MFE Math Foundations
Assignment 3 Solutions

1. (a) By the way the sequence f_J was constructed, we will be able to show that $\|f_{J+1} - f_J\| \leq c \cdot 2^{-J/2}$. Then by the argument in the problem description, f_J would be a Cauchy sequence (and would thus converge since the space is complete under the sup-norm). Indeed, recall that f_{J+1} is defined from f_J via $f_{J+1} = f_J$ on the set X^J, and

$$f_{J+1}(t) = f_J(t) + (-1)^{k+J+1} 2^{-J/2-1}, \quad t = (2k-1)2^{-J-1} \in X^{J+1} \setminus X^J$$

It then follows that

$$\begin{aligned} \|f_{J+1} - f_J\| &= \sup_{t \in [0,1]} |f_{J+1}(t) - f_J(t)| \\ &\leq \sup_{t \in X^{J+1}} |f_{J+1}(t) - f_J(t)| \\ &\leq \tfrac{1}{2} \cdot 2^{-J/2} \end{aligned}$$

so we are done.

(b) Let $t = k \cdot 2^{-J}$. First consider $J' = J + 1$. We need to examine the quantity

$$Z := |f_{J+1}(t + 2^{-J-1}) - f_{J+1}(t)|$$

Recall that $f_{J+1}(t) = f_J(t)$ by construction. As for $f_{J+1}(t + 2^{-J-1})$, note that f_J is linear on the interval $[t, t + 2^{-J}]$. Thus, if we define $a = f_J(t)$ and $b = f_J(t + 2^{-J})$, then we have (by the construction of f_{J+1} from f_J)

$$\begin{aligned} f_{J+1}(t + 2^{-J-1}) &= f_J(t + 2^{-J-1}) \pm 2^{-J/2-1} \\ &= \frac{a+b}{2} \pm 2^{-J/2-1} \end{aligned}$$

Combining the above, we have

$$\begin{aligned} Z = |f_{J+1}(t + 2^{-J-1}) - f_{J+1}(t)| &= \left| \frac{a+b}{2} \pm 2^{-J/2-1} - a \right| \\ &= \left| \frac{b-a}{2} \pm 2^{-J/2-1} \right| \end{aligned}$$

Now, we are done if $Z \geq \frac{2^{-(J+1)/2}}{8}$. If not, then $Z < \frac{2^{-(J+1)/2}}{8}$, and we use this fact to show that we get what we need with $J' = J + 2$ in this case. Specifically, recall that $f_{J+2}(t) = f_J(t)$. We have

$$
\begin{aligned}
|f_{J+2}(t+2^{-J-2}) - f_J(t)| &= \left| \frac{a + (\frac{a+b}{2} \pm 2^{-J/2-1})}{2} \mp 2^{-\frac{J+1}{2}-1} - a \right| \\
&= \left| \frac{\frac{b-a}{2} \pm 2^{-J/2-1})}{2} \mp 2^{-(J+3)/2} \right| \\
&\geq \left| \frac{Z}{2} - 2^{-(J+3)/2} \right| \\
&= 2^{-(J+3)/2} - \frac{Z}{2} \\
&> 2^{-(J+3)/2} - \frac{2^{-(J+1)/2}}{16} \\
&= \frac{2^{-(J+2)/2}}{\sqrt{2}} - \frac{2^{-(J+2)/2}}{8\sqrt{2}} \\
&= 2^{-(J+2)/2} \left(\frac{1}{\sqrt{2}} - \frac{1}{8\sqrt{2}} \right) \\
&\geq \frac{2^{-(J+2)/2}}{8}
\end{aligned}
$$

Thus, the desired result either holds for $J' = J+1$ or $J' = J+2$.

(c) Let $\alpha > \frac{1}{2}$. For any dyadic $t = k \cdot 2^{-J}$, with $0 < k < 2^J$, we will show that

$$\sup_{x \neq t} \frac{|f(x) - f(t)|}{|x-t|^\alpha} = +\infty.$$

Note that we can always write t as $m2^{-J'}$ for any $J' \geq J$ by choosing $m = k2^{J'-J}$. This fact, together with our work in part (b) implies that there are arbitrarily large $J' > J$ and $x = t + 2^{-J'}$, such that

$$\frac{|f(x) - f(t)|}{|x-t|^\alpha} = \frac{|f_{J'}(x) - f_{J'}(t)|}{|x-t|^\alpha} \geq \frac{2^{-J'/2}}{8 \cdot 2^{-J'\alpha}} = \frac{2^{J'(\alpha - \frac{1}{2})}}{8}$$

We observe that since $\alpha > \frac{1}{2}$, the right side of the above inequality is unbounded as J' grows. Thus one lets $J' \to \infty$ to obtain the desired result.

