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The EPR-Bohm Paradox and Kent's One-World Solution

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by

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Introduction

Many ideas in quantum physics are expressed in mathematical terms. I will do my best to avoid unnecessary mathematical jargon, but in order to explain the ideas of this thesis, a certain amount of mathematics is unavoidable. I will therefore endeavor to explain all the mathematical terminology as I go along. However, there will be some sections which may be very challenging to readers who do not have a mathematics or physics background. These sections will be marked with and asterisk *. There is also a lot of terminology from theoretical physics that we will need to invoke. To aid the reader, I will therefore use the convention of putting terminology in italic typeface whenever the terminology is first defined.

Chapter 1

Confronting the EPR-Bohm Paradox

In recent times, it has become increasingly common for popularizers of quantum physics to tell us that we need to let go of our naïve common sense understanding of reality. We're told we must replace this common sense understanding with something that at first seems very bizarre and counter-intuitive: a many-worlds interpretation of reality. This is the idea that whenever there is quantum indeterminacy among several possibilities, then all these possibilities are realized, and the actualization of these possibilities can be extrapolated up to the macroscopic level. Thus, many-worlds advocates, when reflecting on the famous Schrödinger's Cat thought experiment do not question the foundations of quantum mechanics on which the thought experiment is based, but rather they embrace the seemingly absurd conclusion of Schrödinger that a cat could be both dead and alive. They thus speak of the cat being dead in one world and the cat being alive in another world that is just as real as the first. We will be examining the many-worlds interpretation of quantum physics in chapter 2, but in this chapter, we will consider why some people are so keen to reject a one-world interpretation of quantum physics.

The central challenge that one-world interpretations of quantum physics must face is how to account for the mysterious correlations of the so-called EPR-Bohm paradox in a way that is consistent with Einstein's theory of special relativity. In this chapter, no prior knowledge of quantum theory will be assumed. We will therefore need to describe some key ideas of quantum theory, and this we will do in the context of the Stern-Gerlach experiment. We will then describe the EPR-Bohm paradox and the difficulty the traditional Copenhagen interpretation of quantum physics has in dealing with this paradox. A seemingly natural way to overcome this paradox is to supplement standard quantum theory with hidden variables. We will thus describe one way in which this can be done, and we will show how this leads to the remarkable inequality first derived by Bell. However, Bell's inequality is known to be experimentally violated. This means there must be something wrong with Bell's assumptions. We will therefore consider Shimony's analysis of the proof of Bell's inequality in which Shimony draws a distinction between parameter independence and outcome independence. Finally, following Butterfield, we will briefly explain why Shimony's analysis does not adequately resolve the EPR-Bohm Paradox.

1.1 Quantum Physics Basics and the Stern-Gerlach Experiment

Some of the key features of quantum physics are exhibited in the Stern-Gerlach experiment (see figure 1.1). In this experiment, silver atoms are heated in a furnace which randomly emerge from the furnace with various velocities. By aligning two plates with circular holes near the furnace, it is possible to select a subset of the emerging silver atoms having (approximately) the same momentum to form a beam in one direction, the other silver atoms having been absorbed by the two plates. This beam of silver atoms is then directed between two magnets with the north pole of one magnet being aligned toward the south pole of the other magnet as shown in figure 1.1. Now silver atoms have a property somewhat analogous to the classical notion of

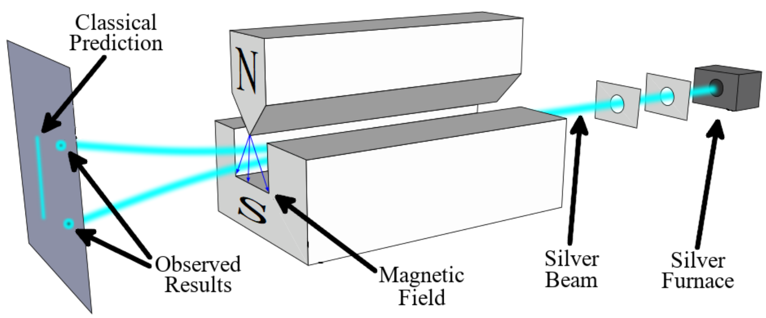

Figure 1.1: The Stern-Gerlach Experiment.[1]

angular momentum. For instance, a spinning top has angular momentum as shown in figure 1.2. Angular momentum is a vector, so it has direction and magnitude. In the case of a spinning top, the direction of the angular momentum would be parallel to the axis of rotation, pointing one way or the other depending on whether the rotation was clockwise or counterclockwise. The magnitude of the angular momentum would then be proportional to the angular velocity of the spinning top.

[1]Original diagram drawn by Theresa Knott. Labeling was modified for use in this dissertation. This image is licensed under the Creative Commons Attribution-Share Alike 4.0 International license. Source: https://commons.wikimedia.org/wiki/File:Stern-Gerlach_experiment_svg.svg

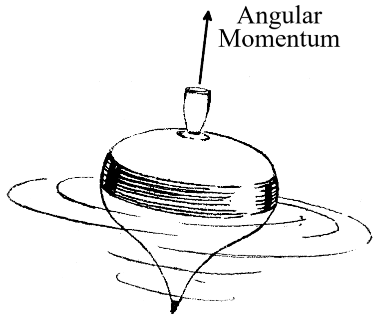

Figure 1.2: Angular Momentum of a Spinning Top.[2]

Now if we tried to understand the angular momentum of a silver atom classically, we would expect the magnetic field of the two magnets to interact with the silver atom in a way that was determined by the relative direction of the silver atom's angular momentum compared to the direction of the magnetic field. Since we would expect the silver atom to have an entirely random angular momentum, we would expect it to be deflected by varying degrees either up or down in the direction of the magnetic field. Thus, if a detection screen were placed beyond the two magnets which the silver atoms would hit, we would expect there to be a whole continuum of possible locations where the silver atoms would be detected. However, in reality, it is found that there are precisely two locations where the silver atoms hit the screen. It is as though the particles can spin either clockwise or anticlockwise, but that there is absolutely no variance in the angular speed at which they rotate. This is surprising. The angular momentum appears to be *quantized* in one of two directions, either parallel to the magnetic field or antiparallel to it.[3] Corresponding to this quantization of angular

[2]Drawing by Pearson Scott Foresman, Public domain, via Wikimedia Commons. Labeling was added for use in this dissertation. Original: https://commons.wikimedia.org/wiki/File:Top_(PSF).png.

[3]See figure 1.3 for what is meant be antiparallel.

momentum, we say that the atom is either in the spin up state or the spin down state with respect to the direction of the magnetic field.

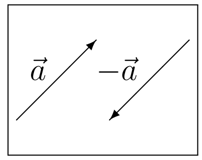

Figure 1.3: Meaning of antiparallel: the arrows in opposite directions are said to be antiparallel to one another.

If the direction of the magnetic field is implicitly understood, we write $|+\rangle$ and $|-\rangle$ for the spin up and spin down states of the atom respectively. We refer to the symbols $|+\rangle$ and $|-\rangle$ as *ket-vectors*, or simply as kets. We can think of the ket $|+\rangle$ for instance as shorthand for the proposition "the particle is in the spin up state." If we knew this proposition to be true, we would know which of the two locations on the detection screen the particle would end up if it were to travel between the two magnets of the Stern-Gerlach apparatus. If we need to specify the spin with respect to a particular direction of the magnetic field, say in the $\mathbf{\hat{a}}$-direction, we write the corresponding spin up and down states as $|\mathbf{\hat{a}}+\rangle$ and $|\mathbf{\hat{a}}-\rangle$. For convenience, we write $\mathbf{\hat{a}}+$ and $\mathbf{\hat{a}}-$ respectively for the location that the particle would hit the detection screen.

The question then arises as to what happens when we rotate the magnetic field around the axis of the particle beam in the Stern-Gerlach experimental setup. It turns out that when we do this, the atoms are again detected in only one of two locations (see figure 1.4).

So suppose we knew the particle was in a spin state such that it was on course to arrive at location $\mathbf{\hat{a}}+$ because we had previously directed it through another magnetic

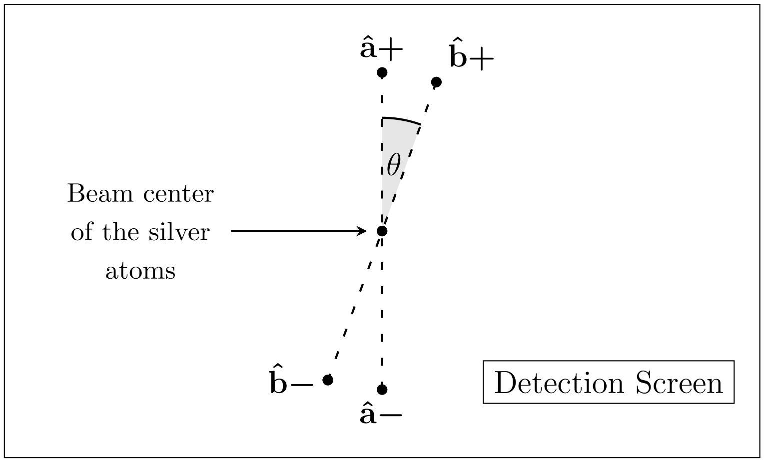

Figure 1.4: Locations of detections before and after rotating the magnetic field by an angle θ. Before rotation, the particles can be detected at either location $\hat{\mathbf{a}}+$ or location $\hat{\mathbf{a}}-$. After the rotation, particles can be detected at either location $\hat{\mathbf{b}}+$ or $\hat{\mathbf{b}}-$.

field in the $\hat{\mathbf{a}}$-direction. For example, see figure 1.5 for how this might be done. In

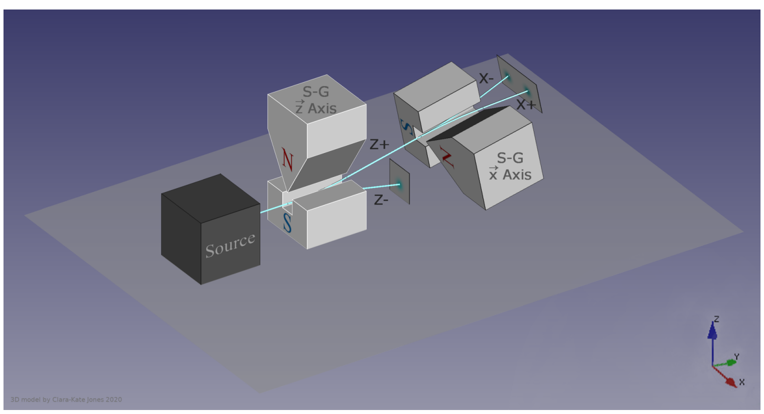

Figure 1.5: Two Stern-Gerlach experiements in sequence. By directing the beam of particles through one magnetic field first, the particles emerging in one of the two beams will be in a known spin state before they enter the second magnetic field.[4]

this experimental setup, the second magnetic field has been rotated by an angle of $90°$ with respect to the first magnetic field. But suppose we just rotated the second magnetic field by a very small angle θ with respect to the first magnetic field. Then

[4]Diagram by MJasK. This file is licensed under the Creative Commons Attribution-Share Alike 4.0 International license. Source: https://commons.wikimedia.org/wiki/File:Stern-Gerlach_Analyzer_Sequential_Series_E2.png.

we would expect the particle now to arrive at a location $\hat{\mathbf{b}}+$ close by to $\hat{\mathbf{a}}+$ as shown in figure 1.4. And this is what we notice experimentally for the most part. However, occasionally, the particle will arrive at location $\hat{\mathbf{b}}-$. The frequency of this occurrence becomes less and less the less and less the magnetic field is rotated (i.e. the smaller θ is), so that if the magnetic field is not rotated at all, i.e. $\theta = 0$, the particle will always arrive at location $\hat{\mathbf{a}}+$. To capture the probabilistic nature of these outcomes, we use the bra-ket notation. Thus, if $|\psi\rangle$ stands for either the $|\hat{\mathbf{a}}+\rangle$ or the $|\hat{\mathbf{a}}-\rangle$-state, and $|\chi\rangle$ stands for either the $|\hat{\mathbf{b}}+\rangle$ or the $|\hat{\mathbf{b}}+\rangle$-state, then we define the bra-ket $\langle\psi|\chi\rangle \in \mathbb{C}$ to be a complex number[5] that satisfies the *Born Rule*, namely the rule that $|\langle\psi|\chi\rangle|^2$ is the probability that the particle will be found to be in state $|\psi\rangle$ given that we know that the particle is in state $|\chi\rangle$. For example, if $|\langle\psi|\chi\rangle|^2 = \frac{1}{4}$, and we performed the experiment a 1000 times with the particle initially prepared in the $|\chi\rangle$-state, then we would expect the particle to be found in the $|\psi\rangle$-state in around 250 runs of the experiment. We would thus expect $|\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle|^2$ to be 0, from which it will follow that $\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle$ has to be 0. The Born Rule also implies that $|\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle|^2 = 1$. We will also insist that $\langle\psi|\chi\rangle = \overline{\langle\chi|\psi\rangle}$,[6] from which it will follow that $\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle$ is a real number of modulus 1 (i.e. $+1$ or -1). By convention, we

[5] With regards to the set of complex numbers $\mathbb{C}$, we will use the notation $i = \sqrt{-1}$. Complex conjugation will be denoted by an overline so that $\overline{x+iy} = x - iy$ for real numbers x and y. The modulus of a complex number $z = x + iy$ will then be given by $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$. Since the defining property of $\langle\psi|\chi\rangle$ is that $|\langle\psi|\chi\rangle|^2$ is the probability that the particle will be found to be in state $|\psi\rangle$ given that we know that the particle is in state $|\chi\rangle$, we have to choose an arbitrary phase to fully determine $\langle\psi|\chi\rangle$.

[6] Note that this conditions implies time symmetry: the probability a particle transitions from a state $|\chi\rangle$ to a state $|\psi\rangle$ will be the same as the probability a particle transitions from the state $|\psi\rangle$ to the state $|\chi\rangle$. This is in accord with the observation that closed quantum systems are symmetric on time-reversal. This might at first seem surprising in the light of the fact that phenomena such as radioactive decay are not obviously time-symmetric. However, it turns out that this time asymmetry results from the quantum system not being closed. For more details, see Saverio Pascazio, "All you ever wanted to know about the quantum Zeno effect in 70 minutes," *44th Symposium on Mathematical Physics on New Developments in the Theory of Open Quantum Systems*, 2013, https://doi.org/10.1142/S1230161214400071, eprint: arXiv:1311.6645v1[quant-ph].

choose $\langle\psi|\chi\rangle$ such that $\langle\psi|\psi\rangle$ is a real and positive number, in which case we must have $\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle = 1$. If we now rotate the magnetic field by an angle θ as indicated in figure 1.4, the particle will be detected either at location $\hat{\mathbf{b}}+$ or location $\hat{\mathbf{b}}-$. We can then ask the question "given that the particle is in state $|\hat{\mathbf{a}}+\rangle$, what is the probability that the particle will be found to be in state $|\hat{\mathbf{b}}+\rangle$?" According to the notation discussed above, this probability will be $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle|^2$ where $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle$ is a complex number such that $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = 1$ when $\theta = 0$ and $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = 0$ when $\theta = 180°$. We would likewise expect $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = 0$ when $\theta = 0$ and $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = 1$ when $\theta = 180°$. Since $\cos 0° = \sin 90° = 1$ and $\cos 90° = \sin 0° = 0$, we might guess that in general $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle| = |\cos(\theta/2)|$ and $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle| = |\sin(\theta/2)|$. Experimentation shows us that this guess is correct. This suggests that we can express the state $|\hat{\mathbf{b}}+\rangle$ in terms of the states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$. We thus suppose that

$$|\hat{\mathbf{b}}+\rangle = \alpha\,|\hat{\mathbf{a}}+\rangle + \beta\,|\hat{\mathbf{a}}-\rangle \tag{1.1a}$$

$$|\hat{\mathbf{b}}-\rangle = \bar{\alpha}\,|\hat{\mathbf{a}}-\rangle - \bar{\beta}\,|\hat{\mathbf{a}}+\rangle \tag{1.1b}$$

for complex numbers $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, and we suppose that the bra-ket has the *linearity* property so that $\langle\psi|\hat{\mathbf{b}}+\rangle = \alpha\,\langle\psi|\hat{\mathbf{a}}+\rangle + \beta\,\langle\psi|\hat{\mathbf{a}}-\rangle$ and $\langle\psi|\hat{\mathbf{b}}-\rangle = \bar{\alpha}\,\langle\psi|\hat{\mathbf{a}}-\rangle - \bar{\beta}\,\langle\psi|\hat{\mathbf{a}}+\rangle$ for any state $|\psi\rangle$. Then it will follow that $\langle\hat{\mathbf{b}}+|\hat{\mathbf{b}}-\rangle = 0$,[7] and that $\langle\hat{\mathbf{b}}+|\hat{\mathbf{b}}+\rangle = \langle\hat{\mathbf{b}}-|\hat{\mathbf{b}}-\rangle = 1$.[8] If we then put $\alpha = \cos(\theta/2)$ and $\beta = \sin(\theta/2)$, it will follow that $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle| = |\cos(\theta/2)|$ and $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle| = |\sin(\theta/2)|$,[9] and so with

[7] To see this, by putting $|\psi\rangle = |\hat{\mathbf{b}}+\rangle$, we will have $\langle\hat{\mathbf{b}}+|\hat{\mathbf{b}}-\rangle = \bar{\alpha}\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle - \bar{\beta}\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle$ by equation (1.1b). Since $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = \overline{\langle\hat{\mathbf{a}}-|\hat{\mathbf{b}}+\rangle}$ we have $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = \bar{\beta}$ by equation (1.1a), and likewise, since $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = \overline{\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}+\rangle}$, we have $\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = \bar{\alpha}$. Therefore $\langle\hat{\mathbf{b}}+|\hat{\mathbf{b}}-\rangle = \overline{\alpha}\overline{\beta} - \overline{\beta}\overline{\alpha} = 0$.

[8] To see this, by putting $|\psi\rangle = |\hat{\mathbf{b}}+\rangle$ and using equation (1.1a), we will have $\langle\hat{\mathbf{b}}+|\hat{\mathbf{b}}+\rangle = \alpha\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle + \beta\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = \alpha\overline{\alpha} + \beta\overline{\beta} = |\alpha|^2 + |\beta|^2 = 1$. By a similar calculation, we also see $\langle\hat{\mathbf{b}}+|\hat{\mathbf{b}}+\rangle = 1$.

[9] To satisfy these criteria, α and β are only determined up to rotation by a complex number. Rotating a complex number $z \in \mathbb{C}$ just means multiplying it by a complex number λ of modulus 1 (i.e. $|\lambda| = 1$) to get λz. We would need to take into account this rotation factor if we considered the three-

these values for α and β we will have

$$|\hat{\mathbf{b}}+\rangle = \cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\,, \tag{1.2a}$$

$$|\hat{\mathbf{b}}-\rangle = \cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\,. \tag{1.2b}$$

dimensional situation. Then, without loss of generality, $\alpha = \cos(\theta/2)$ and $\beta = e^{i\phi}\sin(\theta/2)$ where θ and ϕ are the polar and azimuthal angles respectively.

1.2 The Copenhagen Interpretation and the EPR-Bohm Paradox

Given the equations (1.2a) and (1.2b) that relate the states $|\hat{\mathbf{a}}\pm\rangle$ and $|\hat{\mathbf{b}}\pm\rangle$ to each other, we can calculate probabilities such as the probability a particle will be measured to be in the $|\hat{\mathbf{a}}+\rangle$-state given that it is in the $|\hat{\mathbf{b}}+\rangle$-state. There however arises the question of what the physical meaning of one of these states is. Clearly, the $|\hat{\mathbf{b}}+\rangle$-state says something about the spin of a particle; but is this a complete description of the particle's spin state? For the $|\hat{\mathbf{b}}+\rangle$-state only tells us what the outcome of a spin measurement would be along one particular axis $\hat{\mathbf{b}}$. For a spin measurement along another axis $\hat{\mathbf{a}} \neq \pm\hat{\mathbf{b}}$, $|\hat{\mathbf{b}}+\rangle$ only tells us the probabilities (via equation (1.2a) and the Born Rule) that the measurement outcome would be $|\hat{\mathbf{a}}+\rangle$ or $|\hat{\mathbf{a}}-\rangle$ – but the $|\hat{\mathbf{b}}+\rangle$-state doesn't determine either of these outcomes. So we want to know whether this indetermination of the measurement outcome along the $\hat{\mathbf{a}}$-axis is merely a reflection of our lack of knowledge of a more complete specification of the particle's spin state, or alternatively, whether the $|\hat{\mathbf{b}}+\rangle$-state is a complete description of the spin state of the particle so that there is no fact of the matter about what spin state the particle would be found to be in along the $\hat{\mathbf{a}}$-axis until a measurement of spin along the $\hat{\mathbf{a}}$-axis is made.

Now Bohr and Heisenberg believed the latter to be the case. This was because their mathematical formalism of quantum physics implied that there are physical quantities of particles that couldn't be simultaneously determined. For example, their mathematical formalism is incapable of representing a particle which has a definite spin in both the $\hat{\mathbf{a}}$-direction and the $\hat{\mathbf{b}}$-direction when $\hat{\mathbf{a}} \neq \pm\hat{\mathbf{b}}$. So when a particle that is in the $|\hat{\mathbf{b}}+\rangle$-state is measured along the $\hat{\mathbf{a}}$-axis and is found to be in the $|\hat{\mathbf{a}}+\rangle$-state, there is a so-called collapse of the $|\hat{\mathbf{b}}+\rangle$-state:

$$|\hat{\mathbf{b}}+\rangle = \cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle \xrightarrow{\text{Collapse!!}} |\hat{\mathbf{a}}+\rangle$$

so that after the measurement, the particle is no longer in the $|\hat{\mathbf{b}}+\rangle$-state. This interpretation of the quantum state as a complete physical description in which the state collapses to another state upon measurement is known as the *Copenhagen Interpretation*.

Einstein, Podolsky, and Rosen, however, strongly objected to the Copenhagen Interpretation, and they introduced their EPR paradox to explain what troubled them.[10] The EPR paradox was originally expressed in terms of the position and momentum of a particle, but it was Bohm who translated the EPR paradox to the context of spin,[11] and this is the version we will consider here.

The EPR-Bohm paradox arises in the context of particle pairs known as spin singlets. A *spin singlet* describes the state of two particles which a single particle of zero spin has decayed into. For example, a high energy *photon*, that is, a particle of light, can decay into a negatively charged electron, and a positively charged positron (where a *positron* is a fundamental particle like an electron but of opposite charge). Since spin is a conserved physical quantity, the spin of the two particles q_A and q_B of a spin singlet state must be equal and opposite when measured along the same axis, no matter what direction this axis happens to point in. The existence of spin singlet states thus raises the question of what the physical mechanism or principle

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[10]See A. Einstein, B. Podolsky, and N. Rosen, “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?,” *Physical review* 47, no. 10 (1935): 777–780.

[11]e.g. see D. Bohm, *Quantum Theory* (Englewood Cliffs: Prentice-Hall, 1951), p. 29, Ch. 5 sec. 3, and Ch. 22 sec. 19.

is that ensures two experimenters, Alice and Bob say, will always obtain opposite spin measurement results if Alice measures the spin of particle q_A, and Bob measures the spin of particle q_B along the same axis. There is of course no experiment that could prove that Alice and Bob will always obtain opposite spin measurements, and there are some interpretations of quantum physics such as the GRW spontaneous collapse theory[12] which predict that very occasionally Alice and Bob would obtain the same spin measurement result. But in this dissertation, we will assume that all measurements are consistent with *standard quantum theory*. In other words, we assume that the physical world can be described by quantum states[13] that evolve over time according to the Schrödinger equation, and that the probability of a system being found to be in one state given that it is in another state will be given by the Born Rule. In particular, under the assumption of standard quantum theory, it will follow that Alice and Bob will always obtain opposite spin results when performing measurements along the same axis of two particles in the spin singlet state.

Now naively, one would expect that if the spin of q_A were to be measured, then this would have no effect on any spin-measurement of q_B. This assumption is a special case of *Einstein's locality principle*: For two spatially separated systems S_1 and S_2, the real factual situation of the system S_2 should be independent of what is done to the system S_1.[14] If Einstein's locality principle holds, we would be able to attribute a state

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[12]See Giancarlo Ghirardi and Angelo Bassi, "Collapse Theories," in *The Stanford Encyclopedia of Philosophy*, Summer 2020, ed. Edward N. Zalta (Metaphysics Research Lab, Stanford University, 2020).

[13]In standard quantum theory, we remain agnostic as to whether a quantum state provides a complete description of the physical world, or whether it needs to be supplemented by some additional information in order to obtain a complete description.

[14]Einstein expressed this locality principle in his autobiographical notes: "But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is

$|\alpha\rangle_A$ to particle q_A, and a state $|\beta\rangle_B$ to particle q_B, so that if Alice were to perform a Stern-Gerlach experiment on particle q_A in which one of the possible outcomes was a spin state $|\alpha'\rangle_A$, then by the Born Rule, the probability Alice would find q_A to be in state $|\alpha'\rangle_A$ would be $|\langle\alpha'|\alpha\rangle_A|^2$. Likewise, if Bob were to perform a Stern-Gerlach experiment on particle q_B in which one of the possible outcomes was a spin state $|\beta'\rangle$, then the probability Bob would find q_B to be in state $|\beta'\rangle_B$ would be $|\langle\beta'|\beta\rangle_B|^2$.

Now in order to decide how to represent the joint state of the particles q_A and q_B, we recall that in probability theory, we say that two events X and Y are *statistically independent* if and only if

$$P(X,Y) = P(X)P(Y) \tag{1.3}$$

where $P(X)$ is the probability that X occurs, $P(Y)$ is the probability that Y occurs, and $P(X,Y)$ is the probability that both X and Y both occur. We define the *conditional probability* $P(X|Y)$ of X given Y to be

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}. \tag{1.4}$$

From (1.3), it is easy to see that when X and Y are independent, $P(X|Y) = P(X)$. We also say that two events X and Y are *conditionally independent* given some third event Z if and only if

$$P(X,Y|Z) = P(X|Z)P(Y|Z) \tag{1.5}$$

where $P(X|Z)$ is the conditional probability that X occurs given Z, $P(Y|Z)$ is the conditional probability that Y occurs given Z, and $P(X,Y|Z)$ is the conditional probability that both X and Y occur given Z.

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independent of what is done with the system S_1, which is spatially separated from the former."

Now if q_A can be described by the $|\alpha\rangle_A$-state and q_B can be described by the $|\beta\rangle_B$-state, the Born Rule implies that the conditional probability that Alice would measure her particle to be in the state $|\alpha'\rangle_A$ is not going to depend on $|\beta\rangle_B$, and so $P(\alpha'|\alpha,\beta) = P(\alpha'|\alpha)$. Likewise, $P(\beta'|\alpha,\beta) = P(\beta'|\beta)$. Therefore, if Alice and Bob's measurement outcomes are conditionally independent, then according to (1.5) and the Born Rule, we would obtain the conditional probability

$$P_{AB}(\alpha',\beta'|\alpha,\beta) = |\langle\alpha'|\alpha\rangle_A|^2 \times |\langle\beta'|\beta\rangle_B|^2 = |\langle\alpha'|\alpha\rangle_A \langle\beta'|\beta\rangle_B|^2. \tag{1.6}$$

This suggests that if we write $|\alpha\rangle_A |\beta\rangle_B$ for the state of the composite system of both particles, then the bra-ket of $|\alpha'\rangle_A |\beta'\rangle_B$ and $|\alpha\rangle_A |\beta\rangle_B$ would be

$${}_B\langle\beta'|\,{}_A\langle\alpha'|\alpha\rangle_A\,|\beta\rangle_B = \langle\alpha'|\alpha\rangle_A \langle\beta'|\beta\rangle_B\,, \tag{1.7}$$

and we extend this bracket to sums of states so that it satisfies the linearity property (see page 13). However, when the particles q_A and q_B form a spin singlet, it will not be possible to express their joint state as $|\alpha\rangle_A |\beta\rangle_B$ because otherwise, according to (1.6), we will always be able to find a direction $\hat{\mathbf{a}}$ such that $P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{a}}+|\alpha,\beta) \neq 0$, whereas in reality, the state Z describing the singlet has to satisfy $P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{a}}+|Z) = 0$ for all $\hat{\mathbf{a}}$. But it turns out that the summation of states:

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B). \tag{1.8}$$

can describe the singlet state of the two particles. We refer to the state (1.8) as a *Bell state*.[15] If the composite system is in the Bell state $|\Psi_{\text{Bell}}\rangle$, then according to the Born Rule, the probability that Alice measures her particle to be in state $|\alpha\rangle$ and

Albert Einstein, *Albert Einstein, Philosopher Scientist*, ed. P. A. Schilp (Evanston, Illinois: Library of Living Philosophers, 1949), p. 85.

[15] By convention, the states $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B + |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B)$, $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}+\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}-\rangle_B)$, and $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}+\rangle_B + |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}-\rangle_B)$ are also referred to as Bell states.

Bob measures his particle to be in the state $|\beta\rangle$ will be:

$$\begin{aligned} P_{AB}(\alpha,\beta|\Psi_{\text{Bell}}) &= |\,_B\langle\beta|\,_A\langle\alpha|\Psi_{\text{Bell}}\rangle|^2 \\ &= \frac{1}{2}|\,\langle\alpha|\hat{\mathbf{a}}+\rangle_A\,\langle\beta|\hat{\mathbf{a}}-\rangle_B - \langle\alpha|\hat{\mathbf{a}}-\rangle_A\,\langle\beta|\hat{\mathbf{a}}+\rangle_B\,|^2 \end{aligned} \tag{1.9}$$

This means that whatever axis Bob decides to measure along, if Alice measures her particle along the $\hat{\mathbf{a}}$-axis, then the Born Rule predicts that she will measure the particle to be in either the $|\hat{\mathbf{a}}+\rangle_A$-state or the $|\hat{\mathbf{a}}-\rangle_A$-state, each with probability of $\frac{1}{2}$.[16] But also, the Born Rule implies that if both Alice and Bob measure their respective particles along the same $\hat{\mathbf{a}}$-axis, then the probability Bob will measure his particle to have the same spin as Alice's particle will be zero, and the probability that Bob measures his particle to have the opposite spin from Alice's particle will be one.[17] These probabilities predicted by the Born Rule using the Bell state $|\Psi_{\text{Bell}}\rangle$ correspond

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[16] To see this, suppose Bob performs his measurement along an arbitrary axis $\hat{\mathbf{b}}$. Then the probability Alice measures her particle to be in the $|\hat{\mathbf{a}}+\rangle$-state will be

$$\begin{aligned} &P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+|\Psi_{\text{Bell}}) + P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-|\Psi_{\text{Bell}}) = \\ &\quad=\frac{1}{2}|\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle_A\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle_B - \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle_A\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle_B\,|^2 \\ &\quad\quad+\frac{1}{2}|\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle_A\,\langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}-\rangle_B - \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle_A\,\langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}+\rangle_B\,|^2 \\ &\quad=\frac{1}{2}|\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle_B\,|^2 + \frac{1}{2}|\,\langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}-\rangle_B\,|^2 = \frac{1}{2} \end{aligned} \tag{1.10}$$

where on the last line we have used the fact that

$$|\hat{\mathbf{a}}-\rangle_B = |\hat{\mathbf{b}}+\rangle_B\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle_B + |\hat{\mathbf{b}}-\rangle\,\langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}-\rangle_B$$

and

$$\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}-\rangle_B = 1, \quad \langle\hat{\mathbf{b}}\pm|\hat{\mathbf{b}}\pm\rangle_B = 1, \quad \text{and} \quad \langle\hat{\mathbf{b}}\pm|\hat{\mathbf{b}}\mp\rangle_B = 0$$

so that

$$|\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle_B\,|^2 + \langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}-\rangle_B\,|^2 = 1.$$

[17] This is because the probability Alice and Bob will measure their particles to have the same spin will be

$$\begin{aligned} &P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{a}}+|\Psi_{\text{Bell}}) + P_{AB}(\hat{\mathbf{a}}-,\hat{\mathbf{a}}-|\Psi_{\text{Bell}}) = \\ &\quad=\frac{1}{2}|\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle_A\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle_B - \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle_A\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle_B\,|^2 \\ &\quad\quad+\frac{1}{2}|\,\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle_A\,\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}-\rangle_B - \langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}-\rangle_A\,\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle_B\,|^2 = 0, \end{aligned}$$

to the probabilities observed experimentally. Also note that despite the of appearance the formula (1.8), $|\Psi_{\text{Bell}}\rangle$ is independent of the axis $\hat{\mathbf{a}}$. That is, for any other direction $\hat{\mathbf{b}}$, it can be shown that,

$$\frac{1}{\sqrt{2}}\left(|\hat{\mathbf{a}}+\rangle_A\ |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\ |\hat{\mathbf{a}}+\rangle_B\right) = \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{b}}+\rangle_A\ |\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A\ |\hat{\mathbf{b}}+\rangle_B\right).^{18} \quad (1.11)$$

Therefore, if Alice had chosen to measure her particle along the $\hat{\mathbf{b}}$-axis rather than the $\hat{\mathbf{a}}$-axis, she would still obtain equal probabilities for finding her particle to be in either the state $|\hat{\mathbf{b}}+\rangle_A$ or $|\hat{\mathbf{b}}-\rangle_A$, and the same equal probabilities for Bob's measurement outcomes hold as well.

Now although attributing the state $|\Psi_{\text{Bell}}\rangle$ successfully determines the experimentally observed probabilities for Alice and Bob's spin measurement outcomes, this fact spells trouble for the Copenhagen interpretation. For if we stipulate that $|\Psi_{\text{Bell}}\rangle$ is a

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and the probability Alice and Bob will measure their particles to have different spins will be

$$\begin{aligned}
&P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{a}}-|\Psi_{\text{Bell}}) + P_{AB}(\hat{\mathbf{a}}-,\hat{\mathbf{a}}+|\Psi_{\text{Bell}}) = \\
&\quad = \frac{1}{2}\left|\ \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle_A\ \langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}-\rangle_B - \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle_A\ \langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle_B\ \right|^2 \\
&\quad + \frac{1}{2}\left|\ \langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle_A\ \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle_B - \langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}-\rangle_A\ \langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle_B\ \right|^2 = 1.
\end{aligned}$$

[18] To see this, using the transformation rules given in equation (1.2) we have

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left(|\hat{\mathbf{b}}+\rangle\ |\hat{\mathbf{b}}-\rangle - |\hat{\mathbf{b}}-\rangle\ |\hat{\mathbf{b}}+\rangle\right) \\
&= \frac{1}{\sqrt{2}}\big((\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle)(\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle) \\
&\quad - (\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle)(\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle)\big) \\
&= \frac{1}{\sqrt{2}}\big(\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle \\
&\quad + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle \\
&\quad - \cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle - \cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle \\
&\quad + \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big((\cos^2(\theta/2) + \sin^2(\theta/2))\,|\hat{\mathbf{a}}+\rangle\ |\hat{\mathbf{a}}-\rangle - (\cos^2(\theta/2) + \sin^2(\theta/2))\,|\hat{\mathbf{a}}-\rangle\ |\hat{\mathbf{a}}+\rangle\big) \\
&= \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{a}}+\rangle\ |\hat{\mathbf{a}}-\rangle - |\hat{\mathbf{a}}-\rangle\ |\hat{\mathbf{a}}+\rangle\right).
\end{aligned}$$

complete description of the spin state of the spin-singlet, then assuming Alice makes her measurement first and obtains the measurement outcome $|\hat{\mathbf{a}}+\rangle_A$, then as this measurement outcome is established there will be a collapse of the state

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B) \xrightarrow{\text{Collapse!!}} |\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B\,, \tag{1.12}$$

whereas if Alice makes her measurement first and obtains the measurement outcome $|\hat{\mathbf{a}}-\rangle_A$, there will be a state collapse

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B) \xrightarrow{\text{Collapse!!}} |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B\,. \tag{1.13}$$

Thus, according to the Copenhagen interpretation, prior to Alice's measurement, the real factual situation of particle q_B will be indeterminate, but once Alice has made her measurement, if she obtains the outcome $|\hat{\mathbf{a}}+\rangle_A$, the real factual situation of particle q_B will be expressed by the determinate state $|\hat{\mathbf{a}}-\rangle_B$, and if she obtains the outcome $|\hat{\mathbf{a}}-\rangle_A$, the real factual situation of particle q_B will be expressed by the determinate state $|\hat{\mathbf{a}}+\rangle_B$. But in either case, in the state collapse that Alice brings about via her measurement, there is a violation of Einstein's locality principle.

But things get even worse when we take Einstein's theory of special relativity into consideration. For according to Einstein's theory of relativity, the order in which Alice and Bob perform their measurements is going to depend on which inertial frame of reference one is in.[19] Thus, if we are moving at one velocity, it may appear that Alice makes her measurement first meaning that Alice causes the state collapse and thus

[19]In special relativity, an inertial frame of reference is a spacetime coordinate system (t, x, y, z) in which all objects which have no forces acting on them have trajectories that are straight lines. Thus, we can move to another inertial frame by moving to a reference frame with constant velocity $\mathbf{v}$ with respect to the first reference frame. In the case when $\mathbf{v} = (v, 0, 0)$, Einstein's theory of special relativity tells us that under such a "boost", spacetime coordinates will transform as $(t, \mathbf{x}) \to (t', x', y', z') = (\gamma(t - \frac{vx}{c^2}), \gamma(x - vt), y, z)$ where c is the speed of light and $\gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1}$.

affects the state of Bob's particle, whereas if we are moving at another velocity, it may appear that Bob makes his measurement first causing the state to collapse and thus affects the state of Alice's particle. Once the collapse has occurred, there is no further collapse when the second experimenter makes his or her measurement along the same axis as the first experimenter. Thus, the thesis of the Copenhagen interpretation that it is the act of measurement that causes the state to collapse is not compatible with Einstein's theory of special relativity, for if special relativity is correct, then one and the same measurement both will and will not cause the state to collapse depending upon which frame of reference one is in.

Therefore, if one is more convinced (as most physicists are) of the truth of Einstein's theory of relativity than the truth of the Copenhagen interpretation, then one must reject the Copenhagen interpretation. Thus, in the light of the EPR-Bohm paradox we need an alternative to the Copenhagen interpretation of quantum states that can account for how the two experimenters of the paradox always obtain opposite results when they make their measurement along the same axis. The obvious approach to take is to suppose that the quantum state $|\Psi_{\text{Bell}}\rangle$ is an incomplete description of the spin state of the two particles. This is the hidden variables approach, but as we will see shortly, this approach is also problematic. But before we consider the hidden variables approach, we will turn to what is traditionally considered to be one of the most troubling aspect of the Copenhagen interpretation, namely the possibility of scenarios exemplified by Schrödinger's cat thought experiment.

1.3 Schrödinger's Cat

Schrödinger's famous thought experiment was first discussed in a 1935 paper in the context of the EPR paradox and the inherent problems it raised for the Copenhagen interpretation.[20] According to the Copenhagen interpretation, given the Bell state

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\ |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\ |\hat{\mathbf{a}}+\rangle_B), \qquad \text{(1.8 revisited)}$$

there is no fact of the matter as to whether particle q_A is spin up and particle q_B is spin down (or vice versa) until an observation is made.

Now the strange goings-on at subatomic level might not initially give us much cause for concern about the nature of reality. For one might say that although it is an interesting curiosity that facts about subatomic particles are a bit fuzzy, our observation of definite facts in everyday life should convince us that such fuzziness can be brought into focus when we zoom out from the subatomic level to the macroscopic level. That is, we might suppose that the very many indefinite things on the small scale average out to give us something definite on the large scale. However, the Schrödinger's Cat thought experiment suggests that our confidence in there being definite facts at the macroscopic level is seriously undermined if the Copenhagen interpretation is correct. Schrödinger himself described the scenario in his thought experiment as ridiculous[21] indicating that he didn't think we should doubt the definiteness of facts at the macroscopic

[20]For Schrödinger's original reference to his thought experiment, see E. Schrödinger, "Die gegenwärtige Situation in der Quantenmechanik," *Die Naturwissenschaften* 23, no. 48 (November 1935): 807–812. For an English translation, see John D. Trimmer, "The Present Situation in Quantum Mechanics: A Translation of Schrödinger's "Cat Paradox" Paper" [in eng], *Proceedings of the American Philosophical Society* 124, no. 5 (1980): 323–338.

[21]See Trimmer, p. 328.

level; rather, we should call into question the reasonableness of the Copenhagen interpretation.

The Schrödinger's cat thought experiment invites us to consider a scenario like the one depicted in (1.12) and (1.13), but instead of having two microscopic particles coupled together, we have a microscopic particle such as a radioactive atom coupled together with a macroscopic object such as a cat. Schödinger suggests how this might be done. A cat is enclosed in a steel chamber in which there is a Geiger counter that is directed towards a small radioactive source, so that in the course of an hour, there is a probability of $\frac{1}{2}$ that the Geiger counter will click indicating that one of the radioactive atoms has decayed, and there is a probability of $\frac{1}{2}$ that the Geiger counter doesn't click because none of the radioactive atoms decay over the course of an hour. The Geiger counter itself is hooked up to a relay such that if the Geiger counter clicks, it releases a hammer which shatters a small flask of hydrocyanic acid causing the cat to die. The two possibilities are depicted in figure 3.5.

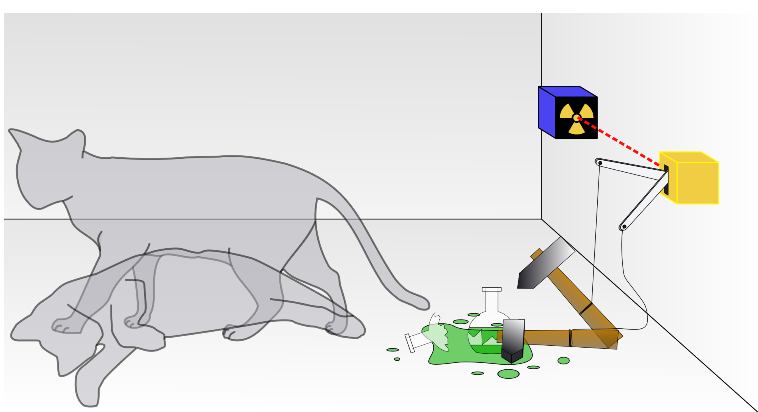

Figure 1.6: A depiction of Schrödinger's cat being both dead and alive.[22]

[22]Original by Dhatfield. This image is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. Source: https://commons.wikimedia.org/wiki/File:Schrodingers_cat.svg

According to the Copenhagen interpretation, the cat will only enter into a determinate state once the box is opened at the end of the hour and an observation is made. Thus, there are two possibilities analogous to (1.12) and (1.13): either

$$\frac{1}{\sqrt{2}}\Big(|\text{No atoms decay}\rangle \; |\text{Cat Alive}\rangle - |\text{Atom decays}\rangle \; |\text{Cat Dead}\rangle \Big)$$
$$\xrightarrow{\text{Collapse!!}} |\text{No atoms decay}\rangle \; |\text{Cat Alive}\rangle \, ,$$

or

$$\frac{1}{\sqrt{2}}\Big(|\text{No atoms decay}\rangle \; |\text{Cat Alive}\rangle - |\text{Atom decays}\rangle \; |\text{Cat Dead}\rangle \Big)$$
$$\xrightarrow{\text{Collapse!!}} |\text{Atom decays}\rangle \; |\text{Cat Dead}\rangle \, .$$

But before the box is opened and any measurement is made, the state of the cat and the atom will be not be determinate, just as in the case of the spin-singlet where the spin states of the two particles are not determined to a definite value before any measurement is made. The Schrödinger's Cat thought experiment thus highlights that under the Copenhagen interpretation, there is no good reason to restrict indeterminacy to the microscopic level. If there is any indeterminacy at the microscopic level, then we should expect it at the macroscopic level as well, in which case it should be possible for a cat to be in an indeterminate state of being alive and dead. It is this possibility that Schrödinger found ridiculous.

1.4 Hidden Variables and Bell's Inequality

Given the problem with the Copenhagen interpretation that the EPR-Bohm paradox and Schrödinger's Cat highlight, it is tempting to suppose that states such as $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ merely represents our limited knowledge of a more complete physical state that would also include a specification of the particle's spin state along other axes besides the $\hat{\mathbf{a}}$-axis. If we were to make this supposition, there would be a fact of the matter, albeit unknown to us, concerning what spin state the particle would be found to be in were we to measure its spin along some $\hat{\mathbf{b}}$-axis for $\hat{\mathbf{b}} \neq \hat{\mathbf{a}}$. And even though we might decide not to measure the spin of the particle along the $\hat{\mathbf{b}}$-axis, there would still be this hidden fact about the particle's spin in this direction. Furthermore, since there would be no reason to suppose there was anything special about the $\hat{\mathbf{a}}$ or $\hat{\mathbf{b}}$-axes, it would then be reasonable to suppose that there were hidden facts about what spin direction the particle would be found to be in for every possible axis orientation. This would mean that a complete description of the particle's spin state would require an infinite list of outcomes for all the possible orientations we could configure the magnetic field of our Stern-Gerlach apparatus to be in. For example, the complete spin state of a particle which according to our limited knowledge was in the $|\hat{\mathbf{b}}+\rangle$-state could be depicted as $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \ldots\rangle$ or $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \ldots\rangle$, etc. where the ellipses would range over one of the two possible measurement outcomes for every other magnetic field orientation. However, because we would never in practice be able to perform all these experiments, and since only one such experiment would be needed to alter this infinite list,[23] nearly all of the entries in this infinite list would remain forever hidden.

[23]In other words, it is assumed that directly measuring the particle will involve perturbing it so that its state will change.

Hence, this would be an example of a *hidden variables* interpretation of quantum theory. Moreover, if we're assuming Einstein's locality principle, it follows that any changes in these hidden spin outcomes for possible measurements of a specific particle can't affect the hidden variables of any other spatially separated localized particles. Therefore, given Einstein's locality principle, it is appropriate to refer to these hidden variables as *local* hidden variables.

Now although a local hidden variables theory seems rather intuitive, in 1964, John Bell derived an inequality based on the local hidden variables theory just described.[24] Moreover, it is now known that Bell's inequality can be violated experimentally.[25] This doesn't mean we must abandon all hidden variables theories. Rather it just means we need to abandon local hidden variables theories like the one Bell used in deriving his inequality. It is nevertheless somewhat paradoxical that local hidden variables cannot account for the correlations between the measurement outcomes of spin singlets in the EPR-Bohm paradox. Local hidden variables were meant to resolve the EPR-Bohm paradox, and so the violation of Bell's inequality really heightens this paradox. Therefore, any satisfactory interpretation of quantum physics has to face up to and resolve this paradox by offering a suitable alternative to a local hidden variables theory.

In order to describe Bell's inequality, we again consider two experimenters Alice and Bob making spin measurements on a spin singlet as described in section 1.2. So in

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[24]In providing a derivation of Bell's inequality, we will follow Jim J. Napolitano and J. J. Sakurai, *Modern Quantum Mechanics* (Pearson Education, 2013), 241–249.

[25]Aspect, Clauser and Zeilinger received the 2022 physics Nobel Prize for establishing the experimental violation of Bell's inequality. See "Press Release: The Nobel Prize in Physics 2022," *Nobel Prize*

each run of the experiment, a spin singlet consisting of two particles q_A and q_B will be generated with particle q_A being sent to Alice who measures q_A's spin in a direction of her choosing, and with particle q_B being sent to Bob who measures q_B's spin in a direction of his choosing. But now, instead of describing the two particles by the state $|\Psi_{\text{Bell}}\rangle$, we describe particles q_A and q_B in terms of all the spin outcomes one would obtain for every possible measurement axis in such a way that if q_A is spin up with respect to an axis $\hat{\mathbf{a}}$, then q_B would be spin down with respect to this axis. For example, if the complete spin state for q_A was given by $|\Psi_A\rangle = |\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-, \ldots\rangle_A$, then the complete spin state for q_B would be given by $|\Psi_B\rangle = |\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+, \ldots\rangle_B$.

We also assume that in each run of the experiment, Alice and Bob independently measure the spin of their particles along one of three possible directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and that Einstein's locality principle holds. Furthermore, we assume that in each run of the experiment, the outcome of Alice's measurement will be statistically independent of any of the other measurement outcomes for different runs of the experiment, and for any of the three axes she measures along, she will get a spin up outcome or a spin down outcome with equal probability of $\frac{1}{2}$. Likewise, we assume Bob's measurement outcomes are also similarly independent between different runs of the experiment. We also assume that the $8 = 2^3$ states $|\hat{\mathbf{a}}\pm, \hat{\mathbf{b}}\pm, \hat{\mathbf{c}}\pm\rangle_A$ exhaust all the possible states for Alice's particles that can be distinguished from one another by making one of the three possible measurement choices available. Thus, Alice can distinguish between the $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$-state and the $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_A$-state by making a measurement along the $\hat{\mathbf{c}}$-axis, though if she happened to make her measurement along the $\hat{\mathbf{a}}$ or $\hat{\mathbf{b}}$-axis,

Outreach, October 2022, accessed January 2, 2023, https://www.nobelprize.org/prizes/physics/2022/summary/.

she wouldn't be able to distinguish between these two states. But in principle, she can distinguish between these two states if she happens to make her measurement along the right axis, in this case the $\hat{\mathbf{c}}$-axis. We similarly assume the states $|\hat{\mathbf{a}}\pm, \hat{\mathbf{b}}\pm, \hat{\mathbf{c}}\pm\rangle_B$ exhaust all the possible states for Bob's particles that he can distinguish between, and we assume that if Alice and Bob measure the particle along the same axis, they will always obtain opposite results from one another. For instance, if Alice's particle is in state $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$, then Bob's particle must be in state $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_B$. Now suppose the experiment is run N times for large N,[26] and let N_i be the number of times particle q_A is in the ith state so that[27] $N = \sum_{i=1}^{8} N_i$ as shown in table 1.1.

Table 1.1: Spin-components of particles q_A and q_B in the hidden-variable theory

Population	*Particle* $\boldsymbol{q_A}$	*Particle* $\boldsymbol{q_B}$
N_1	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_B$
N_2	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_A$	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_B$
N_3	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_A$	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_B$
N_4	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_A$	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_B$
N_5	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_B$
N_6	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_A$	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_B$
N_7	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_A$	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_B$
N_8	$\vert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_A$	$\vert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_B$

As in section 1.2, we define $P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+)$ to be the probability that Alice measures particle q_A to be at location $\hat{\mathbf{a}}+$ on her detection screen and Bob measures particle q_B to be at location $\hat{\mathbf{b}}+$ on his detection screen. We similarly define the probabilities for all other combinations of detection locations. It is relatively easy to calculate all these probabilities in terms of the values N_i from table 1.1,[28] or alternatively by simply measuring the frequency of these different outcomes for where Alice and Bob detect

[26] N has to be large since a frequentist definition of probability is being assumed.

[27] The notation $\sum_{i=1}^{8} N_i$ is shorthand for $N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8$.

[28] e.g. $P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) = \frac{N_3+N_4}{N}$, $P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) = \frac{N_2+N_4}{N}$, $P_{AB}(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+) = \frac{N_3+N_7}{N}$

their particles. Note that the values of N_i will be unknown, but on the assumption that there is a fact of the matter of which states in table 1.1 obtain, and on the assumption that the states to which the N_i correspond exhaust all the possible states for Alice's and Bob's particles, we can show that[29]

$$P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) \leq P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+). \tag{1.14}$$

This inequality is known as *Bell's inequality*, and it follows from Einstein's locality principle. However, as already mentioned, when this experiment is actually performed, we can choose the three axes so that Bell's inequality is violated. Nevertheless, it also turns out that this violation of Bell's inequality is entirely predictable if we assume that the state of the spin singlet consisting of the two particles q_A and q_B is given by the Bell state:

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\;|\hat{\mathbf{a}}-\rangle_B - \,|\hat{\mathbf{a}}-\rangle_A\;|\hat{\mathbf{a}}+\rangle_B). \tag{1.15}$$

When the spin singlet is in the $|\Psi_{\text{Bell}}\rangle$-state, it can be shown that

$$P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) = \frac{1}{2}\sin^2(\theta/2) \tag{1.16}$$

where θ is the angle between the $\hat{\mathbf{a}}$-axis and $\hat{\mathbf{b}}$-axis.[30] Then taking the angle between the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$-axes to be 90°, and the $\hat{\mathbf{c}}$-axis to be at 45° to both the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$-axes, we

[29]This inequality follows since

$$P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) = \frac{N_3 + N_4}{N} \leq \frac{N_2 + N_4 + N_3 + N_7}{N} = P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+).$$

[30]To see why this is, let $P_A(\hat{\mathbf{a}}+)$ be the probability that Alice would detect her particle at location $\hat{\mathbf{a}}+$ given that she is making a measurement along the $\hat{\mathbf{a}}$-axis, and let $P_{BA}(\hat{\mathbf{b}}+|\hat{\mathbf{a}}+)$ be the probability that Bob will detect his particle at location $\hat{\mathbf{b}}+$ given that he is making a measurement along the $\hat{\mathbf{b}}$-axis and Alice has detected her particle at location $\hat{\mathbf{a}}+$. Given that the joint state of the particles is given by equation (1.8), $P_A(\hat{\mathbf{a}}+) = \frac{1}{2}$. But also note that if Alice has detected her particle at location $\hat{\mathbf{a}}+$, then Bob's particle must be in state $|\hat{\mathbf{a}}-\rangle$. From the Born Rule (see page 12) and equation (1.2a) it follows that

$$P_{BA}(\hat{\mathbf{b}}+|\hat{\mathbf{a}}+) = |\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle\,|^2 = \sin^2(\theta/2).$$

would find that $P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) = \frac{1}{4}$ and $P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+) = 0.1464\ldots$, and so Bell's inequality would be violated if we assumed that the probability of each outcome is determined by the Bell state (1.8).

Therefore,

$$P_{AB}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) = P_A(\hat{\mathbf{a}}+)P_{BA}(\hat{\mathbf{b}} + |\hat{\mathbf{a}}+) = \frac{1}{2}\sin^2(\theta/2).$$

1.5 Isolating the Culprit

Given the experimental violation of Bell's inequality, a strategy some philosophers of physics take is to reexamine the assumptions that lead to Bell's Inequality. Because of the violation of Bell's inequality, one of these assumptions will have to be discarded. The false assumption that is used to prove Bell's Inequality is sometimes referred to as *the culprit*.[31] We therefore need to isolate the culprit, that is, we need to decide which assumption we should discard while keeping in mind that we wish to maintain a theory that is compatible with the experimental findings of quantum physics and special relativity.

Shimony noticed that there are two key assumptions in the proof of Bell's Inequality that might be identified as the culprit. He refers to one assumption as Outcome Independence (OI), and to the other assumption as Parameter Independence (PI).[32] Shimony argued that if we only denied OI, then the proof of Bell's Inequality would fail to go through. Yet by continuing to assume PI, there is a sense in which special relativity is not obviously violated. Shimony therefore thought that denying OI and assuming PI was sufficient to ensure peaceful coexistence between quantum theory and special relativity. In other words, Shimony thought OI was the culprit.

[31]This is the terminology Butterfield uses following Abner Shimony. e.g. see Jeremy Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 2017, 1, eprint: arXiv:1710.07844.

[32]See A. Shimony, "Events and processes in the quantum world," in *Search for a Naturalistic World View: Volume II: natural science and metaphysics* (1986; Cambridge: Cambridge University Press, 1993), 146–147.

