# Mathematical Foundations

# MFE Spring 2021

# Assignment 1

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Solved in team: No

# 1. Bond Pricing

> **a. Derive a formula for the value of a growing perpetuity with growth rate g and discount rate r where:**

$$0 < g < r$$
$$B = \sum_{k=0}^{\infty} \frac{C(1+g)^k}{(1+r)^{k+1}}$$

$$B = C \sum_{k=0}^{\infty} \frac{(1+g)^k}{(1+r)^{k+1}}$$

$$B = \frac{C}{(1+r)} \sum_{k=0}^{\infty} \left(\frac{1+g}{1+r}\right)^k$$

$$0 < g < r \implies \left(\frac{1+g}{1+r}\right) < 1$$

$$B = \frac{C}{(1+r)}\left(\frac{1}{1-\frac{1+g}{1+r}}\right) \quad \because \quad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$B = \frac{C}{(1+r)}\left(\frac{1+r}{r-g}\right)$$

$$B = \frac{C}{r-g}$$

**b. As of December 31, 2014, calculate the value of the comet perpetuity. First payment at end of 2061 and occurs every 76 years, the value being the year,** $r = 1\%$

$$B = \sum_{k=0}^{\infty} \frac{2061 + 76k}{(1.01)^{47+76k}}$$

$$B = \frac{1}{1.01^{47}} \sum_{k=0}^{\infty} \frac{2061 + 76k}{1.01^{76k}}$$

$$B = \frac{1}{1.01^{47}}\left(\sum_{k=0}^{\infty} \frac{2061}{1.01^{76k}} + \sum_{k=0}^{\infty} \frac{76k}{1.01^{76k}}\right) \tag{1}$$

I will solve each of the sums in parens individually

$$* * *$$

$$\sum_{k=0}^{\infty} \frac{2016}{1.01^{76k}} = 2061 \sum_{k=0}^{\infty} \left(\frac{1}{1.01^{76}}\right)^k = 2061\left(\frac{1}{1 - \frac{1}{1.01^{76}}}\right) = 2061\left(\frac{1.01^{76}}{1.01^{76} - 1}\right) \approx 3884.5392$$

$$* * *$$

$$\text{Let } V = \sum_{k=0}^{\infty} \frac{76k}{1.01^{76k}}$$

$$V = \sum_{k=0}^{\infty} \frac{76k}{1.01^{76k}} = \sum_{k=1}^{\infty} \frac{76(k-1)}{1.01^{76(k-1)}} = \sum_{k=1}^{\infty} \frac{76(k-1)}{1.01^{(76k-76)}} = (1.01^{76}) \sum_{k=1}^{\infty} \left( \frac{76(k-1)}{1.01^{76k}} \right)$$

$$V = (1.01^{76})\left( \sum_{k=1}^{\infty} \frac{76k}{1.01^{(76k)}} - \sum_{k=1}^{\infty} \frac{76}{1.01^{(76k)}} \right)$$

$$V = (1.01^{76})\left( V - \sum_{k=1}^{\infty} \frac{76}{1.01^{(76k)}} \right) \quad \because \quad \sum_{k=0}^{\infty} \frac{76k}{1.01^{76k}} = \sum_{k=1}^{\infty} \frac{76k}{1.01^{76k}}$$

$$V = (1.01^{76})V - 76(1.01^{76}) \sum_{k=1}^{\infty} \frac{1}{1.01^{(76k)}}$$

$$V - (1.01^{76})V = -76(1.01^{76}) \sum_{k=1}^{\infty} \frac{1}{1.01^{(76k)}}$$

$$V[1 - (1.01^{76})] = -76 \sum_{k=1}^{\infty} \frac{1}{1.01^{76k-76}}$$

$$V[1 - (1.01^{76})] = -76 \sum_{k=1}^{\infty} \frac{1}{1.01^{76(k-1)}}$$

$$V[1 - (1.01^{76})] = -76 \sum_{k=0}^{\infty} \left( \frac{1}{1.01^{76}} \right)^{k}$$

$$V[1 - (1.01^{76})] = -76\left( \frac{1}{1 - \frac{1}{1.01^{76}}} \right)$$

$$V[1 - (1.01^{76})] = -76\left( \frac{1.01^{76}}{1.01^{76} - 1} \right)$$

$$V = 126.7396$$

Plugging the two sums back into (1)