(d) We claim that $f_J(t) = 1 - f_J(1-t)$ for all J and t. We proceed by induction on J; the base case $J = 1$ is trivial since $f_1(t) = t$ and $t = 1 - (1-t)$. Suppose then that the result holds for $1, \ldots, J-1$. Observe that if $t = k \cdot 2^{-J}$, then $1 - t$ is also in X^J, and is given by $m \cdot 2^{-J}$, where $m = 2^J - k$. If k is even, then $t \in X^{J-1}$ also. This implies $1 - t \in X^{J-1}$, and by the inductive hypothesis we would have

$$f_J(t) = f_{J-1}(t) = 1 - f_{J-1}(1-t) = 1 - f_J(1-t)$$

Otherwise, k is odd, which implies that m is odd as well. Suppose that $k = 2k' - 1$ and $m = 2m' - 1$. Then we have

$$
\begin{aligned}
f_J(t) &= f_{J-1}(t) + (-1)^{k'+J+1} 2^{-\frac{J-1}{2}-1} \\
f_J(1-t) &= f_{J-1}(1-t) + (-1)^{m'+J+1} 2^{-\frac{J-1}{2}-1}
\end{aligned}
$$

However, we know that $m = 2^J - k$, which implies that $m' + k' = 2^J + 1$. In particular, one has $(-1)^{k'+J+1} = -(-1)^{m'+J+1}$. Thus, by adding the above equations, we obtain that

$$f_J(t) + f_J(1-t) = f_{J-1}(t) + f_{J-1}(1-t) = 1$$

by using the inductive hypothesis again.
For general t, one can argue that since f_J is extended linearly between dyadic points for which the relationship holds, it will be maintained over these intervals as well.

(e) We can now show that $\int_0^1 f(t)\,dt = \frac{1}{2}$. Using our previous result, we note that for all J, we have

$$1 = \int_0^1 (f_J(t) + f_J(1-t))\,dt = \int_0^1 f_J(t)\,dt + \int_0^1 f_J(1-t)\,dt = 2\int_0^1 f_J(t)\,dt$$

from which it follows that $\int_0^1 f_J(t)\,dt = \frac{1}{2}$ for all J.
Now, since we have $f_J \to f$ in the sup-norm, we can compute

$$\int_0^1 f(t)\,dt = \lim_{J\to\infty} \int_0^1 f_J(t)\,dt = \frac{1}{2}$$

Remember that uniform convergence is required to interchange the integral and the limit!

(f) Denote by V_J the sum corresponding to the Jth quadratic variation:

$$\sum_{k=0}^{2^J-1} \left[f_J((k+1)2^{-J} - f_J(k\cdot 2^{-J})\right]^2$$

Let $t = k\cdot 2^{-J}$ and $s = 2^{-J}$. Note that $t, t+s \in X^J$. Define $f_J(t) = a$ and $f_J(t+s) = b$. Then the contribution to V_J is $(a-b)^2$. In X^{J+1}, the interval $[t, t+s)$ is split into the two halves $[t, t+\frac{s}{2})$ and $[t+\frac{s}{2}, t+s)$. Thus the same interval $[t, t+s)$ contributes two terms to V_{J+1}:

$$\left[f_{J+1}(t+s/2) - f_{J+1}(t)\right]^2 + \left[f_{J+1}(t+s) - f_{J+1}(t+s/2)\right]^2$$

However by the construction of f_{J+1} from f_J, we have

$$\begin{aligned} f_{J+1}(t) &= a \\ f_{J+1}(t+s) &= b \\ f_{J+1}(t+s/2) &= \frac{f_J(t) + f_J(t+s/2)}{2} \pm 2^{-\frac{J+2}{2}} \\ &= \frac{a+b}{2} + c \end{aligned}$$

where we have defined $c = \pm 2^{-\frac{J+2}{2}}$. Thus, the total contribution to V_{J+1} is given by

$$\begin{aligned} \left[\frac{a+b}{2} + c - a\right]^2 + \left[b - \frac{a+b}{2} + c\right]^2 &= \left[\frac{a-b}{2} - c\right]^2 + \left[\frac{a-b}{2} + c\right]^2 \\ &= \frac{(a-b)^2}{2} + 2c^2 \end{aligned}$$

But $a = f_J(t)$ and $b = f_J(t+s)$. Since $c^2 = 2^{-(J+2)}$, the previous line is equivalent to the desired result.

