[Institution]

On Physics and Common Sense

A Dissertation Submitted to

[Faculty]

in Candidacy for the Degree of

[Degree]

[Department]

by

Robert Verrill OP

[City], [State/Province]

July 27, 2022

Contents

Chapter 1

The Appeal of the Many-World's Interpretation of Quantum Physics

In recent times, it has become increasingly common for popularizers of quantum physics to tell us that we need to let go of our naïve common sense understanding of reality. We're told we must replace this common sense understanding with something that at first seems very bizarre and counter-intuitive: a many-worlds account of reality. In this chapter, I will try to explain what is meant by this many-worlds account of reality and why some theoretical physicists find it so appealing.

To begin with, I will give an overview of the Copenhagen interpretation and the hidden variables interpretation of quantum physics. This overview will be helpful since it will not only serve to introduce to the reader the notation of quantum theory, but it will also give us an opportunity to consider the problem with the Copenhagen interpretation and the hidden variables interpretation that the many-worlds interpretation seeks to resolve.

Many ideas in quantum physics are expressed in mathematical terms. I will do my best to avoid unnecessary mathematical jargon, but in order to explain the ideas of this thesis, a certain amount of mathematics is unavoidable. I will endeavor to explain all the mathematical terminology as I go along. However, there will be some sections which may be very challenging to readers who do not have a mathematics or physics background. These sections will be marked with and asterisk *.[asteriskmeaning]

1.1 The Stern-Gerlach Experiment

Some of the key features of quantum physics are exhibited in the Stern-Gerlach experiment (see figure 1.1). In this experiment, silver atoms are heated in a furnace which randomly emerge from the furnace with various velocities. By aligning two plates with circular holes near the furnace, it is possible to select a subset of the emerging silver atoms having the same momentum to form a beam in one direction, the other silver atoms having been absorbed by the two plates. This beam of silver atoms is then directed between two magnets with the north pole of one magnet being aligned toward the south pole of the other magnet as shown in figure 1.1. Now

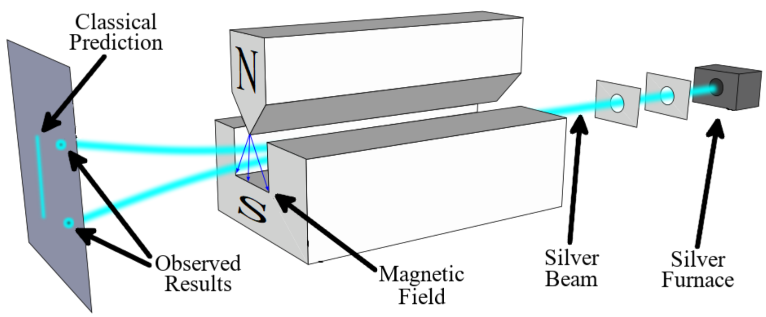


Figure 1.1: The Stern-Gerlach Experiment.[1]

stern

silver atoms have a property somewhat analogous to the classical notion of angular momentum. For instance, a spinning top has angular momentum as shown in figure 1.2. Angular momentum is a vector, so it has direction and magnitude. In the case of a spinning top, the direction of the angular momentum would be parallel to the axis of rotation, pointing one way or the other depending on whether the rotation was clockwise or counterclockwise. The magnitude of the angular momentum would then be proportional to the angular velocity of the spinning top.

[1]Original diagram drawn by Theresa Knott. Labeling was modified for use in this dissertation. This image is licensed under the Creative Commons Attribution-Share Alike 4.0 International license. Source: https://commons.wikimedia.org/wiki/File:Stern-Gerlach_experiment_svg.svg

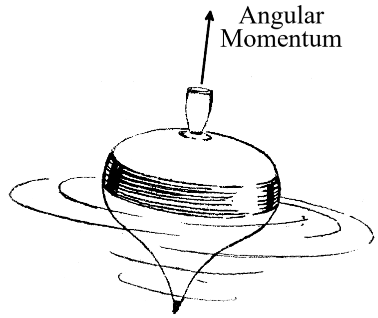


Figure 1.2: Angular Momentum of a Spinning Top.[2] spintop

Now if we tried to understand the angular momentum of a silver atom classically, we would expect the magnetic field of the two magnets to interact with the silver atom in a way that was determined by the relative direction of the silver atom's magnetic momentum compared to the direction of the magnetic field. Since we would expect the silver atom to have an entirely random angular momentum, we would expect it to be deflected by varying degrees either up or down in the direction of the magnetic field. Thus, if a detection screen were placed beyond the two magnets which the silver atoms would hit, we would expect there to be a whole continuum of possible locations where the silver atoms would be detected. However, in reality, it is found that there are precisely two locations where the silver atoms hit the screen. It is as though the particles can spin either clockwise or anticlockwise, but that there is absolutely no variance in the angular speed at which they rotate. This is surprising. The angular momentum appears to be **quantized** in one of two directions, either parallel to the magnetic field or antiparallel to it.[3] Corresponding to this quantization of angular

[2]Drawing by Pearson Scott Foresman, Public domain, via Wikimedia Commons. Labeling was added for use in this dissertation. Original: https://commons.wikimedia.org/wiki/File:Top_(PSF).png.

[3]See figure 1.3 for what is meant be antiparallel.

momentum, we say that the atom is either in the spin up state or the spin down state with respect to the direction of the magnetic field.

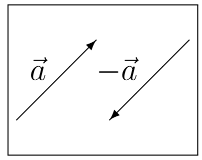


Figure 1.3: Meaning of antiparallel: the arrows in opposite directions are said to be antiparallel to one another.

`antiparallel`

If the direction of the magnetic field is implicitly understood, we write $|+\rangle$ and $|-\rangle$ for the spin up and spin down states of the atom respectively. We refer to the symbols $|+\rangle$ and $|-\rangle$ as **ket-vectors**, or simply as kets. We can think of the ket $|+\rangle$ for instance as shorthand for the proposition "the particle is in the spin up state." If we knew this proposition to be true, we would know which of the two locations on the detection screen the particle would end up if it were to travel between the two magnets of the Stern-Gerlach apparatus. If we need to specify the spin with respect to a particular direction of the magnetic field, say in the $\hat{\mathbf{a}}$-direction, we write the corresponding spin up and down states as $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$. For convenience, we write $\hat{\mathbf{a}}+$ and $\hat{\mathbf{a}}-$ respectively for the location that the particle would hit the detection screen.

The question then arises as to what happens when we rotate the magnetic field around the axis of the particle beam in the Stern-Gerlach experimental setup. It turns out that when we do this, the atoms are again detected in only one of two locations (see figure 1.4).

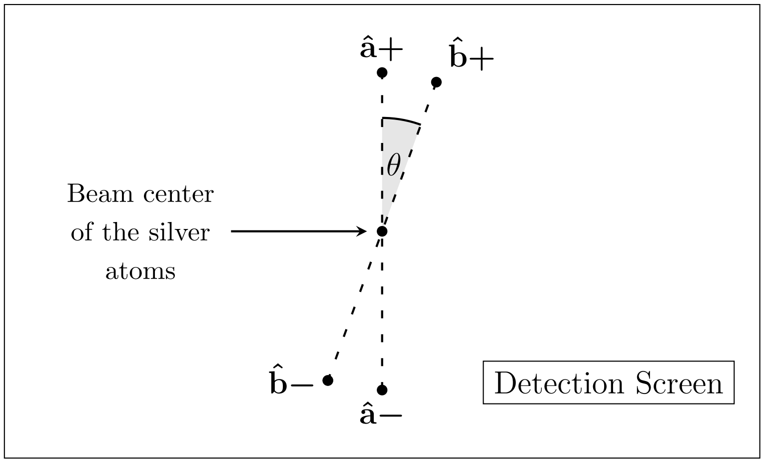


Figure 1.4: Locations of detections before and after rotating the magnetic field by an angle θ. Before rotation, the particles can be detected at either location $\hat{\mathbf{a}}+$ or location $\hat{\mathbf{a}}-$. After the rotation, particles can be detected at either location $\hat{\mathbf{b}}+$ or $\hat{\mathbf{b}}-$.

rotate

So suppose we knew the particle was in a spin state such that it was on course to arrive at location $\hat{\mathbf{a}}+$ because we had previously directed it through another magnetic field in the $\hat{\mathbf{a}}$-direction. For example, see figure 1.5 for how this might be done. In

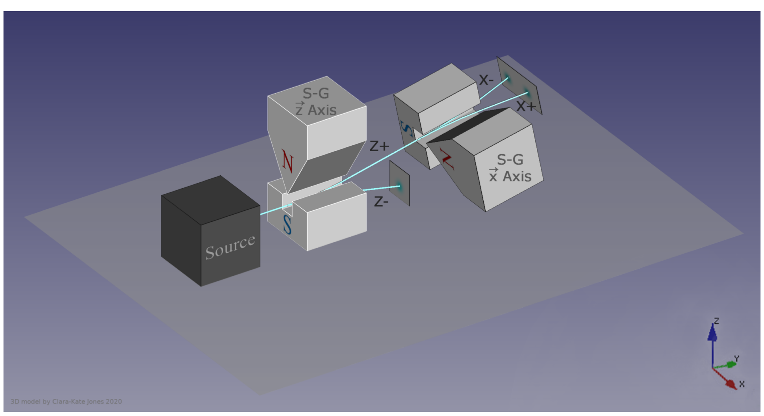


Figure 1.5: Two Stern-Gerlach experiements in sequence. By directing the beam of particles through one magnetic field first, the particles emerging in one of the two beams will be in a known spin state before they enter the second magnetic field.[4]

knownspin

this experimental setup, the second magnetic field has been rotated by an angle of

[4]Diagram by MJasK. This file is licensed under the Creative Commons Attribution-Share Alike 4.0 International license. Source: https://commons.wikimedia.org/wiki/File:Stern-Gerlach_Analyzer_Sequential_Series_E2.png.

$90°$ with respect to the first magnetic field. But suppose we just rotated the second magnetic field by a very small angle θ with respect to the first magnetic field. Then we would expect the particle now to arrive at a location $\hat{\mathbf{b}}+$ close by to $\hat{\mathbf{a}}+$ as shown in figure 1.4. And this is what we notice experimentally for the most part. However, occasionally, the particle will arrive at location $\hat{\mathbf{b}}-$. The frequency of this occurrence becomes less and less the less and less the magnetic field is rotated (i.e. the smaller θ is), so that if the magnetic field is not rotated at all, i.e. $\theta = 0$, the particle will always arrive at location $\hat{\mathbf{a}}+$. To capture the probabilistic nature of these outcomes, we use the bra-ket notation. Thus, if $|\psi\rangle$ stands for either the $|\hat{\mathbf{a}}+\rangle$ or the $|\hat{\mathbf{a}}-\rangle$-state, and $|\chi\rangle$ stands for either the $|\hat{\mathbf{b}}+\rangle$ or the $|\hat{\mathbf{b}}+\rangle$-state, then we define the bra-ket $\langle\psi|\chi\rangle \in \mathbb{C}$ to be a complex number[5] that satisfies the **Born Rule**, bornrule namely $|\langle\psi|\chi\rangle|^2$ is the probability that the particle will be found to be in state $|\psi\rangle$ given that we know that the particle is in state $|\chi\rangle$. For example, if $|\langle\psi|\chi\rangle|^2 = \frac{1}{4}$, and we performed the experiment a 1000 times with the particle initially prepared in the $|\chi\rangle$-state, then we would expect the particle to be found in the $|\psi\rangle$-state in around 250 runs of the experiment. We also insist that $\langle\psi|\chi\rangle = \overline{\langle\chi|\psi\rangle}$.[6] We would thus expect $|\langle\hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle|^2$ to be 1 and $|\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle|^2$ to be 0. It will follow that $\langle\hat{\mathbf{a}}-|\hat{\mathbf{a}}+\rangle$ has to be 0, and that

[5] With regards to the set of complex numbers $\mathbb{C}$, we will use the notation $i = \sqrt{-1}$. Complex conjugation will be denoted by an overline so that $\overline{x+iy} = x - iy$ for real numbers x and y. The modulus of a complex number $z = x + iy$ will then be given by $|z| = \sqrt{z\overline{z}} = \sqrt{x^2+y^2}$. Since the defining property of $\langle\psi|\chi\rangle$ is that $|\langle\psi|\chi\rangle|^2$ is the probability that the particle will be found to be in state $|\psi\rangle$ given that we know that the particle is in state $|\chi\rangle$, we have to choose an arbitrary phase to fully determine $\langle\psi|\chi\rangle$.

[6] Note that this conditions implies time symmetry: the probability a particle transitions from a state $|\chi\rangle$ to a state $|\psi\rangle$ will be the same as the probability a particle transitions from the state $|\psi\rangle$ to the state $|\chi\rangle$. This is in accord with the observation that closed quantum systems are symmetric on time-reversal. This might at first seem surprising in the light of the fact that phenomena such as radioactive decay are not obviously time-symmetric. However, it turns out that this time asymmetry results from the quantum system not being closed. For more details, see cbx@1 Saverio Pascazio, "All you ever wanted to know about the quantum Zeno effect in 70 minutes," *44th Symposium on Mathematical Physics on New Developments in the Theory of Open Quantum Systems*, 2013, https://doi.org/10.1142/S1230161214400071, eprint: arXiv:1311.6645v1[quant-ph].

$\langle \hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle$ has modulus 1, but by convention we choose $\langle \psi|\chi\rangle$ such that $\langle \psi|\psi\rangle$ is a real and positive number, in which case we must have $\langle \hat{\mathbf{a}}+|\hat{\mathbf{a}}+\rangle = 1$. If we now rotate the magnetic field by an angle θ as indicated in figure 1.4, the particle will be detected either at location $\hat{\mathbf{b}}+$ or location $\hat{\mathbf{b}}-$. We can then ask the question "given that the particle is in state $|\hat{\mathbf{a}}+\rangle$, what is the probability that the particle will be found to be in state $|\hat{\mathbf{b}}+\rangle$?" According to the notation discussed above, this probability will be $|\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle|^2$ where $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle$ is a complex number such that $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = 1$ when $\theta = 0$ and $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = 0$ when $\theta = 180°$. We would likewise expect $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = 0$ when $\theta = 0$ and $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = 1$ when $\theta = 180°$. Since $\cos 0° = \sin 90° = 1$ and $\cos 90° = \sin 0° = 0$, we might guess that in general $|\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle| = |\cos(\theta/2)|$ and $|\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle| = |\sin(\theta/2)|$. Experimentation shows us that this guess is correct. This suggests that we can express the state $|\hat{\mathbf{b}}+\rangle$ in terms of the states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$. We thus suppose that vectoradd

{vectoradd1}
$$|\hat{\mathbf{b}}+\rangle = \alpha\,|\hat{\mathbf{a}}+\rangle + \beta\,|\hat{\mathbf{a}}-\rangle \tag{1.1a}$$

{vectoradd2}
$$|\hat{\mathbf{b}}-\rangle = \bar{\alpha}\,|\hat{\mathbf{a}}-\rangle - \bar{\beta}\,|\hat{\mathbf{a}}+\rangle \tag{1.1b}$$

for complex numbers $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, and we suppose that the bra-ket has the **linearity** property so that $\langle \psi|\hat{\mathbf{b}}+\rangle = \alpha\,\langle \psi|\hat{\mathbf{a}}+\rangle + \beta\,\langle \psi|\hat{\mathbf{a}}-\rangle$ and $\langle \psi|\hat{\mathbf{b}}-\rangle = \bar{\alpha}\,\langle \psi|\hat{\mathbf{a}}-\rangle - \bar{\beta}\,\langle \psi|\hat{\mathbf{a}}+\rangle$ for any state $|\psi\rangle$. Then it will follow that $\langle \hat{\mathbf{b}}+|\hat{\mathbf{b}}-\rangle = 0$,[7] and that $\langle \hat{\mathbf{b}}+|\hat{\mathbf{b}}+\rangle = \langle \hat{\mathbf{b}}-|\hat{\mathbf{b}}-\rangle = 1$.[8] If we then put $\alpha = \cos(\theta/2)$ and $\beta = \sin(\theta/2)$, it

---

[7] To see this, by putting $|\psi\rangle = |\hat{\mathbf{b}}+\rangle$, we will have $\langle \hat{\mathbf{b}}+|\hat{\mathbf{b}}-\rangle = \bar{\alpha}\,\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle - \bar{\beta}\,\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle$ by equation (1.1b). Since $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = \overline{\langle \hat{\mathbf{a}}-|\hat{\mathbf{b}}+\rangle}$ we have $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = \bar{\beta}$ by equation (1.1a), and likewise, since $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = \overline{\langle \hat{\mathbf{a}}+|\hat{\mathbf{b}}+\rangle}$, we have $\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle = \bar{\alpha}$. Therefore $\langle \hat{\mathbf{b}}+|\hat{\mathbf{b}}-\rangle = \overline{\alpha}\overline{\beta} - \overline{\beta}\overline{\alpha} = 0$.