$$B \approx \frac{1}{1.01^{47}}(3884.5392 + 126.7396)$$

$$B \approx 2512.9178$$

# 2. Portfolio Hedging

> **a. Derive the delta of a vanilla european put option from the formula:**

$$P_k(S) = Ke^{-rT}[1 - N(x - \sigma\sqrt{T})] - S[1 - N(x)]$$
$$x = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$* * *Solution * **$$

$$\Delta x_S = \frac{1}{S\sigma\sqrt{T}}$$

$$\Delta P = -Ke^{-rT}N'(x - \sigma\sqrt{T})\Delta x_S - 1 + N(x) + SN'(x)\Delta x_S$$

$$\Delta P = -Ke^{-rT}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\Delta x_S - 1 + N(x) + SN'(x)\Delta x_S \quad \because \quad N'(x - \sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}$$

$$\Delta P = -Ke^{-rT}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x^2-2x\sigma\sqrt{T}+\sigma^2T)}{2}}\Delta x_S - 1 + N(x) + SN'(x)\Delta x_S$$

$$\Delta P = \frac{-Ke^{-rT}e^{-\frac{x^2}{2}}e^{-\frac{\sigma^2T}{2}}e^{x\sigma\sqrt{T}}}{\sqrt{2\pi}}\Delta x_S + \frac{Se^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Delta x_S + N(x) - 1$$

$$\Delta P = \frac{\Delta x_S e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left( S - K e^{-rT} e^{-\frac{\sigma^2 T}{2}} e^{x\sigma\sqrt{T}} \right) + N(x) - 1$$

$$\Delta P = \frac{\Delta x_S e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left( S - K e^{-rT} e^{-\frac{\sigma^2 T}{2}} e^{ln(S) - ln(k) + (r+\sigma^2/2)T} \right) + N(x) - 1$$

$$\Delta P = \frac{\Delta x_S e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left( S - K e^{-rT} e^{-\frac{\sigma^2 T}{2}} \frac{S}{K} e^{rT} e^{\frac{\sigma^2 T}{2}} \right) + N(x) - 1$$

$$\Delta P = \frac{\Delta x_S e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left( S - S \right) + N(x) - 1$$

$$\Delta P = N(x) - 1$$

**b. Derive the formula for the gamma of a put option**

$$\Gamma_P = \frac{\partial (N(x) - 1)}{\partial S}$$

$$\Gamma_P = N'(x) \Delta x_S \quad \text{where} \quad \Delta x_S = \frac{\partial \left( \frac{ln(S/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \right)}{\partial (S)} = \frac{\partial \left( \frac{ln(S) - ln(K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \right)}{\partial (S)} = \frac{1}{S\sigma\sqrt{T}}$$

$$\Gamma_P = \frac{N'(x)}{S\sigma\sqrt{T}}$$

**c. Use Taylor's formula to derive expressions for $h_1$ and $h_2$ in the delta-gamma neutral hedge as a function of $\Delta_{P_{K_1}}$ , $\Delta_{P_{K_2}}$ , $\Gamma_{P_{K_1}}$ and $\Gamma_{P_{K_2}}$**

$$V(S) = h_S S + h_1 P_1(S) + h_2 P_2(S)$$

$$V(S+\Delta S) = V(S) + \Delta S \frac{\partial V(S)}{\partial S} + \frac{(\Delta S)^2}{2} \frac{\partial^2 V(S)}{\partial S^2} + \mathcal{O}(\Delta S)^3$$

$$V(S+\Delta S) = V(S) + \Delta S(h_S + h_1 \Delta_{P_1} + h_2 \Delta_{P_2}) + \frac{(\Delta S)^2}{2}(h_1 \Gamma_{P_1} + h_2 \Gamma_{P_2}) + \mathcal{O}(\Delta S)^3$$

$$V(S+\Delta S) - V(S) = \Delta S(h_S + h_1 \Delta_{P_1} + h_2 \Delta_{P_2}) + \frac{(\Delta S)^2}{2}(h_1 \Gamma_{P_1} + h_2 \Gamma_{P_2}) + \mathcal{O}(\Delta S)^3$$