1.6 Parameter Independence

To explain Shimony's[33] notion of Parameter Independence, we suppose we have an experimental setup similar to the experimental setup described in the previous section. Thus, we suppose there are two particles labeled q_A, and q_B, and that a measurement can be made on particle q_A at one location (e.g. Alice's laboratory), and a measurement can be made on particle q_B at some other location (e.g. Bob's laboratory). We will assume that Alice can make a choice of one of n measurements to be made. These are labeled $a_1, \ldots, a_n$. For example, a_1 might be a measurement of q_A's spin along the z-axis, whereas a_2 might be the measurement of q_A's spin along an axis that is at a $45°$ angle to the z-axis etc. We use the variable x to denote Alice's choice so that $x = a_i$ for some $i \in \{1, \ldots, n\}$. If Alice chooses to make measurement a_i (i.e. $x = a_i$), the measurement outcome is labeled A_i, and this outcome can take values $+1$ or -1. For example, Alice could use the convention in which $+1$ corresponds to a spin up outcome, and -1 corresponds to a spin down outcome. We will use the variable X to denote the measurement outcome Alice obtains, so for example, if Alice chooses to make the a_1 measurement so that $x = a_1$ and obtains the outcome $A_1 = 1$, then $X = 1$. Similarly, we use the notation b_i, y, and B_i, Y to correspond to the measurement choices and measurement outcomes for Bob.

We now suppose that there is a complete state $\lambda \in \Lambda$ describing both q_A and q_B that is independent of Alice and Bob's measurement choices, but that encodes all other features that would influence the corresponding measurement outcomes. Here, the domain Λ of all such complete states will depend on how the two particles are

[33] See Shimony, "Events and processes in the quantum world," 146–147 and Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 7–9.

prepared and the model we are assuming. We also assume that q_A and q_B are initially coupled together in such a way that Alice and Bob would always get opposite results when they made their measurements in the same direction. For instance, for $n = 3$, we might assume a model in which

$$\Lambda = \{(A_1, A_2, A_3, B_1, B_2, B_3) : A_1,\ A_2,\ A_3 = \pm 1,\ B_i = -A_i\}. \tag{1.17}$$

In this case, $\lambda \in \Lambda$ would fully determine Alice and Bob's measurement outcomes along the three axes. This would be like the model described in the proof of Bell's Inequality with all the states of Λ being described in table 1.1 of section 1.2.

However, in general, we don't insist on such determinism. Rather, we suppose that given a complete state $\lambda \in \Lambda$, and given that Alice makes a measurement choice x and Bob makes a measurement choice y, then there will be a probability $P_{\lambda,x,y}(X,Y)$ representing the probability Alice gets outcome X and Bob gets outcome Y. In deterministic models, $P_{\lambda,x,y}(X,Y)$ will have values restricted to either 0 or 1. In non-deterministic models, there will have to be some situations when $P_{\lambda,x,y}(X,Y)$ has a value strictly between 0 and 1. For example, if we assume the Copenhagen interpretation model, we could take λ to be the Bell state (1.8). Then it follows from equation (1.11) that as long as Alice's and Bob's measurement choices x and y are in the same direction, then $P_{\lambda,x,y}(1,-1) = 1/2$. Incidentally, we also note that equation (1.11) implies the domain Λ consists of a single state:

$$\Lambda = \left\{ \frac{1}{\sqrt{2}} \big(\left|\hat{\mathbf{a}}+\right\rangle_A \left|\hat{\mathbf{a}}-\right\rangle_B - \left|\hat{\mathbf{a}}-\right\rangle_A \left|\hat{\mathbf{a}}+\right\rangle_B \big) \right\}. \tag{1.18}$$

In both models (1.17) and (1.18), we see that if we define

$$P_{A,\lambda,x,y}(X) = P_{\lambda,x,y}(X,1) + P_{\lambda,x,y}(X,-1), \tag{1.19}$$

$$P_{B,\lambda,x,y}(Y) = P_{\lambda,x,y}(1,Y) + P_{\lambda,x,y}(-1,Y), \tag{1.20}$$

then $P_{A,\lambda,x,y}(X)$ is independent of Bob's choice of measurement y, and $P_{B,\lambda,x,y}(Y)$ is independent of Alice's choice of measurement x.[34] In other models, however, it's possible that such independence does not hold. So to distinguish between such possibilities, we say a model satisfies *Parameter Independence* (PI) if and only if $P_{A,\lambda,x,y}(X)$ is independent of y, and $P_{B,\lambda,x,y}(Y)$ is independent of x. In particular, PI holds in the model (1.17) in the Copenhagen interpretation model (1.18). In other words, PI holds if and only if (1.19) and (1.20) hold for all λ, x, y, X, and Y. If PI fails to hold in a model, we say that the model satisfies *Parameter Dependence* (PD).

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[34]To see that this is true for model (1.17), it is obvious that $P_{A,\lambda,x,y}(X) = 1$ or 0 regardless of what y is. As for model (1.18), it is straightforward to show that $P_{A,\lambda,x,y}(X) = 1/2$ and $P_{B,\lambda,x,y}(Y) = 1/2$ for any X, Y. E.g. for $x = \hat{\mathbf{a}}$ and $y = \hat{\mathbf{b}}$, by (1.11), we can assume the two particles are in the state

$$|\zeta\rangle = \frac{1}{\sqrt{2}}\big(|\hat{\mathbf{b}}+\rangle_A\, |\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A\, |\hat{\mathbf{b}}+\rangle_B \big).$$

Since the inner product on the composite system is given by $\langle\xi'|\xi\rangle = \langle\psi'|\psi\rangle_A \langle\chi'|\chi\rangle_B$ for $|\xi\rangle = |\psi\rangle_A |\chi\rangle_B$ and $|\xi'\rangle = |\psi'\rangle_A |\chi'\rangle_B$, it follows that

$${}_A\langle\hat{\mathbf{a}}+|\, {}_B\langle\hat{\mathbf{b}}\pm|\zeta\rangle = \mp\frac{1}{\sqrt{2}} \langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}\mp\rangle_A .$$

Therefore, by the Born Rule (see page 12)

$$P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-) = \frac{1}{2}\left|\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}-\rangle_A\right|^2 + \frac{1}{2}\left|\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}+\rangle_A\right|^2.$$

But since

$$|\hat{\mathbf{a}}+\rangle_A = \langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle_A\, |\hat{\mathbf{b}}+\rangle_A + \langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}+\rangle_A\, |\hat{\mathbf{b}}-\rangle_A$$

it follows that

$$\left|\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}+\rangle_A\right|^2 + \left|\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}-\rangle_A\right|^2 = 1.$$

Therefore,

$$P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) + P_{\lambda,x,y}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-) = \frac{1}{2}.$$

1.7 Parameter Dependence in the Pilot Wave Model*

One model which exhibits PD is the *pilot wave interpretation* of quantum mechanics. In this interpretation, it is assumed that at any instant of time t, the particles q_A and q_B will have definite positions $\mathbf{x}_A$ and $\mathbf{x}_B$ and definite momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ respectively. But in addition to the positions and momenta of the particles, it is also assumed that there is a so-called *pilot wave*

$$\psi(\mathbf{x}_A, \mathbf{x}_B, t) = r(\mathbf{x}_A, \mathbf{x}_B, t)e^{iS(\mathbf{x}_A,\mathbf{x}_B,t)} \tag{1.21}$$

where $r(\mathbf{x}_A, \mathbf{x}_B, t) > 0$ is the modulus of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$, and the real-valued function $S(\mathbf{x}_A, \mathbf{x}_B, t)$ is the complex phase[35] of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$. The time evolution of the pilot wave is deterministically governed by the Schrödinger equation, and the phase $S(\mathbf{x}_A, \mathbf{x}_B, t)$ relates the positions $\mathbf{x}_A$ and $\mathbf{x}_B$ to the momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ via the gradient of S:

$$\mathbf{p}_A = \boldsymbol{\nabla}_A S(\mathbf{x}_A, \mathbf{x}_B), \qquad \mathbf{p}_B = \boldsymbol{\nabla}_B S(\mathbf{x}_A, \mathbf{x}_B). \tag{1.22}$$

In other words, if we fix $\mathbf{x}_B$ and consider S to be just a function of $\mathbf{x}_A$, then the momentum $\mathbf{p}_A$ is in the direction and has the magnitude of the steepest ascent of S considered as a function of $\mathbf{x}_A$. The momentum $\mathbf{p}_B$ is determined similarly.

In reality, we don't know the exact positions of all the particles, but based on what we know about an experimental setup, we can average over our uncertainty and recover exactly the same predictions that quantum mechanics would make.[36] So for instance, our knowledge of the experimental setup above should enable us to know that q_A and

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[35]The *phase* of a complex number z is the angle θ (in radians) such that $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ is a real number called the *modulus*. Since $\cos\theta + i\sin\theta = e^{i\theta}$, it follows from (1.21) that $r(\mathbf{x}_A, \mathbf{x}_B, t) > 0$ is the modulus of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$, and $S(\mathbf{x}_A, \mathbf{x}_B, t)$ is the phase of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$.

[36]See David Bohm, "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I," *Physical review* 85, no. 2 (1952): 166–179 and David Bohm, "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II," *Physical review* 85, no. 2 (1952): 180–193.

q_B are contained within a region V, and the experimental setup should also enable us to work out the probability $p(V_i, V_j)$ that particle q_A will be in a region V_i, and q_B will be in a region V_j, where the V_i are small non-overlapping regions such that $V = \bigcup_i V_i$. If we are interested in some physical quantity $O(\mathbf{x}_A, \mathbf{x}_B)$ that depends on the positions $\mathbf{x}_A$ and $\mathbf{x}_B$ of the two particles, then when the regions V_i are sufficiently small so that almost everywhere,[37] $O(\mathbf{x}_i, \mathbf{x}_j)$ varies negligibly for any $\mathbf{x}_i \in V_i$ and $\mathbf{x}_j \in V_j$, the average value (also known as the *expectation value*)

$$\langle O \rangle = \sum_{i,j} p(V_i, V_j) O(\mathbf{x}_i, \mathbf{x}_j) \tag{1.23}$$

calculated in the pilot wave interpretation turns out to be the same as the expectation value[38] for O predicted by standard quantum theory.[39]

To see why PI fails to hold in the pilot wave interpretation, we first note that since the pilot wave interpretation makes the same predictions as quantum mechanics when averaged over all the hidden variables, the violation of Bell's inequality (1.14) implies there must be some hidden variable λ and choices of measurement directions $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$ such that

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) > P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) + P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+).^{40} \tag{1.24}$$

[37] When $O(\mathbf{x}_A, \mathbf{x}_B)$ is discontinuous, there may be some $V_i \times V_j$ cells in which $O(\mathbf{x}_A, \mathbf{x}_B)$ varies non-negligibly for $(\mathbf{x}_A, \mathbf{x}_B) \in V_i \times V_j$. But we assume that the sum of the $p(V_i, V_j) \max_{(\mathbf{x}_i, \mathbf{x}_j) \in V_i \times V_j} |O(\mathbf{x}_i, \mathbf{x}_j)|$ terms for such cells is negligible.

[38] See (2.1) for the definition of the expectation value of an observable in standard quantum theory.

[39] In this explanation, I've refrained from using measure theory, but basically this explanation is saying that when we construct a measure μ on $V \times V$ based on our knowledge of the experimental setup, $\int_{V \times V} O(\mathbf{x}_i, \mathbf{x}_j) \mathrm{d}\mu$ will be the same as the expectation value for O predicted by standard quantum theory.

Since physics in the pilot wave interpretation is deterministic, probabilities must be either 0 or 1. Therefore, the only way (1.24) can be satisfied is for

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)=1, \tag{1.28}$$

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+)=0, \tag{1.29}$$

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+)=0. \tag{1.30}$$

We suppose that PI holds, and we will try to arrive at a contradiction. If both Alice and Bob make their measurement in the $\hat{\boldsymbol{c}}$-direction, there are two possibilities: either Alice measures q_A to be in the state $\hat{\mathbf{c}}-$ and Bob measures q_B to be in the state $\hat{\mathbf{c}}+$, or Alice measures q_A to be in the state $\hat{\mathbf{c}}+$ and Bob measures q_B to be in the state $\hat{\mathbf{c}}-$. So expressed in terms of probabilities, these two possibilities are equivalent to either

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+)=1 \qquad \text{and} \qquad P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-)=0. \tag{1.31}$$

or

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-)=1 \qquad \text{and} \qquad P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+)=0 \tag{1.32}$$

Let's first consider case (1.31). Note that

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}-)=0. \tag{1.33}$$

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[40]To see why this is, we consider the observable $O_{\hat{\mathbf{a}}+,\hat{\mathbf{b}}+}(\mathbf{x}_A,\mathbf{x}_B)$ which returns 1 if and only if the particle q_A will end up at location $\hat{\mathbf{a}}+$ and particle q_B will end up at location $\hat{\mathbf{b}}+$ as determined by the pilot wave, and otherwise $O_{\hat{\mathbf{a}}+,\hat{\mathbf{b}}+}(\mathbf{x}_A,\mathbf{x}_B)$ returns 0. Then by (1.23),

$$P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)=\sum_{\lambda} p_\lambda P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) \tag{1.25}$$

where $P_{AB}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)$ is the probability described in section 1.4, λ ranges over the pairs of indices (i,j) corresponding to the cell $V_i\times V_j$, $p_\lambda=p(V_i,V_j)$ for $\lambda=(i,j)$, and we have used the fact that $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)=O_{\hat{\mathbf{a}}+,\hat{\mathbf{b}}+}(\mathbf{x}_i,\mathbf{x}_j)$ for almost all $(\mathbf{x}_i,\mathbf{x}_j)\in V_i\times V_j$. Now if (1.24) is false, then for all λ, $\hat{\boldsymbol{a}}$, and $\hat{\boldsymbol{b}}$,

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)\le P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+) \tag{1.26}$$

Therefore, since we are assuming PI,

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}-)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}-,\hat{\mathbf{c}}-)=0. \tag{1.34}$$

In particular,

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}-)=0. \tag{1.35}$$

But by (1.28), we know that

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-)=1, \tag{1.36}$$

so using this together with PI, we must have

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}-)=1. \tag{1.37}$$

But by (1.29) and (1.35)

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}-)=0. \tag{1.38}$$

Since (1.37) contradicts (1.38), the assumption (1.31) must be false if PI is to hold.

So we now consider the alternative case when (1.32) holds. We will again see that this assumption leads to a contradiction. First note that

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}-)=0. \tag{1.39}$$

By PI

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}-)=0. \tag{1.40}$$

In particular,

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}+)=0. \tag{1.41}$$

Therefore, since $p_\lambda \geq 0$, it follows that

$$\sum_\lambda p_\lambda P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)\leq \sum_\lambda p_\lambda P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+)+\sum_\lambda p_\lambda P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+). \tag{1.27}$$

But by (1.28), we know that

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}-,\hat{\mathbf{b}}+)=1, \tag{1.42}$$

so using this together with PI, we must have

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}+)=1. \tag{1.43}$$

But by (1.30) and (1.41)

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}+)=0. \tag{1.44}$$

Since (1.43) contradicts (1.44), the assumption (1.32) must also be false if PI is to hold. So we can only conclude that PI fails to hold in the pilot wave interpretation. But we can conclude even more than that: any deterministic hidden variable model that gives the same predictions as quantum mechanics when averaged over the hidden variables must violate PI.

Now the violation of PI in the pilot wave interpretation does not sit easily with Einstein's theory of relativity, for according to Einstein's theory, it should be impossible to send signals faster than the speed of light. However, if PI is violated, then if Alice happened to know what λ was for each run of the experiment, and if Bob made the same measurement, then because the distribution of Alice's outcomes will depend on Bob's choice of measurement, with enough runs of the experiment, Alice should be able to work out what measurement Bob is making. And this should be possible even if Alice and Bob are separated by many light years. So it seems faster than light communication would be possible. The only thing preventing such communication would be Alice's lack of knowledge of λ.

One might still respond to this argument against PD by saying that in reality, Alice does not know anything about λ and hence it won't be possible for Bob to send signals faster than the speed of light to Alice. However, it would nevertheless be very strange if the validity of Einstein's theory of relativity hung on what human beings were capable of knowing rather than on the laws that actually governed physical reality itself. But even if it was metaphysically impossible for human beings to know λ, this would not allay the fears that adherents of Einstein's theory of relativity would have against PD. For the real issue is not so much the possibility that Alice could translate a message that was transmitted from Bob at superluminal speed. Rather, the issue is that Bob could do something whose effect was propagated at superluminal speed. If PI fails, then superluminal propagation of effects will be possible. It's just that if Alice is to know the cause of this effect, she will need to know λ. But even if she doesn't know λ, it still seems as though some unknown effect has been transmitted to her superluminally.

It would therefore follow from (1.27) and (1.25) that Bell's inequality (1.14) would hold. But since (1.14) is experimentally violated, it is not the case that (1.26) holds for all λ, $\hat{\boldsymbol{a}}$, and $\hat{\boldsymbol{b}}$. Hence, there must be some λ, $\hat{\boldsymbol{a}}$, and $\hat{\boldsymbol{b}}$, for which (1.24) holds.

1.8 Outcome Independence

Although a PI violation can account for the violation of Bell's Inequality, this is not the only possible culprit to consider. Another assumption of Bell's Inequality that might be violated is *Outcome Independence* (OI). Outcome independence is the assumption

$$P_{\lambda,x,y}(X,Y) = P_{A,\lambda,x,y}(X) \cdot P_{B,\lambda,x,y}(Y), \tag{1.45}$$

where $P_{A,\lambda,x,y}(X)$ and $P_{B,\lambda,x,y}(Y)$ are defined in equations (1.19) and (1.20) respectively. So the difference between OI and PI is the following: with OI, given Alice and Bob's choice of measurements x and y, and the hidden variable λ, Alice and Bob's measurement outcomes will be statistically independent from one another, whereas with PI, given Alice's choice of measurement x and the hidden variable λ, whatever measurement choice Bob makes, this will have absolutely no effect on the probabilities of Alice's measurement outcomes, and similarly, Alice's choice of measurement will have absolutely no effect on the probabilities of Bob's measurement outcomes.

Now we can see that if OI holds in any model which gives the same predictions as standard quantum theory when averaged over the hidden variables, then PI must be violated in such a model. For if both PI and OI hold, then for any measurement choices $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and hidden variable λ, we have

$$\begin{aligned}
P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+) &= P_{A,\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+)\cdot P_{B,\lambda,\hat{a},\hat{c}}(\hat{\mathbf{c}}+)\\
&= \Big(P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+) + P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}-)\Big)\cdot\Big(P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+) + P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}-,\hat{\mathbf{c}}+)\Big)\\
&= \Big(P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+) + P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}-)\Big)\cdot\Big(\cancelto{0}{P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}+)} + P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+)\Big)\\
&= \Big(P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-)\Big)\cdot P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+)\\
&\geq P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)\cdot P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+)
\end{aligned} \tag{1.46}$$

Similarly, we have

$$\begin{aligned}
P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+) &= P_{A,\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+)\cdot P_{B,\lambda,\hat{c},\hat{b}}(\hat{\mathbf{b}}+)\\
&= \Big(P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}-)\Big)\cdot\Big(P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}+)\Big)\\
&= \Big(\cancelto{0}{P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}+)} + P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-)\Big)\cdot\Big(P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}-,\hat{\mathbf{b}}+)\Big)\\
&= P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-)\cdot\Big(P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}-,\hat{\mathbf{b}}+)\Big)\\
&\geq P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-)\cdot P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+).
\end{aligned} \tag{1.47}$$

But since the hidden variable λ is assumed to be independent of Alice and Bob's measurement, and since Alice and Bob will always get opposite results when they make the same choice of measurement, it follows that

$$P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}+,\hat{\mathbf{c}}-) + P_{\lambda,\hat{c},\hat{c}}(\hat{\mathbf{c}}-,\hat{\mathbf{c}}+) = 1 \tag{1.48}$$

Therefore, putting (1.46), (1.47), and (1.48) together, we have

$$P_{\lambda,\hat{a},\hat{c}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+) + P_{\lambda,\hat{c},\hat{b}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+) \geq P_{\lambda,\hat{a},\hat{b}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+). \tag{1.49}$$

We have thus proved that OI and PI together imply Bell's Inequality (1.24). But since Bell's Inequality does not hold in reality, it follows that if OI is always true, then PI must be violated.

In the case of deterministic models in which there are outcomes,[41] OI necessarily holds. To see why, we first note that for deterministic models, either $P_{\lambda,x,y}(X, Y) = 1$ or $P_{\lambda,x,y}(X, Y) = 0$. When $P_{\lambda,x,y}(X, Y) = 1$, then by (1.19), $P_{A,\lambda,x,y}(X) = 1$, and by (1.20), $P_{B,\lambda,x,y}(Y) = 1$, so (1.45) is seen to hold in this case. On the other hand, when $P_{\lambda,x,y}(X, Y) = 0$, if $P_{A,\lambda,x,y}(X) = 1$, then by (1.19), $P_{\lambda,x,y}(X, -Y) = 1$ so that by (1.20), $P_{B,\lambda,x,y}(-Y) = 1$, and hence $P_{B,\lambda,x,y}(Y) = 0$ in which case (1.45) holds. And similarly, if $P_{B,\lambda,x,y}(Y) = 1$, by (1.20), $P_{\lambda,x,y}(-X, Y) = 1$ so that by (1.19), $P_{A,\lambda,x,y}(-X) = 1$, and hence $P_{A,\lambda,x,y}(X) = 0$, so again (1.45) holds. And (1.45) obviously holds when $P_{A,\lambda,x,y}(X) = P_{B,\lambda,x,y}(Y) = 0$. It therefore follows that OI holds in any deterministic model. In particular, OI holds under the pilot wave interpretation.

When it comes to the Copenhagen interpretation of quantum physics, however, OI fails to hold. For instance, if $x = y = \hat{\mathbf{a}}$, then $P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{a}}}(\hat{\mathbf{a}}+, \hat{\mathbf{a}}+) = 0$, but $P_{A,\lambda,\hat{\mathbf{a}},\hat{\mathbf{a}}}(\hat{\mathbf{a}}+) = P_{B,\lambda,\hat{\mathbf{a}},\hat{\mathbf{a}}}(\hat{\mathbf{a}}+) = 1/2$.[42] Hence, OI fails.

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[41] For deterministic models in which there are outcomes, the probability of a particular outcome given a complete description of a system will be either 0 or 1. For models in which there are no outcomes, such as the many-worlds interpretation which will be discussed in the next chapter, we can't speak of outcome independence.

[42] See footnote 34.

1.9 Peaceful Coexistence of Special Relativity and Quantum Physics

So far we've seen that PI holds in the Copenhagen interpretation (section 1.6), that OI holds in the pilot wave interpretation (section 1.8), and that due to the violation of Bell's inequality, it is not possible for both OI and PI to hold in a model that is consistent with physical reality (section 1.8). Nevertheless, as long as PI holds, the failure of OI does not enable Bob to send messages to Alice faster than light because Bob only has control over the measurement choice he makes rather than the outcome he observes. Assuming Bob's mental states have no effect on the measurement outcome, there is nothing he can do to influence his outcome, so although Alice will be able to work out Bob's measurement outcome if she already happens to know which choice of measurement he has made, she will not be able to work out which measurement Bob makes (or even whether Bob has made a measurement at all) by measuring the outcome of her particle. For Shimony,[43] this inability to send super-luminal messages between Alice and Bob when PI holds and OI is violated was deemed sufficient for the theories of standard quantum physics and special relativity to peacefully coexist.

However, Butterfield is not satisfied with Shimony's solution to peaceful coexistence.[44] Firstly, he notes that proofs of non-super-luminal signaling[45] make no assumptions about spacetime locations. One would have thought that any proof that super-luminal signalling between two points is impossible would have to show that a signal cannot

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[43] See Shimony, "Events and processes in the quantum world," 146–147.

[44] See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," p. 12.

[45] e.g. see Michael Redhead, *Incompleteness, nonlocality, and realism : a prolegomenon to the philosophy of quantum mechanics* (Oxford : New York: Clarendon Press ; Oxford University Press, 1987), p. 113–116; David Bohm and B. J Hiley, *The undivided universe : an ontological interpretation of quantum theory* (London: Routledge, 1993), p. 139–140

be transmitted from one point to the other in less time than the time it takes light to travel between the two points. But if nothing is said about the location of these two points or what is so special about the speed of light compared to the speed of any other particle, then there does not seem to be enough information in the premises to draw the desired conclusion that super-luminal signaling is impossible in quantum physics.

Secondly, Butterfield notes that Shimony thinks peaceful coexistence of quantum physics and special relativity is guaranteed by the denial of OI and the acceptance of PI. However, OI itself depends on the (often) rather vague notion of what an outcome really is. For instance, in the many-worlds interpretation, it is not clear that there are any outcomes at all. Rather, there is just a universal quantum state that tells us the probability of certain outcomes, if there were such things as outcomes – it doesn't tell us that there really are any outcomes. In the next chapter, we will discuss the many-worlds interpretation and why denying the reality of outcomes is such an unsatisfactory way of guarantying peaceful coexistence of quantum physics and special relativity. But as is clear from other interpretations of quantum physics, the notion of what an outcome is doesn't have to be vague. In the pilot wave interpretation of quantum physics, it is very clear what an outcome of an experiment is since all the particles have definite positions and momenta. Because of this, the pointers and displays of measuring devices which are made up of particles will have definite readouts which will correspond to the definite positions of particles being measured (assuming the measurement device is working properly). So unlike the many-worlds interpretation, measurements in the pilot wave interpretation have definite outcomes, and hence there is only a single world in the pilot wave interpretation of quantum physics. But as we've just seen, the problem with the pilot wave interpretation is the violation of PI.

Thus, a satisfactory account of the peaceful coexistence of quantum physics and special relativity requires an interpretation of quantum physics in which not only PI holds, but also an interpretation of quantum physics that has special relativity built into it (thus satisfying Butterfield's first objection), and in which we can make sense of what it means to be an outcome (thus satisfying Butterfield's second objection).

To fully address Butterfield's first objection would require quantum field theory, and this would be beyond the scope of this dissertation. But a more modest aspiration that would go some way to address Butterfield's first objection would be to insist on an interpretation of quantum physics which has a clear notion of outcome and which also has a property known as Lorentz invariance. This provides a motivation for the consideration of Kent's theory of quantum physics that has this property of Lorentz invariance. But before we consider Kent's theory, in the next chapter we will examine the many-worlds interpretation of quantum physics discussing both its appeal and its drawbacks.

1.10 Summary

In this chapter, we have considered the EPR-Bohm paradox and the problem it raises of how to account for the mysterious correlations between the measurement outcomes of two observers measuring the spin properties of spin singlets. The question the

EPR-Bohm paradox raises is how one can account for these correlations in a way that is consistent with special relativity and the predictions of standard quantum theory.

The Copenhagen interpretation in which the act of observation causes the quantum state to collapse does not seem to be consistent with special relativity. Theories such as the GRW interpretation which posit the spontaneous collapse of quantum states make predictions that violate the predictions of standard quantum theory, and to date, there is no experimental evidence for such violations.

If the proposed theory posits the existence of hidden variables in addition to the traditional quantum state of standard quantum theory, then it is not possible for the theory to satisfy both Parameter Independence (PI) and Outcome Independence (OI) because otherwise, Bell's inequality would hold in this theory, and this is not consistent with the violation of Bell's inequality in physical realty. A denial of PI would allow for the superluminal propagation of signals if one knew the hidden variables, and so without a compelling reason for why these hidden variables must be unknown, denying PI does not sit easily with special relativity in which superluminal signalling is impossible. The denial of OI seems more promising, but this approach by itself is not sufficient to provide an adequate account of the mysterious correlations of the EPR-Bohm paradox since it does not address the thorny issue of what we mean by outcome or whether there is in fact any physical reality to outcomes at all.

One theory that denies the reality of experimental outcomes is the many-worlds interpretation. In the next chapter, we will discuss the many-worlds interpretation in some detail, and we will see why it does not provide a satisfactory account of the correlations of the EPR-Bohm paradox.

Chapter 2

The Measurement Problem and the Many-Worlds Interpretation

In the previous chapter we discussed the EPR-Bohm paradox which exhibits the mysterious correlations of measurement outcomes made on spin singlets by two different observers. We saw that the standard Copenhagen interpretation of quantum physics (which posits that upon measurement an instantaneous state collapse to a definite measurement outcome occurs) must be rejected if one is to accept Einstein's theory of special relatively. Furthermore, we saw that the experimental violation of Bell's inequality implies that a local hidden variables theory must also be rejected. We also discussed Shimony's suggestion that it would be acceptable to have non-local hidden variables so long as they satisfied Parameter Independence (PI) but failed to satisfy Outcome Independence (OI), since Bell's inequality wouldn't then follow from these two assumptions, and the denial of OI is a sufficiently weak form of non-locality that it doesn't imply superluminal singaling. Thus, Shimony argued that accepting PI and denying OI was sufficient for the peaceful coexistence of quantum physics and special relativity. However, Shimony's solution rests on there being a clear notion of what an outcome is. But the question of what an outcome is and even whether there are such things as measurement outcomes is very controversial and forms an important part of what is known as the measurement problem.

The measurement problem actually consists of three parts: (1) the preferred basis problem, (2) the nonobservability of interference at the macroscopic level, and (3) the problem of outcomes.[1] Decoherence theory is able to resolve parts (1) and (2) of the measurement problem, and following Schlosshauer, we will explain how this is done, but as we will see (also following Schlosshauer), decoherence theory is not able to resolve the part (3) of the measurement problem, the problem of outcomes.

The problem of outcomes arises when we suppose that the quantum state gives a complete description of the system it is describing.[2] Throughout this chapter we will make this assumption, that is, we will assume there are no hidden variables, whether local or otherwise; rather, a quantum state whose evolution is described by the Schrödinger equation provides the most complete description possible of a physical system.

In this chapter, we will be discussing the measurement problem in some detail, and in particular, we will be examining the many-worlds interpretation of quantum physics which attempts to sidestep the problem of outcomes by refusing to acknowledge the reality of outcomes. Although the many-worlds interpretation is mathematically appealing, we'll see that from a philosophical point of view, it is woefully inadequate. Hence, any account of the mysterious correlations of the EPR-Bohm paradox (as described in chapter 1) that depended on the many-worlds interpretation would be unsatisfactory. Nevertheless, an understanding of the many-worlds interpretation and

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[1]See Maximilian Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Berlin: Springer-Verlag, 2007), 50

[2]We need to make a small caveat here – the example of the quantum states we have described so far such as $|\hat{\mathbf{a}}\rangle$ and $|\Psi_{\mathrm{Bell}}\rangle$ only give a complete description of the spin state of a system. These states don't say anything about the position of the system's particles. Nevertheless, we could in principle write down an expression for the state that did include this positional information. We will therefore think of the expressions such as $|\hat{\mathbf{a}}\rangle$ and $|\Psi_{\mathrm{Bell}}\rangle$ as abbreviations for quantum states that also include a complete specification of the particle positions which in the context of quantum physics will be a wave function that determines the probability the particles can be detected in a particular region.

how it relates to the problem of outcomes will prove helpful when we come to evaluate Kent's theory of quantum physics in the following chapter.

2.1 A preliminary consideration of the many-worlds interpretation

In the previous chapter, we saw that in order to calculate the probabilities of the various measurement outcomes of an experiment, we can do this by positing that the state describing the item being measured is in a superposition of two states each of which corresponds to a definite measurement outcome. For example, in the case of a spin singlet, the quantum state of the two particles can be expressed as the Bell state

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\;|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\;|\hat{\mathbf{a}}+\rangle_B). \qquad \text{(1.8 revisited)}$$

which is a superposition of the two definite measurement outcome states $|\hat{\mathbf{a}}+\rangle_A\;|\hat{\mathbf{a}}-\rangle_B$ and $|\hat{\mathbf{a}}-\rangle_A\;|\hat{\mathbf{a}}+\rangle_B$. But at this stage in our line of reasoning, it is too early to resort to a many-worlds interpretation of the Bell state where the first component corresponds to a world in which Alice detects her particle at location $\hat{\mathbf{a}}+$ and Bob detects his particle at location $\hat{\mathbf{a}}-$, and where the second component corresponds to a world in which Alice detects her particle at location $\hat{\mathbf{a}}-$ and Bob detects his particle at location $\hat{\mathbf{a}}+$. Such an interpretation would be premature because as mentioned on page 21, for any other axis $\hat{\mathbf{b}}$, the transformation rules in equation (1.2) imply that

$$\begin{aligned}&\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\;|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\;|\hat{\mathbf{a}}+\rangle_B)\\ &\qquad\qquad = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{b}}+\rangle_A\;|\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A\;|\hat{\mathbf{b}}+\rangle_B).\end{aligned} \qquad \text{(1.11 revisited)}$$

Similarly, given the transformation rules in equation (1.2), we should resist the temptation to interpret a state of the form $\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ as representing two worlds, one in which the particle is in the state $|\hat{\mathbf{a}}+\rangle$, and another in which the particle is in the state $|\hat{\mathbf{a}}-\rangle$. For according to equation (1.2a), the much more obvious interpretation is that this state just describes one world in which the particle is in the state $|\hat{\mathbf{b}}+\rangle$ where the angle between the $\hat{\mathbf{a}}$ and the $\hat{\mathbf{b}}$ axis is $90°$.[3]

In order to make a case for a many-worlds interpretation, we need to discuss decoherence theory. Decoherence theory considers how a system interacts with its environment, and it allows us to understand what kinds of measurements can be made on the system. In order to discuss decoherence theory and its relevance to the many-worlds interpretation, we first need to introduce the mathematical formalism of quantum mechanics.

[3]This is because when $\theta = 90°$, $\sin(\theta/2) = \cos(\theta/2) = \frac{1}{\sqrt{2}}$, so $|\hat{\mathbf{b}}+\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ in equation (1.2a) with $\theta = 90°$.

2.2 The mathematical formalism of quantum mechanics

Given a possible kind of measurement (e.g. measuring the spin of a particle along a particular axis), there will be a mathematical object called an *observable* which encodes all the possible measurement outcomes for this particular kind of measurement. The precise mathematical definition of an observable is as follows: an observable of a physical system is a Hermitian operator that acts on the Hilbert space of states describing the physical system. In order to understand what this definition means, there are a number of things we need to explain: what a Hilbert space is, what a Hermitian operator is, and how a Hermitian operator relates to a particular kind of measurement. In order to explain all this, it will be helpful to keep in mind the simple example of an experimenter Alice making a spin measurement on a particle along an axis $\hat{\mathbf{a}}$. In performing this measurement, we suppose she has a physical measurement device which we denote as $O_{\hat{\mathbf{a}}+}$ and which outputs 1 if the particle is in the spin $|\hat{\mathbf{a}}+\rangle$-state and 0 if the particle is in the spin $|\hat{\mathbf{a}}-\rangle$-state. This measurement device will have a corresponding observable which we will denote by $\hat{O}_{\hat{\mathbf{a}}+}$ and which we will describe shortly once we have defined what a Hilbert space is.[4]

To motivate the definition of a Hilber space, recall that the states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ representing the spin of a particle can be added to give the spin state $|\hat{\mathbf{b}}+\rangle$ and $|\hat{\mathbf{b}}-\rangle$ as seen in equation (1.2). Also, recall that if we have two states $|\psi\rangle$ and $|\chi\rangle$, we can define their bra-ket $\langle\chi|\psi\rangle$ to be a complex number satisfying the Born Rule so that $|\langle\chi|\psi\rangle|^2$ is the probability $P(\chi|\psi)$ that the particle will be found to be in state $|\chi\rangle$

[4]Note that in contrast to the physical object $O_{\hat{\mathbf{a}}+}$, we're taking $\hat{O}_{\hat{\mathbf{a}}+}$ to be a mathematical object, a Hermitian operator on a Hilbert space.

given that we know that the particle is in state $|\psi\rangle$. We thus imposed the assumption that $\langle\psi|\psi\rangle = 1$ for any state $|\psi\rangle$.

Now in order to arrive at a definition of a Hilbert space, we first need to relax this normalization condition $\langle\psi|\psi\rangle = 1$. Thus, if $|\psi\rangle$ is a state and $\lambda \in \mathbb{C}$ is any non-zero complex number, then we allow $|\psi'\rangle = \lambda\,|\psi\rangle$ also to be a state with the caveat that $|\psi'\rangle$ represents exactly the same physical state as $|\psi\rangle$, and that $\langle\psi'|\psi'\rangle = |\lambda|^2\,\langle\psi|\psi\rangle$. We define $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ and we say that $|\psi\rangle$ has been *normalized* when $\|\psi\| = 1$. Now when calculating probabilities, we need to remember to include a normalization factor. Thus, the probability that the particle will be found to be in state $|\chi\rangle$ given that we know that the particle is in state $|\psi\rangle$ will now be $P(\chi|\psi) = \frac{|\langle\chi|\psi\rangle|^2}{\|\psi\|\|\chi\|}$.[5] It is dropping the assumption $\|\psi\| = 1$ on the states of a physical system that gives rise to the mathematical structure known as a Hilbert Space.

A *Hilbert space* is a set H in which

1. any two members of H can be added to obtain another member of H,
2. any member of H can be multiplied by any complex number to obtain another member of H,
3. one can take the bra-ket of any two members of H to obtain a complex number

subject to some natural axioms.[6]

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[5] If there is no such normalization factor because $\|\psi\| = 0$, then $|\psi\rangle$ does not represent a physical state, so the probability the system is ever in this state will be zero, and so in this case we will set $P(\chi|\psi) = P(\psi|\chi) = 0$.

A very simple example of a Hilbert space would be the set of states

$$\{\alpha\,|\mathbf{\hat{a}}+\rangle + \beta\,|\mathbf{\hat{a}}-\rangle : \alpha, \beta \in \mathbb{C}\}.$$

As we will soon see, the observable corresponding to the measurement device $O_{\mathbf{\hat{a}}+}$ will be the operator $\hat{O}_{\mathbf{\hat{a}}+}$ that sends the state $\alpha\,|\mathbf{\hat{a}}+\rangle + \beta\,|\mathbf{\hat{a}}-\rangle$ to the state $\alpha\,|\mathbf{\hat{a}}+\rangle$.

More generally, suppose we have an experimental setup (for example the Stern-Gerlach experiment) where a physical system can be in one of several measurable states $|\psi_1\rangle, \ldots, |\psi_N\rangle \in H$. The physical system could also be in a state described by a sum of some of the $|\psi_1\rangle, \ldots, |\psi_N\rangle$, but by saying the system is in one of these $|\psi_1\rangle, \ldots, |\psi_N\rangle$ measurable states, we mean that there is a measuring device that will always give the same measurement outcome whenever the system is in the same

[6]More formally, a complex Hilbert space H is a complex vector space possessing a bra-ket. By a *complex vector space*, we mean a set V such that the following axioms are satisfied

- $\psi + (\chi + \zeta) = (\psi + \chi) + \zeta$, $\forall \lambda, \chi, \zeta \in V$
- $\psi + \chi = \chi + \psi$, $\forall \psi, \chi \in V$
- there exists an element $\mathbf{0} \in V$ such that $\psi + \mathbf{0} = \psi$, $\forall \psi \in V$.
- $\forall \psi \in V$ there exists an element $-\psi \in V$ such that $\psi + (-\psi) = \mathbf{0}$.
- $\forall \lambda, \mu \in \mathbb{C}$ (i.e. in the set of complex numbers – this is why it is called a *complex* vector space), and $\psi \in V$, $\lambda(\mu\psi) = (\lambda\mu)\psi$.
- for the scalar $1 \in \mathbb{C}$, $1\psi = \psi$, $\forall \psi \in V$
- $\lambda(\psi + \chi) = \lambda\psi + \lambda\chi, \forall \psi, \chi \in V$ and $\lambda \in \mathbb{C}$
- $(\lambda + \mu)\psi = \lambda\psi + \mu\psi$, $\forall \lambda, \mu \in \mathbb{C}$ and $\psi \in V$.

A *Hilbert space* H is a complex vector space possessing a bra-ket. Strictly speaking, a Hilbert space also has a property called completeness, but this property need not concern us here. In quantum theory, elements of H are expressed in terms of kets, $|\cdot\rangle$. Kets behave like vectors, so for $|\psi\rangle, |\chi\rangle \in H$ and $\lambda, \mu \in \mathbb{C}$, we have $\lambda\,|\psi\rangle + \mu\,|\chi\rangle = |\lambda\psi + \mu\chi\rangle$. The bra-ket of $|\psi\rangle$ and $|\chi\rangle$ is then written as $\langle\psi|\chi\rangle$, and it satisfies the following axioms:

- $\langle\psi|\chi\rangle \in \mathbb{C}$, $\forall \psi, \chi \in H$.
- $\langle\psi|\chi\rangle = \overline{\langle\chi|\psi\rangle}$, $\forall \psi, \chi \in H$.
- $\langle\psi|\psi\rangle \geq 0$, $\forall\,|\psi\rangle \in H$ and $\langle\psi|\psi\rangle = 0$ if and only if $|\psi\rangle = \mathbf{0}$.
- $\langle\zeta|\lambda\psi + \mu\chi\rangle = \lambda\,\langle\zeta|\psi\rangle + \mu\,\langle\zeta|\chi\rangle$, $\forall\,|\psi\rangle, |\chi\rangle, |\zeta\rangle \in H$ and $\lambda, \mu \in \mathbb{C}$.

measurable state.[7] We also assume *orthonormality*, that is we assume $\langle\psi_i|\psi_i\rangle = 1$ and $\langle\psi_i|\psi_j\rangle = 0$ for $i \neq j$ so that if the system is measured to be in the $|\psi_j\rangle$-state, then there would be zero probability that it could then be measured to be in the $|\psi_i\rangle$-state for $i \neq j$.

Now suppose that for each measurable state $|\psi_i\rangle$, we assign a real number o_i. There might be a very natural way of doing this, such as assigning o_i to be the angle by which a pointer of a measurement device is deflected when the system is in the state $|\psi_i\rangle$, but the assignment could be as ad hoc as we wished – we can just think of it as the measurement value an experimenter records when he or she observes a particular measurement outcome. Given such an assignment of measurement values, the corresponding observable $\hat{O}$ would be a mapping of states to states satisfying the following rules:

1. $\hat{O}\,|\psi_i\rangle = o_i\,|\psi_i\rangle$
2. $\hat{O}(\lambda\,|\psi\rangle + \mu\,|\chi\rangle) = \lambda\hat{O}\,|\psi\rangle + \mu\hat{O}\,|\chi\rangle$ for all states $|\psi\rangle\,, |\chi\rangle \in H$ and complex numbers $\lambda, \mu \in \mathbb{C}$.

When a mapping $\hat{O}$ satisfies rule 2., we refer to $\hat{O}$ as an *operator* on H. Since the measurement device $O_{\hat{\mathbf{a}}+}$ outputs a value of 1 when the particle is in the $|\hat{\mathbf{a}}+\rangle$-state and 0 when the particle is in the $|\hat{\mathbf{a}}-\rangle$-state, it is now clear from rule 1 and 2 why the corresponding observable $\hat{O}_{\hat{\mathbf{a}}+}$ will be the operator that sends the state $\alpha\,|\hat{\mathbf{a}}+\rangle + \beta\,|\hat{\mathbf{a}}-\rangle$ to $\alpha\,|\hat{\mathbf{a}}+\rangle$.

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[7]When the state is described as a non-trivial sum of the measurable states, we no longer have such certainty, and instead we can only speak of probabilities based on the coefficients on the measurable states.

For a given physical state $|\psi\rangle \in H$, we define the expected measurement value (usually referred to as the *expectation value*) of $\hat{O}$ to be

$$\langle \hat{O} \rangle_\psi \stackrel{\text{def}}{=} \sum_{i=1}^{N} p_i o_i, \tag{2.1}$$

where p_i for $i = 1, \ldots, N$ is the probability the particle will be found to be in the state $|\psi_i\rangle$ given that it is in the state $|\psi\rangle$ so that $\sum_{i=1}^{N} p_i = 1$. If we were to perform the same measurement corresponding to the observable $\hat{O}$ on many systems that were in the state $|\psi\rangle$ and calculated the average measurement outcome from all these measurements, then this average should be approximately equal to the value given in (2.1) if the standard theory of quantum mechanics is correct. Because this expectation value $\langle \hat{O} \rangle_\psi$ will depend on the state $|\psi\rangle$ the system is in, we include the subscript ψ. For instance, if the system is in the $|\psi_i\rangle$-state, then the expectation value of the measurement will be o_i since the probability that the state is in the state $|\psi_i\rangle$ will be 1, so that $p_i = 1$ and $p_j = 0$ for all $j \neq i$. But more generally, if the system was in an arbitrary state $|\psi\rangle = \sum_{i=1}^{N} \alpha_i |\psi_i\rangle$ with $\|\psi\| = 1$, then it turns out that

$$\langle \hat{O} \rangle_\psi = \langle \psi | \hat{O} | \psi \rangle .^{8} \tag{2.2}$$

We can see that this formula is correct in the simple example when the system is in the $|\psi_i\rangle$-state, for in this case, the expectation value should be o_i, and we clearly have $\langle \psi_i | \hat{O} | \psi_i \rangle = o_i$ since $\hat{O} |\psi_i\rangle = o_i |\psi_i\rangle$ and $\langle \psi_i | \psi_i \rangle = 1$. Hence, $\langle \hat{O} \rangle_{\psi_i} = \langle \psi_i | \hat{O} | \psi_i \rangle$ as expected.

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[8] To see this, note that $\alpha_i = \langle \psi_i | \psi \rangle$ from which it follows that if we define the mapping $I = \sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i|$ then $I |\psi\rangle = \sum_{i=1}^{N} |\psi_i\rangle \langle \psi_i | \psi \rangle = \sum_{i=1}^{N} \langle \psi_i | \psi \rangle |\psi_i\rangle = |\psi\rangle$. Therefore,

$$\begin{aligned}
\langle \psi | \hat{O} | \psi \rangle &= \langle \psi | \hat{O} I | \psi \rangle = \sum_{i=1}^{N} \langle \psi | \hat{O} | \psi_i \rangle \langle \psi_i | \psi \rangle = \sum_{i=1}^{N} o_i \langle \psi | \psi_i \rangle \langle \psi_i | \psi \rangle \\
&= \sum_{i=1}^{N} o_i |\langle \psi_i | \psi \rangle|^2 = \sum_{i=1}^{N} o_i p_i = \langle \hat{O} \rangle_\psi .
\end{aligned}$$

To say that $\hat{O}$ is *Hermitian* is to say that $\langle\psi|\hat{O}|\psi\rangle$ is a real number for any arbitrary state $|\psi\rangle$. Thus, the observable $\hat{O}$ defined by the two criteria above is a Hermitian operator acting on the Hilbert space of states H since we are assuming the o_i are all real numbers. Roughly speaking, we can assume[9] that given a Hermitian operator $\hat{O}$ on a Hilbert space of states H, any state $|\psi\rangle \in H$ can be expressed as a (possibly infinite) sum

$$|\psi\rangle = \sum_{i=1}^{N} \alpha_i |\psi_i\rangle \tag{2.3}$$

where $\hat{O}|\psi_i\rangle = o_i |\psi_i\rangle$ for some set of states $|\psi_i\rangle$ referred to as *eigenstates* of $\hat{O}$, and real numbers o_i referred to as *eigenvalues* of $\hat{O}$. We will typically assume that the $|\psi_i\rangle$ are orthonormal. Orthonormality of the $|\psi_i\rangle$ will entail that the coefficients α_i will be uniquely determined by the formula $\alpha_i = \langle\psi_i|\psi\rangle$. Thus, this set of eigenstates $\{|\psi_1\rangle, \ldots |\psi_N\rangle\}$ satisfies the criterion for being a *basis* of the Hilbert space of states H, namely, every state $|\psi\rangle \in H$ can be uniquely expressed by a summation of the form given in equation (2.3).[10] We refer to an expression of the form (2.3) as a *linear combination* of the basis $\{|\psi_1\rangle, \ldots |\psi_N\rangle\}$.

[9]Strictly speaking, we require a Hermitian operator to have a property known as compactness for this assumption to hold.

[10]Note that although for a given basis, equation (2.3) will be unique, there will be many different bases, and the α_i coefficients will depend on which basis is chosen.

2.3 The Preferred Basis Problem[11]

Now just because we can have an observable $\hat{O}$, there is no guarantee that there is a measuring device that could determine whether the system was in one of the eigenstates of $\hat{O}$. For instance, if $|\text{Cat Alive}\rangle$ is the physical state in which a cat is alive, and $|\text{Cat Dead}\rangle$ is the physical state in which the same cat is dead, then although there are measuring devices that can distinguish between the $|\text{Cat Alive}\rangle$-state and the $|\text{Cat Dead}\rangle$-state,[12] there are no known measuring devices that can distinguish between the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$-state and the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$-state. On the other hand, there are measuring devices that can distinguish between the $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$-state and the $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle - |\hat{\mathbf{a}}-\rangle)$-state in a Stern-Gerlach experiment. Why the difference?

This question is at the heart of the *preferred basis problem*. As mentioned already, a basis is just a set of states via which all other states of the system can be uniquely expressed. For instance, we can express the state $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ uniquely as a sum of elements from the basis $\{|\text{Cat Alive}\rangle, |\text{Cat Dead}\rangle\}$, and thus we think of $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ as being a superposition of the $|\text{Cat Alive}\rangle$ and $|\text{Cat Dead}\rangle$ basis states. However, we can also uniquely express $|\text{Cat Alive}\rangle$ in terms of the basis $\{\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle), \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\}$.[13] Nevertheless, we would not tend to think of $|\text{Cat Alive}\rangle$ as being in a superposition of the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$ basis states. That is, we have a preference for the basis $\{|\text{Cat Alive}\rangle, |\text{Cat Dead}\rangle\}$ over the basis

[11] See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 53–55.

[12] For example, we assume that human beings can be thought of as such measuring devices.

[13] i.e. $|\text{Cat Alive}\rangle = \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)\Big) + \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\Big)$.

$\{\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle), \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\}$. We refer to a basis whose basis states our measuring device can distinguish between as a *preferred basis*. As will be shown in section 2.5, decoherence theory offers a very elegant solution to the preferred basis problem.

2.4 Decoherence theory*[14]

Before we can show how decoherence theory solves the preferred basis problem, we will first need to look at decoherence theory in general. To understand what's going on in decoherence theory, there are a number of things we need to discuss, namely

1. composite systems
2. entanglement
3. density matrices and traces
4. coherence
5. partial traces and reduced density matrices
6. the von Neumann measurement scheme
7. decoherence

2.4.1 Composite Systems

First we need to consider *composite systems*. We thus assume there is a distinction between what is being measured and the rest of physical reality. We denote the system that is being measured by $\mathcal{S}$ and the rest of physical reality by $\mathcal{E}$. We will refer to $\mathcal{E}$ as the environment, and we will denote the composite system of $\mathcal{S}$ and $\mathcal{E}$ by $\mathcal{U}$. We will often indicate that $\mathcal{U}$ is a composite of systems $\mathcal{S}$ and $\mathcal{E}$ by writing $\mathcal{U} = \mathcal{S} + \mathcal{E}$. The system $\mathcal{S}$ could be something microscopic like a silver atom, or something much bigger such as a cat or even a planet or galaxy.

*As mentioned in the introduction on page 5, sections marked with an asterisk may be challenging to readers who do not have a mathematics or physics background.

[14]For more details see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, ch. 2.

Now suppose we have an observable (i.e. any Hermitian operator) $\hat{O}_{\mathcal{S}}$ that acts on the Hilbert space $H_{\mathcal{S}}$ of states of $\mathcal{S}$. As already mentioned, this means that we can find orthonormal eigenstates $|\psi_i\rangle_{\mathcal{S}}$ of $\hat{O}_{\mathcal{S}}$ and corresponding eigenvalues o_i such that any state $|\psi\rangle_{\mathcal{S}} \in H_{\mathcal{S}}$ can be uniquely expressed as a sum $|\psi\rangle_{\mathcal{S}} = \sum_{i=1}^{M} \alpha_i |\psi_i\rangle_{\mathcal{S}}$. We will often include the subscript $\mathcal{S}$ on the ket-vectors in order to make it clear that these ket-vectors belong to the Hilbert space $H_{\mathcal{S}}$. At other times we will omit these subscripts when it is clear what system we are talking about, but for the time being, we will keep these subscripts in place.

Now let us suppose we have a basis of normalized (but not necessarily orthonormal) states $\{|\chi_i\rangle_{\mathcal{E}} : i\}$ for the state space $H_{\mathcal{E}}$ of $\mathcal{E}$. In other words, every state $|\chi\rangle_{\mathcal{E}} \in H_{\mathcal{E}}$ can be uniquely expressed as a linear combination $|\chi\rangle_{\mathcal{E}} = \sum_{i=1}^{N} \beta_i |\chi_i\rangle_{\mathcal{E}}$. It is then assumed we will be able to express every state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ of the composite system $\mathcal{U}$ as a linear combination

$$|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}} . \tag{2.4}$$

Thus, we assume there no are emergent physical properties describing the composite system $\mathcal{U}$ that couldn't be expressed in terms of the component subsystems $\mathcal{S}$ and $\mathcal{E}$. The Hilbert space $H_{\mathcal{U}}$ is endowed with the bra-ket $\langle \xi' | \xi \rangle_{\mathcal{U}}$ such that if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ and $|\xi'\rangle_{\mathcal{U}} = |\psi'\rangle_{\mathcal{S}} |\chi'\rangle_{\mathcal{E}}$, then $\langle \xi' | \xi \rangle_{\mathcal{U}} = \langle \psi' | \psi \rangle_{\mathcal{S}} \langle \chi' | \chi \rangle_{\mathcal{E}}$ where we have again used subscripts to indicate which Hilbert space the bra-ket corresponds to.

2.4.2 Entanglement

By defining the bra-ket on the Hilbert space $H_{\mathcal{U}}$ as we have done, we are making the assumption that if we define $P(\psi', \chi' | \psi, \chi)$ to be the probability the composite system would be measured to be in the $|\psi'\rangle_{\mathcal{S}}|\chi'\rangle_{\mathcal{E}}$-state given that the composite system was known to be in the $|\psi\rangle_{\mathcal{S}}|\chi\rangle_{\mathcal{E}}$-state, then $P(\psi',\chi'|\psi,\chi) = P(\psi'|\psi)P(\chi'|\chi)$. A consequence of this assumption is that if the composite system is in the $|\psi\rangle_{\mathcal{S}}|\chi\rangle_{\mathcal{E}}$-state, the probability of finding system $\mathcal{S}$ to be in any particular state in $H_{\mathcal{S}}$ will be independent[15] of the state in $H_{\mathcal{E}}$ describing $\mathcal{E}$. For this reason, when the state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ describing the composite system $\mathcal{U}$ is expressible as a product state $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}|\chi\rangle_{\mathcal{E}}$, we say that $\mathcal{S}$ and $\mathcal{E}$ are *not entangled* with one another. On the other hand, when the state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ of the composite system $\mathcal{U}$ cannot be expressed as a product state, we say that $\mathcal{S}$ and $\mathcal{E}$ are *entangled*. For example, if $|\xi\rangle_{\mathcal{U}} = \frac{1}{\sqrt{2}}(|\psi_1\rangle_{\mathcal{S}}|\chi_1\rangle_{\mathcal{E}} + |\psi_2\rangle_{\mathcal{S}}|\chi_2\rangle_{\mathcal{E}})$ with $|\psi_1\rangle_{\mathcal{S}} \not\propto |\psi_2\rangle_{\mathcal{S}}$ and $|\chi_1\rangle_{\mathcal{E}} \not\propto |\chi_2\rangle_{\mathcal{E}}$,[16] then $\mathcal{S}$ and $\mathcal{E}$ would be entangled with one another.