[8] To see this, by putting $|\psi\rangle = |\hat{\mathbf{b}}+\rangle$ and using equation (1.1a), we will have $\langle \hat{\mathbf{b}}+|\hat{\mathbf{b}}+\rangle = \alpha\,\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle + \beta\,\langle \hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle = \alpha\overline{\alpha} + \beta\overline{\beta} = |\alpha|^2 + |\beta|^2 = 1$. By a similar calculation, we also see $\langle \hat{\mathbf{b}}+|\hat{\mathbf{b}}+\rangle = 1$.

will follow that $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle| = |\cos(\theta/2)|$ and $|\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle| = |\sin(\theta/2)|$,[9] and so with these values for α and β we will have spintrans

{spintrans1}

$$|\hat{\mathbf{b}}+\rangle = \cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\,, \tag{1.2a}$$

{spintrans2}

$$|\hat{\mathbf{b}}-\rangle = \cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\,. \tag{1.2b}$$

[9] To satisfy these criteria, α and β are only determined up to rotation by a complex number. Rotating a complex number $z \in \mathbb{C}$ just means multiplying it by a complex number λ of modulus 1 (i.e. $|\lambda| = 1$) to get λz. We would need to take into account this rotation factor if we considered the three-dimensional situation. Then, without loss of generality, $\alpha = \cos(\theta/2)$ and $\beta = e^{i\phi}\sin(\theta/2)$ where θ and ϕ are the polar and azimuthal angles respectively.

1.2 Hidden Variables

Now it is tempting to suppose that the expression of $|\hat{\mathbf{b}}+\rangle$ in terms of $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ merely represents our knowledge of the true spin state of a particle along the $\hat{\mathbf{a}}$ axis given our knowledge that it would be detected at location $\hat{\mathbf{b}}+$ with probability 1 should we decide to measure the particle's state along the $\hat{\mathbf{b}}$-axis. If we were to make this supposition, there would be a fact of the matter, albeit unknown to us, concerning what spin state the particle would be found to be in were we to measure its spin along the $\hat{\mathbf{a}}$-axis. And even though we might decide not to measure the spin of the particle along the $\hat{\mathbf{a}}$-axis, there would still be this hidden fact about the particle's spin in this direction. And given this supposition, since there would be no reason to suppose there was anything special about the $\hat{\mathbf{a}}$-axis, it would then be reasonable to suppose that there were hidden facts about what spin direction the particle would be found to be in for every possible axis orientation. This would mean that a complete description of the particle's spin state would require an infinite list of outcomes for all the possible orientations we could configure the magnetic field of our Stern-Gerlach apparatus. Given this assumption, as well as the assumption that it is already known that the particle would be detected at $\hat{\mathbf{b}}+$, a complete description of the particle's state could be depicted as $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \ldots\rangle$ or $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \ldots\rangle$, etc. where the ellipses would range over one of the two possible measurement outcomes for every other magnetic field orientation. However, because we would never in practice be able to perform all these experiments, and since only one such experiment would be needed to alter this infinite list,[10] nearly all of the entries in this infinite list would remain forever hidden. Hence, this would be an example of a **hidden variables** interpretation of quantum theory.

[10]In other words, it is assumed that directly measuring the particle will involve perturbing it so that its state will change.

BellSection

1.3 Bell's Inequality

Now although a hidden variables interpretation seems rather intuitive, a problem arises when two spin particles are coupled together with each other. This problem is known as the **EPR paradox.**[11] To explain how this problem arises, we need to consider two identical fermionic particles that are coupled together. Fermions have the property that no two particles that are coupled together can be in exactly the same spin state. Thus, if we call our fermionic particles q_A and q_B, and suppose particle q_A was in the state $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-, \ldots\rangle$, then on the assumption that making a measurement on particle q_A has no effect on the state of particle q_B, it would follow that particle q_B would be in the state $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+, \ldots\rangle$. It is thus appropriate to refer to the spin directions describing q_A and q_B as **local hidden variables**. It seems very natural to assume these hidden variables are local. This assumption is a special case of **Einstein's locality principle**: For two spatially separated systems S_1 and S_2, the real factual situation of the system S_2 should be independent of what is done to the system S_1.[12]

We now suppose we have an experimental setup so that in each run of the experiment, we have two fermionic particles q_A and q_B, and particle q_A is sent to Alice who measures q_A's spin in a direction of her choosing, and particle q_B is sent to Bob who measures q_B's spin in a direction of his choosing. We assume that in each run of the

cbx@2

[11] For more details of this problem, see Jim J. Napolitano and J. J. Sakurai, *Modern Quantum Mechanics* (Pearson Education, 2013), 241–249.

[12] Einstein expressed this locality principle in his autobiographical notes: "But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former."

cbx@3

Albert Einstein, *Albert Einstein, Philosopher Scientist*, ed. P. A. Schilp (Evanston, Illinois: Library of Living Philosophers, 1949), p. 85.

experiment, Alice and Bob independently measure the spin of their particles along one of three possible directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and that Einstein's locality principle holds. Furthermore, we assume that in each run of the experiment, the outcome of Alice's measurement will be statistically independent of any of the other measurement outcomes for different runs of the experiment, and for any of the three axes she measures along, she will get a spin up outcome or a spin down outcome with equal probability of $\frac{1}{2}$. Likewise, we assume Bob's measurement outcomes are also similarly independent between different runs of the experiment. We also assume that the $8 = 2^3$ states $|\hat{\mathbf{a}}\pm, \hat{\mathbf{b}}\pm, \hat{\mathbf{c}}\pm\rangle_A$ exhaust all the possible states for Alice's particles that can be distinguished from one another by one of the three possible measurements she can make. Thus, Alice can distinguish between the $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$-state and the $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_A$-state by making a measurement along the $\hat{\mathbf{c}}$-axis, though if she happened to make her measurement along the $\hat{\mathbf{a}}$ or $\hat{\mathbf{b}}$-axis, she wouldn't be able to distinguish between these two states. But in principle, she can distinguish between these two states if she happens to make her measurement along the right axis, in this case the $\hat{\mathbf{c}}$-axis. We similarly assume the states $|\hat{\mathbf{a}}\pm, \hat{\mathbf{b}}\pm, \hat{\mathbf{c}}\pm\rangle_B$ exhaust all the possible states for Bob's particles that he can distinguish between, and we assume that if Alice and Bob measure the particle along the same axis, they will always obtain opposite results from one another. For instance, if Alice's particle is in state $|\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$, then Bob's particle must be in state $|\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_B$. Now suppose the experiment is run N times for large N,[13] and let N_i be the number of times particle q_A is in the ith state so that[14] $N = \sum_{i=1}^{8} N_i$ as shown in table 1.1.

---

[13] N has to be large since a frequentist definition of probability is being assumed.

[14] The notation $\sum_{i=1}^{8} N_i$ is shorthand for $N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8$.

Table 1.1: Spin-components of particles q_A and q_B in the hidden-variable theory

Population	**Particle q_A**	**Particle q_B**
N_1	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_B$
N_2	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_A$	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_B$
N_3	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_A$	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_B$
N_4	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_A$	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_B$
N_5	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_A$	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_B$
N_6	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_A$	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_B$
N_7	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+\rangle_A$	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-\rangle_B$
N_8	$\lvert\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-\rangle_A$	$\lvert\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+\rangle_B$

hiddentable

We define $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+)$ to be the probability that Alice measures particle q_A to be at location $\hat{\mathbf{a}}+$ on her detection screen and Bob measures particle q_B to be at location $\hat{\mathbf{b}}+$ on his detection screen. We similarly define the probabilities for all other combinations of detection locations. It is relatively easy to calculate all these probabilities in terms of the values N_i from table 1.1,[15] or alternatively by simply measuring the frequency of these different outcomes for where Alice and Bob detect their particles. Although the values of N_i are unknown, on the assumption that there is a fact of the matter of which states in table 1.1 obtain, and on the assumption that the states to which the N_i correspond exhaust all the possible states for Alice's and Bob's particles, we can show that[16]

{bellinequality}

$$P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) \leq P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+). \tag{1.3}$$

This inequality is known as **Bell's inequality**, and it follows from Einstein's locality principle. However, it turns out that when this experiment is actually performed, Bell's inequality is violated. Because of this violation, the most natural conclusion

[15]e.g. $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{N_3+N_4}{N}$, $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) = \frac{N_2+N_4}{N}$, $P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+) = \frac{N_3+N_7}{N}$

[16]This inequality follows since $P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{N_3+N_4}{N} \leq \frac{N_2+N_4+N_3+N_7}{N} = P_{AB}(\hat{\mathbf{a}}+; \hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+; \hat{\mathbf{b}}+)$.

to draw is that it is wrong to suppose that there any hidden variables describing possible spin measurement outcomes. But it also turns out that this violation of Bell's inequality is entirely predictable if we assume that when the two particles q_A and q_B are coupled together, everything that can be said about their spins is encoded in the so-called **Bell state**:

{bell}

$$\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B). \tag{1.4}$$

This state means that if both Alice and Bob measure their particles along the measurement axis $\hat{\mathbf{a}}$, then with probability $\frac{1}{2}$, Alice will detect her particle at location $\hat{\mathbf{a}}+$ and Bob will detect his particle at location $\hat{\mathbf{a}}-$. Likewise, with probability $\frac{1}{2}$, Alice will detect her particle at location $\hat{\mathbf{a}}-$ and Bob will detect his particle at location $\hat{\mathbf{a}}+$. Prima facie, it looks like the Bell state depends on the direction of $\hat{\mathbf{a}}$. However, it can be shown that for any other direction $\hat{\mathbf{b}}$,

{bellstate2}

$$\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\,|\hat{\mathbf{a}}+\rangle_B) = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{b}}+\rangle_A\,|\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A\,|\hat{\mathbf{b}}+\rangle_B).\text{[17]} \tag{1.5}$$

Thus, without loss of generality we can write the joint state of both Alice and Bob's particles as in equation (1.4), from which it would follow that

$$P_{AB}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = \frac{1}{2}\sin^2(\theta/2)$$

---

[17] bellstate2pf To see this, using the transformation rules given in equation (1.2) we have

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{b}}+\rangle\,|\hat{\mathbf{b}}-\rangle - |\hat{\mathbf{b}}-\rangle\,|\hat{\mathbf{b}}+\rangle)\\
&\quad= \frac{1}{\sqrt{2}}((\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle)(\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle)\\
&\qquad - (\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle)(\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle))\\
&\quad= \frac{1}{\sqrt{2}}(\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\\
&\qquad + \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle - \sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\\
&\qquad - \cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle - \cos(\theta/2)\,|\hat{\mathbf{a}}-\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle\\
&\qquad + \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\cos(\theta/2)\,|\hat{\mathbf{a}}+\rangle + \sin(\theta/2)\,|\hat{\mathbf{a}}+\rangle\sin(\theta/2)\,|\hat{\mathbf{a}}-\rangle)
\end{aligned}$$

where θ is the angle between the $\hat{\mathbf{a}}$-axis and $\hat{\mathbf{b}}$-axis.[18] Then taking the angle between the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$-axes to be 90°, and the $\hat{\mathbf{c}}$-axis to be at 45° to both the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$-axes, we would find that $P_{AB}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = \frac{1}{4}$ and $P_{AB}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+) + P_{AB}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+) = 0.1464\ldots$, and so Bell's inequality would be violated if we assumed that the probability of each outcome is determined by the Bell state (1.4).

1.4 The Copenhagen Interpretation

The experiment described in the previous section implies that the behavior of Alice's and Bob's particles can't be explained in terms of local hidden variables. But this experiment also calls into question the Copenhagen interpretation of quantum physics. To explain the Copenhagen interpretation and what is problematic about it, suppose Alice has a measurement device which we will denote by $\Lambda_{\hat{\mathbf{a}}+}$ Lambdaa and which outputs the number 1 when her particle is detected at location $\hat{\mathbf{a}}+$, and outputs 0 when her particle is detected at location $\hat{\mathbf{a}}-$. Given that Alice knows that the state of both particles together is given by equation (1.4), she can work out the expectation value of her measurement $\langle\Lambda_{\hat{\mathbf{a}}+}\rangle$ by summing up the product of each probability measurement

$$= \frac{1}{\sqrt{2}}((\cos^2(\theta/2) + \sin^2(\theta/2))\,|\hat{\mathbf{a}}+\rangle\,|\hat{\mathbf{a}}-\rangle - (\cos^2(\theta/2) + \sin^2(\theta/2))\,|\hat{\mathbf{a}}-\rangle\,|\hat{\mathbf{a}}+\rangle)$$
$$= \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle\,|\hat{\mathbf{a}}-\rangle - |\hat{\mathbf{a}}-\rangle\,|\hat{\mathbf{a}}+\rangle).$$

[18] To see why this is, let $P_A(\hat{\mathbf{a}}+)$ be the probability that Alice would detect her particle at location $\hat{\mathbf{a}}+$ given that she is making a measurement along the $\hat{\mathbf{a}}$-axis, and let $P_{BA}(\hat{\mathbf{b}}+|\hat{\mathbf{a}}+)$ be the probability that Bob will detect his particle at location $\hat{\mathbf{b}}+$ given that he is making a measurement along the $\hat{\mathbf{b}}$-axis and Alice has detected her particle at location $\hat{\mathbf{a}}+$. Given that the joint state of the particles is given by equation (1.4), $P_A(\hat{\mathbf{a}}+) = \frac{1}{2}$. But also note that if Alice has detected her particle at location $\hat{\mathbf{a}}+$, then Bob's particle must be in state $|\hat{\mathbf{a}}-\rangle$. From the Born Rule (see page 9) and equation (1.2a) it follows that

$$P_{BA}(\hat{\mathbf{b}}+|\hat{\mathbf{a}}+) = |\,\langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}-\rangle\,|^2 = \sin^2(\theta/2).$$

Therefore,

$$P_{AB}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = P_A(\hat{\mathbf{a}}+)P_{BA}(\hat{\mathbf{b}}+|\hat{\mathbf{a}}+) = \frac{1}{2}\sin^2(\theta/2).$$

outcome with the value of each measurement. This will give an expectation value of $\langle \Lambda_{\hat{\mathbf{a}}+} \rangle = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$. More generally, if Alice had a measuring device Λ with N measurement outcome values $o_1, \ldots, o_N$ and with respective probabilities $p_1, \ldots, p_N$ so that $\sum_{i=1}^{N} p_i = 1$, then the expectation $\langle \Lambda \rangle$ would be given by the formula

$$\langle \Lambda \rangle = \sum_{i=1}^{N} p_i o_i. \tag{1.6}$$

{expectation}

Now given that there are no hidden variables and that equation (1.4) encodes everything that can be said about the spins of the two particle system, it is tempting to suppose that the expectation value $\langle \Lambda_{\hat{\mathbf{a}}+} \rangle$ tells us something objective about the system rather than just something about Alice's state of knowledge about the system. Given this assumption, there then arises the question of what happens when a measurement is made. According to the Copenhagen interpretation, when Bob makes his measurement, the quantum state collapses to a component of the quantum state corresponding to the measurement Bob makes. Thus, if Bob's measurement device $\Lambda_{\hat{\mathbf{a}}-}$ outputs the number 1, (i.e. Bob's particle is detected at location $\hat{\mathbf{a}}-$), then the state of the combined system would change accordingly as:

$$\frac{1}{\sqrt{2}}\left(|\hat{\mathbf{a}}+\rangle_A \; |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A \; |\hat{\mathbf{a}}+\rangle_B \right) \xrightarrow{\text{Collapse!!}} |\hat{\mathbf{a}}+\rangle_A \; |\hat{\mathbf{a}}-\rangle_B$$

If Bob makes his measurement first with his measurement device $\Lambda_{\hat{\mathbf{a}}-}$ outputting 1, then with probability 1 Alice's measurement device $\Lambda_{\hat{\mathbf{a}}+}$ will output 1. Hence, once Bob has made this measurement, then the expectation value for Alice's measurement will be $\langle \Lambda_{\hat{\mathbf{a}}+} \rangle = 1$. Thus, the expectation value for Alice's measurement device changes from $\frac{1}{2}$ to 1 when Bob makes his measurement.