We want to set the change in portfolio value to zero

$$\Delta S(h_S + h_1 \Delta_{P_1} + h_2 \Delta_{P_2}) + \frac{(\Delta S)^2}{2}(h_1 \Gamma_{P_1} + h_2 \Gamma_{P_2}) = 0$$

I will split these up to set the first order effect to zero and set the second order effect to zero to solve the system for $h_1$ and $h_2$

$$\Delta S(h_S + h_1 \Delta_{P_1} + h_2 \Delta_{P_2}) = 0 \qquad \frac{(\Delta S)^2}{2}(h_1 \Gamma_{P_1} + h_2 \Gamma_{P_2}) = 0$$

$$h_1 \Delta_{P_1} + h_2 \Delta_{P_2} = -h_S \quad (1) \qquad h_1 \Gamma_{P_1} + h_2 \Gamma_{P_2} = 0 \quad (2)$$

I will solve for $h_1$ in (1) then plug it into (2) to solve for $h_2$

$$h_1 = -\frac{(h_S + h_2 \Delta_{P_2})}{\Delta_{P_1}} \quad (3) \qquad -\frac{(h_S + h_2 \Delta_{P_2})}{\Delta_{P_1}} \Gamma_{P_1} + h_2 \Gamma_{P_2} = 0$$

$$(h_S + h_2 \Delta_{P_2})\Gamma_{P_1} - \Delta_{P_1} h_2 \Gamma_{P_2} = 0$$

$$h_S \Gamma_{P_1} + h_2 \Delta_{P_2} \Gamma_{P_1} - \Delta_{P_1} h_2 \Gamma_{P_2} = 0$$

$$h_2(\Delta_{P_2} \Gamma_{P_1} - \Delta_{P_1} \Gamma_{P_2}) = -h_S \Gamma_{P_1}$$

$$h_2 = \frac{-h_S \Gamma_{P_1}}{(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2})}$$

Now plugging $h_2$ back into (3) to solve for $h_1$

$$h_1 = \frac{-(h_S + \frac{-h_S\Gamma_{P_1}}{(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2})}\Delta_{P_2})}{\Delta_{P_1}}$$

$$h_1 = \frac{-(h_S(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2}) - h_S\Gamma_{P_1}\Delta_{P_2})}{\Delta_{P_1}(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2})}$$

$$h_1 = \frac{h_S(\Gamma_{P_1}\Delta_{P_2} - \Delta_{P_2}\Gamma_{P_1} + \Delta_{P_1}\Gamma_{P_2})}{\Delta_{P_1}(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2})}$$

$$h_1 = \frac{h_S(\Gamma_{P_1}\Delta_{P_2} - \Delta_{P_2}\Gamma_{P_1} + \Delta_{P_1}\Gamma_{P_2})}{\Delta_{P_1}(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2})} \qquad h_2 = \frac{-h_S\Gamma_{P_1}}{(\Delta_{P_2}\Gamma_{P_1} - \Delta_{P_1}\Gamma_{P_2})}$$

# 3. Trading Model

**a. What is the t = 0 price of the portfolio $h = (1, 1, 0)^T$**

$$\overline{D} = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 1 & 2 \\ -16 & 8 & 9 \end{bmatrix}$$

$$p_0 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -16 \end{bmatrix} = -\$5$$

**b. Is this market complete?**

For completeness we look at the reachable payoff space which is the rowspace of D which is the columnspace of $D^T$

$$D^T = \begin{bmatrix} 1 & 1 & 8 \\ 1 & 2 & 9 \end{bmatrix}$$

We can easily see that column 3 is just 7 times column 1 + column 2, however column 1 and column 2 are linearly independent and span the whole of $R^2$ which is our outcome space, thus the market is complete.

**c. Does the law of one price (LOOP) hold in this market?**

No it does not hold. For example, if we hold 7 shares of asset 1, and 1 share of asset 2, the payoff will be:

$$\begin{bmatrix} 7 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 9 \end{bmatrix}$$

This is the same payoff as holding asset 3. Asset 3 has a cost of 16, but the above portfolio would cost 17 so we have the same outcome but with different prices at time t = 0.