Now given the observable $\hat{O}_{\mathcal{S}}$ acting on $H_{\mathcal{S}}$, we can naturally define the observable $\hat{O}_{\mathcal{U}}$ acting on $H_{\mathcal{U}}$ so that

$$\hat{O}_{\mathcal{U}}|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^{M}\sum_{j=1}^{N}\gamma_{i,j}\hat{O}_{\mathcal{S}}|\psi_i\rangle_{\mathcal{S}}|\chi_j\rangle_{\mathcal{E}} = \sum_{i=1}^{M}\sum_{j=1}^{N}\gamma_{i,j}o_i|\psi_i\rangle_{\mathcal{S}}|\chi_j\rangle_{\mathcal{E}}. \tag{2.5}$$

Just as in equation (2.2), for a given normalized state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$, the expectation value of the observable $\hat{O}_{\mathcal{U}}$ will be $\langle\hat{O}_{\mathcal{U}}\rangle_{\xi} = \langle\xi|\hat{O}_{\mathcal{U}}|\xi\rangle_{\mathcal{U}}$. It is easy to see that if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}|\chi\rangle_{\mathcal{E}}$ (i.e. $\mathcal{S}$ and $\mathcal{E}$ are not entangled), then $\langle\hat{O}_{\mathcal{U}}\rangle_{\xi} = \langle\hat{O}_{\mathcal{S}}\rangle_{\psi}$.[17] Thus, when $\mathcal{S}$ and $\mathcal{E}$ are not entangled with one another, it is possible to say things about $\mathcal{S}$

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[15]Here we are using the standard probabilistic definition of independence as given by (1.3) on page 18: two events X and Y are independent if and only if $P(X \text{ and } Y \text{ occur}) = P(X \text{ occurs})P(Y \text{ occurs})$

[16]Here the notation $|\psi_1\rangle_{\mathcal{S}} \propto |\psi_2\rangle_{\mathcal{S}}$ means there exists some α such that $|\psi_1\rangle_{\mathcal{S}} = \alpha|\psi_2\rangle_{\mathcal{S}}$, in which case $|\xi\rangle_{\mathcal{U}} = \frac{1}{\sqrt{2}}|\psi_2\rangle_{\mathcal{S}}(\alpha|\chi_1\rangle_{\mathcal{E}} + |\chi_2\rangle_{\mathcal{E}})$. Thus, if $|\psi_1\rangle_{\mathcal{S}} \propto |\psi_2\rangle_{\mathcal{S}}$, then $\mathcal{S}$ and $\mathcal{E}$ would not be entangled. This is why in the above example, we assume $|\psi_1\rangle_{\mathcal{S}} \not\propto |\psi_2\rangle_{\mathcal{S}}$, that is, we assume there is no such α such that $|\psi_1\rangle_{\mathcal{S}} = \alpha|\psi_2\rangle_{\mathcal{S}}$, and for the same reason we assume $|\chi_1\rangle_{\mathcal{S}} \not\propto |\chi_2\rangle_{\mathcal{E}}$.

[17]This is because by definition, if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}|\chi\rangle_{\mathcal{E}}$ and $|\xi'\rangle_{\mathcal{U}} = |\psi'\rangle_{\mathcal{S}}|\chi'\rangle_{\mathcal{E}}$, then $\langle\xi'|\xi\rangle_{\mathcal{U}} = \langle\psi'|\psi\rangle_{\mathcal{S}}\langle\chi'|\chi\rangle_{\mathcal{E}}$. We will also have $\hat{O}_{\mathcal{U}}|\xi\rangle = \hat{O}_{\mathcal{S}}|\psi\rangle_{\mathcal{S}}|\chi\rangle_{\mathcal{E}}$. Thus, assuming both $|\psi\rangle_{\mathcal{S}}$ and $|\chi\rangle_{\mathcal{E}}$ are normalized, we have $\langle\hat{O}_{\mathcal{U}}\rangle_{\xi} = \langle\xi|\hat{O}_{\mathcal{U}}|\xi\rangle = \langle\psi|\hat{O}_{\mathcal{S}}|\psi\rangle_{\mathcal{S}}\langle\chi|\chi\rangle_{\mathcal{E}} = \langle\hat{O}_{\mathcal{S}}\rangle_{\psi}$.

independently of the current state of the environment $\mathcal{E}$. In this case we need have no knowledge of the information about $\mathcal{E}$ encapsulated in the state $|\chi\rangle_{\mathcal{E}}$ to determine the expectation value $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi}$.

However, for a general entangled state $|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}$, $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi}$ will typically depend on the $|\chi_j\rangle_{\mathcal{E}}$-states and the coefficients $\gamma_{i,j}$. Nevertheless, despite there being a huge amount of information contained within these $|\chi_j\rangle_{\mathcal{E}}$-states and the $\gamma_{i,j}$, if we are only interested in making measurements on the system $\mathcal{S}$, nearly all this information can be discarded. In order to see how this is done, we need to generalize the notion of a state to that of a density matrix.

2.4.3 Density Matrices and Traces

Given a normalized state $|\psi\rangle$ in any Hilbert space H, its density matrix will be the operator $\hat{\rho}_{\psi} \stackrel{\text{def}}{=} |\psi\rangle\langle\psi|$ which acts on H by sending an arbitrary state $|\psi'\rangle$ to $\langle\psi|\psi'\rangle |\psi\rangle$. Note that $\hat{\rho}_{\psi}$ is a Hermitian operator.[18] Also note that if we had a measuring device that returned the output 1 if a system was in the state $|\psi\rangle$ and 0 if the system was in a state $|\chi\rangle$ with $\langle\psi|\chi\rangle = 0$, the density matrix $\hat{\rho}_{\psi}$ would be the observable corresponding to this measurement. The expectation value of this measurement for an initial normalized state $|\psi'\rangle$ would then be $\langle\psi'|\hat{\rho}_{\psi}|\psi'\rangle = |\langle\psi|\psi'\rangle|^2 = P(\psi|\psi')$. In particular, if the system was initially in the state $|\psi\rangle$, the expectation value of this measurement would be 1.

[18]This is because for any arbitrary state $|\psi'\rangle$, $\langle\psi'|\hat{\rho}_{\psi}|\psi'\rangle = \langle\psi'|\psi\rangle \langle\psi|\psi'\rangle = \overline{\langle\psi|\psi'\rangle} \langle\psi|\psi'\rangle = |\langle\psi|\psi'\rangle|^2$, and so $\langle\psi'|\hat{\rho}_{\psi}|\psi'\rangle$ is real, and from this it follows that $\hat{\rho}_{\psi}$ is Hermitian.

Now it turns out that if we have an arbitrary orthonormal basis $\{|\phi_i\rangle : i\}$ of H and any observable $\hat{O}$ on H, then

$$\langle \hat{O} \rangle_\psi = \sum_i \langle \phi_i | \hat{\rho}_\psi \hat{O} | \phi_i \rangle .^{19} \tag{2.6}$$

Since this expression can be shown to be independent of which basis we choose,[20] we have a well-defined function called the *trace*, written as $\mathrm{Tr}(\cdot)$, which maps any operator $\hat{A}$ acting on H to a value in $\mathbb{C}$ according to the formula

$$\mathrm{Tr}(\hat{A}) = \sum_i \langle \phi_i | \hat{A} | \phi_i \rangle . \tag{2.7}$$

Thus, it follows from equations (2.6) and (2.7) that

$$\langle \hat{O} \rangle_\psi = \mathrm{Tr}(\hat{\rho}_\psi \hat{O}). \tag{2.8}$$

So far we have defined the density matrix $\hat{\rho}_\psi$ corresponding to a normalized state $|\psi\rangle$. More generally, a *density matrix* $\hat{\rho}$ is defined to be a Hermitian operator with positive eigenvalues such that $\mathrm{Tr}(\hat{\rho}) = 1$. We will write $M(H)$ for the set of all density matrices on H. Since we are assuming[21] that for any Hermitian operator,

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[19]To see this, as we saw in footnote 8, if we define the mapping $I = \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|$ then $I|\psi\rangle = |\psi\rangle$. Therefore, $\langle \hat{O} \rangle_\psi = \langle \psi | \hat{O} | \psi \rangle = \langle \psi | \hat{O} I | \psi \rangle = \sum_i \langle \psi | \hat{O} | \phi_i \rangle \langle \phi_i | \psi \rangle = \sum_i \langle \phi_i | \psi \rangle \langle \psi | \hat{O} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{\rho}_\psi \hat{O} | \phi_i \rangle$.

[20]To see this, we first note that for any orthonormal basis $\{|\phi_i\rangle : i\}$ of H, and any two operators $\hat{A}$ and $\hat{B}$ acting on H, using the fact that $I = \sum_i |\phi_i\rangle\langle\phi_i|$ is the identity operator on H, we have the commutativity property

$$\sum_i \langle \phi_i | \hat{A}\hat{B} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{A} \sum_j |\phi_j\rangle\langle\phi_j| \, \hat{B} | \phi_i \rangle = \sum_{ij} \langle \phi_j | \hat{B} | \phi_i \rangle \, \langle \phi_i | \hat{A} | \phi_j \rangle = \sum_j \langle \phi_j | \hat{B}\hat{A} | \phi_j \rangle .$$

Now suppose that $\{|\phi'_i\rangle : i\}$ is another orthonormal basis of H. Then we can define the operator $\hat{U}$ such that $\hat{U}|\phi_i\rangle = |\phi'_i\rangle$. We can also define the operator $\hat{U}^*$ such that $\langle \phi'_i | \psi \rangle = \langle \phi_i | \hat{U}^* | \psi \rangle$ for any state $|\psi\rangle \in H$. Since $\langle \phi'_i | \phi'_j \rangle = \langle \phi_i | \hat{U}^* | \phi'_j \rangle$ for all i, j, it will follow that $\hat{U}^* |\phi'_j\rangle = |\phi_j\rangle$. Therefore, $\hat{U}\hat{U}^* = I$. Using this fact together with the commutativity property, we have

$$\sum_i \langle \phi'_i | \hat{O} | \phi'_i \rangle = \sum_i \langle \phi_i | \hat{U}^* \hat{O} \hat{U} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{O} \hat{U} \hat{U}^* | \phi_i \rangle = \sum_i \langle \phi_i | \hat{O} | \phi_i \rangle .$$

[21]As mentioned earlier, we are making the assumption that Hermitian operators are compact.

there is an orthonormal basis of the Hilbert space consisting of eigenstates of the Hermitian operator, we can find an orthonormal basis $\{|\psi_i\rangle : i\}$ of eigenstates of $\hat{\rho}$ with corresponding eigenvalues p_i such that

$$\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \, . \tag{2.9}$$

The condition $\mathrm{Tr}(\hat{\rho}) = 1$ will then imply that $\sum_i p_i = 1$. Now we could think of the operator $\hat{\rho}$ as corresponding to a measurement which gave the output p_i when the system was in the state $|\psi_i\rangle$. However, we can alternatively think of $\hat{\rho}$ as describing a system which is known to be in one of the $|\psi_i\rangle$-states, but that we only know it is in the $|\psi_i\rangle$-state with probability p_i. Then given that $\hat{\rho}$ describes all we know about the system, the expectation value $\langle\hat{O}\rangle_\rho$ for an observable $\hat{O}$ on the system can be shown to be

$$\langle\hat{O}\rangle_\rho = \mathrm{Tr}(\hat{\rho}\hat{O}),^{22} \tag{2.10}$$

which is a generalization of (2.8). Thus, we can think of a density matrix $\hat{\rho} \in M(H)$ as a generalization of a state ket-vector $|\psi\rangle \in H$, since for every $|\psi\rangle \in H$ there corresponds a density matrix $\hat{\rho} = |\psi\rangle\langle\psi| \in M(H)$. Because of this identification, $\hat{\rho} = |\psi\rangle\langle\psi|$ is referred to as a *pure state*. On the other hand, the converse does not hold: if $\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \in M(H)$ with more than one of the $p_i > 0$, then there will not be a corresponding $|\psi\rangle \in H$ such that $\hat{\rho} = |\psi\rangle\langle\psi|$. In this case, when $\hat{\rho}$ is interpreted as describing a system that is definitely in one of the $|\psi_i\rangle$-states with probability p_i, then we will refer to $\hat{\rho}$ as a *mixed state*.

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[22]This follows since $\mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_i p_i \, \mathrm{Tr}(|\psi_i\rangle\langle\psi_i| \, \hat{O}) = \sum_i p_i \, \langle\hat{O}\rangle_{\psi_i}$ which will be the expectation value of $\hat{O}$ given that $\hat{\rho}$ encapsulates our knowledge of the system.

2.4.4 Coherence

Now suppose that the system $\mathcal{S}$ is initially in a superposition state $|\psi\rangle = \sum_i c_i |s_i\rangle$ with $\sum_i |c_i|^2 = 1$. Then the corresponding density matrix on $\mathcal{S}$ will be

$$|\psi\rangle\langle\psi| = \sum_{ij} c_i \overline{c_j}\, |s_i\rangle\langle s_j| \,.$$

When a density matrix has non-zero $|s_i\rangle\langle s_j|$-components for $i \neq j$, we say that there is *coherence* between the $|s_i\rangle$ and $|s_j\rangle$-states.[23] Thus, for the density matrix $|\psi\rangle\langle\psi|$ there will be coherence between the $|s_i\rangle$ and $|s_j\rangle$-states so long as both c_i and c_j are non-zero. Decoherence is a process (to be described shortly) by which the $|s_i\rangle\langle s_j|$-components (for $i \neq j$) of a density matrix restricted to a subsystem of a composite system appear to vanish.

2.4.5 Partial Traces and Reduced Density Matrices

As already mentioned, if we have a general entangled state on a composite system $\mathcal{U} = \mathcal{S}+\mathcal{E}$ of the form $|\xi\rangle_{\mathcal{U}} = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}$, there is a huge amount of information in all the $\gamma_{i,j}$. However, most of this information can be discarded if we are only interested in making measurements on the system $\mathcal{S}$. We can't typically encapsulate this information in the form of a state $|\psi\rangle \in H_{\mathcal{S}}$, but we can encapsulate this information in the form of a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which as mentioned on

[23] The fact that a density matrix can be written out in terms of $|s_i\rangle\langle s_j|$-components explains why we refer to a density matrix as a density *matrix*. For example, if our state space has a basis of just two states $\{|s_1\rangle, |s_2\rangle\}$, and if $\hat{\rho} = a|s_1\rangle\langle s_1| + b|s_1\rangle\langle s_2| + c|s_2\rangle\langle s_1| + d|s_2\rangle\langle s_2|$, then we can identify $\hat{\rho}$ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we then identify the state $|\psi\rangle = x|s_1\rangle + y|s_2\rangle$ with the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, then the state $\hat{\rho}|\psi\rangle$ would be identified with the column vector under matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. The trace of a density matrix is then just the sum of the diagonal elements (top left to bottom right) of the matrix. Decoherence with respect to a particular basis occurs when the off-diagonal elements of the density matrix vanish.

page 69 can be thought of as a generalization of a state ket-vector $|\psi\rangle \in H_{\mathcal{S}}$. In this subsection, we will show how the density matrix $\hat{\rho} = |\xi\rangle\langle\xi| \in M(H_{\mathcal{U}})$ can be reduced to a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which encapsulates all the information needed to calculate expectation values of observables on $\mathcal{S}$. The reduced density matrix $\hat{\rho}_{\mathcal{S}}$ is derived from $\hat{\rho}$ via an operation called the partial trace.

In the context of a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$, when taking traces, we will need to be more specific over which basis we are taking the trace over. If $\{|\psi_i\rangle : i\}$ is an orthonormal basis of $H_{\mathcal{S}}$ and $\{|\chi_j\rangle : j\}$ is an orthonormal basis of $H_{\mathcal{E}}$, then $\{|\xi_{ij}\rangle \stackrel{\text{def}}{=} |\psi_i\rangle\,|\chi_j\rangle : i, j\}$ will be an orthonormal basis of $H_{\mathcal{U}}$. For an operator $\hat{A}_{\mathcal{S}}$ of $H_{\mathcal{S}}$, we define

$$\mathrm{Tr}_{\mathcal{S}}(\hat{A}_{\mathcal{S}}) = \sum_i \langle\psi_i|\hat{A}_{\mathcal{S}}|\psi_i\rangle\,,$$

and for an operator $\hat{A}_{\mathcal{U}}$ of $H_{\mathcal{U}}$, we define

$$\mathrm{Tr}_{\mathcal{U}}(\hat{A}_{\mathcal{U}}) = \sum_{ij} \langle\xi_{ij}|\hat{A}_{\mathcal{U}}|\xi_{ij}\rangle\,.$$

This is just what we would expect the traces to be for operators on $H_{\mathcal{S}}$ and on $H_{\mathcal{U}}$ respectively. But we also need the notion of a *partial trace* for an operator $\hat{A}_{\mathcal{U}}$ on $H_{\mathcal{U}}$:

$$\mathrm{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}}) = \sum_j \langle\chi_j|\hat{A}_{\mathcal{U}}|\chi_j\rangle\,. \tag{2.11}$$

Note that whereas $\mathrm{Tr}_{\mathcal{U}}(\hat{A}_{\mathcal{U}})$ is just a number, the partial trace $Tr_{\mathcal{E}}(\hat{A}_{\mathcal{U}})$ is an operator that acts on $H_{\mathcal{S}}$. To see why this is, consider the simple example of when $\hat{A}_{\mathcal{U}} = |\xi_{ij}\rangle\langle\xi_{lk}|$. The operator $\hat{A}_{\mathcal{U}}$ would send the state $|\xi_{lk}\rangle$ to $|\xi_{ij}\rangle$ and all the other $|\xi_{l'k'}\rangle$-states of $H_{\mathcal{U}}$ to 0. But in order to define the partial trace as given in equation (2.11), we need to know what $\hat{A}_{\mathcal{U}}\,|\chi\rangle$ is and then what $\langle\chi|\hat{A}_{\mathcal{U}}|\chi\rangle$ is. In the case when $\hat{A}_{\mathcal{U}} = |\xi_{ij}\rangle\langle\xi_{lk}|$, we stipulate that $\hat{A}_{\mathcal{U}}\,|\chi\rangle$ is the operator that sends the state $|\psi\rangle \in H_{\mathcal{S}}$ to the state

$\langle\psi_l|\psi\rangle\,\langle\chi_k|\chi\rangle\,|\xi_{ij}\rangle \in H_{\mathcal{U}}$. Furthermore, if we stipulate that $\langle\chi|\xi_{ij}\rangle = \langle\chi|\chi_j\rangle\,|\psi_i\rangle \in H_{\mathcal{S}}$, it follows that $\langle\chi|\hat{A}_{\mathcal{U}}|\chi\rangle$ will be the operator $\langle\chi|\chi_j\rangle\,\langle\chi_k|\chi\rangle\,|\psi_i\rangle\langle\psi_l|$ that sends the state $|\psi\rangle \in H_{\mathcal{S}}$ to the state $\langle\chi|\chi_j\rangle\,\langle\chi_k|\chi\rangle\,\langle\psi_l|\psi\rangle\,|\psi_i\rangle \in H_{\mathcal{S}}$. By (2.11), we therefore find that

$$\mathrm{Tr}_{\mathcal{E}}(|\xi_{ij}\rangle\langle\xi_{lk}|) = \begin{cases} |\psi_i\rangle\langle\psi_l| & \text{if } j = k, \\ \\ 0 & \text{if } j \neq k. \end{cases} \tag{2.12}$$

And since any arbitrary operator $\hat{A}_{\mathcal{U}}$ on $H_{\mathcal{U}}$ can be expressed as a sum

$$\hat{A}_{\mathcal{U}} = \sum_{ijkl} \mu_{ijkl}\,|\xi_{ij}\rangle\langle\xi_{lk}|\,,$$

we can use equation (2.12) to find that

$$\mathrm{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}}) = \sum_{ijl} \mu_{ijjl}\,|\psi_i\rangle\langle\psi_l|\,.$$

Now it turns out that given a density matrix $\hat{\rho}$ on $H_{\mathcal{U}}$ and an observable $\hat{O}_{\mathcal{S}}$ of $H_{\mathcal{S}}$ (which induces an observable $\hat{O}_{\mathcal{U}}$ on $H_{\mathcal{U}}$ in the obvious way, e.g. $\hat{O}_{\mathcal{U}}\,|\xi_{ij}\rangle = (\hat{O}_{\mathcal{S}}\,|\psi_i\rangle)\,|\chi_j\rangle)$, we have the important formula

$$\langle\hat{O}_{\mathcal{U}}\rangle_\rho = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}) \tag{2.13}$$

where $\hat{\rho}_{\mathcal{S}} = \mathrm{Tr}_{\mathcal{E}}(\hat{\rho})$.[24] We refer to $\hat{\rho}_{\mathcal{S}}$ as the *reduced density matrix* of $\hat{\rho}$.

Note that if $\hat{\rho} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = |\psi\rangle\,|\chi\rangle$ so that $\mathcal{S}$ and $\mathcal{E}$ are not entangled, then $\hat{\rho}_{\mathcal{S}} = |\psi\rangle\langle\psi|$.[25] This is what we should expect, since if $\mathcal{S}$ and $\mathcal{E}$ are not entangled, then

[24] To see this, following Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 46, we have

$$\begin{aligned} \langle\hat{O}_{\mathcal{U}}\rangle_\rho &= \mathrm{Tr}_{\mathcal{U}}(\hat{\rho}\hat{O}_{\mathcal{U}}) = \sum_{ij} \langle\xi_{ij}|\hat{\rho}\hat{O}_{\mathcal{U}}|\xi_{ij}\rangle = \sum_i \langle\psi_i|\Big(\sum_j \langle\chi_j|\hat{\rho}|\chi_j\rangle\Big)\hat{O}_{\mathcal{S}}\,|\psi_i\rangle \\ &= \sum_i \langle\psi_i|\,\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}\,|\psi_i\rangle = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}). \end{aligned}$$

[25] To see this, we recall that the partial trace $\mathrm{Tr}_{\mathcal{E}}$ is independent of which orthonormal basis $\{|\chi_j\rangle : j\}$ we choose for $\mathcal{E}$. Therefore, if $\hat{\rho} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = |\psi\rangle\,|\chi\rangle$, we can choose $|\chi_1\rangle = |\chi\rangle$ and all other $|\chi_i\rangle$ such that $\langle\chi_i|\chi\rangle = 0$. Then $\mathrm{Tr}_{\mathcal{E}}(\hat{\rho}) = \sum_j \langle\chi_j|\hat{\rho}|\chi_j\rangle = |\psi\rangle\langle\psi|$.

the expectation values of observables defined on $\mathcal{S}$ should be independent of the state of $\mathcal{E}$, and by equation (2.13), this independence is seen to hold when $\hat{\rho}_{\mathcal{S}}$ is independent of any states on $\mathcal{E}$.

More generally, for an entangled state $|\xi\rangle = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle |\chi_j\rangle$, from equations (2.10) and (2.13), we have $\langle \hat{O}_{\mathcal{U}} \rangle_{\rho} = \langle \hat{O}_{\mathcal{S}} \rangle_{\rho_{\mathcal{S}}}$. This means that when it comes to taking expectation values of measurements on a subsystem $\mathcal{S}$ that is part of a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ which is in the state $|\xi\rangle \in H_{\mathcal{U}}$, the subsystem $\mathcal{S}$ behaves as though it was described by the density matrix $\hat{\rho}_{\mathcal{S}}$. However, there is a rather subtle point one needs to be aware of here.[26] For in general, as we saw in equation (2.9), any density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ can be expressed as a sum $\hat{\rho}_{\mathcal{S}} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, and this can be *thought of* as corresponding to the system $\mathcal{S}$ being in one of the $|\psi_i\rangle$-states, but that we only know it is in the $|\psi_i\rangle$-state with probability p_i. If this was the correct interpretation of $\hat{\rho}_{\mathcal{S}}$, then as explained on page 69, we would refer to $\hat{\rho}_{\mathcal{S}}$ as a *mixed state*. But just because we can think of $\hat{\rho}_{\mathcal{S}}$ in this way, it doesn't follow that $\mathcal{S}$ really is in one of these $|\psi_i\rangle$-states and that we are only ignorant of which state it is. When $\mathcal{S}$ is entangled with $\mathcal{E}$ there is no fact of the matter regarding which state $\mathcal{S}$ is in. Rather, there are only facts of the matter for the composite system $\mathcal{U}$, e.g. the fact of the matter is that $\mathcal{U}$ is in the state $|\xi\rangle$ rather than some other state of $H_{\mathcal{U}}$. Therefore, $\mathcal{U}$ is really in a pure state with density matrix $|\xi\rangle\langle\xi|$. Because we cannot give an ignorance interpretation to $\hat{\rho}_{\mathcal{S}}$, d'Espagnat referred to density matrices of this sort as

[26] It is unfortunate that many physicists fail to pick up on this subtlety with the result that they form the erroneous belief that decoherence can by itself solve the measurement problem (of outcomes) when in fact it can't. For a further discussion of the problem of outcomes, see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 57–60.

being *improper mixtures*.[27] But despite this subtle distinction between mixed states and improper mixtures, we have nevertheless succeeded in showing how a density matrix $\hat{\rho} = |\xi\rangle\langle\xi| \in M(H_{\mathcal{U}})$ can be reduced to a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which encapsulates all the information needed to calculate expectation values of observables on $\mathcal{S}$.

2.4.6 The von Neumann Measurement Scheme

We are now in a position to consider the *von Neumann measurement scheme*.[28] Instead of considering the whole of physical reality, for the time being, we just consider a physical system $\mathcal{S}$ and a measuring device $\mathcal{A}$. This division reflects the fact that a scientist doesn't measure the system $\mathcal{S}$ directly, but rather observes a measuring device $\mathcal{A}$ that is affected by $\mathcal{S}$. The measuring device $\mathcal{A}$ has the characteristic that it has a normalized ready state $|a_r(t_0)\rangle$ at initial time t_0 and that there is an orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$, and normalized states $|a_i(t)\rangle$ of $\mathcal{A}$ such that

1. for any $t \geq t_0$ we have the evolution of the states $|s_i\rangle\, |a_r(t_0)\rangle \xrightarrow{\text{time evolution}} |s_i\rangle\, |a_i(t)\rangle$ so that $\mathcal{S}$ and $\mathcal{A}$ do not become entangled when $\mathcal{S}$ is initially in state $|s_i\rangle$ and $\mathcal{A}$ is initially in state $|a_i(t_0)\rangle$.
2. there exists $\delta > 0$ such that if $t > t_0 + \delta$, then $\langle a_i(t)|a_j(t)\rangle \approx 0$ for $i \neq j$.[29]

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[27] See Bernard d' Espagnat, *Conceptual foundations of quantum mechanics*, 2nd ed., completely rev., enl., reset., Mathematical physics monograph series ; 20 (Reading, Mass.; London: W. A. Benjamin, 1976), ch. 6.2 – cited in Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," p. 19.

[28] See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 50–53 for more details.

[29] More precisely, we should say that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $t > t_0 + \delta$, then $|\,\langle a_i(t)|a_j(t)\rangle\,| < \epsilon$ for $i \neq j$.

These two criteria characterize the von Neumann measurement scheme. The orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$ for which these two criteria hold are called *pointer states* of $\mathcal{A}$. These pointer states will be determined by the dynamics of the composite system $\mathcal{S}+\mathcal{A}$ as well as the relative configuration of $\mathcal{S}$ with respect to $\mathcal{A}$. For instance, if $\mathcal{S}$ is a silver atom and $\mathcal{A}$ is a Stern-Gerlach apparatus, then the configuration and dynamics of the system will determine a fixed axis $\hat{\mathbf{a}}$ relative to the Stern-Gerlach configuration $\mathcal{A}$ such that the states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ of $\mathcal{S}$ don't get entangled with $\mathcal{A}$, that is, there exists $\delta > 0$ such that $|\hat{\mathbf{a}}\pm\rangle\,|a_r(t_0)\rangle \xrightarrow{\text{time evolution}} |\hat{\mathbf{a}}\pm\rangle\,|a_\pm(t)\rangle$ with $\langle a_+(t)|a_-(t)\rangle \approx 0$ for $t > t_0+\delta$.[30, 31] Since no entanglement occurs with the silver atom and the Stern-Gerlach apparatus when the silver atom is in the $|\hat{\mathbf{a}}\pm\rangle$-state, then in this situation, we can interact with the apparatus to find out whether the particle is in the $|\hat{\mathbf{a}}+\rangle$-state or the $|\hat{\mathbf{a}}-\rangle$-state without changing the spin state of the silver atom. Indeed, we should expect an experimental apparatus to have this property of non-entanglement with the measurement outcomes it reports, for otherwise, every scientist who looked at the measurement device couldn't be sure that the spin state of the silver atom being measured remained unchanged whenever the apparatus was observed, and so the scientists couldn't expect there to be any agreement among themselves regarding which spin-state the silver atom was in. Thus, the basis of $H_{\mathcal{S}}$ for which entanglement doesn't occur is a preferred basis. However, if we were to consider

[30]Strictly speaking, we would need more information to describe states in $H_{\mathcal{S}}$ besides the spin, so we should really express this scenario in terms of $\{|s_{i,+}\rangle \stackrel{\text{def}}{=} |\hat{\mathbf{a}}+, i\rangle : i\} \cup \{|s_{i,-}\rangle \stackrel{\text{def}}{=} |\hat{\mathbf{a}}-, i\rangle : i\}$ and $\{|a_{i,+}(t)\rangle : i\} \cup \{|a_{i,-}(t)\rangle : i\}$ where the i-indices encode all the additional information beyond spin.

[31]Although we only require that $\langle a_+(t)|a_-(t)\rangle \approx 0$ for $t > t_0+\delta$ rather than demanding $\langle a_+(t)|a_-(t)\rangle = 0$, we can think of the scientist who observes the apparatus as determining whether the apparatus is either in one of two normalized state $|a'_+(t)\rangle$ or $|a'_-(t)\rangle$ where $\langle a'_+(t)|a'_-(t)\rangle = 0$ and $\langle a'_\pm(t)|a'_\pm(t)\rangle \approx 1$, so that the scientist can confidently assert that the particle is in the state $|\hat{\mathbf{a}}+\rangle$ if for instance the measurement device is found to be in the state $|a'_+(t)\rangle$. Because $\langle a'_+(t)|a_-(t)\rangle$ is only very small, but not identically zero, in theory, the particle could be in the $|\hat{\mathbf{a}}-\rangle$-state, but we're assuming that such a possibility would be as likely as a violation of the Second Law of Thermodynamics, say.

a different basis, say $\{\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle), \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle - |\hat{\mathbf{a}}-\rangle)\}$,[32] then assuming that the configuration of $\mathcal{A}$ remained unchanged, entanglement between $\mathcal{S}$ and $\mathcal{A}$ would occur since then $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle \pm |\hat{\mathbf{a}}-\rangle)\,|a_r(t_0)\rangle \xrightarrow{\text{time evolution}} \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle\,|a_+(t)\rangle \pm |\hat{\mathbf{a}}-\rangle\,|a_-(t)\rangle)$. Thus, $\{\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle), \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle - |\hat{\mathbf{a}}-\rangle)\}$ would not be a preferred basis. In this case, if $\mathcal{S}$ was in the $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$-state, a scientist would measure $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state with probability $\frac{1}{2}$. But having measured $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state, the scientist would continue to observe $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state because of the subsequent non-entanglement of $\mathcal{S}$ with $\mathcal{A}$ when $\mathcal{S}$ is in the $|\hat{\mathbf{a}}+\rangle$-state and $\mathcal{A}$ is in the $|a_+(t)\rangle$-state. Note that this situation is somewhat analogous to when we have the Bell-state (1.8), so that when Bob measures his particle to be in the $|\hat{\mathbf{a}}-\rangle$-state, he knows that Alice's particle is in the $|\hat{\mathbf{a}}+\rangle$-state. Likewise, in the von Neumann measurement scheme, if the scientist measures $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state for $t > t_0+\delta$, he will then (almost certainly) know[33] that the system $\mathcal{S}$ will be in the $|\hat{\mathbf{a}}+\rangle$-state.

In the case where $\mathcal{S}$ has more than two states, we can write a generic normalized state of the composite system $\mathcal{U} = \mathcal{S} + \mathcal{A}$ as $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle$ where $|\xi_i(t)\rangle = |s_i\rangle\,|a_i(t)\rangle$. There will then be coherence between $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ for the density matrix $\hat{\rho}(t) \stackrel{\text{def}}{=} |\Psi(t)\rangle\langle\Psi(t)|$ so long as both c_i and c_j are non-zero. However, if we are only interested in observables $\hat{O}_{\mathcal{S}}$ on $H_{\mathcal{S}}$, then we only need to consider the reduced density matrix $\hat{\rho}_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{A}}(\hat{\rho}(t))$. Initially, at time t_0 we have $|a_i(t_0)\rangle = |a_r(t_0)\rangle$ so $|\Psi(t_0)\rangle = |\psi\rangle\,|a_r(t_0)\rangle$ where $|\psi\rangle = \sum_i c_i\,|s_i\rangle$ which we assume to be normalized. Thus, initially, $\mathcal{S}$ would not be entangled with $\mathcal{A}$, and therefore the density matrix describing

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[32]According to equation (1.2), this basis would correspond to measuring the spin in an axis at right angles to $\hat{\mathbf{a}}$.

[33]We say that Bob is *almost* certain rather than *completely* certain because $\langle a_+(t)|a_-(t)\rangle$ will be very nearly zero rather than identically zero as discussed in footnote 31.

$\mathcal{S}$ would be $\hat{\rho}_{\mathcal{S}}(t_0) = |\psi\rangle\langle\psi|$.[34] Hence, if we consider $\hat{O}_{\mathcal{S}} = |\psi\rangle\langle\psi|$ as an observable on $\mathcal{S}$ corresponding to a measurement[35] that records the value 1 if the system is in the $|\psi\rangle$ and 0 if the system is in a state $|\psi'\rangle$ with $\langle\psi'|\psi\rangle = 0$, then both intuitively[36] and by equation (2.13),[37] we would have $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t_0)} = 1$. But if the scientist is to measure $\mathcal{S}$ to be in the $|\psi\rangle$-state, the expectation value $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t)}$ would have to be 1 for times t discernibly greater than t_0.

However, if more than one of the c_i are non-zero, then the scientist will not be able to measure the system $\mathcal{S}$ to be in the $|\psi\rangle$-state for any discernible length of time. To see why this is, we first note that

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_i |c_i|^2 |s_i\rangle\langle s_i| + \sum_{i\neq j} c_i\overline{c_j} \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j| .^{38} \tag{2.14}$$

Now because $\langle a_j(t)|a_i(t)\rangle \approx 0$ for $t > t_0 + \delta$, it follows that $\hat{\rho}_{\mathcal{S}} \approx \sum_i |c_i|^2 |s_i\rangle\langle s_i|$ for $t > t_0 + \delta$. It will then follow that $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t)} = \sum_i |c_i|^4$,[39] and this will only be 1 if only

[34]Recall if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ (i.e. $\mathcal{S}$ and $\mathcal{E}$ are not entangled), then $\langle\hat{O}_{\mathcal{U}}\rangle_{\xi} = \langle\hat{O}_{\mathcal{S}}\rangle_{\psi}$ as explained in footnote 25.

[35]This is a measurement we conduct by some means other than looking at the apparatus $\mathcal{A}$.

[36]I.e. we would expect the expectation value of $\hat{O}_{\mathcal{U}}$ to be 1 if we knew that $\mathcal{S}$ was in the state $|\psi\rangle$ with probability 1.

[37]I.e. given that $\hat{\rho}_{\mathcal{S}}(t_0) = |\psi\rangle\langle\psi| = \hat{O}_{\mathcal{S}}$, and that $\hat{O}_{\mathcal{S}}^2 = \hat{O}_{\mathcal{S}}$, and $\mathrm{Tr}_{\mathcal{S}}(\hat{O}_{\mathcal{S}}) = 1$, it follows that $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t_0)} = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}(t_0)\hat{O}_{\mathcal{S}}) = \mathrm{Tr}_{\mathcal{S}}(\hat{O}_{\mathcal{S}}) = 1$.

[38]To see this, it is sufficient to show that $\mathrm{Tr}_{\mathcal{A}}(|\xi_i(t)\rangle\langle\xi_j(t)|) = \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|$ for then we will obtain the first summand of $\hat{\rho}_{\mathcal{S}}$ from the fact that $|a_i(t)\rangle$ are normalized, and we will obtain the second summand by linearity of $\mathrm{Tr}_{\mathcal{A}}(\cdot)$. Well, taking $\{|\phi_k\rangle : k\}$ to be an orthonormal basis of $H_{\mathcal{A}}$, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{A}}(|\xi_i(t)\rangle\langle\xi_j(t)|) &= \sum_k \langle\phi_k| \Big(|\xi_i(t)\rangle\langle\xi_j(t)| \Big) |\phi_k\rangle = \sum_k \langle\phi_k|a_i(t)\rangle \langle a_j(t)|\phi_k\rangle |s_i\rangle\langle s_j| \\ &= \langle a_j(t)| \Big(\sum_k |\phi_k\rangle\langle\phi_k| \Big) |a_i(t)\rangle |s_i\rangle\langle s_j| = \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j| ,\end{aligned}$$

where we have used the fact that $I = \sum_k |\phi_k\rangle\langle\phi_k|$ is the identity operator on $H_{\mathcal{A}}$.

[39]This is because

$$\begin{aligned}\mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}} |\psi\rangle\langle\psi|) &\approx \mathrm{Tr}_{\mathcal{S}}(\sum_i |c_i|^2 |s_i\rangle\langle s_i| \sum_{jk} c_j\overline{c_k} |s_j\rangle\langle s_k|) \\ &= \mathrm{Tr}_{\mathcal{S}}(\sum_{ik} |c_i|^2 c_i\overline{c_k} |s_i\rangle\langle s_k|) = \sum_l \sum_{ik} \langle s_l| |c_i|^2 c_i\overline{c_k} |s_i\rangle \langle s_k|s_l\rangle = \sum_i |c_i|^4.\end{aligned}$$

one of the c_i is 1 and all the other c_i are 0. Hence, if more than one of the c_i are non-zero, the scientist will not be able to measure the system $\mathcal{S}$ to be in the $|\psi\rangle$-state for any discernible length of time.

2.4.7 Decoherence

Note that although for the original density matrix $|\psi\rangle\langle\psi|$ there is coherence between the states $|s_i\rangle$ and $|s_j\rangle$, this coherence effectively disappears when the system $\mathcal{S}$ interacts with the measuring device $\mathcal{A}$ (i.e. the $|s_i\rangle\langle s_j|$-coefficients of $\hat{\rho}_{\mathcal{S}}$ are approximately zero for $t > t_0 + \delta$). This is what we mean by *decoherence*: the coherence has effectively disappeared. The *decoherence time* δ which is the time it takes for $\langle a_i(t)|a_j(t)\rangle$ to go from 1 when $t = t_0$ to approximately zero when $t = t_0 + \delta$ will depend on what situation we are considering, but very often this time will be extremely small. For instance if we were measuring neurons firing in the brain, the decoherence time will typically be of the order $\delta = 10^{-19}\,\mathrm{s}$.[40] It is because of decoherence that we can't expect the system $\mathcal{S}$ to remain in the state $|\psi\rangle = \sum_i c_i\,|s_i\rangle$ for any discernible length of time, unless $|\psi\rangle$ is proportional to one of the $|s_i\rangle$-states.

Also note that when decoherence occurs, we say the coherence *effectively* disappears, insofar as the coherence will not be measurable if we only consider observables just acting on $H_{\mathcal{S}}$. Thus, after decoherence has taken place, if we restrict our attention to the system $\mathcal{S}$ alone, it will be experimentally indistinguishable[41] from the situation where $\mathcal{S}$ is known to be in one of the $|s_i\rangle$-states, but that we only know that it is in the

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[40]For details of this estimate see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 370.

[41]Recall the discussion following equation (2.9) on page 69 as well as the discussion on page 73.

$|s_i\rangle$-state with probability $|c_i|^2$. Nevertheless, the coherence is still there, since if we chose to consider more general observables on the composite $\mathcal{S}+\mathcal{A}$, the $|\xi_i(t)\rangle\langle\xi_j(t)|$-coefficients of $\hat{\rho}$ will continue to be $c_i\overline{c_j}$ which will in general will be non-zero, and as a whole, at time t the composite system will be in the state $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle$ with probability 1.

2.5 A Solution to the Preferred Basis Problem[42]

We can now see how decoherence theory solves the preferred basis problem. Although up to this point we have been focusing on how a system $\mathcal{S}$ interacts with a measuring apparatus $\mathcal{A}$, we can generalize to the situation in which a system $\mathcal{S}$ interacts with its environment $\mathcal{E}$. We can still define pointer states in the same way as we did on page 75. These pointer states will then make up the preferred basis. The two defining criteria of pointer states entail that pointer states will remain stable and immune to decoherence effects.

Since physicists have a good understanding of how different systems interact, they are able to explain what it is about a basis that makes it a preferred basis. The details of their analysis need not concern us here, but it's possible to show that for macroscopic and mesoscopic objects, states specified in terms of position decohere with one another very rapidly.[43] This explains why we don't detect $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$-states, but we do detect $|\text{Cat Alive}\rangle$ and $|\text{Cat Dead}\rangle$-states. Also note that $|\text{Cat Alive}\rangle$ does indeed have the property that it is immune to decoherence effects, for if we were to express $|\text{Cat Alive}\rangle$ in

[42]For more details, see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 71–84

[43]e.g. See the discussion in Schlosshauer, 94.

terms of the basis $\{|\psi_+\rangle, |\psi_-\rangle\}$ where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle \pm |\text{Cat Dead}\rangle)$, then $|\text{Cat Alive}\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)$. The corresponding density matrix would then be

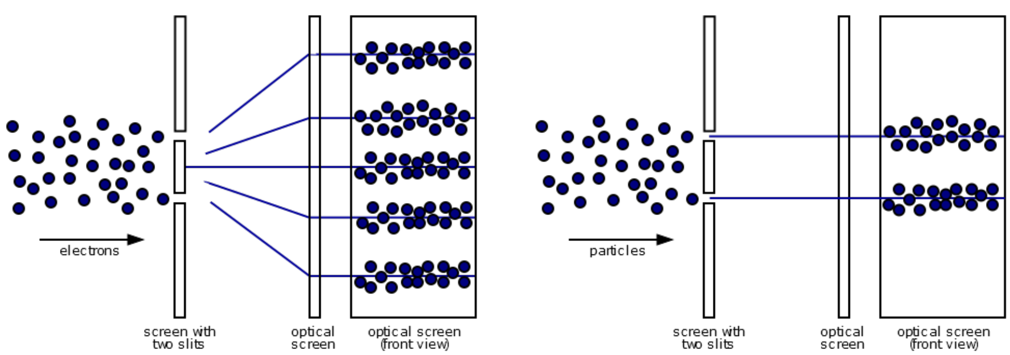
$$|\text{Cat Alive}\rangle\langle\text{Cat Alive}| = \frac{1}{\sqrt{2}}(|\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|). \quad (2.15)$$

Since in normal situations, the left-hand side of equation (2.15) will remain unperturbed by the environment, the coefficients of the off-diagonal terms $|\psi_\pm\rangle\langle\psi_\mp|$ will also remain as they are; that is, $|\psi_\pm\rangle$ and $|\psi_\mp\rangle$ will not decohere with one another. It is only in very contrived situations such as when the cat's environment is a poison releasing device coupled to a radioactive atom that $|\text{Cat Alive}\rangle$ will no longer be a pointer state with respect to this environment.

2.6 The Problem of Nonobservability of Interference and its Solution[44]

Before we consider the many-worlds interpretation in detail, it will be helpful to consider the role that decoherence plays in the removal of quantum interference at the macroscopic scale, as it is this lack of quantum interference between mutually exclusive states that justifies our belief that a system is in a definite state rather than in a superposition of alternative realities. The question of why quantum interference typically disappears at macroscopic scales is referred to as the problem of the nonobservability of interference.

We can explain this problem in the context of the double slit experiment: As figure 2.1



(A) Particles exhibiting interference. (B) Particles not exhibiting interference.

Figure 2.1: The Double-Slit Experiment. Particles are incident on a double slit. In diagram (A), the particles are exhibiting an interference pattern, whereas in diagram (B), the particles are not exhibiting an interference pattern. Whether or not there is interference will depend on factors such as the size of the particles and whether it can be ascertained which slit the particle went through. The larger the particles are, or the more information available as to which slit the particle went through, the less likely the particles will exhibit interference.[45]

indicates, when a beam of particles is incident on a double slit, the particles that are detected on the detection screen are distributed according to a distribution pattern which either exhibits quantum interference as shown on the left in the figure, or does

[44]See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 55–57, 63–65.

not exhibit such interference as shown on the right. Small particles like electrons and photons will tend to exhibit quantum interference, whereas mesoscopic particles will not typically exhibit quantum interference.[46]

To explain what is going on, we suppose that when just the top slit is open, the normalized state of the particle is $|\psi_1\rangle$, whereas if just the bottom slit is open, we suppose that the normalized state of the particle is $|\psi_2\rangle$, and when both slits are open, we suppose that the state of the particle will be $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Now let the variable x describe the position on the detection screen. For instance, we might take $x = 0$ to be the center of the detection screen, and take positive values of x as corresponding to positions on the upper part of the screen, and negative values of x as corresponding to positions on the lower part of the screen, but the precise convention we adopt won't matter. Then we define the $|x\rangle$-state[47] as the physical state describing the particle to be exactly located at position x on the screen. Note that the state $|x\rangle$ is indexed by a continuous parameter, x. This is in contrast to the basis of states $|s_i\rangle$ which we have been considering up until now which are indexed by discrete values of i such as $i = 1, 2, \ldots$. Because of this difference, we need to use calculus to deal with $|x\rangle$-states in a rigorous manner, but such details will not concern us here. In reality, because of the Heisenberg uncertainty principle, a particle is never in just one $|x\rangle$-state, but rather the particle will be in a superposition of many $|x\rangle$-states, which may or may

[46]There are exceptions to this rule. For example, a *Superconducting Quantum Interference Devices (SQUID)* can demonstrate quantum interference even at macroscopic scales where a large superconductor can enter into a superposition of two states with the current flowing in opposite directions in each state. See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, ch. 6.

[46]Diagrams (A) and (B) are by inductiveload, and are Public domain, via Wikimedia Commons. Sources: https://commons.wikimedia.org/wiki/File:Two-Slit_Experiment_Electrons.svg and https://commons.wikimedia.org/wiki/File:Two-Slit_Experiment_Particles.svg.

not be concentrated around a particular location, x_0 say. The more concentrated these $|x\rangle$-states of this superposition are concentrated around a particular location x_0, the more the particle will have the particle-like characteristic of being localized in one place. But if the $|x\rangle$-states of this superposition are more spread out, the particle will have more wave-like characteristics. So when physicists speak of particles, often they are not thinking of physical entities that are very localized in position, as non-physicists would think. Nevertheless, at the moment the particle is detected on the detection screen, it does seem to be highly localized.

Given a state $|\psi\rangle$ for a so-called particle, we define the function $\psi(x) = \langle x|\psi\rangle$. Because of the continuous nature of the variable x (in contrast to the discrete nature of i in a basis $\{|s_i\rangle : i\}$), the function

$$\rho(x) = |\psi(x)|^2 \tag{2.17}$$

determines a probability density for a range of outcomes rather than a probability for a specific outcome. Here, we do not need to go into the details of probability densities,[48] but roughly speaking, the greater the value of $\rho(x)$, the greater will be the relative probability of detecting the particle in the vicinity of location x. Thus, if

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[47] $|x\rangle$ is not really a state in the proper sense. With the states we've seen so far, when $|\phi\rangle$ and $|\psi\rangle$ have been normalized, then $|\langle\phi|\psi\rangle|^2$ will be a conditional probability, and hence at most 1. However, $|x\rangle$ cannot be normalized. This is because the bracket $\langle x|y\rangle$ is defined to be $\langle x|y\rangle = \delta(x-y)$ where $\delta(x)$ is the Dirac delta function such that

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \tag{2.16}$$

and has the property that $\int \mathrm{d}x\, \delta(x)\, f(x) = f(0)$ for any continuous function $f(x)$. The theory of distributions allows one to deal rigorously with Dirac delta functions. E.g. see Walter Rudin, *Functional Analysis*, Second Edition (McGraw-Hill, 1991), ch. 6.

[48] But if you are interested, a probability density $\rho(x)$ for a random variable X that has real values is a function such that $\rho(x) \geq 0\ \forall x \in \mathbb{R}$, and that $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$ and the probability that X has a value in the subset $U \subset \mathbb{R}$ is $\int_U \rho(x)\mathrm{d}x$.

$\rho(x') = 0$ for all x' in the vicinity of x, then the particle would not be detected in the vicinity of location x.

Now if $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$, then $\psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$. Therefore, the corresponding probability density will be

$$|\psi(x)|^2 = \frac{1}{2}(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}(\overline{\psi_1(x)}\psi_2(x)).$$ [49]

Now when the detection screen is far away from the double slits, we will have $|\psi_1(x)|^2 \approx |\psi_2(x)|^2$ for x near the center point on the screen. However, depending on slight changes in the value of x from the center point on the screen, sometimes $\psi_1(x)$ and $\psi_2(x)$ will be in phase so that $\psi_1(x) \approx \psi_2(x)$, in which case $|\psi(x)|^2 \approx 2|\psi_1(x)|^2$. But sometimes $\psi_1(x)$ and $\psi_2(x)$ will be out of phase so that $\psi_1(x) \approx -\psi_2(x)$, in which case $|\psi(x)|^2 \approx 0$. Hence, we get the interference pattern as shown in figure 2.1 (A).

Now in order to consider how decoherence affects interference, we let

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle\,|E_1(t)\rangle + |\psi_2\rangle\,|E_2(t)\rangle)$$

be the state of the composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ where $\mathcal{S}$ is a particle that has gone through the double slit and will be detected on the detection screen, and $\mathcal{E}$ is the local environment of the experimental set up. The expression for $|\Psi(t)\rangle$ indicates that we are assuming $\mathcal{S}$ doesn't become entangled with $\mathcal{E}$ when $\mathcal{S}$ is in the state $|\psi_1\rangle$ or $|\psi_2\rangle$. Corresponding to $|\Psi(t)\rangle$, we can define the density matrix $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. We can

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[49] Here Re means the real part of a complex number. Thus, if the complex number $z = \alpha + i\beta$ for real numbers α and β, then $\mathrm{Re}(z) = \alpha$. To see why the above equation holds, we recall that $|z|^2 = z\overline{z}$ and that $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$. Therefore, if $z = \frac{1}{\sqrt{2}}(v + w)$ for complex number v and w, then $|z|^2 = \frac{1}{\sqrt{2}}(v+w)\frac{1}{\sqrt{2}}\overline{(v + w)} = \frac{1}{2}(v+w)\overline{(v + w)} = \frac{1}{2}(v\overline{v}+w\overline{w}+w\overline{v}+v\overline{w}) = \frac{1}{2}(|v|^2+|w|^2+w\overline{v}+\overline{w\overline{v}}) = \frac{1}{2}(|v|^2 + |w|^2 + 2\,\mathrm{Re}(\overline{v}w))$.

also define the observable[50] $|x\rangle\langle x|_{\mathcal{S}}$ for the system $\mathcal{S}$ so that $|x\rangle\langle x|_{\mathcal{S}}\,|\psi\rangle_{\mathcal{S}} = \psi(x)\,|x\rangle_{\mathcal{S}}$. As we saw in equation (2.5) on page 66, we can naturally extend the action of $|x\rangle\langle x|_{\mathcal{S}}$ to $H_{\mathcal{U}}$.[51] This allows us to define

$$\rho_{\mathcal{U}}(x,t) \stackrel{\text{def}}{=} \mathrm{Tr}_{\mathcal{U}}(\hat{\rho}(t)_{\mathcal{U}}\,|x\rangle\langle x|_{\mathcal{U}}).$$

In the specific case when $\mathcal{S}$ and $\mathcal{E}$ are not entangled so that $\hat{\rho}_{\mathcal{U}}(t) = |\xi(t)\rangle\langle\xi(t)|_{\mathcal{U}}$ with $|\xi(t)\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}\,|E(t)\rangle_{\mathcal{E}}$ for normalized states $|\psi\rangle_{\mathcal{S}}$ and $|E(t)\rangle_{\mathcal{E}}$, we have $\rho_{\mathcal{U}}(x,t) = |\psi(x)|^2$ which is equal to the probability density function $\rho(x)$ we saw in equation (2.17).[52] However, if $|E_1(t)\rangle_{\mathcal{E}}$ is not proportional to $|E_2(t)\rangle_{\mathcal{E}}$, then $|\Psi(t)\rangle$ will be an entangled state of $\mathcal{S}$ and $\mathcal{E}$. But whether or not $\mathcal{S}$ and $\mathcal{E}$ are entangled, we can still use equation (2.14) to calculate the partial trace:

$$\hat{\rho}_{\mathcal{S}}(t) = \frac{1}{2}(|\psi_1\rangle\langle\psi_1|_{\mathcal{S}} + |\psi_2\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,|\psi_1\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}}\,|\psi_2\rangle\langle\psi_1|_{\mathcal{S}}).$$

By equations (2.10) and (2.13), we therefore have

$$\rho_{\mathcal{U}}(x,t) = \frac{1}{2}\Big(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}\left(\langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\overline{\psi_2(x)}\psi_1(x)\right)\Big).^{53} \qquad (2.18)$$

[50]Note that we only call $|x\rangle\langle x|_{\mathcal{S}}$ an observable in an analogical sense since it is not a compact Hermitian operator acting on the Hilbert space of states $H_{\mathcal{S}}$. If we were being more rigorous, we would need to consider a Hermitian operator of the form $\int \sigma(x)\,|x\rangle\langle x|_{\mathcal{S}}\,\mathrm{d}x$ for an appropriate test function $\sigma(x)$.

[51]Strictly speaking, it is not $|x\rangle\langle x|_{\mathcal{S}}$ that is extended to act on $H_{\mathcal{U}}$, but rather a Hermitian operator of the form $\int \sigma(x)\,|x\rangle\langle x|_{\mathcal{S}}\,\mathrm{d}x$ for an appropriate test function $\sigma(x)$ that is extended to $H_{\mathcal{U}}$. For a state $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}\,|E\rangle_{\mathcal{E}}$, the action of $|x\rangle\langle x|_{\mathcal{U}}$ on $|\xi\rangle_{\mathcal{U}}$ gives the 'state' $|x\rangle\langle x|_{\mathcal{U}}\,|\xi\rangle_{\mathcal{U}} \stackrel{\text{def}}{=} \psi(x)\,|x\rangle_{\mathcal{S}}\,|E\rangle_{\mathcal{E}}$, but since this is not normalizable, we have to 'smear' it by integrating it with respect to the test function $\sigma(x)$.

[52]To see this, note that we can ignore $\mathcal{E}$ in calculating $\langle |x\rangle\langle x|_{\mathcal{U}}\rangle_{\xi}$ since when $\mathcal{S}$ and $\mathcal{E}$ are not entangled, $\langle |x\rangle\langle x|_{\mathcal{U}}\rangle_{\xi} = \langle |x\rangle\langle x|_{\mathcal{S}}\rangle_{\psi}$ as explained in footnote 17. We can therefore just consider $\mathcal{S}$ and drop the subscripts. Furthermore, as we saw in equation (2.10), $\langle \hat{O}\rangle_{\psi} = \mathrm{Tr}(\hat{\rho}\hat{O})$ where $\hat{\rho} = |\psi\rangle\langle\psi|$. We can thus take an orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle, \ldots\}$ of $H_{\mathcal{S}}$ with $|\psi_1\rangle = |\psi\rangle$. Then $\rho(x) = \mathrm{Tr}(\hat{\rho}\,|x\rangle\langle x|) = \mathrm{Tr}(|\psi\rangle\langle\psi|\,|x\rangle\langle x|) = \sum_i \langle\psi_i|\psi\rangle\,\langle\psi|x\rangle\,\langle x|\psi_i\rangle = \langle\psi_1|\psi\rangle\,\langle\psi|x\rangle\,\langle x|\psi_1\rangle = \langle\psi|x\rangle\,\langle x|\psi\rangle = \overline{\langle x|\psi\rangle}\,\langle x|\psi\rangle = |\langle x|\psi\rangle|^2 = |\psi(x)|^2$.

Thus, if $\langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \approx 0$ then $\rho_{\mathcal{U}}(x,t) \approx \frac{1}{2}\Big(|\psi_1(x)|^2 + |\psi_2(x)|^2\Big)$ and so we would observe a distribution pattern not exhibiting interference as shown in figure 2.1 (B). On the other hand, if $\langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \not\approx 0$ we would get a distribution pattern exhibiting interference as shown in figure 2.1 (B). Therefore, since it is often possible to determine the asymptotic behavior of $\langle E_1(t)|E_2(t)\rangle_{\mathcal{E}}$,[54] decoherence theory gives us a means of determining whether or not quantum interference will be exhibited.