Copenhagenproblem

Now the problem with this change in expectation value for Alice's measurement is that it will depend on whether Bob performs his measurement first or whether Alice performs her measurement first. But according to Einstein's theory of relativity, who performs their measurement first will depend on which inertial frame of reference one is in.[19] Thus, if we are moving at one velocity, it may appear that Alice makes her measurement first, whereas if we are moving at another velocity, it may appear that Bob makes his measurement first. This suggests the expectation values for Alice and Bob's measuring devices will depend on which frame of reference we are in. However, Einstein's theory of relativity tells us that scalar quantities such as the expectation values for Alice and Bob's measuring devices should be independent of which inertial frame we are in.

Now many physicists would be loath to reject Einstein's theory of relativity. At the same time, many physicists are also convinced by the violation of Bell's inequality that there are no hidden variables for the spin states of particles, and hence they are convinced that the Bell state is not just a description of someone's epistemic state: rather it is a complete physical description of two coupled fermionic particles with regard to their spins. The way many physicists seek to resolve this tension between Einstein's theory of relativity and the violation of Bell's inequality is to deny the Copenhagen interpretation of quantum physics so that there is no quantum state collapse. But if one denies that there is any quantum state collapse and denies that

[19] In special relativity, an inertial frame of reference is a spacetime coordinate system (t, x, y, z) in which all objects which have no forces acting on them have trajectories that are straight lines. Thus, we can move to another inertial frame by moving to a reference frame with constant velocity $\mathbf{v}$ with respect to the first reference frame. In the case when $\mathbf{v} = (v, 0, 0)$, Einstein's theory of special relativity tells us that under such a "boost", spacetime coordinates will transform as $(t, \mathbf{x}) \to (t', x', y', z') = (\gamma\big(t - \frac{vx}{c^2}\big), \gamma(x - vt), y, z)$ where c is the speed of light and $\gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1}$.

there are any hidden variables, the question then arises of how is one meant to interpret the quantum state?

1.5 A preliminary consideration of the many-worlds interpretation

At this stage in the line of reasoning, it is too early to resort to a many-worlds interpretation of the Bell state where the first component corresponds to a world in which Alice detects her particle at location $\hat{\mathbf{a}}+$ and Bob detects his particle at location $\hat{\mathbf{a}}-$, and where the second component corresponds to a world in which Alice detects her particle at location $\hat{\mathbf{a}}-$ and Bob detects his particle at location $\hat{\mathbf{a}}+$. The reason such an interpretation would be premature is because as mentioned on page 17, for any other axis $\hat{\mathbf{b}}$, the transformation rules in equation (1.2) imply that

$$\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle_A\;|\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A\;|\hat{\mathbf{a}}+\rangle_B) = \frac{1}{\sqrt{2}}(\,|\hat{\mathbf{b}}+\rangle_A\;|\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A\;|\hat{\mathbf{b}}+\rangle_B).$$

Similarly, given the transformation rules in equation (1.2), we should resist the temptation to interpret a state of the form $\frac{1}{\sqrt{2}}(\,|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ as representing two worlds, one in which the particle is in the state $|\hat{\mathbf{a}}+\rangle$, and another in which the particle is in the state $|\hat{\mathbf{a}}-\rangle$. For according to equation (1.2a), the much more obvious interpretation is that this state just describes one world in which the particle is in the state $|\hat{\mathbf{b}}+\rangle$ where the angle between the $\hat{\mathbf{a}}$ and the $\hat{\mathbf{b}}$ axis is 90°.[20]

In order to make a case for a many-worlds interpretation, we need to discuss decoherence theory. Decoherence theory considers how a system interacts with its environment, and it allows us to understand what kinds of measurements can be made on the system. In order to discuss decoherence theory and its relevance to the many-worlds interpretation, we first need to introduce the mathematical formalism of quantum mechanics.

1.6 The mathematical formalism of quantum mechanics

Given a possible kind of measurement (e.g. measuring the spin of a particle along a particular axis), there will be a mathematical object called an **observable** which encodes all the possible measurement outcomes for this particular kind of measurement. The precise mathematical definition of an observable is as follows: an observable of a physical system is a Hermitian operator that acts on the Hilbert space of states describing the physical system. In order to understand what this definition means, there are a number of things we need to explain: what a Hilbert space is, what a Hermitian operator is, and how a Hermitian operator relates to a particular kind of measurement. In order to explain all this, it will be helpful to keep in mind the simple example of the measurement device $O_{\hat{\mathbf{a}}+}$ described on page 18 which returns 1 if a particle is in the spin $|\hat{\mathbf{a}}+\rangle$-state and 0 if the particle is in the spin $|\hat{\mathbf{a}}-\rangle$-state. This measurement will have a corresponding observable which we will denote by $\hat{O}_{\hat{\mathbf{a}}+}$.

Now we have seen that states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ representing the spin of a particle can be added to give the spin state $|\hat{\mathbf{b}}+\rangle$ and $|\hat{\mathbf{b}}-\rangle$ as seen in equation (1.2). We have also seen that if we have two states $|\psi\rangle$ and $|\chi\rangle$, we can define their bra-ket $\langle\chi|\psi\rangle$ to be a complex number satisfying the Born Rule so that $|\langle\chi|\psi\rangle|^2$ is the probability $P(\chi|\psi)$ that the particle will be found to be in state $|\chi\rangle$ given that we know that the particle is in state $|\psi\rangle$. We thus imposed the assumption that $\langle\psi|\psi\rangle = 1$ for any state $|\psi\rangle$.

---

[20] This is because when $\theta = 90°$, $\sin(\theta/2) = \cos(\theta/2) = \frac{1}{\sqrt{2}}$, so $|\hat{\mathbf{b}}+\rangle = \frac{1}{\sqrt{2}}(|\hat{\mathbf{a}}+\rangle + |\hat{\mathbf{a}}-\rangle)$ in equation (1.2a) with $\theta = 90°$.

Now in order to arrive at a definition of a Hilbert space, we first need to relax this normalization condition $\langle\psi|\psi\rangle = 1$. Thus, if $|\psi\rangle$ is a state and $\lambda \in \mathbb{C}$ is any non-zero complex number, then we allow $|\psi'\rangle = \lambda\,|\psi\rangle$ also to be a state with the caveat that $|\psi'\rangle$ represents exactly the same physical state as $|\psi\rangle$, and that $\langle\psi'|\psi'\rangle = |\lambda|^2\,\langle\psi|\psi\rangle$. We define $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ and we say that $|\psi\rangle$ has been **normalized** when $\|\psi\| = 1$. Now when calculating probabilities, we need to remember to include a normalization factor. Thus, the probability that the particle will be found to be in state $|\chi\rangle$ given that we know that the particle is in state $|\psi\rangle$ will now be $P(\chi|\psi) = \frac{|\langle\chi|\psi\rangle|^2}{\|\psi\|\|\chi\|}$.[21] It is dropping the assumption $\|\psi\| = 1$ on the states of a physical system that gives rise to the mathematical structure known as a Hilbert Space.

A **Hilbert space** is a set H in which

1. any two members of H can be added to obtain another member of H,
2. any member of H can be multiplied by any complex number to obtain another member of H,
3. one can take the bra-ket of any two members of H to obtain a complex number

subject to some natural axioms.[22]

A very simple example of a Hilbert space would be the set of states

$$\{\alpha\,|\hat{\mathbf{a}}+\rangle + \beta\,|\hat{\mathbf{a}}-\rangle : \alpha, \beta \in \mathbb{C}\}.$$

---

[21] If there is no such normalization factor because $\|\psi\| = 0$, then $|\psi\rangle$ does not represent a physical state, so the probability the system is ever in this state will be zero, and so in this case we will set $P(\chi|\psi) = 0$.

As we will soon see, the observable corresponding to the measurement device $O_{\hat{\mathbf{a}}+}$ will be the operator $\hat{O}_{\hat{\mathbf{a}}+}$ that sends the state $\alpha\,|\hat{\mathbf{a}}+\rangle + \beta\,|\hat{\mathbf{a}}-\rangle$ to the state $\alpha\,|\hat{\mathbf{a}}+\rangle$.

More generally, suppose we have an experimental setup (for example the Stern-Gerlach experiment) where a physical system can be in one of several measurable states $|\psi_1\rangle, \ldots, |\psi_N\rangle \in H$. The physical system could also be in a state described by a sum of some of the $|\psi_1\rangle, \ldots, |\psi_N\rangle$, but by saying the system is in one of these $|\psi_1\rangle, \ldots, |\psi_N\rangle$ measurable states, we mean that there is a measuring device that will always give the same measurement outcome whenever the system is in the same measurable state.[23] We also assume **orthonormality**, that is we assume $\langle\psi_i|\psi_i\rangle = 1$ and $\langle\psi_i|\psi_j\rangle = 0$ for $i \neq j$ so that if the system is measured to be in the $|\psi_j\rangle$-state,

---

[22]More formally, a complex Hilbert space H is a complex vector space possessing a bra-ket. By a **complex vector space**, we mean a set V such that the following axioms are satisfied

- $\psi + (\chi + \zeta) = (\psi + \chi) + \zeta$, $\forall \lambda, \chi, \zeta \in V$
- $\psi + \chi = \chi + \psi$, $\forall \psi, \chi \in V$
- there exists an element $\mathbf{0} \in V$ such that $\psi + \mathbf{0} = \psi$, $\forall \psi \in V$.
- $\forall \psi \in V$ there exists an element $-\psi \in V$ such that $\psi + (-\psi) = \mathbf{0}$.
- $\forall \lambda, \mu \in \mathbb{C}$ (i.e. in the set of complex numbers – this is why it is called a *complex* vector space), and $\psi \in V$, $\lambda(\mu\psi) = (\lambda\mu)\psi$.
- for the scalar $1 \in \mathbb{C}$, $1\psi = \psi$, $\forall \psi \in V$
- $\lambda(\psi + \chi) = \lambda\psi + \lambda\chi, \forall \psi, \chi \in V$ and $\lambda \in \mathbb{C}$
- $(\lambda + \mu)\psi = \lambda\psi + \mu\psi$, $\forall \lambda, \mu \in \mathbb{C}$ and $\psi \in V$.

A **Hilbert space** H is a complex vector space possessing a bra-ket. Strictly speaking, a Hilbert space also has a property called completeness, but this property need not concern us here. In quantum theory, elements of H are expressed in terms of kets, $|\cdot\rangle$. Kets behave like vectors, so for $|\psi\rangle, |\chi\rangle \in H$ and $\lambda, \mu \in \mathbb{C}$, we have $\lambda\,|\psi\rangle + \mu\,|\chi\rangle = |\lambda\psi + \mu\chi\rangle$. The bra-ket of $|\psi\rangle$ and $|\chi\rangle$ is then written as $\langle\psi|\chi\rangle$, and it satisfies the following axioms:

- $\langle\psi|\chi\rangle \in \mathbb{C}$, $\forall \psi, \chi \in H$.
- $\langle\psi|\chi\rangle = \overline{\langle\chi|\psi\rangle}$, $\forall \psi, \chi \in H$.
- $\langle\psi|\psi\rangle \geq 0$, $\forall\,|\psi\rangle \in H$ and $\langle\psi|\psi\rangle = 0$ if and only if $|\psi\rangle = \mathbf{0}$.
- $\langle\zeta|\lambda\psi + \mu\chi\rangle = \lambda\,\langle\zeta|\psi\rangle + \mu\,\langle\zeta|\chi\rangle$, $\forall\,|\psi\rangle, |\chi\rangle, |\zeta\rangle \in H$ and $\lambda, \mu \in \mathbb{C}$.

[23]When the state is described as a non-trivial sum of the measurable states, we no longer have such certainty, and instead we can only speak of probabilities based on the coefficients on the measurable states.

then there would be zero probability that it could then be measured to be in the $|\psi_i\rangle$-state for $i \neq j$.

Now suppose that for each measurable state $|\psi_i\rangle$, we assign a real number o_i. There might be a very natural way of doing this, such as assigning o_i to be the angle by which a pointer of a measurement device is deflected when the system is in the state $|\psi_i\rangle$, but the assignment could be as ad hoc as we wished – we can just think of it as the measurement value an experimenter records when he or she observes a particular measurement outcome. Given such an assignment of measurement values, the corresponding observable $\hat{O}$ would be a mapping of states to states satisfying the following rules:

1. $\hat{O}\,|\psi_i\rangle = o_i\,|\psi_i\rangle$
2. $\hat{O}(\lambda\,|\psi\rangle + \mu\,|\chi\rangle) = \lambda\hat{O}\,|\psi\rangle + \mu\hat{O}\,|\chi\rangle$ for all states $|\psi\rangle\,, |\chi\rangle \in H$ and complex numbers $\lambda, \mu \in \mathbb{C}$.

When a mapping $\hat{O}$ satisfies rule 2., we refer to $\hat{O}$ as an **operator** on H. Since the measurement device $O_{\hat{\mathbf{a}}+}$ outputs a value of 1 when the particle is in the $|\hat{\mathbf{a}}+\rangle$-state and 0 when the particle is in the $|\hat{\mathbf{a}}-\rangle$-state, it is now clear from rule 1 and 2 why the corresponding observable $\hat{O}_{\hat{\mathbf{a}}+}$ will be the operator that sends the state $\alpha\,|\hat{\mathbf{a}}+\rangle + \beta\,|\hat{\mathbf{a}}-\rangle$ to $\alpha\,|\hat{\mathbf{a}}+\rangle$.

For a given physical state $|\psi\rangle \in H$, we define the expected measurement value of $\hat{O}$

{expectation2}

$$\langle\hat{O}\rangle_\psi \overset{\text{def}}{=} \sum_{i=1}^{N} p_i o_i, \tag{1.7}$$

just as in the formula (1.6). Clearly, this expectation value will depend on the state the system is in, hence the subscript ψ. For instance, if the system is in the $|\psi_i\rangle$-state, then the expectation value of the measurement will be o_i since the probability that the state is in the state $|\psi_i\rangle$ will be 1, so that $p_i = 1$ and $p_j = 0$ for all $j \neq i$. But if the system was in an arbitrary state $|\psi\rangle = \sum_{i=1}^{N} \alpha_i |\psi_i\rangle$ with $\|\psi\| = 1$, then it turns out that

{evev}

$$\langle \hat{O} \rangle_\psi = \langle \psi | \hat{O} | \psi \rangle .\tag{1.8}$$ [24]

We can see that this formula is correct in the simple example when the system is in the $|\psi_i\rangle$-state, for in this case, the expectation value should be o_i, and we clearly have $\langle \psi_i | \hat{O} | \psi_i \rangle = o_i$ since $\hat{O} |\psi_i\rangle = o_i |\psi_i\rangle$ and $\langle \psi_i | \psi_i \rangle = 1$. Hence, $\langle \hat{O} \rangle_{\psi_i} = \langle \psi_i | \hat{O} | \psi_i \rangle$ as expected.