# 4. Law of One Price

**a. Use your knowledge of linear algebra to prove part 1 of the theorem: The LOOP holds if and only if there exists a state price vector.**

Let $\mathbf{D} \in \mathbb{R}^{NxM}$ where N = number of securities and M = number of possible outcomes.

Let $\mathbf{s}_0 \in \mathbb{R}^N$ be the initial price vector of assets

Let $\mathbf{h} \in \mathbb{R}^N$ be vector of purchased holdings for each asset.

LOOP holds if $\forall \mathbf{h} : \mathbf{h}^T \mathbf{D} = 0 \implies \mathbf{h}^T \mathbf{s}_0 = 0$ **(1)**

We define the state price vector, $\psi \in \mathbb{R}^m$ if $\forall \mathbf{h} : \mathbf{h}^T \mathbf{s}_0 = \mathbf{h}^T D\psi$ **(2)**

In **(2)** if $\mathbf{h}^T \mathbf{D} = \mathbf{0} \implies \mathbf{h}^T \mathbf{s}_0 = 0$ but this equality holds only where the state price vector exists.

$$\mathbf{h}^T \mathbf{s}_0 = \mathbf{h}^T \mathbf{D}\psi$$

$$0 = \mathbf{h}^T - \mathbf{s}_0 + \mathbf{h}^T \mathbf{D}\psi$$

$$0 = \mathbf{h}^T [-\mathbf{s}_0; \mathbf{D}] \begin{bmatrix} 1 \\ \psi \end{bmatrix}$$

Let $\overline{\mathbf{D}} = [-\mathbf{s}_0|\mathbf{D}] \in \mathbb{R}^{NxM+1}$ be the augemented payoff matrix.

$$\implies \begin{bmatrix} 1 \\ \psi \end{bmatrix} \in Nullspace\left(\overline{\mathbf{D}}\right)$$

Result of fundamental theorem of linear algebra: $Nullspace\left(\overline{\mathbf{D}}\right) \oplus range\left(\overline{\mathbf{D}}^T\right)$ = M + 1

$\implies Nullspace(\mathbf{D})$ exists when $range\left(\overline{\mathbf{D}}^T\right) <= M$

Conversely if $\psi$ doesn't exist then neither does **(2)** and **(1)** doesn't hold

> **b. Use your knowledge of linear algebra to prove part 2 of the theorem: In a market in which LOOP holds, the state price vector is unique if and only if the market is complete.**

Let $\overline{\mathbf{D}} = [-\mathbf{s}_0|\mathbf{D}] \in \mathbb{R}^{NxM+1}$ be the augemented payoff matrix.

We can alternatively define the state price vector, $\psi$ if $\forall \mathbf{h} : \mathbf{h}^T \overline{\mathbf{D}} \begin{bmatrix} 1 \\ \psi \end{bmatrix} = \mathbf{0}$

$$\implies \begin{bmatrix} 1 \\ \psi \end{bmatrix} \in Nullspace\left(\overline{\mathbf{D}}\right)$$

Markets are complete if $rank(\mathbf{D}) = M$

$\mathbf{s}_0 = \mathbf{D}\psi \implies \mathbf{s}_0$ is a linear combination of the columns of $\mathbf{D} \implies$ rank$\left(\overline{\mathbf{D}}\right) = M$

Result of fundamental theorem of linear algebra: $Nullspace\left(\overline{\mathbf{D}}\right) \oplus range\left(\overline{\mathbf{D}}^T\right)$ = M + 1

$$\implies Dim(Nullspace\left(\mathbf{D}\right)) = 1 \because rank\left(\overline{\mathbf{D}}^T\right) = M$$

$\implies \psi$ is unique and only scalar multiples of $\begin{bmatrix} 1 \\ \psi \end{bmatrix}$ exist

Conversely, if the market is not complete, $\implies$ rank($\overline{\mathbf{D}}$) $< M \implies Dim(Nullspace(\overline{\mathbf{D}})) > 1 \implies \psi$ exists in a plane or higher space with multiple solutions.