2.7 The Problem of Outcomes

In the last two sections we have seen how decoherence theory solves the preferred basis problem and the problem of the nonobservability of interference. However, there is a third fundamental problem in quantum physics which decoherence theory is unable to solve. This is the problem of outcomes. As discussed in subsection 2.4.6, in the von Neumann measurement scheme, it is supposed that for the measurement of a physical system $\mathcal{S}$ to take place, it must interact with a measuring device $\mathcal{A}$ which

[53]The calculation is as follows:

$$\begin{aligned}
\rho_{\mathcal{U}}(x,t) &= \mathrm{Tr}_{\mathcal{U}}(\hat{\rho}(t)_{\mathcal{U}}\,|x\rangle\langle x|_{\mathcal{U}}) = \left\langle |x\rangle\langle x|_{\mathcal{U}}\right\rangle_{\rho(t)} = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}(t)\,|x\rangle\langle x|_{\mathcal{S}}) \\
&= \mathrm{Tr}_{\mathcal{S}}\Big(\frac{1}{2}(|\psi_1\rangle\langle\psi_1|_{\mathcal{S}} + |\psi_2\rangle\langle\psi_2|_{\mathcal{S}} \\
&\qquad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,|\psi_1\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}}\,|\psi_2\rangle\langle\psi_1|_{\mathcal{S}})\,|x\rangle\langle x|_{\mathcal{S}}\Big) \\
&= \mathrm{Tr}_{\mathcal{S}}\Big(\frac{1}{2}(\langle\psi_1|x\rangle_{\mathcal{S}}\,|\psi_1\rangle\langle x|_{\mathcal{S}} + \langle\psi_2|x\rangle_{\mathcal{S}}\,|\psi_2\rangle\langle x|_{\mathcal{S}} \\
&\qquad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\langle\psi_2|x\rangle_{\mathcal{S}}\,|\psi_1\rangle\langle x|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}}\,\langle\psi_1|x\rangle_{\mathcal{S}}\,|\psi_2\rangle\langle x|_{\mathcal{S}})\Big) \\
&= \frac{1}{2}\Big(\langle\psi_1|x\rangle_{\mathcal{S}}\,\langle x|\psi_1\rangle_{\mathcal{S}} + \langle\psi_2|x\rangle_{\mathcal{S}}\,\langle x|\psi_2\rangle_{\mathcal{S}} \\
&\qquad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\langle\psi_2|x\rangle_{\mathcal{S}}\,\langle x|\psi_1\rangle_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}}\,\langle\psi_1|x\rangle_{\mathcal{S}}\,\langle x|\psi_2\rangle_{\mathcal{S}}\Big) \\
&= \frac{1}{2}\Big(\overline{\langle x|\psi_1\rangle_{\mathcal{S}}}\,\langle x|\psi_1\rangle_{\mathcal{S}} + \overline{\langle x|\psi_2\rangle_{\mathcal{S}}}\,\langle x|\psi_2\rangle_{\mathcal{S}} \\
&\qquad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\overline{\langle x|\psi_2\rangle_{\mathcal{S}}}\,\langle x|\psi_1\rangle_{\mathcal{S}} + \overline{\langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\overline{\langle x|\psi_2\rangle_{\mathcal{S}}}\,\langle x|\psi_1\rangle_{\mathcal{S}}}\Big) \\
&= \frac{1}{2}\Big(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}\,\big(\langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\overline{\psi_2(x)}\psi_1(x)\big)\Big).
\end{aligned}$$

[54]i.e. whether or not $\langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \to 0$ as $t \to \infty$ and how fast this convergence might take place.

together satisfy the conditions 1. and 2. on page 74. If $\mathcal{S}$ is initially in a superposition of states $|\psi\rangle = \sum_i c_i |s_i\rangle$ then for $\mathcal{A}$ to measure $\mathcal{S}$, it is necessary for the combined system $\mathcal{S} + \mathcal{A}$ to enter into a superposition

$$|\psi\rangle |a_r(t_0)\rangle \xrightarrow{\text{time evolution}} \sum_i c_i |s_i\rangle |a_i(t)\rangle \,. \tag{2.19}$$

However, although the evolution described in (2.19) must take place if $\mathcal{A}$ is to measure $\mathcal{S}$, it is not sufficient. When one takes the partial trace of $|\psi\rangle\langle\psi|$ over $\mathcal{A}$, then according to (2.14),

$$\mathrm{tr}_{\mathcal{A}}(|\psi\rangle\langle\psi|) \xrightarrow{\text{time evolution}} \sum_i |c_i|^2 |s_i\rangle\langle s_i| \,. \tag{2.20}$$

But as noted on page 73, we cannot give an ignorance interpretation to $\sum_i |c_i|^2 |s_i\rangle\langle s_i|$ for as d'Espagnat puts it, this is an improper mixture. When considered together, the system $\mathcal{S}$ and the apparatus $\mathcal{A}$ remain in the superposition described by (2.19), and so none of the measurement outcomes from the set of possible outcomes $\{|s_i\rangle : i\}$ have actually occurred. The problem of explaining how the composite system $\mathcal{S} + \mathcal{A}$ goes from being in the state $\sum_i c_i |s_i\rangle |a_i(t)\rangle$ to a state $|s_i\rangle |a_i(t)\rangle$ is known as the *problem of outcomes*.

2.8 The Many-Worlds Interpretation

Not everyone is convinced that the problem of outcomes is a genuine problem. In particular, people who endorse the many-worlds interpretation of quantum physics effectively argue that there are no outcomes in the traditional sense. In this section, we give an account of the many-worlds interpretation of quantum physics and why physicists find it attractive. To this end, let us consider a physical universe $\mathcal{U} = \mathcal{S} + \mathcal{A} + \mathcal{P}_A + \mathcal{P}_B + \mathcal{E}$ consisting of subsystems $\mathcal{S}, \mathcal{A}, \mathcal{P}_A, \mathcal{P}_B$ and $\mathcal{E}$. $\mathcal{S}$ is the physical system under investigation; $\mathcal{A}$ is some measuring apparatus that interacts with $\mathcal{S}$; $\mathcal{P}_A$ and $\mathcal{P}_B$ are the physical systems corresponding to two scientists, Alice and Bob who observed the apparatus $\mathcal{A}$; and $\mathcal{E}$ is the remainder of the physical universe $\mathcal{U}$. For convenience, we define the composite subsystem $\mathcal{V} = \mathcal{S}+\mathcal{A}+\mathcal{P}_A+\mathcal{P}_B$ so that $\mathcal{U} = \mathcal{V}+\mathcal{E}$. As above on page 75, we assume that there is an orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$ which we again refer to as pointer states, but now we assume that there are ready states $|a_r(t)\rangle \in H_{\mathcal{A}}$, $|p_{r,A}(t)\rangle \in H_{\mathcal{P}_A}$, $|p_{r,B}(t)\rangle \in H_{\mathcal{P}_B}$, and $|E_r(t)\rangle \in H_{\mathcal{E}}$ and that for each i, there are normalized states $|a_i(t)\rangle \in H_{\mathcal{A}}$, $|p_{i,A}(t)\rangle \in H_{\mathcal{P}_A}$, $|p_{i,B}(t)\rangle \in H_{\mathcal{P}_B}$, and $|E_i(t)\rangle \in H_{\mathcal{E}}$ such that

1. for any $t \geq 0$ we have the evolution of the states

$$|s_i\rangle\,|a_r(t)\rangle\,|p_{r,A}(t)\rangle\,|p_{r,B}(t)\rangle\,|E_r(t)\rangle \xrightarrow{\text{time evolution}} |s_i\rangle\,|a_i(t)\rangle\,|p_{i,A}(t)\rangle\,|p_{i,B}(t)\rangle\,|E_i(t)\rangle\,,$$

2. there exists $\delta > 0$ such that if $t > t_0 + \delta$, then for $i \neq j$, $\langle a_i(t)|a_j(t)\rangle \approx 0$, $\langle p_{i,A}(t)|p_{j,A}(t)\rangle \approx 0$, $\langle p_{i,B}(t)|p_{j,B}(t)\rangle \approx 0$ and $\langle E_i(t)|E_j(t)\rangle \approx 0$.[55]

[55]Again, recall footnote 29.

We also suppose that the $|p_{i,A}(t)\rangle$-state would describe actions of Alice consistent with her observing the apparatus being in the $|a_i(t)\rangle$-state, for example, her writing down in her log book that the apparatus is in the $|a_i(t)\rangle$-state or telling her colleague that this is the case. Likewise, we assume the $|p_{i,B}(t)\rangle$-state is consistent with Bob also observing the apparatus to be in the $|a_i(t)\rangle$-state.

Now suppose the initial (normalized) state of $\mathcal{S}$ is $|\psi\rangle = \sum_i c_i |s_i\rangle$, so that the state for the composite system $\mathcal{U}$ is $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ where $|\xi_i(t)\rangle = |s_i\rangle |a_i(t)\rangle |p_{i,A}(t)\rangle |p_{i,B}(t)\rangle$. We also define the corresponding density matrix for the composite system $\hat{\rho} = |\Psi\rangle\langle\Psi|$. If we were unable to make any observations on $\mathcal{E}$, then the partial trace $\hat{\rho}_{\mathcal{V}}(t) = \mathrm{Tr}_{\mathcal{E}}(\hat{\rho}(t))$ will contain all the information we need to work out the expectation values for any observables of $\mathcal{V}$. So just as with equation (2.14), we will have

$$\begin{aligned} \hat{\rho}_{\mathcal{V}}(t) &= \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)| + \sum_{i\neq j} c_i\overline{c_j} \langle E_j(t)|E_i(t)\rangle |\xi_i(t)\rangle\langle\xi_j(t)| \\ &\approx \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)| \end{aligned} \tag{2.21}$$

for $t > t_0 + \delta$. Then the expectation values of any observables on $\mathcal{V}$ will be indistinguishable from the scenario in which $\mathcal{V}$ is actually in one of the $|\xi_i(t)\rangle$-states with probability $|c_i|^2$.[56] It would nevertheless be incorrect for us to conclude on the basis of decoherence theory alone that $\mathcal{V}$ actually was in one of those $|\xi_i(t)\rangle$-states, since equation (2.21) is based on a subjective distinction between $\mathcal{V}$ and $\mathcal{E}$ in the decomposition $\mathcal{U} = \mathcal{V} + \mathcal{E}$. Human scientists make this distinction to reflect the fact that they

[56] Again recall the discussion following equation (2.9) on page 69. There is the question of uniqueness of $\hat{\rho}_{\mathcal{V}} = \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)|$. If all the $|c_i|^2$ are unique, then if we have another decomposition $\hat{\rho}_{\mathcal{S}+\mathcal{A}+\mathcal{P}_A+\mathcal{P}_B} = \sum_i |c_i|^2 |\xi_i'(t)\rangle\langle\xi_i'(t)|$ it follows that $|\xi_i(t)\rangle \propto |\xi_i'(t)\rangle$. But even if some of the $|c_i|^2$ are the same, criteria 1 and 2 above will ensure that states with the same value of $|c_i|^2$ will be determined up to permutation.

can only perform measurements on $\mathcal{V}$ and can't measure $\mathcal{E}$. But if a super-intelligent being could measure everything in $\mathcal{U}$, then such a being would not say that $\mathcal{V}$ was in one of the $|\xi_i(t)\rangle$-states, but rather that $\mathcal{U}$ was in the state $|\Psi(t)\rangle$. As we have already discussed on pages 73–74, the density matrix $\hat{\rho}_{\mathcal{V}}(t)$ is not a mixed state, but is an improper mixture.

Now if we define the observables $\hat{O}_{i,\mathcal{A}}(t) = |p_{i,A}(t)\rangle\langle p_{i,A}(t)|$ that would measure the behavior of Alice, and the observables $\hat{O}_{i,\mathcal{B}}(t) = |p_{i,B}(t)\rangle\langle p_{i,B}(t)|$ that would measure the behavior of Bob, then for $t > t_0 + \delta$, we see that $\hat{O}_{i,\mathcal{A}}(t)\hat{O}_{j,\mathcal{B}}(t)\,|\Psi(t)\rangle \approx 0$, when $i \neq j$. This means that when we consider $\hat{O}_{i,\mathcal{A}}(t)\hat{O}_{j,\mathcal{B}}(t)$ as an observable acting on $\mathcal{V}$, the expectation value $\langle \hat{O}_{i,\mathcal{A}}(t)\hat{O}_{j,\mathcal{B}}(t)\rangle_{\rho_{\mathcal{V}}(t)}$ will be approximately zero for $i \neq j$. What this means is that if we consider ourselves as observing Alice and Bob observing the apparatus, then after time $t_0 + \delta$, the probability we would see Alice and Bob disagreeing with each other concerning their observations of the apparatus would be approximately 0. On the other hand, since $\hat{O}_{i,\mathcal{A}}(t)\hat{O}_{i,\mathcal{B}}(t)\,|\Psi(t)\rangle \approx c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ for $t > t_0 + \delta$, it follows that $\langle \hat{O}_{i,\mathcal{A}}(t)\hat{O}_{i,\mathcal{B}}(t)\rangle_{\rho_{\mathcal{V}}(t)} = |c_i|^2$. We would thus observe Alice and Bob observing the apparatus to be in the $|a_i(t)\rangle$-state with probability $|c_i|^2$.

But note that on the assumption that there are no hidden variables, if we did actually make such an observation and this observation corresponded to reality, then the quantum state $|\Psi(t)\rangle$ would have had to have changed to $|\xi_i(t)\rangle\,|E_i(t)\rangle$, since before our observation when $|\Psi(t)\rangle$ was a complete description of $\mathcal{U}$, we would say Alice and Bob will measure the $|a_i(t)\rangle$-state with probability $|c_i|^2$, but when we are actually seeing them measuring the $|a_i(t)\rangle$-state, we would have to say that now the probability they are measuring the $|a_i(t)\rangle$-state is 1, and hence we would say that the system was in the $|\xi_i(t)\rangle\,|E_i(t)\rangle$-state. Whether or not the process of the state going from $|\Psi(t)\rangle$ to $|\xi_i(t)\rangle\,|E_i(t)\rangle$ was instantaneous or took a non-infinitesimal amount of time, this interpretation would be susceptible to the problems already discussed with the Copenhagen interpretation in section 1.2.

But in the *many-worlds interpretation*, rather than assuming that $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ is the complete description of $\mathcal{U}$ that enables us to work the probability of certain outcomes, we simply say that $|\Psi(t)\rangle$ is a complete description of the state of $\mathcal{U}$, and we drop the assumption that we need to interpret this state as describing probabilities of outcomes. Thus a many-worlds adherent would say we can understand what the state of $\mathcal{U}$ is on its own terms without the need to appeal to any other extrinsic principle such as measurement. Just as we don't puzzle over how to interpret what a sphere is in terms of an extrinsic principle, we don't need to puzzle over how to interpret the space of states of $\mathcal{U}$. We can think of the mathematical formalism $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ describing the state of $\mathcal{U}$ as being somewhat akin to a point lying on a sphere given by the equation $x^2 + y^2 + z^2 = 1$.[57] Although we might be tempted to interpret $|\Psi(t)\rangle$ as describing the probability of outcomes, we are not obliged to do so, since these probabilities can instead be understood to be grounded in the symmetries the system possesses rather than in terms of the frequency of how many measurement outcomes are likely to occur. For instance, when we see a coin and judge that it will come up heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$, we intuit this by looking at the symmetry of the coin rather than tossing the coin millions

[57] On the assumption that $|\Psi(t)\rangle$ is normalized, we could think of $|\Psi(t)\rangle$ as specifying a point on the (possibly infinite-dimensional) hypersphere $\{(c_1, c_2 \ldots) : \sum_i |c_i|^2 = 1\}$.

of times and counting how often it comes up heads and how often it comes up tails. Thus, we might suppose $|\Psi(t)\rangle$ has analogous symmetries that allow us to determine its c_i coefficients without the need to posit any of the $|\xi_i(t)\rangle\,|E_i(t)\rangle$ measurement outcomes being realized.

As for the decomposition $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ in terms of the $|\xi_i(t)\rangle\,|E_i(t)\rangle$ basis states, decoherence theory gives us a natural account of why we should choose this basis rather than any other. When $\mathcal{U}$ is in the state $|\Psi(t_0)\rangle = \Big(\sum_i c_i\,|\xi_i(t_0)\rangle\Big)\,|E_r(t)\rangle$, we can think of this state as describing one world, W say. But once $t > t_0 + \delta$ so that $\langle E_i(t)|E_j(t)\rangle \approx 0$ for $i \neq j$, we can think of each $|\xi_i(t)\rangle\,|E_i(t)\rangle$-component as a different world W_i. Thus, for $t > t_0 + \delta$, we say the world W has *branched* into as many-worlds W_i for which the c_i are non-zero.

But why should we think that there are literally many worlds? Well, from an ontological point of view, one might very well think that there is really only one world and that this world is described by $|\Psi(t)\rangle$; it would be a rather weird world since the entanglement between $\mathcal{V}$ and $\mathcal{E}$ would mean there wouldn't be any absolute matters of fact describing $\mathcal{V}$. But it might not be a bad thing to say that the "many" in the many-worlds interpretation is really just a figure of speech that we shouldn't take too literally. After all, a common objection to the many-worlds interpretation is that it is ontologically extravagant and that we should appeal to Occam's Razor. But if we just say that there is actually only one world described by $|\Psi(t)\rangle$ then this "many"-worlds interpretation is actually rather parsimonious from an ontological point of view.

But if by literal, we mean descriptive rather than ontological, it does seem rather natural to say that there are literally many worlds. For even though we might initially suspect that the worlds W_i and W_j are not clearly delineated given the fact that $\langle E_i(t)|E_j(t)\rangle$ is very small but not zero for $i \neq j$, we can nevertheless expect $\langle \xi_i(t)|\xi_j(t)\rangle$ to be identically zero for $i \neq j$, just as we can expect $\langle \hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle$ to be identically zero.[58] Thus, if we define $|W_i(t)\rangle = |\xi_i(t)\rangle\,|E_i(t)\rangle$, then the $\langle W_i(t)|W_j(t)\rangle$ will be identically zero for $i \neq j$, and so we would in fact be able to clearly delineate these worlds.

Still, the supposition that $|\Psi(t)\rangle = \sum_i c_i\,|W_i(t)\rangle$ with $\langle W_i|W_j\rangle = 0$ for $i \neq j$ is not of itself sufficient justification for describing the state $|\Psi(t)\rangle$ as a composition of mutually exclusive world descriptions given by the $|W_i(t)\rangle$. After all, the fact that

$$\begin{aligned}|\text{Cat Alive}\rangle =& \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)\Big)\\ &+ \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\Big).\end{aligned}$$

does not incline us to think of the state $|\text{Cat Alive}\rangle$ as being composed of the mutually exclusive cat states $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$.

But the key justification for describing the state $|\Psi(t)\rangle$ as a composition of the mutually exclusive $|W_i(t)\rangle$-states is the fact that the states $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ decohere for $i \neq j$, that is, the off-diagonal entries $|\xi_i(t)\rangle\langle\xi_j(t)|$ of the reduced density matrix $\hat{\rho}_{\mathcal{V}}(t)$ will tend to zero, and as we saw in section 2.6, it will follow that quantum interference effects between $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ will then tend to zero. Thus, when it comes to

[58] It is also reasonable to suppose that in situations such as the double-slit experiment described on page 82 that $\langle\psi_1(t)|\psi_2(t)\rangle$ is identically zero. This is because $\langle\psi_1(t)|\psi_2(t)\rangle$ when t is the time at which the particle is going through the slit, and this will remain zero because of a property of the time evolution operator known as unitarity.

observables defined on $\mathcal{V}$, using equations (2.13) and (2.21), we can calculate the expectation value of an observable $\hat{O}_{\mathcal{V}}$ as a weighted sum of expectation values for each of the states $|\xi_i(t)\rangle$:

$$\langle \hat{O}_{\mathcal{U}} \rangle_{\Psi(t)} \approx \sum_i |c_i|^2 \langle \hat{O}_{\mathcal{V}} \rangle_{\xi_i(t)} . \tag{2.22}$$

The fact that (2.22) is only an approximation suggests that the time at which branching occurs is not well-defined. All that we can do is choose a sufficiently large time interval δ such that for $t > t_0 + \delta$, the approximation (2.22) meets our desired level of accuracy.

Despite this vagueness on when branching occurs, we can still form a natural and well-defined notion of worlds according to the following definition: a set $\{W_i : i\}$ is the set of worlds for a universe $\mathcal{U} = \mathcal{V} + \mathcal{E}$ when

1. W_i is a description of $\mathcal{U}$ given by $|W_i(t)\rangle = |\xi_i(t)\rangle |E_i(t)\rangle$ where $|\xi_i(t)\rangle$ is a state of $\mathcal{V}$ and $|E_i(t)\rangle$ is a state of $\mathcal{E}$,
2. $\mathcal{U}$ is in the state $|\Psi(t)\rangle = \sum_i c_i |W_i(t)\rangle$ with all $c_i \neq 0$.
3. $\langle \xi_i(t)|\xi_j(t)\rangle = 0$ for $i \neq j$,
4. $\langle E_i(t)|E_j(t)\rangle \to 0$ as $t \to \infty$ for $i \neq j$, and the convergence is such that for any observable $\hat{O}_{\mathcal{V}}$ defined on $\mathcal{V}$, $\langle \hat{O}_{\mathcal{U}} \rangle_{\Psi(t)} \to \sum_i |c_i|^2 \langle \hat{O}_{\mathcal{V}} \rangle_{\xi_i(t)}$.

Note that according to this definition, the description $|\Psi(t)\rangle$ is rather trivially a world – we just take the environment $\mathcal{E}$ to be empty so that there would be only one $|E_i(t)\rangle$ which would be the vacuum state. So there is at least one world according to this definition. There is a question of whether there could be more than one world, and this would depend on whether we could really have a non-trivial decomposition $\mathcal{U} = \mathcal{V} + \mathcal{E}$, for the supposition that there is such a decomposition requires that it is possible to distinguish $\mathcal{V}$ and $\mathcal{E}$. But this might not in fact be possible. For instance, if the ultimate fate of the universe was that it would collapse into a singularity, then there would come a point at which it wouldn't be possible to make a distinction between $\mathcal{V}$ and $\mathcal{E}$. But despite this possible concern, the above definition makes it seem plausible that there could be many well-defined worlds W_i.[59]

When we look at a particular $|\xi_i(t)\rangle$, it will look like it is describing a fairly classical world with scientists performing their measurements and agreeing about what they measure. And as long as the $|\xi_i(t)\rangle$-states remain pointer states with respect to $|E_i(t)\rangle$, no branching will occur. But typically, a $|\xi_i(t)\rangle$-state will not indefinitely remain a pointer state with respect to $|E_i(t)\rangle$. We can think of how this happens with the Stern-Gerlach experiment. For if one Stern-Gerlach apparatus has its magnetic field orientated in the $\hat{\mathbf{a}}$-direction, then $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ will be pointer states for a silver atom in the vicinity of this apparatus. But if the same silver atom then travels onward to another Stern-Gerlach apparatus with its magnetic field now orientated in the $\hat{\mathbf{b}}$-direction with $\hat{\mathbf{b}} \neq \hat{\mathbf{a}}$, $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ will no longer be pointer states with respect to their environment, and so branching will occur. But this is not necessarily a problem for the definition of many-worlds given above on page 94, for when a $|\xi_i(t)\rangle$-state does not indefinitely remain a pointer state with respect to $|E_i(t)\rangle$, we can just rewrite $|\xi_i(t)\rangle$ as a sum of pointer states $|\xi_{ij}(t)\rangle$ and $|E_i(t)\rangle$ as a sum of their respective environments $|E_{ij}(t)\rangle$, and then $|E_i(t)\rangle$ will be like a ready state for the

[59] For the purposes of this chapter, plausibility is enough. I am only trying to show why physicists might find the many-worlds interpretation of quantum physics attractive. I am certainly not trying to show that the many-worlds interpretation is the most convincing and satisfactory interpretation of quantum physics.

$|E_{ij}(t)\rangle$. Assuming we can do this so that the $|\xi_{ij}(t)\rangle$ are orthogonal to the $|\xi_{i'j'}(t)\rangle$ when $i' \neq i$ or $j' \neq j$, then we would still be able to have well-defined worlds according to the definition given above.

2.9 The Many-Worlds Solution to the EPR-Bohm Paradox

We are now in a position to consider how proponents of the many-worlds interpretation attempt to resolve the EPR-Bohm paradox. We thus suppose there is a spin-singlet as described in section 1.2 consisting of two particles q_A and q_B that are in the entangled Bell state

$$|\Psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B). \tag{1.8 revisited}$$

If we assume that the two experimenters Alice and Bob (themselves constituting physical systems $\mathcal{P}_A$ and $\mathcal{P}_B$ respectively) set their Stern-Gerlach apparatuses $\mathcal{A}_A$ and $\mathcal{A}_B$ to measure their respective particles along the same axis, then by (1.11), we can assume they both perform their measurements along the $\hat{\mathbf{a}}$-axis. This means that $|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B$ and $|\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B$ will be pointer states of the composite system $\mathcal{A}_A + \mathcal{A}_B + \mathcal{P}_A + \mathcal{P}_B$. There will thus be ready states $|a_{r,A}(t)\rangle \in H_{\mathcal{A}_A}$, $|a_{r,B}(t)\rangle \in H_{\mathcal{A}_B}$, $|p_{r,A}(t)\rangle \in H_{\mathcal{P}_A}$ and $|p_{r,B}(t)\rangle \in H_{\mathcal{P}_B}$, and normalized states $|a_{\pm,A}(t)\rangle \in H_{\mathcal{A}_A}$, $|a_{\pm,B}(t)\rangle \in H_{\mathcal{A}_B}$, $|p_{\pm,A}(t)\rangle \in H_{\mathcal{P}_A}$ and $|p_{\pm,B}(t)\rangle \in H_{\mathcal{P}_B}$ such that

$$\begin{aligned}&|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B\,|a_{r,A}(t)\rangle\,|a_{r,B}(t)\rangle\,|p_{r,A}(t)\rangle\,|p_{r,B}(t)\rangle\\ &\qquad\xrightarrow{\text{time evolution}}\;|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B\,|a_{+,A}(t)\rangle\,|a_{-,B}(t)\rangle\,|p_{+,A}(t)\rangle\,|p_{-,B}(t)\rangle\,,\end{aligned}$$

and similarly

$$\begin{aligned}&|\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B\,|a_{r,A}(t)\rangle\,|a_{r,B}(t)\rangle\,|p_{r,A}(t)\rangle\,|p_{r,B}(t)\rangle\\ &\qquad\xrightarrow{\text{time evolution}}\;|\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B\,|a_{-,A}(t)\rangle\,|a_{+,B}(t)\rangle\,|p_{-,A}(t)\rangle\,|p_{+,B}(t)\rangle\,.\end{aligned}$$

It will therefore follow that

$$\begin{aligned}
&|\Psi_{\text{Bell}}\rangle\,|a_{r,A}(t)\rangle\,|a_{r,B}(t)\rangle\,|p_{r,A}(t)\rangle\,|p_{r,B}(t)\rangle \\
&\quad\xrightarrow{\text{time evolution}} \frac{1}{\sqrt{2}}\,|\hat{\mathbf{a}}+\rangle_A\;|\hat{\mathbf{a}}-\rangle_B\,|a_{+,A}(t)\rangle\,|a_{-,B}(t)\rangle\,|p_{+,A}(t)\rangle\,|p_{-,B}(t)\rangle \qquad (2.23)\\
&\qquad\qquad + \frac{1}{\sqrt{2}}\,|\hat{\mathbf{a}}-\rangle_A\;|\hat{\mathbf{a}}+\rangle_B\,|a_{-,A}(t)\rangle\,|a_{+,B}(t)\rangle\,|p_{-,A}(t)\rangle\,|p_{+,B}(t)\rangle\,.
\end{aligned}$$

Thus, in the language of the many-worlds interpretation, the first summand of (2.23) corresponds to a world in which Alice observes her particle to be spin up, and Bob observes his particle to be spin down, and the second summand of (2.23) corresponds to a world in which Alice observes her particle to be spin down, and Bob observes his particle to be spin up. So in each world, Alice and Bob obtain opposite results. But if on the other hand, Bob chooses to make his measurement along a different axis, then $|\hat{\mathbf{a}}+\rangle_B$ and $|\hat{\mathbf{a}}-\rangle_B$ won't be pointer states for the composite system $\mathcal{A}_B + \mathcal{P}_B$, and so if $\left|a'_{r,B}(t)\right\rangle \left|p'_{r,B}(t)\right\rangle$ is the ready state for Bob's measurement choice, we must assume that

$$|\hat{\mathbf{a}}\pm\rangle_B \left|a'_{r,B}(t)\right\rangle \left|p'_{r,B}(t)\right\rangle \xrightarrow{\text{time evolution}} |E_{\pm,B}(t)\rangle$$

for some entangled state $|E_{\pm,B}(t)\rangle$ of the composite system $q_B + \mathcal{A}_B + \mathcal{P}_B$ with $\langle E_{\pm,B}(t)|E_{\pm,B}(t)\rangle = 1$ and $\langle E_{\pm,B}(t)|E_{\mp,B}(t)\rangle = 0$. It will then follow that

$$\begin{aligned}
&|\Psi_{\text{Bell}}\rangle\,|a_{r,A}(t)\rangle\,\left|a'_{r,B}(t)\right\rangle\,|p_{r,A}(t)\rangle\,\left|p'_{r,B}(t)\right\rangle \\
&\qquad\qquad\xrightarrow{\text{time evolution}} \frac{1}{\sqrt{2}}\,|\hat{\mathbf{a}}+\rangle_A\,|a_{+,A}(t)\rangle\,|p_{+,A}(t)\rangle\;|E_{-,B}(t)\rangle \qquad (2.24)\\
&\qquad\qquad\qquad + \frac{1}{\sqrt{2}}\,|\hat{\mathbf{a}}-\rangle_A\,|a_{-,A}(t)\rangle\,|p_{-,A}(t)\rangle\;|E_{+,B}(t)\rangle\,.
\end{aligned}$$

Thus, treating $q_B + \mathcal{A}_B + \mathcal{P}_B$ as though it were an environment of $q_A + \mathcal{A}_A + \mathcal{P}_A$, then according to the definition of a world on page 94, the first summand of (2.24) will correspond to a world in which Alice observes her particle to be spin up, and the second summand of (2.24) will correspond to a world in which Alice observes her particle to be spin down, and this will be the case regardless of what axis Bob chooses to make his measurement along. Moreover, since there is no state collapse when Bob makes his measurement, and since Bob's choice has no effect on the state

$$|\xi_{\pm,A}(t)\rangle = |\hat{\mathbf{a}}\pm\rangle_A \, |a_{\pm,A}(t)\rangle \, |p_{\pm,A}(t)\rangle$$

describing Alice's observation, the many-worlds interpretation gives us no reason to worry about there being a violation of special relativity. So that is how proponents of the many-worlds interpretation attempt to resolve the EPR-Bohm paradox.

2.10 Evaluating the Many-Worlds Interpretation

Given the account in the previous two sections of the many-worlds interpretation of quantum physics and how it attempts to resolve the EPR-Bohm paradox, it does seem understandable why physicists would find it so attractive. Although we can't specify an exact moment at which branching occurs, the idea of branching and of there being many worlds itself is not particularly mysterious. This can all be explained in terms of the dynamics of the system and the environment, and decoherence theory allows us to understand why the interference effects that are the hallmark of quantum physics generally disappear on the macroscopic level.

There are other advantages of the many-worlds interpretation besides these which we need not discuss here.[60] But for all the advantages of the many-worlds hypothesis, there is one fundamental problem, and that is its patent absurdity. It seems that we should be able to say whether a cat is alive or dead without having to say what state the rest of the universe is in. However, the many-worlds interpretation suggests that for any subsystem of the universe, we will in general only be able to say what state it is in with respect to the state of the rest of the universe. For example, if the state $\mathcal{S}$ is the system constituting a cat-wise configuration of particles and $\mathcal{E}$ is the rest of the universe, then given that the composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ is described by the state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\big(|\text{Cat Alive}\rangle_{\mathcal{S}} \, |E_{\text{Cat Alive}}\rangle_{\mathcal{E}} + |\text{Cat Dead}\rangle_{\mathcal{S}} \, |E_{\text{Cat Dead}}\rangle_{\mathcal{E}} \big),$$

then we are in no position to make an absolute matter of fact claim about the system $\mathcal{S}$ and say the cat is dead or the cat is alive. Rather we have to say with respect to

[60] More details can be found in Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* and E. Joos et al., *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer Berlin Heidelberg, 2013).

the environment described by $|E_{\text{Cat Alive}}\rangle_{\mathcal{E}}$, the cat is alive, and with respect to the environment $|E_{\text{Cat Dead}}\rangle_{\mathcal{E}}$, the cat is dead. Moreover, according to the many-worlds hypothesis, the branching into multiple worlds doesn't just occur in rare instances, such as in Schrödinger's cat-type experiments. On the contrary, branching is supposed to be happening all the time.

In order to convey how ubiquitous branching is in the many-worlds interpretation, one just has to consider the behavior of an electron. If we suppose that a free electron is initially described by a wave packet whose width is around $10^{-10}\,\text{m}$ (which is of the order of the width of and atom), then according to the Schrödinger equation which governs how wave packets evolve over time, after one second the width of the wave packet will have spread to a width of around $1000\,\text{km}$.[61] The only reason electrons remain localized rather than spreading out to such vast distances is because the electron is continually interacting with its environment. But according to the many-worlds interpretation, these continual interactions of the electron with its environment will result in a continual branching out of worlds corresponding to the possible locations the environment localizes the particle to. Therefore, because the electron rapidly gets entangled with its environment, we cannot establish matter of fact claims about where the electron is localized to – we can only establish matter of fact claims about the composite system of the electron and its environment which expands with astonishing rapidity.

Because of the ubiquity of branching in the many-worlds interpretation, this interpretation appears to undermine the conditions for the possibility of doing science,

[61] See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 117.

for surely one of the conditions for the possibility of doing science is that measuring devices exist, but it doesn't look like there are such things as measuring devices in the many-worlds interpretation. To see why, consider the properties a measuring device should possess. Firstly, it must be capable of interacting either directly or indirectly[62] with another entity which is the thing to be measured. Secondly, there must be some kind of correspondence between the range of states the measuring device can be in and the range of states the thing being measured can be in. Thirdly, when the measuring device interacts with the thing being measured, the measuring device should enter into the state that corresponds to the state of the thing being measured.[63] But according to the many-worlds interpretation, what is taken to be a measuring device will in general become entangled with the thing that is being measured, and so there will be no fact of the matter regarding what state the measuring device is in. Rather there will at the very most only be a fact of the matter regarding the state of the composite system that includes both the measuring device and the thing being measured.[64] Thus, in the many-worlds interpretation, there are no measuring devices satisfying the criteria one would expect a measuring device to satisfy. And so without such measuring devices, it does not appear to be possible to do science according to what we normally mean by science.

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[62]It is acceptable for a measuring device to interact with an environment that has interacted with the thing that is being measured.

[63]Although the act of measuring may change the state of the thing being measured, a measuring device should still be able to tell us what the state of the thing being measured is in immediately after the measuring device has interacted with it.

[64]In fact, it is questionable whether we can even make a matter of fact claim about there being a measuring device at all – rather, in the many-worlds interpretation, we can only say there is a superposition of states in which there is a measuring device in existence with respect to some environments, and of states in which there is not a measuring device with respect to other environments.

Another reason for rejecting the many-worlds interpretation is that intuitively, it seems obvious that I can know I am alive without needing to know the state of the rest of the universe, but the many-worlds interpretation does not allow me to make this absolute matter of fact claim. So from both a common sense point of view and a scientific point of view, the many-worlds interpretation really is absurd.

Of course some hypotheses may initially seem absurd, but once the hypothesis has been fully explained, it can appear far more plausible. For instance, time dilation in special relativity might initially sound absurd to some people, but once one has a better grasp of special relativity and is open to the possibility that systems moving close to the speed of light with respect to ourselves might have properties rather different to systems that move with much slower speeds, then special relativity doesn't seem absurd at all. However, the many-worlds interpretation as presented here is different in this regard since it is not hypothesizing about some extreme situation. It is hypothesizing about ordinary situations. In order to accept the many-worlds interpretation, the arguments in its favor would have to be at least as convincing as the common sense beliefs it is calling into question such as the belief that we can do science and my own personal belief that I am alive. But arguments for the many-worlds interpretation clearly fail to meet this criterion. Some people may choose to embrace the absurdity of the many-worlds interpretation and reject the most basic notions of common sense. But when a hypothesis entails an absurd conclusion, a reasonable person would surely think it better to reject the hypothesis rather than embrace the absurdity.

But in rejecting a hypothesis because of its absurd consequences, it doesn't mean that absolutely everything in the hypothesis needs to be rejected, for a hypothesis might be formulated in terms of sub-hypotheses, some of which might be very plausible and which don't of themselves entail absurdities, in which case something of the original hypothesis might be salvageable. In the case of the many-worlds hypothesis, I believe it does have something that is salvageable, namely decoherence theory. In the next chapter I will consider Adrian Kent's one-world interpretation of quantum physics in which the basic ideas of decoherence theory remain intact.

Chapter 3

A description of Kent's Theory of Quantum Physics

In this chapter, I will describe Kent's theory of quantum physics, but before doing this, it is worth briefly reminding ourselves of the problem in quantum physics that we wish to address.

In chapter 1, we discussed the EPR-Bohm paradox and the problem of trying to account for the mysterious correlation of spin measurements on two spatially separated particles. We saw that the Copenhagen interpretation of quantum physics is unable to satisfactorily resolve this paradox because it posits that there is an instantaneous collapse of the state upon measurement, but the idea of an instantaneous collapse does not make sense in special relativity where there is no such thing as an instant of time.

We also saw that any local hidden variables theory, that is, any theory in which parameter independence (PI) and outcome independence (OI) hold, will imply Bell's inequality, and this inequality is known to be experimentally violated. Shimony proposed that quantum theory and special relatively could peacefully coexist if we accepted PI and rejected OI, but as Butterfield[1] points out, Shimony's proposal does not address the problem of what an outcome is despite his proposal relying on there being outcomes.

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[1]See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem"

As discussed in chapter 2, the problem of outcomes remains an unresolved part of the measurement problem, and the many-worlds interpretation that attempts to sidestep the problem of outcomes is deeply unsatisfactory. But as well as critiquing Shimony's proposal, Butterfield thinks that a suitable interpretation of quantum physics could provide what is missing in Shimony's account. It is for this reason that Butterfield highlights Kent's theory of quantum physics.

In this chapter, we will just focus on describing Kent's theory, and we will postpone our evaluation of whether Kent's theory can adequately resolve the EPR-Bohm paradox until chapter 4. In describing Kent's theory of quantum physics, we will focus on the ideas Kent presents in his 2014 paper.[2]

Kent's theory of quantum physics has some similarities in common with the pilot wave interpretation. Firstly, there is no quantum state collapse in Kent's theory. Secondly, some additional values beyond standard quantum theory (i.e. in addition to the quantum state) are included in Kent's theory. And thirdly, Kent's theory is a one-world interpretation of quantum physics. I'll consider these three features of Kent's theory in some detail as I describe his theory. I'll then present an account of his toy model that provides a simple example of how the ideas of his theory fit together.

[2]Adrian Kent, "Lorentzian Quantum Reality: Postulates and Toy Models," 2014, https://doi.org/10.1098/rsta.2014.0241, eprint: arXiv:1411.2957.

3.1 The No-collapse Feature of Kent's theory

We first consider the no-collapse feature of Kent's theory. This is a feature that belongs both to the many-worlds interpretation and to the pilot wave interpretation. In all three interpretations, the quantum state deterministically evolves according to the Schrödinger equation. The Schrödinger equation itself describes how a quantum state evolves over time when there are no outside influences. The precise formula for the Schrödinger equation need not concern us here, but all we need to know is that the Schrödinger equation determines a so-called *unitary operator* $U(t', t)$. What this means is that if a system is in a state $|\psi\rangle$ at time t, then it will be in the state $|\psi'\rangle = U(t', t)\,|\psi\rangle$ at time t'. A unitary operator U has the property that if $|\psi'\rangle = U\,|\psi\rangle$ and $|\chi'\rangle = U\,|\chi\rangle$, then

$$\langle \chi' | \psi' \rangle = \langle \chi | \psi \rangle .^{3} \tag{3.1}$$

Under the Copenhagen interpretation, a system will evolve unitarily for the most part, but there will typically be a non-unitary change in the state describing the system whenever there is a measurement.[4] However, in non-collapse models such as the pilot wave interpretation, the many-worlds interpretation, and Kent's theory, the quantum state always evolves unitarily.

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[3]A unitary operator U must also be linear so that for any two states $|\psi\rangle$ and $|\phi\rangle$ and complex numbers α and β, we have

$$U(\alpha\,|\psi\rangle + \beta\,|\phi\rangle) = \alpha U\,|\psi\rangle + \beta U\,|\phi\rangle\,,$$

and furthermore, a unitary operator must have the property that it is invertible: there is a linear operator U^{-1} such that UU^{-1} and $U^{-1}U$ are the identity operator I, i.e. $U^{-1}U\,|\psi\rangle = UU^{-1}\,|\psi\rangle = |\psi\rangle$ for any state $|\psi\rangle$.

[4]Note that to say that the change in a state is non-unitary when a measurement is made is not to say that there is a non-unitary collapse operator that maps the quantum state to an eigenstate of some observable. Such a mapping would not make sense, since the collapse is not deterministic given the initial state. However, one could have a well-defined mapping from a time value t to the quantum state of the system $|\psi(t)\rangle$ at time t. We then say that a system changes unitarily if and only if there is a unitary operator $U(t_1, t_0)$ for any two times t_0 and t_1 such that whenever the state of the system at time t_0 is given by $|\psi(t_0)\rangle$, then the state of the system at time t_1 must be given by

3.2 The Additional Values of Kent's theory*

The second similarity Kent's theory has in common with the pilot wave interpretation is that it posits the reality of some additional values beyond standard quantum theory (i.e. in addition to the quantum state[5]). In the pilot wave interpretation, these additional values are the positions and momenta of all the particles, whereas in Kent's

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$|\psi(t_1)\rangle = U(t_1,t_0)\,|\psi(t_0)\rangle$, and that for an intermediate time t, $U(t_1,t_0) = U(t_1,t)U(t,t_0)$. So to say that the change in a state is non-unitary when a measurement is made is to say that the state $|\psi(t)\rangle$ describing the system does not change unitarily in the process of making a measurement. Now to see why this is the case under the Copenhagen interpretation, we suppose that at time t_0 a system is in the state $|\psi(t_0)\rangle$ and that as long as no measurements are made up until a time $t \geq t_0$, the state evolves to a state $|\psi^{(U)}(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle$ where $U(t,t_0)$ is a unitary operator determined by Schrödinger's equation. Furthermore, we suppose that there is a measurable quantity with which we associate an observable $\hat{\boldsymbol{O}}$ so that whenever the state of the system is an eigenstate of $\hat{\boldsymbol{O}}$, the value of the measurable quantity for the system will be a determinate value and equal to the corresponding eigenvalue of $\hat{\boldsymbol{O}}$. At time t_0, we can express $|\psi(t_0)\rangle$ as a linear combination

$$|\psi(t_0)\rangle = \sum_i c_i\,|s_i(t_0)\rangle$$

where the $|s_i(t_0)\rangle$ are eigenstates of $\hat{\boldsymbol{O}}$ with distinct eigenvalues. As long as no measurement is made, this will evolve as

$$|\psi^{(U)}(t)\rangle = \sum_i c_i U(t,t_0)\,|s_i(t_0)\rangle\,.$$

We assume that as the state $|s_i(t_0)\rangle$ evolves to the state $|s_i(t_1)\rangle$ from time t_0 to t_1, it remains an eigenstates of $\hat{\boldsymbol{O}}$ with approximately the same eigenvalue. This assumption is based on the principle that in practice, performing a measurement is not instantaneous, but rather must take place over a time interval, and so the eigenstate and eigenvalue must be stable enough over this time interval so as to specify a definite outcome. We also assume that when the system is already in an eigenstate $|s_i(t_0)\rangle$ of the observable $\hat{\boldsymbol{O}}$, it will evolve unitarily as $|s_i(t)\rangle = U(t,t_0)\,|s_i(t_0)\rangle$ for t between t_0 and t_1, and that performing the measurement corresponding to $\hat{\boldsymbol{O}}$ will have no effect on the system when it is an eigenstate $|s_i(t)\rangle$ of $\hat{\boldsymbol{O}}$ – otherwise we couldn't be sure that whenever we looked at the measurement readout that we weren't changing the value of the quantity we were trying to measure.

Now according to the Copenhagen interpretation, when the measurement corresponding to $\hat{\boldsymbol{O}}$ is made, the system must enter into one of the eigenstates of the observable $\hat{\boldsymbol{O}}$, and at time t_1 shortly after the measurement has been made, the probability the system will be in the $|s_i(t_1)\rangle$-state given that it was in the $|\psi(t_0)\rangle$-state at time t_0 will be $|\left\langle s_i(t_1)\middle|\psi^{(U)}(t_1)\right\rangle|^2$ in accordance with the Born rule. So taking $|\psi(t_1)\rangle$ to be proportional to $|s_i(t_1)\rangle$ for some i, we see that for $j \neq i$, $\langle s_j(t_1)|\psi(t_1)\rangle = 0$. This is because eigenstates of a Hermitian operator that have different eigenvalues must be orthogonal. However, since $U(t_1,t_0)$ is unitary,

$$\langle s_j(t_1)|\psi^{(U)}(t_1)\rangle = \langle s_j(t_0)|\psi^{(U)}(t_0)\rangle = c_j.$$

So we see that $|\psi^{(U)}(t_1)\rangle \neq |\psi(t_1)\rangle$ if $\psi(t_0)$ is not initially in an eigenstate of $\hat{\boldsymbol{O}}$, and hence $|\psi(t)\rangle$ doesn't evolve unitarily up to time t_1 as $|\psi^{(U)}(t)\rangle$ does.

[5]We may wish to think of these additional values as hidden variables, but we are not obliged to since we don't speculate on whether these additional variables are necessarily unknowable. Rather, we just see them as supplementing the quantum state so as to provide a complete description of the system.

theory, the additional values specify the mass-energy density on a three-dimensional distant future spacelike hypersurface in spacetime to be described shortly. We let S denote this spacelike hypersurface.

To describe the nature of this three-dimensional hyperspace S, we will need some terminology and notation used in special relativity. A *spacetime location* is a point (x^1, x^2, x^3) in three-dimensional space at a particular instant of time t, and hence described by four numbers (x^0, x^1, x^2, x^3) where $x^0 = ct$ and where c is the speed of light.[6] We will use the convention of boldface type to depict spatial locations, e.g. $\boldsymbol{x} = (x^1, x^2, x^3)$, and non-boldface type to depict a spacetime location, e.g. $x = (x^0, x^1, x^2, x^3)$.

Now a key insight of special relativity is that there is no such thing as absolute time. So for instance, two spacetime locations might seem to be simultaneous from one frame of reference, but another person travelling at a different velocity would judge with equal propriety the same two spacetime locations to be non-simultaneous. But it is not the case that for any two spacetime locations we can always find a frame of reference in which the two spacetime locations are simultaneous – sometimes this is not possible. But we refer to spacetime locations that could be simultaneous in some frame of references as being *spacelike-separated*. For example, the two spacetime locations o and a in figure 3.1 are spacelike-separated.

There are also spacetime locations in spacetime that could be connected by a beam of light such as the two spacetime locations o and b in figure 3.1. Such spacetime

[6]Multiplying time by the speed of light means that x^0 is a distance like x^1, x^2, and x^3.

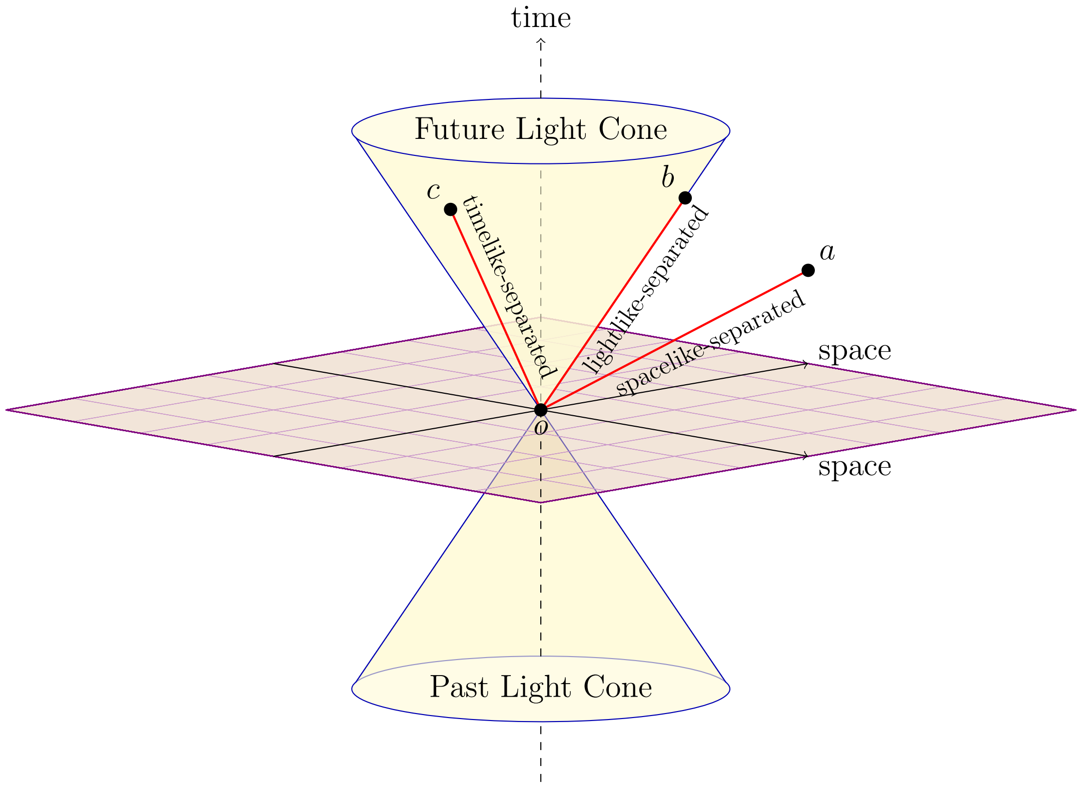

Figure 3.1: The meaning of spacelike, timelike and lightlike-separation when there are two space dimensions and one time dimension.

locations are referred to as being *lightlike-separated*. For any given spacetime location, the spacetime locations that are lightlike-separated from it form two cones[7] called the future light cone and the past light cone as shown in figure 3.1. Because light appears to travel at the same speed no matter what frame of reference one uses, the light cone of a spacetime location remains invariant when one changes from one reference frame to another. In other words, if another spacetime location lies on the light cone of a spacetime location in one frame of reference, then it must lie on the light cone of this spacetime location in every frame of reference.

[7]Strictly speaking, the set of spacetime locations that are lightlike-separated from a give spacetime location form the surface of a cone rather than a cone (which is a convex object). But among physicists, the terminology light cone has stuck.

Figure 3.1 also depicts two spacetime locations o and c that are *timelike-separated*. Such spacetime locations lie within the light cones of each other, and when two spacetime locations are timelike-separated, it is always possible to choose a frame of reference in which the two spacetime locations are located at the same point in space, but with one spacetime location occuring after the other depending on which spacetime location is in the future light cone of the other.

Now a three-dimensional spacelike hypersurface S in spacetime is a maximal[8] three-dimensional surface in which all the spacetime locations of S are spacelike-separated. Kent assumes that this spacelike hypersurface S is in the distant future of an expanding universe so that nearly all the particles that can decay have already done so, and that all the particles that are not bound together are very far from each other so that the probability of any particle collisions is very small. In other words, all the interesting physics in the universe has played its course before S.

At every spacetime location $x \in S$, there is a quantity $T_S(x)$ called the *mass-energy density*.[9] The important thing to note about $T_S(x)$ is that it does not depend on which frame of reference one is in.[10] This property is in contrast to many physical properties that do depend on which frame of reference one is in. For example, the kinetic energy of an object will depend on the calculated velocity of the object, and this velocity will in turn depend on the frame of reference in which this calculation is done.

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[8]That is, it cannot be extended any further along any of its three dimensions, so it is not a small local surface contained within a boundary.

[9]The definition of $T_S(x)$ will be discussed in section 3.2.

[10]The reason for why this is will be discussed in section 3.2.

Now in order to specify the additional values that Kent's theory requires, we need to discuss the Tomonaga-Schwinger picture of relativistic quantum physics.[11] In order to explain their formulation, it is helpful to consider first the distinction between the Heisenberg picture and the Schrödinger picture of quantum mechanics.

In the *Heisenberg picture*, the states describing a system do not change over time. Rather, the observables change over time. So for instance, if there is a time-independent state $|\Phi\rangle$ describing a system and there is some measurable quantity whose expectation value we wish to know at time t given the state $|\Phi\rangle$, then we will need a time dependent observable $\hat{\boldsymbol{O}}(t)$,[12] say, corresponding to the measurable quantity at time t from which we can calculate the expectation value $\langle\Phi|\hat{\boldsymbol{O}}(t)|\Phi\rangle$ at time t given the system is in state $|\Phi\rangle$. In the context of quantum field theory, any observable $\hat{\boldsymbol{O}}(t)$ in the Heisenberg picture will be expressible as a sum (or integral) of observables of the form $\hat{\boldsymbol{O}}(t, \boldsymbol{x})$, where $\hat{\boldsymbol{O}}(t, \boldsymbol{x})$ is an observable of some quantity at a particular time t and spatial location $\boldsymbol{x}$.[13]

The Heisenberg picture is contrasted with the *Schrödinger picture* in which the observables do not change over time, but rather the states change over time. So for instance, if there is a time-dependent state $|\Phi(t)\rangle$ describing a system at a specific time

[11] See Julian Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation," *Physical review* 74, no. 10 (1948): 1439–1461; S. Tomonaga, "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields," *Progress of theoretical physics* (Tokyo) 1, no. 2 (1946): 27–42

[12] See footnote 15 for an explanation of the convention of using a boldface font for this observable.

[13] For example, in quantum electrodynamics (which is one kind of quantum field theory), the observables will be expressible in terms of fields such as the four-vector potential $A^\mu(x)$ and the bispinor field $\psi(x)$ which are defined at all spacetime locations $(t, \boldsymbol{x}) = (t, x^1, x^2, x^3)$. The four-vector potential $A^\mu(x)$ can be used to determine the electromagnetic field, and the bispinor field $\psi(x)$ can be used to determine the electric current density. In the Heisenberg picture, these fields will have corresponding Hilbert space operators at each spacetime location x from which expectation values can be calculated at the spacetime location x for a given time-independent state.

t and there is some measurable quantity whose expectation value we wish to know at time t given the state $|\Phi(t)\rangle$, then we will only require a time-independent observable $\hat{\boldsymbol{O}}$, say, corresponding to the measurable quantity from which we can calculate the expectation value $\langle\Phi(t)|\hat{\boldsymbol{O}}|\Phi(t)\rangle$. As in the Heisenberg picture, we can introduce a spatial dependence into the observables so that any observable $\hat{\boldsymbol{O}}$ is expressible as a sum (or integral) of observables of the form $\hat{\boldsymbol{O}}(\boldsymbol{x})$ where $\hat{\boldsymbol{O}}(\boldsymbol{x})$ is an observable of some quantity at a particular spatial location $\boldsymbol{x}$.

