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Chapter 1

The Appeal of the Many-World's Interpretation of Quantum Physics

In recent times, it has become increasingly common for popularisers of quantum physics to tell us that we need to let go of our naïve common sense understanding of reality. We're told we must replace this common sense understanding with something that at first seems very bizarre and counter-intuitive: a many-worlds account of

reality. In this chapter, I will try to explain what is meant by this many-worlds account of reality and why some theoretical physicists find it so appealing.

To begin with, I will give an overview of the Copenhagen interpretation and the hidden variables interpretation of quantum physics. This overview will be helpful since it will not only serve to introduce to the reader the notation of quantum theory, but it will also give us an opportunity to consider the problem with the Copenhagen interpretation and the hidden variables interpretation that the many-worlds interpretation seeks to resolve.

1.1 The Stern-Gerlach Experiment

Some of the key features of quantum physics are exhibited in the Stern-Gerlach experiment (see figure 1.1. In this experiment, silver atoms are heated in a furnace which randomly emerge from the furnace with various velocities. By aligning two plates with circular holes near the furnace, it is possible to select a subset of the emerging silver atoms having the same momentum to form a beam in one direction, the other silver atoms having been absorbed by the two plates. This beam of silver atoms is then directed between two magnets with the north pole of one magnet being aligned toward the south pole of the other magnet as shown in figure 1.1. Now

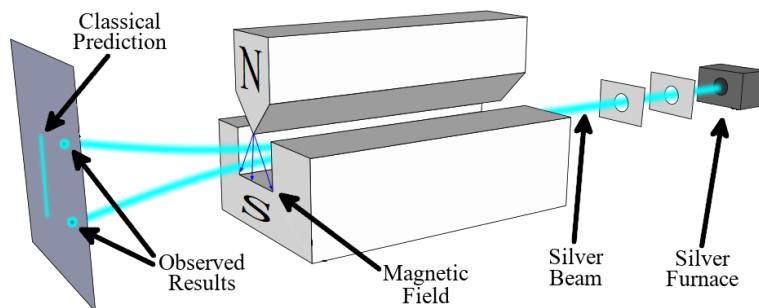


Figure 1.1: The Stern-Gerlach Experiment.¹

stern

silver atoms have a property somewhat analogous to the classical notion of angular momentum. For instance, a spinning top has angular momentum as shown in figure 1.2 Angular momentum is a vector, so it has direction and magnitude. In the case of a spinning top, the direction of the angular momentum would be parallel to the axis of rotation, pointing one way or the other depending on whether the rotation was clockwise or counterclockwise. The magnitude of the angular momentum would then be proportional to the angular velocity of the spinning top.

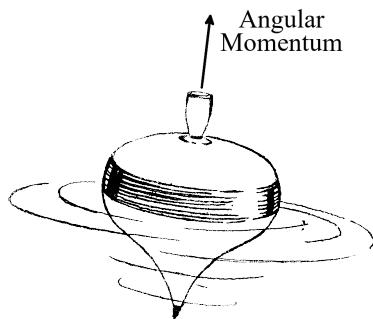


Figure 1.2: Angular Momentum of a Spinning Top.² spintop

Now if we tried to understand the angular momentum of a silver atom classically, we would expect the magnetic field of the two magnets to interact with the silver atom in a way that was determined by the relative direction of the silver atom's magnetic momentum compared to the direction of the magnetic field. Since we would expect the silver atom to have an entirely random angular momentum, we would expect it to be deflected by varying degrees either up or down in the direction of the magnetic field. Thus, if a detection screen were placed beyond the two magnets which the silver atoms would hit, we would expect there to be a whole continuum of possible locations

¹Original diagram drawn by Theresa Knott. Labeling was modified for use in this dissertation. This image is licensed under the Creative Commons Attribution-Share Alike 4.0 International license. Source: https://commons.wikimedia.org/wiki/File:Stern-Gerlach_experiment.svg

where the silver atoms would be detected. However, in reality, it is found that there are precisely two locations where the silver atoms hit the screen. It is as though the particles can spin either clockwise or anticlockwise, but that there is absolutely no variance in the angular speed at which they rotate. This is surprising. The angular momentum appears to be **quantized** in one of two directions, either parallel to the magnetic field or antiparallel to it.³ Corresponding to this quantization of angular momentum, we say that the atom is either in the spin up state or the spin down state with respect to the direction of the magnetic field.

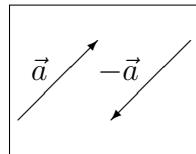


Figure 1.3: Meaning of antiparallel: the arrows in opposite directions are said to be antiparallel to one another.

antiparallel

If the direction of the magnetic field is implicitly understood, we write $|+\rangle$ and $|-\rangle$ for the spin up and spin down states of the atom respectively. We refer to the symbols $|+\rangle$ and $|-\rangle$ as **ket-vectors**, or simply as kets. We can think of the ket $|+\rangle$ for instance as shorthand for the proposition “the particle is in the spin up state.” If we knew this proposition to be true, we would know which of the two locations on the detection screen the particle would end up if it were to travel between the two magnets of the Stern-Gerlach apparatus. If we need to specify the spin with respect to a particular direction of the magnetic field, say in the \hat{a} -direction, we write the corresponding spin up and down states as $|\hat{a}+\rangle$ and $|\hat{a}-\rangle$. For convenience, we write

²Drawing by Pearson Scott Foresman, Public domain, via Wikimedia Commons. Labeling was added for use in this dissertation. Original: [https://commons.wikimedia.org/wiki/File:Top_\(PSF\).png](https://commons.wikimedia.org/wiki/File:Top_(PSF).png).

³See figure 1.3 or what is meant be antiparallel.

$\hat{a}+$ and $\hat{a}-$ respectively for the location that the particle would hit the detection screen.

The question then arises as to what happens when we rotate the magnetic field around the axis of the particle beam in the Stern-Gerlach experimental setup. It turns out that when we do this, the atoms are again detected in only one of two locations (see figure 1.4).

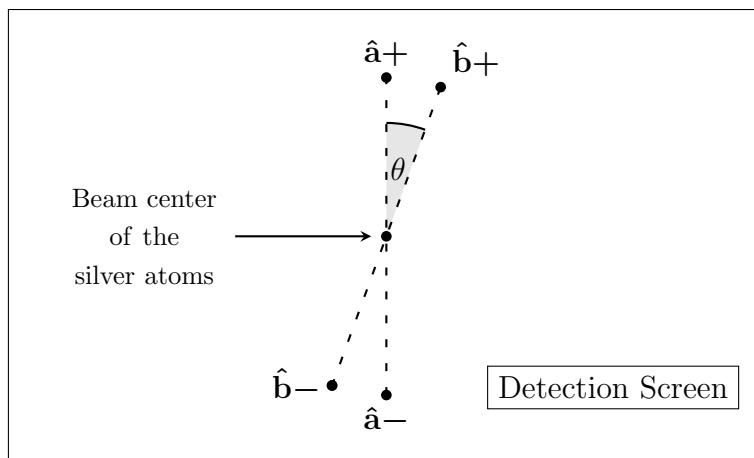


Figure 1.4: Locations of detections before and after rotating the magnetic field by an angle θ . Before rotation, the particles can be detected at either location $\hat{a}+$ or location $\hat{a}-$. After the rotation, particles can be detected at either location $\hat{b}+$ or $\hat{b}-$.

So suppose we knew the particle was in a spin state such that it was on course to arrive at location $\hat{a}+$ because we had previously directed it through another magnetic field in the \hat{a} -direction. For example, see figure 1.5 or how this might be done. In this experimental setup, the second magnetic field has been rotated by an angle of 90° with respect to the first magnetic field. But suppose we just rotated the second magnetic field by a very small angle θ with respect to the first magnetic field. Then

⁴Diagram by MJasK. This file is licensed under the Creative Commons Attribution-Share Alike 4.0 International license. Source: https://commons.wikimedia.org/wiki/File:Stern-Gerlach_Analyzer_Sequential_Series_E2.png.

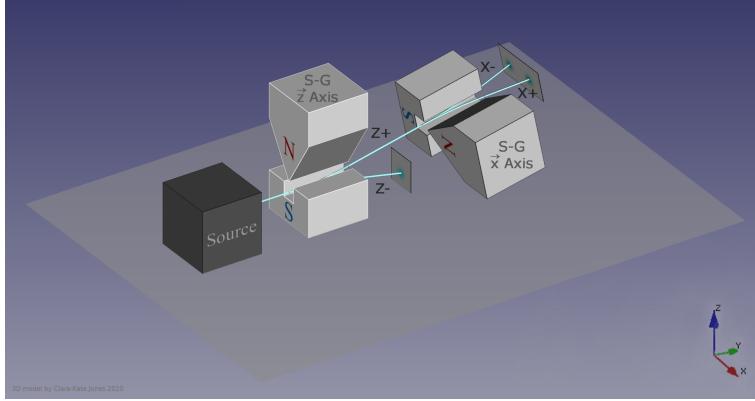


Figure 1.5: Two Stern-Gerlach experiments in sequence. By directing the beam of particles through one magnetic field first, the particles emerging in one of the two beams will be in a known spin state before they enter the second magnetic field.⁴

we would expect the particle now to arrive at a location $\hat{\mathbf{b}}+$ close by to $\hat{\mathbf{a}}+$ as shown in figure 1.4 And this is what we notice experimentally for the most part. However, occasionally, the particle will arrive at location $\hat{\mathbf{b}}-$. The frequency of this occurrence becomes less and less the less and less the magnetic field is rotated (i.e. the smaller θ is), so that if the magnetic field is not rotated at all, i.e. $\theta = 0$, the particle will always arrive at location $\hat{\mathbf{a}}+$. To capture the probabilistic nature of these outcomes, we use the bra-ket notation. Thus, if $|\psi\rangle$ stands for either the $|\hat{\mathbf{a}}+\rangle$ or the $|\hat{\mathbf{a}}-\rangle$ -state, and $|\chi\rangle$ stands for either the $|\hat{\mathbf{b}}+\rangle$ or the $|\hat{\mathbf{b}}-\rangle$ -state, then we define the bra-ket $\langle\psi|\chi\rangle \in \mathbb{C}$ to be a complex number⁵ that satisfies the **Born Rule**,^{bornrule} namely $|\langle\psi|\chi\rangle|^2$ is the probability that the particle will be found to be in state $|\psi\rangle$ given that we know that the particle is in state $|\chi\rangle$. For example, if $|\langle\psi|\chi\rangle|^2 = \frac{1}{4}$, and we performed the experiment a 1000 times with the particle initially prepared in the $|\chi\rangle$ -state, then

⁵With regards to the set of complex numbers \mathbb{C} , we will use the notation $i = \sqrt{-1}$. Complex conjugation will be denoted by an overline so that $\overline{x+iy} = x-iy$ for real numbers x and y . The modulus of a complex number $z = x+iy$ will then be given by $|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$. Since the defining property of $\langle\psi|\chi\rangle$ is that $|\langle\psi|\chi\rangle|^2$ is the probability that the particle will be found to be in state $|\psi\rangle$ given that we know that the particle is in state $|\chi\rangle$, we have to choose an arbitrary phase to fully determine $\langle\psi|\chi\rangle$.

we would expect the particle to be found in the $|\psi\rangle$ -state in around 250 runs of the experiment. We also insist that $\langle\psi|\chi\rangle = \overline{\langle\chi|\psi\rangle}$.⁶ We would thus expect $|\langle\hat{a}+|\hat{a}+\rangle|^2$ to be 1 and $|\langle\hat{a}-|\hat{a}+\rangle|^2$ to be 0. It will follow that $\langle\hat{a}-|\hat{a}+\rangle$ has to be 0, and that $\langle\hat{a}+|\hat{a}+\rangle$ has modulus 1, but by convention we choose $\langle\psi|\chi\rangle$ such that $\langle\psi|\psi\rangle$ is a real and positive number, in which case we must have $\langle\hat{a}+|\hat{a}+\rangle = 1$. If we now rotate the magnetic field by an angle θ as indicated in figure 1.4 the particle will be detected either at location $\hat{b}+$ or location $\hat{b}-$. We can then ask the question “given that the particle is in state $|\hat{a}+\rangle$, what is the probability that the particle will be found to be in state $|\hat{b}+\rangle$?”. According to the notation discussed above, this probability will be $|\langle\hat{b}+|\hat{a}+\rangle|^2$ where $\langle\hat{b}+|\hat{a}+\rangle$ is a complex number such that $\langle\hat{b}+|\hat{a}+\rangle = 1$ when $\theta = 0$ and $\langle\hat{b}+|\hat{a}+\rangle = 0$ when $\theta = 180^\circ$. We would likewise expect $\langle\hat{b}+|\hat{a}-\rangle = 0$ when $\theta = 0$ and $\langle\hat{b}+|\hat{a}-\rangle = 1$ when $\theta = 180^\circ$. Since $\cos 0^\circ = \sin 90^\circ = 1$ and $\cos 90^\circ = \sin 0^\circ = 0$, we might guess that in general $|\langle\hat{b}+|\hat{a}+\rangle| = |\cos(\theta/2)|$ and $|\langle\hat{b}+|\hat{a}-\rangle| = |\sin(\theta/2)|$. Experimentation shows us that this guess is correct. This suggests that we can express the state $|\hat{b}+\rangle$ in terms of the states $|\hat{a}+\rangle$ and $|\hat{a}-\rangle$.

We thus suppose that

$$|\hat{b}+\rangle = \alpha |\hat{a}+\rangle + \beta |\hat{a}-\rangle \quad \text{\{vectoradd1\}} \quad (1.1a)$$

$$|\hat{b}-\rangle = \bar{\alpha} |\hat{a}-\rangle - \bar{\beta} |\hat{a}+\rangle \quad \text{\{vectoradd2\}} \quad (1.1b)$$

for complex numbers $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, and we suppose that the bracket has the **linearity** property so that $\langle\psi|\hat{b}+\rangle = \alpha\langle\psi|\hat{a}+\rangle + \beta\langle\psi|\hat{a}-\rangle$ and $\langle\psi|\hat{b}-\rangle =$

⁶Note that this conditions implies time symmetry: the probability a particle transitions from a state $|\chi\rangle$ to a state $|\psi\rangle$ will be the same as the probability a particle transitions from the state $|\psi\rangle$ to the state $|\chi\rangle$. This is in accord with the observation that closed quantum systems are symmetric on time-reversal. This might at first seem surprising in the light of the fact that phenomena such as radioactive decay are not obviously time-symmetric. However, it turns out that this time asymmetry results from the quantum system not being closed. For more details, see Saverio Pascazio, “All you ever wanted to know about the quantum Zeno effect in 70 minutes,”

$\bar{\alpha} \langle \psi | \hat{\mathbf{a}}- \rangle - \bar{\beta} \langle \psi | \hat{\mathbf{a}}+ \rangle$ for any state $|\psi\rangle$. Then it will follow that $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{b}}- \rangle = 0$,⁷ and that $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{b}}+ \rangle = \langle \hat{\mathbf{b}}- | \hat{\mathbf{b}}- \rangle = 1$.⁸ If we then put $\alpha = \cos(\theta/2)$ and $\beta = \sin(\theta/2)$, it will follow that $|\langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}+ \rangle| = |\cos(\theta/2)|$ and $|\langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}- \rangle| = |\sin(\theta/2)|$,⁹ and so with these values for α and β we will have^{spintrans}

$$|\hat{\mathbf{b}}+\rangle = \cos(\theta/2) |\hat{\mathbf{a}}+\rangle + \sin(\theta/2) |\hat{\mathbf{a}}-\rangle, \quad \text{(1.2a)}$$

$$|\hat{\mathbf{b}}-\rangle = \cos(\theta/2) |\hat{\mathbf{a}}-\rangle - \sin(\theta/2) |\hat{\mathbf{a}}+\rangle. \quad \text{(1.2b)}$$

1.2 Hidden Variables

Now it is tempting to suppose that the expression of $|\hat{\mathbf{b}}+\rangle$ in terms of $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ merely represents our knowledge of the true spin state of a particle along the $\hat{\mathbf{a}}$ axis given our knowledge that it would be detected at location $\hat{\mathbf{b}}+$ with probability 1 should we decide to measure the particle's state along the $\hat{\mathbf{b}}$ -axis. If we were to make this supposition, there would be a fact of the matter, albeit unknown to us, concerning what spin state the particle would be found to be in were we to measure its spin along the $\hat{\mathbf{a}}$ -axis. And even though we might decide not to measure the spin of the particle along the $\hat{\mathbf{a}}$ -axis, there would still be this hidden fact about the

44th Symposium on Mathematical Physics on New Developments in the Theory of Open Quantum Systems, 2013, <https://doi.org/10.1142/S1230161214400071>, eprint: arXiv:1311.6645v1[quant-ph].

⁷To see this, by putting $|\psi\rangle = |\hat{\mathbf{b}}+\rangle$, we will have $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{b}}- \rangle = \bar{\alpha} \langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}- \rangle - \bar{\beta} \langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}+ \rangle$ by equation (1.1b). Since $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}- \rangle = \langle \hat{\mathbf{a}}- | \hat{\mathbf{b}}+ \rangle$ we have $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}- \rangle = \bar{\beta}$ by equation (1.1a), and likewise, since $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}+ \rangle = \langle \hat{\mathbf{a}}+ | \hat{\mathbf{b}}+ \rangle$, we have $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}+ \rangle = \bar{\alpha}$. Therefore $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{b}}- \rangle = \bar{\alpha}\bar{\beta} - \bar{\beta}\bar{\alpha} = 0$.

⁸To see this, by putting $|\psi\rangle = |\hat{\mathbf{b}}+\rangle$ and using equation (1.1a), we will have $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{b}}+ \rangle = \alpha \langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}+ \rangle + \beta \langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}- \rangle = \alpha\bar{\alpha} + \beta\bar{\beta} = |\alpha|^2 + |\beta|^2 = 1$. By a similar calculation, we also see $\langle \hat{\mathbf{b}}+ | \hat{\mathbf{b}}+ \rangle = 1$.

⁹To satisfy these criteria, α and β are only determined up to rotation by a complex number. Rotating a complex number $z \in \mathbb{C}$ just means multiplying it by a complex number λ of modulus 1 (i.e. $|\lambda| = 1$) to get λz . We would need to take into account this rotation factor if we considered the three-dimensional situation. Then, without loss of generality, $\alpha = \cos(\theta/2)$ and $\beta = e^{i\phi} \sin(\theta/2)$ where θ and ϕ are the polar and azimuthal angles respectively.

particle's spin in this direction. And given this supposition, since there would be no reason to suppose there was anything special about the $\hat{\mathbf{a}}$ -axis, it would then be reasonable to suppose that there were hidden facts about what spin direction the particle would be found to be in for every possible axis orientation. This would mean that a complete description of the particle's spin state would require an infinite list of outcomes for all the possible orientations we could configure the magnetic field of our Stern-Gerlach apparatus. Given this assumption, as well as the assumption that it is already known that the particle would be detected at $\hat{\mathbf{b}}+$, a complete description of the particle's state could be depicted as $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \dots\rangle$ or $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \dots\rangle$, etc. where the ellipses would range over one of the two possible measurement outcomes for every other magnetic field orientation. However, because we would never in practice be able to perform all these experiments, and since only one such experiment would be needed to alter this infinite list,¹⁰ nearly all of the entries in this infinite list would remain forever hidden. Hence, this would be an example of a **hidden variables** interpretation of quantum theory.

BellSection

1.3 Bell's Inequality

Now although a hidden variables interpretation seems rather intuitive, a problem arises when two spin particles are coupled together with each other. This problem is known as the **EPR paradox**.¹¹ To explain how this problem arises, we need to consider two identical fermionic particles that are coupled together. Fermions have the property that no two particles that are coupled together can be in exactly the

¹⁰In other words, it is assumed that directly measuring the particle will involve perturbing it so that its state will change.

¹¹For more details of this problem, see Jim J. Napolitano and J. J. Sakurai, *Modern Quantum Mechanics* (Pearson Education, 2013), 241–249.

same spin state. Thus, if we call our fermionic particles q_A and q_B , and suppose particle q_A was in the state $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-, \dots\rangle$, then on the assumption that making a measurement on particle q_A has no effect on the state of particle q_B , it would follow that particle q_B would be in the state $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+, \dots\rangle$. It is thus appropriate to refer to the spin directions describing q_A and q_B as **local hidden variables**. It seems very natural to assume these hidden variables are local. This assumption is a special case of **Einstein's locality principle**: For two spatially separated systems S_1 and S_2 , the real factual situation of the system S_2 should be independent of what is done to the system S_1 .¹²

We now suppose we have an experimental setup so that in each run of the experiment, we have two fermionic particles q_A and q_B , and particle q_A is sent to Alice who measures q_A 's spin in a direction of her choosing, and particle q_B is sent to Bob who measures q_B 's spin in a direction of his choosing. We assume that in each run of the experiment, Alice and Bob independently measure the spin of their particles along one of three possible directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and that Einstein's locality principle holds. Furthermore, we assume that in each run of the experiment, the outcome of Alice's measurement will be statistically independent of any of the other measurement outcomes for different runs of the experiment, and for any of the three axes she measures along, she will get a spin up outcome or a spin down outcome with equal probability of $\frac{1}{2}$. Likewise, we assume Bob's measurement outcomes are also similarly

¹²Einstein expressed this locality principle in his autobiographical notes: “But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with the system S_1 , which is spatially separated from the former.”^{GBX03}Albert Einstein, *Albert Einstein, Philosopher Scientist*, ed. P. A. Schilp (Evanston, Illinois: Library of Living Philosophers, 1949), p. 85.

independent between different runs of the experiment. We also assume that the $8 = 2^3$ states $|\hat{a}\pm, \hat{b}\pm, \hat{c}\pm\rangle_A$ exhaust all the possible states for Alice's particles that can be distinguished from one another by one of the three possible measurements she can make. Thus, Alice can distinguish between the $|\hat{a}+, \hat{b}+, \hat{c}+\rangle_A$ -state and the $|\hat{a}+, \hat{b}+, \hat{c}-\rangle_A$ -state by making a measurement along the \hat{c} -axis, though if she happened to make her measurement along the \hat{a} or \hat{b} -axis, she wouldn't be able to distinguish between these two states. But in principle, she can distinguish between these two states if she happens to make her measurement along the right axis, in this case the \hat{c} -axis. We similarly assume the states $|\hat{a}\pm, \hat{b}\pm, \hat{c}\pm\rangle_B$ exhaust all the possible states for Bob's particles that he can distinguish between, and we assume that if Alice and Bob measure the particle along the same axis, they will always obtain opposite results from one another. For instance, if Alice's particle is in state $|\hat{a}+, \hat{b}+, \hat{c}+\rangle_A$, then Bob's particle must be in state $|\hat{a}-, \hat{b}-, \hat{c}-\rangle_B$. Now suppose the experiment is run N times for large N ,¹³ and let N_i be the number of times particle q_A is in the i th state so that¹⁴ $N = \sum_{i=1}^8 N_i$ as shown in table 1.1

Table 1.1: Spin-components of particles q_A and q_B in the hidden variable theory

Population	Particle q_A	Particle q_B
N_1	$ \hat{a}+, \hat{b}+, \hat{c}+\rangle_A$	$ \hat{a}-, \hat{b}-, \hat{c}-\rangle_B$
N_2	$ \hat{a}+, \hat{b}+, \hat{c}-\rangle_A$	$ \hat{a}-, \hat{b}-, \hat{c}+\rangle_B$
N_3	$ \hat{a}+, \hat{b}-, \hat{c}+\rangle_A$	$ \hat{a}-, \hat{b}+, \hat{c}-\rangle_B$
N_4	$ \hat{a}+, \hat{b}-, \hat{c}-\rangle_A$	$ \hat{a}-, \hat{b}+, \hat{c}+\rangle_B$
N_5	$ \hat{a}-, \hat{b}+, \hat{c}+\rangle_A$	$ \hat{a}+, \hat{b}-, \hat{c}-\rangle_B$
N_6	$ \hat{a}-, \hat{b}+, \hat{c}-\rangle_A$	$ \hat{a}+, \hat{b}-, \hat{c}+\rangle_B$
N_7	$ \hat{a}-, \hat{b}-, \hat{c}+\rangle_A$	$ \hat{a}+, \hat{b}+, \hat{c}-\rangle_B$
N_8	$ \hat{a}-, \hat{b}-, \hat{c}-\rangle_A$	$ \hat{a}+, \hat{b}+, \hat{c}+\rangle_B$

¹³ N has to be large since a frequentist definition of probability is being assumed.

¹⁴The notation $\sum_{i=1}^8 N_i$ is shorthand for $N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8$.

We define $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+)$ to be the probability that Alice measures particle q_A to be at location $\hat{\mathbf{a}}+$ on her detection screen and Bob measures particle q_B to be at location $\hat{\mathbf{b}}+$ on his detection screen. We similarly define the probabilities for all other combinations of detection locations. It is relatively easy to calculate all these probabilities in terms of the values N_i from table 1.1¹⁵ or alternatively by simply measuring the frequency of these different outcomes for where Alice and Bob detect their particles. Although the values of N_i are unknown, on the assumption that there is a fact of the matter of which states in table 1.1btain, and on the assumption that the states to which the N_i correspond exhaust all the possible states for Alice's and Bob's particles, we can show that¹⁶

$$P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) \leq P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+). \quad \text{(1.3)} \quad \{\text{bellinequality}\}$$

This inequality is known as **Bell's inequality**, and it follows from Einstein's locality principle. However, it turns out that when this experiment is actually performed, Bell's inequality is violated. Because of this violation, the most natural conclusion to draw is that it is wrong to suppose that there any hidden variables describing possible spin measurement outcomes. But it also turns out that this violation of Bell's inequality is entirely predictable if we assume that when the two particles q_A and q_B are coupled together, everything that can be said about their spins is encoded in the so called **Bell state**:

$$\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A |\hat{\mathbf{a}}+\rangle_B). \quad \text{(1.4)} \quad \{\text{bell}\}$$

¹⁵e.g. $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{N_3+N_4}{N}$, $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) = \frac{N_2+N_4}{N}$, $P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) = \frac{N_3+N_7}{N}$

¹⁶This inequality follows since

$P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{N_3+N_4}{N} \leq \frac{N_2+N_4+N_3+N_7}{N} = P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+).$

This state means that if both Alice and Bob measure their particles along the measurement axis $\hat{\mathbf{a}}$, then with probability $\frac{1}{2}$, Alice will detect her particle at location $\hat{\mathbf{a}}+$ and Bob will detect his particle at location $\hat{\mathbf{a}}-$. Likewise, with probability $\frac{1}{2}$, Alice will detect her particle at location $\hat{\mathbf{a}}-$ and Bob will detect his particle at location $\hat{\mathbf{a}}+$. Prima facie, it looks like the Bell state depends on the direction of $\hat{\mathbf{a}}$. However it can be shown that for any other direction $\hat{\mathbf{b}}$,

$$\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A |\hat{\mathbf{a}}+\rangle_B) = \frac{1}{\sqrt{2}}(|\hat{\mathbf{b}}+\rangle_A |\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A |\hat{\mathbf{b}}+\rangle_B).^{17} \quad (1.5)$$

Thus, without loss of generality we can write the joint state of both Alice and Bob's particles as in equation (1.4, from which it would follow that

$$P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{1}{2} \sin^2(\theta/2)$$

where θ is the angle between the $\hat{\mathbf{a}}$ -axis and $\hat{\mathbf{b}}$ -axis.¹⁸ Then taking the angle between the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ -axes to be 90° , and the $\hat{\mathbf{c}}$ -axis to be at 45° to both the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ -axes, we would find that $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{1}{4}$ and $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) = 0.1464\dots$

¹⁷To see this, using the transformation rules given in equation (1.2 we have

$$\begin{aligned} \frac{1}{\sqrt{2}}(|\hat{\mathbf{b}}+\rangle |\hat{\mathbf{b}}-\rangle - |\hat{\mathbf{b}}-\rangle |\hat{\mathbf{b}}+\rangle) &= \frac{1}{\sqrt{2}}((\cos(\theta/2)|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)|\hat{\mathbf{a}}-\rangle)(\cos(\theta/2)|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)|\hat{\mathbf{a}}+\rangle) \\ &\quad - (\cos(\theta/2)|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)|\hat{\mathbf{a}}+\rangle)(\cos(\theta/2)|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)|\hat{\mathbf{a}}-\rangle)) \\ &= \frac{1}{\sqrt{2}}(\cos(\theta/2)|\hat{\mathbf{a}}+\rangle \cos(\theta/2)|\hat{\mathbf{a}}-\rangle - \cos(\theta/2)|\hat{\mathbf{a}}+\rangle \sin(\theta/2)|\hat{\mathbf{a}}+\rangle \\ &\quad + \sin(\theta/2)|\hat{\mathbf{a}}-\rangle \cos(\theta/2)|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)|\hat{\mathbf{a}}-\rangle \sin(\theta/2)|\hat{\mathbf{a}}+\rangle \\ &\quad - \cos(\theta/2)|\hat{\mathbf{a}}-\rangle \cos(\theta/2)|\hat{\mathbf{a}}+\rangle - \cos(\theta/2)|\hat{\mathbf{a}}-\rangle \sin(\theta/2)|\hat{\mathbf{a}}-\rangle \\ &\quad + \sin(\theta/2)|\hat{\mathbf{a}}+\rangle \cos(\theta/2)|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)|\hat{\mathbf{a}}+\rangle \sin(\theta/2)|\hat{\mathbf{a}}-\rangle) \\ &= \frac{1}{\sqrt{2}}((\cos^2(\theta/2) + \sin^2(\theta/2))|\hat{\mathbf{a}}+\rangle |\hat{\mathbf{a}}-\rangle \\ &\quad - (\cos^2(\theta/2) + \sin^2(\theta/2))|\hat{\mathbf{a}}-\rangle |\hat{\mathbf{a}}+\rangle) \\ &= \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle |\hat{\mathbf{a}}-\rangle - |\hat{\mathbf{a}}-\rangle |\hat{\mathbf{a}}+\rangle). \end{aligned}$$

¹⁸To see why this is, let $P_A(\hat{\mathbf{a}}+)$ be the probability that Alice would detect her particle at location $\hat{\mathbf{a}}+$ given that she is making a measurement along the $\hat{\mathbf{a}}$ -axis, and let $P_{BA}(\hat{\mathbf{b}}+ | \hat{\mathbf{a}}+)$ be the probability that Bob will detect his particle at location $\hat{\mathbf{b}}+$ given that he is making a measurement along the $\hat{\mathbf{b}}$ -axis and Alice has detected her particle at location $\hat{\mathbf{a}}+$. Given that the joint state of

and so Bell's inequality would be violated if we assumed that the probability of each outcome is determined by the Bell state (1.4).

1.4 The Copenhagen Interpretation

The experiment described in the previous section implies that the behavior of Alice's and Bob's particles can't be explained in terms of local hidden variables. But this experiment also calls into question the Copenhagen interpretation of quantum physics.

To explain the Copenhagen interpretation and what is problematic about it, suppose

Alice has a measurement device which we will denote by $\Lambda_{\hat{a}+}$ ^{Lambda} and which outputs the number 1 when her particle is detected at location $\hat{a}+$, and outputs 0 when her particle is detected at location $\hat{a}-$. Given that Alice knows that the state of both particles together is given by equation (1.4), she can work out the expectation value of her measurement $\langle \Lambda_{\hat{a}+} \rangle$ by summing up the product of each probability measurement outcome with the value of each measurement. This will give an expectation value of $\langle \Lambda_{\hat{a}+} \rangle = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$. More generally, if Alice had a measuring device Λ with N measurement outcome values $\lambda_1, \dots, \lambda_N$ and with respective probabilities p_1, \dots, p_N so that $\sum_{i=1}^N p_i = 1$, then the expectation $\langle \Lambda \rangle$ would be given by the formula

$$\langle \Lambda \rangle = \sum_{i=1}^N p_i \lambda_i. \quad \text{\{expectation\}} \quad (1.6)$$

Now given that there are no hidden variables and that equation (1.4) encodes everything that can be said about the spins of the two particle system, it is tempting to

the particles is given by equation (1.4), $P_A(\hat{a}+) = \frac{1}{2}$. But also note that if Alice has detected her particle at location $\hat{a}+$, then Bob's particle must be in state $|\hat{a}-\rangle$. From the Born Rule (see page 8 and equation (1.2a) it follows that

$$P_{BA}(\hat{b}+ | \hat{a}+) = |\langle \hat{b}+ | \hat{a}- \rangle|^2 = \sin^2(\theta/2).$$

Therefore,

$$P_{AB}(\hat{a}+; \hat{b}+) = P_A(\hat{a}+) P_{BA}(\hat{b}+ | \hat{a}+) = \frac{1}{2} \sin^2(\theta/2).$$

suppose that the expectation value $\langle \Lambda_{\hat{\mathbf{a}}+} \rangle$ tells us something objective about the system rather than just something about Alice's state of knowledge about the system. Given this assumption, there then arises the question of what happens when a measurement is made. According to the Copenhagen interpretation, when Bob makes his measurement, the quantum state collapses to a component of the quantum state corresponding to the measurement Bob makes. Thus, if Bob's measurement device $\Lambda_{\hat{\mathbf{a}}-}$ outputs the number 1, (i.e. Bob's particle is detected at location $\hat{\mathbf{a}}-$), then the state of the combined system would change accordingly as:

$$\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A |\hat{\mathbf{a}}+\rangle_B) \xrightarrow{\text{Collapse!!}} |\hat{\mathbf{a}}+\rangle_A |\hat{\mathbf{a}}-\rangle_B$$

If Bob makes his measurement first with his measurement device $\Lambda_{\hat{\mathbf{a}}-}$ outputting 1, then with probability 1 Alice's measurement device $\Lambda_{\hat{\mathbf{a}}+}$ will output 1. Hence, once Bob has made this measurement, then the expectation value for Alice's measurement will be $\langle \Lambda_{\hat{\mathbf{a}}+} \rangle = 1$. Thus, the expectation value for Alice's measurement device changes from $\frac{1}{2}$ to 1 when Bob makes his measurement.

