Mathematical Foundations

MFE Spring 2021

Assignment 5

Student Name: Rick Wuebker

ID: richard_wuebker

Solved in team: No

1. Probability Spaces

> **Consider the state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with the associated σ-algebra $\mathcal{F} = 2^{\Omega}$, and probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined by:**

i	1	2	3	4
$\mathbb{Q}(\omega_i)$	0.1	0	0.5	0.4
$\mathbb{P}(\omega_i)$	0.4	0.1	0.1	0.4

> **(a) Are $\mathbb{P}$ and $\mathbb{P}$ equivalent probability measures?**

No, $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent because they disagree on which events have 0 probability, specifically $\mathbb{Q}(\omega_2) = 0$ however $\mathbb{P}(\omega_2) = 0.1$.

> **What is the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$?**

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\begin{aligned}
L(\{\emptyset\}) &= 0 & L(\{\omega_1\}) &= \tfrac{0.1}{0.4} = 0.25 \\
L(\{\omega_2\}) &= \tfrac{0}{0.1} = 0 & L(\{\omega_3\}) &= \tfrac{0.5}{0.1} = 5 \\
L(\{\omega_4\}) &= \tfrac{0.4}{0.4} = 1.00 & L(\{\omega_1, \omega_2\}) &= \tfrac{0.1+0.0}{0.4+0.1} = 0.2 \\
L(\{\omega_1, \omega_3\}) &= \tfrac{0.1+0.5}{0.4+0.1} = 1.2 & L(\{\omega_1, \omega_4\}) &= \tfrac{0.1+0.4}{0.4+0.4} = 0.625 \\
L(\{\omega_2, \omega_3\}) &= \tfrac{0.0+0.5}{0.1+0.1} = 2.5 & L(\{\omega_2, \omega_4\}) &= \tfrac{0.0+0.4}{0.1+0.4} = 0.8 \\
L(\{\omega_3, \omega_4\}) &= \tfrac{0.5+0.4}{0.1+0.4} = 1.80 & L(\{\omega_1, \omega_2, \omega_3\}) &= \tfrac{0.1+0.0+0.5}{0.4+0.1+0.4} = 1.00 \\
L(\{\omega_1, \omega_2, \omega_4\}) &= \tfrac{0.1+0.0+0.4}{0.4+0.1+0.4} = 0.556 & L(\{\omega_1, \omega_3, \omega_4\}) &= \tfrac{0.1+0.5+0.4}{0.4+0.1+0.4} = 1.111 \\
L(\{\omega_2, \omega_3, \omega_4\}) &= \tfrac{0+0.5+0.4}{0.1+0.1+0.4} = 1.5 & L(\{\Omega\}) &= 1
\end{aligned}
$$

2. Measurability

Consider the state space $\Omega = \mathbb{R}$, with the σ-algebra $\mathcal{F} = \{(-\infty, 0], (0, \infty), \emptyset, \mathbb{R}\}$, and the random variable $X : \Omega \rightarrow \mathbb{R}$ defined by:

$$X(\omega) = \begin{cases} 3 & \omega < 0 \\ 5 & \omega \geq 0 \end{cases}$$

(a) Is X $\mathcal{F}$ measurable?

No. Consider the interval $I = [4, 10]$, then $X^{-1}(I) = [0, \infty) \notin \mathcal{F}$

3. Stochastic Integration

(a) Given that the company chooses a fixed investment strategy $I \equiv k$, between time 0 and T, derive an expression for what the value of of the project at time T is. The value should be expressed as a function of W_T, k, μ and σ.

$$V_T = \mu k \int_0^T dt + \sigma k \oint_0^T dW_t$$

$$V_T = \mu k T + \sigma k W_T$$

(b) Assume that the company must determine a constant investment strategy at $t = 0$, $I \equiv k$. What value of k should the company choose?

$$U = E(V_T) - \frac{\gamma}{2} Var(V_T)$$

$$U = \mu k T - \frac{\gamma}{2} \sigma^2 k^2 T$$

$$\frac{dU}{dk} = \mu T - \gamma \sigma^2 k T = 0$$

$$\gamma \sigma^2 k T = \mu T$$

$$k = \frac{\mu}{\gamma \sigma^2}$$

$\frac{d^2U}{(dk)^2} = -\gamma \sigma^2 T < 0$ **so we have found a local maximum.**

4. Ito's Lemma

(a) Consider the stochastic process:

$$dX_t = \mu dt + \sigma dW_t$$

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Find a closed form expression for Q, defined by the stochastic integral:

$$Q = \oint_0^T 3X_t(\mu X_t + \sigma^2)dt + 3X_t \sigma dW_t$$

by postulating a function $G(t, W_t)$ that satisfies:

$$dG_t = 3X_t(\mu X_t + \sigma^2)dt + 3X_t^2 \sigma dW_t$$

and using the definition of the stochastic integral in class to get:

$$Q = G(T, W_T) - G(0, W_0)$$

There are only a few terms in the differential so I will guess $G_t = 0$.

$$dG_t = 3\mu X_t^2 dt + 3\sigma^2 X_t dt + 3X_t^2 \sigma dW_t = G_X dX + \frac{1}{2} G_{XX} (dX)^2$$

$$dG_t = 3X_t^2(\mu dt + \sigma dW_t) + \frac{1}{2} 6X_t \sigma^2 dt = G_X dX + \frac{1}{2} G_{XX} (dX)^2$$

$$dG_t = 3X_t^2 dX + \frac{1}{2} 6X_t (dX)^2 = G_X dX + \frac{1}{2} G_{XX} (dX)^2$$

$$\implies G_X = 3X^2 \qquad G_{XX} = 6X$$

Possible solution: $G(t, W_t) = X_t^3 = (X(0) + \mu t + \sigma W_t)^3$

$$Q = G(T, W_T) - G(0, W_0) = (X(0) + \mu T + \sigma W_T)^3 - X(0)^3$$

In [ ]: