Mathematical Foundations
MFE Spring 2021
Assignment 5
Due: Tuesday, February 16, 9am P.T.

Student name: **Student ID:**

Solved in team with another student (please underline): **Yes** **No**

Name of team member: **ID of team member:**

Instructions: You should submit this assignment electronically via bCourses, as a PDF file. The preferred format is to use a text editor or other program with formula handling capacity, e.g., Scientific Word, MikTex, or MS Word with installed equation package, to create a an electronic document, and then save as PDF. We also accept handwritten, scanned, solutions, but these must be extremely clear and well written, or they will receive point deductions (see available sample files on bCourses).

Please contact us at HAASMFE.MATH@gmail.com if you have any questions about the submission format!

You should receive a confirmation e-mail within 24 hours of sending your solution. Assignments received up to 24 hours late will get a point deduction of 50% of the points. Assignments received more than 24 hours late get a 100% point deduction.

You can send questions to HAASMFE.MATH@gmail.com if you get stuck.

All answers need to be justified!

1. *Probability spaces*: Consider the state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with associated σ-algebra $\mathcal{F} = 2^\Omega$, and probability measures, $\mathbb{P}$, and $\mathbb{Q}$, defined by:

i	1	2	3	4
$\mathbb{Q}(\omega_i)$	0.1	0	0.5	0.4
$\mathbb{P}(\omega_i)$	0.4	0.1	0.1	0.4

   (a) Are $\mathbb{P}$ and $\mathbb{Q}$ equivalent probability measures?

   (b) What is the Radon-Nikodym derivative, $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$?

2. *Measurability*: Consider the state space $\Omega = \mathbb{R}$, the σ-algebra, $\mathcal{F} = \{(-\infty, 0], (0, \infty), \emptyset, \mathbb{R}\}$, and the random variable $X : \Omega \to \mathbb{R}$ defined by

$$X(\omega) = \begin{cases} 3, & \omega < 0, \\ 5, & \omega \geq 0. \end{cases}$$

   Is X $\mathcal{F}$-measurable?

3. *Stochastic integration*: A small company is investing resources in a risky project that it hopes will be profitable. The project could, for example, represent the manufacturing and selling of a gadget in a local market. The performance of the market is measured by the Brownian motion $Z_t = \mu t + \sigma W_t$, where W_t is a standardized Brownian motion, as discussed in class. Here, the randomness of market performance could for example be driven by uncertain market demand.

   The small change $Z_{t+\Delta t} - Z_t = \mu \Delta t + \sigma(W_{t+\Delta t} - W_t)$, or on differential form:

$$dZ_t = \mu dt + \sigma dW_t,$$

   represents how well the market performs over a short time period at time t. We call the variable Z the "market performance measure." We assume that $\mu \geq 0$ and $\sigma > 0$, representing that the measure is (weakly) expected to increase and that is has a random component.

   At each point in time, the company chooses how much to invest in the market, I. Here, I can in general be a function of t, as well as of W_t. We allow I to be negative for technical reasons.[1]

   Let V_t denote the value created by the project up until time t. The market performance measure Z then has the following interpretation: If the company chooses to invest a constant,

[1] Although allowing for negative investments in this specific application may seem a bit far-fetched, in many real-world investments applications negative investments are indeed possible.

k, between t and $t + \Delta t$, i.e., if $I_s = k$ for $t \leq s \leq t + \Delta t$, then

$$V_{t+\Delta t} - V_t = k(Z_{t+\Delta t} - Z_t),$$

i.e., the change in value depends on how well the market performed and how much the company invested. Written as a stochastic integral, this then leads to the formula

$$\begin{aligned} V_T &= \oint_0^T I_t dZ_t \\ &= \oint_0^T I_t(\mu dt + \sigma dW_t) \\ &= \mu \int_0^T I_t dt + \sigma \oint_0^T I_t \, dW_t. \end{aligned}$$

Here, the stochastic (second) part of the integral is defined in the sense of Ito, as discussed in class.

(a) Given that the company chooses a fixed investment strategy, $I \equiv k$ between time 0 and T, derive an expression for what the value of the project at time T is. The value should be expressed as a function of W_T, k, μ and σ.

The company wants to choose an investment strategy, I, that leads to as high a value of the project as possible, but it is also risk averse, in that it dislikes high variations of the project's value. Specifically, we use the mean-variance utility model from a previous assignment, and assume that the "value" the company associates (at time 0) with an investment strategy leading to the random value V_T at time T is

$$U = E[V_T] - \frac{\gamma}{2} Var[V_T],$$

where $\gamma > 0$ is a risk aversion parameter that governs how the company trades off uncertainty in terms of variance against expected value.[2]

(b) Assume that the company has to determine a constant investment strategy at $t = 0$, $I \equiv k$. What value of k should the company choose?

It should follow from your result in (b) that in the special case when $\mu = 0$, the company should choose $k = 0$. This is not surprising since when $\mu = 0$, the performance measure has an expected value of 0 and is risky, so a risk averse investor sees no benefit in investing, but only costs in the form of risk.

[2] This is in most real-world cases a too simplistic model for the objective function of (the owners of) a company. We choose it for simplicity.

4. *Ito's lemma*: Consider the stochastic process

$$dX_t = \mu dt + \sigma dW_t.$$

Find a closed form expression for Q, defined by the stochastic integral

$$Q = \int_0^T 3X_t(\mu X_t + \sigma^2)dt + 3X_t^2 \sigma \, dW_t,$$

by postulating a function $G(t, W_t)$ that satisfies $dG_t = 3X_t(\mu X_t + \sigma^2)dt + 3X_t^2\sigma \, dW_t$, and using the definition of the stochastic integral in class to get $Q = G(T, W_T) - G(0, W_0)$.

Extra credit problem[3]

5. *Optimal control*: Consider an investor with utility function

$$U = \sum_{t=0}^{T} \rho^t \ln(c_t), \qquad 0 < \rho < 1.$$

As in class, each period the investor chooses consumption c_t and investment $I_t = W_t - c_t$. Investments grow at the stochastic rate R_t between t and $t+1$, where R_t depends on the value of s_{t+1}, which follows an N-state Markov process with transition matrix $\Phi \in \mathbb{R}_{+}^{N \times N}$. As in class, returns are summarized by the N-vector $\mathbf{R} \in \mathbb{R}_{++}^{N}$. The agent faces the constraint that $W_t \geq 0$ at each point in time.

   (a) Find the optimal consumption policy, $\mathbf{c}(t, W_t)$ and indirect utility function $\mathbf{J}(t, W_t)$, for all $t \in \{0, 1, \ldots, T\}$.

   (b) How does the consumption policy depend on investment returns, $\mathbf{R}$? Discuss whether the formula is reasonable.

   (c) Calculate the optimal consumption policy and indirect utility function in the infinite horizon case, i.e., in the limit when $T \to \infty$.

[3] This voluntary problem is for extra credit. Students who are at risk of not passing the course are recommended to complete this problem.