Now despite the Schrödinger and Heisenberg pictures taking these different perspectives, they are nevertheless physically equivalent. This is because in both pictures, there is a unitary operator $U(\Delta t)$ for any time interval Δt such that $U(\Delta t)\,|\Phi(t)\rangle = |\Phi(t+\Delta t)\rangle$, and $U(\Delta t)\hat{\boldsymbol{O}}(t,\boldsymbol{x})U(\Delta t)^{-1} = \hat{\boldsymbol{O}}(t+\Delta t,\boldsymbol{x})$. Thus, given the Schrödinger picture, to get the Heisenberg picture, all we need to do is the following: firstly, we fix a time t_0 and let all the states of the form $|\Phi(t_0)\rangle$ at time t_0 in the Schrödinger picture be the state space for the Heisenberg picture; then for any Schrödinger picture observable $\hat{\boldsymbol{O}}(\boldsymbol{x})$, we define the corresponding Heisenberg picture time-dependent observable

$$\hat{\boldsymbol{O}}(t,\boldsymbol{x}) = U(t-t_0)\hat{\boldsymbol{O}}(\boldsymbol{x})U(t-t_0)^{-1}.$$

Conversely, to move from the Heisenberg picture to the Schrödinger, we first fix a reference time t_0. Then for any state $|\Phi\rangle$ and observable $\hat{\boldsymbol{O}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \hat{\boldsymbol{O}}(t,\boldsymbol{x})$ in the Heisenberg picture, the corresponding Schrödinger picture time-dependent state at time t will be $U(t-t_0)\,|\Phi\rangle$, and the corresponding Schrödinger picture time-independent observable will be $\hat{\boldsymbol{O}}(t_0,\boldsymbol{x})$.

Now if there is a quantity we wish to measure at time t_0 with corresponding observable $\hat{\boldsymbol{O}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \hat{\boldsymbol{O}}(t_0, \boldsymbol{x})$, then in both pictures, the expectation value of this measurable quantity given $|\Phi\rangle \stackrel{\text{def}}{=} |\Phi(t_0)\rangle$ will be

$$\langle \Phi(t_0)|\hat{\boldsymbol{O}}(\boldsymbol{x})|\Phi(t_0)\rangle = \langle \Phi(t_0)|\hat{\boldsymbol{O}}(t_0, \boldsymbol{x})|\Phi(t_0)\rangle = \langle \Phi|\hat{\boldsymbol{O}}(t_0, \boldsymbol{x})|\Phi\rangle \tag{3.2}$$

Since the left-hand side of (3.2) is the Schrödinger picture expectation value of $\hat{\boldsymbol{O}}(\boldsymbol{x})$, and the right-hand side of (3.2) is the Heisenberg picture expectation value of $\hat{\boldsymbol{O}}(t_0, \boldsymbol{x})$, it follows that whatever picture we choose, it will make no difference to the calculated expectation values of observables – in other words, the two pictures are physically equivalent.

Now although it is easy to move between both the Schrödinger and Heisenberg pictures, they both give a privileged status to spacelike hypersurfaces of the form $t = \text{const}$. However, according to special relativity, there are no privileged spacelike hypersurfaces. One of the advantages of the Tomonaga-Schwinger picture is that it gives no privileged status to any class of spacelike hypersurfaces, but rather all spacelike hypersurfaces are placed on the same footing. We are going to see that the expectation value $\langle \Phi(t_0)|\hat{\boldsymbol{O}}(t_0, \boldsymbol{x})|\Phi(t_0)\rangle$ of equation (3.2) is a special case of what Tomonaga and Schwinger consider more generally.

We first note that if we can calculate $\langle \Phi(t_0)|\hat{\boldsymbol{O}}(t_0, \boldsymbol{x})|\Phi(t_0)\rangle$ for any t_0 and any $\boldsymbol{x}$, then we can calculate all the expectation values that might interest us. But we also note that in the expectation value $\langle \Phi(t_0)|\hat{\boldsymbol{O}}(t_0, \boldsymbol{x})|\Phi(t_0)\rangle$, the $|\Phi(t_0)\rangle$-state is the state of a spacelike hypersurface $t = t_0$, and $(t_0, \boldsymbol{x})$ is a spacetime location on this spacelike hypersurface. Now if we are to place all spacelike hypersurfaces on the same footing, then in specifying expectation values, we should be just as content in specifying expectation values of the form $\langle\Psi[S]|\hat{O}(x)|\Psi[S]\rangle$, where S is any hypersurface, $|\Psi[S]\rangle$ is any state of this spacelike hypersurface,[14] x is any spacetime location on the spacelike hypersurface S, and where $\hat{O}(x)$ is any observable of S.[15] The Tomonaga-Schwinger picture thus works with states of the form $|\Psi[S]\rangle$ for any spacelike hypersurface S, and observables of the form $\hat{O}(x)$ with $x \in S$ acting on the state space of the spacelike hypersurface S from which one can calculate the expectation value $\langle\Psi[S]|\hat{O}(x)|\Psi[S]\rangle$.

In order to construct $|\Psi[S]\rangle$ and $\hat{O}(x)$, Schwinger introduces a unitary operator[16] $U[S]$ that maps the $|\Phi\rangle$-state of the Heisenberg picture to the corresponding $|\Psi[S]\rangle$-state that describes the state of the spacelike hypersurface S, i.e. $|\Psi[S]\rangle = U[S]\,|\Phi\rangle$. Schwinger then defines the observable

$$\hat{O}(x) = U[S]\hat{\boldsymbol{O}}(x)U[S]^{-1} \tag{3.3}$$

on S where x is any spacetime location on S, and where $\hat{\boldsymbol{O}}(x)$ is any Heisenberg picture observable. Ostensibly, $\hat{O}(x)$ depends on the surface S, but Schwinger shows that under conditions that are readily satisfied, $\hat{O}(x)$ is independent of the spacelike

[14]The convention of using square brackets such as in $|\Psi[S]\rangle$ indicates that the thing in question is a functional. Functions and functionals are closely related. A function f is a mapping from one set (the domain) to another set (the codomain), such that each input yields a single output. The typical convention is to use round brackets to denote the output, e.g. $f(x)$ where x is the input. A functional g, on the other hand, is a function that maps a space of functions or other mathematical objects (such as surfaces or volumes) to a some value. The typical convention is to use square brackets to denote the output, e.g. $g[y]$ where y is the input function or other mathematical object. So in the present case, $|\Phi[\cdot]\rangle$ is a functional that takes a surface S as input to produce a state $|\Phi[S]\rangle$ as output.

[15]Here I am following the convention of Schwinger of always using non-boldface type to indicate Tomonaga-Schwinger picture observables, and boldface type to indicate Heisenberg picture and Schrödinger picture observables. See Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation," p. 1448.

[16]See Schwinger, p. 1448.

hypersurface S.[17] Also, since

$$\langle \Psi[S] | \hat{O}(x) | \Psi[S] \rangle = \langle \Phi | \hat{\boldsymbol{O}}(x) | \Phi \rangle \,, \tag{3.4}$$

the Tomonaga-Schwinger picture will give the same physics as the Heisenberg and Schrödinger picture. In order to avoid our notation becoming cluttered, we will write $|\Psi\rangle$ instead of $|\Psi[S]\rangle$, and say that $|\Psi\rangle$ is a state of the spacelike hypersurface S, and we will speak of the Hilbert space H_S of all such states of the spacelike hypersurface S so that we can write $|\Psi\rangle \in H_S$.[18]

We are now in a position to come back to the question of what the additional values of Kent's theory are. As mentioned on page 110, for a given spacelike hypersurface S, there will be a mass-energy density $T_S(x)$. Corresponding to this, there will be a Heisenberg picture observable $\hat{\boldsymbol{T}}_S(x)$, and from this we can construct the Tomonaga-Schwinger observable $\hat{T}_S(x) = U[S]\hat{\boldsymbol{T}}_S(x)U[S]^{-1}$.[19] These mass-energy density observables have

[17] The required condition is that

$$i\hbar \frac{\delta U[S]}{\delta S(x)} = \mathcal{H}(x) U[S]$$

where $\mathcal{H}(x)$ is a Hermitian operator that is a Lorentz invariant function of the field quantities at the spacetime location x and has the dimensions of an energy density, and where the functional derivative $U[S]$ is given by

$$\frac{\delta U[S]}{\delta S(x)} = \lim_{\delta\omega \to 0} \frac{U[S'] - U[S]}{\delta\omega}$$

where S' is a surface that only differs from S in the vicinity of x, and where $\delta\omega$ is the volume enclosed by S and S'. The Hermitian operator

$$\mathcal{H}(x) = -(1/c) j^\mu(x) A_\mu(x)$$

has the desired property where $j^\mu(x)$ is the current density and where $A^\mu(x)$ is the four-vector potential of the electromagnetic field. With this choice for $\hat{H}(x)$, Schwinger shows that $\Box A^\mu(x) = 0$ and $\partial_\mu A^\mu(x)\,|\Psi[S]\rangle = 0$, where $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator – see Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation," p. 1449-1450.

[18] Though to be clear, the H_S are really identical for all spacelike hypersurfaces S since each H_S is the image of the unitary operator $U[S]$ acting on the Heisenberg-picture Hilbert space, and the image of a unitary operator is always equal to the Hilbert space it is acting on.

[19] Note that $\hat{T}_S(x)$ will depend on S. The remark above about $\hat{O}(x)$ not depending on S does not apply here since the independence of $\hat{O}(x)$ from S assumes that the Heisenberg picture observable $\hat{\boldsymbol{O}}(x)$ is independent of S, but this is not the case for $\hat{\boldsymbol{T}}_S(x)$. However, $\hat{T}_S(x)$ will only depend on S in the vicinity of x, so if S' only differs from S outside a neighborhood of x, then $\hat{T}_S(x) = \hat{T}_{S'}(x)$.

the property that if x and y are any two spacetime locations of S, then $\hat{T}_S(x)$ and $\hat{T}_S(y)$ will commute. In other words,

$$\hat{T}_S(x)\hat{T}_S(y) = \hat{T}_S(y)\hat{T}_S(x).$$

The commutativity of all the $\hat{T}_S(x)$ for $x \in S$ means that if $|\Psi\rangle \in H_S$ is an eigenstate of $\hat{T}_S(x)$, then for any $y \in S$, $\hat{T}_S(y)\,|\Psi\rangle$ is also an eigenstate of $\hat{T}_S(x)$ with the same eigenvalue as $|\Psi\rangle$. The invariance of any $\hat{T}_S(x)$-*eigenspace*[20] under the action of $\hat{T}_S(y)$ means that we can create an orthonormal basis of H_S consisting of simultaneous eigenstates of both $\hat{T}_S(x)$ and $\hat{T}_S(y)$, albeit with different eigenvalues.[21] Moreover, for reasons that we need not go into, these eigenvalues must be greater than or equal to 0.[22] But because x and y are arbitrary points of S, this means we can construct an orthonormal basis $\{\,|\Psi^{(i)}\rangle : i\}$ of H_S such that $\hat{T}_S(x)\left|\Psi^{(i)}\right\rangle = \tau_S^{(i)}(x)\left|\Psi^{(i)}\right\rangle$ for all $x \in S$, where $\tau_S^{(i)}(x) \geq 0$ is a possible energy-density measurement defined for every x in S. We will refer to a state $|\Psi\rangle$ as a *simultaneous* $\hat{T}_S$*-eigenstate*, and a real valued function τ_S defined on the whole of S as a *simultaneous* $\hat{T}_S$*-eigenvalue* if and only if $\hat{T}_S(x)\,|\Psi\rangle = \tau_S(x)\,|\Psi\rangle$ for all $x \in S$.

At this point, it is worth clarifying the different meanings of $T_S(x)$, $\hat{T}_S(x)$, and $\tau_S(x)$. We use $T_S(x)$ to refer to the description of the physical quantity that is being observed. Thus, $T_S(x)$ is shorthand for the description "the mass-energy density of the spacelike hypersurface S observed at spacetime location x". The function $\tau_S(x)$ stands for a

[20]An eigenspace of a Hermitian operator $\hat{O}$ acting on a Hilbert space H is just the space of all the eigenstates of $\hat{O}$ in H which have the same eigenvalue.

[21]This is because any $\hat{T}_S(x)$-eigenspace is itself a Hilbert space on which $\hat{T}_S(y)$ acts as a Hermitian operator, so by (2.3), we can find an orthonormal basis of states $\{|\psi_1\rangle, \ldots |\psi_N\rangle\}$ of the $\hat{T}_S(x)$-eigenspace and real numbers $\tau^{(1)}(y), \ldots, \tau^{(N)}(y)$ such that $\hat{T}_S(y)\,|\psi_i\rangle = \tau^{(i)}(y)\,|\psi_i\rangle$ for $i = 1, \ldots, N$. Hence each of the $|\psi_i\rangle$ will be a simultaneous eigenstate of both $\hat{T}_S(x)$ and $\hat{T}_S(y)$.

[22]In other words, we are not going to concern ourselves with theories that allow for negative mass-energy densities.

particular range of values for each $x \in S$ of the physical quantity described by $T_S(x)$. And for each $x \in S$, $\hat{T}_S(x)$ is the observable (i.e. Hermitian operator) such that if an observer deems S to be in an eigenstate $|\psi\rangle$ of $\hat{T}_S(x)$ with eigenvalue τ (a real number), then the observer would observe the physical quantity described by $T_S(x)$ to have the value τ. We will add a further clarification to this when we come to consider different observers in section 4.2.

Now in describing simultaneous $\hat{T}_S$-eigenstates and $\hat{T}_S$-eigenvalues as we've done above, it might be objected that there will be uncountably many simultaneous $\hat{T}_S$-eigenstates and $\hat{T}_S$-eigenvalues so that we won't be able to form an orthonormal basis of states $\{\, |\Psi^{(i)}\rangle : i\}$ with the index i being taken over the whole numbers. However, we can overcome this objection to some extent by supposing that in physical reality there will be a limit on how great the mass-energy density $\tau_S(x)$ can be and how rapidly it can change with respect to x. Although such an assumption will mean this theory will break down in extreme situations such as in the case of black holes, this theory won't be any worse off than quantum field theory which also breaks down in extreme situations.

We can then suppose that with a suitably fine mesh on S,[23] any mass energy density $\tau_S(x)$ (e.g. such as the one depicted in figure 3.2) can be approximated to a function (e.g. like the one depicted in figure 3.3) that has constant values on each cell of this mesh and such that the approximation value at a cell belongs to a finite pool of possible values. For instance, if c_x is the cell which contains $x \in S$, and $\tau_{\max}$ is the

[23]Note that the mesh is only a mesh in S, so the cells of the mesh are cube-like subsets of S. The time might not be constant across each cell because of the possible curvature of S.

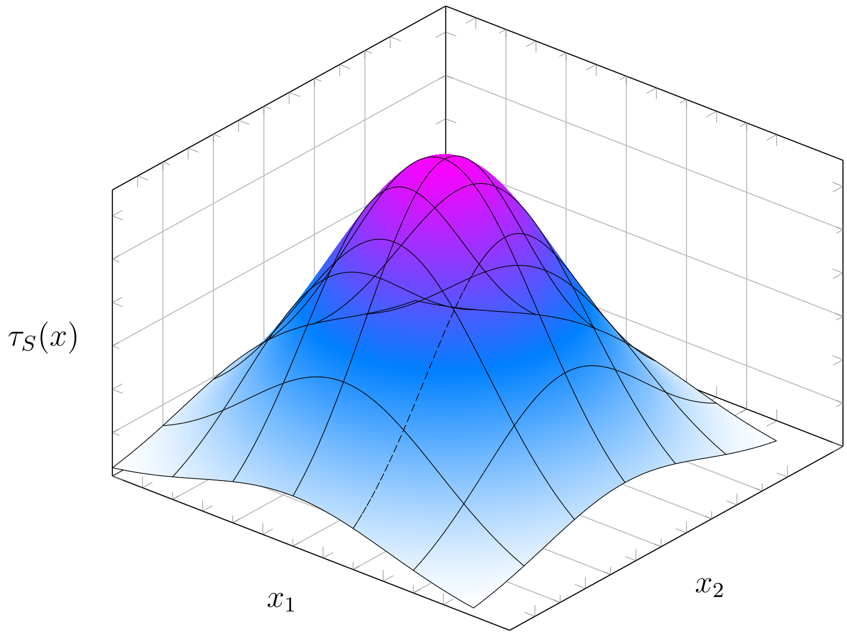

Figure 3.2: An example of an arbitrary mass-energy density τ_S plotted against two dimension x_1 and x_2 belonging to the spacelike hypersurface S.

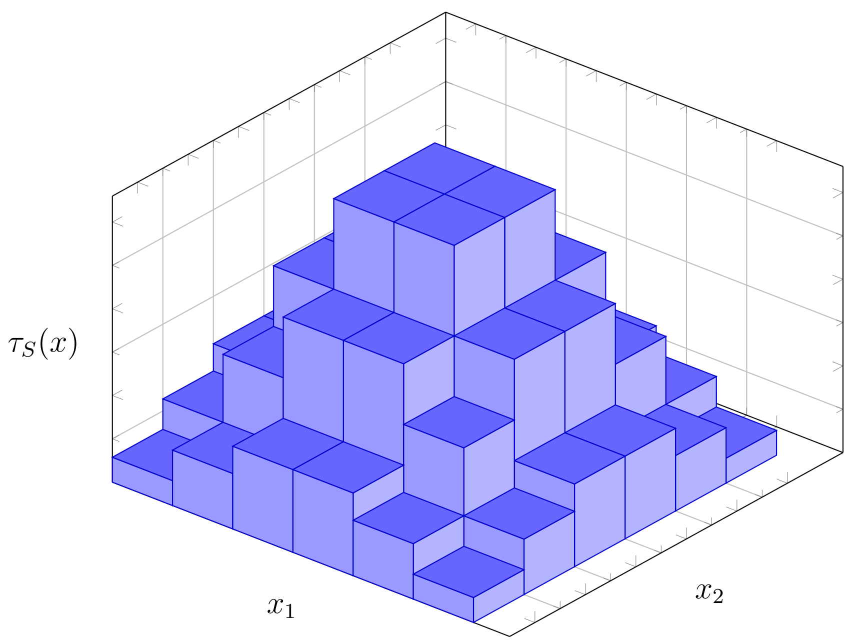

Figure 3.3: Here, the arbitrary mass-energy density τ_S depicted in figure 3.2 has been approximated by a function which is constant in each mesh cell.

maximum possible value the mass-energy density could be, then we could define the approximation to $\tau_S(x)$ at cell c_x to be

$$\tau_S(c_x) = \frac{\tau_{\max}}{N}\left\lfloor N\left(\frac{\text{average of } \tau_S \text{ over } c_x}{\tau_{\max}}\right) + 0.5\right\rfloor \tag{3.5}$$

where $\lfloor z \rfloor$ is the biggest integer $n \leq z$, and where N is a fixed large number. Then $\tau_S(c_x)$ will have $N+1$ possible values between 0 and $\tau_{\max}$. There will then only need to be a countable number of approximations $\tau_S^{(i)}$ to approximate any arbitrary mass-energy density τ_S on S. As long as we choose the cells in the mesh to be sufficiently small and N to be sufficiently large, we can describe physical reality up to our desired level of accuracy. Thus, we assume that for each of the countable states in the orthonormal basis $\{\, |\Psi^{(i)}\rangle : i\}$, there will be a corresponding function $\tau_S^{(i)}$ defined on S which is constant on every cell of S and in which

$$\hat{T}_S(c_x) \left|\Psi^{(i)}\right\rangle = \tau_S^{(i)}(c_x) \left|\Psi^{(i)}\right\rangle \tag{3.6}$$

for all $x \in S$ where $\tau_S^{(i)}(c_x) = \tau_S^{(i)}(x)$ and where $\hat{T}_S(c_x)$ is the observable corresponding to the average approximated value of the physical quantity $T_S(x)$ over the cell c_x (approximated as in (3.5) with τ_S replaced by T_S), but we will normally just write

$$\hat{T}_S(x) \left|\Psi^{(i)}\right\rangle = \tau_S^{(i)}(x) \left|\Psi^{(i)}\right\rangle \tag{3.7}$$

with the implicit understanding that by (3.7) we really mean (3.6), and that when we speak of $\left|\Psi^{(i)}\right\rangle$ and $\tau_S^{(i)}$ as simultaneous $\hat{T}_S$-eigenstates and simultaneous $\hat{T}_S$-eigenvalues respectively, we implicitly understand $\hat{T}_S$ and $\tau_S^{(i)}$ to be defined over cells of the form $c_x \subset S$ rather than over spacetime locations $x \in S$.

Now the additional values beyond standard quantum theory that are included in Kent's theory are given by one of these simultaneous $\hat{T}_S$-eigenvalues $\tau_S^{(i)}$ that (approximately) describe a possible outcome for an energy-density measurement over the whole of S. But although we speak of the measurement of $T_S(x)$ as being $\tau_S^{(i)}(x)$, this is only a notional measurement. Thus, we speak of the measurement of $T_S(x)$ only to mean that $T_S(x)$ has a determinate value for every $x \in S$ despite the quantum state of S in general being in a superposition of $\hat{T}_S(x)$-eigenstates for any given $x \in S$. How this determination of $T_S(x)$ comes about is up to one's philosophical preferences. For example, one could suppose that it was simply by divine fiat that this determination of $T_S(x)$ came about.[24]

Nevertheless, the particular density $\tau_S(x)$ which is found to describe S can't be absolutely anything. Rather, we suppose there is a much earlier spacelike hypersurface S_0 which is described by a state $|\Psi_0\rangle$ belonging to a Hilbert space H_{S_0} as shown in figure 3.4. It is assumed that all physics that we wish to describe takes place between these two spacelike hypersurfaces S_0 and S. In figure 3.4, we therefore let y depicts a generic spacetime location that we wish to describe.

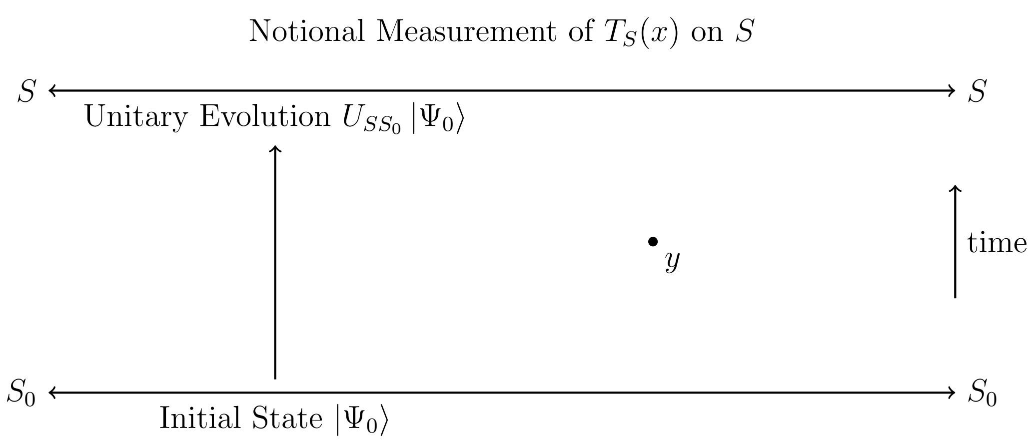

Figure 3.4: A notional measurement of $T_S(x)$ is made for all $x \in S$. The simultaneous $\hat{T}_S$-eigenstate $|\Psi\rangle$ with $\hat{T}_S(x)\,|\Psi\rangle = \tau_S(x)\,|\Psi\rangle$ is selected with probability $|\langle\Psi|U_{SS_0}|\Psi_0\rangle|^2$. The values $\tau_S(x)$ obtained for $T_S(x)$ are then used to calculate the physical properties at the spacetime location y.

If we now define

$$U_{SS_0} = U[S]U[S_0]^{-1} \tag{3.8}$$

[24]I will discuss my philosophical preference in the final chapter.

then U_{SS_0} will be a unitary operator that maps states in H_{S_0} such as $|\Psi_0\rangle$ to states in H_S. Then the probability $P(\Psi|\Psi_0)$ that S will be found to be in the state $|\Psi\rangle$ with mass-energy density $\tau_S(x)$ given that S_0 was initially in the state $|\Psi_0\rangle$ will be given by the Born Rule (see page 12):

$$P(\Psi|\Psi_0) = |\langle\Psi|U_{SS_0}|\Psi_0\rangle|^2. \tag{3.9}$$

It's possible that there might be several different states of H_S that have the same mass-energy density $\tau_S(x)$ for all $x \in S$, but one of these states is still realized with probability given by equation (3.9). But it is the mass-energy density τ_S itself rather than one of the eigenstates with mass-energy density τ_S that constitute the additional values that Kent adds to standard quantum theory.

Also note that if every simultaneous $\hat{T}_S$-eigenstate $|\Psi\rangle$ with simultaneous $\hat{T}_S$-eigenvalue τ_S satisfies $|\langle\Psi|U_{SS_0}|\Psi_0\rangle| = 0$, then by (3.9), τ_S will not be a possible measurement outcome for T_S given $|\Psi_0\rangle$. It is for this reason that we can't expect the measurement outcome of T_S on S to be absolutely anything.

3.3 The One-World Feature of Kent's theory

The third similarity Kent's theory shares with the pilot wave interpretation is that it is a one-world interpretation of quantum physics. It will be helpful to contrast this with the many-worlds interpretation.

Unlike the many-worlds interpretation, Kent's theory does not allow for indeterminate states of macroscopic objects such as cats. In the many-worlds interpretation, Schrödinger will still only observe his cat to be either dead or alive, and not both dead and alive. However, Schrödinger himself goes into a superposition of observing his cat to be alive and his cat to be dead. In the many-worlds interpretation, there is thus a difference between observing something to be so, and something actually being so: the observation is of a particular physical scenario, but the reality is a superposition of different physical scenarios.

To capture this distinction between observation and reality, Bell speaks of *beables*. Bell introduces the term beable when speculating on what would be a more satisfactory physical theory than what quantum physics currently has to offer.[25] Bell says that such a theory should be able to say of a system not only that such and such is observed to be so, but that such and such be so. In other words, a more satisfactory theory would be a theory of beables rather than a theory of observables. On the macroscopic level, these beables should be the underlying reality that gives rise to all the familiar things in the world around us, things like cats, laboratories, procedures, and so on. For example proponents of the pilot wave interpretation believe that the beables are all the particles each with their precise position and momentum. But whatever these beables are, it is because of them that a scientist can observe a physical system to be in such and such a state. Thus, observables are ontologically dependent on beables.

Now the beables in Kent's one world interpretation are expressed in terms of a physical quantity called the *stress-energy tensor* $T^{\mu\nu}(y)$. For any spacetime location y, the stress-energy tensor $T^{\mu\nu}(y)$ is an array of 16 values corresponding to each combination

[25] See J. S. Bell, "Subject and Object," in *Speakable and unspeakable in quantum mechanics*, 2nd ed. (Cambridge: Cambridge University Press, 2004), 40–44

of $\mu, \nu = 0, 1, 2$, or 3. The value $T^{00}(y)$ is the energy density at y divided by c^2,[26] whereas the other values of $T^{\mu\nu}(y)$ indicate how much energy and momentum flow across different surfaces in the neighborhood of y.

It was mentioned in the previous section that for any spacetime location $x \in S$, there is an observable $\hat{T}_S(x)$ acting on H_S corresponding to the mass-energy density of the surface S at x. It turns out that for any $\mu, \nu = 0, 1, 2$, or 3, there is also an observable $\hat{T}^{\mu\nu}(x)$ acting on H_S, such that if $|\Psi\rangle \in H_S$ is a simultaneous eigenstate of $\hat{T}^{\mu\nu}(x)$ with eigenvalue $\tau^{\mu\nu}(x)$ for all $x \in S$, then $|\Psi\rangle$ corresponds to a state of S in which $T^{\mu\nu}(x)$ is $\tau^{\mu\nu}(x)$ for all $x \in S$.[27] Moreover, the observable $\hat{T}_S(x)$ is expressible in terms of the $\hat{T}^{\mu\nu}(x)$-observables.[28,29] Now the beables in Kent's theory are defined at each spacetime location y that occurs after S_0 and before S. For such a spacetime location y, the beables will be determinate values of the stress-energy tensor $T^{\mu\nu}(y)$, but calculated from the expectation of the observable $\hat{T}^{\mu\nu}(y)$ conditional on the energy-density on S being given by $\tau_S(x)$ for all $x \in S$ but outside the light cone of y. In section 3.4, we will come back to the question of why we can't include any information about $\tau_S(x)$ for $x \in S$ within the light cone of y when we discuss how these conditional expectations are calculated. But before we do that, we first consider

[26]This is not to be confused with the mass-energy density $T_S(x)$ defined for x on a spacelike hypersurface S. As will be shown in section 4.1, all 16 elements of $T^{\mu\nu}(x)$ will typically be needed to calculate $T_S(x)$.

[27]Note however, that such a simultaneous eigenstate is only for a fixed choice of μ and ν, since in general, $\hat{T}^{\mu\nu}(x)$ and $\hat{T}^{\mu'\nu'}(x)$ will not commute for $\mu \neq \mu'$ or $\nu \neq \nu'$.

[28]See section 4.1 for an explanation for why this is so.

[29]As in (3.7), we have the same implicit understanding of $\hat{T}^{\mu\nu}(x)$ and $\tau^{\mu\nu}(x)$ as being defined over cells $c_x \subset S$ rather than at spacetime locations $x \in S$, though we will often speak of them as being defined at spacetime locations.

why we should need conditional expectations at all in order to provide a one world description of reality.

To this end, we recall the definition of expectation in equation (2.1) and the expectation formula (2.2) for an observable. In a theory that posited the beables to be the expectation values of $\hat{T}^{\mu\nu}(y)$ for any y located between S_0 and S without conditioning on the value of the energy-density on S, then the $T^{\mu\nu}(y)$-beable would just be $\langle\Psi'|\hat{T}^{\mu\nu}(y)|\Psi'\rangle$ where $|\Psi'\rangle = U_{S'S_0}|\Psi_0\rangle$ for any spacelike hypersurface S' that goes through y.[30] However, such a beable would give a description of reality that was very different from what we observe. For instance, in a Schrödinger cat-like experiment (see section 1.3), there would be a stress-energy tensor distribution corresponding to both the cat being alive and the cat being dead in the same world as depicted in figure 3.5.

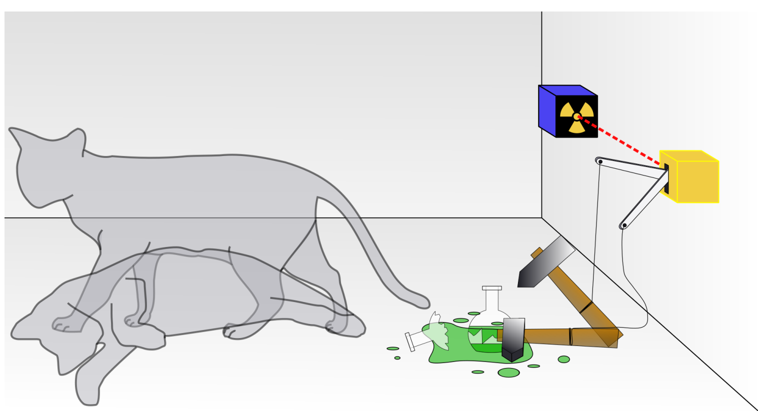

Figure 3.5: A depiction of Schrödinger's cat being both dead and alive.[31]

Such a distribution arises in this context because initially there is an atom that is in a superposition of decayed and non-decayed states, and so the expectation of $\hat{T}^{\mu\nu}(y)$ will have non-zero components both in the location where the non-decayed atom would

[30]This can be done such that $\langle\Psi'|\hat{T}^{\mu\nu}(y)|\Psi'\rangle$ does not depend on the spacelike hypersurface S' other than the fact that it contains y. For more details see Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation."

[31]Original by Dhatfield. This image is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. Source: https://commons.wikimedia.org/wiki/File:Schrodingers_cat.svg

be, and also in the locations of the decayed atom and the particle the atom emitted. As the decayed atom part of the state interacts with the poison releasing device, this device will also enter into a superposition so that in both the location of the poison containing flask and in the locations of all the poison atoms in the container containing the cat and into which the poison is released, the expectation of $\hat{T}^{\mu\nu}(y)$ will have non-zero components. And then the cat will enter into a superposition of being in a dead state and an alive state, and so that the expectation of $\hat{T}^{\mu\nu}(y)$ will have non-zero components in locations where the dead cat ends up and where the living cat happens to be. So the expectation of $\hat{T}^{\mu\nu}(y)$ in the locations of the container containing the cat will be very different from what someone would actually observe.

To overcome this defect, information about the mass-energy density on S is used, specifically the values of $\tau_S(x)$ for $x \in S^1(y)$ where $S^1(y)$ is defined to consist of all the spacetime locations of S outside the light cone of y as depicted in figure 3.6.

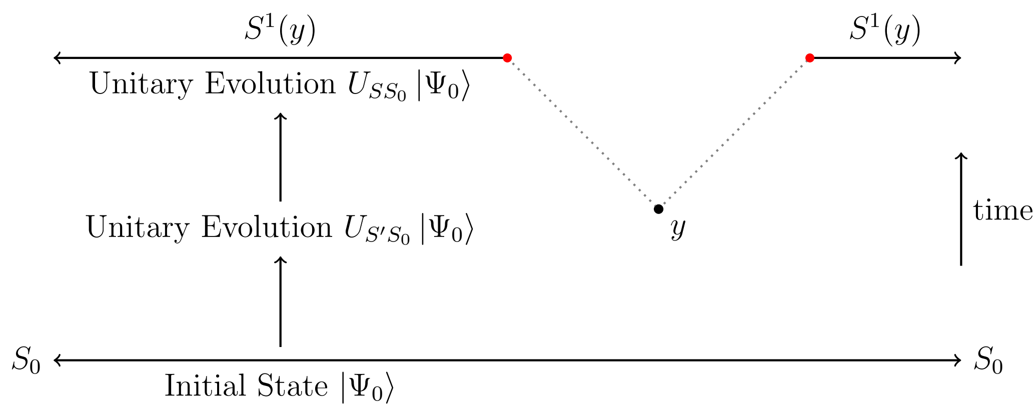

Figure 3.6: The set $S^1(y)$ consists of all the spacetime locations of S outside the light cone of y. The $T^{\mu\nu}(y)$-beables are calculated using the initial state $|\Psi_0\rangle$ together with the values of $\tau_S(x)$ for $x \in S^1(y)$.

So in the case of Schrödinger's cat, if the cat were dead, light reflecting off the dead cat and going off into outer space would eventually intersect the spacelike hypersurface

S, and the light distribution on S would register the inanimate status of the cat. On the other hand, if the cat were alive, the light reflecting off the living cat and going off into outer space would also intersect S, but now the light distribution on S would register the different locations the living cat was in as it moved about. Because light travels at a constant speed in a vacuum, the state of the cat at earlier times would be described by light distributions in regions on S that were further away from the cat than those light distributions in regions of S that described the cat in more recent times.

Now if the cat was in a superposition of dead and alive states, then assuming there is no intermediate collapse of the global quantum state, the spacelike hypersurface S would also enter into a superposition of different states corresponding to these different distributions of light registered on S. But if a notional measurement on S is made that determines which of these distributions is actually realized on S, then this determination will determine which history was actualized, and hence determine whether the cat actually survived Schrödinger's experiment or whether it perished. Thus, by conditioning on one of these two distributions on S being actualized, the conditional expectation of the stress-energy tensor in the vicinity of where Schrödinger's cat might be will not describe a situation like the one depicted in figure 3.5. Rather, it will either describe a situation like the one depicted in figure 3.7, or it will describe a situation like the one depicted in figure 3.8. Which of these two situations occur will be determined by whether the measurement outcome on S corresponds to a light distribution reflected from a living cat, or to a light distribution reflected from a dead cat.

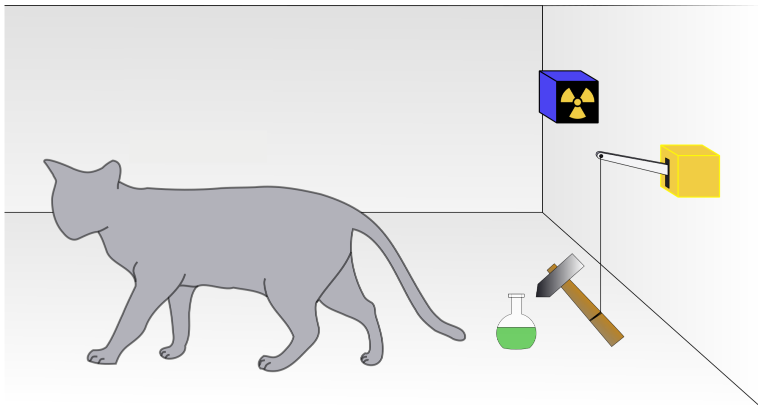

Figure 3.7: A depiction of Schrödinger's cat being alive.[32]

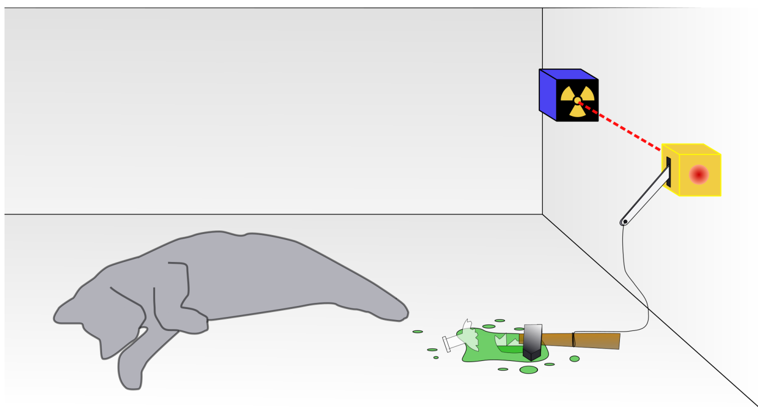

Figure 3.8: A depiction of Schrödinger's cat being dead.[33]

We assume that the outcome of the notional measurement (which determines which mass-energy distribution on S is realized) occurs with a probability given by the Born rule. Under this assumption, we would then expect that in the history conditioned on this notional measurement, any scientists who performed measurements (in the normal sense of measurement) would measure average values of physical quantities consistent with the expectation values predicted by standard quantum mechanics. This intuition will be discussed in more detail in section 4.6.

[32]Original by Dhatfield. This image is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. Source: https://upload.wikimedia.org/wikipedia/commons/archive/9/91/20080627113554!Schrodingers_cat.svg

[33]Original by Dhatfield. Altered by removing numbers and making into two separate figures. This image is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. Source: https://upload.wikimedia.org/wikipedia/commons/archive/9/91/20080627113554!Schrodingers_cat.svg

3.4 Calculating Kent's $T^{\mu\nu}(y)$-beables*

Having given a qualitative description in the last section of how a measurement outcome on S determines which facts obtain in reality such as whether Schrödinger's cat is alive or dead, we now give a more quantitative description of how Kent's beables are calculated. Kent's beables specify $T^{\mu\nu}(y)$ values for all y between S_0 and S, and for all $\mu, \nu = 0, 1, 2,$ and 3. Kent's $T^{\mu\nu}(y)$-beables are conditional expectation values, and the conditional expectation that we need to calculate depends on the notion of *conditional probability*. In probability theory, the conditional probability $P(q|r)$ that a statement q is true given that a statement r is true is given by the formula

$$P(q|r) = \frac{P(q \,\&\, r)}{P(r)}, \tag{3.10}$$

where $P(r)$ is the probability r is true, and $P(q \,\&\, r)$ is the probability both q and r are true. If we now define $q(\tau)$ to be the statement that some quantity T takes the value τ, then the *conditional expectation* of T given r will be given by the formula

$$\langle T \rangle_r \stackrel{\text{def}}{=} \sum_\tau P(q(\tau)|r)\tau \tag{3.11}$$

where the summation is over all the possible values τ that T can take.

The recipe for defining Kent's $T^{\mu\nu}(y)$-beable is first to select an outcome τ_S that is defined over all of S. The outcome τ_S is selected with probability determined by the Born rule using equation (3.9).[34] Then Kent's $T^{\mu\nu}(y)$-beable for any y between S_0 and

---

[34]If there is only one state $|\Psi\rangle$ such that $\hat{T}_S(x)\,|\Psi\rangle = \tau_S(x)\,|\Psi\rangle$ for all $x \in S$, then the probability $P(\tau_S)$ that τ_S is selected will be precisely the probability given by equation (3.9). But if there are several states $\{\,|\Psi_\alpha\rangle : \alpha\}$ such that $\hat{T}_S(x)\,|\Psi_\alpha\rangle = \tau_S(x)\,|\Psi_\alpha\rangle$, then the probability $P(\tau_S)$ that τ_S is selected will be

$$P(\tau_S) = \sum_\alpha |\,\langle \Psi_\alpha | U_{SS_0} | \Psi_0 \rangle|^2.$$

Also note that while no one outcome for τ_S is going to be very likely, having the outcome τ_S shouldn't be highly improbable relative to other possible mass-energy density outcomes. That is, if $P(\tau_S) \ll P(\tau'_S)$, then τ_S shouldn't be selected.

S is defined to be $\langle T^{\mu\nu}(y)\rangle_{\tau_S} \stackrel{\text{def}}{=} \langle T^{\mu\nu}(y)\rangle_{r(\tau_S,y)}$ where $r(\tau_S, y)$ is the statement that $T_S(x)$ has the determinate value $\tau_S(x)$ for all $x \in S^1(y)$,[35] and where $q(\tau)$ in equation (3.11) is the statement that $T^{\mu\nu}(y)$ (understood in the conventional non-Kentian sense) takes the value τ.[36] It is these $T^{\mu\nu}(y)$-beables $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ that give a one-world picture of reality in Kent's theory.

We can see how the formula (3.11) relates to the Schödinger's cat scenario. The distribution of light reflected off the cat that intersects $S^1(y)$ when "measured" will determine a definite statement $r(\tau_S, y)$ about the mass-energy density on $S^1(y)$. This in turn will determine the range of τ for which $P(q(\tau)|r(\tau_S, y))$ is not close to zero, and hence where the stress-energy distribution $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ is not zero. This stress-energy distribution will then correspond either to that of a living cat or to that of a dead cat, but not both.

Coming back to the question of why we don't include any information about $\tau_S(x)$ for $x \in S$ from within the light cone of y, we need to consider in more detail how we would calculate $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$. From (3.10) and (3.11), we will be able to perform this calculation so long as we can calculate $P(q(\tau)\,\&\, r(\tau_S, y))$ and $P(r(\tau_S, y))$.

Calculating $P(r(\tau_S, y))$ is relatively straightforward. As described on page 116, we can find an orthonormal basis $\{\,|\Psi^{(i)}\rangle : i\}$ of H_S consisting of simultaneous $\hat{T}_S$-eigenstates and simultaneous $\hat{T}_S$-eigenvalues $\tau_S^{(i)}$ respectively. The probability $P(r(\tau_S, y))$ will

[35] Strictly speaking, we should say that $r(\tau_S, y)$ is the statement that the approximation of the mass-energy density $T_S(x)$ given by equation (3.5) has the value $\tau_S(c_x)$ for every cell c_x in S outside the light cone of y.

[36] Again, we assume that $T^{\mu\nu}(y)$ is averaged over a small three-dimensional cell c_y of spacelike separated spacetime locations (with $y \in c_y$), and approximated to a finite pool of values as in equation (3.5).

then be

$$P(r(\tau_S, y)) = \sum_{\substack{i \text{ such that} \\ \tau_S^{(i)}(x)=\tau_S(x) \\ \text{for all } x\in S^1(y)}} \left| \left\langle \Psi^{(i)} \middle| U_{SS_0} \middle| \Psi_0 \right\rangle \right|^2 \tag{3.12}$$

where we have used equation (3.9).

But calculating $P(q(\tau)\,\&\,r(\tau_S, y))$ is a bit more involved because in the Tomonaga-Schwinger picture, the definition of observables via

$$\hat{O}(x) = U[S]\hat{\boldsymbol{O}}(x)U[S]^{-1} \tag{3.3 revisted}$$

requires that $x \in S$. This means that we can't define $\hat{T}^{\mu\nu}(y)$ according to (3.3) since $y \notin S$.[37] However, we do not face such restrictions in the Heisenberg picture, so one approach would be to calculate $P(q(\tau)\,\&\,r(\tau_S, y))$ in the Heisenberg picture. As we will see shortly, this is not the approach that Kent takes, but nevertheless, in the Heisenberg picture, it is easier to see why we don't include information from S within the light cone (without begging the question of why we don't) when calculating $P(q(\tau)\,\&\,r(\tau_S, y))$. To see why this is so, consider the simpler case of just two measurable quantities F and G which we assume to have a discrete range of possible values and for which we wish to calculate the joint probability $P((F = f)\,\&\,(G = g))$. To do this in the Heisenberg picture, we need an orthonormal basis of the state space $\{\,|\Phi^{(i)}\rangle : i\}$ consisting of simultaneous eigenstate of the observables $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{G}}$ with eigenvalues $f^{(i)}$ and $g^{(i)}$ respectively so that $\hat{\boldsymbol{F}}\,|\Phi^{(i)}\rangle = f^{(i)}\,|\Phi^{(i)}\rangle$ and $\hat{\boldsymbol{G}}\,|\Phi^{(i)}\rangle = g^{(i)}\,|\Phi^{(i)}\rangle$, and when the system is in the state $|\Phi^{(i)}\rangle$, the quantity F will have the value $f^{(i)}$, and the quantity G will have the value $g^{(i)}$. Given that the system is in the state $|\Phi\rangle$, the joint

[37] If we did attempt to use (3.3) to define $\hat{T}^{\mu\nu}(y) = U[S]\hat{\boldsymbol{T}}^{\mu\nu}(y)U[S]^{-1}$, then $\hat{T}^{\mu\nu}(y)$ would have a (non-local) dependence on S, and such a dependence would not be desirable.

probability $P((F = f) \,\&\, (G = g))$ can then be calculated using the Born rule to get

$$P((F = f) \,\&\, (G = g)) = \sum_{\substack{i \text{ such that} \\ f^{(i)}=f \text{ and } g^{(i)}=g}} \left|\left\langle \Phi^{(i)} \middle| \Phi \right\rangle\right|^2.$$

But in order for such an orthonormal basis to exist, it is necessary that $\hat{F}$ and $\hat{G}$ commute.[38] This means that if $\hat{F}$ and $\hat{G}$ do not commute, then we cannot define the joint probability $P((F = f) \,\&\, (G = g))$.

Now quantum field theory is so constructed that $\hat{\boldsymbol{T}}^{00}(x)$ and $\hat{\boldsymbol{T}}^{\mu\nu}(y)$ will not commute when x and y are not spacelike separated, but $\hat{\boldsymbol{T}}^{\mu'\nu'}(x)$ and $\hat{\boldsymbol{T}}^{\mu\nu}(y)$ will commute when x and y are spacelike separated.[39] As we will see on page 153, $T_S(x)$ will have a $T^{00}(x)$ component, and so we can only be sure that $\hat{\boldsymbol{T}}_S(x)$ will commute with $\hat{\boldsymbol{T}}^{\mu\nu}(y)$ if x and y are spacelike separated. In other words, $\hat{\boldsymbol{T}}_S(x)$ and $\hat{\boldsymbol{T}}^{\mu\nu}(y)$ will commute if x is outside the light cone of y. Extending this argument to multiple $x \in S$, we see that we can only guarantee that the conditional expectation of $T^{\mu\nu}(y)$ is definable if we restrict our conditioning on the value of $T_S(x)$ to $x \in S^1(y)$, that is, to x in S outside the light cone of y.

Having explained why we don't include any information about $\tau_S(x)$ for $x \in S$ from within the light cone of y, we can now proceed to calculate $P(q(\tau) \,\&\, r(\tau_S, y))$. Now

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[38]This is because given such an orthonormal basis $\{\,|\Phi^{(i)}\rangle : i\}$ of simultaneous eigenstates of $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{G}}$, we have

$$\hat{\boldsymbol{F}}\hat{\boldsymbol{G}}\,|\Phi^{(i)}\rangle = f^{(i)}g^{(i)}\,|\Phi^{(i)}\rangle = g^{(i)}f^{(i)}\,|\Phi^{(i)}\rangle = \hat{\boldsymbol{G}}\hat{\boldsymbol{F}}\,|\Phi^{(i)}\rangle$$

so for any arbitrary state $|\Phi\rangle = \sum_i c_i\,|\Phi^{(i)}\rangle$, we have

$$\hat{\boldsymbol{F}}\hat{\boldsymbol{G}}\,|\Phi\rangle = \sum_i c_i \hat{\boldsymbol{F}}\hat{\boldsymbol{G}}\,|\Phi^{(i)}\rangle = \sum_i c_i \hat{\boldsymbol{G}}\hat{\boldsymbol{F}}\,|\Phi^{(i)}\rangle = \hat{\boldsymbol{G}}\hat{\boldsymbol{F}}\,|\Phi\rangle\,.$$

[39]The proof of this statement need not concern us, but one can see that this is the case by considering the four potential commutation relations and the decomposition of the stress-energy tensors as in terms of the four-potentials – see Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation," p. 1443–1444.

although there is no spacelike hypersurface that contains both y and $S^1(y)$, we can find a sequence of spacelike hypersurfaces $S_n(y)$ each of which contains[40] y such that $S_n(y) \subset S_{n'}(y)$ for $n < n'$, and such that for any $x \in S^1(y)$, there exists n and an open subset $U_n(x) \subset S$ containing x such that $U_n(x) \subset S_n(y)$. It will also be convenient to require that $S \setminus S_n(y)$ is bounded. An example of one such $S_n(y)$ is shown in figure 3.9. When there is no ambiguity, we will drop the y and write S_n instead of $S_n(y)$. Such a sequence S_n of hypersurfaces will be sufficient to calculate $P(q(\tau)\,\&\,r(\tau_S, y))$.

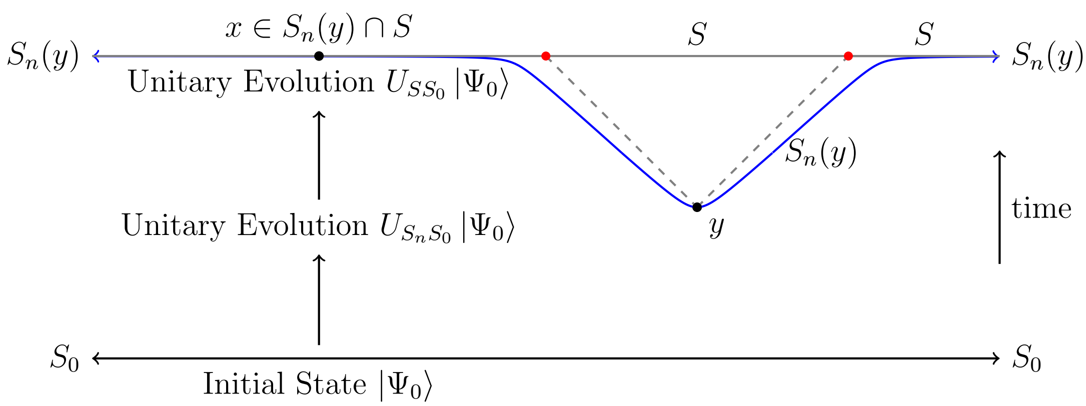

Figure 3.9: $S_n \stackrel{\text{def}}{=} S_n(y)$ is a spacelike hypersurface containing y and all of $S^1(y)$ in the limit as $n \to \infty$.

So let us define r_n to be the statement that $T_S(x)$ has the determinate value $\tau_S(x)$ for all $x \in S_n \cap S$.[41] We recall that in the Tomonaga-Schwinger formulation of

[40]More precisely, we should say each hypersurface $S_n(y)$ contains c_y where c_y is the cell contain y mentioned in footnote 36. We make c_y sufficiently small so that the c_x cells of S outside the light cone of y are identical to the c_x cells of S outside the light cone of c_y. We can do this on the assumption that c_y and the c_x are closed sets, since outside the light cone of y is an open set, so none of the c_x will touch the boundary of the light cone in S.

[41]Strictly speaking, we should say the condition of r_n holds for all $x \in S_n \cap S$ at which S_n and S are tangential to each other. For a possible worry someone might have about the statement r_n without this qualification is that $\tau_S(x)$ is the value of the beable $T_S(x)$ for x in $S^1(y) \cap S$ but it's not the value of the beable $T_{S_n}(x)$ for x in $S_n \cap S$. Such a worry would be valid if the beable $T_S(x)$ depended on the whole of S and the beable $T_{S_n}(x)$ depended on the whole of S_n. However, as we shall see on page 153 in section 4.2, the physical quantity $T_S(x)$ which is defined by equation (4.20) to be $T_S(x) = T^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x)$ will only have a local dependence on S via the future directed four-vector $\eta^\mu(x)$. Therefore, so long as the future directed four-vector for S_n at x is the same as the one for S at x, then the beables $T_S(x)$ and $T_{S_n}(x)$ will be identical. We therefore require that S_n and S are tangential to each other at x since this is a necessary and sufficient condition for the respective future directed four-vectors of S_n and S to be identical.

relativistic quantum physics, the operators $\hat{T}_S(x)$ and $\hat{T}^{\mu\nu}(y)$ for fixed μ, ν commute when x and y are spacelike-separated. It therefore follows that we can express any state of H_{S_n} as a superposition of simultaneous eigenstates of $\hat{T}^{\mu\nu}(y)$ and $\hat{T}_S(x)$ for $x \in S_n \cap S$.[42] For a particular choice of μ, ν, we can then form an orthonormal basis $\{\,|\Psi_n^{(i)}\rangle : i\}$ of H_{S_n} consisting of simultaneous $\hat{T}^{\mu\nu}(y)$, $\hat{T}_S(x)$-eigenstates so that $\hat{T}^{\mu\nu}(y)\,|\Psi_n^{(i)}\rangle = \tau^{(i)}\,|\Psi_n^{(i)}\rangle$ and $\hat{T}_S(x)\,|\Psi_n^{(i)}\rangle = \tau_S^{(i)}(x)\,|\Psi_n^{(i)}\rangle$ for $x \in S_n \cap S$, where $\tau^{(i)}$ and $\tau_S^{(i)}(x)$ are the corresponding eigenvalues. The probability $P(q(\tau)\,\&\,r_n)$ will then be

$$P(q(\tau)\,\&\,r_n) = \sum_{\substack{i \text{ such that } \tau^{(i)}=\tau \\ \text{and } \tau_S^{(i)}(x)=\tau_S(x) \\ \text{for all } x\in S_n\cap S}} \left|\left\langle \Psi_n^{(i)} \middle| U_{SS_0} \middle| \Psi_0 \right\rangle\right|^2.$$

Taking the limit as n tends to infinity, we can calculate the probability $P(q(\tau)\,\&\,r(\tau_S, y))$ to be

$$P(q(\tau)\,\&\,r(\tau_S, y)) = \lim_{n\to\infty} P(q(\tau)\,\&\,r_n).^{43} \tag{3.13}$$

Assuming τ_S is selected in accordance with the Born rule so that $P(r) > 0$, we can plug (3.12) and (3.13) into (3.10) to calculate the conditional probability $P(q(\tau)|r(\tau_S, y))$,

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Nevertheless, we might still worry that the observables $\hat{T}_S(x)$ and $\hat{T}_{S_n}(x)$ corresponding to these two beables aren't identical because $\hat{T}_S(x)$ acts on the Hilbert space H_S whereas $\hat{T}_{S_n}(x)$ acts on the Hilbert space H_{S_n}. However, at this point we need to recall footnote 18 on page 115 that H_S and H_{S_n} are really the same Hilbert space, but just interpreted differently. Now on this one Hilbert space, it turns out that $\hat{T}_S(x)$ and $\hat{T}_{S_n}(x)$ are identical. To see why this is, let $\hat{\boldsymbol{T}}_S(x)$ and $\hat{\boldsymbol{T}}_{S_n}(x)$ be the Heisenberg picture observables. Since $T_S(x) = T^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x) = T_{S_n}(x)$ for $x \in S_n \cap S$ where S_n and S are tangential to one another, we must have $\hat{\boldsymbol{T}}_S(x) = \hat{\boldsymbol{T}}^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x) = \hat{\boldsymbol{T}}_{S_n}(x)$. Now by definition (see equation (3.3)), $\hat{T}_S(x) = U[S]\hat{\boldsymbol{T}}_S(x)U[S]^{-1}$, and $\hat{T}_{S_n}(x) = U[S_n]\hat{\boldsymbol{T}}_{S_n}(x)U[S_n]^{-1}$. But as Schwinger shows, under conditions that are readily satisfied (see footnote 17 for details), for any Heisenberg operator $\hat{\boldsymbol{F}}(x)$, as long as x belongs to S the operator $\hat{F}(x) = U[S]\hat{\boldsymbol{F}}(x)U[S]^{-1}$ is independent of S. Therefore, since $x \in S_n \cap S$ where S_n and S are tangential to one another, we not only have $\hat{\boldsymbol{T}}_S(x) = \hat{\boldsymbol{T}}_{S_n}(x)$, but we must also have $\hat{T}_S(x) = \hat{T}_{S_n}(x)$.