# 5. Eigenvalue Decomposition

> Consider the matrix:
> $$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

> **a. What are the eigenvectors and eigenvalues?**

$$det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right)$$

$$= (3-\lambda)(3-\lambda) - 1$$

$$= 9 - 6\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

$$(\lambda - 4)(\lambda - 2)$$

The eigenvalues = {4,2}

For $\lambda = 4$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

Both of these equations are solved when $x_1 = x_2$ so I will use $x_1 = x_2 = 1$

The corresponding eigenvector for $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 + x_2 = 0$$

Both of these equations are solved when $x_1 = -x_2$ so I will use $x_1 = 1, x_2 = -1$

The corresponding eigenvector for $\lambda = 2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

**b. What are the matricies Q and $\Lambda$ in the spectral decomposition $A = Q\Lambda Q^T$ ?**

The $\Lambda$ matrix has the eigenvalues on the diagonal and zero otherwise:

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$Q$ is just the normalized eigenvectors together

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

In [1]:
```
# check
import numpy as np
A = np.array([[3, 1], [1, 3]])
lam = np.array([[4, 0], [0, 2]])
Q = np.array([[1/np.sqrt(2), 1/np.sqrt(2)],[1/np.sqrt(2), -1/np.sqrt(2
)]])
decomp_A = np.dot(Q, lam).dot(Q.T)
np.allclose(A, decomp_A) # tolerance = 1e-08
```

Out[1]:
```
True
```

**c. Is A positive definite?**

Yes. A is symmetric and all the eigenvalues are positive so it is positive definite.

# 5. Principal Component Analysis

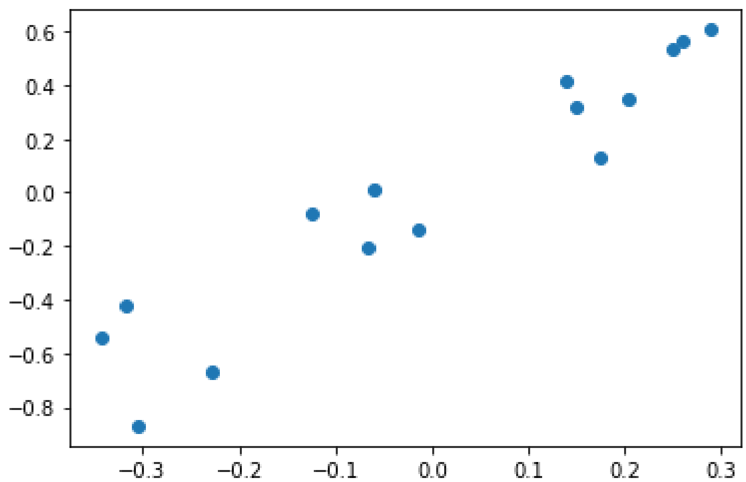
Complete a principal component analysis of the following data, where we decompose $\mathbf{X} = \mathbf{xc}$ where $\mathbf{X} \in \mathbb{R}^{2x15}$ and $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{c} \in \mathbb{R}^{15}$

In [2]:
```
row1 = [-0.066, -0.124, 0.259, 0.289, -0.318, -0.015, -0.060, 0.140,
        0.203, 0.249, -0.229, 0.174, 0.149, -0.343, -0.305]
row2 = [-0.204, -0.079, 0.565, 0.607, -0.425, -0.135, 0.011, 0.411,
        0.350, 0.535, -0.668, 0.128, 0.317, -0.543, -0.870]
X = np.array([row1, row2])
X.shape
```

Out[2]:
```
(2, 15)
```

In [3]:
```
import matplotlib.pyplot as plt
plt.scatter(row1, row2)
```

Out[3]:
```
<matplotlib.collections.PathCollection at 0x12024aa20>
```




We want to find the largest eigenvalue associated with the symmetric matrix $XX^T$ and the associated eigenvector will be our **x** vector

$$A = XX^T$$

In [4]:
```
A = np.dot(X, X.T)
A
```

Out[4]:
```
array([[0.713685, 1.417414],
       [1.417414, 3.126994]])
```

In [5]:
```
evalues, evectors = np.linalg.eig(A)
evalues
```

Out[5]:
```
array([0.05886763, 3.78181137])
```

**a. What is the optimal x?**

3.7818 is the largest eigenvalue so we will use the associated eigenvector as $\mathbf{x}$

In [6]:
```
evectors
```

Out[6]:
```
array([[-0.9078067 , -0.41938883],
       [ 0.41938883, -0.9078067 ]])
```