Now the problem with this change in expectation value for Alice's measurement is that it will depend on whether Bob performs his measurement first or whether Alice performs her measurement first. But according to Einstein's theory of relativity, who performs their measurement first will depend on which inertial frame of reference one is in.¹⁹ Thus, if we are moving at one velocity, it may appear that Alice makes her

¹⁹In Special Relativity, an inertial frame of reference is a space-time coordinate system (t, x, y, z) in which all objects which have no forces acting on them have trajectories that are straight lines. Thus, we can move to another inertial frame by moving to a reference frame with constant velocity \mathbf{v} with respect to the first reference frame. In the case when $\mathbf{v} = (v, 0, 0)$, Einstein's theory of Special Relativity tells us that under such a “boost”, space-time coordinates will transform as $(t, \mathbf{x}) \rightarrow (t', x', y', z') = (\gamma(t - \frac{vx}{c^2}), \gamma(x - vt), y, z)$ where c is the speed of light and $\gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1}$.

measurement first, whereas if we are moving at another velocity, it may appear that Bob makes his measurement first. This suggests the expectation values for Alice and Bob's measuring devices will depend on which frame of reference we are in. However, Einstein's theory of relativity tells us that scalar quantities such as the expectation values for Alice and Bob's measuring devices should be independent of which inertial frame we are in.

Now many physicists would be loath to reject Einstein's theory of relativity. At the same time, many physicists are also convinced by the violation of Bell's inequality that there are no hidden variables for the spin states of particles, and hence they are convinced that the Bell state is not just a description of someone's epistemic state: rather it is a complete physical description of two coupled fermionic particles with regard to their spins. The way many physicists seek to resolve this tension between Einstein's theory of relativity and the violation of Bell's inequality is to deny the Copenhagen interpretation of quantum physics so that there is no quantum state collapse. But if one denies that there is any quantum state collapse and denies that there are any hidden variables, the question then arises of how is one meant to interpret the quantum state?

1.5 A preliminary consideration of the many-worlds interpretation

At this stage in the line of reasoning, it is too early to resort to a many-worlds interpretation of the Bell state where the first component corresponds to a world in which Alice detects her particle at location $\hat{a}+$ and Bob detects his particle at location $\hat{a}-$, and where the second component corresponds to a world in which Alice

detects her particle at location $\hat{\mathbf{a}}-$ and Bob detects his particle at location $\hat{\mathbf{a}}+$. The reason such an interpretation would be premature is because as mentioned on page 15 for any other axis $\hat{\mathbf{b}}$, the transformation rules in equation (1.2) imply that

$$\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle_A |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A |\hat{\mathbf{a}}+\rangle_B) = \frac{1}{\sqrt{2}}(|\hat{\mathbf{b}}+\rangle_A |\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A |\hat{\mathbf{b}}+\rangle_B).$$

Similarly, given the transformation rules in equation (1.2), we should resist the temptation to interpret a state of the form $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ as representing two worlds, one in which the particle is in the state $|\hat{\mathbf{a}}+\rangle$, and another in which the particle is in the state $|\hat{\mathbf{a}}-\rangle$. For according to equation (1.2a), the much more obvious interpretation is that this state just describes one world in which the particle is in the state $|\hat{\mathbf{b}}+\rangle$ where the angle between the $\hat{\mathbf{a}}$ and the $\hat{\mathbf{b}}$ axis is 90° .²⁰

In order to make a case for a many-worlds interpretation, we need to discuss decoherence theory. Decoherence theory considers how a system interacts with its environment, and it allows us to understand what kinds of measurements can be made on the system. In order to discuss decoherence theory and its relevance to the many-worlds interpretation, we first need to introduce the mathematical formalism of quantum mechanics.

1.6 The Preferred Basis Problem²¹

Now just because we can have an observable $\hat{\Lambda}$, there is no guarantee that there is a measuring device that could determine whether the system was in one of the eigenstates of $\hat{\Lambda}$. For instance, if $|\text{Cat Alive}\rangle$ is the physical state in which a cat is alive, and $|\text{Cat Dead}\rangle$ is the physical state in which the same cat is dead, then although

²¹See Maximilian Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Berlin: Springer-Verlag, 2007), 53–55.

there are measuring devices that can distinguish between the $|\text{Cat Alive}\rangle$ -state and the $|\text{Cat Dead}\rangle$ -state,²² there are no known measuring devices that can distinguish between the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ -state and the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$ -state. On the other hand, there are measuring devices that can distinguish between the $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ -state and the $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle - |\hat{\mathbf{a}}-\rangle)$ -state in a Stern-Gerlach experiment. Why the difference?

This question is at the heart of the preferred basis problem. As mentioned already, a basis is just a set of states via which all other states of the system can be uniquely expressed. For instance, we can express the state $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ uniquely as a sum of elements from the basis $\{|\text{Cat Alive}\rangle, |\text{Cat Dead}\rangle\}$, and thus we think of $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ as being a superposition of the $|\text{Cat Alive}\rangle$ and $|\text{Cat Dead}\rangle$ basis states. However, we can also uniquely express $|\text{Cat Alive}\rangle$ in terms of the basis $\{\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle), \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\}$.²³ Nevertheless, we would not tend to think of $|\text{Cat Alive}\rangle$ as being in a superposition of the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$ basis states. That is, we have a preference for the basis $\{|\text{Cat Alive}\rangle, |\text{Cat Dead}\rangle\}$ over the basis $\{\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle), \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\}$. As will be shown in section 1.8 decoherence theory offers a very elegant solution to the preferred basis problem.

²²For example, we assume that human beings can be thought of as such measuring devices.

²³i.e. $|\text{Cat Alive}\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\right)$.

1.7 Decoherence theory²⁴

Before we can show how decoherence theory solves the preferred basis problem, we will first need to look at decoherence theory in general. To understand what's going on in decoherence theory, there are a number of things we need to discuss, namely

1. composite systems
2. entanglement
3. density matrices and traces
4. coherence
5. partial traces and reduced density matrices
6. the von Neumann measurement scheme
7. decoherence

1.7.1 Composite Systems

First we need to consider **composite systems**. We thus assume there is a distinction between what is being measured and the rest of physical reality. We denote the system that is being measured by \mathcal{S} and the rest of physical reality by \mathcal{E} . We will refer to \mathcal{E} as the environment, and we will denote the composite system of \mathcal{S} and \mathcal{E} by \mathcal{U} . We will often indicate that \mathcal{U} is a composite of systems \mathcal{S} and \mathcal{E} by writing $\mathcal{U} = \mathcal{S} + \mathcal{E}$. The system \mathcal{S} could be something microscopic like a silver atom, or something much bigger such as a cat or even a planet.

* As mentioned in the introduction on page ??, sections marked with an asterisk may be challenging to readers who do not have a mathematics or physics background.

²⁴For more details see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, ch. 2.

Now suppose we have an observable (i.e. any Hermitian operator) $\hat{\Lambda}_{\mathcal{S}}$ that acts on the Hilbert space $H_{\mathcal{S}}$ of states of \mathcal{S} . As already mentioned, this means that we can find orthonormal eigenstates $|\psi_i\rangle_{\mathcal{S}}$ of $\hat{\Lambda}_{\mathcal{S}}$ and corresponding eigenvalues λ_i such that any state $|\psi\rangle_{\mathcal{S}} \in H_{\mathcal{S}}$ can be uniquely expressed as a sum $|\psi\rangle_{\mathcal{S}} = \sum_{i=1}^M \alpha_i |\psi_i\rangle_{\mathcal{S}}$. We will often include the subscript \mathcal{S} on the ket-vectors in order to make it clear that these ket-vectors belong to the Hilbert space $H_{\mathcal{S}}$. At other times we will omit these subscripts when it is clear what system we are talking about, but for the time being, we will keep these subscripts in place.

Now let us suppose we have a basis of normalized (but not necessarily orthonormal) states $\{|\chi_i\rangle_{\mathcal{E}} : i\}$ for the state space $H_{\mathcal{E}}$ of \mathcal{E} . In other words, every state $|\chi\rangle_{\mathcal{E}} \in H_{\mathcal{E}}$ can be uniquely expressed as a linear combination $|\chi\rangle_{\mathcal{E}} = \sum_{i=1}^N \beta_i |\chi_i\rangle_{\mathcal{E}}$. It is then assumed we will be able to express every state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ of the composite system \mathcal{U} as a linear combination

$$|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^M \sum_{j=1}^N \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}. \quad \text{(1.7)}$$

Thus, we assume there are no emergent physical properties describing the composite system \mathcal{U} that couldn't be expressed in terms of the component subsystems \mathcal{S} and \mathcal{E} . The Hilbert space $H_{\mathcal{U}}$ is endowed with the bra-ket $\langle \xi' | \xi \rangle_{\mathcal{U}}$ such that if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ and $|\xi'\rangle_{\mathcal{U}} = |\psi'\rangle_{\mathcal{S}} |\chi'\rangle_{\mathcal{E}}$, then $\langle \xi' | \xi \rangle_{\mathcal{U}} = \langle \psi' | \psi \rangle_{\mathcal{S}} \langle \chi' | \chi \rangle_{\mathcal{E}}$ where we have again used subscripts to indicate which Hilbert space the bra-ket corresponds to.

1.7.2 Entanglement

By defining the bra-ket on the Hilbert space $H_{\mathcal{U}}$ as we have done, we are making the assumption that if we define $P(\psi', \chi' | \psi, \chi)$ to be the probability the composite

system would be measured to be in the $|\psi'\rangle_{\mathcal{S}} |\chi'\rangle_{\mathcal{E}}$ -state given that the composite system was known to be in the $|\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ -state, then $P(\psi', \chi'|\psi, \chi) = P(\psi'|\psi)P(\chi'|\chi)$. A consequence of this assumption is that if the composite system is in the $|\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ -state, the probability of finding system \mathcal{S} to be in any particular state in $H_{\mathcal{S}}$ will be independent²⁵ of the state in $H_{\mathcal{E}}$ describing \mathcal{E} . For this reason, when the state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ describing the composite system \mathcal{U} is expressible as a product state $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$, we say that \mathcal{S} and \mathcal{E} are **not entangled** with one another. On the other hand, when the state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ of the composite system \mathcal{U} cannot be expressed as a product state, we say that \mathcal{S} and \mathcal{E} are **entangled**. For example, if $|\xi\rangle_{\mathcal{U}} = \frac{1}{\sqrt{2}}(|\psi_1\rangle_{\mathcal{S}} |\chi_1\rangle_{\mathcal{E}} + |\psi_2\rangle_{\mathcal{S}} |\chi_2\rangle_{\mathcal{E}})$ with $|\psi_1\rangle_{\mathcal{S}} \not\propto |\psi_2\rangle_{\mathcal{S}}$ and $|\chi_1\rangle_{\mathcal{E}} \not\propto |\chi_2\rangle_{\mathcal{E}}$,²⁶ then \mathcal{S} and \mathcal{E} would be entangled with one another.

Now given the observable $\hat{\Lambda}_{\mathcal{S}}$ acting on $H_{\mathcal{S}}$, we can naturally define the observable $\hat{\Lambda}_{\mathcal{U}}$ acting on $H_{\mathcal{U}}$ so that

$$\hat{\Lambda}_{\mathcal{U}} |\xi\rangle_{\mathcal{U}} = \sum_{i=1}^M \sum_{j=1}^N \gamma_{i,j} \hat{\Lambda}_{\mathcal{S}} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}} = \sum_{i=1}^M \sum_{j=1}^N \gamma_{i,j} \lambda_i |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}. \quad (1.8)$$

Just as in equation (??), for a given normalized state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$, the expectation value of the observable $\hat{\Lambda}_{\mathcal{U}}$ will be $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\xi} = \langle \xi | \hat{\Lambda}_{\mathcal{U}} | \xi \rangle_{\mathcal{U}}$. It is easy to see that if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ (i.e. \mathcal{S} and \mathcal{E} are not entangled), then $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\xi} = \langle \hat{\Lambda}_{\mathcal{S}} \rangle_{\psi}$.²⁷ Thus, when \mathcal{S} and \mathcal{E} are not entangled with one another, it is possible to say things about

²⁵Here we are using the standard probabilistic definition of independence: two events X and Y are independent if and only if $P(X \text{ and } Y \text{ occur}) = P(X \text{ occurs})P(Y \text{ occurs})$

²⁶Here the notation $|\psi_1\rangle_{\mathcal{S}} \propto |\psi_2\rangle_{\mathcal{S}}$ means there exists some α such that $|\psi_1\rangle_{\mathcal{S}} = \alpha |\psi_2\rangle_{\mathcal{S}}$, in which case $|\xi\rangle_{\mathcal{U}} = \frac{1}{\sqrt{2}} |\psi_2\rangle_{\mathcal{S}} (\alpha |\chi_1\rangle_{\mathcal{E}} + |\chi_2\rangle_{\mathcal{E}})$. Thus if $|\psi_1\rangle_{\mathcal{S}} \propto |\psi_2\rangle_{\mathcal{S}}$, then \mathcal{S} and \mathcal{E} would not be entangled. This is why in the above example, we assume $|\psi_1\rangle_{\mathcal{S}} \not\propto |\psi_2\rangle_{\mathcal{S}}$, that is, we assume there is no such α such that $|\psi_1\rangle_{\mathcal{S}} = \alpha |\psi_2\rangle_{\mathcal{S}}$, and for the same reason we assume $|\chi_1\rangle_{\mathcal{E}} \not\propto |\chi_2\rangle_{\mathcal{E}}$.

²⁷This is because by definition, if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ and $|\xi'\rangle_{\mathcal{U}} = |\psi'\rangle_{\mathcal{S}} |\chi'\rangle_{\mathcal{E}}$, then $\langle \xi' | \xi \rangle_{\mathcal{U}} = \langle \psi' | \psi \rangle_{\mathcal{S}} \langle \chi' | \chi \rangle_{\mathcal{E}}$. We will also have $\hat{\Lambda}_{\mathcal{U}} |\xi\rangle = \hat{\Lambda}_{\mathcal{S}} |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$. Thus, assuming both $|\psi\rangle_{\mathcal{S}}$ and $|\chi\rangle_{\mathcal{E}}$ are normalized, we have $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\xi} = \langle \xi | \hat{\Lambda}_{\mathcal{U}} | \xi \rangle = \langle \psi | \hat{\Lambda}_{\mathcal{S}} | \psi \rangle_{\mathcal{S}} \langle \chi | \chi \rangle_{\mathcal{E}} = \langle \hat{\Lambda}_{\mathcal{S}} \rangle_{\psi}$.

\mathcal{S} independently of the current state of the environment \mathcal{E} . In this case we need have no knowledge of the information about \mathcal{E} encapsulated in the state $|\chi\rangle_{\mathcal{E}}$ to determine the expectation value $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\xi}$.

However, for a general entangled state $|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^M \sum_{j=1}^N \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}$, $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\xi}$ will typically depend on the $|\chi_j\rangle_{\mathcal{E}}$ -states and the coefficients $\gamma_{i,j}$. Nevertheless, despite there being a huge amount of information contained within these $|\chi_j\rangle_{\mathcal{E}}$ -states and the $\gamma_{i,j}$, if we are only interested in making measurements on the system \mathcal{S} , nearly all this information can be discarded. In order to see how this is done, we need to generalize the notion of a state to that of a density matrix.

1.7.3 Density Matrices and Traces

Given a normalized state $|\psi\rangle$ in any Hilbert space H , its density matrix will be the operator $\hat{\rho} \stackrel{\text{def}}{=} |\psi\rangle\langle\psi|$ which acts on H by sending an arbitrary state $|\psi'\rangle$ to $\langle\psi|\psi'\rangle |\psi\rangle$. Note that $\hat{\rho}$ is a Hermitian operator.²⁸ Also note that if we had a measuring device that returned the output 1 if a system was in the state $|\psi\rangle$ and 0 if the system was in a state $|\chi\rangle$ with $\langle\psi|\chi\rangle = 0$, the density matrix $\hat{\rho}$ would be the observable corresponding to this measurement. The expectation value of this measurement for an initial normalized state $|\psi'\rangle$ would then be $\langle\psi'|\hat{\rho}|\psi'\rangle = |\langle\psi|\psi'\rangle|^2 = P(\psi|\psi')$. In particular, if the system was initially in the state $|\psi\rangle$, the expectation value of this measurement would be 1.

²⁸This is because for any arbitrary state $|\psi'\rangle$, $\langle\psi'|\hat{\rho}|\psi'\rangle = \langle\psi'|\psi\rangle \langle\psi|\psi'\rangle = \overline{\langle\psi|\psi'\rangle} \langle\psi|\psi'\rangle = |\langle\psi|\psi'\rangle|^2$, and so $\langle\psi'|\hat{\rho}|\psi'\rangle$ is real, and from this it follows that $\hat{\rho}$ is Hermitian.

Now it turns out that if we have an arbitrary orthonormal basis $\{|\phi_i\rangle : i\}$ of H and any observable $\hat{\Lambda}$ on H , then

$$\langle \hat{\Lambda} \rangle_{\psi} = \sum_i \langle \phi_i | \hat{\rho} \hat{\Lambda} | \phi_i \rangle. \quad \text{(1.9)} \quad \text{\{exptrace\}}$$

Since this expression can be shown to be independent of which basis we choose,³⁰ we have a well-defined function called the **trace**, written as $\text{Tr}(\cdot)$, which maps any operator \hat{A} acting on H to a value in \mathbb{C} according to the formula

$$\text{Tr}(\hat{A}) = \sum_i \langle \phi_i | \hat{A} | \phi_i \rangle. \quad \text{(1.10)} \quad \text{\{tracedef\}}$$

Thus, it follows from equations (1.9) and (1.10) that

$$\langle \hat{\Lambda} \rangle_{\psi} = \text{Tr}(\hat{\rho} \hat{\Lambda}). \quad \text{(1.11)} \quad \text{\{traceev\}}$$

In general, a **density matrix** $\hat{\rho}$ will be a Hermitian operator with positive eigenvalues such that $\text{Tr}(\hat{\rho}) = 1$. We will write $M(H)$ for the set of all density matrices on H . Since we are assuming³¹ that for any Hermitian operator, there is an orthonormal basis of the Hilbert space consisting of eigenstates of the Hermitian operator, we can find an orthonormal basis $\{|\psi_i\rangle : i\}$ of eigenstates of $\hat{\rho}$ with corresponding eigenvalues

²⁹To see this, as we saw in footnote ??, if we define the mapping $I = \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|$ then $I|\psi\rangle = |\psi\rangle$. Therefore

$\langle \hat{\Lambda} \rangle_{\psi} = \langle \psi | \hat{\Lambda} I | \psi \rangle = \sum_i \langle \psi | \hat{\Lambda} | \phi_i \rangle \langle \phi_i | \psi \rangle = \sum_i \langle \phi_i | \psi \rangle \langle \psi | \hat{\Lambda} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{\rho} \hat{\Lambda} | \phi_i \rangle$.

³⁰To see this, we first note that for any orthonormal basis $\{|\phi_i\rangle : i\}$ of H , and any two operators \hat{A} and \hat{B} acting on H , using the fact that $I = \sum_i |\phi_i\rangle\langle\phi_i|$ is the identity operator on H , we have the commutativity property

$$\sum_i \langle \phi_i | \hat{A} \hat{B} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{A} \sum_j |\phi_j\rangle\langle\phi_j| \hat{B} | \phi_i \rangle = \sum_{ij} \langle \phi_j | \hat{B} | \phi_i \rangle \langle \phi_i | \hat{A} | \phi_j \rangle = \sum_j \langle \phi_j | \hat{B} \hat{A} | \phi_j \rangle.$$

Now suppose that $\{|\phi'_i\rangle : i\}$ is another orthonormal basis of H . Then we can define the operator \hat{U} such that $\hat{U}|\phi_i\rangle = |\phi'_i\rangle$. We can also define the operator \hat{U}^* such that $\langle \phi'_i | \psi \rangle = \langle \phi_i | \hat{U}^* | \psi \rangle$ for any state $|\psi\rangle \in H$. Since $\langle \phi'_i | \phi'_j \rangle = \langle \phi_i | \hat{U}^* | \phi'_j \rangle$ for all i, j , it will follow that $\hat{U}^*|\phi'_j\rangle = |\phi_j\rangle$. Therefore, $\hat{U}\hat{U}^* = I$. Using this fact together with the commutativity property, we have

$$\sum_i \langle \phi'_i | \hat{\Lambda} | \phi'_i \rangle = \sum_i \langle \phi_i | \hat{U}^* \hat{\Lambda} \hat{U} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{\Lambda} \hat{U} \hat{U}^* | \phi_i \rangle = \sum_i \langle \phi_i | \hat{\Lambda} | \phi_i \rangle.$$

³¹As mentioned earlier, we are making the assumption that Hermitian operators are compact.

p_i such that

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \quad \text{rhoddiag}\{(\text{I.12})\}$$

The condition $\text{Tr}(\hat{\rho}) = 1$ will then imply that $\sum_i p_i = 1$. Now we could think of the operator $\hat{\rho}$ as corresponding to a measurement which gave the output p_i when the system was in the state $|\psi_i\rangle$. However, we can alternatively think of $\hat{\rho}$ as describing a system which is known to be in one of the $|\psi_i\rangle$ -states, but that we only know it is in the $|\psi_i\rangle$ -state with probability p_i . Then given that $\hat{\rho}$ describes all we know about the system, the expectation value $\langle\hat{\Lambda}\rangle_\rho$ for an observable $\hat{\Lambda}$ on the system can be shown to be

$$\langle\hat{\Lambda}\rangle_\rho = \text{Tr}(\hat{\rho}\hat{\Lambda}). \quad \text{expdensity}\{(\text{I.13})\}$$

We can think of a density matrix $\hat{\rho} \in M(H)$ as a generalization of a state ket-vector $|\psi\rangle \in H$, since for every $|\psi\rangle \in H$ there corresponds a density matrix $\hat{\rho} = |\psi\rangle\langle\psi| \in M(H)$. Because of this identification, $\hat{\rho} = |\psi\rangle\langle\psi|$ is referred to as a **pure state**. On the other hand, the converse does not hold: if $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in M(H)$ with more than one of the $p_i > 0$, then there will not be a corresponding $|\psi\rangle \in H$ such that $\hat{\rho} = |\psi\rangle\langle\psi|$. In this case, when $\hat{\rho}$ is interpreted as describing a system that is definitely in one of the $|\psi_i\rangle$ -states with probability p_i , then we will refer to $\hat{\rho}$ as a **mixed state**.^{mixedstate}

³²This follows since $\text{Tr}(\hat{\rho}\hat{\Lambda}) = \sum_i p_i \text{Tr}(|\psi_i\rangle\langle\psi_i| \hat{\Lambda}) = \sum_i p_i \langle\Lambda\rangle_{\psi_i}$ which will be the expectation value of $\hat{\Lambda}$ given that $\hat{\rho}$ encapsulates our knowledge of the system.

1.7.4 Coherence

Now suppose that the system \mathcal{S} is initially in a superposition state $|\psi\rangle = \sum_i c_i |s_i\rangle$ with $\sum_i |c_i|^2 = 1$. Then the corresponding density matrix on \mathcal{S} will be

$$|\psi\rangle\langle\psi| = \sum_{ij} c_i \bar{c}_j |s_i\rangle\langle s_j|.$$

When a density matrix has non-zero $|s_i\rangle\langle s_j|$ -components for $i \neq j$, we say that there is **coherence** between the $|s_i\rangle$ and $|s_j\rangle$ -states.³³ Thus, for the density matrix $|\psi\rangle\langle\psi|$ there will be coherence between the $|s_i\rangle$ and $|s_j\rangle$ -states so long as both c_i and c_j are non-zero. Decoherence is a process (to be described shortly) by which the $|s_i\rangle\langle s_j|$ -components of a density matrix restricted to a subsystem of a composite system appear to vanish.

1.7.5 Partial Traces and Reduced Density Matrices

As already mentioned, if we have a general entangled state on a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ of the form $|\xi\rangle_{\mathcal{U}} = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}$, there is a huge amount of information in all the $\gamma_{i,j}$. However, most of this information can be discarded if we are only interested in making measurements on the system \mathcal{S} . We can't typically encapsulate this information in the form of a state $|\psi\rangle \in H_{\mathcal{S}}$, but we can encapsulate this information in the form of a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which as mentioned on

³³The fact that a density matrix can be written out in terms of $|s_i\rangle\langle s_j|$ -components explains why we refer to a density matrix as a density *matrix*. For example, if our state space has a basis of just two states $\{|s_1\rangle, |s_2\rangle\}$, and if $\hat{\rho} = a |s_1\rangle\langle s_1| + b |s_1\rangle\langle s_2| + c |s_2\rangle\langle s_1| + d |s_2\rangle\langle s_2|$, then we can identify $\hat{\rho}$ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we then identify the state $|\psi\rangle = x |s_1\rangle + y |s_2\rangle$ with the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, then the state $\hat{\rho} |\psi\rangle$ would be identified with the column vector under matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. The trace of a density matrix is then just the sum of the diagonal elements (top left to bottom right) of the matrix. Decoherence with respect to a particular basis occurs when the off-diagonal elements of the density matrix vanish.

page 26an be thought of as a generalization of a state ket-vector $|\psi\rangle \in H_{\mathcal{S}}$. In this subsection, we will show how the density matrix $\hat{\rho} = |\xi\rangle\langle\xi| \in M(H_{\mathcal{U}})$ can be reduced to a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which encapsulates all the information needed to calculate expectation values of observables on \mathcal{S} . The reduced density matrix $\hat{\rho}_{\mathcal{S}}$ is derived from $\hat{\rho}$ via an operation call the partial trace.

In the context of a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$, when taking traces, we will need to be more specific over which basis we are taking the trace over. If $\{|\psi_i\rangle : i\}$ is an orthonormal basis of $H_{\mathcal{S}}$ and $\{|\chi_j\rangle : j\}$ is an orthonormal basis of $H_{\mathcal{E}}$, then $\{|\xi_{ij}\rangle \stackrel{\text{def}}{=} |\psi_i\rangle|\chi_j\rangle : i, j\}$ will be an orthonormal basis of $H_{\mathcal{U}}$. For an operator $\hat{A}_{\mathcal{S}}$ of $H_{\mathcal{S}}$, we define

$$\text{Tr}_{\mathcal{S}}(\hat{A}_{\mathcal{S}}) = \sum_i \langle \psi_i | \hat{A}_{\mathcal{S}} | \psi_i \rangle,$$

and for an operator $\hat{A}_{\mathcal{U}}$ of $H_{\mathcal{U}}$, we define

$$\text{Tr}_{\mathcal{U}}(\hat{A}_{\mathcal{U}}) = \sum_{ij} \langle \xi_{ij} | \hat{A}_{\mathcal{U}} | \xi_{ij} \rangle.$$

This is just what we would expect the traces to be for operators on $H_{\mathcal{S}}$ and on $H_{\mathcal{U}}$ respectively. But we also need the notion of a **partial trace** for an operator $\hat{A}_{\mathcal{U}}$ on $H_{\mathcal{U}}$:

$$\text{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}}) = \sum_j \langle \chi_j | \hat{A}_{\mathcal{U}} | \chi_j \rangle. \quad \text{\{partialtrace\}} \quad (1.14)$$

Note that whereas $\text{Tr}_{\mathcal{U}}(\hat{A}_{\mathcal{U}})$ is just a number, the partial trace $\text{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}})$ is an operator that acts on $H_{\mathcal{S}}$. To see why this is, consider the simple example of when $\hat{A}_{\mathcal{U}} = |\xi_{ij}\rangle\langle\xi_{lk}|$. The operator $\hat{A}_{\mathcal{U}}$ would send the state $|\xi_{lk}\rangle$ to $|\xi_{ij}\rangle$ and all the other $|\xi_{l'k'}\rangle$ -states of $H_{\mathcal{U}}$ to 0. But in order to define the partial trace as given in equation (1.14), we need to know what $\hat{A}_{\mathcal{U}}|\chi\rangle$ is and then what $\langle\chi|\hat{A}_{\mathcal{U}}|\chi\rangle$ is. In the case when

$\hat{A}_{\mathcal{U}} = |\xi_{ij}\rangle\langle\xi_{lk}|$, we stipulate that $\hat{A}_{\mathcal{U}}|\chi\rangle$ is the operator that sends the state $|\psi\rangle \in H_{\mathcal{S}}$ to the state $\langle\psi_l|\psi\rangle\langle\chi_k|\chi\rangle|\xi_{ij}\rangle \in H_{\mathcal{U}}$. Furthermore, if we stipulate that $\langle\chi|\xi_{ij}\rangle = \langle\chi|\chi_j\rangle|\psi_i\rangle \in H_{\mathcal{S}}$, it follows that $\langle\chi|\hat{A}_{\mathcal{U}}|\chi\rangle$ will be the operator $\langle\chi|\chi_j\rangle\langle\chi_k|\chi\rangle|\psi_i\rangle\langle\psi_l|$ that sends the state $|\psi\rangle \in H_{\mathcal{S}}$ to the state $\langle\chi|\chi_j\rangle\langle\chi_k|\chi\rangle\langle\psi_l|\psi\rangle|\psi_i\rangle \in H_{\mathcal{S}}$. We therefore find that

$$\text{Tr}_{\mathcal{E}}(|\xi_{ij}\rangle\langle\xi_{lk}|) = \begin{cases} |\psi_i\rangle\langle\psi_l| & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \quad \text{\{partialtrace2\}} \quad (1.15)$$

And since any arbitrary operator $\hat{A}_{\mathcal{U}}$ on $H_{\mathcal{U}}$ can be expressed as a sum $\hat{A}_{\mathcal{U}} = \sum_{ijkl} \mu_{ijkl} |\xi_{ij}\rangle\langle\xi_{lk}|$, we can use equation (1.15) to find that $\text{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}}) = \sum_{ijl} \mu_{ijjl} |\psi_i\rangle\langle\psi_l|$.

Now it turns out that given a density matrix $\hat{\rho}$ on $H_{\mathcal{U}}$ and an observable $\hat{\Lambda}_{\mathcal{S}}$ of $H_{\mathcal{S}}$ (which induces an observable $\hat{\Lambda}_{\mathcal{U}}$ on $H_{\mathcal{U}}$ in the obvious way, e.g. $\hat{\Lambda}_{\mathcal{U}}|\xi_{ij}\rangle = (\hat{\Lambda}_{\mathcal{S}}|\psi_i\rangle)|\chi_j\rangle$), we have the important formula

$$\langle\hat{\Lambda}_{\mathcal{U}}\rangle_{\rho} = \text{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{\Lambda}_{\mathcal{S}}) \quad \text{\{reducedev\}} \quad (1.16)$$

where $\hat{\rho}_{\mathcal{S}} = \text{Tr}_{\mathcal{E}}(\hat{\rho})$.³⁴ We refer to $\hat{\rho}_{\mathcal{S}}$ as the **reduced density matrix** of $\hat{\rho}$.

