MFE Math Foundations
Assignment 1 Solutions

1. Let the per-period discount rate be given by r.

   (a) To compute the value of the growing perpetuity, we need the present value of all future cash flows, which is given by

   $$P = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots = \frac{C}{1+r}\sum_{n=1}^{\infty}\left[\frac{1+g}{1+r}\right]^n .$$

   Note that the series only converges if $g < r$, in which case we have

   $$P = \frac{C}{1+r}\sum_{n=0}^{\infty}\left[\frac{1+g}{1+r}\right]^n = \frac{C}{1+r}\cdot\frac{1}{1-\frac{1+g}{1+r}} = \frac{C}{1+r}\cdot\frac{1+r}{r-g} = \frac{C}{r-g}.$$

   (b) For simplicity, suppose that the end of the year 2061 A.D. is exactly 47 years away. Let $m = 76$ and let $C = 2061$. To simplify later calculations we will also define $\alpha = (1+r)^{-m}$. Then the present value of all future cash flows is given by

   $$\begin{aligned} P &= \frac{C}{(1+r)^{47}} + \frac{C+m}{(1+r)^{47+m}} + \frac{C+2\cdot m}{(1+r)^{47+2\cdot m}} + \cdots \\ &= \frac{1}{(1+r)^{47}}\sum_{k=0}^{\infty}\frac{C+km}{(1+r)^{mk}} \\ &= \frac{1}{(1+r)^{47}}\left[C\sum_{k=0}^{\infty}\alpha^k + m\sum_{k=1}^{\infty}k\alpha^k\right] \\ &= \frac{1}{(1+r)^{47}}\left[\frac{C}{1-\alpha} + \frac{m\alpha}{(1-\alpha)^2}\right] \\ &\approx 2512.92\,. \end{aligned}$$

   Note that we have used the result of Problem 2(b). In Figure 1, one can see that the sequence of partial sums converges quite rapidly to the exact value.

2. Let $P_K(S)$ denote the Black-Scholes value of a European put option. Define $x = \frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$ as in the lecture notes.

   (a) We first compute

   $$\frac{\partial x}{\partial S} = \frac{1}{S\sigma\sqrt{T}}.$$

Using the chain and product rules, we compute

$$
\begin{aligned}
\Delta_{P_K} &= Ke^{-rT}\left[-N'(x-\sigma\sqrt{T})\tfrac{\partial x}{\partial S}\right] + N(x) - 1 - S[-N'(x)\tfrac{\partial x}{\partial S}] \\
&= N(x) - 1 + \frac{N'(x)}{\sigma\sqrt{T}} - \frac{Ke^{-rT}}{S\sigma\sqrt{T}}N'(x-\sigma\sqrt{T}) \\
&= N(x) - 1 + \frac{N'(x)}{\sigma\sqrt{T}}\left[1 - \frac{Ke^{-rT}}{S}e^{x\sigma\sqrt{T}-\frac{\sigma^2 T}{2}}\right].
\end{aligned} \tag{1}
$$

where we have used the fact that $N'(x-\sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}+x\sigma\sqrt{T}-\frac{\sigma^2 T}{2}}$. However, from the definition of x we have

$$x\sigma\sqrt{T} = \ln(S/K) + rT + \tfrac{\sigma^2 T}{2},$$

from which it follows that $e^{x\sigma\sqrt{T}-\frac{\sigma^2 T}{2}} = \frac{S}{K}e^{rT}$. Inserting this into the final expression in (1) yields the desired result.

(b) Thankfully, most of our work has been done in part (a): to compute the gamma of the put Γ_{P_K}, we observe that $\Gamma_{P_K} = \frac{\partial^2}{\partial S^2}P_K = \frac{\partial}{\partial S}\Delta_{P_K}$. Thus, by the chain rule, one has

$$\Gamma_{P_K} = N'(x)\tfrac{\partial x}{\partial S} = \frac{N'(x)}{S\sigma\sqrt{T}}$$

by our previous calculations.

