MFE Math Foundations
Assignment 2 Solutions

1. Corporate Bonds

   (a) From our class discussion, we know that the transition probability of going from state 1 (AAA) to state 8 (Default) over 2 years is given by $\Phi^2_{1,8}$. We can use a software package (e.g., Matlab) to calculate Φ^2.

   Alternatively, given the sparseness of Φ, which contains many zero elements, we can identify that there are just three ways to transition from state 1 to state 8 in 2 years, and calculate the total probability for one of these three events to occur.

   - The first is to transition from state 1 to state 3 in the first year and then from state 3 to state 8 in the second year. The total probability for this event is
   $$0.007164 \times 0.00103 = 7.3748 \times 10^{-6} = 0.00073748\%.$$
   - The second is to transition from state 1 to 4 in the first year and then from state 4 to state 8 in the second year. The total probability for this event is
   $$0.00102 \times 0.00212 = 2.1624 \times 10^{-6} = 0.00021624\%.$$
   - The third is to transition from state 1 to 5 in the first year and then from state 5 to state 8 in the second year. The total probability for this event is
   $$0.00102 \times 0.01209 = 1.23318 \times 10^{-5} = 0.00123318\%.$$

   The total probability of transitioning from state 1 to 8 in two years is then
   $$\Phi^2_{1,8} = 0.00073748\% + 0.00021624\% + 0.00123318\% = 0.0021869\%.$$

   (b) As is the case for question (a), we may use a software package to study Φ^n for large n—or the eigenvectors of Φ^*—to identify stationary distributions. As a note, Φ is actually not irreducible in this case, since there is no way to move out of the default state, state 8, and therefore $\Phi^n_{8,k} = 0$ for all $n > 0$ and $k \in \{1, 2, \ldots, 7\}$. So the Perron-Frobenius theorem does not apply. However, one verifies that Φ has a unique eigenvector with associated largest eigenvalue 1, so there is still a unique stationary distribution in this case.

   In fact, due to the absorbing nature of the default state, we do not need to carry out the formal eigenvector analysis to identify this stationary distribution. Specifically, whichever is the current state, there is always a positive probability to move to state 8 within two periods. Indeed, one can verify that the lowest such probability, q, is to go from state 1 to 8, which from (a) gives us $q = 0.0021869\%$.

   The probability of not ending up in state 8 within the next $2N$ periods is then at least $(1-q)^N$, regardless of the current state, which for large N tends to zero, $\lim_{N\to\infty}(1-q)^N = 0$. We therefore conclude that the unique long-term stationary distribution is
   $$\mathbf{p} = (0, 0, 0, 0, 0, 0, 0, 1),$$

i.e., that all long-lived corporate bonds eventually would end up in default according to this model.

2. (a) We want to maximize the objective function $U(\mathbf{h}) := \mathbf{h}^T\boldsymbol{\mu} - \frac{\gamma}{2}\mathbf{h}^T\Sigma\mathbf{h}$, subject to the constraints $g_1(\mathbf{h}) := \mathbf{h}^T\boldsymbol{\mu} - \mu_0 = 0$ and $g_2(\mathbf{h}) := \mathbf{h}^T\mathbf{1} - 1 = 0$. The Lagrange first order conditions are given by

$$\begin{aligned} \nabla U(\mathbf{h}) + \lambda_1 \nabla g_1(\mathbf{h}) + \lambda_2 \nabla g_2(\mathbf{h}) &= 0 \\ g_1(\mathbf{h}) = g_2(\mathbf{h}) &= 0 \end{aligned}$$

We compute

$$\nabla U = -\gamma\Sigma\mathbf{h} \qquad \nabla g_1 = \boldsymbol{\mu} \qquad \nabla g_2 = \mathbf{1}$$