To say that $\hat{O}$ is **Hermitian** is to say that $\langle \psi | \hat{O} | \psi \rangle$ is a real number for any arbitrary state $|\psi\rangle$. Thus, the observable $\hat{O}$ defined by the two criteria above is a Hermitian operator acting on the Hilbert space of states H. Roughly speaking, we can assume[25] that given a Hermitian operator $\hat{O}$ on a Hilbert space of states H, any state $|\psi\rangle \in H$ can be expressed as a (possibly infinite) sum

{span}

$$|\psi\rangle = \sum_{i=1}^{N} \alpha_i |\psi_i\rangle \tag{1.9}$$

[24] footnote_1 To see this, note that $\alpha_i = \langle \psi_i | \psi \rangle$ from which it follows that if we define the mapping $I = \sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i|$ then $I|\psi\rangle = \sum_{i=1}^{N} |\psi_i\rangle \langle \psi_i | \psi \rangle = \sum_{i=1}^{N} \langle \psi_i | \psi \rangle |\psi_i\rangle = |\psi\rangle$. Therefore,

$$\begin{aligned} \langle \psi | \hat{O} | \psi \rangle &= \langle \psi | \hat{O} I | \psi \rangle = \sum_{i=1}^{N} \langle \psi | \hat{O} | \psi_i \rangle \langle \psi_i | \psi \rangle = \sum_{i=1}^{N} o_i \langle \psi | \psi_i \rangle \langle \psi_i | \psi \rangle \\ &= \sum_{i=1}^{N} o_i |\langle \psi_i | \psi \rangle|^2 = \sum_{i=1}^{N} o_i p_i = \langle \hat{O} \rangle_\psi . \end{aligned}$$

[25] Strictly speaking, we require a Hermitian operator to have a property known as compactness for this assumption to hold.

where $\hat{O}\,|\psi_i\rangle = o_i\,|\psi_i\rangle$ for some set of states $|\psi_i\rangle$ referred to as **eigenstates** of $\hat{O}$, and real numbers o_i referred to as **eigenvalues** of $\hat{O}$. We will typically assume that the $|\psi_i\rangle$ are orthonormal. Orthonormality of the $|\psi_i\rangle$ will entail that the coefficients α_i will be uniquely determined by the formula $\alpha_i = \langle\psi_i|\psi\rangle$. Thus, this set of eigenstates $\{|\psi_1\rangle, \ldots |\psi_N\rangle\}$ satisfies the criterion for being a **basis** of the Hilbert space of states H, namely, every state $|\psi\rangle \in H$ can be uniquely expressed by a summation of the form given in equation (1.9).[26] We refer to an expression of the form (1.9) as a **linear combination** of the basis $\{|\psi_1\rangle, \ldots |\psi_N\rangle\}$.

[26]Note that although for a given basis, equation (1.9) will be unique, there will be many different bases, and the α_i coefficients will depend on which basis is chosen.

1.7 The Preferred Basis Problem[27]

Now just because we can have an observable $\hat{O}$, there is no guarantee that there is a measuring device that could determine whether the system was in one of the eigenstates of $\hat{O}$. For instance, if $|\text{Cat Alive}\rangle$ is the physical state in which a cat is alive, and $|\text{Cat Dead}\rangle$ is the physical state in which the same cat is dead, then although there are measuring devices that can distinguish between the $|\text{Cat Alive}\rangle$-state and the $|\text{Cat Dead}\rangle$-state,[28] there are no known measuring devices that can distinguish between the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$-state and the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$-state. On the other hand, there are measuring devices that can distinguish between the $\frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle + |\mathbf{\hat{a}}-\rangle)$-state and the $\frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle - |\mathbf{\hat{a}}-\rangle)$-state in a Stern-Gerlach experiment. Why the difference?

This question is at the heart of the preferred basis problem. As mentioned already, a basis is just a set of states via which all other states of the system can be uniquely expressed. For instance, we can express the state $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ uniquely as a sum of elements from the basis $\{|\text{Cat Alive}\rangle, |\text{Cat Dead}\rangle\}$, and thus we think of $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ as being a superposition of the $|\text{Cat Alive}\rangle$ and $|\text{Cat Dead}\rangle$ basis states. However, we can also uniquely express $|\text{Cat Alive}\rangle$ in terms of the basis $\{\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle), \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\}$.[29] Nevertheless, we would not tend to think of $|\text{Cat Alive}\rangle$ as being in a superposition of the $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$ basis states. That is, we have a preference for the basis $\{|\text{Cat Alive}\rangle, |\text{Cat Dead}\rangle\}$ over the basis

cbx@4

[27] See Maximilian Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Berlin: Springer-Verlag, 2007), 53–55.

[28] For example, we assume that human beings can be thought of as such measuring devices.

[29] i.e. $|\text{Cat Alive}\rangle = \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)\Big) + \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\Big)$.

$\{\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle), \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\}$. As will be shown in section 1.9, decoherence theory offers a very elegant solution to the preferred basis problem.

1.8 Decoherence theory*[30] decotheory

Before we can show how decoherence theory solves the preferred basis problem, we will first need to look at decoherence theory in general. To understand what's going on in decoherence theory, there are a number of things we need to discuss, namely

1. composite systems
2. entanglement
3. density matrices and traces
4. coherence
5. partial traces and reduced density matrices
6. the von Neumann measurement scheme
7. decoherence

1.8.1 Composite Systems

First we need to consider **composite systems**. We thus assume there is a distinction between what is being measured and the rest of physical reality. We denote the system that is being measured by $\mathcal{S}$ and the rest of physical reality by $\mathcal{E}$. We will refer to $\mathcal{E}$ as the environment, and we will denote the composite system of $\mathcal{S}$ and $\mathcal{E}$ by $\mathcal{U}$. We will often indicate that $\mathcal{U}$ is a composite of systems $\mathcal{S}$ and $\mathcal{E}$ by writing $\mathcal{U} = \mathcal{S} + \mathcal{E}$. The system $\mathcal{S}$ could be something microscopic like a silver atom, or something much bigger such as a cat or even a planet.

---

*As mentioned in the introduction on page 4, sections marked with an asterisk may be challenging to readers who do not have a mathematics or physics background.

[30]For more details see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, ch. 2.

Now suppose we have an observable (i.e. any Hermitian operator) $\hat{O}_{\mathcal{S}}$ that acts on the Hilbert space $H_{\mathcal{S}}$ of states of $\mathcal{S}$. As already mentioned, this means that we can find orthonormal eigenstates $|\psi_i\rangle_{\mathcal{S}}$ of $\hat{O}_{\mathcal{S}}$ and corresponding eigenvalues o_i such that any state $|\psi\rangle_{\mathcal{S}} \in H_{\mathcal{S}}$ can be uniquely expressed as a sum $|\psi\rangle_{\mathcal{S}} = \sum_{i=1}^{M} \alpha_i \, |\psi_i\rangle_{\mathcal{S}}$. We will often include the subscript $\mathcal{S}$ on the ket-vectors in order to make it clear that these ket-vectors belong to the Hilbert space $H_{\mathcal{S}}$. At other times we will omit these subscripts when it is clear what system we are talking about, but for the time being, we will keep these subscripts in place.

Now let us suppose we have a basis of normalized (but not necessarily orthonormal) states $\{|\chi_i\rangle_{\mathcal{E}} : i\}$ for the state space $H_{\mathcal{E}}$ of $\mathcal{E}$. In other words, every state $|\chi\rangle_{\mathcal{E}} \in H_{\mathcal{E}}$ can be uniquely expressed as a linear combination $|\chi\rangle_{\mathcal{E}} = \sum_{i=1}^{N} \beta_i \, |\chi_i\rangle_{\mathcal{E}}$. It is then assumed we will be able to express every state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ of the composite system $\mathcal{U}$ as a linear combination

$$|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_{i,j} \, |\psi_i\rangle_{\mathcal{S}} \, |\chi_j\rangle_{\mathcal{E}} \, . \qquad \texttt{\{entangled\}} \tag{1.10}$$

Thus, we assume there no are emergent physical properties describing the composite system $\mathcal{U}$ that couldn't be expressed in terms of the component subsystems $\mathcal{S}$ and $\mathcal{E}$. The Hilbert space $H_{\mathcal{U}}$ is endowed with the bra-ket $\langle \xi' | \xi \rangle_{\mathcal{U}}$ such that if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} \, |\chi\rangle_{\mathcal{E}}$ and $|\xi'\rangle_{\mathcal{U}} = |\psi'\rangle_{\mathcal{S}} \, |\chi'\rangle_{\mathcal{E}}$, then $\langle \xi' | \xi \rangle_{\mathcal{U}} = \langle \psi' | \psi \rangle_{\mathcal{S}} \, \langle \chi' | \chi \rangle_{\mathcal{E}}$ where we have again used subscripts to indicate which Hilbert space the bra-ket corresponds to.

1.8.2 Entanglement

By defining the bra-ket on the Hilbert space $H_{\mathcal{U}}$ as we have done, we are making the assumption that if we define $P(\psi', \chi' | \psi, \chi)$ to be the probability the composite system would be measured to be in the $|\psi'\rangle_{\mathcal{S}} |\chi'\rangle_{\mathcal{E}}$-state given that the composite system was known to be in the $|\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$-state, then $P(\psi', \chi'|\psi, \chi) = P(\psi'|\psi)P(\chi'|\chi)$. A consequence of this assumption is that if the composite system is in the $|\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$-state, the probability of finding system $\mathcal{S}$ to be in any particular state in $H_{\mathcal{S}}$ will be independent[31] of the state in $H_{\mathcal{E}}$ describing $\mathcal{E}$. For this reason, when the state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ describing the composite system $\mathcal{U}$ is expressible as a product state $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$, we say that $\mathcal{S}$ and $\mathcal{E}$ are **not entangled** with one another. On the other hand, when the state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$ of the composite system $\mathcal{U}$ cannot be expressed as a product state, we say that $\mathcal{S}$ and $\mathcal{E}$ are **entangled**. For example, if $|\xi\rangle_{\mathcal{U}} = \frac{1}{\sqrt{2}}(|\psi_1\rangle_{\mathcal{S}} |\chi_1\rangle_{\mathcal{E}} + |\psi_2\rangle_{\mathcal{S}} |\chi_2\rangle_{\mathcal{E}})$ with $|\psi_1\rangle_{\mathcal{S}} \not\propto |\psi_2\rangle_{\mathcal{S}}$ and $|\chi_1\rangle_{\mathcal{E}} \not\propto |\chi_2\rangle_{\mathcal{E}}$,[32] then $\mathcal{S}$ and $\mathcal{E}$ would be entangled with one another.

Now given the observable $\hat{O}_{\mathcal{S}}$ acting on $H_{\mathcal{S}}$, we can naturally define the observable $\hat{O}_{\mathcal{U}}$ acting on $H_{\mathcal{U}}$ so that

{extension}

$$\hat{O}_{\mathcal{U}} |\xi\rangle_{\mathcal{U}} = \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_{i,j} \hat{O}_{\mathcal{S}} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}} = \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_{i,j} o_i |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}} . \tag{1.11}$$

Just as in equation (1.8), for a given normalized state $|\xi\rangle_{\mathcal{U}} \in H_{\mathcal{U}}$, the expectation value of the observable $\hat{O}_{\mathcal{U}}$ will be $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi} = \langle \xi | \hat{O}_{\mathcal{U}} | \xi \rangle_{\mathcal{U}}$. It is easy to see that if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ (i.e. $\mathcal{S}$ and $\mathcal{E}$ are not entangled), then $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi} = \langle \hat{O}_{\mathcal{S}} \rangle_{\psi}$.[33] Thus, when $\mathcal{S}$ and $\mathcal{E}$ are not entangled with one another, it is possible to say things about $\mathcal{S}$

[31] Here we are using the standard probabilistic definition of independence: two events X and Y are independent if and only if $P(X \text{ and } Y \text{ occur}) = P(X \text{ occurs})P(Y \text{ occurs})$

[32] Here the notation $|\psi_1\rangle_{\mathcal{S}} \propto |\psi_2\rangle_{\mathcal{S}}$ means there exists some α such that $|\psi_1\rangle_{\mathcal{S}} = \alpha |\psi_2\rangle_{\mathcal{S}}$, in which case $|\xi\rangle_{\mathcal{U}} = \frac{1}{\sqrt{2}} |\psi_2\rangle_{\mathcal{S}} (\alpha |\chi_1\rangle_{\mathcal{E}} + |\chi_2\rangle_{\mathcal{E}})$. Thus, if $|\psi_1\rangle_{\mathcal{S}} \propto |\psi_2\rangle_{\mathcal{S}}$, then $\mathcal{S}$ and $\mathcal{E}$ would not be entangled. This is why in the above example, we assume $|\psi_1\rangle_{\mathcal{S}} \not\propto |\psi_2\rangle_{\mathcal{S}}$, that is, we assume there is no such α such that $|\psi_1\rangle_{\mathcal{S}} = \alpha |\psi_2\rangle_{\mathcal{S}}$, and for the same reason we assume $|\chi_1\rangle_{\mathcal{S}} \not\propto |\chi_2\rangle_{\mathcal{E}}$.

untangledobservable

[33] This is because by definition, if $|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$ and $|\xi'\rangle_{\mathcal{U}} = |\psi'\rangle_{\mathcal{S}} |\chi'\rangle_{\mathcal{E}}$, then $\langle \xi' | \xi \rangle_{\mathcal{U}} = \langle \psi' | \psi \rangle_{\mathcal{S}} \langle \chi' | \chi \rangle_{\mathcal{E}}$. We will also have $\hat{O}_{\mathcal{U}} |\xi\rangle = \hat{O}_{\mathcal{S}} |\psi\rangle_{\mathcal{S}} |\chi\rangle_{\mathcal{E}}$. Thus, assuming both $|\psi\rangle_{\mathcal{S}}$ and $|\chi\rangle_{\mathcal{E}}$ are normalized, we have $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi} = \langle \xi | \hat{O}_{\mathcal{U}} | \xi \rangle = \langle \psi | \hat{O}_{\mathcal{S}} | \psi \rangle_{\mathcal{S}} \langle \chi | \chi \rangle_{\mathcal{E}} = \langle \hat{O}_{\mathcal{S}} \rangle_{\psi}$.

independently of the current state of the environment $\mathcal{E}$. In this case we need have no knowledge of the information about $\mathcal{E}$ encapsulated in the state $|\chi\rangle_{\mathcal{E}}$ to determine the expectation value $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi}$.

However, for a general entangled state $|\xi\rangle_{\mathcal{U}} = \sum_{i=1}^{M} \sum_{j=1}^{N} \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}$, $\langle \hat{O}_{\mathcal{U}} \rangle_{\xi}$ will typically depend on the $|\chi_j\rangle_{\mathcal{E}}$-states and the coefficients $\gamma_{i,j}$. Nevertheless, despite there being a huge amount of information contained within these $|\chi_j\rangle_{\mathcal{E}}$-states and the $\gamma_{i,j}$, if we are only interested in making measurements on the system $\mathcal{S}$, nearly all this information can be discarded. In order to see how this is done, we need to generalize the notion of a state to that of a density matrix.

1.8.3 Density Matrices and Traces

Given a normalized state $|\psi\rangle$ in any Hilbert space H, its density matrix will be the operator $\hat{\rho} \stackrel{\text{def}}{=} |\psi\rangle\langle\psi|$ which acts on H by sending an arbitrary state $|\psi'\rangle$ to $\langle\psi|\psi'\rangle\, |\psi\rangle$. Note that $\hat{\rho}$ is a Hermitian operator.[34] Also note that if we had a measuring device that returned the output 1 if a system was in the state $|\psi\rangle$ and 0 if the system was in a state $|\chi\rangle$ with $\langle\psi|\chi\rangle = 0$, the density matrix $\hat{\rho}$ would be the observable corresponding to this measurement. The expectation value of this measurement for an initial normalized state $|\psi'\rangle$ would then be $\langle\psi'|\hat{\rho}|\psi'\rangle = |\langle\psi|\psi'\rangle|^2 = P(\psi|\psi')$. In particular, if the system was initially in the state $|\psi\rangle$, the expectation value of this measurement would be 1.

[34]This is because for any arbitrary state $|\psi'\rangle$, $\langle\psi'|\hat{\rho}|\psi'\rangle = \langle\psi'|\psi\rangle\, \langle\psi|\psi'\rangle = \overline{\langle\psi|\psi'\rangle}\, \langle\psi|\psi'\rangle = |\,\langle\psi|\psi'\rangle|^2$, and so $\langle\psi'|\hat{\rho}|\psi'\rangle$ is real, and from this it follows that $\hat{\rho}$ is Hermitian.