[42] Strictly speaking we should say simultaneous eigenstates of $\hat{T}^{\mu\nu}(c_y)$ and $\hat{T}_S(c_x)$ for all $c_x \subset S_n \cap S$.

[43] In fact, there will be a finite n' such that $P(q(\tau)\,\&\,r(\tau_S, y)) = P(q(\tau)\,\&\,r_{n'})$. This is because for any n we are assuming $S \setminus S_n$ is bounded, so there will be a finite number of c_x cells of S outside the light cone of c_y that are not contained in S_n, and so the union of all these cells U will be compact. But for any $x \in U$, we can find n'' such that the open set $U_{n''}(x)$ containing x with $U_{n''}(x) \subset S$ and

and hence the conditional expectation $\langle T^{\mu\nu}(y)\rangle_{\tau_S} \stackrel{\text{def}}{=} \langle T^{\mu\nu}(y)\rangle_{r(\tau_S,y)}$ via equation (3.11). We thus obtain

$$\langle T^{\mu\nu}(y)\rangle_{\tau_S} = \sum_{\tau} P(q(\tau)|r(\tau_S,y))\tau = \lim_{n\to\infty}\sum_{\tau}\frac{P(q(\tau)\,\&\,r_n)\tau}{P(r_n)}. \tag{3.14}$$

In section 4.1, we will give a more detailed description of this calculation in order to show that the predictions Kent's theory makes are consistent with standard quantum physics.

3.5 Kent's toy example

To get a feel for how all the elements of Kent's theory fit together, we will conclude this chapter by describing Kent's toy model example that he discusses in his 2014 paper.[44] In his toy model, Kent considers a system in one spatial dimension which is the superposition of two localized states/wave functions[45] $\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$ where ψ_1^{sys} is localized at spatial location z_1, ψ_2^{sys} is localized at spatial location z_2, and $|c_1|^2 + |c_2|^2 = 1$. According to the Copenhagen interpretation, a measurement on this system would collapse the wave function of ψ_0^{sys} to the wave function of ψ_1^{sys} with probability $|c_1|^2$, and to the wave function of ψ_2^{sys} with probability $|c_2|^2$. The purpose

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$U_{n''}(x) \subset S_n$. These $U_{n''}(x)$ will then form an open cover of U, and by the definition of compactness, every open cover has a finite subcover. If we therefore choose n' to be the maximum n'' of this finite subcover, then $S_{n'}$ will contain all of U since $S_{n''} \subset S_{n'}$ for $n'' < n'$. Then by definition of the statements $r(\tau_S, y)$ and r_n, it follows that $r(\tau_S, y) = r_{n'}$.

[44]See Kent, "Lorentzian Quantum Reality: Postulates and Toy Models," p. 3–4.

[45]So far in this chapter, we have been describing systems in terms of their quantum states rather than their *quantum wave functions*. It is easiest to understand what a quantum wave function is in the context of a single particle system. In the Schrödinger picture, a particle in state $|\psi(t)\rangle$ allows us to calculate the expectation value of a quantity O belonging to the particle via the formula $\langle\psi(t)|\hat{\boldsymbol{O}}|\psi(t)\rangle$ where $\hat{\boldsymbol{O}}$ is the observable corresponding to O. One such quantity is the particle's spatial location. If the particle is at spatial location $\boldsymbol{x}$, then the particle will be in the state $|\boldsymbol{x}\rangle$ (c.f. the definition of $|x\rangle$ in footnote 47 on page 83.) where the position observable $\hat{\boldsymbol{X}}_i$ satisfies $\hat{\boldsymbol{X}}_i\,|\boldsymbol{x}\rangle = x_i\,|\boldsymbol{x}\rangle$ for $i = 1, 2$, or 3. The corresponding wave function for this particle is then $\psi(\boldsymbol{x}, t) = \langle\boldsymbol{x}|\psi(t)\rangle$. For a particle restricted to one spatial dimension, the particle would have a wave function $\psi(z,t) = \langle z|\psi(t)\rangle$ where z is now just a single number that specifies the particle's possible position. We will write ψ to denote the wave function itself, and $\psi(z,t)$ to denote the value of the wave function ψ at spacetime location (z,t).

of Kent's toy model is to show that within his interpretation, there is something analogous to wave function collapse. In order for this "collapse" to happen, one needs to consider how the system interacts with light. Thus, Kent supposes that a photon (which is modelled as a point particle) comes in from the left, and as it interacts with the two states ψ_1^{sys} and ψ_2^{sys}, the photon enters into a superposition of states, corresponding to whether the photon reflects off the localized ψ_1^{sys}-state at time t_1 or the localized ψ_2^{sys}-state at time t_2. The photon in superposition then travels to the left and eventually reaches the one dimensional spacelike hypersurface S at locations γ_1 and γ_2 as shown in figure 3.10.

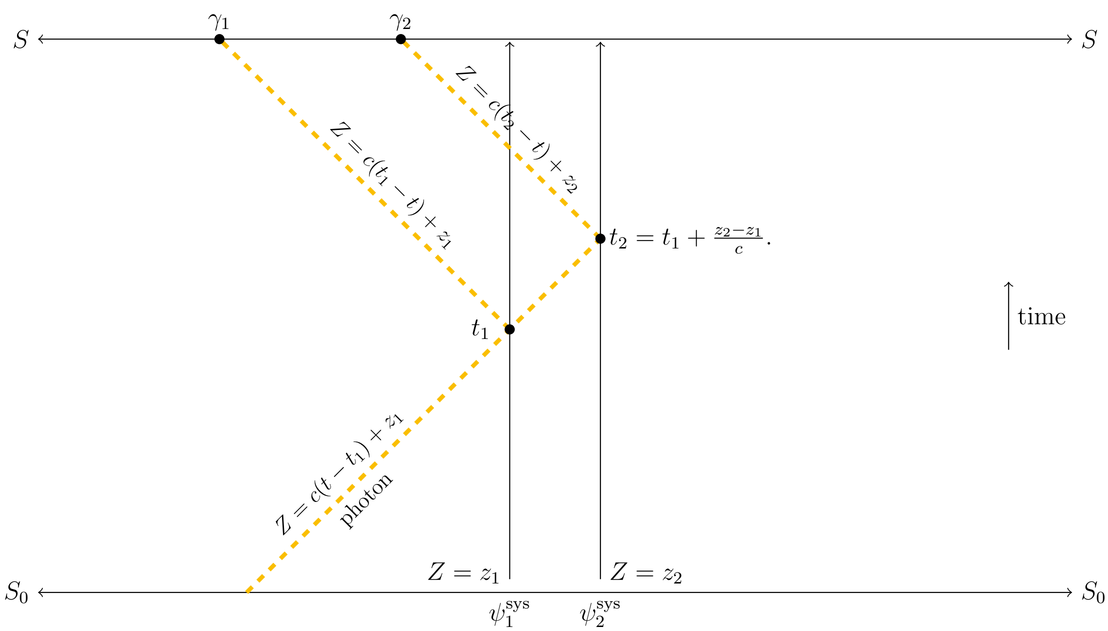
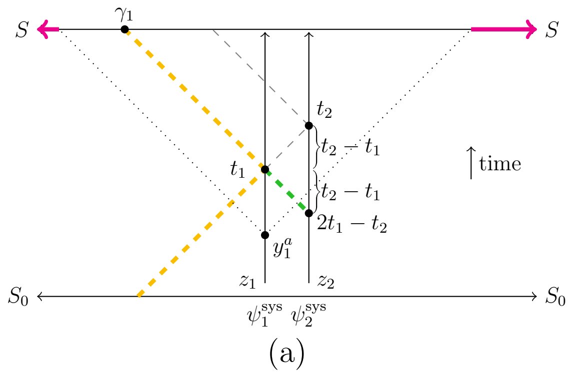


Figure 3.10: Kent's toy model

We now suppose that when the mass-energy density S is "measured", the energy of the photon is found to be at γ_1 rather than at γ_2. We then consider the mass-density at early spacetime locations $y_1^a = (z_1, t_a)$ and $y_2^a = (z_2, t_a)$ as shown in figure 3.11 (a) and (b).



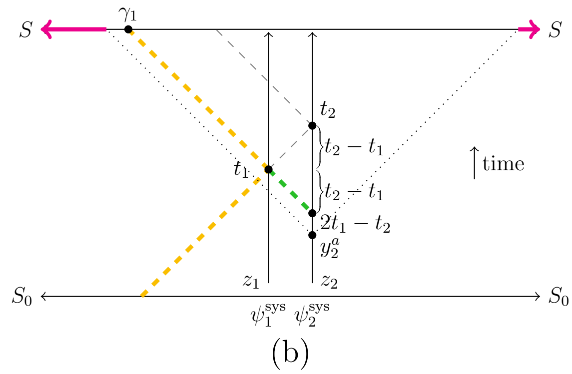


Figure 3.11: (a) highlights the part of S used to calculate the energy density at y_1^a whose time is less than $2t_1 - t_2$. (b) highlights the part of S used to calculate the energy density at y_2^a whose time is less than $2t_1 - t_2$.

By early, we mean that $t_a < 2t_1 - t_2$. This will mean that the possible detection locations γ_1 and γ_2 will be inside the forward light cones of y_1^a and y_2^a. Hence, $S^1(y_1^a) \cap S$ and $S^1(y_2^a) \cap S$ contain no additional information beyond standard quantum theory by which we could calculate the conditional expectation values of the energy at y_1^a and y_2^a. Hence, according to Kent's theory, the total energy at time t_a will be divided between the two spatial locations with a proportion of $|c_1|^2$ at z_1 and a proportion of $|c_2|^2$ at z_2.

However, the situation is different for two spacetime locations $y_1^b = (z_1, t_b)$ and $y_2^b = (z_2, t_b)$ with t_b slightly after $2t_1 - t_2$ as depicted in figure 3.12.

In this situation, when we consider the location y_1^b, there is no additional information in $S^1(y_1^b) \cap S$ beyond standard quantum theory, so there will be a proportion of $|c_1|^2$ of the total initial energy of the system at y_1^b. But at location y_2^b, the information in $S^1(y_2^b) \cap S$ shows that the photon has reflected from the localized $\psi_1^{\rm sys}$-state, and so this additional information tells us that after time t_b, there is no energy localized at z_2 since from the perspective of y_2^b, the energy is known to be localized at z_1. So it

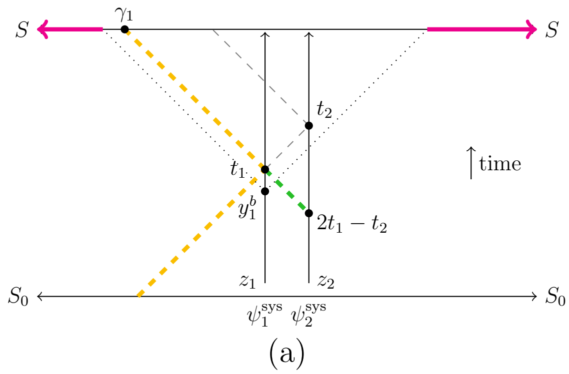


Figure 3.12: (a) highlights the part of S used to calculate the energy density at y_1^b whose time is greater than $2t_1 - t_2$. (b) highlights the part of S used to calculate the energy density at y_2^b whose time is greater than $2t_1 - t_2$.

is as though the information of $S^1(y_2^b) \cap S$ has determined that we are in a world in which there is an energy density of zero at y_2^b, and there are no other worlds in which the energy density at y_2^b is non-zero since all worlds have to be consistent with the notional measurement made on S. So for a short time the total energy of the system is reduced by $|c_1|^2$.

However, as shown in figure 3.13, for times t_c greater than t_1, the total energy of the system is once again restored to the initial energy the system had when in the state ψ_0^{sys}.

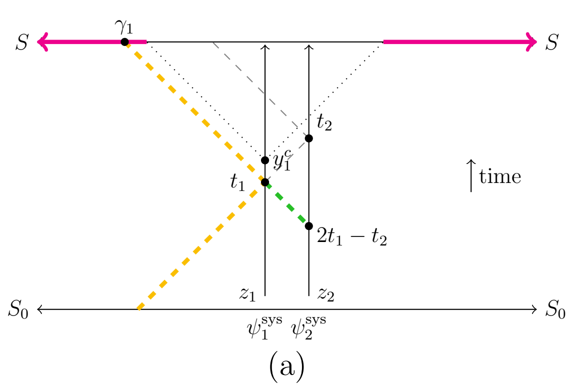


Figure 3.13: (a) highlights the part of S used to calculate the energy density at y_1^c whose time is greater than t_1. (b) highlights the part of S used to calculate the energy density at y_2^c whose time is greater than t_1.

In this situation, there is now information in $S^1(y_1^c) \cap S$ that determines that the photon reflected off the localized ψ_1^{sys}-state. This means that when the conditional expectation of the energy density of y_1^c is calculated, the extra information in $S^1(y_1^c) \cap S$ determines that all the energy of the system is located at location z_1 for times t_c greater than t_1, and the energy is equal to the initial energy of the system so that energy is conserved.

Chapter 4

Evaluating Kent's Theory

In order to evaluate Kent's theory of quantum physics, it will first be helpful to remind ourselves of the problem we are trying to solve.

In chapter 1, we discussed the EPR-Bohm paradox and the problem of explaining the mysterious correlations between measurement outcomes of spin singlets in a way consistent with special relativity and the predictions of standard quantum theory. We saw that the Copenhagen interpretation does not seem to be consistent with special relativity. We also discussed Shimony's distinction between Outcome Independence (OI) and Parameter Independence (PI) and Shimony's idea that we should only accept a theory in which OI is false and PI is true. Since PI is false in the pilot-wave theory, we should reject it according to Shimony's criterion.

But although Shimony's criterion is a promising line of inquiry, by itself, it is not sufficient to resolve the EPR-Bohm paradox. This is because Shimony's criterion doesn't address the controversial issue of what is meant by an outcome. In chapter 2, we discussed this controversy over outcomes and why the many-worlds interpretation that denies the reality of outcomes is unsatisfactory. This motivated the discussion of Kent's theory in chapter 2 in the hope that it might provide a satisfactory solution to the EPR-Bohm paradox. In the previous chapter, we only got as far as describing the key features of Kent's theory such as it being a one-world theory which posits additional variables to standard quantum theory.

So having now reminded ourselves of the problem at hand, we see that there are several issues that we need to consider in order to evaluate whether Kent's theory provides a satisfactory solution to this problem. Firstly, we should consider whether the predictions of Kent's theory are consistent with the predictions of quantum theory that have been experimentally validated. Since standard quantum theory (that is a theory of states whose evolution is determined by the Schrödinger equation) predicts the correlations observed in the EPR-Bohm paradox, then if Kent's theory is consistent with standard quantum theory, these EPR-Bohm correlations will also be exhibited in Kent's theory.

Secondly, since a satisfactory solution to our problem must be consistent with special relativity, we need to consider whether such consistency holds in Kent's theory. Consistency with special relativity is guaranteed in a theory if and only if it is invariant under a group of symmetries called Lorentz transformations. We therefore need to consider whether Kent's theory satisfies Lorentz invariance.

Thirdly, since a satisfactory theory must be one in which there are outcomes, we need to consider whether Kent succeeds in giving us a convincing account of what an outcome is. In doing this, we will examine how Kent's theory ties in with decoherence theory and d'Espagnat's objection about improper mixtures.

Butterfield has also emphasized the need to understand Kent's theory in the light of an important theorem proved in recent years, the so-called Collbeck-Renner Theorem[1]

[1]See Gijs Leegwater, "An impossibility theorem for parameter independent hidden-variable theories," *Studies in History and Philosophy of Modern Physics* 54 (2016): 18–34, Roger Colbeck and Renato Renner, "No extension of quantum theory can have improved predictive power," *Nature communications* (England) 2, no. 1 (2011): 411–411, Roger Colbeck and Renato Renner, "The completeness of

which says that if a theory satisfies PI together with what is called a ‘no conspiracy’ criterion, then this theory is reducible to standard quantum theory without any hidden variables. On the assumption that a violation of ‘no conspiracy’ would not be acceptable, this then suggests that any theory that satisfactorily addresses the EPR-Bohm paradox can’t be a hidden-variables theory. We will therefore need to consider whether Kent’s model can be an interpretation of quantum physics without it being a hidden-variables theory.

I will argue that in the light all these considerations, Kent’s theory does provide a satisfactory solution to the EPR-Bohm paradox. There are still questions concerning the nature of Kent’s beables, and Kent’s theory may also strike us as rather counter-intuitive given that his theory posits that present events should be conditioned on far-distant future states of affairs. We will therefore conclude this chapter by discussing how Kent’s theory could be made to appear less counter-intuitive.

4.1 Consistency of Kent’s Theory with Standard Quantum Physics*

If we are to take Kent’s theory seriously, it should be just as good at making predictions as standard quantum theory, and it had better not contradict empirical observations. Over the last century, standard quantum theory has been firmly established scientifically, and so far, it has not been contradicted by any experimental observations. Standard quantum theory allows us to form a quantum state description of a physical

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quantum theory for predicting measurement outcomes,” 2012, K Landsman, “On the Colbeck–Renner theorem,” *Journal of mathematical physics* (United States) 56, no. 12 (2015): 122103, and Klaas Landsman, *Foundations of Quantum Theory : From Classical Concepts to Operator Algebras (Volume 188.0)*, vol. 188, Fundamental Theories of Physics (Cham: Springer Open, 2017).

*As mentioned in the introduction on page 5, sections marked with an asterisk may be challenging to readers who do not have a background in physics.

system based on how the system was set up in an experimental environment, and then Schrödinger's equation can be used to evolve this state forwards in time, and finally, we can calculate expectation values for various physical quantities belonging to this physical system, and these agree with the average values measured on the system when the experiment is performed many times. In other words, standard quantum theory is *empirically adequate*[2] in its domain of applicability. Thus, if we can show that Kent's theory is just as good at making predictions as standard quantum theory, then it too will be empirically adequate to the same degree. This doesn't necessarily mean that Kent's theory will make exactly the same predictions as standard quantum physics, for the additional information that Kent's theory requires beyond standard quantum theory may alter these predictions. Indeed, if this additional information made absolutely no difference to the predictions of standard quantum theory, then it would seem rather redundant. But we should nevertheless be able to derive the predictions of standard quantum theory from Kent's theory by averaging over the unknown variables that describe the additional information in Kent's theory.

In order to show that Kent's theory is just as good as standard quantum theory, we recall the preliminary calculation of $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ given in section 3.4:

$$\langle T^{\mu\nu}(y)\rangle_{\tau_S} = \sum_{\tau} P(q(\tau)|r(\tau_S, y))\tau = \lim_{n\to\infty}\sum_{\tau}\frac{P(q(\tau)\,\&\, r_n)\tau}{P(r_n)}. \qquad \text{(3.14 revisited)}$$

We proceed to calculate $P(r_n)$ and $P(q(\tau)\,\&\, r_n)$ in order to calculate $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$. We first note that since S_n is a spacelike hypersurface, there will exist a unitary operator $U_{S_nS_0}$ defined by equation (3.8) which maps the Hilbert space[3] of states H_{S_0} describing

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[2]See p. 181 for a more formal definition of empirical adequacy.

[3]The Hilbert space H_S for any spacelike hypersurface S is defined on page 115.

S_0 to the Hilbert space of states H_{S_n} describing S_n in accord with how the states of H_{S_0} evolve over time. Now let $H_{S_n,\tau_S} \subset H_{S_n}$ be the subspace of states $|\xi\rangle$ for which $\hat{T}_S(x)\,|\xi\rangle = \tau_S(x)\,|\xi\rangle$ for all $x \in S_n \cap S$, and let $\{|\xi_1\rangle\,,|\xi_2\rangle\,,\ldots\}$ be an orthonormal basis of H_{S_n,τ_S}. Given that the initial state of the world is $|\Psi_0\rangle$, the probability $P(r_n)$ of "measuring" the value of $T_S(x)$ on $S_n \cap S$ to be $\tau_S(x)$ will be

$$P(r_n) = \sum_j |\langle\xi_j|\Psi_n\rangle|^2, \tag{4.1}$$

where $|\Psi_n\rangle = U_{S_nS_0}\,|\Psi_0\rangle$, and this probability will be independent of the particular orthonormal basis $\{|\xi_j\rangle : j\}$ of H_{S_n,τ_S}.[4] If we define

$$\pi_n = \sum_j |\xi_j\rangle\langle\xi_j|\,, \tag{4.2}$$

then it is easy to see that

$$P(r_n) = \langle\Psi_n|\pi_n|\Psi_n\rangle\,. \tag{4.3}$$

We also see that π_n is Hermitian (i.e. has real eigenvalues) and that $\pi_n\pi_n = \pi_n$. Any Hermitian operator π with $\pi^2 = \pi$ is called a *projection*. We thus see that π_n is a projection.

Turning to the calculation of $P(q(\tau)\,\&\,r_n)$, note that for the Tomonaga-Schwinger formulation of relativistic quantum physics, the operators $\hat{T}_S(x)$ and $\hat{T}^{\mu\nu}(y)$ for

[4] To see why this is, we note that we can extend the orthonormal set $\{|\xi_1\rangle\,,|\xi_2\rangle\,,\ldots\}$ to an orthonormal basis $\{|\xi_1\rangle\,,|\xi_2\rangle\,,\ldots\} \cup \{|\zeta_1\rangle\,,|\zeta_2\rangle\,,\ldots\}$ of H_{S_n} which consists entirely of $\hat{T}_S(x)$-eigenstates for all $x \in S_n \cap S$. We can think of each of the states of this orthonormal basis as the possible measurement outcomes when making the notional measurement of $T_S(x)$ on $S_n \cap S$. By the Born Rule, it therefore follows that $P(r_n) = \sum_j |\langle\xi_j|\Psi_n\rangle|^2$. But to see that this probability is independent of the particular basis, we can uniquely write $|\Psi_n\rangle$ as a sum $|\Psi_n\rangle = |\xi\rangle + |\zeta\rangle$ where $|\xi\rangle$ belongs to the span of $\{|\xi_j\rangle : j\}$ and $|\zeta\rangle$ belongs to the span of $\{|\zeta_j\rangle : j\}$. Then since $|\xi\rangle = \sum_j \langle\xi_j|\Psi_n\rangle\,|\xi_j\rangle$, it follows that

$$\langle\xi|\xi\rangle = \sum_j |\langle\xi_j|\Psi_n\rangle|^2 = P(r_n).$$

Therefore, since $\langle\xi|\xi\rangle$ is independent of the particular basis chosen of H_{S_n,τ_S}, so is $P(r_n)$.

fixed μ, ν commute when x and y are spacelike-separated. It therefore follows that we can express any state of H_{S_n} as a superposition of simultaneous eigenstates of $\hat{T}^{\mu\nu}(y)$ and $\hat{T}_S(x)$ for $x \in S_n \cap S$.[5] For a particular choice of μ, ν, we can then form an orthonormal basis $\{|\eta_j\rangle : j\}$ of H_{S_n} consisting of simultaneous $\hat{T}^{\mu\nu}(y)$, $\hat{T}_S(x)$-eigenstates so that $\hat{T}^{\mu\nu}(y)\,|\eta_j\rangle = \tau^{(j)}\,|\eta_j\rangle$ and $\hat{T}_S(x)\,|\eta_j\rangle = \tau_S^{(j)}(x)\,|\eta_j\rangle$ for $x \in S_n \cap S$, where $\tau^{(j)}$ and $\tau_S^{(j)}(x)$ are the corresponding eigenvalues. If we define $\pi_{n,\tau} = \sum_j |\chi_{j,\tau}\rangle\langle\chi_{j,\tau}|$ where $\{|\chi_{j,\tau}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ such that $\hat{T}^{\mu\nu}(y)\,|\chi_{j,\tau}\rangle = \tau\,|\chi_{j,\tau}\rangle$ and $\hat{T}_S(x)\,|\chi_{j,\tau}\rangle = \tau_S(x)\,|\chi_{j,\tau}\rangle$ for all $x \in S_n \cap S$, then

$$P(q(\tau)\,\&\,r_n) = \sum_j |\langle\chi_{j,\tau}|\Psi_n\rangle|^2 = \langle\Psi_n|\pi_{n,\tau}|\Psi_n\rangle\,.^6 \qquad (4.4)$$

But if we define $\pi_\tau = \sum_j |\eta_{j,\tau}\rangle\langle\eta_{j,\tau}|$ where $\{|\eta_{j,\tau}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ with $\hat{T}^{\mu\nu}(y)\,|\eta_{j,\tau}\rangle = \tau\,|\eta_{j,\tau}\rangle$, then we also have $\pi_{n,\tau} = \pi_n\pi_\tau$.[7] Hence,

$$P(q(\tau)\,\&\,r_n) = \langle\Psi_n|\pi_n\pi_\tau|\Psi_n\rangle\,. \qquad (4.5)$$

But clearly $\hat{T}^{\mu\nu}(y) = \sum_\tau \tau\pi_\tau$. Therefore, combining (3.14), (4.3), and (4.5), we have

$$\langle T^{\mu\nu}(y)\rangle_{\tau_S} = \lim_{n\to\infty} \frac{\sum_\tau \langle\Psi_n|\pi_n\pi_\tau|\Psi_n\rangle\,\tau}{\langle\Psi_n|\pi_n|\Psi_n\rangle} = \lim_{n\to\infty} \frac{\langle\Psi_n|\pi_n\hat{T}^{\mu\nu}(y)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle}. \qquad (4.6)$$

We are now in a position to show that Kent's theory is consistent with standard quantum theory. First let us consider what we need to show.

In the pilot wave interpretation, its consistency with standard quantum theory requires that if one averages the expectation values of an observable over the hidden variables (i.e. the positions and the momenta of all the particles) then one obtains the expectation

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[5] We make the same approximation as depicted in figure 3.3 on page 118.

[6] The proof of this is very similar to the proof given in footnote 4.

[7] To see why this is, we first show that $\pi_n = \sum_j |h_{n,j}\rangle\langle h_{n,j}|$ where $\{|h_{n,j}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ for which $|h_{n,j}\rangle \in H_{S_n,\tau_S}$. Note that $\pi_n\,|h_{n,j}\rangle = |h_{n,j}\rangle$ since $\{|\xi_j\rangle : j\}$ is a basis for H_{S_n,τ_S} and $|h_{n,j}\rangle \in H_{S_n,\tau_S}$. Therefore, $\pi_n\pi_{n,h} = \pi_{n,h}$ where $\pi_{n,h} = \sum_j |h_{n,j}\rangle\langle h_{n,j}|$. But $\pi_{n,h}\,|\xi_j\rangle = |\xi_j\rangle$ since

value of the observable given by standard quantum theory as indicated in equation (1.23).

Now in Kent's theory, the hidden variables on which his beables $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ depend are the values $\tau_S(x)$ of $T_S(x)$ for $x \in S^1(y) \cap S$. The operator π_n in equation (4.6) in the limit as $n \to \infty$ encapsulates this hidden information. To remind ourselves of π_n's dependency on τ_S restricted to $S_n \cap S$, we will now write $\pi_n(\tau_{S_n \cap S})$ for π_n where $\tau_{S_n \cap S}$ is the function τ_S restricted to $S_n \cap S$. Likewise, we will write $r_n(\tau_{S_n \cap S})$ for r_n, the statement that $T_S(x) = \tau_S(x)$ for all $x \in S_n(y) \cap S$. If we let j index all possible functions $\tau^{(j)}_{S_n \cap S}$ taking real values on $S_n \cap S$,[8] then the analogue of (1.23) requires us to show that

$$\langle \Psi_0 | \hat{\boldsymbol{T}}^{\mu\nu}(y) | \Psi_0 \rangle = \lim_{n \to \infty} \sum_j P\big(r_n(\tau^{(j)}_{S_n \cap S})\big) \, \langle T^{\mu\nu}(y) \rangle_{\tau^{(j)}_{S_n \cap S}} \tag{4.7}$$

for all y lying between S_0 and S, where the left-hand side of (4.7) is just the expectation value of $\hat{\boldsymbol{T}}^{\mu\nu}(y)$ in the Heisenberg picture as predicted by standard quantum theory.[9] Equation (4.7) is sufficient to establish consistency with standard quantum theory because ultimately, all observables are going to be reducible to expressions dependent on $\hat{T}^{\mu\nu}(y)$, since once we know what to expect for $\hat{T}^{\mu\nu}(y)$, we will know what to expect for the energy and momentum densities for all measuring apparatus readouts etc. and hence what to expect for all measurement outcomes. But from (4.3) and (4.6), we

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$\{|h_{n,j}\rangle : j\}$ is a basis for H_{S_n,τ_S} and $|\xi_j\rangle \in H_{S_n,\tau_S}$. Therefore, $\pi_{n,h}\pi_n = \pi_n$. But $\pi_{n,h}\pi_n = \pi_n\pi_{n,h}$ since π_n and $\pi_{n,h}$ are Hermitian. Hence, $\pi_n = \pi_{n,h}$. Now the summands of $\pi_n\pi_\tau$ are only going to consist of those $|\eta_j\rangle\langle\eta_j|$ for which $\hat{T}^{\mu\nu}(y)\,|\eta_j\rangle = \tau\,|\eta_j\rangle$ and for which $\hat{T}_S(x)\,|\eta_j\rangle = \tau_S(x)\,|\eta_j\rangle$ for all $x \in S_n \cap S$, and these are just the $|\chi_{j,\tau}\rangle\langle\chi_{j,\tau}|$ which are the summands of $\pi_{n,\tau}$. Hence, $\pi_n\pi_\tau = \pi_{n,\tau}$.

[8]Recall that we are implicitly using an approximation scheme described by equation (3.5), so we are really considering functions on the cells of a mesh over $S_n \cap S$ taking values from a finite pool of possible values. Also see footnote 42. This is why we can use an index j to index all the $\tau^{(j)}_{S_n \cap S}$.

[9]Where we are using Schwinger's bold typeface convention as described in footnote 15.

have

$$\lim_{n\to\infty}\sum_j P\big(r_n(\tau^{(j)}_{S_n\cap S})\big)\,\langle T^{\mu\nu}(y)\rangle_{\tau^{(j)}_{S_n\cap S}} = \lim_{n\to\infty}\sum_j\ \langle\Psi_n|\pi_n(\tau^{(j)}_{S_n\cap S})\hat{T}^{\mu\nu}(y)|\Psi_n\rangle \tag{4.8}$$

Since there is an orthonormal basis $\{|\eta_j\rangle : j\}$ of H_{S_n} consisting of simultaneous $\hat{T}_S(x)$-eigenstates so that $\hat{T}_S(x)\,|\eta_j\rangle = \tau^{(j)}_{S_n\cap S}(x)\,|\eta_j\rangle$ for all $x\in S_n\cap S$, it follows that $\sum_j \pi_n(\tau^{(j)}_{S_n\cap S}) = I$. Combining this with (4.8) we get

$$\lim_{n\to\infty}\sum_j P\big(r_n(\tau^{(j)}_{S_n\cap S})\big)\,\langle T^{\mu\nu}(y)\rangle_{\tau^{(j)}_{S_n\cap S}} = \lim_{n\to\infty}\ \langle\Psi_n|\hat{T}^{\mu\nu}(y)|\Psi_n\rangle \tag{4.9}$$

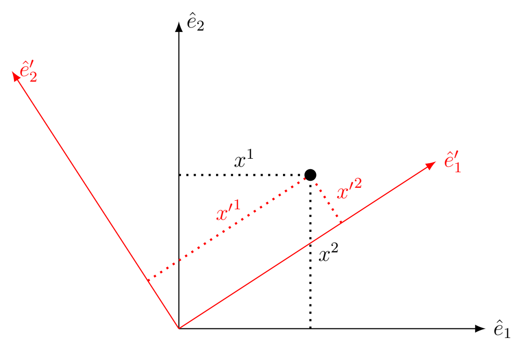
In the notation of equation (3.4), we have $|\Psi_n\rangle = |\Psi[S_n]\rangle$, and according to (3.4) the expectation value $\langle\Psi[S_n]|\hat{T}^{\mu\nu}(y)|\Psi[S_n]\rangle$ will be independent of the hypersurface S_n so long as it contains y, and the result will be equal to the expectation value $\langle\Psi_0|\hat{\boldsymbol{T}}^{\mu\nu}(y)|\Psi_0\rangle$ in the Heisenberg picture which is the value that is predicted by standard quantum mechanics. Therefore, equation (4.7) follows from (3.4) and (4.9) which is what we were aiming to show for standard quantum consistency to hold.

4.2 Kent's Theory and Lorentz Invariance*

In order to explain what it means for Kent's theory to be Lorentz invariant, we first need to explain how spacetime coordinates look to different observers.

Up until now we have been representing a spacetime location by a four-tuple (x^0, x^1, x^2, x^3) where $(x^i)_{i=1}^3$ are spatial coordinates, and where $x^0 = ct$ with c being equal to the speed of light and t being the time. It will be convenient to represent spacetime locations using a more concise notation. So we let $(1,0,0,0)$ correspond to the spacetime location $\hat{e}_0$, $(0,1,0,0)$ correspond to the spacetime location $\hat{e}_1$, etc.. Then we can express any other spacetime location as a sum $\sum_{\mu=0}^3 x^\mu\hat{e}_\mu$. We will use the so-called *Einstein summation convention* of dropping the summation sign and implicitly assuming that there is a summation whenever an upper index and a lower index are the same so that we can write $x^\mu \hat{e}_\mu$ instead of $\sum_{\mu=0}^{3} x^\mu \hat{e}_\mu$. We also use the convention of letting Greek letters range over 0, 1, 2, and 3 (e.g. the μ in $x^\mu \hat{e}_\mu$), and of letting Roman letters range over 1, 2, and 3 (e.g. the i in $(x^i)_{i=1}^3$).

Now suppose an observer O expresses spacetime locations in terms of $\{\hat{e}_\mu : \mu = 0, \ldots, 3\}$ and hence uses the coordinates (x^0, x^1, x^2, x^3) to describe various spacetime locations. For another observer O', it may be more natural to express spacetime locations in terms of a different set $\{\hat{e}'_\mu : \mu = 0, \ldots, 3\}$ so that the location described by O as (x^0, x^1, x^2, x^3) would be described by O' as (x'^0, x'^1, x'^2, x'^3) where $x'^\mu \hat{e}'_\mu = x^\mu \hat{e}_\mu$. For instance if O and O' are moving with respect to each other, they may both want to use coordinates in which their own spatial coordinates are fixed and in which the spatial coordinates of the other observer are changing. As another example, figure 4.1 shows how the (x^1, x^2)-coordinates transform under a spatial rotation.



(a)

Figure 4.1: Shows how a location (marked as $\bullet$) can be expressed either in coordinates (x^1, x^2) with respect to the basis $\{\hat{e}_1, \hat{e}_2\}$ or in coordinates (x'^1, x'^2) with respect to the basis $\{\hat{e}'_1, \hat{e}'_2\}$.

Now the key fact about all observers is that they must always observe light in a vacuum to have a constant speed c. Thus, for a photon that goes through the spacetime locations $(0,0,0,0)$ and (x^0,x^1,x^2,x^3) in the coordinates of O, we must have $(x^0,x^1,x^2,x^3)=(ct,tv^1,tv^2,tv^3)$ where

$$\sqrt{(v^1)^2+(v^2)^2+(v^3)^2}=c.$$

But if $(0,0,0,0)$ and (x^0,x^1,x^2,x^3) corresponds to $(0,0,0,0)$ and (x'^0,x'^1,x'^2,x'^3) respectively in the coordinates of another observer O', then we must also have $(x'^0,x'^1,x'^2,x'^3)=(ct',t'v'^1,t'v'^2,t'v'^3)$ where

$$\sqrt{(v'^1)^2+(v'^2)^2+(v'^3)^2}=c.$$

In either case, we must have

$$(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=(x'^0)^2-(x'^1)^2-(x'^2)^2-(x'^3)^2=0. \tag{4.10}$$

If we define $\eta_{00}=1$, $\eta_{ii}=-1$ for $i=1,2,3$ and $\eta_{\mu\nu}=0$ for $\mu\neq\nu$, then using the Einstein summation convention as well as the convention of lowering indices so that we define $x_\mu \stackrel{\text{def}}{=} \eta_{\mu\nu}x^\nu$, then (4.10) is equivalent to

$$x_\mu x^\mu = {x'}_\mu x'^\mu = 0.$$

Thus, for any coordinate transformation $x\to x'$ such that $x_\mu x^\mu={x'}_\mu x'^\mu$, if the speed of light is c in the x-coordinates, then the speed of light is also guaranteed to be c in the x'-coordinates. A *Lorentz transformation* Λ is any coordinate transformation of the form $x'^\mu=\Lambda^\mu{}_\nu x^\nu$ such that $x_\mu x^\mu={x'}_\mu x'^\mu$. Since a Lorentz transformation must satisfy

$$x_\mu x^\mu=\eta_{\mu\rho}\Lambda^\rho{}_\sigma x^\sigma\Lambda^\mu{}_\nu x^\nu$$

for all x, it follows that

$$\Lambda^\rho{}_\mu\eta_{\rho\sigma}\Lambda^\sigma{}_\nu=\eta_{\mu\nu}.^{10} \tag{4.11}$$

Having considered how the coordinates of a spacetime location viewed by one observer relate to the coordinates of the same spacetime location viewed by a different observer, we can now consider how physical quantities viewed by different observers relate to each other. The simplest kind of physical quantity is called a *scalar*. A scalar defined at a particular spacetime location has the same value no matter what frame of reference an observer uses. One example of a scalar is an object's *rest mass* which is the mass an object would have if it had no velocity. There is still a transformation rule for scalars since the spacetime location at which the scalar is measured is usually expressed in terms of an observer's coordinate system, and the coordinates of such a location will differ for different observers. Thus, if $\phi(x) \stackrel{\text{def}}{=} \phi(x^0, x^1, x^2, x^3)$ is the value of a scalar defined at the spacetime location (x^0, x^1, x^2, x^3) as described by an observer O, then another observer O' using a different set of coordinate (x'^0, x'^1, x'^2, x'^3) to describe the spacetime location (x^0, x^1, x^2, x^3) will describe this same scalar as $\phi'(x') \stackrel{\text{def}}{=} \phi'(x'^0, x'^1, x'^2, x'^3)$ where $\phi'(x') = \phi(x)$. Therefore

$$\phi'(x') = \phi(\Lambda^{-1}x') \tag{4.12}$$

where Λ^{-1} is the inverse Lorentz transformation that takes the coordinates $x' = (x'^0, x'^1, x'^2, x'^3)$ of a spacetime location to the coordinates $x = (x^0, x^1, x^2, x^3)$ describing that spacetime location. Thus, equation (4.12) shows us how a scalar transforms under a Lorentz transformation Λ.

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[10] To see why this is, note that if $x_\mu x^\mu = x'_\mu x'^\mu$ for all x, then for any other spacetime location y, we have $(x+y)_\mu (x+y)^\mu = (x'+y')_\mu (x'+y')^\mu$. If we expand this out and cancel $x_\mu x^\mu$ with $x'_\mu x'^\mu$ and cancel $y_\mu y^\mu$ with $y'_\mu y'^\mu$, and using the fact that $y_\mu x^\mu = x_\mu y^\mu$, etc. we find that $x_\mu y^\mu = x'_\mu y'^\mu$ for all x and y. Hence,

$$\eta_{\nu\mu} x^\mu y^\nu = x_\mu y^\mu = \eta_{\sigma\rho} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu x^\mu y^\nu.$$

Since we can choose x such that $x^\mu = 1$ and $x^\alpha = 0$ for $\alpha \neq \mu$, and can choose y such that $y^\nu = 1$ and $y^\beta = 0$ for $\beta \neq \nu$. Then we get

$$\eta_{\mu\nu} = \eta_{\sigma\rho} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu,$$

Many physical quantities, however, are not scalars and so will look different to different observers. For instance, the energy of an object has a kinetic component that depends on the velocity the object has relative to an observer. However, it turns out that if an observer O considers an object's energy E together with its three components of momentum p^1, p^2, and p^3 (in the directions $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ respectively) to form the four-tuple $p \stackrel{\text{def}}{=} (E/c, p^1, p^2, p^3)$ known as the object's *four-momentum*, then p transforms in the same way as spacetime coordinates transform between different observers. In other words, a different observer O' whose coordinates are given by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ would observe the object's four-momentum to be $p'^\mu = \Lambda^\mu{}_\nu p^\nu$.[11] More generally, any list of four physical quantities $(\varphi^0, \varphi^1, \varphi^2, \varphi^3)$ that transforms as $\varphi \to \varphi'$ with $\varphi'^\mu = \Lambda^\mu{}_\nu \varphi^\nu$ is called a *four-vector*. Figure 4.2 shows how (two of) the components of a four-vector φ at a particular location will differ for different observers under a spatial rotation of the coordinates. A four-vector $\varphi^\mu(x)$ defined at every spacetime location x is called a *four-vector field*. Thus, at each spacetime location x, the four-vector field $\varphi^\mu(x)$

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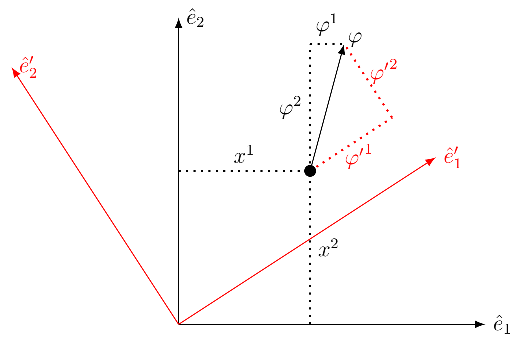
and hence the result follows.

[11]In order for p to transform in this way, we have to redefine what we mean by energy and momentum. In classical mechanics, the momentum of an object is the product of the object's mass and its velocity. In the context of special relativity, however, the four-momentum of an object is defined to be the product of its rest mass m_0 and its *four-velocity* where the four velocity of an object is a four-tuple (u^0, u^1, u^2, u^3) with $u_\mu u^\mu = c^2$ such that the object's velocity (in the classical sense) is the vector $(c\frac{u^1}{u^0}, c\frac{u^2}{u^0}, c\frac{u^3}{u^0})$. The motivation for this definition can be seen by considering an object whose classical velocity is $\mathbf{v} = (v^1, v^2, v^3)$ that goes through $(0,0,0,0)$. It will have a spacetime trajectory $x(t) = (ct, v^1 t, v^2 t, v^3 t)$. u is just the four-vector proportional to $x(1)$ with $u_\mu u^\mu = c^2$, so if γ is the constant of proportionality such that $u^0 = \gamma c$ and $u^i = \gamma v^i$, then by eliminating γ we get $v^i = c\frac{u^i}{u^0}$. We can then easily work out the constant of proportionality γ and hence the four-velocity u of an object whose classical velocity is $\mathbf{v}$. For we must have $u^i = \frac{v^i u^0}{c}$, for $i = 1$ to 3. Therefore, since $u_\mu u^\mu = c^2$, we must have $(u^0)^2(1 - \frac{v^2}{c^2}) = c^2$ where $v = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}$. Thus, if we define $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, then $u^0 = \gamma c$ and $u^i = \gamma v^i$ for $i = 1$ to 3, and hence the four-velocity of the object must be $u = \gamma(c, v^1, v^2, v^3)$. From this, we see that the object's four-momentum will be $\gamma m_0(c, v^1, v^2, v^3)$. If the object's velocity is very small compared to the speed of light, then $\gamma \approx 1 + \frac{v^2}{2c^2}$, and hence the object's four-momentum $(E/c, p^1, p^2, p^3)$ will be approximately $(m_0 c + \frac{1}{2} m_0 v^2/c, m_0 v^1, m_0 v^2, m_0 v^3)$. Therefore, (p^1, p^2, p^3) is approximately equal to the classical momentum. However, the energy is now $E = m_0 c^2 + \frac{1}{2} m_0 v^2$. Thus, in addition to the kinetic energy term $\frac{1}{2} m_0 v^2$, there is a rest mass energy $m_0 c^2$. If we define the *relativistic mass* $m = \gamma m_0$, then we obtain Einstein's famous formula $E = mc^2$.



(b)

Figure 4.2: Shows how a four-vector φ (of which only two components are shown) defined at a spacetime location (indicated by $\bullet$) can be expressed either as (φ^1, φ^2) with respect to the basis $\{\hat{e}_1, \hat{e}_2\}$ or as (φ'^1, φ'^2) with respect to the basis $\{\hat{e}'_1, \hat{e}'_2\}$.

assigns four numbers, $\varphi^0(x)$, $\varphi^1(x)$, $\varphi^2(x)$, and $\varphi^3(x)$. If O observes this vector-field $\varphi^\mu(x)$, and O' is another observer whose coordinates are related to the coordinates O via the Lorentz transformation Λ, then O' will describe the same physical reality that O describes by assigning four numbers $\varphi'^0(x')$, $\varphi'^1(x')$, $\varphi'^2(x')$, and $\varphi'^3(x')$ at every spacetime location x', and the relationship between the description O gives and the description O' gives will be given by the formula

$$\varphi'^\mu(x') = \Lambda^\mu{}_\nu \varphi^\nu(x).$$

Hence, under the Lorentz transformation Λ, a vector field $\varphi^\mu(x)$ transforms as $\varphi^\mu(x) \to \varphi'^\mu(x')$ where

$$\varphi'^\mu(x') = \Lambda^\mu{}_\nu \varphi^\nu(\Lambda^{-1}x'). \tag{4.13}$$

From a four-vector φ^μ, we can also define the so-called *four-covector*:

$$\varphi_\mu \stackrel{\text{def}}{=} \eta_{\mu\nu}\varphi^\nu. \tag{4.14}$$

To see how four-covectors transform under a Lorentz transformation Λ, it will be helpful to define

$$\Lambda_\mu{}^\nu \stackrel{\text{def}}{=} \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma \tag{4.15}$$

where $\eta^{\nu\sigma} = \eta_{\nu\sigma}$. If we also define the *Kronecker-delta* δ_μ^ν such that $\delta_\mu^\nu = 1$ when $\mu = \nu$ and $\delta_\mu^\nu = 0$ otherwise, then using the fact that $\eta_{\mu\rho}\eta^{\nu\rho} = \delta_\mu^\nu$ together with equation (4.11), we have

$$\Lambda^\rho{}_\mu\Lambda_\rho{}^\nu = \delta_\mu^\nu. \tag{4.16}$$

Since by definition, the inverse of Λ^{-1} satisfies $(\Lambda^{-1})^\nu{}_\rho\Lambda^\rho{}_\mu = \delta_\mu^\nu$, we have $(\Lambda^{-1})^\nu{}_\rho = \Lambda_\rho{}^\nu$. From (4.13), (4.14), and (4.15), we therefore see that under a Lorentz transformation Λ, a four-covector field $\varphi_\mu(x)$ transforms as $\varphi_\mu(x) \to \varphi'_\mu(x')$ where

$$\varphi'_\mu(x') = \Lambda_\mu{}^\nu\varphi_\nu(\Lambda^{-1}x') \tag{4.17}$$

Besides scalars, four-vectors, and four-covectors, we also need to consider physical quantities called rank-two tensors. The stress-energy tensor $T^{\mu\nu}$ introduced on page 122 is an example of a rank-two tensor. The defining property of a rank-two tensor field $\varphi^{\mu\nu}(x)$ is that under a Lorentz transformation Λ, it transforms as $\varphi^{\mu\nu}(x) \to \varphi'^{\mu\nu}(x')$ where

$$\varphi'^{\mu\nu}(x') = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\varphi^{\rho\sigma}(\Lambda^{-1}x'). \tag{4.18}$$

On page 110, we introduced the mass-energy density $T_S(x)$ on a spacelike hypersurface S. As explained in section 3.2, the values of $T_S(x)$ for all $x \in S$ are the additional values that Kent uses to supplement standard quantum theory. It was mentioned in passing that $T_S(x)$ does not depend on which frame of reference one is in. In other words, $T_S(x)$ is a scalar. I will now explain why this is so.

We first need to consider the precise definition of $T_S(x)$. At each spacetime location on the spacelike hypersurface S which an observer O describes as having coordinates $x = (x^\mu)_{\mu=0}^3$, we define $\eta^\mu(x)$ to be the future-directed unit four-vector at x that is orthogonal to S. In other words, $\eta^0(x) > 0$, $\eta_\mu(x)\eta^\mu(x) = 1$, and if $y \in S$ is very close to x, then

$$\frac{(x-y)_\mu \eta^\mu(x)}{\sqrt{(x-y)_\nu (x-y)^\nu}} \approx 0. \tag{4.19}$$

$T_S(x)$ is then given by the formula

$$T_S(x) = T^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x). \tag{4.20}$$

For example, if S was the spacelike hypersurface consisting of all spacetime locations $x = (0, x^1, x^2, x^3)$, then $\big(\eta^0(x), \eta^1(x), \eta^3(x), \eta^3(x)\big) = (1, 0, 0, 0)$, and hence $T_S(x) = T^{00}(x)$ which is the density of relativistic mass at x, i.e. the energy density at x divided by c^2. Note that the condition that $y \in S$ is very close to x in (4.19) means that $T_S(x)$ only has a local dependence on S in the vicinity of x. i.e. if S' only differs from S outside the vicinity of x, then $T_{S'}(x) = T_S(x)$.

To see why $T_S(x)$ is a scalar, suppose that Λ is a Lorentz transformation such that ${\Lambda^0}_\mu \eta^\mu > 0$ for any future-directed unit four-vector vector η^μ. We refer to a Λ with this property as an *orthochronous* Lorentz transformation. Also, suppose that O and O' are two observers such that spacetime locations that observer O describes as having coordinates $x = (x^\mu)_{\mu=0}^3$ are described by O' as having coordinates $x' = ({\Lambda^\mu}_\nu x^\nu)_{\mu=0}^3$. Then since ${x'}_\mu {y'}^\mu = x_\mu y^\mu$, it follows that the future-directed unit four-vector orthogonal to S at x which O describes as $\eta^\mu(x)$ will be described by O' as $\eta'^\mu(x') = \Lambda^\mu{}_\nu \eta^\nu(x)$. Thus, for any spacetime location in S that O' describes as having coordinates x' with corresponding future-directed S-orthogonal unit four-vector $\eta'^\mu(x')$, O' can construct a function $T'_S(x')$ with

$$T'_S(x') = T'^{\mu\nu}(x')\eta'_\mu(x')\eta'_\nu(x'). \tag{4.21}$$

Then using (4.17) and (4.18) on the right-hand side of (4.21), we have

$$\begin{aligned} T'_S(x') &= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}(x)\Lambda_\mu{}^\alpha \eta_\alpha(x)\Lambda_\nu{}^\beta \eta_\beta(x) \\ &= \Lambda^\mu{}_\rho \Lambda_\mu{}^\alpha \Lambda^\nu{}_\sigma \Lambda_\nu{}^\beta T^{\rho\sigma}(x)\eta_\alpha(x)\eta_\beta(x) \\ &= \delta^\alpha_\rho \delta^\beta_\sigma T^{\rho\sigma}(x)\eta_\alpha(x)\eta_\beta(x) \\ &= T^{\alpha\beta}(x)\eta_\alpha(x)\eta_\beta(x) \\ &= T_S(x) \end{aligned} \tag{4.22}$$

where on the third line we have used (4.16), and on the last line we have used (4.20). To obtain (4.22), we assumed that Λ is orthochronous because definition (4.20) assumes that $\eta^\mu(x)$ is future-directed. But if Λ is non-orthochronous, we would need to take the negations of $\eta'^\mu(x')$ to get the future-directed S-orthogonal unit four-vector. But clearly this will not affect the equality in (4.22), so (4.22) holds for all Lorentz transformations, whether they are orthochronous or non-orthochronous. We thus see that $T_S(x)$ is a scalar.

Let us now consider the Hilbert space H_{S_n} for a spacelike hypersurface S_n as defined on page 143.[12] Given that $\hat{T}^{\mu\nu}(x)$ is the observable in the Tomonaga-Schwinger picture whose eigenstates with eigenvalues τ are the states of S_n for which an observer O

[12] Also see page 115 for the definition of H_S.

observes the stress-energy tensor $T^{\mu\nu}(x)$ to take the value τ at x, it follows from (4.20) that

$$\hat{T}_S(x) \stackrel{\text{def}}{=} \hat{T}^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x) \tag{4.23}$$

will be the observable whose eigenstates with eigenvalues $\tau_S(x)$ are the states of S_n for which an observer O observes $T_S(x)$ to take the value $\tau_S(x)$ at x when $x \in S_n \cap S$.[13]

Now two observers O and O' will typically assign different physical states to S_n based on their frame of reference. E.g. if O and O' are traveling at different speeds, they will attribute different energy levels and momenta to the spacetime locations of S_n. To understand the relationship between the states O assigns to S_n and the states O' assigns, suppose $|\psi\rangle$ and $|\chi\rangle$ are two states that an observer O might judge S_n to be in. As usual, we suppose the coordinates x^μ of observer O transform to coordinates $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ of observer O' for some Lorentz transformation Λ. We also suppose that the states $|\psi\rangle$ and $|\chi\rangle$ that O observes will transform to states $|\psi'\rangle$ and $|\chi'\rangle$ that O' observes. We will denote the Hilbert space of the states on S_n that O' can observe as H'_{S_n}.[14]

Now if O judged S_n to be in the superposition state $|\psi\rangle + |\chi\rangle$, then O' would judge S_n to be proportional to the superposition state $|\psi'\rangle + |\chi'\rangle$. Also recall that $|\psi'\rangle$ and $\lambda\,|\psi'\rangle$ represent the same physical state for any complex number λ, so there is sufficient flexibility as to which state we deem $|\psi\rangle$ transforms to that we can deem the transformation $|\psi\rangle \to |\psi'\rangle$ to be a linear transformation. But also note that if

[13]So long as S_n and S are tangential at x as noted in footnote 41.

[14]As we will discuss shortly, there is a map inner product preserving map $U(\Lambda)$ via which there is a one-to-one correspondence between states in H_{S_n} and states in H'_{S_n} so we can identify H'_{S_n} with H_{S_n} in which case $U(\Lambda)$ will be a unitary operator.

observer O uses the Born rule to calculate the transition probability from state $|\psi\rangle$ to state $|\chi\rangle$, and observer O' uses the Born rule to calculate the transition probability from state $|\psi'\rangle$ to state $|\chi'\rangle$, then they should calculate the same probabilities. We must therefore have

$$|\langle\chi|\psi\rangle|^2 = |\langle\chi'|\psi'\rangle|^2.$$

Using this fact together with the fact that $|\psi\rangle$ and $\lambda|\psi\rangle$ represent the same physical states, it can be shown that there is a unitary operator $U(\Lambda)$ which relates the states $|\psi\rangle$ and $|\psi'\rangle$ via the formulated

$$|\psi'\rangle = U(\Lambda)|\psi\rangle.$$[15]

At this point, it is worth clarifying the different meanings of $T^{\mu\nu}(x)$, $T'^{\mu\nu}(x')$, $\tau^{\mu\nu}(x)$, $\tau'^{\mu\nu}(x')$, $\hat{T}^{\mu\nu}(x)$ and $\hat{T}'^{\mu\nu}(x')$.

- We use $T^{\mu\nu}(x)$ to refer to the description of the physical quantity that is being observed by O. Thus, $T^{\mu\nu}(x)$ is shorthand for the description "the $\mu\nu$-component of the stress-energy tensor that O observes at the spacetime location belonging to S that O describes as x."

- Similarly, $T'^{\mu\nu}(x')$ is shorthand for the description "the $\mu\nu$-component of the stress-energy tensor that O' observes at a spacetime location belonging to S that O' describes as x'."