Note that if $\hat{\rho} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = |\psi\rangle|\chi\rangle$ so that \mathcal{S} and \mathcal{E} are not entangled, then $\hat{\rho}_{\mathcal{S}} = |\psi\rangle\langle\psi|$.³⁵ This is what we should expect, since if \mathcal{S} and \mathcal{E} are not entangled, then the expectation values of observables defined on \mathcal{S} should be independent of the state

³⁴To see this, following Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 46, we have

$$\begin{aligned} \langle\hat{\Lambda}_{\mathcal{U}}\rangle_{\rho} &= \text{Tr}_{\mathcal{U}}(\hat{\rho}\hat{\Lambda}_{\mathcal{U}}) = \sum_{ij} \langle\xi_{ij}|\hat{\rho}\hat{\Lambda}_{\mathcal{U}}|\xi_{ij}\rangle = \sum_i \langle\psi_i| \left(\sum_j \langle\chi_j|\hat{\rho}|\chi_j\rangle \right) \hat{\Lambda}_{\mathcal{S}}|\psi_i\rangle \\ &= \sum_i \langle\psi_i|\hat{\rho}_{\mathcal{S}}\hat{\Lambda}_{\mathcal{S}}|\psi_i\rangle = \text{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{\Lambda}_{\mathcal{S}}). \end{aligned}$$

³⁵**untanglepartialtrace** To see this, we recall that the partial trace $\text{Tr}_{\mathcal{E}}$ is independent of which orthonormal basis $\{|\chi_j\rangle : j\}$ we choose for \mathcal{E} . Therefore, if $\hat{\rho} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = |\psi\rangle|\chi\rangle$, we can choose $|\chi_1\rangle = |\chi\rangle$ and all other $|\chi_i\rangle$ such that $\langle\chi_i|\chi\rangle = 0$. Then $\text{Tr}_{\mathcal{E}}(\hat{\rho}) = \sum_j \langle\chi_j|\hat{\rho}|\chi_j\rangle = |\psi\rangle\langle\psi|$.

of \mathcal{E} , and by equation (1.16), this independence is seen to hold when $\hat{\rho}_{\mathcal{S}}$ is independent of any states on \mathcal{E} .

More generally, ^{subtle} for an entangled state $|\xi\rangle = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle |\chi_j\rangle$, from equations (1.13 and (1.16, we have $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\rho} = \langle \hat{\Lambda}_{\mathcal{S}} \rangle_{\rho_{\mathcal{S}}}$. This means that when it comes to taking expectation values of measurements on a subsystem \mathcal{S} that is part of a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ which is in the state $|\xi\rangle \in H_{\mathcal{U}}$, the subsystem \mathcal{S} behaves as though it was described by the density matrix $\hat{\rho}_{\mathcal{S}}$. However, there is a rather subtle point one needs to be aware of here.³⁶ For in general, as we saw in equation (1.12, any density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ can be expressed as a sum $\hat{\rho}_{\mathcal{S}} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, and this can be *thought of* as corresponding to the system \mathcal{S} being in one of the $|\psi_i\rangle$ -states, but that we only know it is in the $|\psi_i\rangle$ -state with probability p_i . If this was the correct interpretation of $\hat{\rho}_{\mathcal{S}}$, then as explained on page 26 we would refer to $\hat{\rho}_{\mathcal{S}}$ as a *mixed state*. But just because we can think of $\hat{\rho}_{\mathcal{S}}$ in this way, it doesn't follow that \mathcal{S} really is in one of these $|\psi_i\rangle$ -states and that we are only ignorant of which state it is. When \mathcal{S} is entangled with \mathcal{E} there is no fact of the matter regarding which state \mathcal{S} is in. Rather, there are only facts of the matter for the composite system \mathcal{U} , e.g. the fact of the matter is that \mathcal{U} is in the state $|\xi\rangle$ rather than some other state of $H_{\mathcal{U}}$. Therefore \mathcal{U} is really in a pure state with density matrix $|\xi\rangle\langle\xi|$. Because we cannot give an ignorance interpretation to $\hat{\rho}_{\mathcal{S}}$ d'Espagnat^{Espagnat} referred to density matrices of this sort as being **improper mixtures**.³⁷ But despite this subtle distinction between mixed states

³⁶It is unfortunate that many physicists fail to pick up on this subtlety with the result that they form the erroneous belief that decoherence can by itself solve the measurement problem (of outcomes) when in fact it can't. For a further discussion of the problem of outcomes, see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 57–60.

³⁷See Bernard d' Espagnat, *Conceptual foundations of quantum mechanics*, 2nd ed., completely rev., enl., reset., Mathematical physics monograph series ; 20 (Reading, Mass. ; London: W. A.

and improper mixtures, we have nevertheless succeeded in showing how a density matrix $\hat{\rho} = |\xi\rangle\langle\xi| \in M(H_{\mathcal{U}})$ can be reduced to a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which encapsulates all the information needed to calculate expectation values of observables on \mathcal{S} .³⁷

~~vonNeumannMeasurement~~

1.7.6 The von Neumann Measurement Scheme

We are now in a position to consider the **von Neumann measurement scheme**.³⁸ Instead of considering the whole of physical reality, for the time being, we just consider a physical system \mathcal{S} and a measuring device \mathcal{A} . This division reflects the fact that a scientist doesn't measure the system \mathcal{S} directly, but rather observes a measuring device \mathcal{A} that is affected by \mathcal{S} . The measuring device \mathcal{A} has the characteristic that it has a normalized ready state $|a_r(t_0)\rangle$ at initial time t_0 and that there is an orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$, and normalized states $|a_i(t)\rangle$ of \mathcal{A} such that

1. for any $t \geq t_0$ we have the evolution of the states $|\langle s_i | a_r(t_0) \rangle| \xrightarrow{\text{vonNeumannMeasurement1}} |s_i\rangle |a_i(t)\rangle$ so that \mathcal{S} and \mathcal{A} do not become entangled when \mathcal{S} is initially in state $|s_i\rangle$ and \mathcal{A} is initially in state $|a_i(t_0)\rangle$.
2. there exists $\delta > 0$ such that if $t > t_0 + \delta$, then $\langle a_i(t) | a_j(t) \rangle \approx 0$ for $i \neq j$.³⁹

These two criteria characterize the von Neumann measurement scheme. The orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$ for which these two criteria hold are called **pointer states**.⁴⁰ These pointer states will be determined by the dynamics of the composite

³⁷ Benjamin, 1976), ch. 6.2 – cited in Jeremy Butterfield, “Peaceful Coexistence: Examining Kent’s Relativistic Solution to the Quantum Measurement Problem,” 2017, p. 19, eprint: arXiv:1710.07844.

³⁸ See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 50–53 for more details.

³⁹ More precisely, we should say that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $t > t_0 + \delta$, then $|\langle a_i(t) | a_j(t) \rangle| < \epsilon$ for $i \neq j$.

⁴⁰ cbx@9

system $\mathcal{S} + \mathcal{A}$ as well as the relative configuration of \mathcal{S} with respect to \mathcal{A} . For instance, if \mathcal{S} is a silver atom and \mathcal{A} is a Stern-Gerlach apparatus, then the configuration and dynamics of the system will determine a fixed axis $\hat{\mathbf{a}}$ relative to the Stern-Gerlach configuration \mathcal{A} such that the states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ of \mathcal{S} don't get entangled with \mathcal{A} , that is, there exists $\delta > 0$ such that $|\hat{\mathbf{a}}\pm\rangle|a_r(t_0)\rangle \xrightarrow{\text{time evolution}} |\hat{\mathbf{a}}\pm\rangle|a_\pm(t)\rangle$ with $\langle a_+(t)|a_-(t)\rangle \approx 0$ for $t > t_0 + \delta$.⁴⁰⁴¹ Since no entanglement occurs with the silver atom and the Stern-Gerlach apparatus when the silver atom is in the $|\hat{\mathbf{a}}\pm\rangle$ -state, then in this situation, we can interact with the apparatus to find out whether the particle is in the $|\hat{\mathbf{a}}+\rangle$ -state or the $|\hat{\mathbf{a}}-\rangle$ -state without changing the spin state of the silver atom. Indeed, we should expect an experimental apparatus to have this property of non-entanglement with the measurement outcomes it reports, for otherwise, every scientist who looked at the measurement device couldn't be sure that the spin state of the silver atom being measured remained unchanged whenever the apparatus was observed, and so the scientists couldn't expect there to be any agreement among themselves regarding which spin-state the silver atom was in. Thus, the basis of $H_{\mathcal{S}}$ for which entanglement doesn't occur is a preferred basis. However, if we were to consider a different basis, say $\{\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle), \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle - |\hat{\mathbf{a}}-\rangle)\}$,⁴² then assuming that

⁴⁰Strictly speaking, we would need more information to describe states in $H_{\mathcal{S}}$ besides the spin, so we should really express this scenario in terms of $\{|s_{i,+}\rangle \stackrel{\text{def}}{=} |\hat{\mathbf{a}}+, i\rangle : i\} \cup \{|s_{i,-}\rangle \stackrel{\text{def}}{=} |\hat{\mathbf{a}}-, i\rangle : i\}$ and $\{|a_{i,+}(t)\rangle : i\} \cup \{|a_{i,-}(t)\rangle : i\}$ where the i -indices encode all the additional information beyond spin.

⁴¹^{nearzero}Although we only require that $\langle a_+(t)|a_-(t)\rangle \approx 0$ for $t > t_0 + \delta$ rather than demanding $\langle a_+(t)|a_-(t)\rangle = 0$, we can think of the scientist who observes the apparatus as determining whether the apparatus is either in one of two normalized state $|a'_+(t)\rangle$ or $|a'_-(t)\rangle$ where $\langle a'_+(t)|a'_-(t)\rangle = 0$ and $\langle a'_\pm(t)|a'_\pm(t)\rangle \approx 1$, so that the scientist can confidently assert that the particle is in the state $|\hat{\mathbf{a}}+\rangle$ if for instance the measurement device is found to be in the state $|a'_+(t)\rangle$. Because $\langle a'_+(t)|a_-(t)\rangle$ is only very small, but not identically zero, in theory, the particle could be in the $|\hat{\mathbf{a}}-\rangle$ -state, but we're assuming that such a possibility would be as likely as a violation of the Second Law of Thermodynamics, say.

⁴²According to equation (1.2, this basis would correspond to measuring the spin in an axis at right angles to $\hat{\mathbf{a}}$.

the configuration of \mathcal{A} remained unchanged, entanglement between \mathcal{S} and \mathcal{A} would occur since then $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle \pm |\hat{\mathbf{a}}-\rangle) |a_r(t_0)\rangle \xrightarrow{\text{time evolution}} \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle |a_+(t)\rangle \pm |\hat{\mathbf{a}}-\rangle |a_-(t)\rangle)$. Thus, $\{\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle), \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle - |\hat{\mathbf{a}}-\rangle)\}$ would not be a preferred basis. In this case, if \mathcal{S} was in the $\frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ -state, a scientist would measure \mathcal{A} to be in the $|a_+(t)\rangle$ -state with probability $\frac{1}{2}$. But having measured \mathcal{A} to be in the $|a_+(t)\rangle$ -state, the scientist would continue to observe \mathcal{A} to be in the $|a_+(t)\rangle$ -state because of the subsequent non-entanglement of \mathcal{S} with \mathcal{A} when \mathcal{S} is in the $|\hat{\mathbf{a}}+\rangle$ -state and \mathcal{A} is in the $|a_+(t)\rangle$ -state. Note that this situation is somewhat analogous to when we have the Bell-state (1.4, so that when Bob measures his particle to be in the $|\hat{\mathbf{a}}-\rangle$ -state, he knows that Alice's particle is in the $|\hat{\mathbf{a}}+\rangle$ -state. Likewise, in the von Neumann measurement scheme, if the scientist measures \mathcal{A} to be in the $|a_+(t)\rangle$ -state for $t > t_0 + \delta$, he will then (almost certainly) know⁴³ that the system \mathcal{S} will be in the $|\hat{\mathbf{a}}+\rangle$ -state.

In the case where \mathcal{S} has more than two states, we can write a generic normalized state of the composite system $\mathcal{U} = \mathcal{S} + \mathcal{A}$ as $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle$ where $|\xi_i(t)\rangle = |s_i\rangle |a_i(t)\rangle$. There will then be coherence between $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ for the density matrix $\hat{\rho}(t) \stackrel{\text{def}}{=} |\Psi(t)\rangle\langle\Psi(t)|$ so long as both c_i and c_j are non-zero. However, if we are only interested in observables $\hat{\Lambda}_{\mathcal{S}}$ on $H_{\mathcal{S}}$, then we only need to consider the reduced density matrix $\hat{\rho}_{\mathcal{S}}(t) = \text{Tr}_{\mathcal{A}}(\hat{\rho}(t))$. Initially, at time t_0 we have $|a_i(t_0)\rangle = |a_r(t_0)\rangle$ so $|\Psi(t_0)\rangle = |\psi\rangle |a_r(t_0)\rangle$ where $|\psi\rangle = \sum_i c_i |s_i\rangle$ which we assume to be normalized. Thus, initially, \mathcal{S} would not be entangled with \mathcal{A} , and therefore the density matrix describing \mathcal{S} would be $\hat{\rho}_{\mathcal{S}}(t_0) = |\psi\rangle\langle\psi|$.⁴⁴ Hence if we consider $\hat{\Lambda}_{\mathcal{S}} = |\psi\rangle\langle\psi|$ as an

⁴³This is because then $\langle a_+(t)|a_-(t)\rangle$ will be very nearly zero rather than identically zero as discussed in footnote 41

⁴⁴Recall footnote 35

observable on \mathcal{S} corresponding to a measurement⁴⁵ that records the value 1 if the system is in the $|\psi\rangle$ and 0 if the system is in a state $|\psi'\rangle$ with $\langle\psi'|\psi\rangle = 0$, then both intuitively⁴⁶ and by equation (1.16,⁴⁷ we would have $\langle\hat{\Lambda}_{\mathcal{U}}\rangle_{\rho(t_0)} = 1$. But if the scientist is to measure \mathcal{S} to be in the $|\psi\rangle$ -state, the expectation value $\langle\hat{\Lambda}_{\mathcal{U}}\rangle_{\rho(t)}$ would have to be 1 for times t discernibly greater than t_0 .

However, if more than one of the c_i are non-zero, then the scientist will not be able to measure the system \mathcal{S} to be in the $|\psi\rangle$ -state for any discernible length of time. To see why this is, we first note that

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_i |c_i|^2 |s_i\rangle\langle s_i| + \sum_{i \neq j} c_i \bar{c}_j \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|. \quad \text{(1.17)} \quad \text{[reduced]}$$

Now because $\langle a_j(t)|a_i(t)\rangle \approx 0$ for $t > t_0 + \delta$, it follows that $\hat{\rho}_{\mathcal{S}} \approx \sum_i |c_i|^2 |s_i\rangle\langle s_i|$ for $t > t_0 + \delta$. It will then follow that $\langle\hat{\Lambda}_{\mathcal{U}}\rangle_{\rho(t)} = \sum_i |c_i|^4$,⁴⁹ and this will only be 1 if only one of the c_i is 1 and all the other c_i are 0. Hence, if more than one of the c_i are

⁴⁵This is a measurement we conduct by some means other than looking at the apparatus \mathcal{A} .

⁴⁶I.e. we would expect the expectation value of $\hat{\Lambda}_{\mathcal{U}}$ to be 1 if we knew that \mathcal{S} was in the state $|\psi\rangle$ with probability 1.

⁴⁷I.e. given that $\hat{\rho}_{\mathcal{S}}(t_0) = |\psi\rangle\langle\psi| = \hat{\Lambda}_{\mathcal{S}}$, and that $\hat{\Lambda}_{\mathcal{S}}^2 = \hat{\Lambda}_{\mathcal{S}}$, and $\text{Tr}_{\mathcal{S}}(\hat{\Lambda}_{\mathcal{S}}) = 1$, it follows that $\langle\hat{\Lambda}_{\mathcal{U}}\rangle_{\rho(t_0)} = \text{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}(t_0)\hat{\Lambda}_{\mathcal{S}}) = \text{Tr}_{\mathcal{S}}(\hat{\Lambda}_{\mathcal{S}}) = 1$.

⁴⁸To see this, it is sufficient to show that $\text{Tr}_{\mathcal{A}}(|\xi_i(t)\rangle\langle\xi_j(t)|) = \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|$ for then we will obtain the first summand of $\hat{\rho}_{\mathcal{S}}$ from the fact that $|a_i(t)\rangle$ are normalized, and we will obtain the second summand by linearity of $\text{Tr}_{\mathcal{A}}(\cdot)$. Well, taking $\{|\phi_k\rangle : k\}$ to be an orthonormal basis of $H_{\mathcal{A}}$, we have

$$\begin{aligned} \text{Tr}_{\mathcal{A}}(|\xi_i(t)\rangle\langle\xi_j(t)|) &= \sum_k \langle\phi_k| \left(|\xi_i(t)\rangle\langle\xi_j(t)| \right) |\phi_k\rangle = \sum_k \langle\phi_k|a_i(t)\rangle \langle a_j(t)|\phi_k\rangle |s_i\rangle\langle s_j| \\ &= \langle a_j(t)| \left(\sum_k |\phi_k\rangle\langle\phi_k| \right) |a_i(t)\rangle |s_i\rangle\langle s_j| = \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|, \end{aligned}$$

where we have used the fact that $I = \sum_k |\phi_k\rangle\langle\phi_k|$ is the identity operator on $H_{\mathcal{A}}$.

⁴⁹This is because

$$\begin{aligned} \text{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}} |\psi\rangle\langle\psi|) &\approx \text{Tr}_{\mathcal{S}} \left(\sum_i |c_i|^2 |s_i\rangle\langle s_i| \sum_{jk} c_j \bar{c}_k |s_j\rangle\langle s_k| \right) \\ &= \text{Tr}_{\mathcal{S}} \left(\sum_{ik} |c_i|^2 c_i \bar{c}_k |s_i\rangle\langle s_k| \right) = \sum_l \sum_{ik} \langle s_l| |c_i|^2 c_i \bar{c}_k |s_i\rangle \langle s_k| s_l \rangle = \sum_i |c_i|^4. \end{aligned}$$

non-zero, the scientist will not be able to measure the system \mathcal{S} to be in the $|\psi\rangle$ -state for any discernible length of time.

1.7.7 Decoherence

Note that although for the original density matrix $|\psi\rangle\langle\psi|$ there is coherence between the states $|s_i\rangle$ and $|s_j\rangle$, this coherence effectively disappears when the system \mathcal{S} interacts with the measuring device \mathcal{A} (i.e. the $|s_i\rangle\langle s_j|$ -coefficients of $\hat{\rho}_{\mathcal{S}}$ are approximately zero for $t > t_0 + \delta$). This is what we mean by **decoherence**: the coherence has effectively disappeared. The **decoherence time** δ which is the time it takes for $\langle a_i(t)|a_j(t)\rangle$ to go from 1 when $t = t_0$ to approximately zero when $t = t_0 + \delta$ will depend on what situation we are considering, but very often this time will be extremely small. For instance if we were measuring neurons firing in the brain, the decoherence time will typically be of the order $\delta = 10^{-19}$ s.⁵⁰ It is because of decoherence that we can't expect the system \mathcal{S} to remain in the state $|\psi\rangle = \sum_i c_i |s_i\rangle$ for any discernible length of time, unless $|\psi\rangle$ is proportional to one of the $|s_i\rangle$ -states.

Also note that when decoherence occurs, we say the coherence *effectively* disappears, insofar as the coherence will not be measurable if we only consider observables just acting on $H_{\mathcal{S}}$. Thus, after decoherence has taken place, if we restrict our attention to the system \mathcal{S} alone, it will be experimentally indistinguishable⁵¹ from the situation where \mathcal{S} is known to be in one of the $|s_i\rangle$ -states, but that we only know that it is in the $|s_i\rangle$ -state with probability $|c_i|^2$. Nevertheless, the coherence is still there, since if we chose to consider more general observables on the composite $\mathcal{S} + \mathcal{A}$, the $|\xi_i(t)\rangle\langle\xi_j(t)|$ -

⁵⁰For details of this estimate see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 370.

⁵¹Recall the discussion following equation (1.12 on page 26s well as the discussion on page 30

coefficients of $\hat{\rho}$ will continue to be $c_i \bar{c_j}$ which will in general will be non-zero, and as a whole, at time t the composite system will be in the state $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle$ with probability 1.

1.8 sectionPreferredBasis A Solution to the Preferred Basis Problem⁵²

We can now see how decoherence theory solves the preferred basis problem. Although up to this point we have been focusing on how a system \mathcal{S} interacts with a measuring apparatus \mathcal{A} , we can generalize to the situation in which a system \mathcal{S} interacts with its environment \mathcal{E} . We can still define pointer states in the same way as we did on page 31. These pointer states will then make up the preferred basis. The two defining criteria of pointer states entail that pointer states will remain stable and immune to decoherence effects.

Since physicists have a good understanding of how different systems interact, they are able to explain what it is about a basis that makes it a preferred basis. The details of their analysis need not concern us here, but it's possible to show that for macroscopic and mesoscopic objects, states specified in terms of position decohere with one another very rapidly.⁵³ This explains why we don't detect $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$ -states, but we do detect $|\text{Cat Alive}\rangle$ and $|\text{Cat Dead}\rangle$ -states. Also note that $|\text{Cat Alive}\rangle$ does indeed have the property that it is immune to decoherence effects, for if we were to express $|\text{Cat Alive}\rangle$ in terms of the basis $\{|\psi_+\rangle, |\psi_-\rangle\}$ where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle \pm |\text{Cat Dead}\rangle)$, then

⁵²For more details, see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 71–84

⁵³e.g. See the discussion in Schlosshauer, 94.

$|\text{Cat Alive}\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)$. The corresponding density matrix would be

$$|\text{Cat Alive}\rangle\langle\text{Cat Alive}| = \frac{1}{\sqrt{2}}(|\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|) \quad (1.18)$$

Since in normal situations, the left-hand side of equation (1.18) will remain unperturbed by the environment, the coefficients of the off-diagonal terms $|\psi_\pm\rangle\langle\psi_\mp|$ will also remain as they are; that is, $|\psi_\pm\rangle$ and $|\psi_\mp\rangle$ will not decohere with one another. It is only in very contrived situations such as when the cat's environment is a poison releasing device coupled to a radioactive atom that $|\text{Cat Alive}\rangle$ will no longer be a pointer state with respect to this environment.

1.9 ^{Nonobservability} The Problem of Nonobservability of Interference and its Solution⁵⁴

Before we consider the many-worlds interpretation in detail, it will be helpful to consider the role that decoherence plays in the removal of quantum interference on the macroscopic scale, as it is this lack of quantum interference between mutually exclusive states of a system that justifies our thinking of the system as being composed of alternative realities. The question of why quantum interference typically disappears at macroscopic scales is referred to as the problem of the nonobservability of interference.

We can explain this problem in the context of the double slit experiment: As figure 1.6ndicates, when a beam of particles is incident on a double slit, the particles that are detected on the detection screen are distributed according to a distribution pattern which either exhibits quantum interference as shown on the left in the figure, or does not exhibit such interference as shown on the right. Small particles like electrons and

⁵⁴See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 55–57, 63–65.

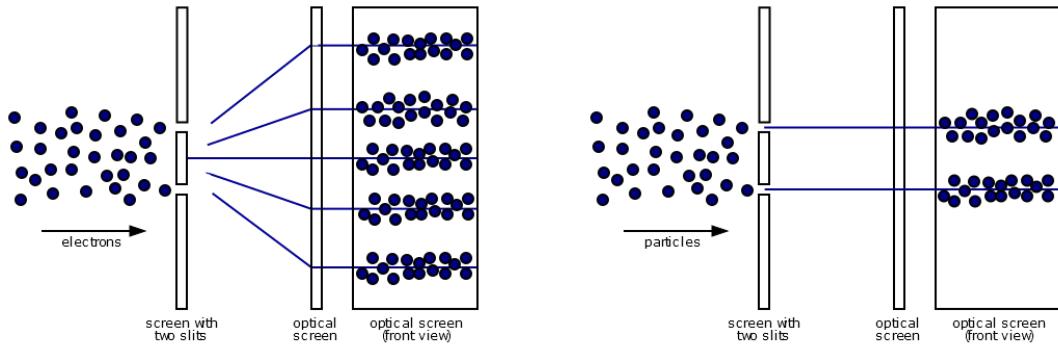


Figure 1.6: The Double-Slit Experiment. Particles are incident on a double slit. In diagram (A), the particles are exhibiting an interference pattern, whereas in diagram (B), the particles are not exhibiting an interference pattern. Whether or not there is interference will depend on factors such as the size of the particles and whether it can be ascertained which slit the particle went through. The larger the particles are or the more information available as to which slit the particle went through, the less likely the particles will exhibit interference.⁵⁵

DoubleSlit

photons will tend to exhibit quantum interference, whereas mesoscopic particles will not typically exhibit quantum interference.

To explain what is going on, we suppose that when just the top slit is open, the normalized state of the particle is $|\psi_1\rangle$,^{psi_slit} whereas if just the bottom slit is open, we suppose that the normalized state of the particle is $|\psi_2\rangle$, and when both slits are open, we suppose that the state of the particle will be $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Now let the variable x describe the position on the detection screen. For instance, we might take $x = 0$ to be the center of the detection screen, and take positive values of x as corresponding to positions on the upper part of the screen, and negative values of x as corresponding to positions on the lower part of the screen, but the precise convention we adopt won't

matter. Then we define the $|x\rangle$ -state⁵⁶ as the physical state describing the particle to be exactly located at position x on the screen. Note that the state $|x\rangle$ is indexed by a continuous parameter, x . This is in contrast to the basis of states $|s_i\rangle$ which we have been considering up until now which are indexed by discrete values of i such as $i = 1, 2, \dots$. Because of this difference, we need to use calculus to deal with $|x\rangle$ -states in a rigorous manner, but such details will not concern us here. In reality, because of the Heisenberg uncertainty principle, a particle is never in just one $|x\rangle$ -state, but rather the particle will be in a superposition of many $|x\rangle$ -states, which may or may not be concentrated around a particular location, x_0 say. The more concentrated these $|x\rangle$ -states of this superposition are concentrated around a particular location x_0 , the more the particle will have the particle-like characteristic of being localized in one place. But if the $|x\rangle$ -states of this superposition are more spread out, the particle will have more wave-like characteristics. So when physicists speak of particles, often they are not thinking of physical entities that are very localized in position, as non-physicists would think. Nevertheless, at the moment the particle is detected on the detection screen, it does seem to be highly localized.

Given a state $|\psi\rangle$ for a so-called particle, we define the function $\psi(x) = \langle x|\psi\rangle$. Because of the continuous nature of the variable x (in contrast to the discrete nature of i in a basis $\{|s_i\rangle : i\}$), the function $\rho(x) = |\psi(x)|^2$ ⁵⁷ determines a probability density for a range of outcomes rather than a probability for a specific outcome. Here, we do

⁵⁵Diagrams (A) and (B) are by inductiveload, and are Public domain, via Wikimedia Commons. Sources: https://commons.wikimedia.org/wiki/File:Two-Slit_Experiment_Electrons.svg and https://commons.wikimedia.org/wiki/File:Two-Slit_Experiment_Particles.svg.

⁵⁶ $|x\rangle$ is not really a state in the proper sense. With the states we've seen so far, when $|\phi\rangle$ and $|\psi\rangle$ have been normalized, then $|\langle\phi|\psi\rangle|^2$ will be a conditional probability, and hence at most 1. However, $|x\rangle$ cannot be normalized. This is because the bracket $\langle x|y\rangle$ is defined to be

not need to go into the details of probability densities,⁵⁷ but roughly speaking, the greater the value of $\rho(x)$, the greater will be the relative probability of detecting the particle at location x . Thus, if $\rho(x) = 0$, then the particle would not be detected at location x .

Now if $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$, then $\psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$. Therefore, the corresponding probability density will be

$$|\psi(x)|^2 = \frac{1}{2}(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2 \operatorname{Re}(\overline{\psi_1(x)}\psi_2(x))).^{58}$$

Now when the detection screen is far away from the double slits, we will have $|\psi_1(x)|^2 \approx |\psi_2(x)|^2$ for x near the center point on the screen. However, depending on slight changes in the value of x from the center point on the screen, sometimes $\psi_1(x)$ and $\psi_2(x)$ will be in phase so that $\psi_1(x) \approx \psi_2(x)$, in which case $|\psi(x)|^2 \approx 2|\psi_1(x)|^2$. But sometimes $\psi_1(x)$ and $\psi_2(x)$ will be out of phase so that $\psi_1(x) \approx -\psi_2(x)$, in which case $|\psi(x)|^2 \approx 0$. Hence we get the interference pattern as shown in figure 1.6A).

Now in order to consider how decoherence affects interference, we let

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle |E_1(t)\rangle + |\psi_2\rangle |E_2(t)\rangle)$$

$\langle x|y\rangle = \delta(x - y)$ where $\delta(x)$ is the Dirac delta function such that

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \quad (1.19)$$

and has the property that $\int dx \delta(x) f(x) = f(0)$ for any continuous function $f(x)$. The theory of distributions allows one to deal rigorously with Dirac delta functions. E.g. see Walter Rudin, *Functional Analysis*, Second Edition (McGraw-Hill, 1991), ch. 6.

⁵⁷But if you are interested, a probability density $\rho(x)$ for a random variable X that has real values is a function such that $\rho(x) \geq 0 \forall x \in \mathbb{R}$, and that $\int_{\mathbb{R}} \rho(x)dx = 1$ and the probability that X has a value in the subset $U \subset \mathbb{R}$ is $\int_U \rho(x)dx$.

⁵⁸Here Re means the real part of a complex number. Thus if the complex number $z = \alpha + i\beta$ for real numbers α and β , then $\operatorname{Re}(z) = \alpha$. To see why the above equation holds, we recall that $|z|^2 = z\bar{z}$ and that $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$. Therefore if $z = \frac{1}{\sqrt{2}}(v + w)$ for complex number v and w ,

be the state of the composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ where \mathcal{S} is a particle that has gone through the double slit and will be detected on the detection screen, and \mathcal{E} is the local environment of the experimental set up. The expression for $|\Psi(t)\rangle$ indicates that we are assuming \mathcal{S} doesn't become entangled with \mathcal{E} when \mathcal{S} is in the state $|\psi_1\rangle$ or $|\psi_2\rangle$. Corresponding to $|\Psi(t)\rangle$ we can define the density matrix $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. We can also define the observable⁵⁹ $|x\rangle\langle x|_{\mathcal{S}}$ for the system \mathcal{S} so that $|x\rangle\langle x|_{\mathcal{S}}|\psi\rangle_{\mathcal{S}} = \psi(x)|x\rangle_{\mathcal{S}}$. As we saw in equation (1.8 on page 23 we can naturally extend the action of $|x\rangle\langle x|_{\mathcal{S}}$ to $H_{\mathcal{U}}$.⁶⁰ This allows us to define

$$\rho_{\mathcal{U}}(x, t) \stackrel{\text{def}}{=} \text{Tr}_{\mathcal{U}}(\hat{\rho}(t)_{\mathcal{U}} |x\rangle\langle x|_{\mathcal{U}}).$$

In the specific case when $\hat{\rho}_{\mathcal{U}}(t) = |\xi(t)\rangle\langle\xi(t)|_{\mathcal{U}}$ where $|\xi(t)\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}|E(t)\rangle_{\mathcal{E}}$ for normalized states $|\psi\rangle_{\mathcal{S}}$ and $|E(t)\rangle_{\mathcal{E}}$, we have $\rho_{\mathcal{U}}(x, t) = |\psi(x)|^2$ which is equal to the probability density function $\rho(x)$ we saw on page 39⁶¹ However, if $|E_1(t)\rangle_{\mathcal{E}}$ is not proportional to $|E_2(t)\rangle_{\mathcal{E}}$, then $|\Psi(t)\rangle$ will be an entangled state of \mathcal{S} and \mathcal{E} . But whether or not \mathcal{S} and \mathcal{E} are entangled, we can still use equation (1.17 to calculate the partial trace:

$$\hat{\rho}_{\mathcal{S}}(t) = \frac{1}{2}(|\psi_1\rangle\langle\psi_1|_{\mathcal{S}} + |\psi_2\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}|\psi_1\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}}|\psi_2\rangle\langle\psi_1|_{\mathcal{S}}).$$

then $|z|^2 = \frac{1}{\sqrt{2}}(v+w)\frac{1}{\sqrt{2}}\overline{(v+w)} = \frac{1}{2}(v+w)\overline{(v+w)} = \frac{1}{2}(v\bar{v} + w\bar{w} + w\bar{v} + v\bar{w}) = \frac{1}{2}(|v|^2 + |w|^2 + w\bar{v} + \bar{w}\bar{v}) = \frac{1}{2}(|v|^2 + |w|^2 + 2\text{Re}(\bar{v}w)).$

⁵⁹Note that we only call $|x\rangle\langle x|_{\mathcal{S}}$ an observable in an analogical sense since it is not a compact Hermitian operator acting on the Hilbert space of states $H_{\mathcal{S}}$. If we were being more rigorous, we would need to consider a Hermitian operator of the form $\int \sigma(x)|x\rangle\langle x|_{\mathcal{S}} dx$ for an appropriate test function $\sigma(x)$.