Now it turns out that if we have an arbitrary orthonormal basis $\{|\phi_i\rangle : i\}$ of H and any observable $\hat{O}$ on H, then

$$\langle \hat{O} \rangle_\psi = \sum_i \langle \phi_i | \hat{\rho} \hat{O} | \phi_i \rangle .^{35} \tag{1.12}$$

{exptrace}

Since this expression can be shown to be independent of which basis we choose,[36] we have a well-defined function called the **trace**, written as $\mathrm{Tr}(\cdot)$, which maps any operator $\hat{A}$ acting on H to a value in $\mathbb{C}$ according to the formula

$$\mathrm{Tr}(\hat{A}) = \sum_i \langle \phi_i | \hat{A} | \phi_i \rangle . \tag{1.13}$$

{tracedef}

Thus, it follows from equations (1.12) and (1.13) that

$$\langle \hat{O} \rangle_\psi = \mathrm{Tr}(\hat{\rho}\hat{O}). \tag{1.14}$$

{traceev}

In general, a **density matrix** $\hat{\rho}$ will be a Hermitian operator with positive eigenvalues such that $\mathrm{Tr}(\hat{\rho}) = 1$. We will write $M(H)$ for the set of all density matrices on H. Since we are assuming[37] that for any Hermitian operator, there is an orthonormal basis of the Hilbert space consisting of eigenstates of the Hermitian operator, we can

---

[35]To see this, as we saw in footnote 24, if we define the mapping $I = \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|$ then $I|\psi\rangle = |\psi\rangle$. Therefore, $\langle \hat{O} \rangle_\psi = \langle \psi | \hat{O} | \psi \rangle = \langle \psi | \hat{O} I | \psi \rangle = \sum_i \langle \psi | \hat{O} | \phi_i \rangle \langle \phi_i | \psi \rangle = \sum_i \langle \phi_i | \psi \rangle \langle \psi | \hat{O} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{\rho} \hat{O} | \phi_i \rangle$.

[36]To see this, we first note that for any orthonormal basis $\{|\phi_i\rangle : i\}$ of H, and any two operators $\hat{A}$ and $\hat{B}$ acting on H, using the fact that $I = \sum_i |\phi_i\rangle\langle\phi_i|$ is the identity operator on H, we have the commutativity property

$$\sum_i \langle \phi_i | \hat{A}\hat{B} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{A} \sum_j |\phi_j\rangle\langle\phi_j| \, \hat{B} | \phi_i \rangle = \sum_{ij} \langle \phi_j | \hat{B} | \phi_i \rangle \, \langle \phi_i | \hat{A} | \phi_j \rangle = \sum_j \langle \phi_j | \hat{B}\hat{A} | \phi_j \rangle .$$

Now suppose that $\{|\phi'_i\rangle : i\}$ is another orthonormal basis of H. Then we can define the operator $\hat{U}$ such that $\hat{U}|\phi_i\rangle = |\phi'_i\rangle$. We can also define the operator $\hat{U}^*$ such that $\langle \phi'_i | \psi \rangle = \langle \phi_i | \hat{U}^* | \psi \rangle$ for any state $|\psi\rangle \in H$. Since $\langle \phi'_i | \phi'_j \rangle = \langle \phi_i | \hat{U}^* | \phi'_j \rangle$ for all i, j, it will follow that $\hat{U}^* |\phi'_j\rangle = |\phi_j\rangle$. Therefore, $\hat{U}\hat{U}^* = I$. Using this fact together with the commutativity property, we have

$$\sum_i \langle \phi'_i | \hat{O} | \phi'_i \rangle = \sum_i \langle \phi_i | \hat{U}^* \hat{O} \hat{U} | \phi_i \rangle = \sum_i \langle \phi_i | \hat{O} \hat{U} \hat{U}^* | \phi_i \rangle = \sum_i \langle \phi_i | \hat{O} | \phi_i \rangle .$$

[37]As mentioned earlier, we are making the assumption that Hermitian operators are compact.

find an orthonormal basis $\{|\psi_i\rangle : i\}$ of eigenstates of $\hat{\rho}$ with corresponding eigenvalues p_i such that

{rhodiag}

$$\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \,. \tag{1.15}$$

The condition $\mathrm{Tr}(\hat{\rho}) = 1$ will then imply that $\sum_i p_i = 1$. Now we could think of the operator $\hat{\rho}$ as corresponding to a measurement which gave the output p_i when the system was in the state $|\psi_i\rangle$. However, we can alternatively think of $\hat{\rho}$ as describing a system which is known to be in one of the $|\psi_i\rangle$-states, but that we only know it is in the $|\psi_i\rangle$-state with probability p_i. Then given that $\hat{\rho}$ describes all we know about the system, the expectation value $\langle\hat{O}\rangle_\rho$ for an observable $\hat{O}$ on the system can be shown to be

{expdensity}

$$\langle\hat{O}\rangle_\rho = \mathrm{Tr}(\hat{\rho}\hat{O}).^{38} \tag{1.16}$$

We can think of a density matrix $\hat{\rho} \in M(H)$ as a generalization genket of a state ket-vector $|\psi\rangle \in H$, since for every $|\psi\rangle \in H$ there corresponds a density matrix $\hat{\rho} = |\psi\rangle\langle\psi| \in M(H)$. Because of this identification, $\hat{\rho} = |\psi\rangle\langle\psi|$ is referred to as a **pure state**. On the other hand, the converse does not hold: if $\hat{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \in M(H)$ with more than one of the $p_i > 0$, then there will not be a corresponding $|\psi\rangle \in H$ such that $\hat{\rho} = |\psi\rangle\langle\psi|$. In this case, when $\hat{\rho}$ is interpreted as describing a system that is definitely in one of the $|\psi_i\rangle$-states with probability p_i, then we will refer to $\hat{\rho}$ as a **mixed state**. mixedstate

[38] This follows since $\mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_i p_i \, \mathrm{Tr}(|\psi_i\rangle\langle\psi_i| \, \hat{O}) = \sum_i p_i \, \langle\hat{O}\rangle_{\psi_i}$ which will be the expectation value of $\hat{O}$ given that $\hat{\rho}$ encapsulates our knowledge of the system.

1.8.4 Coherence

Now suppose that the system $\mathcal{S}$ is initially in a superposition state $|\psi\rangle = \sum_i c_i |s_i\rangle$ with $\sum_i |c_i|^2 = 1$. Then the corresponding density matrix on $\mathcal{S}$ will be

$$|\psi\rangle\langle\psi| = \sum_{ij} c_i \overline{c_j} \, |s_i\rangle\langle s_j| \, .$$

When a density matrix has non-zero $|s_i\rangle\langle s_j|$-components for $i \neq j$, we say that there is **coherence** between the $|s_i\rangle$ and $|s_j\rangle$-states.[39] Thus, for the density matrix $|\psi\rangle\langle\psi|$ there will be coherence between the $|s_i\rangle$ and $|s_j\rangle$-states so long as both c_i and c_j are non-zero. Decoherence is a process (to be described shortly) by which the $|s_i\rangle\langle s_j|$-components of a density matrix restricted to a subsystem of a composite system appear to vanish.

1.8.5 Partial Traces and Reduced Density Matrices

As already mentioned, if we have a general entangled state on a composite system $\mathcal{U} = \mathcal{S}+\mathcal{E}$ of the form $|\xi\rangle_{\mathcal{U}} = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle_{\mathcal{S}} |\chi_j\rangle_{\mathcal{E}}$, there is a huge amount of information in all the $\gamma_{i,j}$. However, most of this information can be discarded if we are only interested in making measurements on the system $\mathcal{S}$. We can't typically encapsulate this information in the form of a state $|\psi\rangle \in H_{\mathcal{S}}$, but we can encapsulate this information in the form of a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which as mentioned on page 36 can be thought of as a generalization of a state ket-vector $|\psi\rangle \in H_{\mathcal{S}}$. In this

[39] The fact that a density matrix can be written out in terms of $|s_i\rangle\langle s_j|$-components explains why we refer to a density matrix as a density *matrix*. For example, if our state space has a basis of just two states $\{|s_1\rangle, |s_2\rangle\}$, and if $\hat{\rho} = a\,|s_1\rangle\langle s_1| + b\,|s_1\rangle\langle s_2| + c\,|s_2\rangle\langle s_1| + d\,|s_2\rangle\langle s_2|$, then we can identify $\hat{\rho}$ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we then identify the state $|\psi\rangle = x\,|s_1\rangle + y\,|s_2\rangle$ with the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, then the state $\hat{\rho}\,|\psi\rangle$ would be identified with the column vector under matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. The trace of a density matrix is then just the sum of the diagonal elements (top left to bottom right) of the matrix. Decoherence with respect to a particular basis occurs when the off-diagonal elements of the density matrix vanish.

subsection, we will show how the density matrix $\hat{\rho} = |\xi\rangle\langle\xi| \in M(H_{\mathcal{U}})$ can be reduced to a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which encapsulates all the information needed to calculate expectation values of observables on $\mathcal{S}$. The reduced density matrix $\hat{\rho}_{\mathcal{S}}$ is derived from $\hat{\rho}$ via an operation call the partial trace.

In the context of a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$, when taking traces, we will need to be more specific over which basis we are taking the trace over. If $\{|\psi_i\rangle : i\}$ is an orthonormal basis of $H_{\mathcal{S}}$ and $\{|\chi_j\rangle : j\}$ is an orthonormal basis of $H_{\mathcal{E}}$, then $\{|\xi_{ij}\rangle \stackrel{\text{def}}{=} |\psi_i\rangle\,|\chi_j\rangle : i, j\}$ will be an orthonormal basis of $H_{\mathcal{U}}$. For an operator $\hat{A}_{\mathcal{S}}$ of $H_{\mathcal{S}}$, we define

$$\mathrm{Tr}_{\mathcal{S}}(\hat{A}_{\mathcal{S}}) = \sum_i \langle\psi_i|\hat{A}_{\mathcal{S}}|\psi_i\rangle\,,$$

and for an operator $\hat{A}_{\mathcal{U}}$ of $H_{\mathcal{U}}$, we define

$$\mathrm{Tr}_{\mathcal{U}}(\hat{A}_{\mathcal{U}}) = \sum_{ij} \langle\xi_{ij}|\hat{A}_{\mathcal{U}}|\xi_{ij}\rangle\,.$$

This is just what we would expect the traces to be for operators on $H_{\mathcal{S}}$ and on $H_{\mathcal{U}}$ respectively. But we also need the notion of a **partial trace** for an operator $\hat{A}_{\mathcal{U}}$ on $H_{\mathcal{U}}$:

{partialtrace}

$$\mathrm{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}}) = \sum_j \langle\chi_j|\hat{A}_{\mathcal{U}}|\chi_j\rangle\,. \tag{1.17}$$

Note that whereas $\mathrm{Tr}_{\mathcal{U}}(\hat{A}_{\mathcal{U}})$ is just a number, the partial trace $Tr_{\mathcal{E}}(\hat{A}_{\mathcal{U}})$ is an operator that acts on $H_{\mathcal{S}}$. To see why this is, consider the simple example of when $\hat{A}_{\mathcal{U}} = |\xi_{ij}\rangle\langle\xi_{lk}|$. The operator $\hat{A}_{\mathcal{U}}$ would send the state $|\xi_{lk}\rangle$ to $|\xi_{ij}\rangle$ and all the other $|\xi_{l'k'}\rangle$-states of $H_{\mathcal{U}}$ to 0. But in order to define the partial trace as given in equation (1.17), we need to know what $\hat{A}_{\mathcal{U}}\,|\chi\rangle$ is and then what $\langle\chi|\hat{A}_{\mathcal{U}}|\chi\rangle$ is. In the case when $\hat{A}_{\mathcal{U}} = |\xi_{ij}\rangle\langle\xi_{lk}|$, we stipulate that $\hat{A}_{\mathcal{U}}\,|\chi\rangle$ is the operator that sends the state $|\psi\rangle \in H_{\mathcal{S}}$ to the state

$\langle\psi_l|\psi\rangle\,\langle\chi_k|\chi\rangle\,|\xi_{ij}\rangle \in H_{\mathcal{U}}$. Furthermore, if we stipulate that $\langle\chi|\xi_{ij}\rangle = \langle\chi|\chi_j\rangle\,|\psi_i\rangle \in H_{\mathcal{S}}$, it follows that $\langle\chi|\hat{A}_{\mathcal{U}}|\chi\rangle$ will be the operator $\langle\chi|\chi_j\rangle\,\langle\chi_k|\chi\rangle\,|\psi_i\rangle\langle\psi_l|$ that sends the state $|\psi\rangle \in H_{\mathcal{S}}$ to the state $\langle\chi|\chi_j\rangle\,\langle\chi_k|\chi\rangle\,\langle\psi_l|\psi\rangle\,|\psi_i\rangle \in H_{\mathcal{S}}$. We therefore find that

{partialtrace2}

$$\mathrm{Tr}_{\mathcal{E}}(|\xi_{ij}\rangle\langle\xi_{lk}|) = \begin{cases} |\psi_i\rangle\langle\psi_l| & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{1.18}$$

And since any arbitrary operator $\hat{A}_{\mathcal{U}}$ on $H_{\mathcal{U}}$ can be expressed as a sum

$$\hat{A}_{\mathcal{U}} = \sum_{ijkl} \mu_{ijkl}\,|\xi_{ij}\rangle\langle\xi_{lk}|\,,$$

we can use equation (1.18) to find that

$$\mathrm{Tr}_{\mathcal{E}}(\hat{A}_{\mathcal{U}}) = \sum_{ijl} \mu_{ijjl}\,|\psi_i\rangle\langle\psi_l|\,.$$

Now it turns out that given a density matrix $\hat{\rho}$ on $H_{\mathcal{U}}$ and an observable $\hat{O}_{\mathcal{S}}$ of $H_{\mathcal{S}}$ (which induces an observable $\hat{O}_{\mathcal{U}}$ on $H_{\mathcal{U}}$ in the obvious way, e.g. $\hat{O}_{\mathcal{U}}\,|\xi_{ij}\rangle = (\hat{O}_{\mathcal{S}}\,|\psi_i\rangle)\,|\chi_j\rangle)$, we have the important formula

{reducedev}

$$\langle\hat{O}_{\mathcal{U}}\rangle_\rho = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}) \tag{1.19}$$

where $\hat{\rho}_{\mathcal{S}} = \mathrm{Tr}_{\mathcal{E}}(\hat{\rho})$.[40] We refer to $\hat{\rho}_{\mathcal{S}}$ as the **reduced density matrix** of $\hat{\rho}$.

Note that if $\hat{\rho} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = |\psi\rangle\,|\chi\rangle$ so that $\mathcal{S}$ and $\mathcal{E}$ are not entangled, then $\hat{\rho}_{\mathcal{S}} = |\psi\rangle\langle\psi|$.[41] This is what we should expect, since if $\mathcal{S}$ and $\mathcal{E}$ are not entangled, then

[40] To see this, following Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 46, we have

$$\begin{aligned}\langle\hat{O}_{\mathcal{U}}\rangle_\rho = \mathrm{Tr}_{\mathcal{U}}(\hat{\rho}\hat{O}_{\mathcal{U}}) &= \sum_{ij}\langle\xi_{ij}|\hat{\rho}\hat{O}_{\mathcal{U}}|\xi_{ij}\rangle = \sum_i \langle\psi_i|\Big(\sum_j \langle\chi_j|\hat{\rho}|\chi_j\rangle\Big)\hat{O}_{\mathcal{S}}\,|\psi_i\rangle \\ &= \sum_i \langle\psi_i|\,\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}\,|\psi_i\rangle = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}).\end{aligned}$$

untanglepartialtrace

[41] To see this, we recall that the partial trace $\mathrm{Tr}_{\mathcal{E}}$ is independent of which orthonormal basis $\{|\chi_j\rangle : j\}$ we choose for $\mathcal{E}$. Therefore, if $\hat{\rho} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = |\psi\rangle\,|\chi\rangle$, we can choose $|\chi_1\rangle = |\chi\rangle$ and all other $|\chi_i\rangle$ such that $\langle\chi_i|\chi\rangle = 0$. Then $\mathrm{Tr}_{\mathcal{E}}(\hat{\rho}) = \sum_j \langle\chi_j|\hat{\rho}|\chi_j\rangle = |\psi\rangle\langle\psi|$.

the expectation values of observables defined on $\mathcal{S}$ should be independent of the state of $\mathcal{E}$, and by equation (1.19), this independence is seen to hold when $\hat{\rho}_{\mathcal{S}}$ is independent of any states on $\mathcal{E}$.