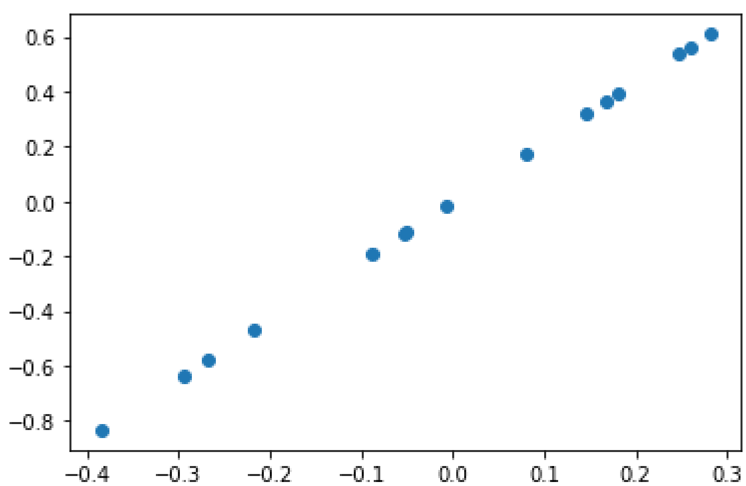
In [7]:
```
x = evectors[:,1].reshape((2,1))
x
```

Out[7]:
```
array([[-0.41938883],
       [-0.9078067 ]])
```

$$\mathbf{x} = \begin{bmatrix} -0.4194 \\ -0.9078 \end{bmatrix}$$

In [8]:
```
proj_op = x.T.dot(X)
reduced_X = x.dot(proj_op)
plt.scatter(reduced_X[0], reduced_X[1])
```

Out[8]:
```
<matplotlib.collections.PathCollection at 0x120352780>
```




**b. What is the error** $e = \frac{1}{M}\sqrt{\Sigma_m \|\mathbf{v}^m - \mathbf{x}\mathbf{c}_m\|_2^2}$

In [9]:
```
m = 15
sum_sq_norm_diffs = 0
for i in range(m):
    diff = X[:,i] - reduced_X[:,i]
    norm_diff = np.linalg.norm(diff)
    sq_norm_diff = norm_diff * norm_diff
    sum_sq_norm_diffs += sq_norm_diff

root_sq_norm_diffs = np.sqrt(sum_sq_norm_diffs)
e = root_sq_norm_diffs/m
e
```

Out[9]: 0.016175101322599528

$$e = 0.0162$$

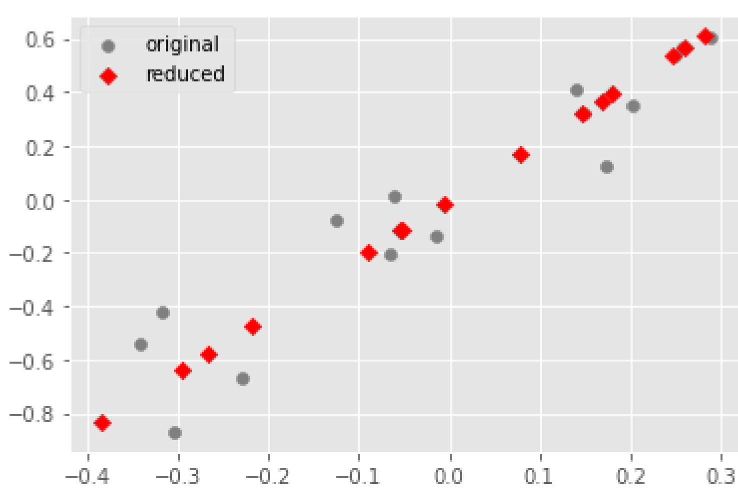
**c. In light of your previous results, is it possible to represent the v's well in a 1-dimensional subspace, $E$?**

**The error above appears to be low. Taking a look at both values plotted on the same axes we can see that the reduced line passes through the original points well.**

In [10]:
```
plt.style.use('ggplot')

fig = plt.figure()
ax1 = fig.add_subplot(111)

ax1.scatter(X[0], X[1], c='gray', marker="o", label='original')
ax1.scatter(reduced_X[0], reduced_X[1], c='red', marker="D", label='redu
ced')
plt.legend(loc='upper left');
plt.show()
```



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