Thus, we must solve the system of equations

$$-\gamma\Sigma\mathbf{h} + (\lambda_1 + 1)\boldsymbol{\mu} + \lambda_2\mathbf{1} = 0 \tag{1}$$

$$\mathbf{h}^T\boldsymbol{\mu} = \mu_0 \tag{2}$$

$$\mathbf{h}^T\mathbf{1} = 1 \tag{3}$$

The first step is to solve (1) for $\mathbf{h}$, which yields

$$\mathbf{h} = \frac{\lambda_1 + 1}{\gamma}\Sigma^{-1}\boldsymbol{\mu} + \frac{\lambda_2}{\gamma}\Sigma^{-1}\mathbf{1} \tag{4}$$

Note that (2) and (3) imply that $\boldsymbol{\mu}^T\mathbf{h} = \mu_0$ and $\mathbf{1}^T\mathbf{h} = 1$, respectively. Using these in conjunction with (1), we obtain (using the notation suggested in the assignment description) the system

$$\begin{aligned} \lambda_1 C + \lambda_2 B &= \gamma\mu_0 - C \\ \lambda_1 B + \lambda_2 A &= \gamma - B \end{aligned}$$

which, in matrix form, is the same as

$$\begin{bmatrix} C & B \\ B & A \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \gamma\mu_0 - C \\ \gamma - B \end{bmatrix}$$

Using our linear algebraic techniques, we solve this system (e.g. by inverting the matrix on the left side) to obtain the solution

$$\begin{aligned} \lambda_1 + 1 &= \frac{\gamma}{\Delta}(A\mu_0 - B) \\ \lambda_2 &= \frac{\gamma}{\Delta}(C - B\mu_0) \end{aligned}$$

Finally, we insert these results back into (4) to obtain the optimal weights

$$\mathbf{h} = \tfrac{B}{\Delta}(A\mu_0 - B)\mathbf{w}_B + \tfrac{A}{\Delta}(C - B\mu_0)\mathbf{w}_A \tag{5}$$

Note also that the Hessian of U is $\nabla^2 U = -\gamma\Sigma$, which is negative definite since $\gamma > 0$. This shows that the solution that satisfies the first-order conditions will be the unique maximum.

(b) We want to compute $\sigma_{\mathbf{h}}^2 = \mathbf{h}^T\Sigma\mathbf{h}$. Recall from the previous part that the optimal $\mathbf{h}$ is given by

$$\mathbf{h} = \alpha\Sigma^{-1}\boldsymbol{\mu} + \beta\Sigma^{-1}\mathbf{1}$$

where $\alpha = \frac{\lambda_1+1}{\gamma} = \frac{1}{\Delta}(A\mu_0 - B)$ and $\beta = \frac{\lambda_2}{\gamma} = \frac{1}{\Delta}(C - B\mu_0)$. It follows then that

$$\begin{aligned}\sigma_{\mathbf{h}}^2 &= \mathbf{h}^T\left(\alpha\boldsymbol{\mu} + \beta\mathbf{1}\right)\\ &= \alpha\mu_0 + \beta\end{aligned}$$

Substituting back in for α and β yields

$$\sigma_{\mathbf{h}}^2 = \frac{1}{\Delta}(A\mu_0^2 - 2B\mu_0 + C) \tag{6}$$

Note that the minimum variance possible is obtained by setting $\frac{d\sigma_{\mathbf{h}}^2}{d\mu_0} = 0$, leading to $\mu_0 = \frac{B}{A}$. This is the minimum variance portfolio, and you can verify that it is obtained by the second part of (4), i.e., by investing in the portfolio $\frac{1}{\mathbf{1}\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}$.

(c) The agent's utility as a function of μ_0 and γ, given that the optimal portfolio for that μ_0 is chosen, is

$$\begin{aligned}U(\mu_0,\gamma) &= \mu_0 - \frac{\gamma}{2}\sigma_{\mathbf{h}}^2\\ &= \mu_0 - \frac{\gamma}{2}\frac{A\mu_0^2 - 2B\mu_0 + C}{\Delta},\end{aligned}$$

following from (6). This is clearly a concave function of μ_0, so by solving for the first-order condition we find the globally optimal portfolio. We have

$$0 = \frac{dU}{d\mu_0} = 1 - \gamma\frac{A\mu_0 - B}{\Delta},$$

leading to the relation

$$\mu_0 = \frac{\Delta}{A}\cdot\frac{1}{\gamma} + \frac{B}{A}.$$

This expression makes a lot of sense: When γ is large, the expected return of the portfolio is close to $\frac{B}{A}$, representing the fact that a very risk averse investor will basically invest in the least risky portfolio available, which is the minimum variance portfolio (see (b)). When γ is small, the investor who is close to risk-neutral, will speculate aggressively and have a very high expected return.