More generally, subtle for an entangled state $|\xi\rangle = \sum_{i,j} \gamma_{i,j} |\psi_i\rangle |\chi_j\rangle$, from equations (1.16) and (1.19), we have $\langle \hat{O}_{\mathcal{U}} \rangle_{\rho} = \langle \hat{O}_{\mathcal{S}} \rangle_{\rho_{\mathcal{S}}}$. This means that when it comes to taking expectation values of measurements on a subsystem $\mathcal{S}$ that is part of a composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ which is in the state $|\xi\rangle \in H_{\mathcal{U}}$, the subsystem $\mathcal{S}$ behaves as though it was described by the density matrix $\hat{\rho}_{\mathcal{S}}$. However, there is a rather subtle point one needs to be aware of here.[42] For in general, as we saw in equation (1.15), any density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ can be expressed as a sum $\hat{\rho}_{\mathcal{S}} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, and this can be *thought of* as corresponding to the system $\mathcal{S}$ being in one of the $|\psi_i\rangle$-states, but that we only know it is in the $|\psi_i\rangle$-state with probability p_i. If this was the correct interpretation of $\hat{\rho}_{\mathcal{S}}$, then as explained on page 36, we would refer to $\hat{\rho}_{\mathcal{S}}$ as a *mixed state*. But just because we can think of $\hat{\rho}_{\mathcal{S}}$ in this way, it doesn't follow that $\mathcal{S}$ really is in one of these $|\psi_i\rangle$-states and that we are only ignorant of which state it is. When $\mathcal{S}$ is entangled with $\mathcal{E}$ there is no fact of the matter regarding which state $\mathcal{S}$ is in. Rather, there are only facts of the matter for the composite system $\mathcal{U}$, e.g. the fact of the matter is that $\mathcal{U}$ is in the state $|\xi\rangle$ rather than some other state of $H_{\mathcal{U}}$. Therefore, $\mathcal{U}$ is really in a pure state with density matrix $|\xi\rangle\langle\xi|$. Because we cannot give an ignorance interpretation to $\hat{\rho}_{\mathcal{S}}$ d'Espagnat Espagnat referred to density matrices of this sort as

[42]It is unfortunate that many physicists fail to pick up on this subtlety with the result that they form the erroneous belief that decoherence can by itself solve the measurement problem (of outcomes) when in fact it can't. For a further discussion of the problem of outcomes, see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 57–60.

being **improper mixtures**.[43] But despite this subtle distinction between mixed states and improper mixtures, we have nevertheless succeeded in showing how a density matrix $\hat{\rho} = |\xi\rangle\langle\xi| \in M(H_{\mathcal{U}})$ can be reduced to a density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ which encapsulates all the information needed to calculate expectation values of observables on $\mathcal{S}$. subtleend

vonNeumannMeasurement

1.8.6 The von Neumann Measurement Scheme

We are now in a position to consider the **von Neumann measurement scheme**.[44] Instead of considering the whole of physical reality, for the time being, we just consider a physical system $\mathcal{S}$ and a measuring device $\mathcal{A}$. This division reflects the fact that a scientist doesn't measure the system $\mathcal{S}$ directly, but rather observes a measuring device $\mathcal{A}$ that is affected by $\mathcal{S}$. The measuring device $\mathcal{A}$ has the characteristic that it has a normalized ready state $|a_r(t_0)\rangle$ at initial time t_0 and that there is an orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$, and normalized states $|a_i(t)\rangle$ of $\mathcal{A}$ such that

vonNeumannMeasurement1

1. for any $t \geq t_0$ we have the evolution of the states $|s_i\rangle |a_r(t_0)\rangle \xrightarrow{\text{time evolution}} |s_i\rangle |a_i(t)\rangle$ so that $\mathcal{S}$ and $\mathcal{A}$ do not become entangled when $\mathcal{S}$ is initially in state $|s_i\rangle$ and $\mathcal{A}$ is initially in state $|a_i(t_0)\rangle$. vonNeumannMeasurement2
2. there exists $\delta > 0$ such that if $t > t_0 + \delta$, then $\langle a_i(t)|a_j(t)\rangle \approx 0$ for $i \neq j$.[45]

---

[43] See cbx@8 Bernard d' Espagnat, *Conceptual foundations of quantum mechanics*, 2nd ed., completely rev., enl., reset., Mathematical physics monograph series ; 20 (Reading, Mass.; London: W. A. Benjamin, 1976), ch. 6.2 – cited in cbx@9 Jeremy Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 2017, p. 19, eprint: arXiv:1710.07844.

[44] See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 50–53 for more details.

[45] approx More precisely, we should say that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $t > t_0 + \delta$, then $|\langle a_i(t)|a_j(t)\rangle| < \epsilon$ for $i \neq j$.

These two criteria characterize the von Neumann measurement scheme. The orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$ for which these two criteria hold are called **pointer states**. pointer These pointer states will be determined by the dynamics of the composite system $\mathcal{S} + \mathcal{A}$ as well as the relative configuration of $\mathcal{S}$ with respect to $\mathcal{A}$. For instance, if $\mathcal{S}$ is a silver atom and $\mathcal{A}$ is a Stern-Gerlach apparatus, then the configuration and dynamics of the system will determine a fixed axis $\hat{\mathbf{a}}$ relative to the Stern-Gerlach configuration $\mathcal{A}$ such that the states $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ of $\mathcal{S}$ don't get entangled with $\mathcal{A}$, that is, there exists $\delta > 0$ such that $|\hat{\mathbf{a}}\pm\rangle\,|a_r(t_0)\rangle \xrightarrow{\text{time evolution}} |\hat{\mathbf{a}}\pm\rangle\,|a_\pm(t)\rangle$ with $\langle a_+(t)|a_-(t)\rangle \approx 0$ for $t > t_0 + \delta$.[46, 47] Since no entanglement occurs with the silver atom and the Stern-Gerlach apparatus when the silver atom is in the $|\hat{\mathbf{a}}\pm\rangle$-state, then in this situation, we can interact with the apparatus to find out whether the particle is in the $|\hat{\mathbf{a}}+\rangle$-state or the $|\hat{\mathbf{a}}-\rangle$-state without changing the spin state of the silver atom. Indeed, we should expect an experimental apparatus to have this property of non-entanglement with the measurement outcomes it reports, for otherwise, every scientist who looked at the measurement device couldn't be sure that the spin state of the silver atom being measured remained unchanged whenever the apparatus was observed, and so the scientists couldn't expect there to be any agreement among themselves regarding which spin-state the silver atom was in. Thus, the basis of $H_{\mathcal{S}}$ for which entanglement doesn't occur is a preferred basis. However, if we were to consider

[46] Strictly speaking, we would need more information to describe states in $H_{\mathcal{S}}$ besides the spin, so we should really express this scenario in terms of $\{|s_{i,+}\rangle \overset{\text{def}}{=} |\hat{\mathbf{a}}+, i\rangle : i\} \cup \{|s_{i,-}\rangle \overset{\text{def}}{=} |\hat{\mathbf{a}}-, i\rangle : i\}$ and $\{|a_{i,+}(t)\rangle : i\} \cup \{|a_{i,-}(t)\rangle : i\}$ where the i-indices encode all the additional information beyond spin.

[47] nearzero Although we only require that $\langle a_+(t)|a_-(t)\rangle \approx 0$ for $t > t_0 + \delta$ rather than demanding $\langle a_+(t)|a_-(t)\rangle = 0$, we can think of the scientist who observes the apparatus as determining whether the apparatus is either in one of two normalized state $|a'_+(t)\rangle$ or $|a'_-(t)\rangle$ where $\langle a'_+(t)|a'_-(t)\rangle = 0$ and $\langle a'_\pm(t)|a'_\pm(t)\rangle \approx 1$, so that the scientist can confidently assert that the particle is in the state $|\hat{\mathbf{a}}+\rangle$ if for instance the measurement device is found to be in the state $|a'_+(t)\rangle$. Because $\langle a'_+(t)|a_-(t)\rangle$ is only very small, but not identically zero, in theory, the particle could be in the $|\hat{\mathbf{a}}-\rangle$-state, but we're assuming that such a possibility would be as likely as a violation of the Second Law of Thermodynamics, say.

a different basis, say $\{\frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle + |\mathbf{\hat{a}}-\rangle), \frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle - |\mathbf{\hat{a}}-\rangle)\}$,[48] then assuming that the configuration of $\mathcal{A}$ remained unchanged, entanglement between $\mathcal{S}$ and $\mathcal{A}$ would occur since then $\frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle \pm |\mathbf{\hat{a}}-\rangle)\,|a_r(t_0)\rangle \xrightarrow{\text{time evolution}} \frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle\,|a_+(t)\rangle \pm |\mathbf{\hat{a}}-\rangle\,|a_-(t)\rangle)$. Thus, $\{\frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle + |\mathbf{\hat{a}}-\rangle), \frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle - |\mathbf{\hat{a}}-\rangle)\}$ would not be a preferred basis. In this case, if $\mathcal{S}$ was in the $\frac{1}{\sqrt{2}}(|\mathbf{\hat{a}}+\rangle + |\mathbf{\hat{a}}-\rangle)$-state, a scientist would measure $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state with probability $\frac{1}{2}$. But having measured $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state, the scientist would continue to observe $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state because of the subsequent non-entanglement of $\mathcal{S}$ with $\mathcal{A}$ when $\mathcal{S}$ is in the $|\mathbf{\hat{a}}+\rangle$-state and $\mathcal{A}$ is in the $|a_+(t)\rangle$-state. Note that this situation is somewhat analogous to when we have the Bell-state (1.4), so that when Bob measures his particle to be in the $|\mathbf{\hat{a}}-\rangle$-state, he knows that Alice's particle is in the $|\mathbf{\hat{a}}+\rangle$-state. Likewise, in the von Neumann measurement scheme, if the scientist measures $\mathcal{A}$ to be in the $|a_+(t)\rangle$-state for $t > t_0+\delta$, he will then (almost certainly) know[49] that the system $\mathcal{S}$ will be in the $|\mathbf{\hat{a}}+\rangle$-state.

In the case where $\mathcal{S}$ has more than two states, we can write a generic normalized state of the composite system $\mathcal{U} = \mathcal{S} + \mathcal{A}$ as $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle$ where $|\xi_i(t)\rangle = |s_i\rangle\,|a_i(t)\rangle$. There will then be coherence between $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ for the density matrix $\hat{\rho}(t) \stackrel{\text{def}}{=} |\Psi(t)\rangle\langle\Psi(t)|$ so long as both c_i and c_j are non-zero. However, if we are only interested in observables $\hat{O}_{\mathcal{S}}$ on $H_{\mathcal{S}}$, then we only need to consider the reduced density matrix $\hat{\rho}_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{A}}(\hat{\rho}(t))$. Initially, at time t_0 we have $|a_i(t_0)\rangle = |a_r(t_0)\rangle$ so $|\Psi(t_0)\rangle = |\psi\rangle\,|a_r(t_0)\rangle$ where $|\psi\rangle = \sum_i c_i\,|s_i\rangle$ which we assume to be normalized. Thus, initially, $\mathcal{S}$ would not be entangled with $\mathcal{A}$, and therefore the density matrix describing

---

[48]According to equation (1.2), this basis would correspond to measuring the spin in an axis at right angles to $\mathbf{\hat{a}}$.

[49]This is because then $\langle a_+(t)|a_-(t)\rangle$ will be very nearly zero rather than identically zero as discussed in footnote 47.

$\mathcal{S}$ would be $\hat{\rho}_{\mathcal{S}}(t_0) = |\psi\rangle\langle\psi|$.[50] Hence, if we consider $\hat{O}_{\mathcal{S}} = |\psi\rangle\langle\psi|$ as an observable on $\mathcal{S}$ corresponding to a measurement[51] that records the value 1 if the system is in the $|\psi\rangle$ and 0 if the system is in a state $|\psi'\rangle$ with $\langle\psi'|\psi\rangle = 0$, then both intuitively[52] and by equation (1.19),[53] we would have $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t_0)} = 1$. But if the scientist is to measure $\mathcal{S}$ to be in the $|\psi\rangle$-state, the expectation value $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t)}$ would have to be 1 for times t discernibly greater than t_0.

However, if more than one of the c_i are non-zero, then the scientist will not be able to measure the system $\mathcal{S}$ to be in the $|\psi\rangle$-state for any discernible length of time. To see why this is, we first note that

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_i |c_i|^2 |s_i\rangle\langle s_i| + \sum_{i\neq j} c_i\overline{c_j} \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|\,.^{54} \qquad \text{\{reduced\}} \quad (1.20)$$

Now because $\langle a_j(t)|a_i(t)\rangle \approx 0$ for $t > t_0 + \delta$, it follows that $\hat{\rho}_{\mathcal{S}} \approx \sum_i |c_i|^2 |s_i\rangle\langle s_i|$ for $t > t_0 + \delta$. It will then follow that $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t)} = \sum_i |c_i|^4$,[55] and this will only be 1 if only

[50]Recall footnote 41.

[51]This is a measurement we conduct by some means other than looking at the apparatus $\mathcal{A}$.

[52]I.e. we would expect the expectation value of $\hat{O}_{\mathcal{U}}$ to be 1 if we knew that $\mathcal{S}$ was in the state $|\psi\rangle$ with probability 1.

[53]I.e. given that $\hat{\rho}_{\mathcal{S}}(t_0) = |\psi\rangle\langle\psi| = \hat{O}_{\mathcal{S}}$, and that $\hat{O}_{\mathcal{S}}^2 = \hat{O}_{\mathcal{S}}$, and $\mathrm{Tr}_{\mathcal{S}}(\hat{O}_{\mathcal{S}}) = 1$, it follows that $\langle\hat{O}_{\mathcal{U}}\rangle_{\rho(t_0)} = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}(t_0)\hat{O}_{\mathcal{S}}) = \mathrm{Tr}_{\mathcal{S}}(\hat{O}_{\mathcal{S}}) = 1$.

[54]To see this, it is sufficient to show that $\mathrm{Tr}_{\mathcal{A}}(|\xi_i(t)\rangle\langle\xi_j(t)|) = \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|$ for then we will obtain the first summand of $\hat{\rho}_{\mathcal{S}}$ from the fact that $|a_i(t)\rangle$ are normalized, and we will obtain the second summand by linearity of $\mathrm{Tr}_{\mathcal{A}}(\cdot)$. Well, taking $\{|\phi_k\rangle : k\}$ to be an orthonormal basis of $H_{\mathcal{A}}$, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{A}}(|\xi_i(t)\rangle\langle\xi_j(t)|) &= \sum_k \langle\phi_k| \Big(|\xi_i(t)\rangle\langle\xi_j(t)| \Big) |\phi_k\rangle = \sum_k \langle\phi_k|a_i(t)\rangle \langle a_j(t)|\phi_k\rangle |s_i\rangle\langle s_j| \\ &= \langle a_j(t)| \Big(\sum_k |\phi_k\rangle\langle\phi_k| \Big) |a_i(t)\rangle |s_i\rangle\langle s_j| = \langle a_j(t)|a_i(t)\rangle |s_i\rangle\langle s_j|\,,\end{aligned}$$

where we have used the fact that $I = \sum_k |\phi_k\rangle\langle\phi_k|$ is the identity operator on $H_{\mathcal{A}}$.

[55]This is because

$$\begin{aligned}\mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}} |\psi\rangle\langle\psi|) &\approx \mathrm{Tr}_{\mathcal{S}}(\sum_i |c_i|^2 |s_i\rangle\langle s_i| \sum_{jk} c_j\overline{c_k} |s_j\rangle\langle s_k|) \\ &= \mathrm{Tr}_{\mathcal{S}}(\sum_{ik} |c_i|^2 c_i\overline{c_k} |s_i\rangle\langle s_k|) = \sum_l \sum_{ik} \langle s_l| |c_i|^2 c_i\overline{c_k} |s_i\rangle \langle s_k|s_l\rangle = \sum_i |c_i|^4.\end{aligned}$$

one of the c_i is 1 and all the other c_i are 0. Hence, if more than one of the c_i are non-zero, the scientist will not be able to measure the system $\mathcal{S}$ to be in the $|\psi\rangle$-state for any discernible length of time.

1.8.7 Decoherence

Note that although for the original density matrix $|\psi\rangle\langle\psi|$ there is coherence between the states $|s_i\rangle$ and $|s_j\rangle$, this coherence effectively disappears when the system $\mathcal{S}$ interacts with the measuring device $\mathcal{A}$ (i.e. the $|s_i\rangle\langle s_j|$-coefficients of $\hat{\rho}_{\mathcal{S}}$ are approximately zero for $t > t_0 + \delta$). This is what we mean by **decoherence**: the coherence has effectively disappeared. The **decoherence time** δ which is the time it takes for $\langle a_i(t)|a_j(t)\rangle$ to go from 1 when $t = t_0$ to approximately zero when $t = t_0 + \delta$ will depend on what situation we are considering, but very often this time will be extremely small. For instance if we were measuring neurons firing in the brain, the decoherence time will typically be of the order $\delta = 10^{-19}\,\mathrm{s}$.[56] It is because of decoherence that we can't expect the system $\mathcal{S}$ to remain in the state $|\psi\rangle = \sum_i c_i\,|s_i\rangle$ for any discernible length of time, unless $|\psi\rangle$ is proportional to one of the $|s_i\rangle$-states.