[15]For more details, see E. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group" [in eng], *Annals of mathematics* 40, no. 1 (1939): 149–204. Since a unitary operator maps a Hilbert space to itself, we first need to identify H_{S_n} and H'_{S_n} in order for $U(\Lambda)$ to be unitary.

- $\tau^{\mu\nu}(x)$ stands for a particular (real) value of the physical quantity described by $T^{\mu\nu}(x)$ that O observes, and

- $\tau'^{\mu\nu}(x')$ stands for a particular (real) value of the physical quantity described by $T'^{\mu\nu}(x')$ that O' observes.

- $\hat{T}^{\mu\nu}(x)$ for $x \in S_n$ is the Tomonaga-Schwinger observable acting on H_{S_n} such that if observer O deemed S_n to be in an eigenstate $|\psi\rangle$ of $\hat{T}^{\mu\nu}(x)$ with eigenvalue τ (a real number), then observer O would observe the physical quantity described by $T^{\mu\nu}(x)$ to have the value τ.[16]

- $\hat{T}'^{\mu\nu}(x')$ is the Tomonaga-Schwinger observable acting on H'_{S_n} such that if observer O' deemed S_n to be in an eigenstate $|\psi'\rangle$ of $\hat{T}'^{\mu\nu}(x')$ with eigenvalue τ', then observer O' would observe the physical quantity described by $T'^{\mu\nu}(x')$ to have the value τ'.

- $T_S(x) = T^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x)$ is shorthand for the description "the mass-energy density of the spacelike hypersurface S observed by observer O at a spacetime location that O describes as x".

- $T'_S(x') = T'^{\mu\nu}(x')\eta'_\mu(x')\eta'_\nu(x')$ is shorthand for the description "the mass-energy density of the spacelike hypersurface S observed by observer O' at a spacetime location that O' describes as x'".

[16]Note that we consider a real number τ here rather than a real valued function $\tau^{\mu\nu}(x)$ of spacetime locations $x \in S$ and indices μ and ν since $\hat{T}^{\mu\nu}(x)$ will not in general commute for different values of μ, ν, so we won't be able to find a state which is a simultaneous eigenstate for all the different observables $\hat{T}^{\mu\nu}(x)$, though we may find a state which is very close to being an eigenstate of all the $\hat{T}^{\mu\nu}(x)$ for different μ and ν.

- The function $\tau_S(x)$ stands for a particular range of values for each $x \in S$ of the physical quantity described by $T_S(x)$ observed by O.

- The function $\tau'_S(x')$ stands for a particular range of values for each $x' \in S$ of the physical quantity described by $T'_S(x')$ observed by O'.

- For each $x \in S$, $\hat{T}_S(x) = \hat{T}^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x)$ is the Tomonaga-Schwinger observable such that if an observer O deems S to be in an eigenstate $|\psi\rangle \in H_{S_n}$ of $\hat{T}_S(x)$ with eigenvalue τ (a real number), then O would observe the physical quantity described by $T_S(x)$ to have the value τ.

- For each $x' \in S$, $\hat{T}'_S(x') = \hat{T}'^{\mu\nu}(x)\eta'_\mu(x')\eta'_\nu(x')$ is the Tomonaga-Schwinger observable such that if an observer O' deems S to be in an eigenstate $|\psi'\rangle \in H'_{S_n}$ of $\hat{T}'_S(x')$ with eigenvalue τ' (a real number), then O' would observe the physical quantity described by $T'_S(x')$ to have the value τ'.

Having clarified this terminology, we see that if $|\psi\rangle \in H_{S_n}$ is a state for which $T^{\mu\nu}(x) \approx \tau^{\mu\nu}(x)$,[17] and if $|\psi'\rangle \in H'_{S_n}$ is a state for which $T'^{\mu\nu}(x') \approx \tau^{\mu\nu}(x')$, then

$$\hat{T}^{\mu\nu}(x)\,|\psi\rangle \approx \tau^{\mu\nu}(x)\,|\psi\rangle\,, \text{ and} \tag{4.24a}$$

$$\hat{T}'^{\mu\nu}(x')\,|\psi'\rangle \approx \tau'^{\mu\nu}(x')\,|\psi'\rangle\,. \tag{4.24b}$$

It then follows from (4.24b) that if $|\psi'\rangle = U(\Lambda)\,|\psi\rangle$, then

$$U(\Lambda)^{-1}\hat{T}'^{\mu\nu}(x')U(\Lambda)\,|\psi\rangle \approx T'^{\mu\nu}(x')\,|\psi\rangle\,. \tag{4.25}$$

[17]We say approximately ($\approx$) here since the operators $\hat{T}^{\mu\nu}$ will not in general commute, so we won't typically be able to find a state $|\psi\rangle$ which is an eigenstate for all the observables $\hat{T}^{\mu\nu}$. It is the non-commutativity of observables that is responsible for Heisenberg's uncertainty principle.

Therefore, in order for (4.18) to hold for $T'^{\mu\nu}(x')$ in the classical limit (where we treat the stress-energy observables as though they commute with each other and replace the approximations by equalities), by plugging an operator form of (4.18) into (4.25), we see that we must have

$$U(\Lambda)^{-1}\hat{T}'^{\mu\nu}(x')U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\hat{T}^{\rho\sigma}(\Lambda^{-1}x'). \tag{4.26}$$

Now to say that Kent's model is Lorentz invariant, is to say that (4.6) defines a rank-two tensor, for then this quantity and the quantities on which it depends will transform in the way that physical quantities should transform under a Lorentz transformation Λ when the spacetime coordinates of two observers O and O' are related by the formula $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$.[18] Thus, in order to show that Kent's model is Lorentz invariant, we need to show that if $\{|\xi_j\rangle : j\}$ is an orthonormal basis of the Hilbert space of states[19] H_{S_n,τ_S} for which O observes $T_S(x)$ to be $\tau_S(x)$ for all $x \in S_n(y) \cap S$, and if $\{|\xi'_j\rangle : j\}$ is an orthonormal basis of the Hilbert space of states[20] H'_{S_n,τ'_S} for which O' observes $T'_S(x')$ to be $\tau'_S(x')$ for all $x' \in S_n(y') \cap S$, then

$$\lim_{n\to\infty}\frac{\langle\Psi'_n|\pi'_n\hat{T}'^{\mu\nu}(y')|\Psi'_n\rangle}{\langle\Psi'_n|\pi'_n|\Psi'_n\rangle} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\lim_{n\to\infty}\frac{\langle\Psi_n|\pi_n\hat{T}^{\rho\sigma}(y)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle} \tag{4.27}$$

[18]Note that having a privileged spacelike hypersurface S in which a notional measurement of T_S is made does not of itself break Lorentz invariance. Just because we are privileging a spacelike hypersurface S, we are not making any assumptions about simultaneity being defined by S. Because spacelike separation is a Lorentz invariant proper, both O and O' will deem the spacetime locations on S to be spacelike separated. The Lorentz transformation itself has absolutely no effect on what S is. It is just that O and O' will use different coordinates to describe a particular spacetime location on S. It maybe that in the coordinate system of O, some of the spacetime locations of S are simultaneous (i.e. have the same x^0 value), but there is no requirement of simultaneity, and there is no claim that a reference frame in which the spacetime locations of S are simultaneous is particularly special. In Kent's theory, it is sufficient for there to be just one hypersurface on which T_S has a determinate value. But if another hypersurface were to be chosen instead, it would make no difference to empirical adequacy since (4.7) will hold regardless of what spacelike hypersurface S is chosen.

[19]Thus, H_{S_n,τ_S} is the subspace of states $|\xi\rangle \in H_{S_n}$ for which $\hat{T}_S(x)\,|\xi\rangle = \tau_S(x)\,|\xi\rangle$ for all $x \in S_n \cap S$ as mentioned on page 143.

[20]Thus, H'_{S_n,τ'_S} is the subspace of states $|\xi'\rangle \in H'_{S_n}$ for which $\hat{T}'_S(x')\,|\xi'\rangle = \tau'_S(x')\,|\xi'\rangle$ for all $x' \in S_n \cap S$.

where $\pi_n = \sum_j |\xi_j\rangle\langle\xi_j|$, $\pi'_n = \sum_j |\xi'_j\rangle\langle\xi'_j|$, and $|\Psi'_n\rangle = U(\Lambda)\,|\Psi_n\rangle$.

To see why (4.27) holds, we first recall that π'_n will be independent of which orthonormal basis we choose for H_{S_n,τ'_S}.[21] Therefore, if we can show that $\{|\xi'_j\rangle \stackrel{\text{def}}{=} U(\Lambda)\,|\xi_j\rangle : j\}$ is an orthonormal basis of H_{S_n,τ'_S}, it will follow that $\pi'_n = U(\Lambda)\pi_n U(\Lambda)^{-1}$.

That the elements of $\{U(\Lambda)\,|\xi_j\rangle : j\}$ are orthonormal follows from the unitarity of $U(\Lambda)$ together with the orthonormality of $\{|\xi_j\rangle : j\}$. It remains for us to show that each $U(\Lambda)\,|\xi_j\rangle \in H'_{S_n,\tau'_S}$, and that any $|\xi'\rangle \in H'_{S_n,\tau'_S}$ can be expressed as a linear combination of the $U(\Lambda)\,|\xi_j\rangle$.

Well, first note that by (4.26) and a calculation similar to (4.22)

$$\begin{aligned} U(\Lambda)^{-1}\hat{T}'_S(x')U(\Lambda) &= U(\Lambda)^{-1}\hat{T}'^{\mu\nu}(x')\eta'_\mu(x')\eta'_\nu(x')U(\Lambda) \\ &= \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\hat{T}^{\rho\sigma}(x)\Lambda_\mu{}^\alpha\eta_\alpha(x)\Lambda_\nu{}^\beta\eta_\beta(x) \\ &= \Lambda^\mu{}_\rho\Lambda_\mu{}^\alpha\Lambda^\nu{}_\sigma\Lambda_\nu{}^\beta\hat{T}^{\rho\sigma}(x)\eta_\alpha(x)\eta_\beta(x) \\ &= \delta^\alpha_\rho\delta^\beta_\sigma\hat{T}^{\rho\sigma}(x)\eta_\alpha(x)\eta_\beta(x) \\ &= \hat{T}^{\alpha\beta}(x)\eta_\alpha(x)\eta_\beta(x) \\ &= \hat{T}_S(x) \end{aligned} \tag{4.28}$$

From (4.28), we see that $U(\Lambda)\hat{T}_S(x) = \hat{T}'_S(x')U(\Lambda)$, so

$$\hat{T}'_S(x')U(\Lambda)\,|\xi_j\rangle = U(\Lambda)\hat{T}_S(x)\,|\xi_j\rangle = \tau_S(x)U(\Lambda)\,|\xi_j\rangle = \tau'_S(x')U(\Lambda)\,|\xi_j\rangle$$

for all $x' \in S_n(y') \cap S$, where we have used the fact $\tau'_S(x') = \tau_S(x)$ since $T_S(x)$ is a scalar. Therefore, $U(\Lambda)\,|\xi_j\rangle \in H'_{S_n,\tau'_S}$.

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[21] We showed this was the case for π_n in footnote 4 on page 143.

Now suppose that $|\xi'\rangle$ is a state for which O' observes $T'_S(x')$ to be $\tau'_S(x')$ for all $x' \in S_n(y') \cap S$, i.e. $\hat{T}'_S(x')\,|\xi'\rangle = \tau'_S(x')\,|\xi'\rangle$. From (4.28) we see that $\hat{T}_S(x)U(\Lambda)^{-1} = U(\Lambda)^{-1}\hat{T}'_S(x')$, so

$$\begin{aligned} \hat{T}_S(x)U(\Lambda)^{-1}\,|\xi'\rangle &= U(\Lambda)^{-1}\hat{T}'_S(x')\,|\xi'\rangle \\ &= \tau'_S(x')U(\Lambda)^{-1}\,|\xi'\rangle \\ &= \tau_S(x)U(\Lambda)^{-1}\,|\xi'\rangle \end{aligned} \tag{4.29}$$

where on the last line we have used the fact that $T_S(x)$ is a scalar. Therefore, $U(\Lambda)^{-1}\,|\xi'\rangle$ can be expressed as a linear combination of the basis elements $\{|\xi_j\rangle : j\}$ of H_{S_n,τ_S}, and hence $|\xi'\rangle$ can be expressed as a linear combination of $\{U(\Lambda)\,|\xi_j\rangle : j\}$.

Thus, we see that $\{\left|\xi'_j\right\rangle \stackrel{\text{def}}{=} U(\Lambda)\,|\xi_j\rangle : j\}$ is a spanning orthonormal subset of H'_{S_n,τ'_S}, so it must therefore be an orthonormal basis of H'_{S_n,τ'_S}. From this it follows that $\pi'_n = U(\Lambda)\pi_n U(\Lambda)^{-1}$. Therefore,

$$\begin{aligned} \frac{\langle\Psi'_n|\pi'_n\hat{T}^{\mu\nu}(y')|\Psi'_n\rangle}{\langle\Psi'_n|\pi'_n|\Psi'_n\rangle} &= \frac{\langle\Psi_n|U(\Lambda)^{-1}U(\Lambda)\pi_n U(\Lambda)^{-1}\hat{T}^{\mu\nu}(y')U(\Lambda)|\Psi_n\rangle}{\langle\Psi_n|U(\Lambda)^{-1}U(\Lambda)\pi_n U(\Lambda)^{-1}U(\Lambda)|\Psi_n\rangle} \\ &= \frac{\langle\Psi_n|\pi_n U(\Lambda)^{-1}\hat{T}^{\mu\nu}(y')U(\Lambda)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle} \\ &= \frac{\langle\Psi_n|\pi_n\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}_{\sigma}\hat{T}^{\rho\sigma}(y)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle} \end{aligned} \tag{4.30}$$

where on the last line we have used (4.26). Thus, equation (4.27) holds, and hence Kent's model is Lorentz invariant.

Note that in this proof of Lorentz invariance, we don't need to take the limit of S_n as $n \to \infty$. That is, we could remove the $\lim_{n\to\infty}$ from equation (4.6) and consider a particular S_n, and the corresponding $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ would still be a rank-two tensor. Butterfield tells us that Kent's theory is Lorentz invariant because his algorithm respects the light cone structure of y.[22] However, this statement could be slightly misleading because we don't need to consider the subset $S^1(y) \subset S$ of locations outside the light cone of y in order to obtain a Lorentz invariant model. Doing the calculation on any Tomonaga-Schwinger spacelike hypersurface is sufficient to guarantee Lorentz invariance since any such spacelike hypersurface (e.g. S_n) is not altered at all by a Lorentz transformation – only its coordinate description changes under a Lorentz transformation, and so the additional information of the scalar $\tau_S(x)$ on $S_n \cap S$ is Lorentz invariant. The only reason we need to consider the limit $\lim_{n\to\infty} S_n$ and hence $S^1(y) = \lim_{n\to\infty} S_n \cap S$ is that it is only in the limit that we use all the available information in $\tau_S(x)$ to calculate $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$.

4.3 Kent's Theory and Decoherence Theory*

In section 2.7 we saw that decoherence theory by itself does not offer a solution to the problem of outcomes. In this section, we consider how the additional information in Kent's theory is sufficient to address this problem. We will explain this by again considering Kent's toy model discussed in section 3.5.

We thus suppose that a system is in a superposition $\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$ of two local states ψ_1^{sys} and ψ_2^{sys} where $|c_1|^2 + |c_2|^2 = 1$, and that there is a photon coming in from the left that interacts with the system.[23] We also suppose that y_1 is a spacetime location with spatial location z_1 between the two spacelike hypersurfaces S_0 and

[22] See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 30.

[23] As in footnote 45 on page 134, we write ψ_i^{sys} for the wave function that corresponds to the state $|\psi_i^{\text{sys}}(t)\rangle$ with $\psi_i^{\text{sys}}(z,t) = \langle z|\psi_i^{\text{sys}}(t)\rangle$.

S, and we consider a spacelike hypersurface $S_n = S_n(y_1)$ in a sequence of spacelike hypersurfaces that each contain y_1 as described on page 132.

In order to obtain a sufficiently simple description of the state $|\Psi_n\rangle \in H_{S_n}$ of S_n for which we can use the formula (4.6) to calculate Kent's beable, we will use a coarse-grained model so that S_n is treated as a mesh of tiny cells[24] labeled by a sequence $(y_k)_{k=1}^{\infty}$. Thus, for each cell y_k there will be a Hilbert space H_k describing the state of that cell. We can think of each of these y_k as systems that can become entangled with one another, but we will assume that y_1 is entangled with only a finite number M of the other y_k which we label as $y_{k_1}, \ldots, y_{k_M}$. What this means is that the most general expression for $|\Psi_n\rangle$ will be of the form

$$|\Psi_n\rangle = \Big(\sum_j \sum_{n \in \mathbb{N}^M} c_{j,n} \, |\xi_{1,j}\rangle \prod_{l=1}^{M} |\xi_{k_l,n_l}\rangle \Big) \, |\Xi\rangle \, . \tag{4.31}$$

In this expression, $\{|\xi_{1,j}\rangle : j\}$ is an orthonormal basis of H_1, $\mathbb{N}^M$ means the set of all lists $(n_1, \ldots, n_M)$ with each $n_l \in \mathbb{N}$ where $\mathbb{N}$ is the set of positive integers greater than 0. The set of states $\{|\xi_{k_l,m}\rangle : m \in \mathbb{N}\}$ form an orthonormal basis of H_{k_l} for each k_l, and the k_l are all distinct from each other and from 1. Also, M is chosen to be as small as possible so that any common factors of $|\Psi_n\rangle$ belong to $|\Xi\rangle$ which is a sum of states of the form $\prod_l |\xi_{\kappa_l}\rangle$ where the states $|\xi_{\kappa_l}\rangle \in H_{\kappa_l}$ range over all the cells of S_n not included in the set $\{k_l : l = 1, \ldots, M\}$. We also assume that each summand $c_{j,n} \, |\xi_{1,j}\rangle \prod_{l=1}^{M} |\xi_{k_l,n_l}\rangle \; |\Xi\rangle$ of $|\Psi_n\rangle$ contains a state in each H_k for every cell k of S_n. In other words, if $k \neq 1$ and does not belong to the set $\{k_l : l = 1, \ldots, M\}$, then k belongs to the set $\{\kappa_l :$ there exists $|\xi_{\kappa_l}\rangle \in H_{\kappa_l}$ appearing in $|\Xi\rangle .\}$. Also, we will

[24]For a more detailed discussion of coarse-graining, see pp. 117 ff.

give H_{S_n} an inner product so that if

$$|\Psi'_n\rangle = \Big(\sum_j \sum_{n\in\mathbb{N}^M} c'_{j,n} |\xi_{1,j}\rangle \prod_{l=1}^{M} |\xi_{k_l,n_l}\rangle\Big) |\Xi'\rangle \,,$$

then

$$\langle\Psi'_n|\Psi_n\rangle = \Big(\sum_j \sum_{n\in\mathbb{N}^M} \overline{c'_{j,n}} c_{j,n}\Big) \langle\Xi'|\Xi\rangle$$

where $\langle\Xi'|\Xi\rangle$ is defined in the obvious way. With this inner product, we will assume that $|\Psi_n\rangle$ is appropriately normalized so that $\langle\Psi_n|\Psi_n\rangle = 1$. If we also assume that $\langle\Xi|\Xi\rangle = 1$, it will follow that $\sum_j \sum_{n\in\mathbb{N}^M} |c_{j,n}|^2 = 1$.

Now in order to see how Kent's model addresses the problem of outcomes, we will need to consider several scenarios from his toy model. In each scenario, we will use the decomposition (4.31) of $|\Psi_n\rangle$ to calculate the reduced density matrix that encapsulates all the information needed to calculate expectation values at different spacetime locations.

First, consider Figure 4.3 which depicts the spacelike hypersurface $S_n(y_1^a)$ for a spacetime location y_1^a that occurs before the photon has interacted with the system.

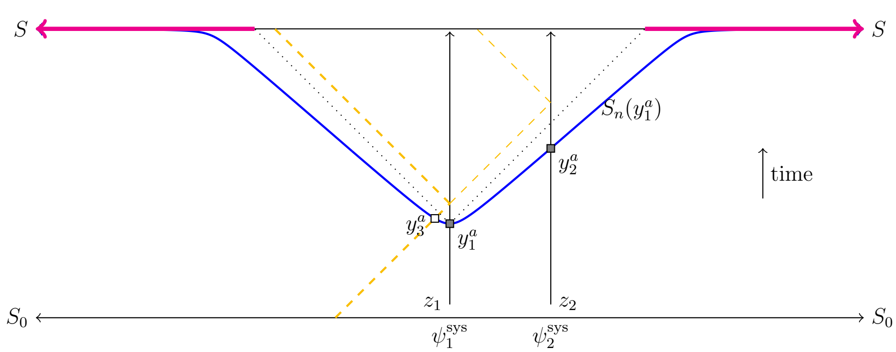

Figure 4.3: Depiction of a superposition of two local states at z_1 and z_2 before the photon has interacted with them. The gray squares indicate cells in $S^1(y_1^a)$ whose states are among the summands in (4.31) rather than in Ξ. The white square indicates a cell in $S_n(y_1^a)$ whose state is a factor in Ξ.

The gray squares correspond to the summands that appear in (4.31). If the system were in the ψ_1^{sys}-state, then the state describing $S_n(y_1^a)$ would have a factor $|\psi_1^{\text{sys}}\rangle \in H_1$ indicating that there is a non-zero mass at the y_1^a-cell, and there would also be a factor $|0_2\rangle \in H_2$ which we use to indicate that there is zero mass/energy at y_2^a. There is also an incoming photon at the y_3^a-cell, and so we use $|\gamma_3\rangle$ to indicate that there is a photon there. Thus, if the system were in the ψ_1^{sys}-state, we would write the state of $S_n(y_1^a)$ as $|\Psi_n\rangle = |\psi_1^{\text{sys}}\rangle\,|0_2\rangle\,|\gamma_3\rangle\;|\Xi'\rangle$, where $|\Xi'\rangle$ describes the states of all the other cells of $S_n(y_1^a)$. In this very simple scenario, $|\Xi'\rangle = \prod_{k\neq 1,2,3}|0_k\rangle$ indicating that there is zero mass/energy at all the other y_k.

On the other hand, if the system were in the state ψ_2^{sys}, then the state describing $S_n(y_1^a)$ would have a factor $|\psi_2^{\text{sys}}\rangle \in H_2$ indicating that there is a non-zero mass at the y_2^a-cell, and there would also be a factor $|0_1\rangle \in H_1$ which we use to indicate that there is zero mass at y_1^a, and again the y_3^a-cell would be in the $|\gamma_3\rangle$-state, and every other cell would be described by $|\Xi'\rangle$ just as if the system had been in the ψ_1^{sys}-state. Therefore, when the system is in the state ψ_2^{sys}, we would write the state of $S_n(y_1^a)$ as $|\Psi_n\rangle = |0_1\rangle\,|\psi_2^{\text{sys}}\rangle\,|\gamma_3\rangle\;|\Xi'\rangle$.

Now since the system is actually in a supposition $\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$, the state of $S_n(y_1^a)$ will be

$$|\Psi_n\rangle = \big(c_1\,|\psi_1^{\text{sys}}\rangle\,|0_2\rangle + c_2\,|0_1\rangle\,|\psi_2^{\text{sys}}\rangle\,\big)\,|\gamma_3\rangle\;|\Xi'\rangle = \big(c_1\,|\psi_1^{\text{sys}}\rangle\,|0_2\rangle + c_2\,|0_1\rangle\,|\psi_2^{\text{sys}}\rangle\,\big)\,|\Xi\rangle$$

where we have absorbed the $|\gamma_3\rangle$-state into $|\Xi\rangle$ (i.e. $|\Xi\rangle = |\gamma_3\rangle\;|\Xi'\rangle$).

Now as it stands, the state $|\Psi_n\rangle$ describing $S_n(y_1^a)$ has a definite mass-energy density $\tau_S(x)$ for $x \in S_n(y_1^a) \cap S$, namely 0. Thus, if π_n is the operator featuring in (4.6) that corresponds to this definite mass-energy density, then $\pi_n |\Psi_n\rangle = |\Psi_n\rangle$. Therefore, equation (4.6) for Kent's beables tells us that

$$\langle T^{\mu\nu}(y_1^a)\rangle_{\tau_S} = \langle \Psi_n|\hat{T}^{\mu\nu}(y_1^a)|\Psi_n\rangle \,, \tag{4.32}$$

where we have also used the fact that $\langle \Psi_n|\Psi_n\rangle = 1$.

Now as we saw in section 2.4, if we are interested only in the expectation values of observables for a system $\mathcal{S}$ contained within a universe $\mathcal{U} = \mathcal{S}+\mathcal{E}$, then the information needed to do this can be encapsulated in the reduced density matrix for $\mathcal{S}$. Thus, if the universe is described by a state $|\Psi\rangle = \sum_j c_j |\psi_j\rangle_{\mathcal{S}} |E_j\rangle_{\mathcal{E}}$ with corresponding density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi| \in M(H_{\mathcal{U}})$, then the reduced density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ is the Hermitian operator acting on the state space $H_{\mathcal{S}}$ with the property that

$$\langle \hat{O}_{\mathcal{U}}\rangle_{\rho} = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}) \tag{2.13 revisited}$$

where $\hat{O}_{\mathcal{S}}$ is an observable on $H_{\mathcal{S}}$, and $\hat{O}_{\mathcal{U}}$ is the corresponding observable on $H_{\mathcal{U}}$. Furthermore, we also have

$$\hat{\rho}_{\mathcal{S}} = \sum_j |c_j|^2 |\psi_j\rangle\langle\psi_j| + \sum_{j\neq k} c_j\overline{c_k} \langle E_k|E_j\rangle |\psi_j\rangle\langle\psi_k| \,. \text{[25]} \tag{4.33}$$

We can thus apply this to the situation at hand by taking S_n to be our universe $\mathcal{U}$ and y_1^a to be the system $\mathcal{S}$, and $S_n \setminus \{y_1^a\}$ to be the environment $\mathcal{E}$. If we assume that $\langle 0_2|\psi_2^{\mathrm{sys}}\rangle = 0$, then by (4.33), the corresponding reduced density matrix $\hat{\rho}_{y_1^a}$ takes the form of an improper mixture

$$\hat{\rho}_{y_1^a} = |c_1|^2 |\psi_1^{\mathrm{sys}}\rangle\langle\psi_1^{\mathrm{sys}}| + |c_2|^2 |0_1\rangle\langle 0_1| \,. \tag{4.34}$$

[25] cf. (2.14)

Therefore, by (4.32) and (4.34), Kent's beable at y_1^a will take the form

$$\begin{aligned}\langle T^{\mu\nu}(y_1^a)\rangle_{\tau_S} &= \langle\Psi_n|\hat{T}^{\mu\nu}(y_1^a)|\Psi_n\rangle \\ &= \mathrm{Tr}_{y_1^a}(\hat{\rho}_{y_1^a}\hat{T}^{\mu\nu}(y_1^a)) \\ &= |c_1|^2\,\langle\psi_1^{\mathrm{sys}}|\hat{T}^{\mu\nu}(y_1^a)|\psi_1^{\mathrm{sys}}\rangle + |c_2|^2\,\langle 0_1|\hat{T}^{\mu\nu}(y_1^a)|0_1\rangle\,.\end{aligned} \tag{4.35}$$

But since $\hat{\rho}_{y_1^a}$ is an improper mixture, we cannot give (4.35) an ignorance interpretation – the universe is still in the superposition state

$$|\Psi_n\rangle = c_1\,|\psi_1^{\mathrm{sys}}\rangle\,|E_1\rangle + c_2\,|0_1\rangle\,|E_2\rangle$$

where $|E_1\rangle = |0_2\rangle\,|\Xi\rangle$ and $|E_2\rangle = |\psi_2^{\mathrm{sys}}\rangle\,|\Xi\rangle$.

In order to solve the problem of outcomes, we need to provide a satisfactory explanation (i.e. an explanation that is consistent with special relativity and the predictions of standard quantum theory) of how the superposition state $|\Psi_n\rangle$ effectively goes to either the state $|\psi_1^{\mathrm{sys}}\rangle\,|E_1\rangle$ or to the state $|0_1\rangle\,|E_2\rangle$.[26] In terms of density operators, this means we need to show how the improper state (4.34), transitions to a pure state of the form $|\psi_1^{\mathrm{sys}}\rangle\langle\psi_1^{\mathrm{sys}}|$ or $|0_1\rangle\langle 0_1|$.

To this end, let us consider Kent's beables at the spacetime location y_1^b depicted in figure 4.4. The state of $S_n(y_1^b)$ will then be

$$|\Psi_n\rangle = \big(c_1\,|\psi_1^{\mathrm{sys}}\rangle\,|0_2\rangle\,|\gamma_3\rangle\,|0_4\rangle + c_2\,|0_1\rangle\,|\psi_2^{\mathrm{sys}}\rangle\,|0_3\rangle\,|\gamma_4\rangle\,\big)\,|\Xi\rangle$$

where the notation is analogous to that in the previous example. Since no photon detections are registered on $S_n(y_1^b)\cap S$, we again have $\pi_n\,|\Psi_n\rangle = |\Psi_n\rangle$ so that the

[26]cf. the initial discussion of the problem of outcomes on page 87. I have used the word 'effectively' to qualify this sentence since it is not necessary to prove that there actually is such a transition from the state $|\Psi_n\rangle$ to either the state $|\psi_1^{\mathrm{sys}}\rangle\,|E_1\rangle$ or to the state $|0_1\rangle\,|E_2\rangle$. Rather, it is sufficient to show that if we consider any observable $\hat{O}_{\mathcal{S}}$ acting on $\mathcal{S}$, then the expectation value takes the form $\langle\psi_1^{\mathrm{sys}}|\hat{O}_{\mathcal{S}}|\psi_1^{\mathrm{sys}}\rangle$ or $\langle 0_1|\hat{O}_{\mathcal{S}}|0_1\rangle$ once there is an outcome, for then the system $\mathcal{S}$ has all the properties consistent with it being in the state $|\psi_1^{\mathrm{sys}}\rangle$ or the state $|0_1\rangle$ respectively.

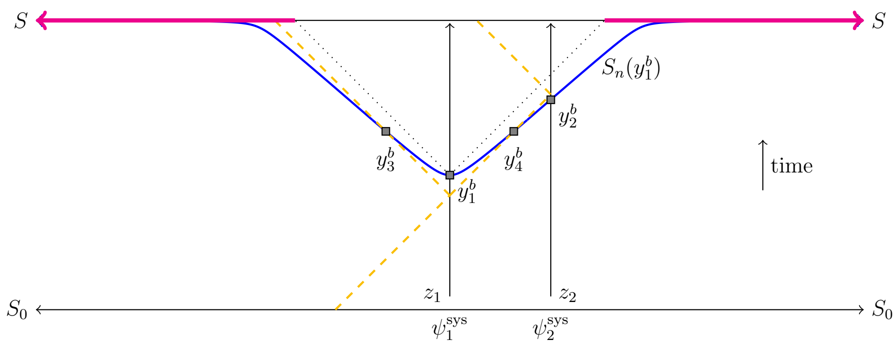

Figure 4.4: Depiction of a superposition of two local states at z_1 and z_2 with $S_n(y_1^b)$ being after the photon has interacted without the photon intersecting $S_n(y_1^b) \cap S$. The gray squares indicate cells in $S^1(y_1^b)$ whose states are among the summands in (4.31).

reduced density matrix $\hat{\rho}_{y_1^b}$ will again be given by (4.34) with y_1^a replaced by y_1^b. However, in this case, Kent's beables $\langle T^{\mu\nu}(y_1^b)\rangle_{\tau_S}$ will not be given by (4.35) because in the limit as $n \to \infty$, the photon *will* be registered on $S_n(y_1^b) \cap S$.

To deal with the case when a photon is registered on $S_n(y_1^b) \cap S$, we consider a third example as depicted in figure 4.5.

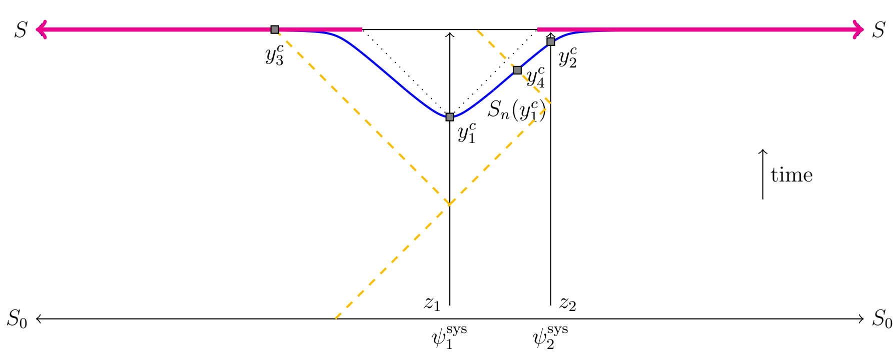

Figure 4.5: Depiction of a superposition of two local states at z_1 and z_2 with y_1^c sufficiently late that the photon intersects $S_n(y_1^c) \cap S$. The gray squares indicate cells in $S^1(y_1^c)$ whose states are among the summands in (4.31)

In this case, the state of $S_n(y_1^c)$ will be

$$|\Psi_n\rangle = \big(c_1 \, |\psi_1^{\text{sys}}\rangle \, |0_2\rangle \, |\gamma_3\rangle \, |0_4\rangle + c_2 \, |0_1\rangle \, |\psi_2^{\text{sys}}\rangle \, |0_3\rangle \, |\gamma_4\rangle \,\big) \, |\Xi\rangle$$

but now we have to consider the fact that the photon intersects $S_n(y_1^c) \cap S$. There are two possible (notional) measurement outcomes that can occur on $S_n(y_1^c) \cap S$: either $T_S = \tau_{S,1}$ where $\tau_{S,1}(y_3^c) \neq 0$, or $T_S = \tau_{S,2}$ where $\tau_{S,2}(y_3^c) = 0$.

The case $T_S = \tau_{S,1}$ indicates that there is a photon detection at y_3^c so that the local state at the y_3^c-cell is $|\gamma_3\rangle$. Therefore, if we write $\pi_{n,1}$ for the operator π_n, we have

$$\pi_{n,1} |\Psi_n\rangle = c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle \ |\Xi\rangle \,.$$

Therefore, $\langle\Psi_n|\pi_{n,1}\hat{T}^{\mu\nu}(y_1^c)|\Psi_n\rangle = |c_1|^2 \langle\psi_1^{\text{sys}}|\hat{T}^{\mu\nu}(y_1^c)|\psi_1^{\text{sys}}\rangle$ and $\langle\Psi_n|\pi_{n,1}|\Psi_n\rangle = |c_1|^2$. Hence, by (4.6), Kent's beables at y_1^c will be

$$\langle T^{\mu\nu}(y_1^c)\rangle_{\tau_{S,1}} = \langle\psi_1^{\text{sys}}|\hat{T}^{\mu\nu}(y_1^c)|\psi_1^{\text{sys}}\rangle \,.$$

From this, it follows that the reduced density matrix at y_1^c will take the form of a pure state:

$$\hat{\rho}_{y_1^c} = |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}| \,. \tag{4.36}$$

On the other hand, for the case when $T_S = \tau_{S,2}$, this indicates that there is no photon detection at y_3^c, so that the local state at the y_3^c-cell will be $|0_3\rangle$. So if we now write $\pi_{n,2}$ for the operator π_n, we have

$$\pi_{n,2} |\Psi_n\rangle = c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |0_3\rangle |\gamma_4\rangle \ |\Xi\rangle \,.$$

Therefore, $\langle\Psi_n|\pi_{n,2}\hat{T}^{\mu\nu}(y_1^c)|\Psi_n\rangle = |c_2|^2 \langle 0_1|\hat{T}^{\mu\nu}(y_1^c)|0_1\rangle$ and $\langle\Psi_n|\pi_{n,2}|\Psi_n\rangle = |c_2|^2$, and so by (4.6), Kent's beables at y_1^c will be

$$\langle T^{\mu\nu}(y_1^c)\rangle_{\tau_{S,2}} = \langle 0_1|\hat{T}^{\mu\nu}(y_1^c)|0_1\rangle \,.$$

In this case, the reduced density matrix at y_1^c will be

$$\hat{\rho}_{y_1^c} = |0_1\rangle\langle 0_1| \,, \tag{4.37}$$

which is again a pure state.

In these examples we have therefore seen how the additional information concerning photon detection on $S_n(y_1) \cap S$ is able to determine whether the reduced density matrix at y_1 is a pure state or an improper mixture. Hence, Kent's theory offers an answer to d'Espagnat's problem of outcomes. As mentioned in section 2.7, d'Espagnat noticed that with decoherence theory alone, we are not entitled to give an ignorance interpretation to the reduced density matrix for a system that is an improper mixture, and thus we are not able to conclude that an outcome has occurred. However, if the reduced density matrix of a system goes from being an improper mixture to a pure state of the form $|\psi\rangle\langle\psi|$ as it does when Kent's additional information is taken into account, then we can say that an outcome has occurred, namely the outcome of the system being in the state $|\psi\rangle$.

4.4 Butterfield's Analysis of Outcome Independence in Kent's theory

Let us now consider Kent's theory in the light of Shimony's notion of Outcome Independence (OI) as defined in section 1.8.

Butterfield[27] tries to answer the question of whether OI holds in Kent's theory by considering an example that builds on Kent's toy model. Butterfield's example is designed to capture the salient features of a Bell experiment where two spatially separated observers always observe opposite outcomes of some measurement. Following Kent, Butterfield thus considers a universe in one spatial dimensional. In this universe,

[27]See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 30–32

there are two entangled systems, a left-system and a right-system as depicted in figure 4.6.

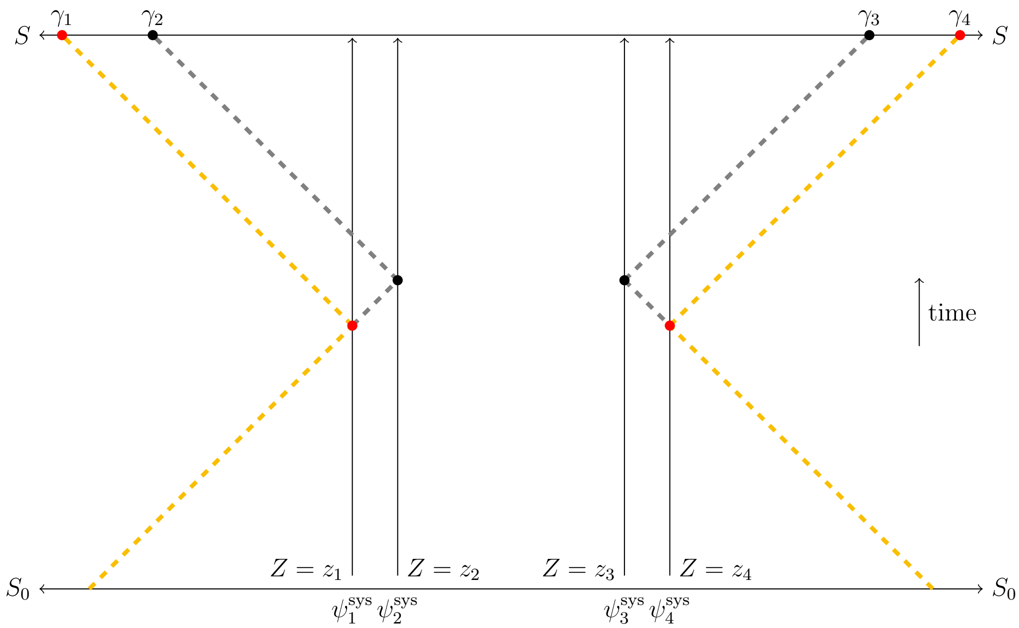

Figure 4.6: Butterfield's thought experiment for analyzing OI

Two locations z_1 and z_2 with $z_2 > z_1$ belong to a left-system, and there are two possible outcomes for a measurement on the left-system: either all the mass/energy of the left-system is localized at z_1 or all the mass/energy of the left-system is localized at z_2. These two possibilities are analogous to a spin up or a spin down measurement outcome in a Stern-Gerlach statement. Likewise, two locations z_3 and z_4 with $z_3 < z_4$ and $z_3 \gg z_2$ belong to a right-system, and again, there are two possible measurement outcomes: either all the mass/energy of the right-system is localized at z_3 or all the mass/energy of the right-system is localized at z_4.

The initial joint state of the two systems is $a\,|\psi_1\rangle\;|\psi_4\rangle + b\,|\psi_2\rangle\;|\psi_3\rangle$. This means that the left-system will be found to be localized at z_1 with probability $|a|^2$, and at z_2 with probability $|b|^2$, and if the left-system is localized at z_1, the right system must be localized at z_4, whereas if the left-system is localized at z_2, then the right system must be localized at z_3.

Now Butterfield supposes that there are two photons, one coming in from the left that interacts with the left system, and one coming in from the right that interacts with the right system. As in Kent's toy model, there is a late time spacelike hypersurface S, on which the photons are "measured". Since the joint state of the two systems is in superposition, there will be two possible measurement outcomes for the two photons that arrive at S. Either the left-photon is measured at γ_1 and the right-photon is measured at γ_4, or the left-photon is measured at γ_2 and the right photon is measured at γ_3. Thus, if we suppose that the (notional) measurement for $T_S(x)$ yields an energy distribution $\tau_S(x)$ that is nonzero at γ_1 and γ_4, but is zero at γ_2 and γ_3, then we can say that the outcome of the measurement on the two systems is that the left system is localized at z_1 and the right system is localized at z_4. Moreover, the probability of this outcome is 1 given that the (notional) measurement of $T_S(x)$ on S is $\tau_S(x)$. In other words, this model is deterministic. But as we saw on page 45, if a model is deterministic, then OI must hold. This is the conclusion that Butterfield draws.

Now if Kent's theory is to be consistent with special relativity, OI being satisfied might initially seem concerning. Indeed, we saw in section 1.8 that OI implies the negation of PI, and the negation of PI is not consistent with special relativity.[28] However, there is one salient feature of a Bell experiment that is not captured in Butterfield's

[28]At this point, one might make the following remark: the argument that a violation of PI constitutes a violation of relativity is based on the idea that if one knew the value of the hidden data, one could transmit messages at superluminal speed. But when the hidden data is grounded in data about the future hypersurface S, then the fact that if one knew this data, one could transmit messages superluminally should not be a cause for concern since all sorts of things become possible if you can know future contingents.

scenario, namely, in a Bell experiment, one can perform different measurements. PI and its negation only make sense when there are parameters that can be changed. Furthermore, in the proof that OI implies the negation of PI,[29] it is assumed that the choice of parameter is not determined by the hidden variable λ. If the choice of parameters was determined by λ, then for $\hat{a} \neq \hat{b}$, at least one of the probabilities $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+,\hat{\mathbf{c}}+)$, $P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+,\hat{\mathbf{b}}+)$ or $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+)$ would not be well-defined.[30] Even though Butterfield is only considering OI in his thought experiment, a proper analysis of OI shouldn't be undertaken without considering an experiment with parameters (e.g. knob settings that correspond to measurement axes of a Stern-Gerlach experiment). This is because the determination of whether OI holds will depend on what one counts as being the hidden variable data of a system, and we need the hidden variable of a system to be such that the notion of PI is well-defined. Otherwise, one's verdict on OI will be irrelevant to Shimony's analysis of why Bell's inequality fails to hold.

To this remark, there are two responses that one could make. Firstly, it would be somewhat misleading in the context of Kent's interpretation to say that a knowledge of data about S implies a knowledge of future contingents since when calculating Kent's $T^{\mu\nu}(y)$-beables $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$, the only knowledge about the data of S that is needed is for regions of S outside the light cone of y, and whether this data is about something in the future or in the past is going to depend on what frame of reference one is in. Only when the data is within the light cone of y will we be able to say categorically that knowledge of this data constitutes knowledge of future contingents. But Kent's beables are not dependent on such data.

Secondly, as mentioned on page 42, the main concern with a violation of PI is not simply the possibility of superluminal signally, but rather the possibility of the propatation of superluminal effects, of which superluminal signallying would be a very clear demonstration. But whether or not there is such a demonstration, a violation of PI seems to be saying that effects can be propagated superluminally, and this is unacceptable to adherents of relativity theory.

[29]The proof that determinism implies the negation of PI (on pages 38 to 41), also assumes that the choice of parameter is not determined by the hidden variable λ.

[30]If λ did determine the choice of measurements, either $P(\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}})=0$, $P(\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}})=0$, or $P(\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})=0$. So for example, if we thought of $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(X,Y)$ as a conditional probability $P(X,Y|\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})$, and the probability $P(\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})=0$, then according to the definition of conditional probability, $P(X,Y|\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})=\frac{0}{0}$.

In the next section I will discuss Leegwater's criteria for what one should count as being the hidden variable data of a system, and in the following section 4.6, I will consider a toy model that takkes parameter settings into account.

4.5 Hidden variables and the Colbeck-Renner theorem

In this section we will consider hidden variables in the light of the Colbeck-Renner theorem. Roughly speaking, the Colbeck-Renner theorem says there are no hidden variables theories that are of interest to quantum physicists – either the hidden variables will be redundant or the hidden variable theory will be incompatible with standard quantum physics. It might therefore seem that the Colbeck-Renner theorem presents a serious challenge to Kent's theory. However, the challenge really hinges on what criteria the data of a theory must satisfy for it to be classified as hidden variable data. A careful analysis of the kind of data Kent's theory relies on reveals that the Colbeck-Renner theorm is not as serious a challenge to Kent's theory as it first might seem.

First we need to consider the criteria a set of data should satisfy if it is to be classified as hidden variable data of a physical system $\mathcal{S}$. The criteria we discuss below can be found either explicitly or implicitly in Leegwater's proof of the Colbeck-Renner theorem.[31] The first criterion we will discuss is the following:

1. all the information of λ is about $\mathcal{S}$ so that a change in λ corresponds to a change in the system $\mathcal{S}$.

[31]See Leegwater, "An impossibility theorem for parameter independent hidden-variable theories"

Notice that Butterfield does not accept this criterion. Butterfield assumes that the hidden variables in Kent's theory consist in the outcome $\tau_S(x)$ of $T_S(x)$ over the whole of S. However, this assumption is going to cause difficulties in the context of Shimony's analysis. This is because in Kent's theory, the information in $\tau_S(x)$ over the whole of S clearly would determine which parameters are chosen in a Bell experiment, for this information would determine where a silver atom coming out of a Stern-Gerlach apparatus would be detected on a detection screen (as depicted in figure 1.4), and from the position of this detection, one could determine the orientation of the magnetic field used in the Stern-Gerlach experiment. So if we stipulated that $\lambda = \tau_S$ is the hidden variable data of every system in Kent's theory, then Kent's theory wouldn't satisfy the preconditions necessary for defining OI and PI. This would make Kent's theory radically different from the pilot wave interpretation where one can define OI and PI because the hidden variables in the pilot wave interpretation, being the positions and momenta of the particles, are independent of the measurement choices. An unfortunate consequence of not being able to define OI and PI is that we wouldn't be able to evaluate Kent's theory in the light of Shimony's analysis of why Bell's inequality fails to hold.

But it is not obvious that we should stipulate that $\lambda = \tau_S$ is the hidden variable data of every system in Kent's theory. Just because we give τ_S a single label λ, it doesn't follow that τ_S is a single piece of information. There is typically going to be a huge amount of information in τ_S, and so for a given system $\mathcal{S}$, we should carefully discern what collection of information in τ_S should be stipulated as being the hidden variable data λ of $\mathcal{S}$, hence criterion 1. Criterion 1 shouldn't be that difficult to satisfy, for if

λ contained information that could change without this corresponding to any change in $\mathcal{S}$, then we should be able to discard this irrelevant information when considering $\mathcal{S}$ and redefine what λ should be for the system.

In the pilot wave interpretation, the positions and momenta of the particles that constitute a system would fulfil criterion 1. On the other hand, all the information in τ_S of Kent's theory would not fulfil this criterion unless, of course, $\mathcal{S}$ was the whole universe. So in order for Kent's theory to satisfy criterion 1, we just need to discard the information of τ_S that is irrelevant to the system $\mathcal{S}$ that is being considered in order to obtain the appropriate hidden variable data λ.

Note, however, that we don't insist that a difference in $\mathcal{S}$ entails a difference in λ. This is because a hidden-variables theory is envisaged as augmenting standard quantum theory. So in the case when $\mathcal{S}$ is not entangled with any other system, there will be a quantum state describing $\mathcal{S}$, and this quantum state can be other than it is (indicating that $\mathcal{S}$ can be in a different physical state) whilst the hidden variable remains the same. We thus impose a second criterion for a hidden-variables theory:

2. If λ is the hidden variable of a system $\mathcal{S}$ and if $|\phi\rangle$ is the quantum state of $\mathcal{S}$ or of some composite system $\mathcal{U}$ that contains $\mathcal{S}$ as a subsystem, then it is possible for there to be a different quantum state $|\phi'\rangle$ of $\mathcal{S}$ (or $\mathcal{U}$) while the hidden variable λ remains unchanged, and it is possible for there to be a different hidden variable λ while $|\phi\rangle$ remains unchanged.

This criterion is satisfied in the pilot wave interpretation, since the quantum state is the pilot wave itself. The pilot wave could be other than it is without any of the positions and momenta of the particles changing, but changing the pilot wave would result in a physical change of the system since the pilot wave governs how the positions and the momenta of the particles subsequently evolve over time. We will discuss the failure of criterion 2 for Kent's theory at the end of this section.

Another criterion implicit in Leegwater's proof for a set of data λ to constitute the hidden variable data of a system $\mathcal{S}$ is the following:

3. it should be possible to change the measurement parameters when measuring $\mathcal{S}$ without this determining what λ should be.

It is worth noting that we used criterion 3 when showing that OI implies the negation of PI. If this criterion doesn't hold, we cannot even begin to consider whether PI holds in a given theory. This is because the criterion for PI depends on the probability $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}\pm,\hat{\mathbf{b}}\pm)$ being well-defined, but if the choice of $\hat{\boldsymbol{a}}$ or $\hat{\boldsymbol{b}}$ determines what λ should be, then one wouldn't be able to define $P_{\lambda',\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}\pm,\hat{\mathbf{b}}\pm)$ for $\lambda' \neq \lambda$.[32] But even without any consideration of PI, a rejection of criterion 3 would still seem undesirable, for it would be rather strange if changing the choice of measurement parameters necessarily changed the hidden variable λ of the system $\mathcal{S}$, and hence by criterion 1 the system $\mathcal{S}$ itself.[33] Moreover, if free will is real and the free choice of the measurement parameter

[32]Cf. footnote 30 on page 173.

[33]Of course, in the quantum world, strangeness is not necessarily an argument against something being true.

was made before it was determined which system was going to be measured, then it wouldn't even be clear which λ of which system we were talking about.[34]

In the pilot wave interpretation, the positions and momenta of the particles that constitute a system would fulfil criterion 3. In the case of Kent's interpretation, we would have to choose carefully the subset of τ_S for the λ of the system being measured if criterion 3 is to hold. But it seems that this should be possible, for one can imagine just changing the part of τ_S that corresponded to where the measurement parameter dial/knob was without this changing Kent's stress-momentum tensor for the system being measured immediately before it interacts with the measurement device.

Closely related to criterion 3 is the following criterion:

4. There is a range of possible values λ for a system $\mathcal{S}$, and for each possible value, we can assign a probability p_λ that λ obtains, and we can do so in such a way that p_λ is independent of any choice of measurement that is to be made on $\mathcal{S}$.

Butterfield refers to criterion 4 as the 'no-conspiracy' assumption, and it is one of the criteria assumed in the Colbeck-Renner theorem. However, Butterfield adds that 'no-conspiracy' is a rather unfair label for this criterion since there wouldn't necessarily be anything conspiratorial if this assumption was violated.[35] But in saying this,

[34] If the reality of free will is denied, then there wouldn't necessarily be this problem of associating the choice of measurement with the λ of the system, because the physical state of the universe could determine which choice of measurement is going to be made as well as which system is going to be measured.

[35] See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 34.

Butterfield is envisaging λ to be the whole of τ_S, and this is in violation of criterion 1. Since we can expect the whole of τ_S (together with the universal state $|\Psi_0\rangle$) to determine Kent's stress-momentum tensor for the measurement parameter dial/knob, then we wouldn't expect criterion 4 to hold. But on the other hand, if we adopt criterion 1 and assume that only information relevant to $\mathcal{S}$ is included in the λ for $\mathcal{S}$, then criterion 4 is more plausible. It ties in with common sense intuitions that scientists have free will and can make choices about which measurements they make, and that these choices are statistically independent of the states of the physical systems they are measuring. However, not everyone shares this intuition. If criterion 4 doesn't hold, then what a physical system does will depend on what property of this system one is about to measure. This kind of dependence is referred to as *superdeterminism*. Bell coined the term superdeterminism in a 1983 BBC interview, where he said:

> There is a way to escape the inference of superluminal speeds and spooky action at a distance. But it involves absolute determinism in the universe, the complete absence of free will. Suppose the world is superdeterministic, with not just inanimate nature running on behind-the-scenes clockwork, but with our behavior, including our belief that we are free to choose to do one experiment rather than another, absolutely predetermined, including the "decision" by the experimenter to carry out one set of measurements rather than another, the difficulty disappears.

However, Sabine Hossenfelder disputes Bell's argument that the only way to escape the inference of superluminal speeds and spooky action at a distance is to violate free will. Rather, all that is needed to escape this inference is a violation of the statistical independence of the choice of measurement parameters and the state of the system being measured, and this violation is what Hossenfelder takes to be superdeterminism.[36]

In addition to these four criteria for the hidden variable data λ of a system $\mathcal{S}$, it is also desirable for a hidden-variables theory to satisfy PI and empirical adequacy. We defined PI for a two-outcome measurement on page 36, but it is easy to generalize the definition of PI for measurements with more than two outcomes. In this more generalized setting, we suppose that $\mathcal{A}$ is any system that is entangled with the system $\mathcal{S}$, and that the quantum state of the composite system $\mathcal{S}+\mathcal{A}$ is $|\phi\rangle_{\mathcal{S}+\mathcal{A}}$. We also suppose that $O_{\mathcal{S}}$ and $O_{\mathcal{A}}$ represent measurement procedures on $\mathcal{S}$ and $\mathcal{A}$ respectively, and that $o_{\mathcal{S}}$ and $o_{\mathcal{S}}$ represent particular measurement outcomes respectively. For a hidden variable λ for the system $\mathcal{S}$,[37] there will be a probability $P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}} = o_{\mathcal{S}} \,\&\, O_{\mathcal{A}} = o_{\mathcal{A}})$ (understandable in a frequentist sense) that the measurement outcomes of $O_{\mathcal{S}}$ and $O_{\mathcal{A}}$ will be $o_{\mathcal{S}}$ and $o_{\mathcal{A}}$ respectively. Given a second measurement procedure $O'_{\mathcal{A}}$ on $\mathcal{A}$, PI states that

$$\sum_{\substack{o_{\mathcal{A}} \text{ an} \\ \text{outcome} \\ \text{of } O_{\mathcal{A}}}} P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}} = o_{\mathcal{S}} \,\&\, O_{\mathcal{A}} = o_{\mathcal{A}}) = \sum_{\substack{o'_{\mathcal{A}} \text{ an} \\ \text{outcome} \\ \text{of } O'_{\mathcal{A}}}} P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}} = o_{\mathcal{S}} \,\&\, O'_{\mathcal{A}} = o'_{\mathcal{A}}). \quad \text{(PI)}$$

If PI holds, then we can define the probability

$$P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}} = o_{\mathcal{S}}) = \sum_{\substack{o_{\mathcal{A}} \text{ an} \\ \text{outcome} \\ \text{of } O_{\mathcal{A}}}} P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}} = o_{\mathcal{S}} \,\&\, O_{\mathcal{A}} = o_{\mathcal{A}}) \tag{4.38}$$

that is independent of the measurement procedure $O_{\mathcal{A}}$ on $\mathcal{A}$.[38]

[36]See Sabine Hossenfelder, "Does Superdeterminism save Quantum Mechanics? Or does it kill free will and destroy science?," Youtube, 2021, https://www.youtube.com/watch?v=ytyjgIyegDI.