(c) Our portfolio value is given by

$$V(S) = h_S S + h_1 P_{K_1} + h_2 P_{K_2}.$$

The first and second order Taylor expansions are given by

$$V(S+\Delta S) = V(S) + V'(S)\Delta S + C\Delta S^2,$$

and

$$V(S+\Delta S) = V(S) + V'(S)\Delta S + \frac{1}{2}V''(S)\Delta S^2 + C'\Delta S^3.$$

for some constants C, C'. Thus for a delta-gamma hedge, we require $V'(S) = V''(S) = 0$ (this will immunize our portfolio to small changes in the stock ΔS, with errors as above). Assuming that h_S is fixed, we compute

$$
\begin{aligned}
V'(S) &= h_S + h_1\Delta_{P_{K_1}} + h_2\Delta_{P_{K_2}} \\
V''(S) &= h_1\Gamma_{P_{K_1}} + h_2\Gamma_{P_{K_2}}
\end{aligned}
$$

Setting these equal to 0 and then solving yields

$$
\begin{aligned}
h_1 &= \frac{-h_S\Gamma_{P_{K_2}}}{\Gamma_{P_{K_2}}\Delta_{P_{K_1}} - \Delta_{P_{K_2}}\Gamma_{P_{K_1}}} \\
h_2 &= \frac{h_S\Gamma_{P_{K_1}}}{\Gamma_{P_{K_2}}\Delta_{P_{K_1}} - \Delta_{P_{K_2}}\Gamma_{P_{K_1}}}
\end{aligned}
$$

3. Let $\bar{\mathbf{D}} = \left[-\mathbf{s}^0, \mathbf{D}\right] = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 1 & 2 \\ -16 & 8 & 9 \end{bmatrix}$.

(a) The time 0 price of the portfolio $\mathbf{h} = (1,1,0)^T$ is $\mathbf{h}^T\mathbf{s}^0 = 5$.

(b) The market is complete since $\mathbf{rank}(\mathbf{D}) = 2$ which is equal to the number of states.

(c) The LOOP does not hold in this market. Note that $\mathbf{rank}(\bar{\mathbf{D}}) = 3$ but $\mathbf{rank}(\mathbf{D}) = 2$, so the system of equations for a state price vector is inconsistent. For example, the portfolio $\mathbf{h}^1 = (7, 1, 0)^T$ costs 17 and generates payoffs of 8 in outcome $m = 1$ and 9 in outcome $m = 2$. This is the same as the payoffs generated by the portfolio $\mathbf{h}^2 = (0, 0, 1)^T$, which at a cost of 16 is cheaper than $\mathbf{h}_1$.

4. (a) An equivalent way of stating that $\psi \in \mathbb{R}^M$ is a state price vector is that $\mathbf{h}^T\bar{\mathbf{D}}[1;\psi] = 0$ for all $\mathbf{h} \in \mathbb{R}^N$, i.e., that $\bar{\mathbf{D}}[1;\psi] = \mathbf{0}$. Recall that $\mathcal{N}(\bar{\mathbf{D}}) = \{\eta \in \mathbb{R}^{M+1} : \bar{\mathbf{D}}\eta = \mathbf{0}\}$, so if we define $\Psi = \mathcal{N}(\bar{\mathbf{D}})$, ψ is a state price vector if $[1;\psi] \in \Psi$.
Recall that $Range(\bar{\mathbf{D}}^T) = \{\bar{\mathbf{D}}^T\mathbf{h} : \mathbf{h} \in \mathbb{R}^N\}$. We define $R = Range(\bar{\mathbf{D}}^T)$, and from the theory in class we then know that $R^\perp = \Psi$, $\mathbb{R}^{M+1} = R \oplus \Psi$, and the rank theorem: $dim(R) + dim(\Psi) = M + 1$.
Now, define the vector $\delta_0 = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^{M+1}$, i.e., the vector with the first element being equal to one and all other elements equal to zero. The law of one price can then equivalently be stated as $\delta_0 \notin R$.
So, the linear algebraic formulation of part 1 is that
$$\delta_0 \notin R \Leftrightarrow [1;\psi] \in \Psi \text{ for some } \psi \in \mathbb{R}^M,$$
which is the result we wish to prove.
$\Rightarrow$: If $\delta_0 \notin R$, $dim(R) \le M$, so $\dim(\Psi) \ge 1$ from the rank theorem. Since $\delta_0 \notin R$, there is a $\eta \in \Psi$ that is not orthogonal to δ_0, i.e., $\langle \delta_0, \eta \rangle \neq 0$, so $\eta_1 \neq 0$. Rescaling the vector, we get $\frac{1}{\eta_1}\eta = [1;\psi] \in \Psi$, so there exists a state price vector.
$\Leftarrow$: If $\delta_0 \in R$, then any vector of the form $\eta = [1;\psi]$ has $\langle \delta_0, \eta \rangle \neq 0$, so it cannot be that $\eta \in \Psi$. Thus, no state price vector exists.