⁶⁰Strictly speaking, it is not $|x\rangle\langle x|_{\mathcal{S}}$ that is extended to act on $H_{\mathcal{U}}$, but rather a Hermitian operator of the form $\int \sigma(x)|x\rangle\langle x|_{\mathcal{S}} dx$ for an appropriate test function $\sigma(x)$ that is extended to $H_{\mathcal{U}}$. For a state $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}}|E\rangle_{\mathcal{E}}$, the action of $|x\rangle\langle x|_{\mathcal{U}}$ on $|\xi\rangle_{\mathcal{U}}$ gives the ‘state’ $|x\rangle\langle x|_{\mathcal{U}}|\xi\rangle_{\mathcal{U}} \stackrel{\text{def}}{=} \psi(x)|x\rangle_{\mathcal{S}}|E\rangle_{\mathcal{E}}$, but since this is not normalizable, we have to ‘smear’ it by integrating it with respect to the test function $\sigma(x)$.

⁶¹To see this, note that we can ignore \mathcal{E} in calculating $\langle|x\rangle\langle x|_{\mathcal{U}}\rangle_{\xi}$ since when \mathcal{S} and \mathcal{E} are not entangled, $\langle|x\rangle\langle x|_{\mathcal{U}}\rangle_{\xi} = \langle|x\rangle\langle x|_{\mathcal{S}}\rangle_{\psi}$ as explained in footnote 27 We can therefore just consider \mathcal{S} and drop the subscripts. Furthermore, as we saw on page 25 $\langle\hat{\Lambda}\rangle_{\psi} = \text{Tr}(\hat{\rho}\hat{\Lambda})$ where $\hat{\rho} = |\psi\rangle\langle\psi|$. We can thus take an orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle, \dots\}$ of $H_{\mathcal{S}}$ with $|\psi_1\rangle = |\psi\rangle$. Then

By equation (1.16), we therefore have

$$\rho_{\mathcal{U}}(x, t) = \frac{1}{2} \left(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2 \operatorname{Re} \left(\langle E_2(t) | E_1(t) \rangle_{\mathcal{E}} \overline{\psi_2(x)} \psi_1(x) \right) \right).^{62} \quad (1.20)$$

Thus, if $\langle E_1(t) | E_2(t) \rangle_{\mathcal{E}} \approx 0$ then $\rho_{\mathcal{U}}(x, t) \approx \frac{1}{2} (|\psi_1(x)|^2 + |\psi_2(x)|^2)$ and so we would observe a distribution pattern not exhibiting interference as shown in figure 1.6B), whereas if $\langle E_1(t) | E_2(t) \rangle_{\mathcal{E}} \not\approx 0$ we would get a distribution pattern exhibiting interference as shown in figure 1.6B). Thus, decoherence theory gives us a means of determining whether or not quantum interference will be exhibited.

`probOutcomes`

1.10 The Problem of Outcomes

In the last two sections we have seen how decoherence theory solves the preferred basis problem and the problem of the nonobservability of interference. However, there is a third fundamental problem in quantum physics which decoherence theory is unable to solve. This is the problem of outcomes. As discussed in subsection 1.7.6 in the von Neumann measurement scheme, it is supposed that for the measurement of a

$$\rho(x) = \operatorname{Tr}(\hat{\rho}|x\rangle\langle x|) = \operatorname{Tr}(|\psi\rangle\langle\psi| |x\rangle\langle x|) = \sum_i \langle\psi_i|\psi\rangle \langle\psi|x\rangle \langle x|\psi_i\rangle = \langle\psi_1|\psi\rangle \langle\psi|x\rangle \langle x|\psi_1\rangle = \langle\psi|x\rangle \langle x|\psi\rangle = \overline{\langle x|\psi\rangle} \langle x|\psi\rangle = |\langle x|\psi\rangle|^2 = |\psi(x)|^2.$$

⁶²The calculation is as follows

$$\begin{aligned} \rho_{\mathcal{U}}(x, t) &= \operatorname{Tr}_{\mathcal{U}}(\hat{\rho}(t)_{\mathcal{U}} |x\rangle\langle x|_{\mathcal{U}}) = \langle|x\rangle\langle x|_{\mathcal{U}}\rangle_{\rho(t)} = \operatorname{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}(t) |x\rangle\langle x|_{\mathcal{S}}) \\ &= \operatorname{Tr}_{\mathcal{S}} \left(\frac{1}{2} (|\psi_1\rangle\langle\psi_1|_{\mathcal{S}} + |\psi_2\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_2(t) | E_1(t) \rangle_{\mathcal{E}} |\psi_1\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_1(t) | E_2(t) \rangle_{\mathcal{E}} |\psi_2\rangle\langle\psi_1|_{\mathcal{S}}) |x\rangle\langle x|_{\mathcal{S}} \right) \\ &= \operatorname{Tr}_{\mathcal{S}} \left(\frac{1}{2} (\langle\psi_1|x\rangle_{\mathcal{S}} |\psi_1\rangle\langle x|_{\mathcal{S}} + \langle\psi_2|x\rangle_{\mathcal{S}} |\psi_2\rangle\langle x|_{\mathcal{S}} \right. \\ &\quad \left. + \langle E_2(t) | E_1(t) \rangle_{\mathcal{E}} \langle\psi_2|x\rangle_{\mathcal{S}} |\psi_1\rangle\langle x|_{\mathcal{S}} + \langle E_1(t) | E_2(t) \rangle_{\mathcal{E}} \langle\psi_1|x\rangle_{\mathcal{S}} |\psi_2\rangle\langle x|_{\mathcal{S}}) \right) \\ &= \frac{1}{2} \left(\langle\psi_1|x\rangle_{\mathcal{S}} \langle x|\psi_1\rangle_{\mathcal{S}} + \langle\psi_2|x\rangle_{\mathcal{S}} \langle x|\psi_2\rangle_{\mathcal{S}} \right. \\ &\quad \left. + \langle E_2(t) | E_1(t) \rangle_{\mathcal{E}} \langle\psi_2|x\rangle_{\mathcal{S}} \langle x|\psi_1\rangle_{\mathcal{S}} + \langle E_1(t) | E_2(t) \rangle_{\mathcal{E}} \langle\psi_1|x\rangle_{\mathcal{S}} \langle x|\psi_2\rangle_{\mathcal{S}} \right) \\ &= \frac{1}{2} \left(\overline{\langle x|\psi_1\rangle_{\mathcal{S}}} \langle x|\psi_1\rangle_{\mathcal{S}} + \overline{\langle x|\psi_2\rangle_{\mathcal{S}}} \langle x|\psi_2\rangle_{\mathcal{S}} \right. \\ &\quad \left. + \langle E_2(t) | E_1(t) \rangle_{\mathcal{E}} \overline{\langle x|\psi_2\rangle_{\mathcal{S}}} \langle x|\psi_1\rangle_{\mathcal{S}} + \overline{\langle E_2(t) | E_1(t) \rangle_{\mathcal{E}}} \overline{\langle x|\psi_2\rangle_{\mathcal{S}}} \langle x|\psi_1\rangle_{\mathcal{S}} \right) \\ &= \frac{1}{2} \left(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2 \operatorname{Re} \left(\langle E_2(t) | E_1(t) \rangle_{\mathcal{E}} \overline{\psi_2(x)} \psi_1(x) \right) \right). \end{aligned}$$

physical system \mathcal{S} to take place, it must interact with a measuring device \mathcal{A} which together satisfy the conditions 1. and 2. on page 31 If \mathcal{S} is initially in a superposition of states $|\psi\rangle = \sum_i c_i |s_i\rangle$ then for \mathcal{A} to measure \mathcal{S} , it is necessary for the combined system $\mathcal{S} + \mathcal{A}$ to enter into a superposition

$$|\psi\rangle |a_r(t_0)\rangle \xrightarrow{\text{time evolution}} \sum_i c_i |s_i\rangle |a_i(t)\rangle. \quad \text{vNevolution}\{1.21\}$$

However, although the evolution described in (1.21) must take place if \mathcal{A} is to measure \mathcal{S} , it is not sufficient. When one takes the partial trace of $|\psi\rangle\langle\psi|$ over \mathcal{A} , then according to (1.17,

$$\text{tr}_{\mathcal{A}}(|\psi\rangle\langle\psi|) \xrightarrow{\text{time evolution}} \sum_i |c_i|^2 |s_i\rangle\langle s_i|. \quad \text{vNevolution2}\{1.22\}$$

But as noted on page 30 we cannot give an ignorance interpretation to $\sum_i c_i |s_i\rangle\langle s_i|$ for as d'Espagnat puts it, this is an improper mixture. When considered together, the system \mathcal{S} and the apparatus \mathcal{A} remain in the superposition described by (1.21), and so none of the measurement outcomes from the set of possible outcomes $\{|s_i\rangle\}$ have actually occurred. The problem of explaining how the composite system $\mathcal{S} + \mathcal{A}$ goes from being in the state $\sum_i c_i |s_i\rangle |a_i(t)\rangle$ to a state $|s_i\rangle |a_i(t)\rangle$ is known as the **problem of outcomes**.

manyworldsinterpretation1 1.11 The Many-Worlds Interpretation

Not everyone is convinced that the problem of outcomes is a genuine problem. In particular, people who endorse the many-worlds interpretation of quantum physics effectively argue that there are no outcomes in the traditional sense. In this section, we now give an account of the many-worlds interpretation of quantum physics and why physicists find it attractive. To this end, let us consider a physical universe

$\mathcal{U} = \mathcal{S} + \mathcal{A} + \mathcal{P}_A + \mathcal{P}_B + \mathcal{E}$ consisting of subsystems $\mathcal{S}, \mathcal{A}, \mathcal{P}_A, \mathcal{P}_B$ and \mathcal{E} . \mathcal{S} is the physical system under investigation; \mathcal{A} is some measuring apparatus that interacts with \mathcal{S} ; \mathcal{P}_A and \mathcal{P}_B are the physical systems corresponding to two scientists, Alice and Bob who observed the apparatus \mathcal{A} ; and \mathcal{E} is the remainder of the physical universe \mathcal{U} . For convenience, we define the composite subsystem $\mathcal{V} = \mathcal{S} + \mathcal{A} + \mathcal{P}_A + \mathcal{P}_B$ so that $\mathcal{U} = \mathcal{V} + \mathcal{E}$. As above on page 31 we assume that there is an orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$ which we again refer to as pointer states, but now we assume that there are ready states $|a_r(t)\rangle \in H_{\mathcal{A}}, |p_{r,A}(t)\rangle \in H_{\mathcal{P}_A}, |p_{r,B}(t)\rangle \in H_{\mathcal{P}_B}$, and $|E_r(t)\rangle \in H_{\mathcal{E}}$ and that for each i , there are normalized states $|a_i(t)\rangle \in H_{\mathcal{A}}, |p_{i,A}(t)\rangle \in H_{\mathcal{P}_A}, |p_{i,B}(t)\rangle \in H_{\mathcal{P}_B}$, and $|E_i(t)\rangle \in H_{\mathcal{E}}$ such that

1. for any $t \geq 0$ we have the evolution of the states

$$\begin{aligned} &|s_i\rangle |a_r(t)\rangle |p_{r,A}(t)\rangle |p_{r,B}(t)\rangle |E_r(t)\rangle \\ &\xrightarrow{\text{time evolution}} |s_i\rangle |a_i(t)\rangle |p_{i,A}(t)\rangle |p_{i,B}(t)\rangle |E_i(t)\rangle, \end{aligned}$$

2. there exists $\delta > 0$ such that if $t > t_0 + \delta$, then for $i \neq j$, $\langle a_i(t)|a_j(t)\rangle \approx 0$,

$$\langle p_{i,A}(t)|p_{j,A}(t)\rangle \approx 0, \langle p_{i,B}(t)|p_{j,B}(t)\rangle \approx 0 \text{ and } \langle E_i(t)|E_j(t)\rangle \approx 0.^{63}$$

We also suppose that the $|p_{i,A}(t)\rangle$ -state would describe actions of Alice consistent with her observing the apparatus being in the $|a_i(t)\rangle$ -state, for example, her writing down in her log book that the apparatus is in the $|a_i(t)\rangle$ -state or telling her colleague that this is the case. Likewise, we assume the $|p_{i,B}(t)\rangle$ -state is consistent with Bob also observing the apparatus to be in the $|a_i(t)\rangle$ -state.

⁶³Again, recall footnote 39

Now suppose the initial (normalized) state of \mathcal{S} is $|\psi\rangle = \sum_i c_i |s_i\rangle$, so that the state for the composite system \mathcal{U} is $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ where $|\xi_i(t)\rangle = |s_i\rangle |a_i(t)\rangle |p_{i,A}(t)\rangle |p_{i,B}(t)\rangle$. We also define the corresponding density matrix for the composite system $\hat{\rho} = |\Psi\rangle\langle\Psi|$. If we were unable to make any observations on \mathcal{E} , then the partial trace $\hat{\rho}_{\mathcal{V}}(t) = \text{Tr}_{\mathcal{E}}(\hat{\rho}(t))$ will contain all the information we need to work out the expectation values for any observables of \mathcal{V} . So just as with equation (1.17), we will have

$$\begin{aligned}\hat{\rho}_{\mathcal{V}}(t) &= \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)| + \sum_{i \neq j} c_i \bar{c_j} \langle E_j(t)|E_i(t)\rangle |\xi_i(t)\rangle\langle\xi_j(t)| \\ &\approx \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)|\end{aligned}\quad \begin{matrix} \{\text{reduced by}\} \\ (1.23) \end{matrix}$$

for $t > t_0 + \delta$. Then the expectation values of any observables on \mathcal{V} will be indistinguishable from the scenario in which \mathcal{V} is actually in one of the $|\xi_i(t)\rangle$ -states with probability $|c_i|^2$.⁶⁴ It would nevertheless be incorrect for us to conclude on the basis of decoherence theory alone that \mathcal{V} actually was in one of those $|\xi_i(t)\rangle$ -states, since equation (1.23) is based on a subjective distinction between \mathcal{V} and \mathcal{E} in the decomposition $\mathcal{U} = \mathcal{V} + \mathcal{E}$. Human scientists make this distinction to reflect the fact that they can only perform measurements on \mathcal{V} and can't measure \mathcal{E} . But if a super-intelligent being could measure everything in \mathcal{U} , then such a being would not say that \mathcal{V} was in one of the $|\xi_i(t)\rangle$ -states, but rather that \mathcal{U} was in the state $|\Psi(t)\rangle$. As we have already discussed on pages 30-31 the density matrix $\hat{\rho}_{\mathcal{V}}(t)$ is not a mixed state, but is an improper mixture.

⁶⁴Again recall the discussion following equation (1.12) on page 26. There is the question of uniqueness of $\hat{\rho}_{\mathcal{V}} = \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)|$. If all the $|c_i|^2$ are unique, then if we have another decomposition $\hat{\rho}_{\mathcal{S}+\mathcal{A}+\mathcal{P}_A+\mathcal{P}_B} = \sum_i |c_i|^2 |\xi'_i(t)\rangle\langle\xi'_i(t)|$ it follows that $|\xi_i(t)\rangle \propto |\xi'_i(t)\rangle$. But even if some of the $|c_i|^2$ are the same, criteria 1 and 2 above will ensure that states with the same value of $|c_i|^2$ will be determined up to permutation.

Now if we define the observables $\hat{\Lambda}_{i,\mathcal{A}}(t) = |p_{i,A}(t)\rangle\langle p_{i,A}(t)|$ that would measure the behavior of Alice, and the observables $\hat{\Lambda}_{i,\mathcal{B}}(t) = |p_{i,B}(t)\rangle\langle p_{i,B}(t)|$ that would measure the behavior of Bob, then for $t > t_0 + \delta$, we see that $\hat{\Lambda}_{i,\mathcal{A}}(t)\hat{\Lambda}_{j,\mathcal{B}}(t)|\Psi(t)\rangle \approx 0$, when $i \neq j$. This means that when we consider $\hat{\Lambda}_{i,\mathcal{A}}(t)\hat{\Lambda}_{j,\mathcal{B}}(t)$ as an observable acting on \mathcal{V} , the expectation value $\langle\hat{\Lambda}_{i,\mathcal{A}}(t)\hat{\Lambda}_{j,\mathcal{B}}(t)\rangle_{\rho_{\mathcal{V}}(t)}$ will be approximately zero for $i \neq j$. What this means is that if we consider ourselves as observing Alice and Bob observing the apparatus, then after time $t_0 + \delta$, the probability we would see Alice and Bob disagreeing with each other concerning their observations of the apparatus would be approximately 0. On the other hand, since $\hat{\Lambda}_{i,\mathcal{A}}(t)\hat{\Lambda}_{i,\mathcal{B}}(t)|\Psi(t)\rangle \approx c_i |\xi_i(t)\rangle |E_i(t)\rangle$ for $t > t_0 + \delta$, it follows that $\langle\hat{\Lambda}_{i,\mathcal{A}}(t)\hat{\Lambda}_{i,\mathcal{B}}(t)\rangle_{\rho_{\mathcal{V}}(t)} = |c_i|^2$. We would thus observe Alice and Bob observing the apparatus to be in the $|a_i(t)\rangle$ -state with probability $|c_i|^2$.

But note that on the assumption that there are no hidden variables, if we did actually make such an observation and this observation corresponded to reality, then the quantum state $|\Psi(t)\rangle$ would have had to have changed to $|\xi_i(t)\rangle |E_i(t)\rangle$, since before our observation when $|\Psi(t)\rangle$ was a complete description of \mathcal{U} , we would say Alice and Bob will measure the $|a_i(t)\rangle$ -state with probability $|c_i|^2$, but when we are actually seeing them measuring the $|a_i(t)\rangle$ -state, we would have to say that now the probability they are measuring the $|a_i(t)\rangle$ -state is 1, and hence we would say that the system was in the $|\xi_i(t)\rangle |E_i(t)\rangle$ -state. Whether or not the process of the state going from $|\Psi(t)\rangle$ to $|\xi_i(t)\rangle |E_i(t)\rangle$ was instantaneous or took a non-infinitesimal amount of time, this interpretation would be susceptible to the problems already discussed with the Copenhagen interpretation on page 17

But in the **many-worlds interpretation**, rather than assuming that $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ is the complete description of \mathcal{U} that enables us to work the probability of certain outcomes, we simply say that $|\Psi(t)\rangle$ is a complete description of the state of \mathcal{U} , and we drop the assumption that we need to interpret this state as describing probabilities of outcomes. Thus a many-worlds adherent would say we can understand what the state of \mathcal{U} is on its own terms without the need to appeal to any other extrinsic principle such as measurement. Just as we don't puzzle over how to interpret what a sphere is in terms of an extrinsic principle, we don't need to puzzle over how to interpret the states of \mathcal{U} . We can think of the mathematical formalism $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ describing the state of \mathcal{U} as being somewhat akin to the equation $x^2 + y^2 + z^2 = 1$ describing a sphere. Although we might be tempted to interpret $|\Psi(t)\rangle$ as describing the probability of outcomes, we are not obliged to do so, since these probabilities can instead be understood to be grounded in the symmetries the system possesses rather than in terms of the frequency of how many measurement outcomes are likely to occur. For instance, when we see a coin and judge that it will come up heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$, we intuit this by looking at the symmetry of the coin rather than tossing the coin millions of times and counting how often it comes up heads and how often it comes up tails.

As for the decomposition $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ in terms of the $|\xi_i(t)\rangle |E_i(t)\rangle$ basis states, decoherence theory gives us a natural account of why we should choose this basis rather than any other. When \mathcal{U} is in the state $|\Psi(t_0)\rangle = \left(\sum_i c_i |\xi_i(t_0)\rangle \right) |E_r(t)\rangle$, we can think of this state as describing one world, W say. But once $t > t_0 + \delta$ so that $\langle E_i(t)|E_j(t)\rangle \approx 0$ for $i \neq j$, we can think of each $|\xi_i(t)\rangle |E_i(t)\rangle$ -component as a

different world W_i . Thus, for $t > t_0 + \delta$, we say the world W has **branched** into as many-worlds W_i for which the c_i are non-zero.

But why should we think that there are literally many worlds? Well, from an ontological point of view, one might very well think that there is really only one world and that this world is described by $|\Psi(t)\rangle$; it would be a rather weird world since the entanglement between \mathcal{V} and \mathcal{E} would mean there wouldn't be any absolute matters of fact describing \mathcal{V} . But it might not be a bad thing to say that the “many” in the many-worlds interpretation is really just a figure of speech that we shouldn't take too literally, since a common objection to the many-worlds interpretation is that it is ontologically extravagant and that we should appeal to Occam's Razor. However, if we just say that there is actually only one world described by $|\Psi(t)\rangle$ then this “many”-worlds interpretation is actually rather parsimonious from an ontological point of view.

But if by literal, we mean descriptive rather than ontological, it does seem rather natural to say that there are literally many worlds. For although we might initially suspect that the worlds W_i and W_j are not well-defined given the fact that $\langle E_i(t)|E_j(t)\rangle$ is very small but not zero for $i \neq j$, we can nevertheless expect $\langle \xi_i(t)|\xi_j(t)\rangle$ to be identically zero for $i \neq j$, just as we can expect $\langle \hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle$ to be identically zero.⁶⁵ Thus, if we define $|W_i(t)\rangle = |\xi_i(t)\rangle |E_i(t)\rangle$, then the $\langle W_i(t)|W_j(t)\rangle$ will be identically zero for $i \neq j$.

⁶⁵It is also reasonable to suppose that in situations such as the double-slit experiment described on page 38hat $\langle \psi_1(t)|\psi_2(t)\rangle$ is identically zero. This is because $\langle \psi_1(t)|\psi_2(t)\rangle$ when t is the time at which the particle is going through the slit, and this will remain zero because of a property of the time evolution operator known as unitarity.

Still, the supposition that $|\Psi(t)\rangle = \sum_i c_i |W_i(t)\rangle$ with $\langle W_i|W_j\rangle = 0$ for $i \neq j$ is not of itself sufficient justification for describing the state $|\Psi(t)\rangle$ as a composition of mutually exclusive world descriptions given by the $|W_i(t)\rangle$. After all, the fact that

$$\begin{aligned} |\text{Cat Alive}\rangle &= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle) \right) \\ &\quad + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle) \right). \end{aligned}$$

does not incline us to think of the state $|\text{Cat Alive}\rangle$ as being composed of the mutually exclusive cat states $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$.

The key justification for describing the state $|\Psi(t)\rangle$ as a composition of the mutually exclusive $|W_i(t)\rangle$ -states is the fact that the states $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ decohere for $i \neq j$, that is, the off-diagonal entries $|\xi_i(t)\rangle\langle\xi_j(t)|$ of the reduced density matrix $\hat{\rho}_{\mathcal{V}}(t)$ will tend to zero, and as we saw in section 1.9 it will follow that quantum interference effects between $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ will then tend to zero. Thus, when it comes to observables defined on \mathcal{V} , using equations (1.16 and (1.23, we can calculate the expectation value of an observable $\hat{\Lambda}_{\mathcal{V}}$ as a weighted sum of expectation values for each of the states $|\xi_i(t)\rangle$:

$$\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\Psi(t)} \approx \sum_i |c_i|^2 \langle \hat{\Lambda}_{\mathcal{V}} \rangle_{\xi_i(t)}. \quad \text{\{manyapprox\}}_{(1.24)}$$

The fact that (1.24 is only an approximation suggests that the time at which branching occurs is not well-defined. All that we can do is choose a time sufficiently large δ so that for $t > t_0 + \delta$, the approximation (1.24 meets our desired level of accuracy.

Despite this vagueness on when branching occurs, we can still form a natural and well-defined notion of worlds according to the following definition: ^{rigorousworld} a set $\{W_i : i\}$ is the set of worlds for a universe $\mathcal{U} = \mathcal{V} + \mathcal{E}$ when

1. W_i is a description of \mathcal{U} given by $|W_i(t)\rangle = |\xi_i(t)\rangle |E_i(t)\rangle$,
2. \mathcal{U} is in the state $|\Psi(t)\rangle = \sum_i c_i |W_i(t)\rangle$ with all $c_i \neq 0$.
3. $\langle \xi_i(t) | \xi_j(t) \rangle = 0$ for $i \neq j$,
4. $\langle E_i(t) | E_j(t) \rangle \rightarrow 0$ as $t \rightarrow \infty$ for $i \neq j$ and the convergence is such that for any observable $\hat{\Lambda}_{\mathcal{V}}$ defined on \mathcal{V} , $\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\Psi(t)} \rightarrow \sum_i |c_i|^2 \langle \hat{\Lambda}_{\mathcal{V}} \rangle_{\xi_i(t)}$.

Note that according to this definition, the description $|\Psi(t)\rangle$ is rather trivially a world – we just take the environment \mathcal{E} to be empty so that there would be only one $|E_i(t)\rangle$ which would be the vacuum state. So there is at least one world according to this definition. There is a question of whether there could be more than one world and this would depend on whether we could really have a non-trivial decomposition $\mathcal{U} = \mathcal{V} + \mathcal{E}$, for the supposition that there is such a decomposition requires that it is possible to distinguish \mathcal{V} and \mathcal{E} , but this might not in fact be possible. For instance, if the ultimate fate of the universe was that it would collapse into a singularity, then there would come a point at which it wouldn't be possible to make a distinction between \mathcal{V} and \mathcal{E} . But despite this possible concern, the above definition makes it seem plausible that there could be many well-defined worlds W_i .⁶⁶

⁶⁶Recall that the purpose of this chapter is only to show why physicists might find the many-worlds interpretation of quantum physics attractive. It is not the purpose of this chapter to show that the many-worlds interpretation is without any problems.

When we look at a particular $|\xi_i(t)\rangle$ it will look like it is describing a fairly classical world with scientists performing their measurements and agreeing about what they measure. And as long as the $|\xi_i(t)\rangle$ -states remain pointer states with respect to $|E_i(t)\rangle$ no branching will occur. But typically, a $|\xi_i(t)\rangle$ -state will not indefinitely remain a pointer state with respect to $|E_i(t)\rangle$. We can think of how this happens with the Stern-Gerlach experiment. For if one Stern-Gerlach apparatus has its magnetic field orientated in the $\hat{\mathbf{a}}$ -direction, then $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ will be pointer states for a silver atom in the vicinity of this apparatus. But if the same silver atom then travels onward to another Stern-Gerlach apparatus with its magnetic field now orientated in the $\hat{\mathbf{b}}$ -direction, $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ will no longer be pointer states with respect to their environment, and so branching will occur. But this is not necessarily a problem for the definition of many-worlds given above on page 50 for when a $|\xi_i(t)\rangle$ -state does not indefinitely remain a pointer state with respect to $|E_i(t)\rangle$, we can just rewrite $|\xi_i(t)\rangle$ as a sum of pointer states $|\xi_{ij}(t)\rangle$ and $|E_i(t)\rangle$ as a sum of their respective environments $|E_{ij}(t)\rangle$, and then $|E_i(t)\rangle$ will be like a ready state for the $|E_{ij}(t)\rangle$. Assuming we can do this so that the $|\xi_{ij}(t)\rangle$ are orthogonal to the $|\xi_{i'j'}(t)\rangle$ when $i' \neq i$ or $j' \neq j$, then we would still be able to have well-defined worlds according to the definition given above.

manyworldsinterpretation2

1.12 Evaluating the Many-Worlds Interpretation

Given the above account of the many-worlds hypothesis of quantum physics, it does seem understandable why physicists would find it so attractive. Although we can't specify an exact moment at which branching occurs, the idea of branching and of there being many worlds itself is not particularly mysterious. This can all be explained in

terms of the dynamics of the system and the environment, and decoherence theory allows us to understand why the interference effects that are the hallmark of quantum physics generally disappear on the macroscopic level.

There are other advantages of the many-worlds hypothesis besides these which we need not discuss here.⁶⁷ But for all the advantages of the many-worlds hypothesis, there is one fundamental problem, and that is its patent absurdity. It seems that we should be able to say whether a cat is alive or dead without having to say what state the rest of the universe is in. However, the many-worlds hypothesis suggests that for any subsystem of the universe, we will in general only be able to say what state it is in with respect to the state of the rest of the universe. For example, if the state \mathcal{S} is the system constituting a cat-wise configuration of particles and \mathcal{E} is the rest of the universe, then given that the composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ is described by the state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle_{\mathcal{S}}|E_{\text{Cat Alive}}\rangle_{\mathcal{E}} + |\text{Cat Dead}\rangle_{\mathcal{S}}|E_{\text{Cat Dead}}\rangle_{\mathcal{E}}),$$

then we are in no position to make an absolute matter of fact claim about the system \mathcal{S} and say the cat is dead or the cat is alive. Rather we have to say with respect to the environment described by $|E_{\text{Cat Alive}}\rangle_{\mathcal{E}}$, the cat is alive, and with respect to the environment $|E_{\text{Cat Dead}}\rangle_{\mathcal{E}}$, the cat is dead. According to the many-worlds hypothesis, the branching into multiple worlds doesn't just occur in rare instances, such as in Schrödinger's cat type experiments. On the contrary, branching is supposed to be happening all the time. Now anyone who has a common sense understanding of science would say that science enables us to understand absolute matters of fact

⁶⁷ More details can be found in Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* and E. Joos et al., *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer Berlin Heidelberg, 2013).

about subsystems of the universe and the principles that govern them. But if there really are no such matters of fact, then we have to abandon this common sense understanding of science. Intuitively, it also seems obvious that I can know I am alive without needing to know the state of the rest of the universe, but the many-worlds hypothesis does not allow me to make this absolute matter of fact claim. So from a common sense point of view, the many-worlds hypothesis really is absurd.

Of course some hypotheses may initially seem absurd, but once the hypothesis has been fully explained, it can appear far more plausible. For instance, time dilation in Special Relativity might initially sound absurd to some people, but once one has a better grasp of Special Relativity and is open to the possibility that systems moving close to the speed of light with respect to ourselves might have properties rather different to systems that move with much slower speeds, then Special Relativity doesn't seem absurd at all.

However, the many-worlds hypothesis as presented here is different in this regard since it is not hypothesizing about some extreme situation. It is hypothesizing about ordinary situations. And one can have a fairly good understanding of the many-worlds hypothesis and still find it absurd. Some people choose to embrace the absurdity and reject common sense. But throughout much of human history, when a hypothesis has entailed an absurd conclusion, reasonable people have usually thought it better to reject the hypothesis rather than embrace the absurdity.

But in rejecting a hypothesis as absurd, it doesn't mean that absolutely everything in the hypothesis needs to be rejected, for a hypothesis might be formulated in terms of

sub-hypotheses, some of which might be very plausible, in which case something of the original hypothesis might be salvageable. In the case of the many-worlds hypothesis, I believe it does have something that is salvageable, namely decoherence theory. In the next chapter I will consider Adrian Kent's one-world interpretation of quantum physics in which the basic ideas of decoherence theory remain intact.

Chapter 2

Evaluating Kent's Solution to the Measurement Problem

In the previous chapter, we saw how a denial of the Copenhagen interpretation of quantum mechanics and a denial of hidden variables leads fairly naturally to a so-called Many-Worlds interpretation of quantum mechanics. However, the Many-Worlds interpretation seems to be radically opposed to the view that we can make common sense truth claims about the physical world. A strategy among some philosophers of physics who do not wish to endorse the Many-Worlds interpretation is therefore to reexamine the assumptions that lead to Bell's Inequality. One of these assumptions will have to be discarded since Bell's Inequality is experimentally violated. The false assumption that is used to prove Bell's Inequality is sometimes referred to as **the culprit**. We therefore need to identify the culprit, that is we need to decide which assumption we should discard while keeping in mind that we wish to maintain a theory that is compatible with the experimental findings of quantum physics and special relativity.

Shimony, noticed that there are two key assumptions in the proof of Bell's Inequality that might be identified as the culprit. He refers to one assumption as Outcome Independence (OI), and to the other assumption as Parameter Independence (PI).¹ Shimony argued that if we only denied OI, then the proof of Bell's Inequality would fail to go through. Yet by continuing to assume PI, there is a sense in which Special Relativity is not obviously violated. Shimony therefore thought that denying OI and assuming PI was sufficient to ensure peaceful coexistence between quantum theory and special relativity. In other words, Shimony thought OI was the culprit. Butterfield², however, argues that although PI is a necessary assumption if there is to be peaceful coexistence between quantum theory and special relativity, PI together with the denial of OI independence is not sufficient to ensure peaceful coexistence, the reason being that Shimony offers no account of what an outcome is. As mentioned in the previous chapter, the problem of outcomes is an important issue that d'Espagnat highlights. It is Shimony's failure to address the problem of outcomes that motivated Butterfield to explore whether Kent's interpretation of quantum physics provides what is lacking in Shimony's account.

