MFE Math Foundations
Assignment 5 Solutions

1. (a) Equivalent probability measures must agree on which sets have measure 0. Thus, $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent measures, since $\mathbb{Q}(\omega_2) = 0 \neq \mathbb{P}(\omega_2)$.

   (b) We have

i	1	2	3	4
$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega_i)$	0.25	0	5	1

2. A random variable $X : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable if for all $a \in \mathbb{R}$, we have

$$X^{-1}((a, \infty)) \in \mathcal{F}$$

Here we are using the notation $X^{-1}(E) = \{\omega \in \Omega \mid X(\omega) \in E\}$. However, note that we have

$$X^{-1}((4, \infty)) = \{\omega \in \Omega \mid X(\omega) > 4\} = [0, \infty) \notin \mathcal{F}$$

Thus, this X is not $\mathcal{F}$-measurable.

3. (a) For $I_t \equiv k$, we have

$$\begin{aligned} V_T &= \mu \int_0^T k\, dt + \sigma \int_0^T k\, dW_t \\ &= \mu k T + \sigma k (W_T - W_0) \\ &= \mu k T + \sigma k W_T \end{aligned}$$

   (b) We compute

$$\mathbb{E}[V_T] = \mu k T$$

and

$$\mathrm{Var}[V_T] = \sigma^2 k^2 \mathrm{Var}[W_T] = \sigma^2 k^2 T$$

Thus, our objective function is

$$U(k) = \mu k T - \frac{\gamma}{2} \sigma^2 k^2 T$$

The first order condition is $0 = U'(k) = \mu T - \gamma \sigma^2 k T$, and $U''(k) = -\gamma \sigma^2 T < 0$ so a unique maximum exists. Solving the equation from the first order condition, we get the solution

$$k = \frac{\mu}{\gamma \sigma^2}$$

4. Note that

$$X_t = X_0 + \mu t + \sigma W_t,$$

from class. Ito's lemma for the function $H(X_t) = X_t^3$ gives us

$$\begin{aligned} dH &= 3X^2\,dX + \frac{1}{2}6X(dX^2) \\ &= 3X^2(\mu dt + \sigma dW) + 3X\sigma^2\,dt \\ &= 3X(\mu X + \sigma^2)dt + 3X^2\sigma dW. \end{aligned}$$

Thus,

$$\begin{aligned} Q = \int_0^T dH &= H(T) - H(0) \\ &= X_T^3 - X_0^3 \\ &= (X_0 + \mu T + \sigma W_T)^3 - X_0^3. \end{aligned}$$

Also, note that if we define $G(t, W_t) = (X_0 + \mu T + \sigma W_T)^3$, then $Q = G(T, W_T) - G(0, W_0)$.