(b) From the previous result, given that a state-price vector exists, $dim(R) \le M$. Also, since a state price vector exists, $\bar{\mathbf{D}}[1;\psi] = \mathbf{0}$, which means that $\mathbf{s}^0 = \mathbf{D}\psi$, so $\mathbf{s}^0$ is within the range of $\mathbf{D}$, and therefore $Rank(\mathbf{D}) = Rank(\bar{\mathbf{D}}) = dim(R)$. The market is complete if and only if $Rank(\mathbf{D}) = M$, which is therefore equivalent to $dim(R) = M$, and via the rank theorem to $dim(\Psi) = 1$, i.e., to uniqueness of the state price vector.

5. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

(a) The eigenvalues are the roots of the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. In this case, the equation reads
$$\lambda^2 - 6\lambda + 8 = 0\,,$$
which has roots $\lambda_1 = 2$, $\lambda_2 = 4$.
The corresponding eigenvectors are found by finding the nullspace (just like we did in the previous assignment) of $\lambda_i\mathbf{I} - \mathbf{A}$ for $i = 1, 2$. In this case, we find $\mathbf{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

(b) $\mathbf{Q} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\mathbf{\Lambda} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

(c) **Yes**, since $\mathbf{A}$ is symmetric with signature $(0, 0, 2)$.

6. Let $\mathbf{X}$ be as in the problem statement.

(a) To find the optimal $\mathbf{x}$, we proceed as suggested in the lecture notes and compute the eigenvalue decomposition $\mathbf{X}\mathbf{X}^T = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where the eigenvalues are listed in decreasing absolute value. We have

$$\mathbf{X}\mathbf{X}^T = \begin{bmatrix} 0.7155 & 1.4200 \\ 1.4200 & 3.127 \end{bmatrix},$$

and, calculating the eigenvalues and eigenvectors, we get $\lambda_1 = 3.784$, $\lambda_2 = 0.058$, and

$$\mathbf{X}\mathbf{X}^T = \mathbf{Q}\Lambda\mathbf{Q}^T = \begin{bmatrix} 0.42 & -0.9075 \\ 0.9075 & 0.42 \end{bmatrix} \begin{bmatrix} 3.874 & 0 \\ 0 & 0.0584 \end{bmatrix} \begin{bmatrix} 0.42 & 0.9075 \\ -0.9075 & 0.42 \end{bmatrix}.$$

Then $\mathbf{x}$ will be (transpose of) the first column of $\mathbf{Q}$. In this case $\mathbf{x} \approx (0.4200, 0.9075)^T$.

(b) The coordinates $\mathbf{c}$ that minimize the squared error given this $\mathbf{x}$ are $c_m = \frac{\langle \mathbf{x}, \mathbf{v}^m \rangle}{\| \mathbf{x} \|_2^2}$. In this case, the error is

