Mathematical Foundations
MFE Spring 2021
Assignment 4
Due: Tuesday, February 9, 9am P.T.

Student name: **Student ID:**

Solved in team with another student (please underline): **Yes** **No**

Name of team member: **ID of team member:**

Instructions: You should submit this assignment electronically via bCourses, as a PDF file. The preferred format is to use a text editor or other program with formula handling capacity, e.g., Scientific Word, MikTex, or MS Word with installed equation package, to create a an electronic document, and then save as PDF. We also accept handwritten, scanned, solutions, but these must be extremely clear and well written, or they will receive point deductions (see available sample files on bCourses).

Please contact us at HAASMFE.MATH@gmail.com if you have any questions about the submission format!

You should receive a confirmation e-mail within 24 hours of sending your solution. Assignments received up to 24 hours late will get a point deduction of 50% of the points. Assignments received more than 24 hours late get a 100% point deduction.

You can send questions to HAASMFE.MATH@gmail.com if you get stuck.

All answers need to be justified!

1. *Black Scholes PDE*: In a market in which the Black Scholes assumptions are satisfied, an asset makes the terminal payoff

$$f(S(T)) = S(T)^n, \qquad n > 1,$$

where $S(T)$ is the value of the underlying stock at time T. The asset's value if the stock price reaches zero is $P(0,t) = 0$.

   (a) Show that the price of this power asset at $0 < t \leq T$ is

$$P(S(t),t) = e^{((n-1)r + n(n-1)\sigma^2/2)(T-t)} S(t)^n.$$

2. *Finite difference method for Black Scholes PDE (in reflected log coordinates)*: Consider the double knock-out power option discussed in class. The underlying stock price at time t is S_t. The initial stock price is $S_0 \in (50, 100)$, and the option's maturity date is $T = 1$, measured in years. The risk-free rate is $r = 0.05$ per year, and the stock's volatility is $\sigma = 0.4$ per year.

   At expiration, if the stock price lies in the interval $S \in (50, 100)$, *and* its price has never been outside of this interval between $t = 0$ and T the option pays out:

$$P(S,T) = (S - 50)(100 - S).$$

   If the stock price has been outside of the interval, the payout is $P(S,T) = 0$, i.e., the option is worthless. Since the price path of the stock is continuous (almost surely), the event this occurs if $\min_{0 \leq t \leq T} S_t > 50$, and $\max_{0 \leq t \leq T} S_t < 100$.

   (a) *PDE*: Write the PDE for the price of this option, $V(s,x)$, including boundary and initial conditions, in log space coordinates, $x = \ln(S)$, and reflected time coordinates, $s = T - t$.

   (b) *Solver*: Implement a finite difference solver for the PDE, using the finite difference method with operators D_{+s}, D_{0x}, and $D_{+x}D_{-x}$ for the different derivative estimates, as discussed in class.

   You may use your favorite programming language (Matlab, Python, etc.), or even Excel. Please submit your source code (or in case of Excel, your spreadsheet) with your submission.

   (c) *Solution*: Using the ratio between time and space step-length $k = \frac{\Delta t}{\Delta x^2} = 2$, calculate the approximate solution at $t = 0$, with $\Delta x = 100$ points.

   - Please include graphs of $V(x,0)$, $V(x,1)$, $P(S,1)$, and $P(S,0)$.
   - Where does the value (in S coordinates) reach its maximum?

- The maximum value of the payouts $P(S,T)$ is reached at $S = 75$. Why do you think the maximum value of $P(S,0)$ is reached at a different value of S?

(d) *Stability*: Vary k. How large can k be while keeping the approximation method stable? Does this bound on k depend on Δx?

(e) *Order of convergence*: Vary Δx, keeping $k = \frac{1}{4}$ constant. Estimate the order of convergence of the approximation method to the true solution.