3. (a) Now we want to maximize the objective $U(\mathbf{h}) = R + \mathbf{h}^T(\mu - R\mathbf{1}) - \frac{\gamma}{2}\mathbf{h}^T\Sigma\mathbf{h}$. This time we impose no constraints on $\mathbf{h}$, so we begin by computing ∇U and checking the first order conditions:

$$\nabla U = \boldsymbol{\mu} - R\mathbf{1} - \gamma\Sigma\mathbf{h}$$

Thus at an optimum, we must have

$$\begin{aligned}\mathbf{h} &= \frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{\mu} - R\mathbf{1})\\ &= \frac{B - AR}{\gamma}\mathbf{w}\end{aligned}$$

so every investor chooses the stock market portfolio $\mathbf{w}$ (one-fund separation), but investors differ in how much they invest in this portfolio, depending on their risk aversion, γ.

(b) Define the total expected return $E_R := R + \mathbf{h}^T(\boldsymbol{\mu} - R\mathbf{1})$. With risk tolerance $\gamma > 0$, we substitute for the optimal $\mathbf{h}$ to obtain

$$\begin{aligned}E_R &= R + \mathbf{h}^T(\boldsymbol{\mu} - R\mathbf{1})\\ &= R + \frac{1}{\gamma}(\boldsymbol{\mu} - R\mathbf{1})^T\Sigma^{-1}(\boldsymbol{\mu} - R\mathbf{1})\\ &= R + \frac{1}{\gamma}(AR^2 - 2BR + C)\end{aligned}$$

From class we know that Σ^{-1} is positive definite, so $(\boldsymbol{\mu}-R\mathbf{1})^T\Sigma^{-1}(\boldsymbol{\mu}-R\mathbf{1}) > 0$ and the expected return of the investor's portfolio is therefore again inversely related to γ.

4. (a) With $\mathbf{w}$ as defined in the assignment and $\boldsymbol{\beta} = \frac{1}{\mathbf{w}^T\Sigma\mathbf{w}}\Sigma\mathbf{w}$, we want to show that

$$\boldsymbol{\mu} - R\mathbf{1} = \boldsymbol{\beta}\mathbf{w}^T(\boldsymbol{\mu} - R\mathbf{1})$$

To simplify our calculations we define $\mathbf{q} = \boldsymbol{\mu} - R\mathbf{1}$. It follows then that $\mathbf{w} = (B-AR)^{-1}\Sigma^{-1}\mathbf{q}$, $\Sigma w = (B-AR)^{-1}\mathbf{q}$, and thus that $\mathbf{w}^T\Sigma\mathbf{w} = (B-AR)^{-2}\mathbf{q}^T\Sigma^{-1}\mathbf{q}$. Beginning with the right hand side, we have

$$\begin{aligned}
\boldsymbol{\beta}\mathbf{w}^T(\boldsymbol{\mu} - R\mathbf{1}) = \boldsymbol{\beta}\mathbf{w}^T\mathbf{q} &= \frac{1}{\mathbf{w}^T\Sigma\mathbf{w}}\Sigma\mathbf{w}\mathbf{w}^T\mathbf{q} \\
&= \frac{(B-AR)^{-2}}{(B-AR)^{-2}\mathbf{q}^T\Sigma^{-1}\mathbf{q}}\mathbf{q}\mathbf{q}^T\Sigma^{-1}\mathbf{q} \\
&= \frac{\mathbf{q}\mathbf{q}^T\Sigma^{-1}\mathbf{q}}{\mathbf{q}^T\Sigma^{-1}\mathbf{q}} \\
&= \mathbf{q} \\
&= \boldsymbol{\mu} - R\mathbf{1}
\end{aligned}$$

Note that we have very carefully used associativity of matrix multiplication here.