In this chapter, I will present and evaluate Kent's interpretation of quantum physics. In order to provide such an evaluation, it will be necessary show that the predictions of Kent's theory do not contradict the predictions of quantum theory that have been experimentally validated. We also need to show that Kent's theory also possesses the

¹See A. Shimony, "Events and processes in the quantum world," in *Search for a Naturalistic World View: Volume II: natural science and metaphysics* (1986; Cambridge: Cambridge University Press, 1993), 146–147.

²See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem."

symmetries, known as Lorentz invariance, that belong to a theory that is consistent with Einstein's theory of Special Relativity. But we also need to address Shimony's question of identifying the culprit, and to this end, we will need to explain what is meant by OI and PI. In recent years, Leegwater et al.³ have also proved an important theorem, the so-called ‘Collbeck-Renner Theorem’ regarding PI which suggest that if PI holds together with what is called a ‘no conspiracy’ criterion, then this implies standard quantum theory without any additional variables. And finally, we will have to consider the extent to which Kent's interpretation yields a convincing account of what an outcome is.

We will first turn our attention to the notions of OI and PI.

2.1 Outcome Independence versus Parameter Independence

To explain Shimony's⁴ notion of Outcome Independence and Parameter Independence, we suppose we have an experimental setup similar to the experimental setup described in section 1.3n Bell's Inequality. Thus, we suppose there are two particles labeled q_A , and q_B , and that a measurement can be made on particle q_A at one location (e.g. Alice's laboratory), and a measurement can be made on particle q_B at some other location (e.g. Bob's laboratory). Alice can make a choice of one of n mea-

³See Gijs Leegwater, “An impossibility theorem for parameter independent hidden variable theories,” *Studies in History and Philosophy of Modern Physics* 54 (2016): 18–34, Roger Colbeck and Renato Renner, “No extension of quantum theory can have improved predictive power,” *Nature communications* (England) 2, no. 1 (2011): 411–411, Roger Colbeck and Renato Renner, “The completeness of quantum theory for predicting measurement outcomes,” 2012, K Landsman, “On the Colbeck–Renner theorem,” *Journal of mathematical physics* (United States) 56, no. 12 (2015): 122103, and Klaas Landsman, *Foundations of Quantum Theory : From Classical Concepts to Operator Algebras (Volume 188.0)*, vol. 188, Fundamental Theories of Physics (Cham: Springer Open, 2017).

⁴See Shimony, “Events and processes in the quantum world,” 146–147 and Butterfield, “Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem,” 7–9.

surements to be made labeled a_1, \dots, a_n . For example, a_1 might be a measurement of q_A 's spin along the z -axis, whereas a_2 might be the measurement of q_A 's spin along an axis that is at a 45° angle to the z -axis etc.. We use the variable x to denote Alice's choice so that $x = a_i$ for some $i \in \{1, \dots, n\}$. If Alice chooses to make measurement a_i (i.e. $x = a_i$), the measurement outcome is labeled A_i and can take values $+1$ or -1 . For example, Alice could use the convention in which $+1$ corresponds to a spin up outcome, and -1 corresponds to a spin down outcome. We will use the variable X to denote the measurement outcome Alice obtains, so for example, if Alice chooses to make the a_1 measurement so that $x = a_1$ and obtains the outcome $A_1 = 1$, then $X = 1$. In a similar way, we use the notation b_i, y and B_i, Y to correspond to the measurement choices and measurement outcomes for Bob.

We now suppose that there is a complete state $\lambda \in \Lambda$ describing both q_A and q_B that is independent of Alice and Bob's measurement choices, but that encodes all other features that would influence the corresponding measurement outcomes. Here, the domain Λ of all such complete states will depend on how the two particles are prepared and the model we are assuming. We also assume that q_A and q_B are initially coupled together in such a way that Alice and Bob would always get opposite results when they made their measurements in the same direction. For instance, for $n = 3$, we might assume a model in which

$$\Lambda = \{(A_1, A_2, A_3, B_1, B_2, B_3) : A_1, A_2, A_3 = \pm 1, B_i = -A_i\}^{\text{bellLambda}} \quad (2.1)$$

In this case, $\lambda \in \Lambda$ would fully determine Alice and Bob's measurement outcomes. This would be like the model described in the proof of Bell's Inequality with all the states of Λ being described in table 1.1f section 1.3

However, in general, we don't insist on such determinism. Rather, we suppose that given a complete state $\lambda \in \Lambda$, and given that Alice makes measurement x and Bob makes measurement y , then there will be a probability $P_{\lambda,x,y}(X, Y)$ representing the probability Alice gets outcome X and Bob gets outcome Y . It is only in deterministic models that $P_{\lambda,x,y}(X, Y)$ will only have values restricted to either 0 or 1. In non-deterministic models, there will have to be some situations when $P_{\lambda,x,y}(X, Y)$ has a value strictly between 0 and 1. For example, if our model was standard quantum mechanics, we could take λ to be the Bell state (1.4). Then it follows from equation (1.5) that as long as Alice's and Bob's measurement choices x and y are in the same direction, then $P_{\lambda,x,y}(1, -1) = 1/2$. Incidentally, we also note that equation (1.5) implies the domain Λ consists of a single state:

$$\Lambda = \left\{ \frac{1}{\sqrt{2}} (|{\hat{a}}+\rangle_A |{\hat{a}}-\rangle_B - |{\hat{a}}-\rangle_A |{\hat{a}}+\rangle_B) \right\}. \quad \{\text{quantumLambda}\}_{(2.2)}$$

In both models (2.1 and (2.2) we see that if we define

$$P_{A,\lambda,x,y}(X) = P_{\lambda,x,y}(X, 1) + P_{\lambda,x,y}(X, -1), \quad \{\text{PIone}\}_{(2.3)}$$

$$P_{B,\lambda,x,y}(Y) = P_{\lambda,x,y}(1, Y) + P_{\lambda,x,y}(-1, Y), \quad \{\text{PItwo}\}_{(2.4)}$$

then $P_{A,\lambda,x,y}(X)$ is independent of Bob's choice of measurement y and $P_{B,\lambda,x,y}(Y)$ is independent of Alice's choice of measurement x .⁵ In other models, however, it's possible that such independence does not hold. So to distinguish between such possibilities, we say a model satisfies **Parameter Independence** (PI) if and only if $P_{A,\lambda,x,y}(X)$ is independent of y and $P_{B,\lambda,x,y}(Y)$ is independent of x . In other words, PI holds if and only if (2.3 and (2.4 hold for all λ, x, y, X , and Y .

One model in which PI fails to hold is the **pilot wave interpretation** of quantum mechanics. In this interpretation, it is assumed that at any instant of time t , the particles q_A and q_B will have definite positions \mathbf{x}_A and \mathbf{x}_B and definite momenta \mathbf{p}_A and \mathbf{p}_B respectively. But in addition, it is also assumed that there is a so-called **pilot wave**

$$\psi(\mathbf{x}_A, \mathbf{x}_B, t) = r(\mathbf{x}_A, \mathbf{x}_B, t) e^{iS(\mathbf{x}_A, \mathbf{x}_B, t)} \quad (2.5)$$

⁵It is easy to see that $P_{A,\lambda,x,y}(X) = 1/2$ and $P_{B,\lambda,x,y}(Y) = 1/2$ for any X, Y . E.g. for $x = \hat{\mathbf{a}}$ and $y = \hat{\mathbf{b}}$, by (1.5), we can assume the two particles are in the state

$$|\zeta\rangle = \frac{1}{\sqrt{2}} (|\hat{\mathbf{b}}+\rangle_A |\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A |\hat{\mathbf{b}}+\rangle_B).$$

Since the inner product on the composite system is given by $\langle \xi' | \xi \rangle = \langle \psi' | \psi \rangle_A \langle \chi' | \chi \rangle_B$ for $|\xi\rangle = |\psi\rangle_A |\chi\rangle_B$ and $|\xi'\rangle = |\psi'\rangle_A |\chi'\rangle_B$, it follows that

$${}_A \langle \hat{\mathbf{a}}+ | {}_B \langle \hat{\mathbf{b}}\pm | \zeta \rangle = \mp \frac{1}{\sqrt{2}} \langle \hat{\mathbf{a}}+ | \hat{\mathbf{b}}\mp \rangle_A$$

Therefore, by the Born Rule (see page 8)

$$P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) + P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+, \hat{\mathbf{a}}-) = \frac{1}{2} | \langle \hat{\mathbf{a}}+ | \hat{\mathbf{b}}+ \rangle_A | + \frac{1}{2} | \langle \hat{\mathbf{a}}+ | \hat{\mathbf{b}}- \rangle_A | .$$

But since

$$| \hat{\mathbf{a}}+ \rangle_A = \langle \hat{\mathbf{b}}+ | \hat{\mathbf{a}}+ \rangle_A | \hat{\mathbf{b}}+ \rangle_A + \langle \hat{\mathbf{b}}- | \hat{\mathbf{a}}+ \rangle_A | \hat{\mathbf{b}}- \rangle_A$$

it follows that

$$| \langle \hat{\mathbf{a}}+ | \hat{\mathbf{b}}+ \rangle_A | + | \langle \hat{\mathbf{a}}+ | \hat{\mathbf{b}}- \rangle_A | = 1.$$

Therefore

$$P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) + P_{\lambda, x, y}(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-) = \frac{1}{2} .$$

where $r(\mathbf{x}_A, \mathbf{x}_B, t) > 0$ is the magnitude of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$, and the real-valued function $S(\mathbf{x}_A, \mathbf{x}_B, t)$ is the phase of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$. The time evolution of the pilot-wave is deterministically governed by the Schrödinger equation, and the phase $S(\mathbf{x}_A, \mathbf{x}_B, t)$ determines the positions \mathbf{x}_A and \mathbf{x}_B to the momenta \mathbf{p}_A and \mathbf{p}_B via the gradient of S :

$$\mathbf{p}_A = \nabla_A S(\mathbf{x}_A, \mathbf{x}_B), \quad \mathbf{p}_B = \nabla_B S(\mathbf{x}_A, \mathbf{x}_B). \quad (2.6)$$

In other words, if we fix \mathbf{x}_B and consider S to be just a function of \mathbf{x}_A , then the momentum \mathbf{p}_A is in the direction and has the magnitude of the steepest ascent of S considered as a function of \mathbf{x}_A , and the momentum \mathbf{p}_B is determined in a similar fashion.

In reality, we don't know the exact positions of all the particles, but based on what we know about an experimental setup, we can average over our uncertainty and recover exactly the same predictions that quantum mechanics would make.⁶ So for instance, our knowledge of the experimental setup above should enable us both to know that q_A and q_B are contained within a region V , and also for us to workout the probability $p(V_i, V_j)$ that particle q_A will be in a region V_i and q_B will be in a region V_j , where the V_i are small non-overlapping regions such that $V = \bigcup_i V_i$. Then given some physical quantity $\Lambda(\mathbf{x}_A, \mathbf{x}_B)$ that depends on the positions \mathbf{x}_A and \mathbf{x}_B of the two particles, then when the regions V_i are sufficiently small so that $\Lambda(\mathbf{x}_i, \mathbf{x}_j)$ varies negligibly for any $\mathbf{x}_i \in V_i$ and $\mathbf{x}_j \in V_j$, the average value

$$\langle \Lambda \rangle = \sum_{i,j} p(V_i, V_j) \Lambda(\mathbf{x}_i, \mathbf{x}_j) \quad \text{\{bohmconsistency\}} \quad (2.7)$$

⁶See David Bohm, “A Suggested Interpretation of the Quantum Theory in Terms of ”Hidden” Variables. I,” *Physical review* 85, no. 2 (1952): 166–179 and David Bohm, “A Suggested Interpretation of the Quantum Theory in Terms of ”Hidden” Variables. II,” *Physical review* 85, no. 2 (1952): 180–193.

will be the same as the expectation value for Λ predicted by standard quantum mechanics.⁷

So to see why PI fails to hold in the pilot wave model,⁸ we first note that since the pilot wave model makes the same predictions as quantum mechanics when averaged over all the hidden variables, the violation of Bell's inequality (2.8) implies there must be some hidden variable λ and choices of measurement directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ such that

$$P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) > P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+). \quad \text{(2.8)}$$

Since physics in the pilot wave model is deterministic, probabilities must be either 0 or 1. Therefore, the only way (2.8) can be satisfied is for

$$P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = 1 \quad \text{(2.9)}$$

$$P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) = 0 \quad \text{(2.10)}$$

$$P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) = 0 \quad \text{(2.11)}$$

We suppose that PI holds and we will try to arrive at a contradiction. If both Alice and Bob make their measurement in the $\hat{\mathbf{c}}$ -direction, there are two possibilities: either Alice measures q_A to be in the state $\hat{\mathbf{c}}-$ and Bob measures q_B to be in the state $\hat{\mathbf{c}}+$, or Alice measures q_A to be in the state $\hat{\mathbf{c}}+$ and Bob measures q_B to be in the state $\hat{\mathbf{c}}-$. So expressed in terms of probabilities, these two possibilities are

⁷In this explanation, I've refrained from using measure theory, but basically this explanation is saying that when we construct a measure μ on $V \times V$ based on our knowledge of the experimental setup, $\int_{V \times V} \Lambda(\mathbf{x}_i, \mathbf{x}_j) d\mu$ will be the same as the expectation value for Λ predicted by standard quantum physics.

⁸This is my own proof

equivalent to either

$$P_{\lambda, \hat{c}, \hat{c}}(\hat{c}-; \hat{c}+) = 1 \quad \text{and} \quad P_{\lambda, \hat{c}, \hat{c}}(\hat{c}+; \hat{c}-) = 0. \quad \{\text{PDproofcase1}\} \quad (2.12)$$

or

$$P_{\lambda, \hat{c}, \hat{c}}(\hat{c}+; \hat{c}-) = 1 \quad \text{and} \quad P_{\lambda, \hat{c}, \hat{c}}(\hat{c}-; \hat{c}+) = 0. \quad \{\text{PDproofcase2}\} \quad (2.13)$$

Let's first consider case (2.12). Note that

$$P_{\lambda, \hat{c}, \hat{c}}(\hat{c}+; \hat{c}-) + P_{\lambda, \hat{c}, \hat{c}}(\hat{c}-; \hat{c}+) = 0. \quad (2.14)$$

Therefore, since we are assuming PI,

$$P_{\lambda, \hat{a}, \hat{c}}(\hat{a}+; \hat{c}-) + P_{\lambda, \hat{a}, \hat{c}}(\hat{a}-; \hat{c}-) = 0. \quad (2.15)$$

In particular,

$$P_{\lambda, \hat{a}, \hat{c}}(\hat{a}+; \hat{c}-) = 0. \quad \{\text{PDproof4}\} \quad (2.16)$$

But by (2.9), we know that

$$P_{\lambda, \hat{a}, \hat{b}}(\hat{a}+; \hat{b}+) + P_{\lambda, \hat{a}, \hat{b}}(\hat{a}+; \hat{b}-) = 1, \quad (2.17)$$

so using this together with PI, we must have

$$P_{\lambda, \hat{a}, \hat{c}}(\hat{a}+; \hat{c}+) + P_{\lambda, \hat{a}, \hat{c}}(\hat{a}+; \hat{c}-) = 1. \quad \{\text{PDproof5}\} \quad (2.18)$$

But by (2.10) and (2.16)

$$P_{\lambda, \hat{a}, \hat{c}}(\hat{a}+; \hat{c}+) + P_{\lambda, \hat{a}, \hat{c}}(\hat{a}+; \hat{c}-) = 0. \quad \{\text{PDproof6}\} \quad (2.19)$$

Since (2.18) contradicts (2.19), the assumption (2.12) must be false if PI is to hold.

So we now consider the alternative case when (2.13 holds. We will again see that this assumption leads to a contradiction. First note that

$$P_{\lambda, \hat{c}, \hat{e}}(\hat{c}-; \hat{c}+) + P_{\lambda, \hat{c}, \hat{e}}(\hat{c}-; \hat{c}-) = 0. \quad (2.20)$$

By PI

$$P_{\lambda, \hat{c}, \hat{b}}(\hat{c}-; \hat{b}+) + P_{\lambda, \hat{c}, \hat{b}}(\hat{c}-; \hat{b}-) = 0. \quad (2.21)$$

In particular,

$$P_{\lambda, \hat{c}, \hat{b}}(\hat{c}-; \hat{b}+) = 0. \quad \{\text{PDproof8}\}_{(2.22)}$$

But by (2.9, we know that

$$P_{\lambda, \hat{a}, \hat{b}}(\hat{a}+; \hat{b}+) + P_{\lambda, \hat{a}, \hat{b}}(\hat{a}-; \hat{b}+) = 1. \quad (2.23)$$

so using this together with PI, we must have

$$P_{\lambda, \hat{c}, \hat{b}}(\hat{c}+; \hat{b}+) + P_{\lambda, \hat{c}, \hat{b}}(\hat{c}-; \hat{b}+) = 1. \quad \{\text{PDproof7}\}_{(2.24)}$$

But by (2.11 and (2.22

$$P_{\lambda, \hat{c}, \hat{b}}(\hat{c}+; \hat{b}+) + P_{\lambda, \hat{c}, \hat{b}}(\hat{c}-; \hat{b}+) = 0. \quad \{\text{PDproof9}\}_{(2.25)}$$

Since (2.24 contradicts (2.25, the assumption (2.13 must also be false if PI is to hold.

So we can only conclude that PI fails to hold in the pilot wave model. But we can conclude even more than that: any deterministic hidden variable model that gives the same predictions as quantum mechanics when averaged over the hidden variables must violate PI.

The violation of PI by the pilot wave model does not sit easily with Einstein's theory of relativity, for according to Einstein's theory, it should be impossible to send signals

faster than the speed of light. However, if PI is violated, then if Alice happened to know what λ was for each run of the experiment and if Bob made the same measurement, then because the distribution of Alice's outcomes will depend of Bob's choice of measurement, with enough runs of the experiment, Alice should be able to work out what measurement Bob is making. And this should be possible even if Alice and Bob are separated by many light years. So it seems faster than light communication would be possible. The only thing preventing such communication would be Alice's lack of knowledge of λ .

But although a PI violation can account for the violation of Bell's Inequality, this is not the only possible culprit to consider. Another assumption of Bell's Inequality that might be violated is **Outcome Independence** (OI). Outcome independence is the assumption

$$P_{\lambda,x,y}(X, Y) = P_{A,\lambda,x,y}(X) \cdot P_{B,\lambda,x,y}(Y), \quad (2.26)$$

We can see that if OI holds any model which gives the same predictions as standard quantum theory when averaged over the hidden variables, then PI must be violated in such a model. For if both PI and OI hold, then for any measurement choices $\hat{\mathbf{a}}, \hat{\mathbf{b}}$,

and $\hat{\mathbf{c}}$ and hidden variable λ , we have

$$\begin{aligned}
P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) &= P_{A, \lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+) \cdot P_{B, \lambda, \hat{\mathbf{c}}, \hat{\mathbf{a}}}(\hat{\mathbf{c}}+) \\
&= \left(P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}-) \right) \cdot \left(P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}-; \hat{\mathbf{c}}+) \right) \\
&= \left(P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}-) \right) \cdot \left(\underbrace{P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}+; \hat{\mathbf{c}}+)}_0 + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}-; \hat{\mathbf{c}}+) \right) \\
&= \left(P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) + P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}-) \right) \cdot P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}-; \hat{\mathbf{c}}+) \\
&\geq P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) \cdot P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}-; \hat{\mathbf{c}}+)
\end{aligned}
\tag{2.27}$$

{OPII1}

Similarly, we have

$$\begin{aligned}
P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) &= P_{A, \lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+) \cdot P_{B, \lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{b}}+) \\
&= \left(P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}-) \right) \cdot \left(P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}-; \hat{\mathbf{b}}+) \right) \\
&= \left(\underbrace{P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}+; \hat{\mathbf{c}}+)}_0 + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}+; \hat{\mathbf{c}}-) \right) \cdot \left(P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}-; \hat{\mathbf{b}}+) \right) \\
&= P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}+; \hat{\mathbf{c}}-) \cdot \left(P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) + P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}-; \hat{\mathbf{b}}+) \right) \\
&\geq P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}+; \hat{\mathbf{c}}-) \cdot P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+).
\end{aligned}
\tag{2.28}$$

{OPII2}

But since the hidden variable λ is assumed to be independent of Alice and Bob's measurement, and since Alice and Bob will always get opposite results when they make the same choice of measurement, it follows that

$$P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}+; \hat{\mathbf{c}}-) + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{c}}}(\hat{\mathbf{c}}-; \hat{\mathbf{c}}+) = 1
\tag{2.29}$$

{OPII3}

Therefore, putting (2.27), (2.28), and (2.29) together, we have

$$P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{c}}}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{\lambda, \hat{\mathbf{c}}, \hat{\mathbf{b}}}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) \geq P_{\lambda, \hat{\mathbf{a}}, \hat{\mathbf{b}}}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+).
\tag{2.30}$$

We have thus proved that OI and PI implies Bell's Inequality (2.8). But since Bell's Inequality does not hold in reality, it follows that if OI is always true, then PI must be violated.

In the case of deterministic models, OI necessarily holds. To see why, we first note that for deterministic models, either $P_{\lambda,x,y}(X, Y) = 1$ or $P_{\lambda,x,y}(X, Y) = 0$. When $P_{\lambda,x,y}(X, Y) = 1$ by (2.3), $P_{A,\lambda,x,y}(X) = 1$, and by (2.4), $P_{B,\lambda,x,y}(Y) = 1$, so (2.26) is seen to hold in this case. On the other hand, when $P_{\lambda,x,y}(X, Y) = 0$, if $P_{A,\lambda,x,y}(X) = 1$, then by (2.3), $P_{\lambda,x,y}(X, -Y) = 1$ so that (2.4), $P_{B,\lambda,x,y}(Y) = 0$ in which case (2.26) holds. And similarly, if $P_{B,\lambda,x,y}(Y) = 1$, by (2.4), $P_{\lambda,x,y}(-X, Y) = 1$ so that (2.3), $P_{A,\lambda,x,y}(X) = 0$ so again (2.26) holds. And (2.26) obviously holds when $P_{A,\lambda,x,y}(X) = P_{B,\lambda,x,y}(Y) = 0$.

It therefore follows that OI holds in any deterministic model.

When it comes to standard quantum mechanics, however, OI fails to hold. For instance, if $x = y = \hat{a}$, then $P_{\lambda,\hat{a},\hat{a}}(\hat{a}+, \hat{a}+) = 0$, but $P_{A,\lambda,\hat{a},\hat{a}}(\hat{a}+) = P_{B,\lambda,\hat{a},\hat{a}}(\hat{a}+) = 1/2$.⁹ Hence, OI fails. Nevertheless, as long as PI holds, the failure of OI does not enable Bob to send message to Alice faster than light because Bob only has control over the measurement he makes. Assuming Bob's mental states have no effect on the measurement outcome, there is nothing he can do to influence his outcome, so although Alice will be able to work out Bob's measurement outcome if she already happens to know which choice of measurement he has made, she will not be able to work out which Bob makes (or even whether he has made a measurement at all) by measuring the outcome of her particle. For Shimony¹⁰ this inability to send

⁹See footnote 5

¹⁰See Shimony, "Events and processes in the quantum world," 146–147.

superluminal messages between Alice and Bob when PI holds and OI is violate was deemed sufficient for the theories of standard quantum physics and special relativity to peacefully coexist.

However, Butterfield is not satisfied with Shimony's solution to peaceful coexistence.¹¹ Firstly, he notes that proofs of non-superluminal signaling¹² make no assumptions about spacetime locations. One would have thought that any proof that superluminal signalling between two points is impossible would have to show that a signal cannot be transmitted from one point to the other in less time than the time it takes light to travel between the two points. But if nothing is said about the location of these two points or what is so special about the speed of light compared to the speed of any other particle, then there does not seem to be enough information in the premises to draw the desired conclusion superluminal signaling is impossible in quantum physics.

Secondly, Butterfield notes that Shimony thinks peaceful coexistence of quantum physics and special relativity is guaranteed by the denial of OI and the acceptance of PI, but OI itself depends on the sometimes rather vague notion of what an outcome really is. For instance, in the Many-Worlds interpretation described in sections 1.11nd 1.12 it is not clear that there are ever any outcomes at all. Rather, there is just a universal wavefunction that tells us the probability of a certain outcomes, if there were such things as outcomes – it doesn't tell us that there really are any outcomes.

¹¹See Butterfield, “Peaceful Coexistence: Examining Kent’s Relativistic Solution to the Quantum Measurement Problem,” p. 12.

¹²e.g. see Michael Redhead, *Incompleteness, nonlocality, and realism : a prolegomenon to the philosophy of quantum mechanics* (Oxford : New York: Clarendon Press ; Oxford University Press, 1987), p. 113–116; David Bohm and B. J Hiley, *The undivided universe : an ontological interpretation of quantum theory* (London: Routledge, 1993), p. 139–140

But as we saw in the previous chapter, the Many-Worlds interpretation does flow rather naturally from the postulate of standard quantum theory.

Still, the notion of what an outcome is doesn't have to be vague. In the pilot wave interpretation of quantum physics, it is very clear what an outcome of an experiment is since all the particles have definite positions and momenta, and so because of this the pointers and displays of measuring devices which are made up of particles will have definite read outs which will correspond to the definite positions of particles being measured (assuming the measurement is working properly). So unlike the Many-Worlds interpretation, measurements in the pilot-wave interpretation have definite outcomes, and hence there is only a single world in the pilot-wave interpretation of quantum physics. But as we've just seen, the problem with the pilot wave interpretation is the violation of PI.

Thus, a satisfactory account of the peaceful coexistence of quantum physics and special relativity requires an interpretation of quantum physics in which not only PI holds, but also an interpretation of quantum physics that has special relativity built into it (thus satisfying Butterfield's first objection), and in which we can make sense of what it means to be an outcome (thus satisfying Butterfield's second objection). To fully address Butterfield's first objection would require quantum field theory and this would be beyond the scope of this dissertation. But a more modest aspiration that would go some way to address Butterfield's first objection would be to insist on an interpretation of quantum physics that has a property known as Lorentz invariance. Thus, in the next section, we will examine Kent's interpretation of quantum physics

which we see has this property of Lorentz invariance and in which the notion of outcome has a clear meaning.

2.2 A description of Kent's Interpretation of Quantum Physics

In this section I will describe Kent's interpretation of quantum focusing on the ideas he presents in his 2014 paper.¹³ Kent's interpretation of quantum physics has some similarities in common with the pilot wave interpretation. Firstly, there is no wave-function collapse in Kent's interpretation. Secondly, some additional values beyond standard quantum orthodoxy (i.e. in addition to the quantum wave function) are included in Kent's interpretation. And thirdly, Kent's interpretation is a one-world interpretation of quantum physics. I'll consider these three features of Kent's interpretation in some detail as I describe his theory. I'll then present an account of his toy model that provides a simple example of how the ideas of his theory fit together. Then I'll show that Kent's interpretation is consistent with standard quantum theory, and I'll also show that it is consistent with special relativity. I'll then consider how Kent's interpretation ties in with decoherence theory and d'Espagnat's objection about improper mixtures. I'll then show that PI holds for Kent's interpretation, and I will show why Collbeck and Renner's theorem does not apply to Kent's interpretation.

2.2.1 The No-collapse Feature of Kent's Interpretation

We first consider the no-collapse feature of Kent's interpretation. This is a feature that belongs both to the Many-World's interpretation as well as to the pilot wave interpretation. In all three interpretations the wave-function deterministically evolves

¹³Adrian Kent, "Lorentzian Quantum Reality: Postulates and Toy Models," 2014, <https://doi.org/10.1098/rsta.2014.0241>, eprint: arXiv:1411.2957.

according to the Schrödinger equation. The Schrödinger equation itself describes how a quantum state evolves over time. The precise formula for the Schrödinger equation need not concern us here, but all we need to know is that the Schrödinger equation determines a so-called **unitary operator** $U(t', t)$, so that if a system is in state $|\psi\rangle$ at time t , then it will be in state $|\psi'\rangle = U(t', t)|\psi\rangle$ at time t' . A unitary operator U has the property that if $|\psi'\rangle = U|\psi\rangle$ and $|\chi'\rangle = U|\chi\rangle$, then

$$\langle \chi' | \psi' \rangle = \langle \chi | \psi \rangle .^{14} \quad \text{unitarycond} \quad (2.31)$$

In non-collapse models such as the pilot-wave, the Many-World's, and Kent's interpretation, the wavefunction always evolves unitarily. In contrast, in the Copenhagen interpretation, the state will evolve unitarily for the most part, but there will typically be a non-unitary change in the state whenever there is a measurement. To see why this is, we recall the situation described on page ?? where a state $|\psi\rangle$ is the sum of eigenstates $|\psi_i\rangle$ of an observable

$$|\psi\rangle = \sum_{i=1}^N \alpha_i |\psi_i\rangle . \quad \text{?? revisited}$$

If $|\psi'_i\rangle = U(t, t_0)|\psi_i\rangle$ and $|\psi'\rangle = U|\psi\rangle$, then the unitary condition (2.31) implies that $\langle \psi'_i | \psi' \rangle = \alpha_i$. But if on measurement at time t , the system collapses to the $|\psi'_j\rangle$ state for $j \neq i$ so that $|\psi'\rangle = |\psi'_j\rangle$, we will have $\langle \psi'_i | \psi' \rangle = 0$. So in the Copenhagen interpretation, the unitary condition (2.31) will fail if $\alpha_i \neq 0$.

2.2.2 The Additional Values of Kent's Interpretation^{additional}

Secondly, like the pilot wave interpretation, some additional values beyond standard quantum orthodoxy (i.e. in addition to the quantum wave function) are included in

¹⁴A unitary operator also has the property that it is invertible: there is an operator U^{-1} such that UU^{-1} and $U^{-1}U$ are the identity operator I , i.e. $U^{-1}U|\psi\rangle = UU^{-1}|\psi\rangle = |\psi\rangle$ for any state $|\psi\rangle$.

Kent's interpretation. In the pilot wave interpretation, these additional values are the positions and momenta of all the particles, whereas in Kent's interpretation, the additional values specify the mass-energy density on a three-dimensional distant future hypersurface in space-time S . To understand the nature of this three-dimensional hyperspace, we recall that in special relativity, there is no such thing as absolute time. So for instance, two events (i.e. two points in space-time) might seem to be simultaneous from one frame of reference, but another person travelling at a different velocity would judge the same two events to be non-simultaneous. But it is not the case that for any two events we can find a frame of reference in which the two events are simultaneous. But we refer to events that could be simultaneous in some frame of references as being **spacelike separated**. The two points O and A in figure 2.1re spacelike separated.

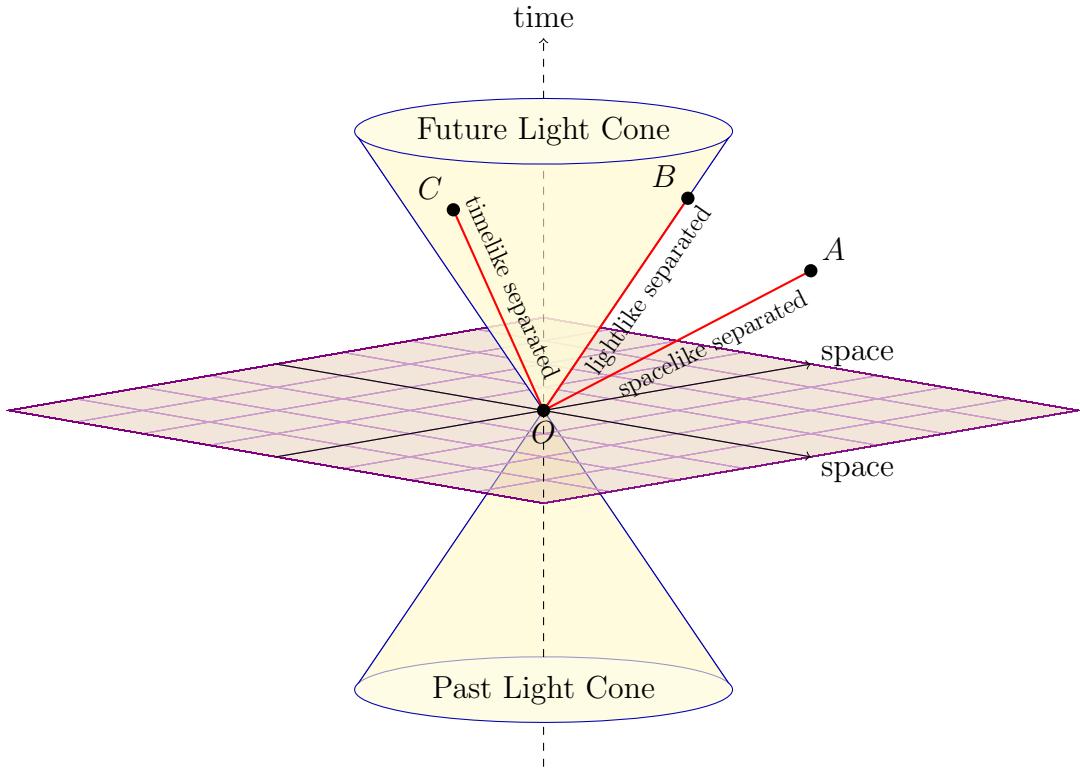


Figure 2.1: The meaning of spacelike, timelike and lightlike separation when there are two space dimensions and one time dimension

cone

There are also events in space-time that could be connected by a beam of light such as the two event O and B in figure 2.1 Such events are referred to as being lightlike separated. For any given event, the events that are lightlike separated from it form two cone called the future light cone and the past light cone as shown in figure 2.1 Because light appear to travel at the same speed no matter what frame of reference one uses, the light cone of an event remains invariant when one change from one reference frame to another. In other words if another event lies on the light cone of an event in one frame of reference, then it must lie on the light cone in every frame of reference.