Also note that when decoherence occurs, we say the coherence *effectively* disappears, insofar as the coherence will not be measurable if we only consider observables just acting on $H_{\mathcal{S}}$. Thus, after decoherence has taken place, if we restrict our attention to the system $\mathcal{S}$ alone, it will be experimentally indistinguishable[57] from the situation where $\mathcal{S}$ is known to be in one of the $|s_i\rangle$-states, but that we only know that it is in the $|s_i\rangle$-state with probability $|c_i|^2$. Nevertheless, the coherence is still there, since if we

[56] For details of this estimate see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 370.

[57] Recall the discussion following equation (1.15) on page 36 as well as the discussion on page 40.

chose to consider more general observables on the composite $\mathcal{S}+\mathcal{A}$, the $|\xi_i(t)\rangle\langle\xi_j(t)|$-coefficients of $\hat{\rho}$ will continue to be $c_i\overline{c_j}$ which will in general will be non-zero, and as a whole, at time t the composite system will be in the state $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle$ with probability 1.

1.9 sectionPreferredBasis A Solution to the Preferred Basis Problem[58]

We can now see how decoherence theory solves the preferred basis problem. Although up to this point we have been focusing on how a system $\mathcal{S}$ interacts with a measuring apparatus $\mathcal{A}$, we can generalize to the situation in which a system $\mathcal{S}$ interacts with its environment $\mathcal{E}$. We can still define pointer states in the same way as we did on page 42. These pointer states will then make up the preferred basis. The two defining criteria of pointer states entail that pointer states will remain stable and immune to decoherence effects.

Since physicists have a good understanding of how different systems interact, they are able to explain what it is about a basis that makes it a preferred basis. The details of their analysis need not concern us here, but it's possible to show that for macroscopic and mesoscopic objects, states specified in terms of position decohere with one another very rapidly.[59] This explains why we don't detect $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$-states, but we do detect $|\text{Cat Alive}\rangle$ and $|\text{Cat Dead}\rangle$-states. Also note that $|\text{Cat Alive}\rangle$ does indeed have the property that it is immune to decoherence effects, for if we were to express $|\text{Cat Alive}\rangle$ in terms of the basis $\{|\psi_+\rangle, |\psi_-\rangle\}$ where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle \pm |\text{Cat Dead}\rangle)$, then

---

[58] For more details, see Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 71–84

[59] e.g. See the discussion in Schlosshauer, 94.

$|\text{Cat Alive}\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)$. The corresponding density matrix would then be

{catdensity}

$$|\text{Cat Alive}\rangle\langle\text{Cat Alive}| = \frac{1}{\sqrt{2}}(|\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|). \tag{1.21}$$

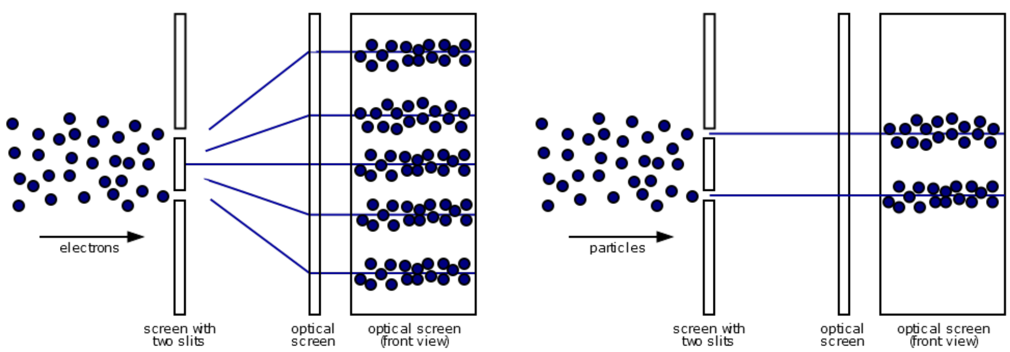
Since in normal situations, the left-hand side of equation (1.21) will remain unperturbed by the environment, the coefficients of the off-diagonal terms $|\psi_\pm\rangle\langle\psi_\mp|$ will also remain as they are; that is, $|\psi_\pm\rangle$ and $|\psi_\mp\rangle$ will not decohere with one another. It is only in very contrived situations such as when the cat's environment is a poison releasing device coupled to a radioactive atom that $|\text{Cat Alive}\rangle$ will no longer be a pointer state with respect to this environment.

Nonobservability

1.10 The Problem of Nonobservability of Interference and its Solution[60]

Before we consider the many-worlds interpretation in detail, it will be helpful to consider the role that decoherence plays in the removal of quantum interference on the macroscopic scale, as it is this lack of quantum interference between mutually exclusive states of a system that justifies our thinking of the system as being composed of alternative realities. The question of why quantum interference typically disappears at macroscopic scales is referred to as the problem of the nonobservability of interference.

We can explain this problem in the context of the double slit experiment: As figure 1.6



(A) Particles exhibiting interference. (B) Particles not exhibiting interference.

Figure 1.6: The Double-Slit Experiment. Particles are incident on a double slit. In diagram (A), the particles are exhibiting an interference pattern, whereas in diagram (B), the particles are not exhibiting an interference pattern. Whether or not there is interference will depend on factors such as the size of the particles and whether it can be ascertained which slit the particle went through. The larger the particles are or the more information available as to which slit the particle went through, the less likely the particles will exhibit interference.[61]

DoubleSlit

indicates, when a beam of particles is incident on a double slit, the particles that are detected on the detection screen are distributed according to a distribution pattern which either exhibits quantum interference as shown on the left in the figure, or does not exhibit such interference as shown on the right. Small particles like electrons and

---

[60]See Schlosshauer, *Decoherence and the Quantum-to-Classical Transition*, 55–57, 63–65.

photons will tend to exhibit quantum interference, whereas mesoscopic particles will not typically exhibit quantum interference.

To explain what is going on, we suppose that when just the top slit is open, the normalized state of the particle is $|\psi_1\rangle$, psi_slit whereas if just the bottom slit is open, we suppose that the normalized state of the particle is $|\psi_2\rangle$, and when both slits are open, we suppose that the state of the particle will be $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Now let the variable x describe the position on the detection screen. For instance, we might take $x = 0$ to be the center of the detection screen, and take positive values of x as corresponding to positions on the upper part of the screen, and negative values of x as corresponding to positions on the lower part of the screen, but the precise convention we adopt won't matter. Then we define the $|x\rangle$-state[62] as the physical state describing the particle to be exactly located at position x on the screen. Note that the state $|x\rangle$ is indexed by a continuous parameter, x. This is in contrast to the basis of states $|s_i\rangle$ which we have been considering up until now which are indexed by discrete values of i such as $i = 1, 2, \ldots$. Because of this difference, we need to use calculus to deal with $|x\rangle$-states in a rigorous manner, but such details will not concern us here. In reality, because of the Heisenberg uncertainty principle, a particle is never in just one $|x\rangle$-state, but rather the particle will be in a superposition of many $|x\rangle$-states, which may or may not be concentrated around a particular location, x_0 say. The more concentrated these $|x\rangle$-states of this superposition are concentrated around a particular location x_0, the more the particle will have the particle-like characteristic of being localized

[61]Diagrams (A) and (B) are by inductiveload, and are Public domain, via Wikimedia Commons. Sources: https://commons.wikimedia.org/wiki/File:Two-Slit_Experiment_Electrons.svg and https://commons.wikimedia.org/wiki/File:Two-Slit_Experiment_Particles.svg.

in one place. But if the $|x\rangle$-states of this superposition are more spread out, the particle will have more wave-like characteristics. So when physicists speak of particles, often they are not thinking of physical entities that are very localized in position, as non-physicists would think. Nevertheless, at the moment the particle is detected on the detection screen, it does seem to be highly localized.

Given a state $|\psi\rangle$ for a so-called particle, we define the function $\psi(x) = \langle x|\psi\rangle$. Because of the continuous nature of the variable x (in contrast to the discrete nature of i in a basis $\{|s_i\rangle : i\}$), the function $\rho(x) = |\psi(x)|^2$ rhodensity determines a probability density for a range of outcomes rather than a probability for a specific outcome. Here, we do not need to go into the details of probability densities,[63] but roughly speaking, the greater the value of $\rho(x)$, the greater will be the relative probability of detecting the particle at location x. Thus, if $\rho(x) = 0$, then the particle would not be detected at location x.

Now if $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$, then $\psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$. Therefore, the corresponding probability density will be

$$|\psi(x)|^2 = \frac{1}{2}(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}(\overline{\psi_1(x)}\psi_2(x)).$$[64]

---

[62] $|x\rangle$ is not really a state in the proper sense. With the states we've seen so far, when $|\phi\rangle$ and $|\psi\rangle$ have been normalized, then $|\langle\phi|\psi\rangle|^2$ will be a conditional probability, and hence at most 1. However, $|x\rangle$ cannot be normalized. This is because the bracket $\langle x|y\rangle$ is defined to be $\langle x|y\rangle = \delta(x-y)$ where $\delta(x)$ is the Dirac delta function such that

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \tag{1.22}$$

and has the property that $\int \mathrm{d}x\,\delta(x)\,f(x) = f(0)$ for any continuous function $f(x)$. The theory of distributions allows one to deal rigorously with Dirac delta functions. E.g. see cbx015 Walter Rudin, *Functional Analysis*, Second Edition (McGraw-Hill, 1991), ch. 6.

[63] But if you are interested, a probability density $\rho(x)$ for a random variable X that has real values is a function such that $\rho(x) \geq 0\ \forall x \in \mathbb{R}$, and that $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$ and the probability that X has a value in the subset $U \subset \mathbb{R}$ is $\int_U \rho(x)\mathrm{d}x$.

Now when the detection screen is far away from the double slits, we will have $|\psi_1(x)|^2 \approx |\psi_2(x)|^2$ for x near the center point on the screen. However, depending on slight changes in the value of x from the center point on the screen, sometimes $\psi_1(x)$ and $\psi_2(x)$ will be in phase so that $\psi_1(x) \approx \psi_2(x)$, in which case $|\psi(x)|^2 \approx 2|\psi_1(x)|^2$. But sometimes $\psi_1(x)$ and $\psi_2(x)$ will be out of phase so that $\psi_1(x) \approx -\psi_2(x)$, in which case $|\psi(x)|^2 \approx 0$. Hence, we get the interference pattern as shown in figure 1.6 (A).

Now in order to consider how decoherence affects interference, we let

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle\,|E_1(t)\rangle + |\psi_2\rangle\,|E_2(t)\rangle)$$

be the state of the composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ where $\mathcal{S}$ is a particle that has gone through the double slit and will be detected on the detection screen, and $\mathcal{E}$ is the local environment of the experimental set up. The expression for $|\Psi(t)\rangle$ indicates that we are assuming $\mathcal{S}$ doesn't become entangled with $\mathcal{E}$ when $\mathcal{S}$ is in the state $|\psi_1\rangle$ or $|\psi_2\rangle$. Corresponding to $|\Psi(t)\rangle$ we can define the density matrix $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. We can also define the observable[65] $|x\rangle\langle x|_{\mathcal{S}}$ for the system $\mathcal{S}$ so that $|x\rangle\langle x|_{\mathcal{S}}\,|\psi\rangle_{\mathcal{S}} = \psi(x)\,|x\rangle_{\mathcal{S}}$. As we saw in equation (1.11) on page 33, we can naturally extend the action of $|x\rangle\langle x|_{\mathcal{S}}$ to $H_{\mathcal{U}}$.[66] This allows us to define

$$\rho_{\mathcal{U}}(x,t) \stackrel{\text{def}}{=} \mathrm{Tr}_{\mathcal{U}}(\hat{\rho}(t)_{\mathcal{U}}\,|x\rangle\langle x|_{\mathcal{U}}).$$

---

[64]Here Re means the real part of a complex number. Thus, if the complex number $z = \alpha + i\beta$ for real numbers α and β, then $\mathrm{Re}(z) = \alpha$. To see why the above equation holds, we recall that $|z|^2 = z\overline{z}$ and that $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$. Therefore, if $z = \frac{1}{\sqrt{2}}(v + w)$ for complex number v and w, then $|z|^2 = \frac{1}{\sqrt{2}}(v+w)\frac{1}{\sqrt{2}}\overline{(v+w)} = \frac{1}{2}(v+w)\overline{(v+w)} = \frac{1}{2}(v\overline{v}+w\overline{w}+w\overline{v}+v\overline{w}) = \frac{1}{2}(|v|^2+|w|^2+w\overline{v}+\overline{w\overline{v}}) = \frac{1}{2}(|v|^2+|w|^2+2\,\mathrm{Re}(\overline{v}w))$.

[65]Note that we only call $|x\rangle\langle x|_{\mathcal{S}}$ an observable in an analogical sense since it is not a compact Hermitian operator acting on the Hilbert space of states $H_{\mathcal{S}}$. If we were being more rigorous, we would need to consider a Hermitian operator of the form $\int \sigma(x)\,|x\rangle\langle x|_{\mathcal{S}}\,\mathrm{d}x$ for an appropriate test function $\sigma(x)$.

[66]Strictly speaking, it is not $|x\rangle\langle x|_{\mathcal{S}}$ that is extended to act on $H_{\mathcal{U}}$, but rather a Hermitian operator of the form $\int \sigma(x)\,|x\rangle\langle x|_{\mathcal{S}}\,\mathrm{d}x$ for an appropriate test function $\sigma(x)$ that is extended to $H_{\mathcal{U}}$. For a state

In the specific case when $\hat{\rho}_{\mathcal{U}}(t) = |\xi(t)\rangle\langle\xi(t)|_{\mathcal{U}}$ where $|\xi(t)\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |E(t)\rangle_{\mathcal{E}}$ for normalized states $|\psi\rangle_{\mathcal{S}}$ and $|E(t)\rangle_{\mathcal{E}}$, we have $\rho_{\mathcal{U}}(x,t) = |\psi(x)|^2$ which is equal to the probability density function $\rho(x)$ we saw on page 50.[67] However, if $|E_1(t)\rangle_{\mathcal{E}}$ is not proportional to $|E_2(t)\rangle_{\mathcal{E}}$, then $|\Psi(t)\rangle$ will be an entangled state of $\mathcal{S}$ and $\mathcal{E}$. But whether or not $\mathcal{S}$ and $\mathcal{E}$ are entangled, we can still use equation (1.20) to calculate the partial trace:

$$\hat{\rho}_{\mathcal{S}}(t) = \frac{1}{2}(|\psi_1\rangle\langle\psi_1|_{\mathcal{S}} + |\psi_2\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} |\psi_1\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} |\psi_2\rangle\langle\psi_1|_{\mathcal{S}}).$$

By equation (1.19), we therefore have

$$\rho_{\mathcal{U}}(x,t) = \frac{1}{2}\Big(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}\left(\langle E_2(t)|E_1(t)\rangle_{\mathcal{E}}\,\overline{\psi_2(x)}\psi_1(x)\right)\Big).^{68} \qquad (1.23)$$

$|\xi\rangle_{\mathcal{U}} = |\psi\rangle_{\mathcal{S}} |E\rangle_{\mathcal{E}}$, the action of $|x\rangle\langle x|_{\mathcal{U}}$ on $|\xi\rangle_{\mathcal{U}}$ gives the 'state' $|x\rangle\langle x|_{\mathcal{U}} |\xi\rangle_{\mathcal{U}} \stackrel{\text{def}}{=} \psi(x) |x\rangle_{\mathcal{S}} |E\rangle_{\mathcal{E}}$, but since this is not normalizable, we have to 'smear' it by integrating it with respect to the test function $\sigma(x)$.

