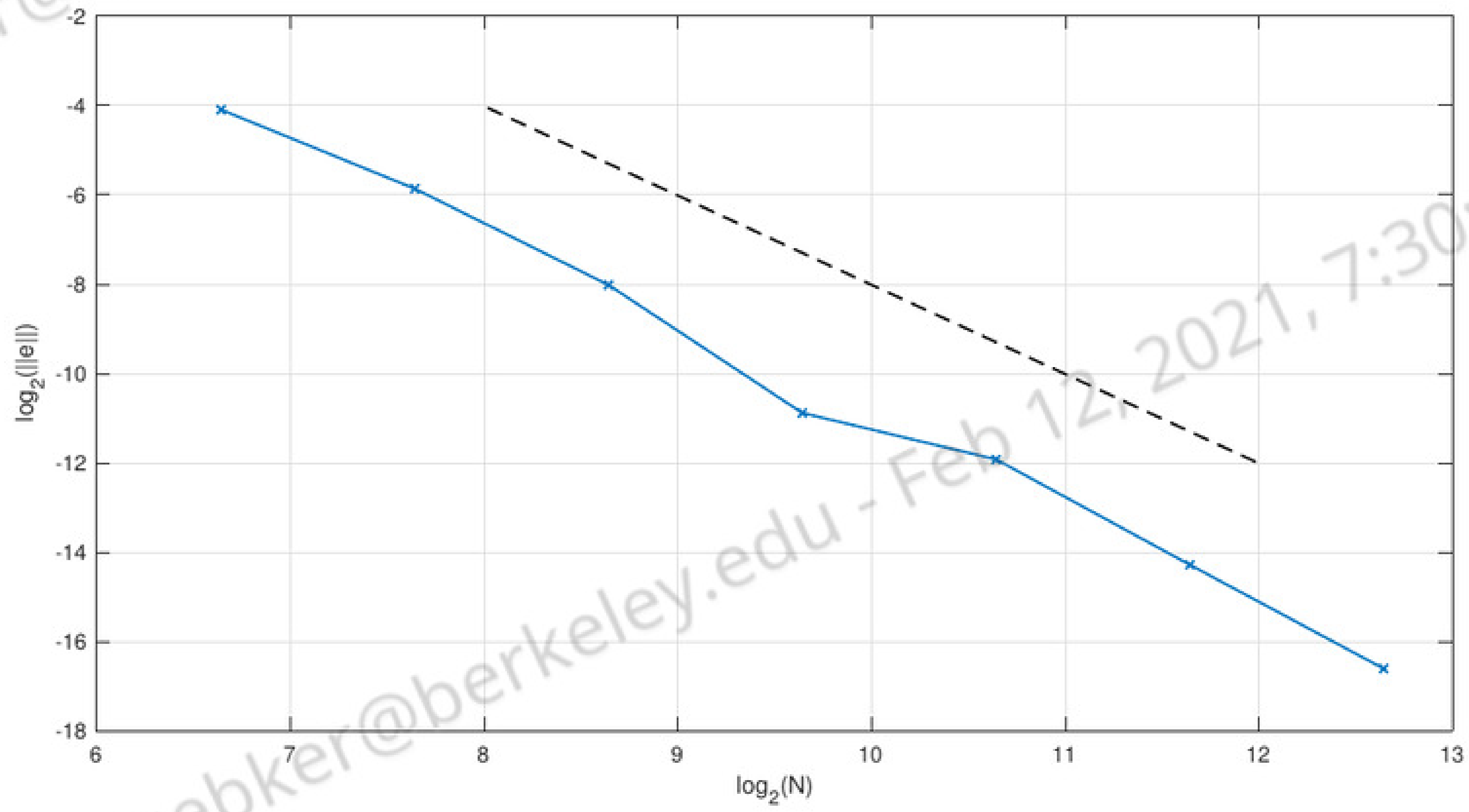

Figure 3: *Estimated convergence order. Slope of log-error as a function of log-number of points is approximately -2, suggesting second order convergence.*

where the constants C_T, C_T' are allowed to depend on T, but not on Δx, Δs, or n.
As was the case for the stability result, the theoretical proof of this result is outside of the scope of this course, but provided in what follows, for instructiveness. The local truncation error introduced by $\mathcal{F}$ is defined by

$$T_n^{m+1} = \hat{V}_n^{m+1} - (\mathcal{F}[\hat{V}^m])_n,$$

i.e., it is the error introduced when applying $\mathcal{F}$ to the exact solution. We note that (2) implies that