Figure 2.1 also depicts two events O and C that are timelike separated. Such events lie within the light cones of each other, and when two events are timelike separated, it is always possible to choose a frame of reference in which the two events are located at the same point in space, but with one event occurring after the other depending on which event is the future light cone of the other.

Now a three-dimensional hypersurface S in space-time is a three-dimensional surface in which all the events of S are spacelike separated. Kent assumes that this hypersurface S is in the distant future of an expanding universe so that all nearly all the particles that can decay have already done so, and that all the particles are very far from each other so that probability any particle collisions is very small. In other words, all the interesting physics in universe has played its course before S .

At every point of $x \in S$ there is a quantity $T_S(x)$ called the **massenergydensity**¹⁵ **mass-energy density**.¹⁵

The important thing to note about $T_S(x)$ is that it does not depend on which frame of reference one is in.¹⁶ This property is in contrast to many physical properties that do depend on which frame of reference one is in. For example, the kinetic energy of an object will depend on the calculated velocity of the object, and this velocity will depend on the frame of reference in which this calculation is done.

Now according to the Tomogana-Schwinger formulation of relativistic quantum physics,¹⁷ for any hypersurface S , there is a Hilbert space H_S of states describing S . One of the properties of H_S is that for any event $x \in S$ there is an observable $\hat{T}_S(x)$ acting on H_S such that if $|\Psi\rangle \in H_S$ is an eigenstate of $\hat{T}_S(x)$ with eigenvalue $t(x)$, then $|\Psi\rangle$ corresponds to a state of S in which the energy-density at x is $t(x)$. This can be done in such a way that $\hat{T}_S(x)$ only depends on x rather than on the hypersurface S that contains x . Furthermore, if x and y are any two events of S , then the observables $\hat{T}_S(x)$ and $\hat{T}_S(y)$ commute. In other words,

$$\hat{T}_S(x)\hat{T}_S(y) = \hat{T}_S(y)\hat{T}_S(x).$$

The commutativity of all the $\hat{T}_S(x)$ for $x \in S$ means that if $|\Psi\rangle$ is an eigenstate $\hat{T}_S(x)$, then for any $y \in S$, $\hat{T}_S(y)|\Psi\rangle$ is also an eigenstate of $\hat{T}_S(x)$ with the same eigenvalue as $|\Psi\rangle$. The invariance of the $\hat{T}_S(x)$ -eigenspace under the action of $\hat{T}_S(y)$ means that we can create simultaneous eigenstates for both $\hat{T}_S(x)$ and $\hat{T}_S(y)$, albeit with different eigenvalues. But because x and y are arbitrary points of S , this means that we can

¹⁵The definition of $T_S(x)$ will be discussed in section 2.3.4

¹⁶The reason for why this is will be discussed in section 2.3.4

¹⁷See Julian Schwinger, “Quantum Electrodynamics. I. A Covariant Formulation,” *Physical review* 74, no. 10 (1948): 1439–1461; S. Tomonaga, “On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields,” *Progress of theoretical physics* (Tokyo) 1, no. 2 (1946): 27–42

express any state H_S as a superposition of simultaneous $\hat{T}_S(x)$ -eigenstates of the form

$|\Psi^{(i)}\rangle$ where $\hat{T}_S(x)|\Psi^{(i)}\rangle = t_S^{(i)}(x)|\Psi^{(i)}\rangle$ for all $x \in S$ where $t_S^{(i)}(x) \geq 0$ is a possible energy-density measurement over the whole of S .¹⁸

The additional values beyond standard quantum orthodoxy that are included in Kent's interpretation are given by one of these possible outcomes for an energy-density measurement over the whole of S . We will denote this outcome as $\tau_S(x)$ and we will let $|\Psi\rangle$ be the state such that $\hat{T}_S(x)|\Psi\rangle = \tau_S(x)|\Psi\rangle$. But although we speak of the measurement of $T_S(x)$ over S as being $\tau_S(x)$, this is only a notional measurement. Thus, we speak of measurement of $T_S(x)$ on S only to mean that the state describing S goes from a state in which $T_S(x)$ is indeterminate to a state in which $T_S(x)$ is determinate, i.e. to an $\hat{T}_S(x)$ -eigenstate for every $x \in S$ with corresponding eigenvalue $\tau_S(x)$. How this determination comes about is up to one's philosophical preferences. For example, one could suppose that it was simply by divine fiat that this determination of $T_S(x)$ came about.¹⁹

Nevertheless, the particular density $\tau_S(x)$ which is found to describe S can't be absolutely anything. Rather, we suppose there is a much earlier hypersurface S_0 which is described by a state $|\Psi_0\rangle$ belonging to a Hilbert space H_{S_0} as shown in figure 2.2. It is assumed that all physics that we wish to describe takes place between these two hypersurfaces S_0 and S . In figure 2.2 we therefore let y depicts a generic event that we wish to describe.

¹⁸We will gloss over the details of how to make this rigorous for continuous variables x and continuous indices i . It is sufficient to approximate to the desired level of accuracy the continuous variables and indices as discrete variables and indices when thinking about the simultaneous $\hat{T}_S(x)$ -eigenspaces.

¹⁹I will discuss my philosophical preference in the final chapter.

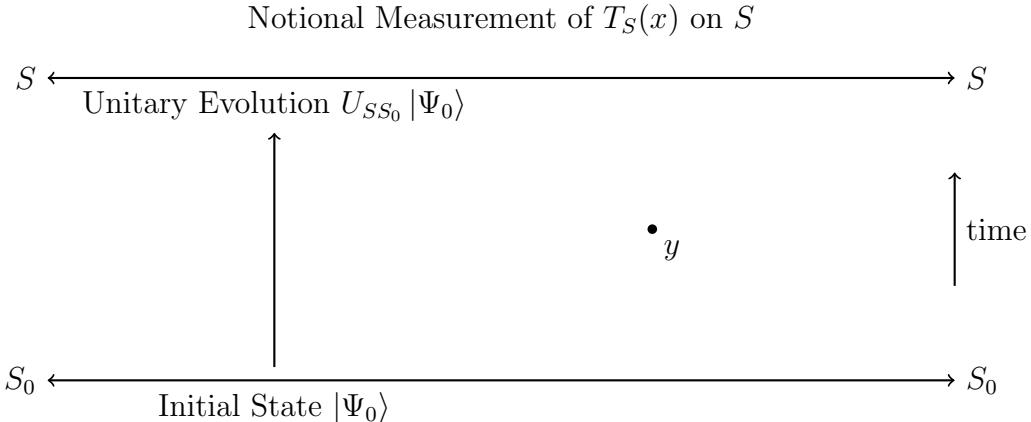


Figure 2.2: A notional measurement of $T_S(x)$ is made for all $x \in S$. The simultaneous \hat{T}_S -eigenstate $|\Psi\rangle$ with $\hat{T}_S(x)|\Psi\rangle = \tau_S(x)|\Psi\rangle$ is selected with probability $|\langle\Psi|U_{SS_0}|\Psi_0\rangle|^2$. The values $\tau_S(x)$ obtained for $T_S(x)$ are then used to calculate the physical properties at the event y .

S1

Now according to Schwinger²⁰ there is a unitary operator U_{SS_0} that maps states in H_{S_0} such as $|\Psi_0\rangle$ to states in H_S . Then the probability $P(\Psi|\Psi_0)$ that S will be found to be in the state $|\Psi\rangle$ with mass-energy density $\tau_S(x)$ given that S_0 was initially in the state $|\Psi_0\rangle$ will be given by the Born rule (see page 8:

$$P(\Psi|\Psi_0) = |\langle\Psi|U_{SS_0}|\Psi_0\rangle|^2 \quad \{ \text{bornpsi} \}_{(2.32)}$$

It's possible that there might be several different states of H_S that have the same mass-energy density $\tau_S(x)$, but one of these states is still realized with probability given by equation (2.32). But it is $\tau_S(x)$ rather than one of the eigenstates with mass-energy density $\tau_S(x)$ that constitute the additional values that Kent adds to standard quantum orthodoxy.

²⁰Schwinger, “Quantum Electrodynamics. I. A Covariant Formulation,” p.1459

2.2.3 The One-World Feature of Kent's Interpretation

The third similarity Kent's interpretation shares with the pilot wave interpretation is that it is a one-world interpretation of quantum physics. So unlike the Many-Worlds interpretation there are no superpositions of living and dead cats in Kent's interpretation. In the Many-Worlds interpretation, Schrödinger still only observes his cat to be either dead or alive, and both dead and alive, but Schrödinger himself goes into a superposition of observing his cat to be alive and his cat to be dead. In the Many-Worlds interpretation, there is thus a difference between observing something to be so and something actually being so: observation is a particular outcome, but the reality is a superposition of different outcomes. To capture this distinction, Bell speaks of **beables**. Bell introduces the term beable when speculating on what would be a more satisfactory physical theory than quantum physics currently has to offer.²¹ Bell says that such a theory should be able to say of a system not only such and such is observed to be so, but that such and such be so. In other words, such a theory is a theory of beables rather than observables. On the macroscopic level, these beables should be the underlying reality that give rise to all the familiar things in the world around us, things like cats, laboratories, procedures, and so on. In the pilot wave interpretation, the beables are all the particles with their precise position and momentum. It is because of such beables, that a scientist can observe a physical system to be in such and such a state, and so observables are made out of beables.

Now the beables in Kent's one world interpretation are expressed in terms of a physical quantity called the **stress-energy tensor** $T^{\mu\nu}(y)$.²² For any event y in spacetime, the

²¹See J. S. Bell, "Subject and Object," in *Speakable and unspeakable in quantum mechanics*, 2nd ed. (Cambridge: Cambridge University Press, 2004), 40–44.

stress-energy tensor $T^{\mu\nu}(y)$ is an array of 16 values corresponding to each combination of $\mu, \nu = 0, 1, 2$, or 3. The value $T^{00}(y)$ is the energy density at y ,²² whereas the other values of $T^{\mu\nu}(y)$ indicate how much energy and momentum flow across different surfaces in the neighborhood of y .

It was mentioned in the previous section that for any event $x \in S$, there is an observable $\hat{T}_S(x)$ acting on H_S . It turns out that for any $\mu, \nu = 0, 1, 2$, or 3, there is also an observable $\hat{T}^{\mu\nu}(x)$ acting on H_S , such that if $|\Psi\rangle \in H_S$ is an eigenstate of $\hat{T}_S(x)$ with eigenvalue $\tau^{\mu\nu}(x)$, then $|\Psi\rangle$ corresponds to a state of S in which the $T^{\mu\nu}(x)$ is $\tau^{\mu\nu}(x)$. Moreover, the observable $\hat{T}_S(x)$ is expressible in terms of the $\hat{T}^{\mu\nu}(x)$ -observables.²³

Now the beables in Kent's interpretation are defined at each event y that occurs after S_0 and before S . For such an event y , the beables will be determinate values of the stress-energy tensor $T^{\mu\nu}(y)$, but calculated from the expectation of the observable $\hat{T}^{\mu\nu}(y)$ conditional on the energy-density on S being given by $\tau_S(x)$ for all x outside the lightcone of y . To explain what this means in more detail, recall the definition of expectation in equation (??) and the expectation formula (??) for an observable. If the beable in question was simply the expectation of $\hat{T}^{\mu\nu}(y)$ without conditioning on the value of the energy-density on S , the $T^{\mu\nu}(y)$ -beable would just be $\langle \Psi' | \hat{T}^{\mu\nu}(y) | \Psi' \rangle$ where $|\Psi'\rangle = U_{S'S_0} |\Psi_0\rangle$ for any hypersurface S' that goes through y .²⁴ However, such

²²This is not to be confused with the mass-energy density $T_S(x)$ defined for x on a hypersurface S . As will be shown in section 2.3.4 typically all 16 elements of $T^{\mu\nu}(x)$ will be needed to calculate $T_S(x)$.

²³See section 2.3.4

²⁴This can be done such that $\langle \Psi' | \hat{T}^{\mu\nu}(y) | \Psi' \rangle$ does not depend on the hypersurface S' other than the fact that it contains y . For more details see Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation."

a beable would give a description of reality that was very different from what we observe – for instance, in a Schrödinger cat-like experiment, there would be energy-densities corresponding to both the cat being alive and the cat being dead in the same world. To overcome this defect, information about the mass-energy density on S is required, specifically the values of $\tau_S(x)$ for $x \in S^1(y)$ where $S^1(y)$ consists of all the events of S outside of the lightcone of y as depicted in figure 2.3

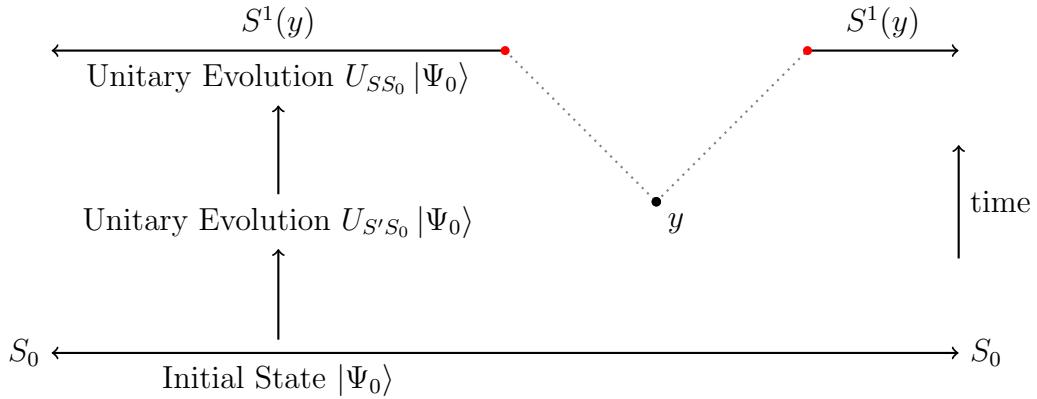


Figure 2.3: The set $S^1(y)$ consists of all the events of S outside of the lightcone of y . The $T^{\mu\nu}(y)$ -beables are calculated using the initial state $|\Psi_0\rangle$ together with the values of $\tau_S(x)$ for $x \in S^1(y)$.

s2

The conditional expectation we need to calculate depends on the notion of **conditional probability**. In probability theory, the conditional probability $P(q|r)$ that a statement q is true given that a statement r is given by the formula

$$P(q|r) = \frac{P(q \& r)}{P(r)}.$$

If we now define $q(\kappa)$ to be the statement that some quantity K takes the value κ . Then the **conditional expectation** of K given r will be given by the formula

$$\langle K \rangle_r \stackrel{\text{def}}{=} \sum_{\kappa} \frac{P(q(\kappa), r)\kappa}{P(r)} \quad \{\text{conditional expectation}\} \quad (2.33)$$

where the summation is over all the possible values κ that K can take.

If we let $\langle T^{\mu\nu}(y) \rangle_{\tau_S}$ stand for Kent's $T^{\mu\nu}(y)$ -beable, then this can be calculated from (2.33 by taking r to be the statement that $T_S(x) = \tau_S(x)$ for all $x \in S^1(y)$, and $q(\tau)$ to be the statement that the universe is found to be in a quantum eigenstate of the observable $\hat{T}^{\mu\nu}(y)$ with eigenvalue τ . It is these $T^{\mu\nu}(y)$ -beables that give a one-world picture of reality.

toysection

2.2.4 Kent's toy example

To get a feel for how all the elements of Kent's interpretation fit together, it is helpful to consider his toy model example that he discusses in his 2014 paper.²⁵ In his toy model, Kent considers a system in one spatial dimension which is the superposition of two localized states $\psi_0^{\text{sys}} = a\psi_1^{\text{sys}} + b\psi_2^{\text{sys}}$ where ψ_1^{sys} is localized at x_1 , ψ_2^{sys} is localized at x_2 , and $|a|^2 + |b|^2 = 1$. According to the Copenhagen interpretation, a measurement on this system would collapse the wavefunction of ψ_0^{sys} to the wavefunction of ψ_1^{sys} with probability $|a|^2$, and to the wavefunction of ψ_2^{sys} with probability $|b|^2$. The purpose of Kent's toy model is to show that within his interpretation, there is something analogous to wavefunction collapse. In order for this “collapse” to happen, one needs to consider how the system interacts with light. Thus Kent supposes that a photon (which is modelled as a point particle) comes in from the left, and as it interacts with the two states ψ_1^{sys} and ψ_2^{sys} , the photon enters in a superposition of states, corresponding to whether the photon reflects off the localized ψ_1^{sys} -state at time t_1 or the localized ψ_2^{sys} -state at time t_2 . The photon in superposition then travels to the right and eventually reaches the one dimensional hypersurface S at locations γ_1 and γ_2 as shown in figure 2.4

²⁵See Kent, “Lorentzian Quantum Reality: Postulates and Toy Models,” p.3–4.

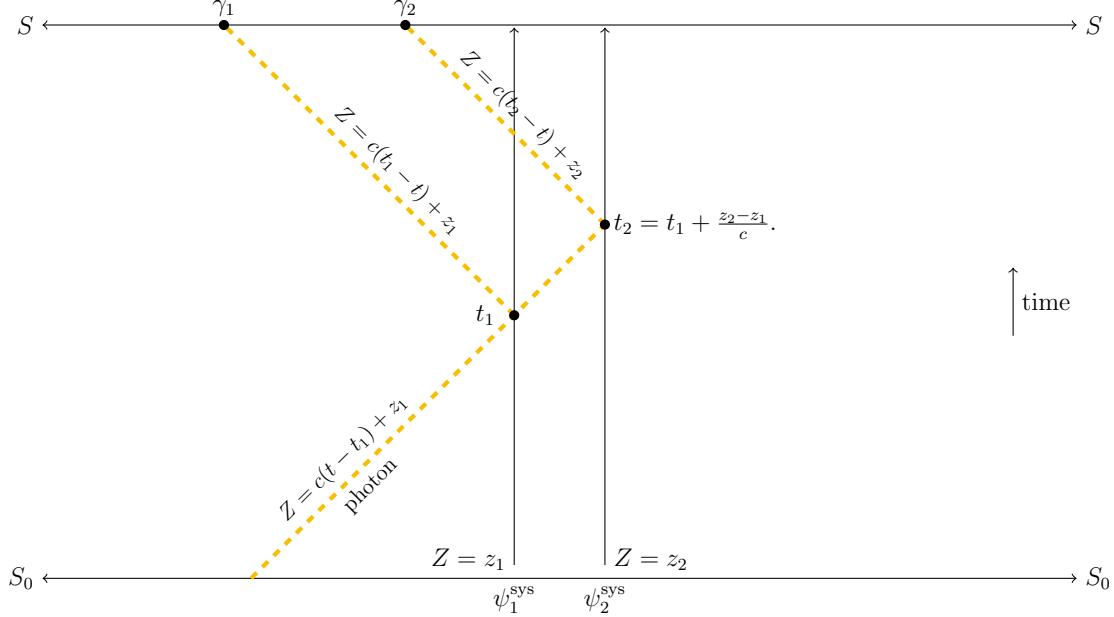


Figure 2.4: Kent's toy model

TM1

We now suppose that when the mass-energy density S is “measured” the energy of the photon is found to be at γ_1 rather than at γ_2 . We then consider the mass-density at early spacetime locations $y_1^a = (z_1, t_d)$ and $y_2^a = (z_2, t_d)$ as show in figure 2.5a) and (b).

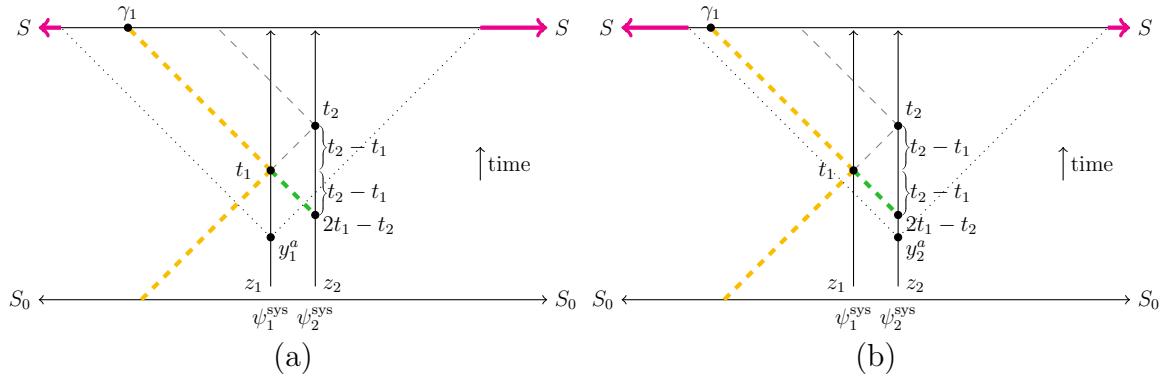


Figure 2.5: (a) highlights the part of S used to calculate the energy density at y_1^a whose time is less than $2t_1 - t_2$. (b) highlights the part of S used to calculate the energy density at y_2^a whose time is greater than $2t_1 - t_2$.

TM2

By early, we mean that $t_d < 2t_1 - t_2$. This will mean that the possible detection locations γ_1 and γ_2 will be outside the forward light cones of y_1^a and y_2^a . Hence

$S^1(y_1^a) \cap S$ and $S^1(y_2^a) \cap S$ contain no additional information beyond orthodox quantum theory by which we could calculate the conditional expectation values of the energy at y_1^a and y_2^a . Hence, according to Kent's interpretation, the total energy will be divided between the two locations with a proportion of $|a|^2$ at y_1^a and a proportion of $|b|^2$ at y_2^a .

However, the situation is different for two events slightly after $2t_1 - t_2$ as depicted in figure 2.6

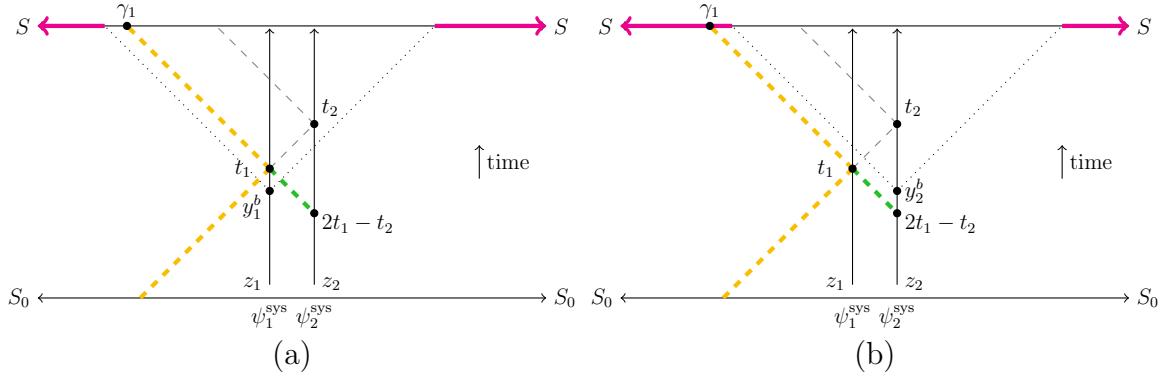


Figure 2.6: (a) highlights the part of S used to calculate the energy density at y_1^b whose time is greater than $2t_1 - t_2$. (b) highlights the part of S used to calculate the energy density at y_2^b whose time is greater than $2t_1 - t_2$.

TM3

In this situation, when we consider the location y_1^b , there is no additional information in $S^1(y_1^b) \cap S$ beyond standard quantum theory, so there will be a proportion of $|b|^2$ of the total initial energy of the system at y_1^b . But at location y_2^b , the information in $S^1(y_2^b) \cap S$ shows that the photon has reflected from the localized ψ_1^{sys} -state, and so this additional information tells us that the photon didn't reflect the localized ψ_2^{sys} -state. So it is as though the information of $S^1(y_2^b) \cap S$ has determined that we are in a world in which there is an energy density of zero at y_2^b , and there are no other worlds in which energy density at y_2^b is non-zero since all worlds have to be consistent

with the notional measurement made on S . So for a short time the total energy of the system is reduced by a factor of $|b|^2$. But the total energy of the system is once again restored to the initial energy at times greater than t_1 as shown in figure 2.7

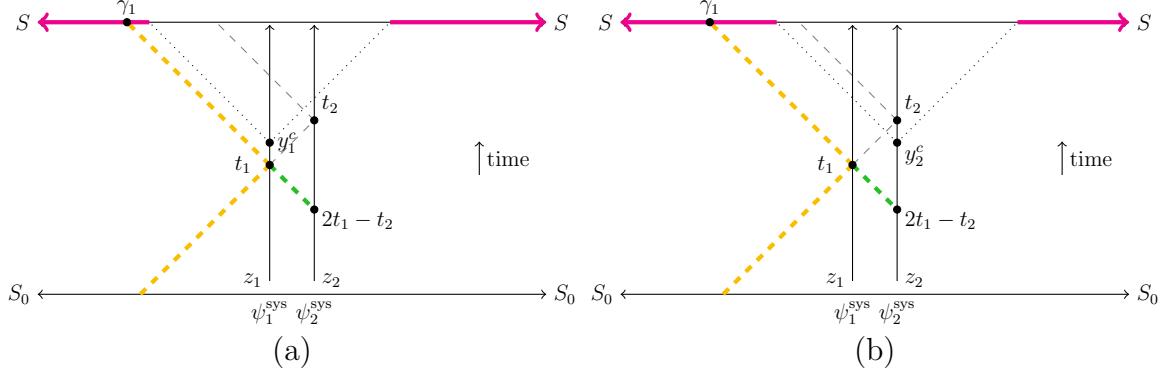


Figure 2.7: (a) highlights the part of S used to calculate the energy density at y_1^c whose time is greater than t_1 . (b) highlights the part of S used to calculate the energy density at y_2^b whose time is greater than t_1 .

TM4

In this situation, there is now information in $S^1(y_1^c) \cap S$ that determines that the photon reflected off the localized ψ_1^{sys} -state. This means that when the conditional expectation of the energy density of y_1^c is calculated, the extra information in $S^1(y_1^c) \cap S$ determines that all the energy of the system is located at location z_1 for times greater than t_1 , and the energy is equal to the initial energy of the system so that energy is conserved.

2.3 Evaluating Kent's Interpretation

2.3.1 The Empirical Adequacy of Kent's interpretation

We will need to express $\langle T^{\mu\nu}(y) \rangle_{\tau_S}$ in terms of the observable $\hat{T}^{\mu\nu}(y)$ and the initial state $|\Psi_0\rangle$ in order to show that Kent's addition to quantum orthodoxy rather contradicts it. To find such an expression we would ideally like to find a hypersurface S' that contains both $S^1(y)$ and y . Then we could consider how the observables $\hat{T}_S(x)$ and $\hat{T}^{\mu\nu}(y)$ act on the state $|\Psi'\rangle = U_{S'S_0} |\Psi_0\rangle$ and use this action to determine

the probabilities $P(q(\tau), r)$ and $P(r)$ needed to define the conditional expectation $P(q(\kappa)|r)$. However, since by definition a hypersurface must be continuous with any two events of it being space-like separated, it is going to be impossible to find a hypersurface S' with the desired properties. Nevertheless, what we can do is find a sequence of hypersurfaces $S_i(y)$ such that $\stackrel{\text{sydef}}{S_i(y)} \subset S_j(y)$ for $i < j$, and that for any $x \in S^1(y)$, there exists i such that $x \in S_i(y)$. An example of one such $S_i(y)$ is shown in figure 2.8 When there is no ambiguity, we will drop the y and write S_i instead of $S_i(y)$.