[37]Although Leegwater, in his proof of the Colbeck-Renner theorem assumes that λ is a hidden variable for the system $\mathcal{S}$ only, it looks like the proof would still go through if λ was a hidden variable for the composite system $\mathcal{S}+\mathcal{A}$.

[38]Cf. (1.19).

As for the definition of *empirical adequacy* (EA), this states that

$$\sum_{\lambda \in \Lambda} p_\lambda P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S} \,\&\, O_\mathcal{A} = o_\mathcal{A}) = P^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S} \,\&\, O_\mathcal{A} = o_\mathcal{A}) \tag{EA}$$

where Λ is the set of all hidden variables so that $\sum_{\lambda \in \Lambda} p_\lambda = 1$, and where

$$P^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S} \,\&\, O_\mathcal{A} = o_\mathcal{A})$$

is the standard probability calculated using the Born Rule with the eigenstates of the observables $\hat{O}_\mathcal{S}$ and $\hat{O}_\mathcal{A}$ and the quantum state $|\phi\rangle_{\mathcal{S}+\mathcal{A}}$. EA is essentially the same as equation (1.23). It also has some similarities with (4.7), though the main difference is the range of the summation – the index of the summands of (4.7) does not parametrize hidden variables that satisfy criteria 1 to 4 above.

Now, as I've been alluding to, criteria 1 to 4 together with the conditions of PI and EA are very restrictive. Leegwater proves a version of the Colbeck-Renner theorem[39] which takes the following form: if one defines hidden variables according to criteria 1 to 4, then in any hidden-variables theory for which PI and EA hold, the hidden variables are redundant. In other words,

$$P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S} \,\&\, O_\mathcal{A} = o_\mathcal{A}) = P^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S} \,\&\, O_\mathcal{A} = o_\mathcal{A}) \tag{4.39}$$

for any measurement $O_\mathcal{S}$ on $\mathcal{S}$ and $O_\mathcal{A}$ on $\mathcal{A}$.[40]

Thus, the Colbeck-Renner theorem means that we cannot hope to make Kent's theory into a hidden-variables theory that satisfies criterion 1 to 4 as well as PI and EA, for the information in Kent's theory is clearly non-redundant.

[39] See Leegwater, "An impossibility theorem for parameter independent hidden-variable theories."

[40] Strictly speaking, we should say that equation (4.39) holds for almost all λ, but we need not concern ourselves here with the details of measure theory that would be needed to make sense of this qualification.

But nevertheless, it still seems that we should be able to make some kind of sense of PI and EA in Kent's theory and that we should be able to evaluate Kent's theory on the basis of whether these notions of PI and EA are true in this context. To achieve this aim, one strategy would be to relax or drop one or more of the four criteria for a hidden variable. Since we still want to be able to make sense of PI and EA, we won't want to relax criteria 3 or 4. That leaves the possibility of relaxing or droping criteria 1 or 2.

Now clearly, we wouldn't be able to drop criterion 1 entirely, for otherwise λ would contain information that would determine the choice of measurement made on $\mathcal{S}$. But it doesn't seem problematic if we relax criterion 1 so that there can be information that can change without this corresponding to a change in the system $\mathcal{S}$ so long as this information doesn't determine the measurement choice that is to be made.

As for criterion 2, there doesn't seem to be any problem with dropping it entirely. On doing this, then instead of thinking of τ_S as an augmentation of standard quantum theory, we can think of τ_S as a rather elaborate way of stipulating the initial quantum states of experiments as well as the quantum states of measurement outcomes. In the next section, we will describe in some more detail how to extract the quantum state of a system from the universal quantum state $|\Psi_0\rangle$ and τ_S, but roughly speaking, if we consider an experimental setup including some measurement apparatus $\mathcal{A}$ and an object to be measured $\mathcal{S}$, then the information in τ_S outside the light cone of the spacetime location of $\mathcal{S} + \mathcal{A}$ before they have interacted will determine the initial quantum states of $\mathcal{S}$ and $\mathcal{A}$ before they interact, and likewise, the information in τ_S

outside the light cone of the spacetime location of $\mathcal{S} + \mathcal{A}$ after they have interacted will determine the quantum outcome states of $\mathcal{S}$ and $\mathcal{A}$ after they have interacted. This will mean that when the information in τ_S that is about $\mathcal{S}$ changes, the quantum state of $\mathcal{S}$ extracted from τ_S and $|\Psi_0\rangle$ will also change, and hence criterion 2 will fail to hold. But the information of τ_S will be non-redundant, for without this information, we would only have the evolution of universal quantum state $|\Psi_0\rangle$ which would continually branch into many worlds. In the many worlds that resulted, the energy density on the spacelike hypersurface S would not be in a definite state, but rather would be in a superposition of definite states. But with the information of τ_S one of these many states in this superposition is selected as actual. If we could then appropriately partition the information in $\tau_S(x)$ on the basis of whether it determined the quantum state of $\mathcal{S}$, or the quantum state of the apparatus $\mathcal{A}$, or the quantum state of the rest of the universe, we could then consider whether Kent's theory gave the same predictions as standard quantum theory. If it did, then PI and EA would hold in Kent's theory, since these both hold in standard quantum theory. And since Kent's theory is formulated in the Lorentz invariant setting of Schwinger and Tomonaga, this would mean that Kent's theory is a solution to the measurement problem! In other words, we would have a one-world interpretation of quantum physics which gave the same probabilities for experimental outcomes that standard quantum physics predicts, and under this interpretation, the physical world would possess the necessary symmetries that guarantee whatever frame of reference one was in, the speed of light would be constant.

4.6 Kent's Theory and standard quantum theory*

In this section, I will describe in more detail how to extract the quantum state of a system at a particular time from the universal quantum state $|\Psi_0\rangle$ and the mass-energy density τ_S, and I will show that Kent's theory does indeed give the same predictions as standard quantum theory in the case of an experimental apparatus $\mathcal{A}$ measuring the properties of a particle $\mathcal{S}$. Again, we will consider a toy model similar to Kent's toy model described in section 3.5 where photons are treated as point particles.

So let τ_S be the notional mass-energy density measurement on S. In order to avoid undue complexity, we will assume that there is no simultaneous $\hat{T}_S$-eigenstate degeneracy so that if $|\Psi\rangle$ and $|\Psi'\rangle$ are normalized[41] simultaneous $\hat{T}_S$-eigenstates with simultaneous eigenvalues τ_S and τ'_S respectively, then

$$(\forall x \in S)\,(\tau_S(x) = \tau'_S(x)) \implies |\Psi\rangle = |\Psi'\rangle\,.$$

This means that corresponding to τ_S, there will be a unique simultaneous $\hat{T}_S$-eigenstate $|\Psi\rangle$, and according to the Born Rule, the notional mass-energy density measurement τ_S will have been selected with a probability

$$P(T_S = \tau_S) = |\,\langle\Psi|U_{SS_0}|\Psi_0\rangle\,|^2$$

where $|\Psi_0\rangle$ is the state of the initial spacelike hypersurface S_0, and where U_{SS_0} is the unitary operator defined by equation (3.8).

Now suppose this notional measurement τ_S indicates that there is some apparatus $\mathcal{A}$ that exists in the vicinity of a spatial location z_0 at time t_i as depicted in figure 4.7. This means that prior to t_i, the apparatus will have interacted with many photons so

[41]In this section we will assume that all states are normalized.

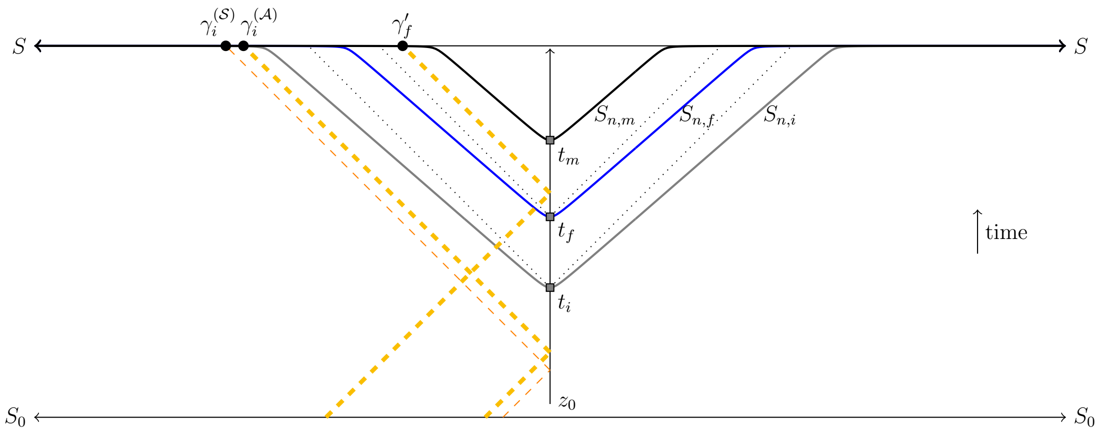

Figure 4.7: Depicts an experiment where the state of some photons $\gamma_i^{(\mathcal{S})}$ and $\gamma_i^{(\mathcal{A})}$ on the spacelike hypersurface S determines the initial conditions of an experimental setup of a particle $\mathcal{S}$ and apparatus $\mathcal{A}$ in the vicinity of the spacetime location (t_i, z_0). The state of the photons γ_f' on the spacelike hypersurface S determines the final state of the apparatus $\mathcal{A}$ after the particle $\mathcal{S}$ has finished interacting with it from time t_f onwards so that the apparatus at time t_m displays a definite measurement outcome. It is assumed that no incoming photons have become entangled with the experiment after the $\gamma_i^{(\mathcal{S})}$ and $\gamma_i^{(\mathcal{A})}$ photons and before the γ_f' photons have become entangled with the experiment.

that the photon detections on S outside the light cone of $y_i = (t_i, z_0)$ will indicate via Kent's stress-energy beables that there is some apparatus in the vicinity of z_0 that we can identify as $\mathcal{A}$. We further assume that Kent's stress-energy beables determine that $\mathcal{A}$ is in a state $|a\rangle$ which encapsulates among other things the measurement parameters of the apparatus. We also suppose that the information in τ_S indicates that there is a particle $\mathcal{S}$ at time t_i in a state $|s\rangle$ that is heading towards the apparatus so that it will interact with it. This means that all the possible[42] simultaneous $\hat{T}_S$-eigenstates on S whose simultaneous $\hat{T}_S$-eigenvalues agree with τ_S for all $x \in S^1(y_i)$[43] are such that their simultaneous $\hat{T}_S$-eigenvalues within the light cone of y_i indicate that at some time after t_i, the particle $\mathcal{S}$ would have interacted with the apparatus $\mathcal{A}$.

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[42]i.e. possible given $|\Psi_0\rangle$ and the Born rule selection criterion.

[43]Recall from page 125 that $S^1(y_i)$ is the subset of S that is outside the light cone of y_i.

At this point, it will be helpful to define a *simultaneous $\hat{T}_S$-eigenstate of a subregion U* of S to be a state $|\Psi_U\rangle \in H_U$ which is an eigenstate of $\hat{T}_S(u)$ for every $u \in U$, where H_U is the Hilbert space of states describing U. We will assume that we can write any state of H_S as a superposition of states of the form $|\Psi_U\rangle\, |\Psi_{S\setminus U}\rangle$, where $|\Psi_U\rangle \in H_U$ is a simultaneous $\hat{T}_S$-eigenstate of the subregion U of S, and $|\Psi_{S\setminus U}\rangle \in H_{S\setminus U}$ is a simultaneous $\hat{T}_S$-eigenstate of the subregion $S \setminus U$ of S.[44] We will accordingly let $|\gamma_i^{(S^1(y_i))}\rangle$ denote the simultaneous $\hat{T}_S$-eigenstate of the subregion $S^1(y_i)$ of S with simultaneous $\hat{T}_S$-eigenvalue $\tau_S(x)$ respectively for all $x \in S^1(y_i)$. Then any simultaneous $\hat{T}_S$-eigenstate over the whole of S whose simultaneous $\hat{T}_S$-eigenvalue agrees with τ_S on the subregion $S^1(y_i)$ will be expressible as $|\gamma_i^{(S^1(y_i))}\rangle\, |\Psi_{S\setminus S^1(y_i)}\rangle$, and any such simultaneous $\hat{T}_S$-eigenstate will determine the apparatus $\mathcal{A}$ and the particle $\mathcal{S}$ to exist in states $|a\rangle$ and $|s\rangle$ respectively given the state $|\Psi_0\rangle$ of the initial hypersurface S_0. In figure 4.7, rather than depicting $|\gamma_i^{(S^1(y_i))}\rangle$, we depict the two components $|\gamma_i^{(\mathcal{S})}\rangle$ and $|\gamma_i^{(\mathcal{A})}\rangle$ of $|\gamma_i^{(S^1(y_i))}\rangle$ which are simultaneous $\hat{T}_S$-eigenstates for subregions $S_{\mathcal{S}}$ and $S_{\mathcal{A}}$ of $S^1(y_i)$ where the states of $\mathcal{S}$ and $\mathcal{A}$ are determined respectively, so that

$$|\gamma_i^{(S^1(y_i))}\rangle = |\gamma_i^{(\mathcal{S})}\rangle\, |\gamma_i^{(\mathcal{A})}\rangle\, |\Xi_i^{(S^1(y_i))}\rangle \tag{4.40}$$

where $|\Xi_i^{(S^1(y_i))}\rangle$ is a simultaneous $\hat{T}_S$-eigenstate over the remainder of $S^1(y_i)$, i.e. for the subregion $S^1(y_i) \setminus (S_{\mathcal{S}} \cup S_{\mathcal{A}})$. Note that it is not necessary that the state $|\gamma_i^{(\mathcal{S})}\rangle$ is caused by photons that have interacted directly with $\mathcal{S}$. Rather, it is sufficient that the photons have been reflected from some system $\mathcal{B}$, say, that has interacted with $\mathcal{S}$ for which $|s\rangle$ is a pointer state, and that enough photons from $\mathcal{B}$ have intersected S so as to distinguish its different pointer states.

[44] We effectively made this assumption in equation (4.31).

We also assume that there are no further interactions of $\mathcal{S}$ with photons that are registered on S until the particle has finished interacting with the apparatus $\mathcal{A}$. In making this assumption, we suppose that there is a time $t_f > t_i$ such that if the notional measurement on S had resulted in the outcome $\tau_S^{(j)}$ rather than τ_S where $\tau_S^{(j)}(x) = \tau_S(x)$ for all $x \in S^1(y_i)$, then for all $x \in S^1(y_f, y_i) = S^1(y_f) \setminus S^1(y_i)$,[45] we would be able to say of the mass-energy density $\tau_S^{(j)}(x)$ at x that it wasn't caused by a photon that had been reflected from $\mathcal{S}$ or from something that had become entangled with $\mathcal{S}$.[46]

We will assume that given that the notional measurement restricted to $S^1(y_i)$ is τ_S, there is sufficiently little interaction of the composite system $\mathcal{S}+\mathcal{A}$ with its environment so that we can assume it evolves unitarily in accordance with the Schrödinger equation. Therefore, given that the apparatus $\mathcal{A}$ is in a state $|a\rangle$ that encapsulates its parameter settings, and the particle $\mathcal{S}$ is on course to interact with $\mathcal{A}$, we can express the state $|s\rangle$ of $\mathcal{S}$ as a superposition $|s\rangle = \sum_j c_j |s_j\rangle$ where $\{|s_j\rangle : j\}$ is the set of pointer states corresponding to a particular parameter setting of the apparatus $\mathcal{A}$ that are encapsulated in the state $|a\rangle$. As described on page 75, this means there are future states $|a_j\rangle$ of the apparatus $\mathcal{A}$ such that $\langle a_j | a_{j'} \rangle = 0$ for $j \neq j'$,[47] and such that the composite system $\mathcal{S} + \mathcal{A}$ will evolve according to the Schrödinger equation as

$$|s_j\rangle |a\rangle \to |s_j\rangle |a_j\rangle$$

[45]i.e $S^1(y_f, y_i)$ is the subset of S that is within the light cone of $y_i = (t_i, z_0)$ but outside the light cone of $y_f = (t_f, z_0)$.

[46]This assumption of our toy model assumes that photons can be treated as point-particles so that they have precise trajectories and that different photons will almost certainly be detected at different locations on S. In physical reality, we can't treat photons as point-particles, and so there will be some ambiguity about where the photons detected on S came from.

[47]Strictly speaking, $\langle a_j | a_{j'} \rangle \approx 0$ for $j \neq j'$, but we will assume $\langle a_j | a_{j'} \rangle = 0$ in this toy model in order to avoid undue complexity.

for each pointer state $|s_j\rangle$, and hence

$$|s\rangle\,|a\rangle \to \sum_j c_j\,|s_j\rangle\,|a_j\rangle\,. \tag{4.41}$$

We will also assume that given that the notional measurement restricted to $S^1(y_i)$ is τ_S, there is a time interval between t_f and $t_m > t_f$ during which photons will reflect off the apparatus $\mathcal{A}$ and ultimately be detected at spacetime locations on the hypersurface S that correspond to one of the definite measurement states $|a_f\rangle$ of the apparatus indicating that $\mathcal{S}$ is in the state $|s_f\rangle$. But whatever the notional measurement on S is, so long as it results in an outcome $\tau_S^{(j)}$ with $\tau_S^{(j)}(x) = \tau_S(x)$ for all $x \in S^1(y_i)$, then we're assuming that for all $x \in S^1(y_m, y_f)$,[48] we would be able to say of the mass-energy density $\tau_S^{(j)}(x)$ at x that it was caused by a photon that had been reflected from the apparatus $\mathcal{A}$ being in some state $|a_j\rangle$ rather than one of the other states.[49] We will let $S'_{\mathcal{A}}$ denote the subregion of $S^1(y_m, y_f)$ where photons coming from the apparatus arrive and hence determine which state the apparatus is in, and we will let $|\gamma'_j\rangle$ denote the simultaneous $\hat{T}_S$-eigenstate of the subregion $S'_{\mathcal{A}}$ of S that corresponds to the apparatus $\mathcal{A}$ being in state $|a_j\rangle$. We will also label the states on the subregion $S \setminus (S^1(y_i) \cup S'_{\mathcal{A}})$ as $|\Xi_j\rangle$ so that we can express the state $|\Psi_S\rangle = U_{SS_0}\,|\Psi_0\rangle$ as a superposition

$$|\Psi_S\rangle = b\,|\gamma_i^{(S^1(y_i))}\rangle \sum_j c_j\,|\gamma'_j\rangle\;|\Xi_j\rangle + \sum_{k\neq 0} b_k\,|\gamma_k^{(S^1(y_i))}\rangle\;|\Xi'_k\rangle \tag{4.42}$$

where the $|\gamma_k^{(S^1(y_i))}\rangle$ for $k \neq i$ are simultaneous $\hat{T}_S$-eigenstates of the subregion $S^1(y_i)$ whose simultaneous $\hat{T}_S$-eigenvalues restricted to $S^1(y_i)$ are distinct from τ_S, where

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[48]where $y_m = (t_m, z_0)$.

[49]This is again an assumption of our toy model, so it may not be true in more realistic models where there are many more object off which photons could reflect.

$|\Xi'_k\rangle$ are states of the subregion $S \setminus S^1(y_i)$, and where b is a complex numbers whose modulus squared gives the probability that the notional measurement on $S^1(y_i)$ will be τ_S, and likewise, the modulus squared of the b_k give the probabilities of the other possible notional measurements on $S^1(y_i)$.

Now we aim to show that within Kent's theory, we can calculate the probability the particle emerges from the measuring apparatus $\mathcal{A}$ in state $|s_f\rangle$ given that it enters $\mathcal{A}$ in state $|s\rangle$, and that this probability is the same as if one ignored S and just applied the Born Rule to $|s\rangle$ and $|s_f\rangle$.

In order to show this, let us choose a sequence of spacelike hypersurfaces $S_{n,i}$ which go through the spacetime location $y_i = (t_i, z_0)$ such that $\lim_{n\to\infty} S_{n,i} \cap S = S^1(y_i)$.[50] Let us assume that n is sufficiently large so that the photons described by $|\gamma_i^{(\mathcal{S})}\rangle$ and $|\gamma_i^{(\mathcal{A})}\rangle$ belong to $S_{n,i}$. The spacelike hypersurface $S_{n,i}$ and the photons being reflected from the vicinity of z_0 just before time t_i are depicted in figure 4.7.

With equation (4.42) in mind, we can express the quantum state $|\Psi_{n,i}\rangle = U_{S_{n,i},S_0}\,|\Psi_0\rangle$ of the spacelike hypersurface $S_{n,i}$ as a superposition

$$|\Psi_{n,i}\rangle = b\,|\gamma_i^{(\mathcal{S})}\rangle\ |\gamma_i^{(\mathcal{A})}\rangle\,|s\rangle\,|a\rangle\,|\xi_{n,i}\rangle + \cdots\,. \tag{4.43}$$

The $|\gamma_i^{(\mathcal{S})}\rangle\,|\gamma_i^{(\mathcal{A})}\rangle$ component of the first summand of (4.43) is extracted from the $|\gamma_i^{(S^1(y_i))}\rangle$ component of (4.42) which we can do because we're assuming $S_{n,i}$ overlaps with S in the subregion $S^1(y_i)$ corresponding to the states $|\gamma_i^{(\mathcal{S})}\rangle$ and $|\gamma_i^{(\mathcal{A})}\rangle$. Corre-

[50]We understand this limit first by giving $S^1(y_i)$ and $S_{n,i}$ topologies that makes them locally homeomorphic to Euclidean space. Then by saying there is this limit of hypersurfaces, we mean that every point $u \in S^1(y_i)$ has a neighborhood $U \subset S^1(y_i)$ for which there is an integer N such that $U \subset S_{n,i}$ for all $n \geq N$.

sponding to the region in the vicinity of y_i on $S_{n,i}$, we have the components $|s\rangle\,|a\rangle$ because we're assuming that the measurement of τ_S on $S^1(y_i)$ guarantees that $\mathcal{S}$ and $\mathcal{A}$ are in the states $|s\rangle$ and $|a\rangle$ respectively. The component $|\xi_{n,i}\rangle$ corresponds to the state of all the other regions of $S_{n,i}$ not determined by $|\gamma_i^{(\mathcal{S})}\rangle$, $|\gamma_i^{(\mathcal{A})}\rangle$, $|s\rangle$ or $|a\rangle$, and the ellipses correspond to the summation over k term of (4.42) suitably modified so that the summands are states on $S_{n,i}$ rather than S.

If we now define the projection $\pi_{n,i}$ corresponding to the measurement outcome $\tau_S(x)$ on $S_{n,i} \cap S$ as in equation (4.2), then

$$\pi_{n,i}\,|\Psi_{n,i}\rangle \approx b\,|\gamma_i^{(\mathcal{S})}\rangle\;|\gamma_i^{(\mathcal{A})}\rangle\,|s\rangle\,|a\rangle\,|\xi_{n,i}\rangle\,, \tag{4.44}$$

and the larger n is, the closer (4.44) will come to being an equality. The key thing to note about (4.44) is that the systems $\mathcal{S}$ and $\mathcal{A}$ are not entangled with each other or with the environment. We can therefore think of the measurement outcome of $\tau_S(x)$ on $S_{n,i} \cap S$ for sufficiently large n as specifying the initial states $|s\rangle$ and $|a\rangle$ of $\mathcal{S}$ and $\mathcal{A}$ before they interact. If n was too small, $S_{n,i} \cap S$ might not contain the subregion for which $\left|\gamma_i^{\mathcal{A}}\right\rangle$ is a state, in which case we could expect the $\mathcal{A}$-component of $\pi_{n,i}\,|\Psi_{n,i}\rangle$ to be entangled with the environment with different environment states being correlated with different parameter settings of $\mathcal{A}$. Likewise, we could expect the $\mathcal{S}$-component of $\pi_{n,i}\,|\Psi_{n,i}\rangle$ to be entangled with the environment with different environment states being correlated with different states of $\mathcal{S}$ if n was too small. In some situations, the rate at which $\mathcal{S}$ interacts with its immediate environment may be greater than the rate at which photons from the immediate environment of $\mathcal{S}$ are registered on S, in which case, it would not be possible to disentangle $\mathcal{S}$ from its environment in

$\pi_{n,i} |\Psi_{n,i}\rangle$. But in controlled experimental settings in which the system $\mathcal{S}$ is prepared to be in a definite state, it should be possible to disentangle $\mathcal{S}$ from its environment in $\pi_{n,i} |\Psi_{n,i}\rangle$. So in this situation, to extract the state of a system $\mathcal{S}$ in the vicinity of z_0 at time t_i from $|\Psi_0\rangle$ and τ_S, we need to take a hypersurface $S_{n,i}$ for sufficiently large n that goes through $y_i = (t_i, z_0)$. The state of $\mathcal{S}$ in the vicinity of y_i will then be the disentangled normalized $\mathcal{S}$-component of $\pi_{n,i} |\Psi_{n,i}\rangle = \pi_{n,i} U_{S_{n,i},S_0} |\Psi_0\rangle$. So in the case of (4.43), the disentangled normalized $\mathcal{S}$-component of $\pi_{n,i} |\Psi_{n,i}\rangle$ will be $|s\rangle$.

Now given that the system $\mathcal{S}$ is in state $|s\rangle$ and that $|s_f\rangle$ was one of the possible measurement outcome states for $\mathcal{S}$, according to standard quantum mechanics, the Born rule would predict that the measurement outcome state $|s_f\rangle$ occurs with a probability of $|\langle s|s_f\rangle|^2$. We will now show that we can obtain this probability in Kent's theory as well. To do this, we recall that in standard quantum theory, if we define the operator $[\psi] = |\psi\rangle\langle\psi|$ for some state $|\psi\rangle$ of a system, then when the system is in some initial state $|\chi\rangle$, the Born Rule implies that $\langle\chi|[\psi]|\chi\rangle = P(\psi|\chi)$, where $P(\psi|\chi)$ is the probability that the system will be found to be in state $|\psi\rangle$ given that it was initially in state $|\chi\rangle$. But by (2.2), $\langle\chi|[\psi]|\chi\rangle$ is just the expectation $\langle[\psi]\rangle_\chi$ of $[\psi]$ when $[\psi]$ is treated as an observable.

Now in equation (4.6), we saw how to calculate the expectation value $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ of the observable $\hat{T}^{\mu\nu}(y)$ given the notional measurement τ_S on S outside the light cone of y. This suggests that the expectation value of any observable $\hat{O}$ defined at spacetime location $y_i = (t_i, z_0)$ given the notional measurement τ_S on S outside the light cone of y_i is going to be

$$\langle \hat{O} \rangle_{\tau_S} = \lim_{n\to\infty} \frac{\langle \Psi_{n,i} | \pi_{n,i} \hat{O} | \Psi_{n,i} \rangle}{\langle \Psi_{n,i} | \pi_{n,i} | \Psi_{n,i} \rangle}. \tag{4.45}$$

By (4.44), $\lim_{n\to\infty} \langle \Psi_{n,i} | \pi_{n,i} | \Psi_{n,i} \rangle = |b|^2$, and so taking $\hat{O}$ to be $[s_f]$ at spacetime location y_i we have

$$\langle [s_f] \rangle_{\tau_S} = \frac{|b|^2 |\langle s_f | s \rangle|^2}{|b|^2} = |\langle s_f | s \rangle|^2.$$

Thus, Kent's conditional expectation $\langle [s_f] \rangle_{\tau_S}$ at spacetime location y_i gives us the same probability $|\langle s_f | s \rangle|^2$ for a particle transitioning from state $|s\rangle$ to state $|s_f\rangle$ as in standard quantum theory.

We can also use Kent's conditional expectation to show that at time t_m, $|s_f\rangle$ occurs with probability 1 given the notional measurement τ_S on S outside the light cone of $y_m = (t_m, z_0)$ has $\left|\gamma'_f\right\rangle$ as a component . To see this, first note that since we are assuming that between times t_i and t_f, $|s\rangle\,|a\rangle$ evolves according to (4.41), we can apply $U_{S_{n,f},S_{n,i}}$ to (4.43) (where $S_{n,f}$ is one of the spacelike hypersurfaces that goes through y_f as depicted in figure 4.7) to get

$$|\Psi_{n,f}\rangle = b\,|\gamma_i^{(\mathcal{S})}\rangle\;|\gamma_i^{(\mathcal{A})}\rangle \sum_j c_j\,|s_j\rangle\,|a_j\rangle\,|\xi_{n,f}\rangle + \cdots . \tag{4.46}$$

In (4.47), the component $|\xi_{n,f}\rangle$ corresponds to the state of all the other regions of $S_{n,f}$ not determined by $|\gamma_i^{(\mathcal{S})}\rangle$, $|\gamma_i^{(\mathcal{A})}\rangle$, or the state of $\mathcal{S}$ and $\mathcal{A}$ in the vicinity of $y_f = (t_f, z_0)$, and the ellipses correspond to the ellipses of (4.43) to which $U_{S_{n,f},S_{n,i}}$ has been applied.

Since we need to calculate Kent's conditional expectation of $[s_f]$ at spacetime location y_m, we need to apply $U_{S_{n,m},S_{n,f}}$ to (4.47), where $S_{n,m}$ is one of the spacelike hypersurfaces that goes through y_m as depicted in figure 4.7. To do this, we continue to assume that no photons interact with $\mathcal{S}$ between times t_f and t_m. However, we do assume that photons will interact with the apparatus, and for large enough n, the $S_{n,m}$ hypersurfaces will contain the subregion $S'_{\mathcal{A}}$ of $S^1(y_m)$ where the photons coming from the apparatus arrive, and so for each j, the $\left|\gamma'_j\right\rangle$-state that corresponds to the $|a_j\rangle$-state of the apparatus $\mathcal{A}$ and which forms a component of one of the summands of $|\Psi_S\rangle$ as shown in equation (4.42) will also appear $|\Psi_{n,m}\rangle$. It therefore follows that

$$|\Psi_{n,f}\rangle = b\,|\gamma_i^{(\mathcal{S})}\rangle\;|\gamma_i^{(\mathcal{A})}\rangle \sum_j c_j\,|s_j\rangle\,|a_j\rangle\,\left|\gamma'_j\right\rangle|\xi_{n,m}\rangle + \cdots\,. \tag{4.47}$$

where the component $|\xi_{n,m}\rangle$ corresponds to the state of all the regions of $S_{n,m}$ not determined by $|\gamma_i^{(\mathcal{S})}\rangle$, $|\gamma_i^{(\mathcal{A})}\rangle$, $S'_{\mathcal{A}}$, or the state of $\mathcal{S}$ and $\mathcal{A}$ in the vicinity of $y_m = (t_m, z_0)$, and the ellipses correspond to the ellipses of (4.47) to which $U_{S_{n,m},S_{n,f}}$ has been applied. Since we are assuming that the notional measurement restricted to $S^1(y_m)$ will correspond to the apparatus being in the definite measurement state $|a_f\rangle$, then defining the projection corresponding to the measurement outcome τ_S on $S_{n,m}\cap S$ as in equation (4.2), we will have

$$\pi_{n,m}\,|\Psi_{n,m}\rangle \approx bc_f\,|\gamma_i^{(\mathcal{S})}\rangle\;|\gamma_i^{(\mathcal{A})}\rangle\,|s_f\rangle\,|a_f\rangle\,\left|\gamma'_f\right\rangle|\xi_{n,m}\rangle \tag{4.48}$$

and the larger n is, the closer (4.48) will come to being an equality.

We can now use (4.7) to calculate Kent's conditional expectation of $\langle[s_f]\rangle_{\tau_S}$ at spacetime location y_m. By (4.48), we have $\lim_{n\to\infty}\langle\Psi_{n,m}|\pi_{n,m}|\Psi_{n,m}\rangle = |b|^2|c_f|^2$, and $\lim_{n\to\infty}\langle\Psi_{n,m}|[s_f]\pi_{n,m}|\Psi_{n,m}\rangle = |b|^2|c_f|^2$, and so by (4.45), at spacetime location y_m we have

$$\langle[s_f]\rangle_{\tau_S} = 1,$$

and so according to Kent's theory, $\mathcal{S}$ will be in state $|s_f\rangle$ by time t_m.

Also note that we can typically expect the $|\gamma_i^{(\mathcal{S})}\rangle$-state to be independent of the $|\gamma_i^{(\mathcal{A})}\rangle$-state. Therefore, since $|\gamma_i^{(\mathcal{A})}\rangle$ will determine the measurement choice, and since $|\gamma_i^{(\mathcal{S})}\rangle$ determines the initial state of the particle, we can expect the state of the particle to be independent of the measurement choice in Kent's theory. Thus, we can fulfil one of the necessary criteria (i.e. criterion 3) for PI to be a well-defined notion.

We can also choose a set of λ so that criterion 4 holds. To do this, we consider equation (4.40) which presupposes there is a subregion $S_{\mathcal{S}}$ of $S^1(y_i)$ which determines the state $\mathcal{S}$ and another non-overlapping subregion $S_{\mathcal{A}}$ that determines the state $\mathcal{A}$. In general, we wouldn't be able to make this association between subregions of $S^1(y_i)$ and states of $\mathcal{S}$ and $\mathcal{A}$ – after all, $\mathcal{A}$ might not even exist. But we should be able to make this association if an appropriate choice for the state of the remainder of $S^1(y_i)$ is made. In (4.40), $|\Xi_i^{(S^1(y_i))}\rangle$ is able to serve this role. We can then suppose that $|\gamma_i^{(\mathcal{S})}\rangle$ is from a basis $\Lambda_{\mathcal{S}} = \{\, |\gamma_{i,1}^{(\mathcal{S})}\rangle\,, |\gamma_{i,2}^{(\mathcal{S})}\rangle\,, \ldots\}$ of states that describe all the states of the subregion $S_{\mathcal{S}}$ of $S^1(y_i)$ corresponding to $\mathcal{S}$ that together with $|\Xi_i^{(S^1(y_i))}\rangle$ and $|\Psi_S\rangle$ determine $\mathcal{S}$ to be in the state $|s\rangle$. We could then take the λ of the system $\mathcal{S}$ in criterion 4 to be one of the basis states in $\Lambda_{\mathcal{S}}$. Given that the interpretation of the states in $\Lambda_{\mathcal{S}}$ presupposes $|\Xi_i^{(S^1(y_i))}\rangle$, we would take the probability p_λ for $\lambda = |\gamma_i^{(\mathcal{S})}\rangle$ to be the probability that the notional measurement on S given $|\Xi_i^{(S^1(y_i))}\rangle$ and $|\Psi_S\rangle$ agreed with the energy-density values specified by $|\gamma_i^{(\mathcal{S})}\rangle$ on the subregion $S_{\mathcal{S}}$ of $S^1(y_1)$ that $|\gamma_i^{(\mathcal{S})}\rangle$ describes. If we let $\{\, |Z_1\rangle\,, |Z_2\rangle\,, \ldots\}$ be a basis of simultaneous $\hat{T}_S$-eigenstates for the subregion $(S \setminus S^1(y_1)) \cup S_{\mathcal{A}}$ so that states of the form $|\gamma_{i,l}^{(\mathcal{S})}\rangle\, |\Xi_i^{(S^1(y_i))}\rangle\, |Z_k\rangle$ will be simultaneous $\hat{T}_S$-eigenstates for the whole of S, then the formula for the probability

p_λ will be

$$p_\lambda = \frac{\sum_k |\langle\Psi_S|\gamma_i^{(\mathcal{S})}\rangle\ |\Xi_i^{(S^1(y_i))}\rangle\ |Z_k\rangle\,|^2}{\sum_{k,l} |\langle\Psi_S|\gamma_{i,l}^{(\mathcal{S})}\rangle\ |\Xi_i^{(S^1(y_i))}\rangle\ |Z_k\rangle\,|^2}. \tag{4.49}$$

We can then show that a version of EA analogous to (EA) on page 181 holds. To express EA in this context, we let $O_\mathcal{S}$ be the observable that returns j if the system $\mathcal{S}$ is measured to be in the state $|s_j\rangle$, and for $\lambda = |\gamma_i^{(\mathcal{S})}\rangle$, we let $P_\lambda^{|s\rangle}(O_\mathcal{S} = j)$ denote the probability that $O_\mathcal{S} = j$ given that the hypersurface is in state $|\Psi_S\rangle$ before the notional energy-density measurement on S, and that this measurement result is only determined up to the state $|\gamma_i^{(\mathcal{S})}\rangle\ |\Xi_i^{(S^1(y_i))}\rangle$ on the subregion $S^1(y_1)\backslash S_\mathcal{A}$ which nevertheless ensures the system $\mathcal{S}$ is in the state $|s\rangle$ before it interacts with the apparatus $\mathcal{A}$. Then to calculate $P_\lambda^{|s\rangle}(O_\mathcal{S} = j)$, we will need to sum over the probabilities that the notional energy density measurement has a $|\gamma'_j\rangle$ component[51] in its simultaneous $\hat{T}_S$-eigenstate for each state in the basis of states $\{\,|Z'_1\rangle\,, |Z'_2\rangle\,, \ldots\}$ corresponding to the subregion $S \setminus (S^1(y_i) \cup S'_\mathcal{A})$. Accordingly, if we slightly redefine our notation by absorbing $|\gamma_i^{(\mathcal{A})}\rangle$ into the $|\gamma'_l\rangle$, we will find that

$$P_\lambda^{|s\rangle}(O_\mathcal{S} = j) = \frac{\sum_k |\langle\Psi_S|\gamma_i^{(\mathcal{S})}\rangle\ |\Xi_i^{(S^1(y_i))}\rangle\ |\gamma'_j\rangle\ |Z'_k\rangle\,|^2}{\sum_{k,l} |\langle\Psi_S|\gamma_i^{(\mathcal{S})}\rangle\ |\Xi_i^{(S^1(y_i))}\rangle\ |\gamma'_l\rangle\ |Z'_k\rangle\,|^2}.$$

Also note that with this slight redefinition of $|\gamma'_l\rangle$, equation (4.42) becomes

$$|\Psi_S\rangle = b\,|\gamma_i^{(\mathcal{S})}\rangle\ |\Xi_i^{(S^1(y_i))}\rangle \sum_l c_l\,|\gamma'_l\rangle\,|\Xi_l\rangle + \cdots\,.$$

We will therefore be able to express $|\Xi_l\rangle$ in terms of the basis $\{\,|Z'_1\rangle\,, |Z'_2\rangle\,, \ldots\}$ so that

$$|\Xi_l\rangle = \sum_k d_{lk}\,|Z'_k\rangle$$

[51]To avoid undue complexity, we assume there is only one $|\gamma'_j\rangle$-state for each $|a_j\rangle$-state of the apparatus $\mathcal{A}$.

with $\sum_k |d_{lk}|^2 = 1$ since we are assuming all states are normalized. Therefore

$$P_\lambda^{|s\rangle}(O_{\mathcal{S}} = j) = \frac{|b|^2|c_j|^2 \sum_k |d_{jk}|^2}{|b|^2 \sum_{k,l} |c_l|^2 |d_{lk}|^2} = |c_j|^2.$$

Therefore

$$P_\lambda^{|s\rangle}(O_{\mathcal{S}} = j) = |\langle s|s_j\rangle|^2 = P^{|s\rangle}(O_{\mathcal{S}} = j).$$

Moreover, since $\sum_\lambda p_\lambda = 1$ we have

$$\sum_\lambda p_\lambda P_\lambda^{|s\rangle}(O_{\mathcal{S}} = j) = P^{|s\rangle}(O_{\mathcal{S}} = j)$$

which is analogous to the EA formula on page 181.

4.7 Kent's Theory and Parameter Independence

In the previous section, we saw how we can generalize Kent's beable $\langle \hat{T}^{\mu\nu}(y)\rangle_{\tau_S}$ to calculate conditional expectations $\langle \hat{O}\rangle_{\tau_S}$ for any observable $\hat{O}$ defined at a particular spacetime location (t_i, z_0), and that in the case of the observable $[s_f] = |s_f\rangle\langle s_f|$, this expectation yields the same probability as standard quantum mechanics for the outcome $|s_f\rangle$ given the initial state $|s\rangle$ of the system. We also saw that EA holds in Kent's theory, and to see this, it was necessary to calculate the probability $P_\lambda^{|s\rangle}(O_{\mathcal{S}} = j)$ appropriately conditioned on the energy-density measurement determined on a subregion of S.

Now the expectation $\langle [s_f]\rangle_{\tau_S}$ and the probability $P_\lambda^{|s\rangle}(O_{\mathcal{S}} = j)$ depend on just one observable for just one spacetime location. However, in order to consider whether PI holds, we need to consider two observables corresponding to two different spacetime locations. In order to do this, we need to make a further adaption to Kent's theory. In this section, we will describe this adaption and show that with it, Kent's theory allows us to calculate probabilities for Bell-type experiments, and that these probabilities are the same as in standard quantum theory. Since PI holds in standard quantum theory, a consequence of Kent's theory agreeing with standard quantum theory is that PI will also hold in Kent's theory.

So let's consider figure 4.8 which depicts a one-dimensional view of a Bell-type experiment. There is a left wing of the experiment located in the vicinity of z_L, and

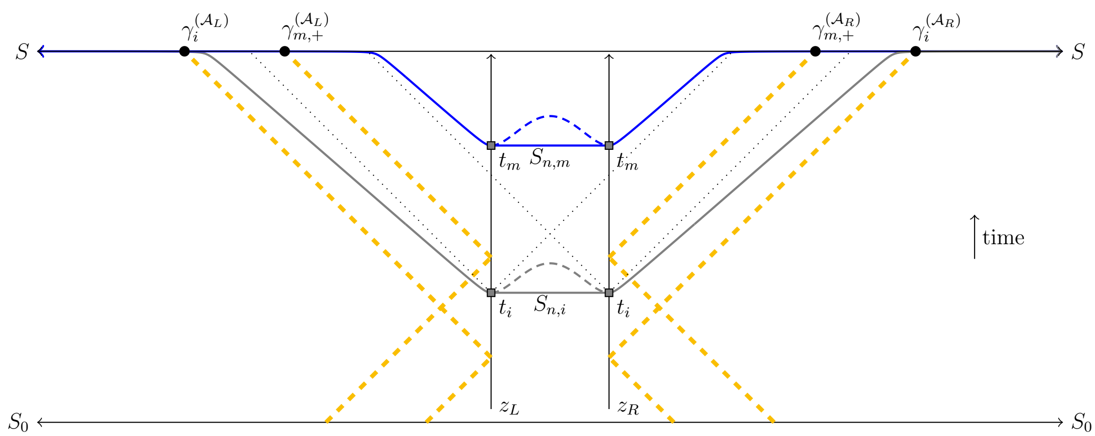

Figure 4.8: Depicts a Bell-type experiment where the state of some photons $\gamma_i^{(\mathcal{A}_L)}$ and $\gamma_i^{(\mathcal{A}_R)}$ on the spacelike hypersurface S determines the choice of measurement parameters of the left wing and right wing of the experiment respectively, and some photons $\gamma_{m,+}^{(\mathcal{A}_L)}$ and $\gamma_{m,+}^{(\mathcal{A}_R)}$ on the spacelike hypersurface S determine the measurement outcome of the experiment on the left wing and the right wing respectively. The dashed lines on the spacelike hypersurfaces $S_{n,m}$ and $S_{n,i}$ indicate other choices for the spacelike hypersurfaces, but they still lead to the same probability being calculated.

a right wing of the experiment located in the vicinity of z_R. Shortly before time t_i, photons interact with a Stern-Gerlach apparatus on the left wing and a Stern-Gerlach apparatus on the right wing, and some of these photons eventually intersect the spacelike hypersurface S so that there are subregions $S_{\mathcal{A}_L}$ and $S_{\mathcal{A}_R}$ of S such that if the notional measurement of the mass-energy density corresponds to simultaneous $\hat{T}_S$-eigenstates $|\gamma_i^{(\mathcal{A}_L)}\rangle$ and $|\gamma_i^{(\mathcal{A}_R)}\rangle$ respectively on these subregions, then this will be sufficient to determine the measurement parameters of the apparatuses on the left wing and the right wing of the experiment respectively.

Now in order to consider whether PI holds, we will need to adapt Kent's sequences of spacelike hypersurfaces so that they can be used to calculate conditional expectation values for observables that depend on two spacetime locations $y_L = (t_i, z_L)$ and $y_R = (t_i, z_R)$. We therefore require that sequences of spacelike hypersurfaces $S_{n,i}$ are chosen so that they all contain the spacetime locations y_L and y_R, and that in the limit, $\lim_{n\to\infty} S_{n,i}$ contains as much of $S^1(y_L)$ and $S^1(y_R)$ as possible, where as usual, $S^1(y)$ denotes the subset of S lying outside the light cone of y. Ultimately, this limit (unlike the limit of Kent's spacelike hypersurfaces) will not contain the whole of $S^1(y_L)$ or $S^1(y_R)$, but only serves to guarantee that we use as much of the information in S as possible in calculating the expectation values of observables at y_L and y_R. There will be some degree of freedom in what we choose for the spacelike hypersurface between y_L and y_R as depicted by the dashed line in figure 4.8. However, such freedom will have no effect on the probabilities calculated, because under the assumption that the spacelike hypersurface is very far into the future, there will be no choice of spacelike hypersurface in this region that would give us more information in S to condition on. Also, we recall that the stress-energy operators of the form $\hat{T}^{\mu\nu}(y)$ in the Tomonaga-Schwinger formulation of relativistic quantum physics are defined so that they are invariant under any perturbation of the spacelike hypersurface (so long as the hypersurface continues to contain y), so under the assumption that all physical observables will be ultimately expressible in terms of the stress-energy operators, the arbitrary choice of the spacelike hypersurfaces in regions that can't intersect with S will have no effect of the probabilities calculated.

On the spacelike hypersurface $S_{n,i}$, we assume that n is sufficiently large that enough of the subregions $S_{\mathcal{A}_L}$ and $S_{\mathcal{A}_R}$ are contained within $S_{n,i}$ so that the simultaneous $\hat{T}_S$-eigenstates $|\gamma_i^{(\mathcal{A}_L)}\rangle$ and $|\gamma_i^{(\mathcal{A}_R)}\rangle$ restricted to $S_{\mathcal{A}_L} \cap S_{n,i}$ and $S_{\mathcal{A}_R} \cap S_{n,i}$ are still able to determine the choice of measurement axes for the left and right wings of the experiment respectively. In order to avoid introducing too much extra notation, we will shrink the subregions $S_{\mathcal{A}_\mathcal{L}}$ and $S_{\mathcal{A}_\mathcal{R}}$ so that they are contained within $S_{n,i}$ for sufficiently large n, but we only shrink them slightly so that the mass-energy density measurement on them is still sufficient to determine the choice of measurement axes for the left and right wings of the experiment.[52]

Let us now assume that the axis of orientation of the right wing Stern-Gerlach apparatus makes an angle θ with the axis of the left wing apparatus. We also assume that there are two particles that together form a Bell-state

$$\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{s}}+\rangle_L\;|\hat{\mathbf{s}}-\rangle_R - \,|\hat{\mathbf{s}}-\rangle_L\;|\hat{\mathbf{s}}+\rangle_R). \tag{4.50}$$

We saw in footnote 18 on page 21 that a Bell state does not depend on the orientation of $\hat{\mathbf{s}}$, so without loss of generality, we can suppose that the $|\hat{\mathbf{s}}+\rangle_L$ and $|\hat{\mathbf{s}}-\rangle_L$ are

[52] For more realistic models, we wouldn't expect the mass-energy density measurement on $S_{n,i}$ to determine the choice of measurement axes with 100% certainty, since there will be a degree of overlap of the different $|\gamma_i^{(\mathcal{A}_L)}\rangle$ for different measurement choices (and likewise for the different $|\gamma_i^{(\mathcal{A}_R)}\rangle$). But this overlap will get smaller and smaller the more that photons interacting with the apparatus intersect $S_{n,i}$, and so the certainty of which measurement is being made will approach 100% as long as there is sufficient enough time from the time the measurement parameters are chosen to time t_i and as long as n is large enough so that there are enough photon interactions with the apparatus that intersect $S_{n,i}$. Nevertheless, the fact that we never reach 100% certainty should not worry us too much in the context of Kent's theory, since it just means that Kent's $\langle \hat{T}^{\mu\nu}(y)\rangle_{\tau_S}$-beables will be perturbed by a very small amount in the vicinity of the apparatus caused by the very small amount of overlap between the the different $|\gamma_i^{(\mathcal{A}_L)}\rangle$ for different measurement choices.

pointer states for the apparatus on the left-wing of the experiment. This means there will be a ready state $|a\rangle_L$ as well as two states $|a{+}\rangle_L$ and $|a{-}\rangle_L$ of the left wing apparatus such that

$$|\hat{\mathbf{s}}\pm\rangle_L \, |a\rangle_L \rightarrow |\hat{\mathbf{s}}\pm\rangle_L \, |a\pm\rangle_L \,.$$

As for the right wing of the experiment, we let $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}{+}\rangle_R$ and $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}{-}\rangle_R$ be pointer states for the apparatus so that there is a ready state $|a\rangle_R$ as well as two states $|a_\theta{+}\rangle_R$ and $|a_\theta{-}\rangle_R$ of the right wing apparatus such that

$$|\hat{\boldsymbol{s}}_\theta\pm\rangle_R \, |a\rangle_R \rightarrow |\hat{\boldsymbol{s}}_\theta\pm\rangle_R \, |a_\theta\pm\rangle_R \,.$$

In a manner similar to equation (4.43), we can express $|\Psi_{n,i}\rangle = U_{S_{n,i},S_0} \, |\Psi_0\rangle$ as a superposition

$$|\Psi_{n,i}\rangle = \frac{b}{\sqrt{2}}\big(|\hat{\mathbf{s}}{+}\rangle_L \, |\hat{\mathbf{s}}{-}\rangle_R - |\hat{\mathbf{s}}{-}\rangle_L \, |\hat{\mathbf{s}}{+}\rangle_R \big) \, |a\rangle_L \, |a\rangle_R \, |\gamma_i^{(\mathcal{A}_L)}\rangle \, |\gamma_i^{(\mathcal{A}_R)}\rangle \, |\xi_{n,i}\rangle + \cdots \,. \tag{4.51}$$

where $|\xi_{n,i}\rangle$ corresponds to the state of the subregion of $S_{n,i}$ not determined by $|\hat{\mathbf{s}}\pm\rangle_L \, |\hat{\mathbf{s}}\mp\rangle_R$, $|a\rangle_L$, $|a\rangle_R$, $|\gamma_i^{(\mathcal{A}_L)}\rangle$ or $|\gamma_i^{(\mathcal{A}_R)}\rangle$. As in equations (1.2a) and (1.2b), we have

$$|\hat{\mathbf{s}}{+}\rangle_R = \alpha_\theta \, |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}{+}\rangle_R + \beta_\theta \, |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}{-}\rangle_R \,,$$

$$|\hat{\mathbf{s}}{-}\rangle_R = \alpha_\theta \, |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}{-}\rangle_R - \beta_\theta \, |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}{+}\rangle_R \,,$$

where $\alpha_\theta = \cos(\theta/2)$, and $\beta_\theta = \sin(\theta/2)$. Substituting this into (4.51), we can express the state of the spacelike hypersurface $S_{n,i}$ that goes through the two particles at spacetime locations y_L and y_R as

$$\begin{aligned}|\Psi_{n,i}\rangle = \frac{b}{\sqrt{2}}\big(&\alpha_\theta\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R - \beta_\theta\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R - \alpha_\theta\,|\hat{\mathrm{s}}-\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R \\ &- \beta_\theta\,|\hat{\mathrm{s}}-\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R\big)\,|a\rangle_L\,|a\rangle_R\,|\gamma_i^{(\mathcal{A}_L)}\rangle\;|\gamma_i^{(\mathcal{A}_R)}\rangle\;|\xi_{n,i}\rangle + \cdots\,.\end{aligned} \tag{4.52}$$

We now let $\pi_{n,i}$ be the projection as defined in equation 4.2 that corresponds to the measurement outcome $\tau_S(x)$ on $S_{n,i}\cap S$. If we apply $\pi_{n,i}$ to $|\Psi_{n,i}\rangle$ as we did in (4.44) then we will get the approximation

$$\begin{aligned}\pi_{n,i}\,|\Psi_{n,i}\rangle \approx \frac{b}{\sqrt{2}}\big(&\alpha_\theta\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R - \beta_\theta\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R - \alpha_\theta\,|\hat{\mathrm{s}}-\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R \\ &- \beta_\theta\,|\hat{\mathrm{s}}-\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R\big)\,|a\rangle_L\,|a\rangle_R\,|\gamma_i^{(\mathcal{A}_L)}\rangle\;|\gamma_i^{(\mathcal{A}_R)}\rangle\;|\xi_{n,i}\rangle\,,\end{aligned} \tag{4.53}$$

and the larger n is, the closer (4.53) will come to being an equality, though in the case of our toy model where we treat photons as point particles, we can expect (4.53) to become an equality for sufficiently large n.

Now in the previous section, we calculated Kent's conditional expectation value of $[s_f]$ and argued that this would give the probability that the measurement outcome of the system $\mathcal{S}$ would be $|s_f\rangle$ given a particular mass-energy density measurement on $S^1(y_i)$. In the situation at hand in which we wish to know the probability of two measurements, we need to consider conditional expectation values of observables such as $[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R$ where the observable $[\hat{\mathrm{s}}+]_L = |\hat{\mathrm{s}}+\rangle_{L\,L}\langle\hat{\mathrm{s}}+|$ depends on spacetime location y_L, and where the observable $[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R = |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_{R\,R}\langle\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+|$ depends on spacetime location y_R. Since we are choosing the sequence of spacelike hypersurfaces $S_{n,i}$ so that both y_L and y_R belong to $S_{n,i}$, and since any two observables for locations that are spacelike separated commute, we can easily see what the eigenvalues for $[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R$ must be: they are 1 and 0, where the eigenvalue of 1 corresponds to all the states of $S_{n,i}$ in which the particle about to be measured by the left wing apparatus is in the $|\hat{\boldsymbol{s}}+\rangle_L$-state and in which the particle about to be measured by the right wing apparatus is in the $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R$-state, and where the eigenvalue of 0 corresponds to all the states of $S_{n,i}$ in which either the particle about to be measured by the left wing apparatus is in a pointer state of the apparatus that is not the $|\hat{\boldsymbol{s}}+\rangle_L$-state or the particle about to be measured by the right wing apparatus is a poitner state of the apparatus that is not the $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R$-state. It therefore follows from the definition of expectation that given the mass-energy density measurement on the subregion $S_{\mathcal{A}_L}$ is $|\gamma_i^{(\mathcal{A}_L)}\rangle$ and the mass-energy density measurement on the subregion $S_{\mathcal{A}_R}$ is $|\gamma_i^{(\mathcal{A}_R)}\rangle$, the probability that the left wing will be measured to be in state $|\hat{\boldsymbol{s}}+\rangle_L$ and that the right wing will be measured to be in $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R$ will be

$$\begin{aligned}\langle[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R\rangle_{\tau_S} &= \lim_{n\to\infty}\frac{\langle\Psi_{n,i}|\pi_{n,i}[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R|\Psi_{n,i}\rangle}{\langle\Psi_{n,i}|\pi_{n,i}|\Psi_{n,i}\rangle} \\ &= \frac{{}_R\langle\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+|\;{}_L\langle\hat{s}+|\overline{\frac{b}{\sqrt{2}}\beta_\theta}[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R\frac{b}{\sqrt{2}}\beta_\theta|\hat{s}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R}{|b|^2} \\ &= \frac{|\beta_\theta|^2}{2} = \frac{1}{2}\sin^2(\theta/2),\end{aligned} \tag{4.54}$$

where we have used (4.53), and this is the same as the probability that standard quantum mechanics predicts as given in equation (1.16). We therefore see that the mass-energy density measurement on S allows us to determine the state of the particles and the apparatus before the particles are measured in an EPR-Bohm type experiement, but the adaption I've made to Kent's model so that it can compute probabilities of the measurement outcomes of this experiement produces the same probabilities as standard quanutm mechanics, and since PI hold's in standard quantum mechanics, it must also hold Kent's adapted model as well.

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