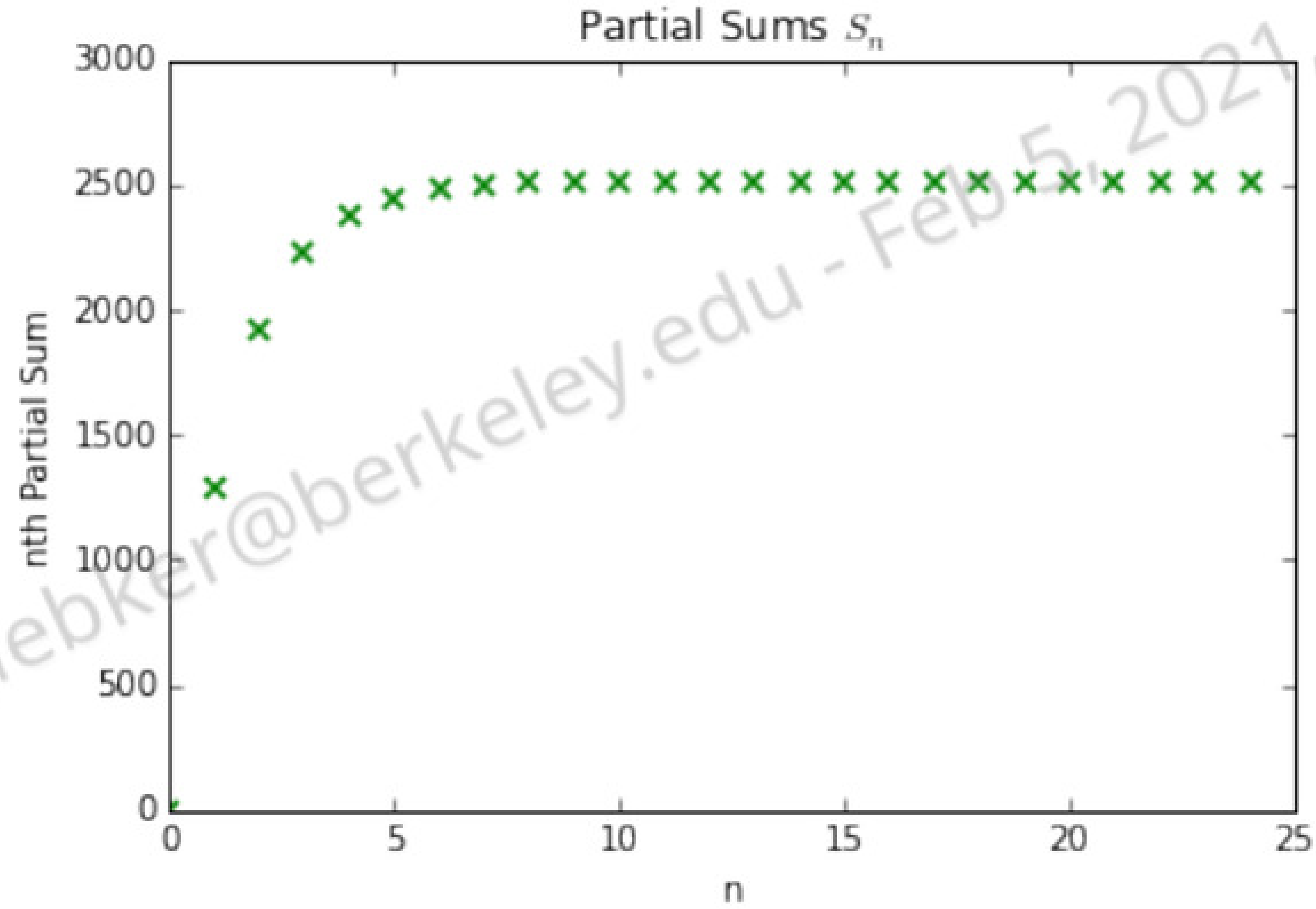
$$\varepsilon = \frac{1}{M}\sqrt{\sum_m \| \mathbf{v}^m - \mathbf{x}c_m \|_2^2} \approx 0.0161$$

(c) The error suggests that this is a fairly good approximation. However, there are a few outliers. In Figure 2, we display a scatter plot of the data points as well as the corresponding approximations in the subspace produced by our PCA analysis.

Plots

Problem 3(b)

Figure 1: Partial sums of the comet perpetuity expansion



Problem 6(c)

Figure 2: Graphical error analysis