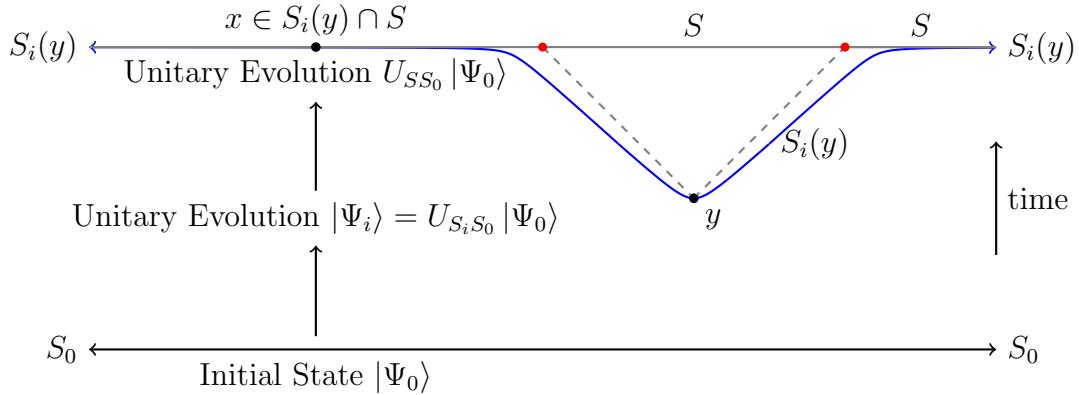


Figure 2.8: $S_i \stackrel{\text{def}}{=} S_i(y)$ is a hypersurface containing y and all of $S^1(y)$ in the limit as $i \rightarrow \infty$.

s3

Now if r_i is the statement that $T_S(x) = \tau_S(x)$ for all $x \in S_i(y) \cap S$, then

$$P(q(\tau)|r) = \lim_{i \rightarrow \infty} P(q(\tau)|r_i),$$

and therefore from the definition of $\langle T^{\mu\nu}(y) \rangle_{\tau_S}$ together with equation (??), we have

$$\langle T^{\mu\nu}(y) \rangle_{\tau_S} = \lim_{i \rightarrow \infty} \sum_{\tau} \frac{P(q(\tau), r_i)\tau}{P(r_i)}. \quad \begin{matrix} \text{\{beable1\}} \\ (2.34) \end{matrix}$$

To calculate $P(q(\tau)|r_i)$, we note that since S_i is a hypersurface, then there will exist a unitary operator $U_{S_i S_0}$ which maps the Hilbert space of states H_{S_0} describing S_0 to the Hilbert space of states H_{S_i} ^{HSidef}describing S_i in accord with how the states of

H_{S_0} evolve over time. Now let $H_{S_i, \tau_S} \subset H_{S_i}$ be subspace of states $|\xi\rangle$ for which $T_S(x)|\xi\rangle = \tau_S(x)|\xi\rangle$ for all $x \in S_i \cap S$, and let $\{|\xi_1\rangle, |\xi_2\rangle, \dots\}$ be an orthonormal basis of the states of H_{S_i, τ_S} . Given that the initial state of the world is $|\Psi_0\rangle$, the probability $P(r_i)$ of “measuring” the value of $T_S(x)$ on $S_i \cap S$ to be $\tau_S(x)$ will be

$$P(r_i) = \sum_j |\langle \xi_j | \Psi_i \rangle|^2, \quad (2.35)^{\{\text{Pr}\}}$$

where $|\Psi_i\rangle = U_{S_i S_0}$, and this probability will be independent of the particular orthonormal basis $\{|\xi_j\rangle : j\}$ of H_{S_i, τ_S} .²⁶ If we define $\pi_i = \sum_j |\xi_j\rangle \langle \xi_j|$, then it is easy to see that

$$P(r_i) = \langle \Psi_i | \pi_i | \Psi_i \rangle.$$

Turning to the calculation of $P(q(\tau), r_i)$, note that for the Tomogana-Schwinger formulation of relativistic quantum physics, the operators $\hat{T}_S(x)$ and $\hat{T}^{\mu\nu}(y)$ for fixed μ, ν commute when x and y are spacelike separated. It therefore follows that we can express any state of H_{S_i} as a superposition of simultaneous eigenstates of $\hat{T}^{\mu\nu}(y)$ and $\hat{T}_S(x)$ for $x \in S_i \cap S$.²⁷ For a particular choice of μ, ν , we can then form an orthonormal basis $\{|\eta_j\rangle : j\}$ of H_{S_i} consisting of simultaneous $\hat{T}^{\mu\nu}(y)$, $\hat{T}_S(x)$ -eigenstates so that $\hat{T}^{\mu\nu}(y)|\eta_j\rangle = \tau^{(j)}|\eta_j\rangle$ and $\hat{T}_S(x)|\eta_j\rangle = \tau_S^{(j)}(x)|\eta_j\rangle$ for $x \in S_i \cap S$, where $\tau^{(j)}$ and $\tau_S^{(j)}(x)$ are the corresponding eigenstates. If we define $\pi_{i,\tau} = \sum_j |\chi_{j,\tau}\rangle \langle \chi_{j,\tau}|$

²⁶To see why this is, we note that we can extend the orthonormal set $\{|\xi_1\rangle, |\xi_2\rangle, \dots\}$ to an orthonormal basis $\{|\xi_1\rangle, |\xi_2\rangle, \dots\} \cup \{|\zeta_1\rangle, |\zeta_2\rangle, \dots\}$ which consists entirely of \hat{T}_S -eigenstates. We can think of each of the states of this orthonormal basis as the possible measurement outcomes when making the notional measurement of $T_S(x)$ on $S_i \cap S$. By the Born rule, it therefore follows that $P(r_i) = \sum_j |\langle \xi_j | \Psi_i \rangle|^2$. But to see that this probability is independent of the particular basis, we can uniquely write $|\Psi_i\rangle$ as a sum $|\Psi_i\rangle = |\xi\rangle + |\zeta\rangle$ where $|\xi\rangle$ belongs to the span of $\{|\xi_j\rangle : j\}$ and $|\zeta\rangle$ belongs to the span of $\{|\zeta_j\rangle : j\}$. Then since $|\xi\rangle = \sum_j \langle \xi_j | \Psi_i \rangle |\xi_j\rangle$, it follows that

$$\langle \xi | \xi \rangle = \sum_j |\langle \xi_j | \Psi_i \rangle|^2 = P(r_i).$$

Therefore, since $\langle \xi | \xi \rangle$ is independent of the particular basis chosen, then so is $P(r_i)$. priproof

²⁷cf. page 74nd footnote 18

where $\{|\chi_{j,\tau}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ such that $\hat{T}^{\mu\nu}(y)|\chi_{j,\tau}\rangle = \tau|\chi_{j,\tau}\rangle$ and $\hat{T}_S(x)|\chi_{j,\tau}\rangle = \tau_S(x)|\chi_{j,\tau}\rangle$ for all $x \in S_i \cap S$, then

$$P(q(\tau), r_i) = \sum_j |\langle \chi_{j,\tau} | \Psi_i \rangle|^2 = \langle \Psi_i | \pi_{i,\tau} | \Psi_i \rangle.^{28} \quad \text{pqtauri}_{(2.36)}$$

But if we define $\pi_\tau = \sum_j |\eta_{j,\tau}\rangle\langle\eta_{j,\tau}|$ where $\{|\eta_{j,\tau}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ with $\hat{T}^{\mu\nu}(y)|\eta_{j,\tau}\rangle = \tau|\eta_{j,\tau}\rangle$, then we also have $\pi_{i,\tau} = \pi_i \pi_\tau$.²⁹ Hence

$$P(q(\tau), r_i) = \langle \Psi_i | \pi_i \pi_\tau | \Psi_i \rangle. \quad \text{pqtauri2}_{(2.37)}$$

But clearly $\hat{T}^{\mu\nu}(y) = \sum_\tau \tau \pi_\tau$. Therefore combining (2.34 and (2.37), we have

$$\langle T^{\mu\nu}(y) \rangle_{\tau_S} = \lim_{i \rightarrow \infty} \frac{\sum_\tau \langle \Psi_i | \pi_i \pi_\tau | \Psi_i \rangle \tau}{\langle \Psi_i | \pi_i | \Psi_i \rangle} = \lim_{i \rightarrow \infty} \frac{\langle \Psi_i | \pi_i \hat{T}^{\mu\nu}(y) | \Psi_i \rangle}{\langle \Psi_i | \pi_i | \Psi_i \rangle}.^{29}_{\text{Kent consistency0}} \quad (2.38)$$

We are now in a position to show that Kent's theory is consistent with standard quantum orthodoxy. First let us consider what we need to show.

In the pilot-wave interpretation, its consistency with standard quantum orthodoxy requires that if one averages the expectation values of an observable over the hidden variables (i.e. the positions and the momenta of all the particles) then one obtains the expectation value of the observable given by standard quantum orthodoxy as indicated in equation (2.7).

Now in Kent's interpretation, the hidden variables on which Kent's beables $\langle T^{\mu\nu}(y) \rangle_{\tau_S}$ are the values $\tau_S(x)$ of $T_S(x)$ for $x \in S^1(y) \cap S$. The operator π_i in equation (in the

²⁸The proof of this is very similar to the proof given in footnote 26

²⁹To see why, we first show that $\pi_i = \sum_j |\eta_{i,j}\rangle\langle\eta_{i,j}|$ where $\{|\eta_{i,j}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ for which $|\eta_{i,j}\rangle \in H_{S_i, \tau_S}$. To see why this holds, note that $\pi_i|\eta_{i,j}\rangle = |\eta_{i,j}\rangle$ since $\{|\xi_j\rangle : j\}$ is a basis for H_{S_i, τ_S} and $|\eta_{i,j}\rangle \in H_{S_i, \tau_S}$. Therefore $\pi_i \pi_{i,\eta} = \pi_{i,\eta}$ where $\pi_{i,\eta} = \sum_j |\eta_{i,j}\rangle\langle\eta_{i,j}|$. But $\pi_{i,\eta}|\xi_j\rangle = |\xi_j\rangle$ since $\{|\eta_{i,j}\rangle : j\}$ is a basis for H_{S_i, τ_S} and $|\xi_j\rangle \in H_{S_i, \tau_S}$. Therefore $\pi_{i,\eta} \pi_i = \pi_i$. But $\pi_{i,\eta} \pi_i = \pi_i \pi_{i,\eta}$ since π_i and $\pi_{i,\eta}$ are Hermitian. Hence $\pi_i = \pi_{i,\eta}$.

Now the summands of $\pi_i \pi_\tau$ are only going to consist of those $|\eta_j\rangle\langle\eta_j|$ for which $\hat{T}^{\mu\nu}(y)|\eta_j\rangle = \tau|\eta_j\rangle$ and for which $\hat{T}_S(x)|\eta_j\rangle = \tau_S(x)|\eta_j\rangle$ for all $x \in S_i \cap S$, and these are just the $|\chi_{j,\tau}\rangle\langle\chi_{j,\tau}|$ which are the summands of $\pi_{i,\tau}$. Hence $\pi_i \pi_\tau = \pi_{i,\tau}$.

limit as $i \rightarrow \infty$) encapsulates this hidden information. To remind ourselves of π_i 's dependency on τ_S restricted to $S_i \cap S$, we will now write $\pi_i(\tau_{S_i \cap S})$ for π_i where $\tau_{S_i \cap S}$ is the function τ_S restricted to $S_i \cap S$. Likewise, we will now write $r_i(\tau_{S_i \cap S})$ for r_i , the statement that $T_S(x) = \tau_S(x)$ for all $x \in S_i(y) \cap S$. If we let j index all possible functions $\tau_{S_i \cap S}^{(j)}$ taking real values on $S_i \cap S$, then the analogue of (2.7) requires us to show that

$$\langle \hat{T}^{\mu\nu}(y) \rangle = \lim_{i \rightarrow \infty} \sum_j P(r_i(\tau_{S_i \cap S}^{(j)})) \langle T^{\mu\nu}(y) \rangle_{\tau_S} \quad \{ \text{kentconsistency} \} \quad (2.39)$$

for all y lying between S_0 and S . Equation (2.39) is sufficient to establish consistency with quantum orthodoxy because ultimately, all observables are going to be reducible to expressions dependent on $\hat{T}^{\mu\nu}(y)$, since once we know what to expect for $\hat{T}^{\mu\nu}(y)$ we will know what to expect for the energy and momentum densities for all measuring apparatus readouts etc. and hence what to expect for all measurement outcomes.

But from (2.38), we have

$$\lim_{i \rightarrow \infty} \sum_j P(r_i(\tau_{S_i \cap S}^{(j)})) \langle T^{\mu\nu}(y) \rangle_{\tau_{S_i \cap S}^{(j)}} = \lim_{i \rightarrow \infty} \sum_j \langle \Psi_i | \pi_i(\tau_{S_i \cap S}^{(j)}) \hat{T}^{\mu\nu}(y) | \Psi_i \rangle \quad \{ \text{kentconsistency1} \} \quad (2.40)$$

Since there is an orthonormal basis $\{|\eta_j\rangle : j\}$ of H_{S_i} consisting of simultaneous $\hat{T}_S(x)$ -eigenstates so that $\hat{T}_S(x)|\eta_j\rangle = \tau_{S_i \cap S}^{(j)}(x)|\eta_j\rangle$ for all $x \in S_i \cap S$, it follows that $\sum_j \pi_i(\tau_{S_i \cap S}^{(j)}) = I$. Therefore equation (2.39) follows from (2.40) which is what we were aiming to show for orthodox quantum consistency to hold.

2.3.2 Parameter Independence in Kent's interpretation
 2.3.3 Kent's interpretation and the Colbeck Renner theorem
 2.3.4 Kent's interpretation and Lorentz Invariance^{massenergydensity}

In order to explain what it means for Kent's interpretation to be compatible with Special Relativity, we first need to explain how spacetime coordinates look to different observers.

A spacetime location is represented by a four-tuples (x^0, x^1, x^2, x^3) where $(x^i)_{i=1}^3$ are spatial coordinates, and where $x^0 = ct$ with c being equal to the speed of light and t being the time. If $(1, 0, 0, 0)$ corresponds to the spacetime location \hat{e}_0 , $(0, 1, 0, 0)$ corresponds to the spacetime location \hat{e}_1 , etc., then we can express any other spacetime location as a sum $\sum_{\mu=0}^3 x^\mu \hat{e}_\mu$. We will use the so-called Einstein convention of dropping the summation sign and implicitly assume there is a summation whenever an upper index and a lower index are the same so that we can write $x^\mu \hat{e}_\mu$ instead of $\sum_{\mu=0}^3 x^\mu \hat{e}_\mu$.

Now suppose an observer O expresses spacetime locations in terms of $\{\hat{e}_\mu\}_{\mu=0}^3$ and hence uses the coordinates (x^0, x^1, x^2, x^3) to describe various spacetime locations. For another observer O' , it may be more natural to express spacetime locations in terms of a different set of spacetime locations $\{\hat{e}'_\mu\}_{\mu=0}^3$ so that the location described by O as (x^0, x^1, x^2, x^3) would be described by O' as (x'^0, x'^1, x'^2, x'^3) where $x'^\mu \hat{e}'_\mu = x^\mu \hat{e}_\mu$. For instance if O and O' are moving with respect to each other, they may both want to use coordinates in which their own spatial coordinates are fixed and in which the spatial coordinates of the other observer are changing. As another example, figure 2.9 shows how the (x^1, x^2) -coordinates transform under a spatial rotation.

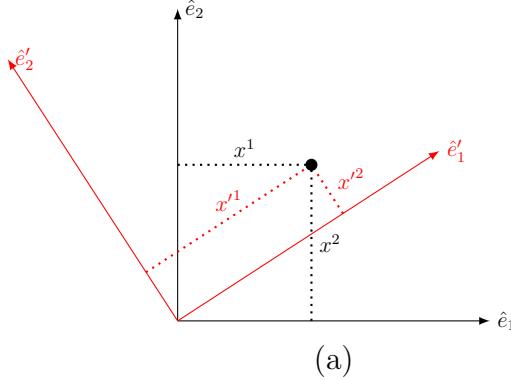


Figure 2.9: Shows how a location (marked as \bullet) can be expressed either in coordinates (x^1, x^2) with respect to the basis $\{\hat{e}_1, \hat{e}_2\}$ or in coordinates (x'^1, x'^2) with respect to the basis $\{\hat{e}'_1, \hat{e}'_2\}$.

rotfigure

Now the key fact about all observer is that they must always observe light to have a constant speed c . Thus, for a photon that goes through the spacetime locations $(0, 0, 0, 0)$ and (x^0, x^1, x^2, x^3) in the coordinates of O , we must have $(x^0, x^1, x^2, x^3) = (ct, tv^1, tv^2, tv^3)$ where

$$\sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2} = c.$$

But if $(0, 0, 0, 0)$ and (x^0, x^1, x^2, x^3) corresponds to $(0, 0, 0, 0)$ and (x'^0, x'^1, x'^2, x'^3) respecetively in the coordinates of another observer O' , then we must also have $(x'^0, x'^1, x'^2, x'^3) = (ct', t'v'^1, t'v'^2, t'v'^3)$ where

$$\sqrt{(v'^1)^2 + (v'^2)^2 + (v'^3)^2} = c.$$

In either case, we must have

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = 0. \quad \text{invariant} \quad (2.41)$$

If we define $\eta_{00} = 1$, $\eta_{ii} = -1$ for $i = 1, 2, 3$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, then using the Einstein summation convention as well as the convention of lowering indices so that

we define $x_\mu = \eta_{\mu\nu}x^\nu$, then (2.41) is equivalent to

$$x_\mu x^\nu = x'_\mu x'^\nu = 0.$$

Thus, for any coordinate transformation $x \rightarrow x'$ such that $x_\mu x^\mu = x'_\mu x'^\mu$, if the speed of light is c in the x -coordinates, then the speed of light is also guaranteed to be c in the x' -coordinates. A **Lorentz Transformation** Λ is any coordinate transformation of the form $x'^\mu = \Lambda^\mu_\nu x^\nu$ such that $x_\mu x^\mu = x'_\mu x'^\mu$. Since a Lorentz transformation must satisfy

$$x_\mu x^\mu = \eta_{\mu\rho} \Lambda^\rho_\sigma x^\sigma \Lambda^\mu_\nu x^\nu$$

for all x , it follows that

$$\Lambda^\rho_\mu \eta_{\rho\sigma} \Lambda^\sigma_\nu = \eta_{\mu\nu}.^{30} \quad \text{\{lorentztrans\}} \quad (2.42)$$

Having considered how the coordinates of a spacetime location viewed by one observers relate to the coordinates of the same spacetime location viewed by a different observer, we can now consider how physical quantities viewed by different observers relate to each other. The simplest kind of physical quantity is called a **scalar**. A scalar defined at a particular spacetime location has the same value no matter what frame of reference an observer uses. One example of a scalar is an object's **rest mass** which is the mass an object would have if it had no velocity.

³⁰To see why this is, note that if $x_\mu x^\mu = x'_\mu x'^\mu$ for all x , then putting for any other spacetime location y , we have $(x+y)_\mu (x+y)^\mu = (x'+y')_\mu (x'+y')^\mu$. If we expand this out and cancel $x_\mu x^\mu$ with $x'_\mu x'^\mu$ and cancel $y_\mu y^\mu$ with $y'_\mu y'^\mu$, and using the fact that $y_\mu x^\mu = x_\mu y^\mu$, etc. we find that $x_\mu y^\mu = x'_\mu y'^\mu$ for all x and y . Hence

$$\eta_{\nu\mu} x^\mu y^\nu = x_\mu y^\mu = \eta_{\sigma\rho} \Lambda^\rho_\mu \Lambda^\sigma_\nu x^\mu y^\nu.$$

Since we can choose x such that $x^\mu = 1$ and $x^\alpha = 0$ for $\alpha \neq \mu$, and choose y such that $y^\nu = 1$ and $y^\beta = 0$ for $\beta \neq \nu$. Then we get

$$\eta_{\mu\nu} = \eta_{\sigma\rho} \Lambda^\rho_\mu \Lambda^\sigma_\nu,$$

and hence the result follows.

There is still a transformation rule for scalars since the spacetime location at which the scalar is measured is usually expressed in terms of an observer's coordinate system, and the coordinates of such a location will differ for different observers. Thus, if $\phi(x) \stackrel{\text{def}}{=} \phi(x^0, x^1, x^2, x^3)$ is the value of a scalar defined at the spacetime location (x^0, x^1, x^2, x^3) as described by an observer O , then another observer O' using a different set of coordinate (x'^0, x'^1, x'^2, x'^3) to describe the location (x^0, x^1, x^2, x^3) will describe this same scalar as $\phi'(x') \stackrel{\text{def}}{=} \phi'(x'^0, x'^1, x'^2, x'^3)$ where $\phi'(x') = \phi(x)$. Since ϕ' is just a function of the four numbers x'^0, x'^1, x'^2 , and x'^3 , we can rename these numbers x^0, x^1, x^2 , and x^3 , and then

$$\phi'(x) = \phi(\Lambda^{-1}x) \quad \text{\{lorentzscalar\}} \quad (2.43)$$

where Λ is the inverse Lorentz transformation that takes (x'^0, x'^1, x'^2, x'^3) to (x^0, x^1, x^2, x^3) . Thus, equation (2.43) shows us how a scalar transforms under a Lorentz transformation Λ .

Many physical quantities, however, are not scalars and so will look different to different observers. For instance, the energy of an object has a kinetic component that depends on the velocity the object has relative to an observer. However, it turns out that if an observer O considers an object's energy E together with its three components of momentum p^1, p^2 , and p^3 (in the directions \hat{e}_1, \hat{e}_2 , and \hat{e}_3 respectively) to form the four-tuple $p \stackrel{\text{def}}{=} (E/c, p^1, p^2, p^3)$ known as the object's **four-momentum**, then p transforms in the same way as spacetime coordinates transform between different observers. In other words, a different observer O' whose coordinates are given

by $x'^\mu = \Lambda^\mu_\nu x^\nu$ would observe the object's four-momentum to be $p'^\mu = \Lambda^\mu_\nu p^\nu$.³¹

More generally, any list of four physical quantities $(\varphi^0, \varphi^1, \varphi^2, \varphi^3)$ that transforms as

$\varphi \rightarrow \varphi'$ with $\varphi'^\mu = \Lambda^\mu_\nu \varphi^\nu$ is called a **four-vector**. Figure 2.9 shows how (two of) the components of a four-vector φ at a particular location will differ for different observers under a spatial rotation of the coordinates. A four-vector $\varphi^\mu(x)$ defined at every spacetime location x is called a **vector field**, and if O observes this vector-field $\varphi^\mu(x)$, and O' is another observer whose coordinates are related to the coordinates O via the Lorentz transformation Λ , then O' will describe this vector-field as $\varphi'^\mu(x') \stackrel{\text{def}}{=} \Lambda^\mu_\nu \varphi^\nu(x)$. Hence under the Lorentz transformation Λ , $\varphi^\mu \rightarrow \varphi'^\mu$ where

$$\varphi'^\mu(x) = \Lambda^\mu_\nu \varphi^\nu(\Lambda^{-1}x). \quad \text{\{lorentzvector\}} \quad (2.44)$$

If φ^μ is a four-vector, then we refer to $\varphi_\mu \stackrel{\text{def}}{=} \eta_{\mu\nu} \varphi^\nu$ as a **four-covector**. To see how covectors transform under a Lorentz transformation Λ , it will be helpful to define

³¹In order for p to transform in this way, we have to redefine what we mean by energy and momentum. In classical mechanics, the momentum of an object is the product of the object's mass and its velocity. In the context of special relativity, however, the four-momentum of an object is defined to be the product of its rest mass m_0 and its **four-velocity** where the four velocity of an object is a four-tuple (u^0, u^1, u^2, u^3) such that the object's velocity (in the classical sense) is the vector $(c \frac{u^1}{u^0}, c \frac{u^2}{u^0}, c \frac{u^3}{u^0})$ and such that $v_\mu v^\mu = c^2$. One can work out an object's four-velocity by first considering its four-velocity relative to an observer who is stationary relative to the object. In this case, the object's four-velocity will be $(c, 0, 0, 0)$. Now suppose the observer is moving at a constant speed v in the \hat{e}_1 direction towards the object. Then according to the transformation rule for four-vectors, the corresponding four-velocity must now be (u^0, u^1, u^2, u^3) with $u^\mu = \Lambda^\mu_0 c$ for some Lorentz transformation Λ^μ_ν such that $(c \frac{u^1}{u^0}, c \frac{u^2}{u^0}, c \frac{u^3}{u^0}) = (-v, 0, 0)$. From this it follows that $u^2 = u^3 = 0$. But in terms of the Λ^μ_ν components, we must also have $(c, 0, 0, 0) \rightarrow (\Lambda_0^0 c, \Lambda_0^1 c, 0, 0)$. Hence, $\Lambda_0^1 = -\Lambda_0^0 v/c$. It is conventional to define $\beta = v/c$ and $\gamma = \Lambda_0^0$ so that $\Lambda_0^1 = -\gamma\beta$. Moreover, since $u_0 u^0 - u_1 u^1 = c^2$, we must also have $(\gamma c)^2 - (\gamma\beta c)^2 = c^2$ from which it follows that $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Thus, in the case of an object moving with velocity $(-v, 0, 0)$, its four-velocity will be $(\gamma c, -\gamma\beta c, 0, 0) = (\gamma c, -\gamma v, 0, 0)$. Without loss of generality, we therefore see that in the case of an object moving with velocity (v^1, v^2, v^3) , then its four-velocity must be $\gamma(c, v^1, v^2, v^3)$ where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = v/c = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}/c$. Hence, the object's four-momentum will be $\gamma m_0(c, v^1, v^2, v^3)$. If the object's velocity is very small compared to the speed of light, then $\gamma \approx 1 + \frac{v^2}{2c^2}$, and hence the object's four-momentum $(E/c, p^1, p^2, p^3)$ will be approximately $(m_0 c + \frac{1}{2} m_0 v^2/c, m_0 v^1, m_0 v^2, m_0 v^3)$. Therefore, (p^1, p^2, p^3) is approximately equal to the classical momentum. However, the energy is now $E = m_0 c^2 + \frac{1}{2} m_0 v^2$. Thus, in addition to the kinetic energy term $\frac{1}{2} m_0 v^2$, there is a rest mass energy $m_0 c^2$. If we define the **relativistic mass** $m = \gamma m_0$, then we obtain Einstein's famous formula $E = mc^2$.

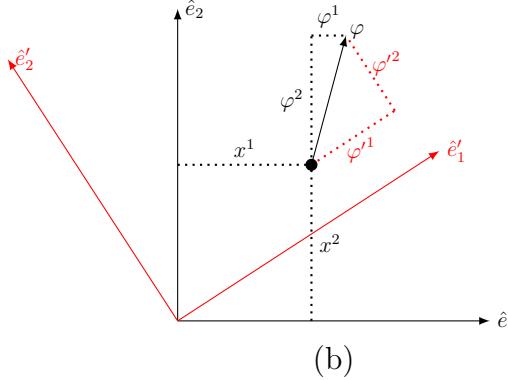


Figure 2.10: Shows how a four-vector φ (of which only two components are shown) defined at a spacetime location (indicated by \bullet) can be expressed either as (φ^1, φ^2) with respect to the basis $\{\hat{e}_1, \hat{e}_2\}$ or as (φ'^1, φ'^2) with respect to the basis $\{\hat{e}'_1, \hat{e}'_2\}$.

$\Lambda_\mu^\nu \stackrel{\text{def}}{=} \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho_\sigma$ where $\eta^{\nu\sigma} = \eta_{\nu\sigma}$. If we also define the **Kronecker-delta** δ_μ^ν such that $\delta_\mu^\nu = 1$ when $\mu = \nu$ and $\delta_\mu^\nu = 0$ otherwise. Then using the fact that $\eta_{\mu\rho}\eta^{\nu\rho} = \delta_\mu^\nu$ together with equation (2.42), we have

$$\Lambda^\rho_\mu \Lambda_\rho^\nu = \delta_\mu^\nu. \quad \text{(2.45)}$$

By definition, the inverse of Λ^{-1} satisfies $(\Lambda^{-1})_\rho^\nu \Lambda^\rho_\mu = \delta_\mu^\nu$, so we have $(\Lambda^{-1})_\rho^\nu = \Lambda_\rho^\nu$.

From (2.42), we see that under a Lorentz transformation Λ , $\varphi_\mu \rightarrow \varphi'_\mu$ where

$$\varphi'_\mu(x) = \Lambda_\mu^\nu \varphi_\nu(\Lambda^{-1}x) \quad \text{(2.46)}$$

Besides scalars, vectors, and covectors we also need to consider physical quantities called rank two tensors. The stress-energy tensor $T^{\mu\nu}$ mentioned on page 2.2.3s is an example of a rank two tensor. The defining property of a rank two tensor $\varphi^{\mu\nu}(x)$ is that under a Lorentz transformation Λ , $\varphi^{\mu\nu} \rightarrow \varphi'^{\mu\nu}$ where

$$\varphi'^{\mu\nu}(x) = \Lambda_\rho^\mu \Lambda_\sigma^\nu \varphi^{\rho\sigma}(\Lambda^{-1}x). \quad \text{(2.47)}$$

On page 87 we introduced the mass-energy density $T_S(x)$ on a hypersurface S . As explained in section 2.3.4 the values of $T_S(x)$ are the additional values that Kent uses

to supplement standard quantum orthodoxy. It was mentioned in passing that $T_S(x)$ does not depend on which frame of reference one is in. In other words, $T_S(x)$ is a scalar. I will now explain why this is so.

We first need to consider the precise definition of $T_S(x)$. At each spacetime location on the hypersurface S which an observer O describes as having coordinates $x = (x^\mu)_{\mu=0}^3$, we define $\eta^\mu(x)$ to be the future-directed unit four-vector at x that is orthogonal to S . In other words, $\eta^0(x) > 0$, $\eta_\mu(x)\eta^\mu(x) = 1$, and if $y \in S$ is very close to x , then

$$\frac{(x-y)_\mu\eta^\mu(x)}{\sqrt{(x-y)_\nu(x-y)^\nu}} \approx 0. T_S(x)$$

is then given by the formula

$$T_S(x) = T^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x). \quad \{TSdef\}_{(2.48)}$$

For example, if S was the hypersurface consisting of all spacetime locations $(0, x^1, x^2, x^3)$, then $(\eta^0(x), \eta^1(x), \eta^2(x), \eta^3(x)) = (1, 0, 0, 0)$, and hence $T_S(x) = T^{00}(x)$ which is the density of relativistic mass at x , i.e. the energy density at x divided by c^2 .

Let us now show that $T_S(x)$ is a scalar. So suppose that Λ is a Lorentz transformation such that $\Lambda^0_\mu\eta^\mu > 0$ for any future-directed unit four-vector vector η^μ . We refer to a Λ with this property as a **orthochronous** Lorentz transformation. Also suppose that O and O' are two observers such that spacetime locations that observer O describes as having coordinates $x = (x^\mu)_{\mu=0}^3$ are described by O' as having coordinates $x' = (\Lambda^\mu_\nu x^\nu)_{\mu=0}^3$. Then since $x'_\mu y'^\mu = x_\mu y^\mu$, it follows that the future-directed unit four-vector orthogonal to S at x which O describes as $\eta^\mu(x)$ will be described by O' as $\eta'^\mu(x') = \Lambda^\mu_\nu\eta^\nu(x)$. Thus, for any location in S that O' describes as having coordinates x' with corresponding future-directed S -orthogonal unit four-vector $\eta'^\mu(x')$,

O' can construct a function $T'_S(x')$ with

$$T'_S(x') = T'^{\mu\nu}(x')\eta'_\mu(x')\eta'_\nu(x'). \quad \{ \text{TSprimeDef} \} \quad (2.49)$$

Then using (2.44, (2.46 and (2.47 on the right hand side of (2.49, we have

$$\begin{aligned} T'_S(x') &= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}(x) \Lambda_\mu{}^\alpha \eta_\alpha(x) \Lambda_\nu{}^\beta \eta_\beta(x) \\ &= \Lambda^\mu{}_\rho \Lambda_\mu{}^\alpha \Lambda^\nu{}_\sigma \Lambda_\nu{}^\beta T^{\rho\sigma}(x) \eta_\alpha(x) \eta_\beta(x) \\ &= \delta_\rho^\alpha \delta_\sigma^\beta T^{\rho\sigma}(x) \eta_\alpha(x) \eta_\beta(x) \quad \{ \text{invariantTS1} \} \quad (2.50) \\ &= T^{\alpha\beta}(x) \eta_\alpha(x) \eta_\beta(x) \\ &= T_S(x) \end{aligned}$$

where on the third line we have used (2.45 and on the last line we have used (2.48. To obtain (2.50, we assumed that Λ is orthochronous, but if Λ is non-orthochronous, we would need to take the negations of $\eta'^\mu(x')$ to get the future-directed S -orthogonal unit four-vector. But clearly this will not affect the equality in (2.50, so (2.50 holds for all Lorentz transformation, whether they are orthochronous or non-orthochronous. We thus see that $T_S(x)$ is a scalar.

Let us now consider the Hilbert space H_{S_i} as defined on page 83 or a hypersurface S_i . Two observers O and O' will typically assign different physical states to S_i based on their frame of reference. E.g. if O and O' are traveling at different speeds, they will attribute different energy levels and momenta to the spacetime locations of S_i . For the Lorentz transformation that relates the coordinates of O' to the coordinates of O , i.e. $x' = \Lambda x$, there will then be a unitary operator $U(\Lambda) : H_{S_i} \rightarrow H_{S_i}$ such that if O observes S_i to be in state $|\psi_i\rangle \in H_{S_i}$, then O' will observe S_i to be in state $U(\Lambda)|\psi_i\rangle$. Also, if $\hat{T}^{\mu\nu}(x)$ is the observable whose eigenstates with eigenvalues τ are the states

of S_i for which O observes the stress-energy tensor $T^{\mu\nu}(x)$ to take the value τ at x , then

$$U(\Lambda)^{-1}\hat{T}^{\mu\nu}(x)U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\hat{T}^{\rho\sigma}(\Lambda^{-1}x). \quad \text{[TUrelation]} \quad (2.51)$$

We insist on this identity because it makes it the case that $\hat{T}^{\mu\nu}(x')$ will be the observable whose eigenstates with eigenvalues τ' are the states of S_i for which O' observes the stress-energy tensor $T'^{\mu\nu}(x')$ to take the value τ' at x' .

We also insist that $U(\Lambda)$ is unitary because this means that if O calculates the probability S_i transitions from state $|\psi_i\rangle$ to state $|\chi_i\rangle$, then O' would calculate the same probability for the corresponding transition from the state $|\psi'_i\rangle = U(\Lambda)|\psi_i\rangle$ to state $|\chi'_i\rangle = U(\Lambda)|\chi_i\rangle$ ³²

Now in order to show that Kent's model is consistent with special relativity, we need to show that (2.38) defines a rank 2 tensor. In other words, if $\{|\xi_j\rangle\}_{j=1}^\infty$ is an orthonormal basis of the Hilbert space of states H_{S_i, τ_S} for which O observes $T_S(x)$ to be $\tau_S(x)$ for all $x \in S_i(y) \cap S$, and if $\{|\xi'_j\rangle\}_{j=1}^\infty$ is an orthonormal basis of the Hilbert space of states H_{S_i, τ'_S} for which O' observes $T'_S(x')$ to be $\tau'_S(x')$ for all $x' \in S_i(y') \cap S$, then

$$\lim_{i \rightarrow \infty} \frac{\langle \Psi'_i | \pi'_i \hat{T}^{\mu\nu}(y') | \Psi'_i \rangle}{\langle \Psi'_i | \pi'_i | \Psi'_i \rangle} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \lim_{i \rightarrow \infty} \frac{\langle \Psi_i | \pi_i \hat{T}^{\rho\sigma}(y) | \Psi_i \rangle}{\langle \Psi_i | \pi_i | \Psi_i \rangle} \quad \text{[kentlorentz]} \quad (2.52)$$

where $\pi_i = \sum_j |\xi_j\rangle\langle\xi_j|$, $\pi'_i = \sum_j |\xi'_j\rangle\langle\xi'_j|$, and $|\Psi'_i\rangle = U(\Lambda)|\Psi_i\rangle$. To see why (2.52) holds, we first recall that π'_i will be independent of which orthonormal basis we choose

³²This follows from (2.31) which implies $|\langle \chi'_i | \psi'_i \rangle|^2 = |\langle \chi_i | \psi_i \rangle|^2$, together with the Born rule given on page 8

for H_{S_i, τ'_S} .³³ Therefore, if we can show that $\{|\xi'_j\rangle \stackrel{\text{def}}{=} U(\Lambda)|\xi_j\rangle\}_{j=1}^\infty$ is an orthonormal basis of H_{S_i, τ'_S} , it will follow that $\pi'_i = U(\Lambda)\pi_i U(\Lambda)^{-1}$.