$$\mathcal{F}[\hat{V}^m] = \hat{V}^m + \Delta s \left(aD_{0x} + bD_{+x}D_{-x} - r\right) V^m,$$

and since D_{0x} and $D_{+x}D_{-x}$ are second order approximations of the first and second order differential operators, a Taylor expansion yields

$$\begin{aligned} T_n^{m+1} = \hat{V}_n^{m+1} - (\mathcal{F}[\hat{V}^m])_n &= \hat{V}_n^{m+1} - \hat{V}_n^m - \Delta s \left(a(\hat{V}_x)_n^m + b(\hat{V}_{xx})_n^m - r\hat{V}_n^m + C_1\Delta x^2 \right) \\ &= (\hat{V}_t)_n^m \Delta s + C_2(\Delta s^2) - \Delta s \left(a(\hat{V}_x)_n^m + b(\hat{V}_{xx})_n^m - r\hat{V}_n^m + +C_1\Delta x^2 \right) \\ &= C_2(\Delta s^2) - C_1(\Delta s \Delta x^2). \end{aligned}$$

Here, $(\hat{V}_t)_n^m = V_t(m\Delta s, n\Delta x)$, $(\hat{V}_x)_n^m = V_x(m\Delta s, n\Delta x)$, and $(\hat{V}_{xx})_n^m = V_{xx}(m\Delta s, n\Delta x)$ represent the partial derivatives of the exact solution evaluated at $s = m\Delta s$, $x = n\Delta x$, we use the fact that $\hat{V}$ is the solution to the PDE (1), and

$$\begin{aligned} |C_1| &\leq K_1 \stackrel{\text{def}}{=} \sup_{x\in A, s\in(0,T]} \frac{1}{6} \left|V_{xxx}(s,x)\right| + \sup_{x\in A, s\in(0,T]} \frac{1}{12} \left|V_{xxxx}(s,x)\right|, \\ |C_2| &\leq K_2 \stackrel{\text{def}}{=} \sup_{x\in A, s\in(0,T]} \frac{1}{2} \left|V_{tt}(s,x)\right|. \end{aligned}$$

So $|T_n^{m+1}| \leq C\Delta s(\Delta s + \Delta x^2)$, where $C = K_1 + K_2$.

We can now expand

$$
\begin{aligned}
\hat{V}^1 &= \mathcal{F}[\hat{V}^0] + T^1 \\
\hat{V}^2 &= \mathcal{F}[\hat{V}^1] + T^2 = \mathcal{F}^2[\hat{V}^0] + \mathcal{F}[T^1] + T^2 \\
\hat{V}^3 &= \mathcal{F}[\hat{V}^2] + T^3 = \mathcal{F}^3[\hat{V}^0] + \mathcal{F}^2[T^1] + \mathcal{F}[T^2] + T^3, \\
&\dots \\
\hat{V}^m &= \mathcal{F}^m[\hat{V}^0] + \sum_{k=1}^{m} \mathcal{F}^{M-k}[T^k] = V^m + \sum_{k=1}^{m} \mathcal{F}^{M-k}[T^k],
\end{aligned}
$$

arriving at

$$
\begin{aligned}
\left\| \hat{V}^m - V^m \right\|_\infty &= \left\| \sum_{k=1}^{m} \mathcal{F}^{M-k}[T^k] \right\|_\infty \\
&\leq \sum_{k=1}^{m} \|\mathcal{F}\|_\infty^{M-k} \left\| T^k \right\|_\infty \\
&\leq \sum_{k=1}^{m} K^{M-k} C \Delta s(\Delta s + \Delta x^2) \\
&\leq C m \Delta s(\Delta s + \Delta x^2).
\end{aligned}
$$

When plugging in $m = \frac{T}{\Delta s}$, this leads to the final bound

$$
\left| \hat{V}_n^{T/\Delta s} - V_n^{T/\Delta s} \right| \leq C \frac{T}{\Delta s} \Delta s(\Delta s + \Delta x^2) = C_T(\Delta x^2),
$$

where $C_T = CT(k+1)$, and $k = \frac{\Delta s}{\Delta x^2}$. We have shown second order convergence in the maximum norm. Note that the fact that $\|\mathcal{F}\|_\infty = K \leq 1$, which was needed to show stability before, was crucial in the final steps of this derivation.