(g) By summing up all the contributions using our results from the previous part, we obtain (for $J \geq 2$)

$$V_{J+1} = \frac{V_J}{2} + 2^J \cdot 2^{-(J+1)} = \frac{V_J}{2} + \frac{1}{2}$$

However, we can use the relationship $V_{J+1} = \frac{V_J}{2} + \frac{1}{2}$ to prove that in fact we have

$$V_J = 1 - 2^{-J}$$

We can easily compute $V_1 = \frac{1}{2}$. Assume this result holds for the first J integers; then we have

$$\begin{aligned} V_{J+1} &= \frac{V_J}{2} + \frac{1}{2} \\ &= \frac{1 - 2^{-J}}{2} + \frac{1}{2} \\ &= \frac{2 - 2^{-J}}{2} \\ &= 1 - 2^{-(J+1)}. \end{aligned}$$

2. (a) Let $V(t)$ denote the value of the contract that, at time $T = 10$, pays out \$1,000. Given the explicit formula for the short rate, we have

$$V(t) = 1000\, e^{-\int_t^T r(s)\, ds}$$

Thus to compute $V(0)$ we need to evaluate $\int_0^{10} r(s)\, ds$. Note that we can also write $r(t) = \frac{2}{40+t}$. For any a, b, we have

$$\int_a^b r(s)\, ds = \int_a^b \frac{2}{40+s}\, ds = 2\log(40+s)\Big|_a^b = 2\log(40+b) - 2\log(40+a) = 2\log\left(\tfrac{40+b}{40+a}\right)$$

Thus $\int_0^{10} r(s)\, ds = 2\log\frac{5}{4}$, so we conclude that

$$V(0) = 1000\, e^{-2\log\frac{5}{4}} = 1000\, e^{\log\frac{16}{25}} = \$640.00$$

(b) Let $t^* = \inf\{t > 0 \mid V(t) = 3312\}$. Then for $t > t^*$, we have $I(t) = 0$. We can solve the equation easily in the domain (t^*, ∞); indeed, for any $t > t^*$ one has

$$\log V(t) - \log V(t^*) = \int_{t^*}^t r(s)\, ds = 2\log(40+s)\Big|_{t^*}^t = 2\log\left(\frac{40+t}{40+t^*}\right)$$

But $V(t^*) = 3312$ (this assumes a priori that V is continuous) from which it follows that (again for $t > t^*$)

$$V(t) = 3312 e^{2\log\left(\frac{40+t}{40+t^*}\right)} = 3312\left(\frac{40+t}{40+t^*}\right)^2$$

In the other region $(0, t^*)$, we have the equation

$$V'(t) - r(t)V(t) = 300$$

Note that if we define $\mu(t) := e^{-\int_0^t r(u)\, du}$, then $\mu'(t) = -r(t)\mu(t)$. Thus, for any $0 < t < t^*$, we have

$$\frac{d}{dt}\left(\mu(t)V(t)\right) = 300\mu(t)$$

Integrating from 0 to t, we obtain

$$\mu(t)V(t) - V(0)\mu(0) = 300\int_0^t \mu(s)\, ds$$

Since $\mu(0) = 1$ and $V(0) = 300$, this is equivalent to

$$\begin{aligned}
V(t) &= 300e^{\int_0^t r(u)\,du} + 300\int_0^t e^{\int_s^t r(u)\,du}\,ds \\
&= 300e^{2\log\left(\frac{40+t}{40}\right)} + 300\int_0^t e^{2\log\left(\frac{40+t}{40+s}\right)}\,ds \\
&= \frac{3}{16}(40+t)^2 + 300(40+t)^2\int_0^t \frac{1}{(40+s)^2}\,ds \\
&= \frac{3}{16}(40+t)^2 + 300(40+t)^2\left[\frac{1}{40+s}\right]_0^t \\
&= \frac{3}{16}(40+t)^2 + 300(40+t)^2\left[\frac{t}{40(40+t)}\right] \\
&= \frac{3}{16}(40+t)^2 + \frac{15}{2}t\,(40+t)
\end{aligned}$$

This formula is valid for $0 < t < t^*$, i.e. until $V(t)$ reaches 3312. We can now compute t^* by solving $V(t) = 3312$ using the formula for $0 < t < t^*$. Solving the equation