That the elements of $\{U(\Lambda)|\xi_j\rangle\}_{j=1}^\infty$ are orthonormal follows from the unitarity of $U(\Lambda)$ together with the orthormality of $\{|\xi_j\rangle\}_{j=1}^\infty$. Since $\hat{T}^{\mu\nu}(x')$ is the observable whose eigenstates with eigenvalue τ' are the states of $S_i(y')$ for which O' observes the stress-energy tensor $T'^{\mu\nu}(x')$ to take the value τ' at x' , it follows that $\hat{T}'_S(x') \stackrel{\text{def}}{=} \eta'_\mu(x')\eta'_\nu(x')\hat{T}^{\mu\nu}(x')$ will be the observable whose eigenstates with eigenvalue τ'_S are the states of $S_i(y')$ for which O' observes $T'_S(x')$ to take the value τ'_S at x' , where as usual, $\eta'^\mu(x')$ is the unit four-vector orthogonal to $S_i(y')$ at x' . Now if $x' \in S_i(y') \cap S$, then $x = \Lambda^{-1}x' \in S_i(y) \cap S$. Using the same calculation as in (2.50) together with (2.46), we have

$$\begin{aligned}\hat{T}_S(x) &= \eta_\mu(x)\eta_\nu(x)\hat{T}^{\mu\nu}(x) \\ &= \eta'_\mu(x')\eta'_\nu(x')\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\hat{T}^{\rho\sigma}(x)\end{aligned}\quad \{\text{TS}\Lambda\}_{(2.53)}$$

Using (2.53) together with (2.51), we therefore have

$$\begin{aligned}\hat{T}_S(x)U(\Lambda)^{-1} &= \eta'_\mu(x')\eta'_\nu(x')\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\hat{T}^{\rho\sigma}(x)U(\Lambda)^{-1} \\ &= U(\Lambda)^{-1}\eta'_\mu(x')\eta'_\nu(x')\hat{T}^{\mu\nu}(x') \\ &= U(\Lambda)^{-1}\hat{T}'_S(x').\end{aligned}\quad \{\text{TSU}\}_{(2.54)}$$

Now suppose that $|\xi'\rangle$ is a state for which O' observes $T'_S(x')$ to be $\tau_S(x')$ for all $x' \in S_i(y') \cap S$. Then $\hat{T}'_S(x')|\xi'\rangle = \tau'_S(x')$, and so by (2.54),

$$\begin{aligned}\hat{T}_S(x)U(\Lambda)^{-1}|\xi'\rangle &= U(\Lambda)^{-1}\hat{T}'_S(x')|\xi'\rangle \\ &= \tau'_S(x')U(\Lambda)^{-1}|\xi'\rangle \\ &= \tau_S(x)U(\Lambda)^{-1}|\xi'\rangle\end{aligned}\quad \{\text{TSUxi}\}_{(2.55)}$$

³³We showed this was the case for π_i in footnote 26n page 84

where on the last line we have used the fact that $T_S(x)$ is a scalar. Therefore, $U(\Lambda)^{-1} |\xi'\rangle$ can be expressed as a linear combination of basis elements $\{|\xi_j\rangle\}_{j=1}^\infty$, and hence $|\xi'\rangle$ is can be expressed as a linear combination of $\{U(\Lambda) |\xi_j\rangle\}_{j=1}^\infty$. From (2.54) we also see that $U(\Lambda)\hat{T}_S(x) = \hat{T}'_S(x')U(\Lambda)$, so

$$\hat{T}'_S(x')U(\Lambda) |\xi_j\rangle = U(\Lambda)\hat{T}_S(x) |\xi_j\rangle = \tau_S(x)U(\Lambda) |\xi_j\rangle = \tau'_S(x')U(\Lambda) |\xi_j\rangle$$

for all $x' \in S_i(y') \cap S$. Therefore $\{|\xi'_j\rangle\}_{j=1}^\infty \stackrel{\text{def}}{=} U(\Lambda) |\xi_j\rangle$ is an orthonormal basis of H_{S_i, τ'_S} , and from this it follows that $\pi'_i = U(\Lambda)\pi_i U(\Lambda)^{-1}$. Therefore

$$\begin{aligned} \frac{\langle \Psi'_i | \pi'_i \hat{T}^{\mu\nu}(y') | \Psi'_i \rangle}{\langle \Psi'_i | \pi'_i | \Psi'_i \rangle} &= \frac{\langle \Psi_i | U(\Lambda)^{-1} U(\Lambda) \pi_i U(\Lambda)^{-1} \hat{T}^{\mu\nu}(y') U(\Lambda) | \Psi_i \rangle}{\langle \Psi_i | U(\Lambda)^{-1} U(\Lambda) \pi_i U(\Lambda)^{-1} U(\Lambda) | \Psi_i \rangle} \\ &= \frac{\langle \Psi_i | \pi_i U(\Lambda)^{-1} \hat{T}^{\mu\nu}(y') U(\Lambda) | \Psi_i \rangle}{\langle \Psi_i | \pi_i | \Psi_i \rangle} \quad \text{\{kentlorentz2\}} \\ &= \frac{\langle \Psi_i | \pi_i \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \hat{T}^{\rho\sigma}(y) | \Psi_i \rangle}{\langle \Psi_i | \pi_i | \Psi_i \rangle} \end{aligned} \tag{2.56}$$

where on the last line we have used (2.51). Thus, equation (2.52) holds, and hence Kent's model is consistent with special relativity.

2.3.5 Kent's Interpretation and Decoherence Theory

In section 1.10e saw that decoherence theory by itself does not offer a solution to the problem of outcomes. In this section, we consider how the additional information in Kent's interpretation is sufficient to address this problem. We suppose that y_1 is an event between the two hypersurfaces S_0 and S , and we consider a hypersurface $S_i = S_i(y_1)$ in a sequence of hypersurfaces that each contain y_1 as described on page 83 To simplify our description of S_i we use a coarse-grained model so that S_i is treated as a mesh of tiny cells labeled by a sequence $(y_k)_{k=1}^\infty$, and for each cell y_k there is a Hilbert space H_k describing the state of that cell. For any y_k we denote its spatial component as z_k and its time component as t_k . Then we suppose that the state $|\Psi_i\rangle$

describing S_i can be decomposed as

$$|\Psi_i\rangle = \left(\sum_{j=1}^J c_j |\xi_{1,j}\rangle \prod_{l=1}^{N_j} |\xi_{k_{l,j},j}\rangle \right) \Xi \quad \{Sistate\}_{(2.57)}$$

where $|\xi_{1,j}\rangle \in H_1$ are normalized states with $\langle \xi_{1,j} | \xi_{1,j'} \rangle = 0$ for $j \neq j'$, where $|\xi_{k_{l,j},j}\rangle \in H_{k_{l,j}}$ are normalized states with all the $k_{l,j}$ for $l = 1$ to N_j being non-zero and unique for a particular j , and where Ξ is the sum of states of the form $\prod_l |\xi_{\kappa_l}\rangle$ describing all the cells of S_i not included in the set $\bigcup_j \bigcup_{l=1}^{N_j} \{k_{l,j}\}$. We also assume each summand $c_j |\xi_{1,j}\rangle \prod_{l=1}^{N_j} |\xi_{k_{l,j},j}\rangle \Xi$ contains a state in each H_k for every cell k of S_i . In other words if k does not belong to the set $\{k_{l,j} : l\}$ then k belongs to the set $\{\kappa_l : l\}$. From this it follows that $\bigcup_{l=1}^{N_j} \{k_{l,j}\} = \bigcup_{l=1}^{N'_j} \{k_{l,j'}\}$ for any $j' \neq j$. We can therefore assume $N_j = N$ and $k_{l,j} = k_l$ are independent of j . We can thus rewrite (2.57) as

$$|\Psi_i\rangle = \left(\sum_{j=1}^J c_j |\xi_{1,j}\rangle \prod_{l=1}^N |\xi_{k_l,j}\rangle \right) \Xi \quad \{Sistate2\}_{(2.58)}$$

We also assume that Ξ describes as much of S_i as possible, so that there is no common factor $|\xi\rangle$ among all the elements of $\{\prod_{l=1}^N |\xi_{k_l,j}\rangle : j\}$. Furthermore, we assume that the k_l are appropriately ordered so that there is some M such that $y_{k_l} \in S \cap S_i$ for $l \geq M$ and $y_{k_l} \notin S \cap S_i$ for $l < M$. Finally, we will assume that if $N = 0$ then $J = 1$, for otherwise we can add the different $|\xi_{1,j}\rangle$ to form a single state in H_1 .

To get a sense of the applicability of the decomposition (2.58), we will consider a few different scenarios inspired by Kent's toy model, and in each scenario, we will use (2.58) to calculate the partial trace encapsulating all the information to calculate expectation values at different space-time locations. We thus suppose that a system is in a superposition $\psi_0^{\text{sys}} = c_1 \psi_1^{\text{sys}} + c_2 \psi_2^{\text{sys}}$ of two local states ψ_1^{sys} and ψ_2^{sys} where $|c_1|^2 + |c_2|^2 = 1$ and that there is a photon coming in from the left that interacts with

the system. Figure 2.11 depicts the hypersurface $S_i(y_1^a)$ for an event y_1^a that occurs before the photon has interacted with the system.

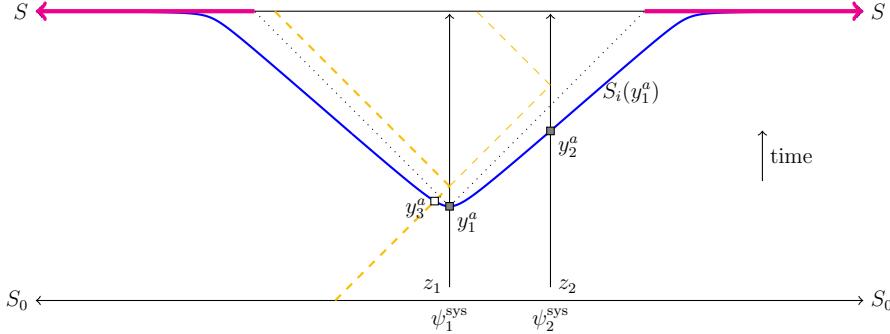


Figure 2.11: Depiction of a superposition of two local states at z_1 and z_2 before the photon has interacted with them. The gray squares indicate cells in $S^1(y_1^a)$ whose states are among the summands in (2.58). The white square indicates a cell in $S_i(y_1^a)$ whose state is a factor in Ξ .

kentdeco1

The gray squares correspond to the summands that appear in (2.58). If the system were in the ψ_1^{sys} -state, then the state describing $S_i(y_1^a)$ would have a factor $|\psi_1^{\text{sys}}\rangle \in H_1$ indicating that there is a non-zero mass at the y_1^a -cell, and there would also be a factor $|0_2\rangle \in H_2$ which we use to indicate that there is zero mass at y_2^a . There is also an incoming photon at the y_3^a -cell, and so we use $|\gamma_3\rangle$ to indicate that there is a photon there. Thus, if the system were in the ψ_1^{sys} -state, we would write the state of $S_i(y_1^a)$ as $|\Psi_i\rangle = |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle \Xi'$ where Ξ' describes the states of all the other cells of $S_i(y_1^a)$. But on the other hand, if the system were in the state ψ_2^{sys} , then the state describing $S_i(y_1^a)$ would have a factor $|\psi_2^{\text{sys}}\rangle \in H_2$ indicating that there is a non-zero mass at the y_2^a -cell, and there would also be a factor $|0_1\rangle \in H_1$ which we use to indicate that there is zero mass at y_1^a , and again the y_3^a -cell would be in the $|\gamma_3\rangle$, and every other cell would be described by Ξ' just as if the system had been in the ψ_1^{sys} -state. Therefore, when the system is in the state ψ_2^{sys} , we would write the state of $S_i(y_1^a)$ as $|\Psi_i\rangle = |0_1\rangle |\psi_2^{\text{sys}}\rangle |\gamma_3\rangle \Xi'$ But since the system is actually in a superposition

$\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$, the state of $S_i(y_1^a)$ will be

$$|\Psi_i\rangle = (c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle) |\gamma_3\rangle \Xi' = (c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle) \Xi$$

where we have absorbed the $|\gamma_3\rangle$ -state into Ξ (i.e. $\Xi = |\gamma_3\rangle \Xi'$).

Now as it stands, the state $|\Psi_i\rangle$ describing $S_i(y_1^a)$ has a definite mass-energy density $\tau_S(x)$ for $x \in S_i(y_1^a) \cap S$. Thus if π_i is the operator featuring in (2.38) that corresponds to this definite mass-energy density, then $\pi_i |\Psi_i\rangle = |\Psi_i\rangle$. Therefore, equation (2.38) for Kent's beables tells us that

$$\langle T^{\mu\nu}(y_1^a) \rangle_{\tau_S} = \langle \Psi_i | \hat{T}^{\mu\nu}(y_1^a) | \Psi_i \rangle.$$

where we have also used the fact that $\langle \Psi_i | \Psi_i \rangle = 1$.

Now as we saw in section 1.7 if we are interested only in the expectation values of observables for a system \mathcal{S} contained within a universe $\mathcal{U} = \mathcal{S} + \mathcal{E}$ then the information needed to do this can be encapsulated in the reduced density matrix for \mathcal{S} . Thus, if the universe is described by a state $|\Psi\rangle = \sum_j c_j |\psi_j\rangle_{\mathcal{S}} |E_j\rangle$ with corresponding density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi| \in M(H_{\mathcal{U}})$, then the reduced density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ is the operator that acts on the Hermitian operators of the state space $H_{\mathcal{S}}$ with the property that

$$\langle \hat{\Lambda}_{\mathcal{U}} \rangle_{\rho} = \text{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}} \hat{\Lambda}_{\mathcal{S}}) \quad (1.16\text{evisited})$$

where $\hat{\Lambda}_{\mathcal{S}}$ is an observable on $H_{\mathcal{S}}$ and $\hat{\Lambda}_{\mathcal{U}}$ is the corresponding observable on $H_{\mathcal{U}}$.

Furthermore, we also have

$$\hat{\rho}_{\mathcal{S}} = \sum_j |c_j|^2 |\psi_j\rangle\langle\psi_j| + \sum_{j \neq k} c_j \overline{c_k} \langle E_k | E_j \rangle |\psi_j\rangle\langle\psi_k|.^{34} \quad \text{reduced2} \quad (2.59)$$

We can thus apply this to the situation at hand by taking S_i to be our universe \mathcal{U} and y_1^a to be the system \mathcal{S} , and $S_i \setminus y_1^a$ to be the environment \mathcal{E} . If we assume that $\langle 0_2 | \psi_2^{\text{sys}} \rangle \approx 0$, then by (2.59), the corresponding reduced density matrix $\hat{\rho}_{y_1^a}$ takes the form of an improper mixture

$$\hat{\rho}_{y_1^a} \approx |c_1|^2 |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}| + |c_2|^2 |0_1\rangle\langle 0_1| \quad \{ \text{kentred} \} \quad (2.60)$$

Kent's beables at y_1^a will thus take the form

$$\begin{aligned} \langle T^{\mu\nu}(y_1^a) \rangle_{\tau_S} &= \text{Tr}_{y_1^a}(\hat{\rho}_{y_1^a} \hat{T}^{\mu\nu}(y_1^a)) \\ &= |c_1|^2 \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^a) | \psi_1^{\text{sys}} \rangle + |c_2|^2 \langle 0_1 | \hat{T}^{\mu\nu}(y_1^a) | 0_1 \rangle. \end{aligned} \quad \{ \text{kentbe} \} \quad (2.61)$$

Let us now consider Kent's beables at the spacetime location y_1^b depicted in figure

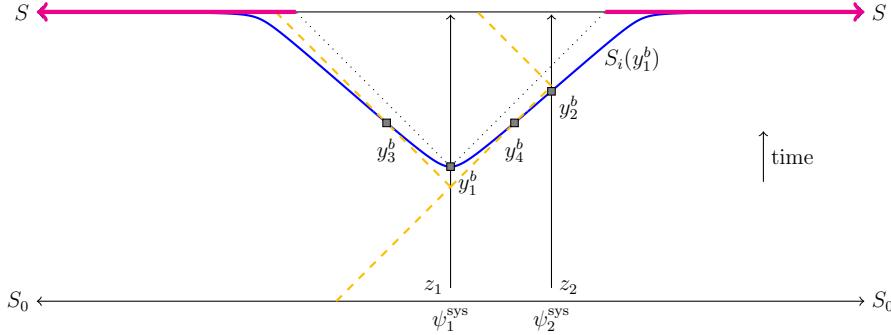


Figure 2.12: Depiction of a superposition of two local states at z_1 and z_2 with $S_i(y_1^b)$ being after the photon has interacted without the photon intersecting $S_i(y_1^b) \cap S$. The gray squares indicate cells in $S^1(y_1^b)$ whose states are among the summands in (2.58) _{kentdecon2}

The state of $S_i(y_1^b)$ will then be

$$|\Psi_i\rangle = (c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |0_3\rangle |\gamma_4\rangle) \Xi$$

where the notation is analogous to that in the previous example. Since no photon detections are registered on $S_i(y_1^b) \cap S$, we again have $\pi_i |\Psi_i\rangle = |\Psi_i\rangle$ so that the

³⁴cf. (1.17)

reduced density matrix $\hat{\rho}_{y_1^b}$ will again be given by (2.60 with y_1^a replaced by y_1^b , and likewise, Kent's beables $\langle T^{\mu\nu}(y_1^b) \rangle_{\tau_S}$ will be given by (2.61.

For the next example, we consider the case

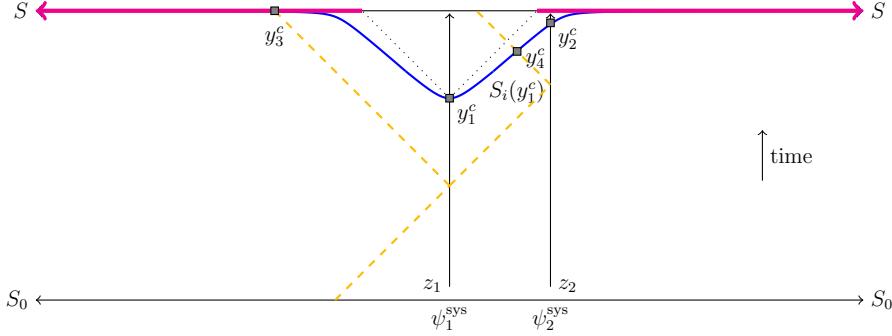


Figure 2.13: Depiction of a superposition of two local states at z_1 and z_2 with y_1^c sufficiently late that the photon intersects $S_i(y_1^c) \cap S$. The gray squares indicate cells in $S^1(y_1^c)$ whose states are among the summands in (2.58)

kentdecoh3

In this case, the state of $S_i(y_1^c)$ will be

$$|\Psi_i\rangle = (c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |0_3\rangle |\gamma_4\rangle) \Xi$$

but this time we have to consider the fact that the photon intersects $S_i(y_1^c) \cap S$. There are two possible (notional) measurements outcomes that can occur on $S_i(y_1^c) \cap S$: either $T_S = \tau_{S,1}(y_3^c) \neq 0$ or $T_S = \tau_{S,2}(y_3^c) = 0$. The case $T_S = \tau_{S,1}$ indicates that there is a photon detection at y_3^c so that the local state at the y_3^c -cell is $|\gamma_3\rangle$. Therefore, if we write $\pi_{i,1}$ for the operator π_i , we have

$$\pi_{i,1} |\Psi_i\rangle = c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle \Xi,$$

Therefore $\langle \Psi_i | \pi_{i,1} \hat{T}^{\mu\nu}(y_1^c) | \Psi_i \rangle = |c_1|^2 \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^c) | \psi_1^{\text{sys}} \rangle$ and $\langle \Psi_i | \pi_{i,1} | \Psi_i \rangle = |c_1|^2$.

Hence, by (2.38, Kent's beables at y_1^c will be

$$\langle T^{\mu\nu}(y_1^c) \rangle_{\tau_{S,1}} = \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^c) | \psi_1^{\text{sys}} \rangle.$$

From this, it follows that the reduced density matrix at y_1^c will take the form of a pure state:

$$\hat{\rho}_{y_1^c} \approx |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}|. \quad (2.62)$$

On the other hand, for the case when $T_S = \tau_{S,2}$, this indicates that there is no photon detection at y_3^c , so that the local state at the y_3^c -cell will be $|0_3\rangle$. So if now we write $\pi_{i,2}$ for the operator π_i , we have

$$\pi_{i,2}|\Psi_i\rangle = c_2|0_1\rangle|\psi_2^{\text{sys}}\rangle|0_3\rangle|\gamma_4\rangle\Xi.$$

Therefore $\langle\Psi_i|\pi_{i,2}\hat{T}^{\mu\nu}(y_1^c)|\Psi_i\rangle = |c_2|^2\langle0_1|\hat{T}^{\mu\nu}(y_1^c)|0_1\rangle$ and $\langle\Psi_i|\pi_{i,2}|\Psi_i\rangle = |c_2|^2$, and so by (2.38), Kent's beables at y_1^c will be

$$\langle T^{\mu\nu}(y_1^c)\rangle_{\tau_{S,2}} = \langle 0_1|\hat{T}^{\mu\nu}(y_1^c)|0_1\rangle.$$

In this case, the reduced density matrix at y_1^c will be

$$\hat{\rho}_{y_1^c} \approx |0_1\rangle\langle 0_1|, \quad (2.63)$$

which is again a pure state.

In the final example, we introduce an atom at a spatial location z_3 initially in its ground state, and suppose that it can't interact with the incoming photon, but that after the photon's interaction with the two local states at z_1 and z_2 , the atom at spacetime location y_3^c can either absorb the photon with probability $|d_+|^2$ or not absorb the photon with probability $|d_-|^2$ as depicted in figure 2.14

If the atom at y_3^d absorbs the photon it will go into a higher energy state $|\psi_3^+\rangle$, whereas if it does not absorb the photon, it will remain in its lower energy state $|\psi_3^-\rangle$. The

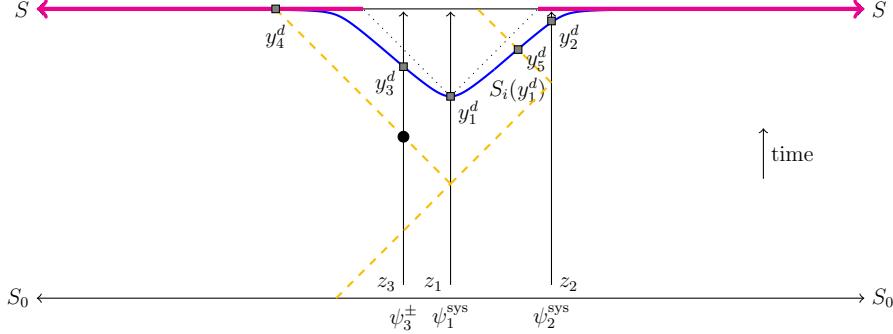


Figure 2.14: Depiction of a superposition of two local states at z_1 and z_2 with y_i^c sufficiently late that the photon can intersect with $S_i(y_i^c) \cap S$, but there is also an atom at z_3 that can absorb the photon. The gray squares indicate cells in $S^1(y_1^d)$ whose states are among the summands in (2.58)

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state of $S_i(y_1^d)$ will thus be

$$|\Psi_i\rangle = (c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle (d_+ |\psi_3^+\rangle |0_4\rangle + d_- |\psi_3^-\rangle |\gamma_4\rangle) |0_5\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |\psi_3^-\rangle |0_4\rangle |\gamma_5\rangle) \Xi.$$

As in the previous example, there are two (notional) measurements that can occur $S_i(y_1^d) \cap S$: either $T_S = \tau_{S,1}$ where $\tau_{S,1}(y_4^d) \neq 0$ or $T_S = \tau_{S,2}$ where $\tau_{S,2}(y_4^d) = 0$. The case $T_S = \tau_{S,1}$ indicates that there is a photon detection at y_4^d so that the local state at the y_3^d -cell is $|\gamma_4\rangle$. Therefore, if we write $\pi_{i,1}$ for the operator π_i , we have

$$\pi_{i,1} |\Psi_i\rangle = c_1 d_- |\psi_1^{\text{sys}}\rangle |0_2\rangle |\psi_3^-\rangle |\gamma_4\rangle |0_5\rangle \Xi.$$

Therefore $\langle \Psi_i | \pi_{i,1} \hat{T}^{\mu\nu}(y_1^d) | \Psi_i \rangle = |c_1 d_-|^2 \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^d) | \psi_1^{\text{sys}} \rangle$ and $\langle \Psi_i | \pi_{i,1} | \Psi_i \rangle = |c_1 d_-|^2$, and so by (2.38, Kent's beables at y_1^d will be

$$\langle T^{\mu\nu}(y_1^d) \rangle_{\tau_{S,1}} = \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^d) | \psi_1^{\text{sys}} \rangle.$$

and the reduced density matrix at y_1^d will be the pure state

$$\hat{\rho}_{y_1^d} \approx |\psi_1^{\text{sys}}\rangle \langle \psi_1^{\text{sys}}|. \quad (2.64)$$

On the other hand, for the case when $T_S = \tau_{S,2}$, writing $\pi_{i,2}$ for the operator π_i , we have

$$\pi_{i,2} |\Psi_i\rangle = (c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle d_+ |\psi_3^+\rangle |0_4\rangle |0_5\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |\psi_3^-\rangle |0_4\rangle |\gamma_5\rangle) \Xi.$$

Therefore $\langle \Psi_i | \pi_{i,2} \hat{T}^{\mu\nu}(y_1^c) | \Psi_i \rangle = |c_1 d_+|^2 \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^d) | \psi_1^{\text{sys}} \rangle + |c_2|^2 \langle 0_1 | \hat{T}^{\mu\nu}(y_1^c) | 0_1 \rangle$ and $\langle \Psi_i | \pi_{i,2} | \Psi_i \rangle = |c_1 d_+|^2 + |c_2|^2$. Hence by (2.38, Kent's beables at y_1^d will be

$$\langle T^{\mu\nu}(y_1^d) \rangle_{\tau_{S,2}} = \frac{|c_1 d_+|^2 \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^d) | \psi_1^{\text{sys}} \rangle + |c_2|^2 \langle 0_1 | \hat{T}^{\mu\nu}(y_1^c) | 0_1 \rangle}{|c_1 d_+|^2 + |c_2|^2},$$

and the reduced density matrix at y_1^d will take the form of an improper mixture:

$$\hat{\rho}_{y_1^d} \approx \frac{|c_1 d_+|^2 |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}| + |c_2|^2 |0_1\rangle\langle 0_1|}{|c_1 d_+|^2 + |c_2|^2}. \quad (2.65)$$

2.3.6 An objection to Kent's beables

In the examples of the last subsection we have seen how the additional information concerning photon detection on $S_i(y_1) \cap S$ is able to determine whether the reduced density matrix at y_1 is a pure state or an improper mixture. As already mentioned, according to d'Espagnat, if the reduced density matrix for a system is an improper mixture, we are not entitled to give it an ignorance interpretation, and thus we are unable to say that an outcome has occurred. However, if the reduced density matrix of a system goes from being an improper mixture to a pure state state of the form $|\psi\rangle\langle\psi|$, then we can say that an outcome has occurred, namely that the system is in the state $|\psi\rangle$.

There is still the question of why Kent decides that the beables of his theory should take the form of expectation values. As an analogy, it seems a bit like saying a six sided dice can actually come up with a 3.5 since this is its expectation value.

We can however address this objection if we can find a way of having multiple throws of the dice, so to speak. My suggestion is to stipulate that the reduced density matrices $\hat{\rho}_y$ as calculated in the previous section are more fundamental beables than the conditional expectation values of $\hat{T}_{\mu,\nu}(y)$, and that the determinate value of T_S is more fundamental still. There are a few things I need to explain here such as what I mean fundamental, what I mean by saying a beable is a reduced density matrix, and how the beable of a reduced density matrix can give rise to Kent's expectation value beables.

By saying that the determinate value of T_S is more fundamental than the reduced density matrices, I mean that if we have statements of the form “the beable at y is $\hat{\rho}_y$ ”, then such a statement is reducible to statements about T_S . Such statements might be simple factual statements such as “there is photon at x_1 , and x_2 but not at x_3 or x_4 ” where the locations are indicated in figure 2.15

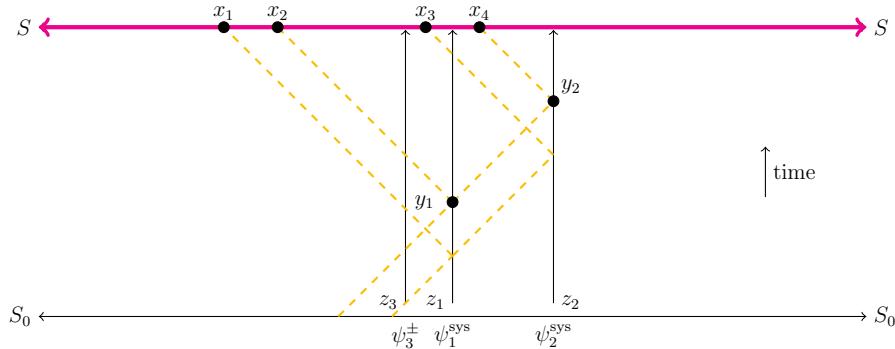


Figure 2.15: Depiction of a superposition of two local states at z_1 and z_2 with y_1^c sufficiently late that the photon intersects $S_i(y_1^c) \cap S$. With enough photon detections on S we can make statements about

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With enough such factual statements, we would have enough information on T_S to say that there is a local state ψ_1^{sys} at spacetime location y_1 and hence conclude that the statement p=“the beable at y_1 is $|\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}|$ ” is true. But we would very likely

need quite a lot more information than knowledge of the fact that a photon is at x_1 , and x_2 , to conclude p since we would need to be able to work out something about time at which the incoming arrived in the vicinity of the system, and this would depend on physics of photon creation. But it seems plausible that from all the information in T_S one could make statements such as p that involve pure states. It is no more controversial than the assumption that we can draw valid conclusions about the physical world based on which cells in our retina are excited. What is controversial is my (and Kent's) suggestion that the information contained in T_S determines the state of physical reality on earlier hypersurfaces rather than the physical state of the earlier hypersurfaces determining the the information contained in T_S . In the final chapter, I will aim to justify this suggestion and show that it is not quite as alien to common sense as it might first seem.

As for statements that involve improper mixtures, for example statements of the form q=“the beable at y_1 is $|c_1|^2 |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}| + |c_2|^2 |0_1\rangle\langle 0_1|$ ” we could take these to be expressible in terms of modal statement about such as “it's possible that a photon could have been detected at either x_1 or x_3 but not both” as depicted in figure 2.15 As in the case for pure states, we would also need other modal statements and declarative propositions about T_S in order to say enough about the times at which incoming photons would arrive at the z_1 and z_2 . But with enough information on T_S it seems plausible that we could build up a picture of such photon interaction on earlier hypersurfaces. Also, by comparing the values of T_S over different region of S , we could detect similar configurations from which a superintelligent being could conclude that these similar configurations correspond to some kind of activity such as

a human being performing an experiment. But these multiple configurations would also have differences, some of which would correspond to different measurement outcomes. By surveying many of these configurations, the superintelligent being could assign probabilities to these outcomes and hence calculate expectation values for observables and the reduced density matrices that would give rise to these expectation values and hence make statements like q which involve improper mixtures.

At this point it is worth reminding ourselves that Kent is not saying that an actual measurement of T_S on S is made, but only a notional measurement which is to say if a measurement were made on T_S it would have a determinate value τ_S , say. One could choose a hypersurface S' even later than S , but Kent supposes that the description of physical reality between S_0 and S is not going to be significantly different if the one used the notional measurements for $T_{S'}$ on S' rather than those on T_S .

So let's now consider how the understanding of beables in terms of reduced density matrices can give rise to Kent's beables as conditional expectation values of $T_{\mu,\nu}(y)$. In this section we have S_i we been using a coarse-grained model so that S_i is treated as a mesh of tiny cells labeled. The state of these one of these tiny cells, y_k would not in general be pure eigenstates of $T_{\mu,\nu}(y_k)$

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