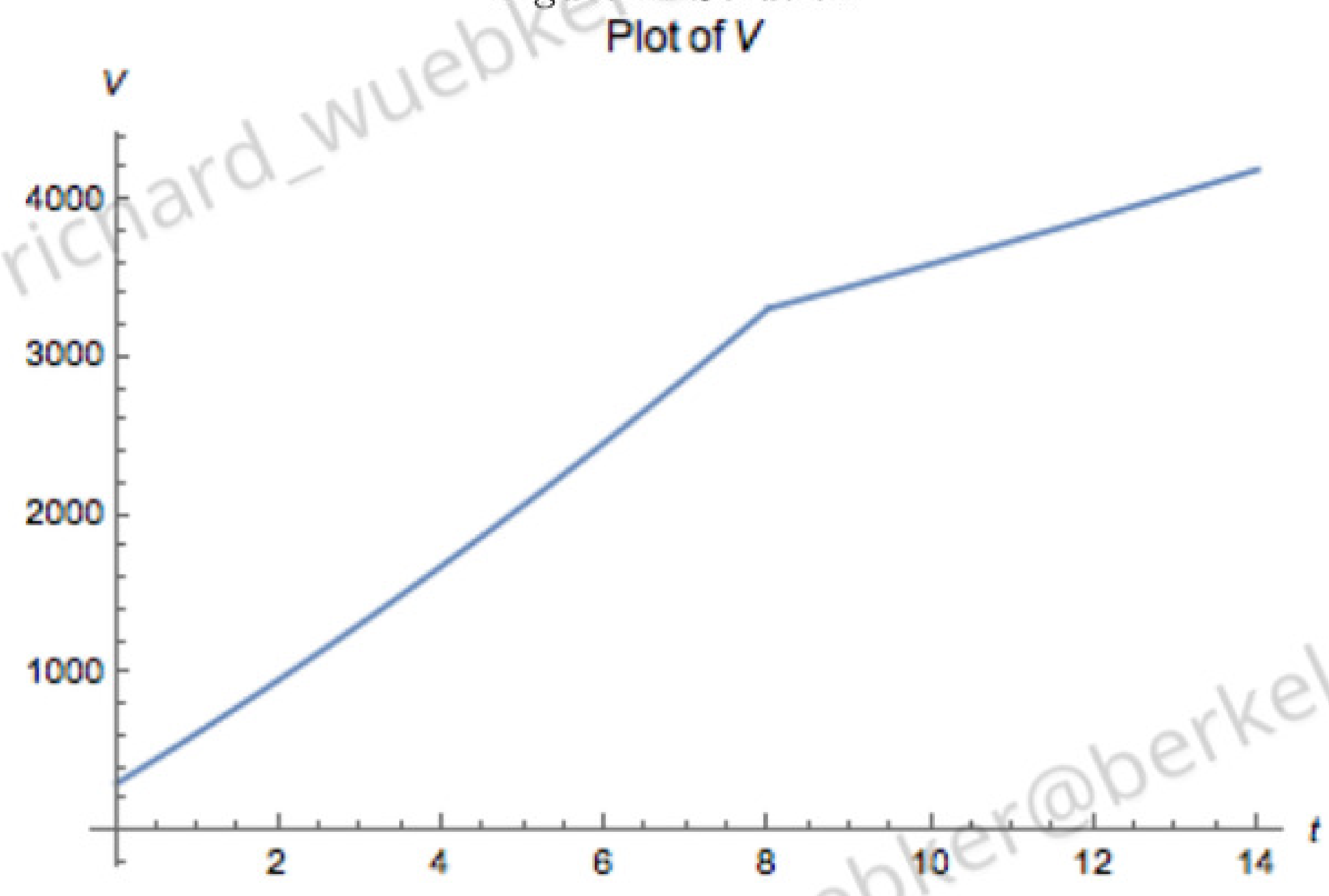
$$3312 = \frac{3}{16}(40+t^*)^2 + \frac{15}{2}t^*\,(40+t^*)$$

yields $t^* = 8$.

Our complete solution to the problem is now given by

$$V(t) = \begin{cases} \frac{3}{16}(40+t)^2 + \frac{15}{2}t\,(40+t), & 0 \le t < 8, \\ \frac{23}{16}(40+t)^2, & t \ge 8 \end{cases}$$

Figure 1: Solution



Note that in Figure 1, we can see that $V(t)$ is actually not differentiable at $t^* = 8$. This is to be expected since the ODE tells us that $V'(t)$ is discontinuous there.