[67] To see this, note that we can ignore $\mathcal{E}$ in calculating $\langle |x\rangle\langle x|_{\mathcal{U}}\rangle_{\xi}$ since when $\mathcal{S}$ and $\mathcal{E}$ are not entangled, $\langle |x\rangle\langle x|_{\mathcal{U}}\rangle_{\xi} = \langle |x\rangle\langle x|_{\mathcal{S}}\rangle_{\psi}$ as explained in footnote 33. We can therefore just consider $\mathcal{S}$ and drop the subscripts. Furthermore, as we saw on page 35, $\langle \hat{O}\rangle_{\psi} = \mathrm{Tr}(\hat{\rho}\hat{O})$ where $\hat{\rho} = |\psi\rangle\langle\psi|$. We can thus take an orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle, \ldots\}$ of $H_{\mathcal{S}}$ with $|\psi_1\rangle = |\psi\rangle$. Then $\rho(x) = \mathrm{Tr}(\hat{\rho}\,|x\rangle\langle x|) = \mathrm{Tr}(|\psi\rangle\langle\psi|\,|x\rangle\langle x|) = \sum_i \langle\psi_i|\psi\rangle\langle\psi|x\rangle\langle x|\psi_i\rangle = \langle\psi_1|\psi\rangle\langle\psi|x\rangle\langle x|\psi_1\rangle = \langle\psi|x\rangle\langle x|\psi\rangle = \overline{\langle x|\psi\rangle}\langle x|\psi\rangle = |\langle x|\psi\rangle|^2 = |\psi(x)|^2$.

Thus, if $\langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \approx 0$ then $\rho_{\mathcal{U}}(x,t) \approx \frac{1}{2}\Big(|\psi_1(x)|^2 + |\psi_2(x)|^2\Big)$ and so we would observe a distribution pattern not exhibiting interference as shown in figure 1.6 (B), whereas if $\langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \not\approx 0$ we would get a distribution pattern exhibiting interference as shown in figure 1.6 (B). Thus, decoherence theory gives us a means of determining whether or not quantum interference will be exhibited.

probOutcomes

1.11 The Problem of Outcomes

In the last two sections we have seen how decoherence theory solves the preferred basis problem and the problem of the nonobservability of interference. However, there is a third fundamental problem in quantum physics which decoherence theory is unable to solve. This is the problem of outcomes. As discussed in subsection 1.8.6, in the von Neumann measurement scheme, it is supposed that for the measurement of a physical system $\mathcal{S}$ to take place, it must interact with a measuring device $\mathcal{A}$ which together satisfy the conditions 1. and 2. on page 41. If $\mathcal{S}$ is initially in a superposition of states $|\psi\rangle = \sum_i c_i |s_i\rangle$ then for $\mathcal{A}$ to measure $\mathcal{S}$, it is necessary for the combined

[68]The calculation is as follows

$$\begin{aligned}
\rho_{\mathcal{U}}(x,t) &= \mathrm{Tr}_{\mathcal{U}}(\hat{\rho}(t)_{\mathcal{U}} |x\rangle\langle x|_{\mathcal{U}}) = \langle |x\rangle\langle x|_{\mathcal{U}}\rangle_{\rho(t)} = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}(t) |x\rangle\langle x|_{\mathcal{S}}) \\
&= \mathrm{Tr}_{\mathcal{S}}\Big(\frac{1}{2}(|\psi_1\rangle\langle\psi_1|_{\mathcal{S}} + |\psi_2\rangle\langle\psi_2|_{\mathcal{S}} \\
&\quad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} |\psi_1\rangle\langle\psi_2|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} |\psi_2\rangle\langle\psi_1|_{\mathcal{S}}) |x\rangle\langle x|_{\mathcal{S}}\Big) \\
&= \mathrm{Tr}_{\mathcal{S}}\Big(\frac{1}{2}(\langle\psi_1|x\rangle_{\mathcal{S}} |\psi_1\rangle\langle x|_{\mathcal{S}} + \langle\psi_2|x\rangle_{\mathcal{S}} |\psi_2\rangle\langle x|_{\mathcal{S}} \\
&\quad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} \langle\psi_2|x\rangle_{\mathcal{S}} |\psi_1\rangle\langle x|_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \langle\psi_1|x\rangle_{\mathcal{S}} |\psi_2\rangle\langle x|_{\mathcal{S}})\Big) \\
&= \frac{1}{2}\Big(\langle\psi_1|x\rangle_{\mathcal{S}} \langle x|\psi_1\rangle_{\mathcal{S}} + \langle\psi_2|x\rangle_{\mathcal{S}} \langle x|\psi_2\rangle_{\mathcal{S}} \\
&\quad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} \langle\psi_2|x\rangle_{\mathcal{S}} \langle x|\psi_1\rangle_{\mathcal{S}} + \langle E_1(t)|E_2(t)\rangle_{\mathcal{E}} \langle\psi_1|x\rangle_{\mathcal{S}} \langle x|\psi_2\rangle_{\mathcal{S}}\Big) \\
&= \frac{1}{2}\Big(\overline{\langle x|\psi_1\rangle_{\mathcal{S}}} \langle x|\psi_1\rangle_{\mathcal{S}} + \overline{\langle x|\psi_2\rangle_{\mathcal{S}}} \langle x|\psi_2\rangle_{\mathcal{S}} \\
&\quad + \langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} \overline{\langle x|\psi_2\rangle_{\mathcal{S}}} \langle x|\psi_1\rangle_{\mathcal{S}} + \overline{\langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} \overline{\langle x|\psi_2\rangle_{\mathcal{S}}} \langle x|\psi_1\rangle_{\mathcal{S}}}\Big) \\
&= \frac{1}{2}\Big(|\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}\,\big(\langle E_2(t)|E_1(t)\rangle_{\mathcal{E}} \overline{\psi_2(x)}\psi_1(x)\big)\Big).
\end{aligned}$$

system $\mathcal{S} + \mathcal{A}$ to enter into a superposition

{vNevolution}

$$\left|\psi\right\rangle \left|a_r(t_0)\right\rangle \xrightarrow{\text{time evolution}} \sum_i c_i \left|s_i\right\rangle \left|a_i(t)\right\rangle . \tag{1.24}$$

However, although the evolution described in (1.24) must take place if $\mathcal{A}$ is to measure $\mathcal{S}$, it is not sufficient. When one takes the partial trace of $|\psi\rangle\langle\psi|$ over $\mathcal{A}$, then according to (1.20),

{vNevolution2}

$$\mathrm{tr}_{\mathcal{A}}(|\psi\rangle\langle\psi|) \xrightarrow{\text{time evolution}} \sum_i |c_i|^2 \, |s_i\rangle\langle s_i| . \tag{1.25}$$

But as noted on page 40, we cannot give an ignorance interpretation to $\sum_i c_i \, |s_i\rangle\langle s_i|$ for as d'Espagnat puts it, this is an improper mixture. When considered together, the system $\mathcal{S}$ and the apparatus $\mathcal{A}$ remain in the superposition described by (1.24), and so none of the measurement outcomes from the set of possible outcomes $\{|s_i\rangle : i\}$ have actually occurred. The problem of explaining how the composite system $\mathcal{S} + \mathcal{A}$ goes from being in the state $\sum_i c_i \left|s_i\right\rangle \left|a_i(t)\right\rangle$ to a state $\left|s_i\right\rangle \left|a_i(t)\right\rangle$ is known as the **problem of outcomes**.

manyworldsinterpretation1

1.12 The Many-Worlds Interpretation

Not everyone is convinced that the problem of outcomes is a genuine problem. In particular, people who endorse the many-worlds interpretation of quantum physics effectively argue that there are no outcomes in the traditional sense. In this section, we now give and account of the many-worlds interpretation of quantum physics and why physicists find it attractive. To this end, let us consider a physical universe $\mathcal{U} = \mathcal{S} + \mathcal{A} + \mathcal{P}_A + \mathcal{P}_B + \mathcal{E}$ consisting of subsystems $\mathcal{S}, \mathcal{A}, \mathcal{P}_A, \mathcal{P}_B$ and $\mathcal{E}$. $\mathcal{S}$ is the physical system under investigation; $\mathcal{A}$ is some measuring apparatus that interacts with $\mathcal{S}$; $\mathcal{P}_A$ and $\mathcal{P}_B$ are the physical systems corresponding to two scientists, Alice and Bob who observed the apparatus $\mathcal{A}$; and $\mathcal{E}$ is the remainder of the physical universe $\mathcal{U}$. For convenience, we define the composite subsystem $\mathcal{V} = \mathcal{S} + \mathcal{A} + \mathcal{P}_A + \mathcal{P}_B$ so that $\mathcal{U} = \mathcal{V} + \mathcal{E}$. As above on page 42, we assume that there is an orthonormal basis $\{|s_i\rangle : i\}$ of $H_{\mathcal{S}}$ which we again refer to as pointer states, but now we assume that there are ready states $|a_r(t)\rangle \in H_{\mathcal{A}}$, $|p_{r,A}(t)\rangle \in H_{\mathcal{P}_A}$, $|p_{r,B}(t)\rangle \in H_{\mathcal{P}_B}$, and $|E_r(t)\rangle \in H_{\mathcal{E}}$ and that for each i, there are normalized states $|a_i(t)\rangle \in H_{\mathcal{A}}$, $|p_{i,A}(t)\rangle \in H_{\mathcal{P}_A}$, $|p_{i,B}(t)\rangle \in H_{\mathcal{P}_B}$, and $|E_i(t)\rangle \in H_{\mathcal{E}}$ such that

1. for any $t \geq 0$ we have the evolution of the states

$$|s_i\rangle\, |a_r(t)\rangle\, |p_{r,A}(t)\rangle\, |p_{r,B}(t)\rangle\, |E_r(t)\rangle$$
$$\xrightarrow{\text{time evolution}} |s_i\rangle\, |a_i(t)\rangle\, |p_{i,A}(t)\rangle\, |p_{i,B}(t)\rangle\, |E_i(t)\rangle\,,$$

2. there exists $\delta > 0$ such that if $t > t_0 + \delta$, then for $i \neq j$, $\langle a_i(t)|a_j(t)\rangle \approx 0$, $\langle p_{i,A}(t)|p_{j,A}(t)\rangle \approx 0$, $\langle p_{i,B}(t)|p_{j,B}(t)\rangle \approx 0$ and $\langle E_i(t)|E_j(t)\rangle \approx 0$.[69]

[69] Again, recall footnote 45.

We also suppose that the $|p_{i,A}(t)\rangle$-state would describe actions of Alice consistent with her observing the apparatus being in the $|a_i(t)\rangle$-state, for example, her writing down in her log book that the apparatus is in the $|a_i(t)\rangle$-state or telling her colleague that this is the case. Likewise, we assume the $|p_{i,B}(t)\rangle$-state is consistent with Bob also observing the apparatus to be in the $|a_i(t)\rangle$-state.

Now suppose the initial (normalized) state of $\mathcal{S}$ is $|\psi\rangle = \sum_i c_i |s_i\rangle$, so that the state for the composite system $\mathcal{U}$ is $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ where $|\xi_i(t)\rangle = |s_i\rangle |a_i(t)\rangle |p_{i,A}(t)\rangle |p_{i,B}(t)\rangle$. We also define the corresponding density matrix for the composite system $\hat{\rho} = |\Psi\rangle\langle\Psi|$. If we were unable to make any observations on $\mathcal{E}$, then the partial trace $\hat{\rho}_{\mathcal{V}}(t) = \mathrm{Tr}_{\mathcal{E}}(\hat{\rho}(t))$ will contain all the information we need to work out the expectation values for any observables of $\mathcal{V}$. So just as with equation (1.20), we will have

$$\begin{aligned} \hat{\rho}_{\mathcal{V}}(t) &= \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)| + \sum_{i\neq j} c_i\overline{c_j} \langle E_j(t)|E_i(t)\rangle |\xi_i(t)\rangle\langle\xi_j(t)| \\ &\approx \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)| \end{aligned} \quad \texttt{\{reducedv\}} \tag{1.26}$$

for $t > t_0 + \delta$. Then the expectation values of any observables on $\mathcal{V}$ will be indistinguishable from the scenario in which $\mathcal{V}$ is actually in one of the $|\xi_i(t)\rangle$-states with probability $|c_i|^2$.[70] It would nevertheless be incorrect for us to conclude on the basis of decoherence theory alone that $\mathcal{V}$ actually was in one of those $|\xi_i(t)\rangle$-states, since equation (1.26) is based on a subjective distinction between $\mathcal{V}$ and $\mathcal{E}$ in the decomposition $\mathcal{U} = \mathcal{V} + \mathcal{E}$. Human scientists make this distinction to reflect the fact that they

[70] Again recall the discussion following equation (1.15) on page 36. There is the question of uniqueness of $\hat{\rho}_{\mathcal{V}} = \sum_i |c_i|^2 |\xi_i(t)\rangle\langle\xi_i(t)|$. If all the $|c_i|^2$ are unique, then if we have another decomposition $\hat{\rho}_{\mathcal{S}+\mathcal{A}+\mathcal{P}_A+\mathcal{P}_B} = \sum_i |c_i|^2 |\xi_i'(t)\rangle\langle\xi_i'(t)|$ it follows that $|\xi_i(t)\rangle \propto |\xi_i'(t)\rangle$. But even if some of the $|c_i|^2$ are the same, criteria 1 and 2 above will ensure that states with the same value of $|c_i|^2$ will be determined up to permutation.

can only perform measurements on $\mathcal{V}$ and can't measure $\mathcal{E}$. But if a super-intelligent being could measure everything in $\mathcal{U}$, then such a being would not say that $\mathcal{V}$ was in one of the $|\xi_i(t)\rangle$-states, but rather that $\mathcal{U}$ was in the state $|\Psi(t)\rangle$. As we have already discussed on pages 40–41, the density matrix $\hat{\rho}_{\mathcal{V}}(t)$ is not a mixed state, but is an improper mixture.

Now if we define the observables $\hat{O}_{i,\mathcal{A}}(t) = |p_{i,A}(t)\rangle\langle p_{i,A}(t)|$ that would measure the behavior of Alice, and the observables $\hat{O}_{i,\mathcal{B}}(t) = |p_{i,B}(t)\rangle\langle p_{i,B}(t)|$ that would measure the behavior of Bob, then for $t > t_0 + \delta$, we see that $\hat{O}_{i,\mathcal{A}}(t)\hat{O}_{j,\mathcal{B}}(t)\,|\Psi(t)\rangle \approx 0$, when $i \neq j$. This means that when we consider $\hat{O}_{i,\mathcal{A}}(t)\hat{O}_{j,\mathcal{B}}(t)$ as an observable acting on $\mathcal{V}$, the expectation value $\langle \hat{O}_{i,\mathcal{A}}(t)\hat{O}_{j,\mathcal{B}}(t)\rangle_{\rho_{\mathcal{V}}(t)}$ will be approximately zero for $i \neq j$. What this means is that if we consider ourselves as observing Alice and Bob observing the apparatus, then after time $t_0 + \delta$, the probability we would see Alice and Bob disagreeing with each other concerning their observations of the apparatus would be approximately 0. On the other hand, since $\hat{O}_{i,\mathcal{A}}(t)\hat{O}_{i,\mathcal{B}}(t)\,|\Psi(t)\rangle \approx c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ for $t > t_0 + \delta$, it follows that $\langle \hat{O}_{i,\mathcal{A}}(t)\hat{O}_{i,\mathcal{B}}(t)\rangle_{\rho_{\mathcal{V}}(t)} = |c_i|^2$. We would thus observe Alice and Bob observing the apparatus to be in the $|a_i(t)\rangle$-state with probability $|c_i|^2$.

But note that on the assumption that there are no hidden variables, if we did actually make such an observation and this observation corresponded to reality, then the quantum state $|\Psi(t)\rangle$ would have had to have changed to $|\xi_i(t)\rangle\,|E_i(t)\rangle$, since before our observation when $|\Psi(t)\rangle$ was a complete description of $\mathcal{U}$, we would say Alice and Bob will measure the $|a_i(t)\rangle$-state with probability $|c_i|^2$, but when we are actually seeing them measuring the $|a_i(t)\rangle$-state, we would have to say that now the probability they are measuring the $|a_i(t)\rangle$-state is 1, and hence we would say that the system was in the $|\xi_i(t)\rangle\,|E_i(t)\rangle$-state. Whether or not the process of the state going from $|\Psi(t)\rangle$ to $|\xi_i(t)\rangle\,|E_i(t)\rangle$ was instantaneous or took a non-infinitesimal amount of time, this interpretation would be susceptible to the problems already discussed with the Copenhagen interpretation on page 20.

But in the **many-worlds interpretation**, rather than assuming that $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ is the complete description of $\mathcal{U}$ that enables us to work the probability of certain outcomes, we simply say that $|\Psi(t)\rangle$ is a complete description of the state of $\mathcal{U}$, and we drop the assumption that we need to interpret this state as describing probabilities of outcomes. Thus a many-worlds adherent would say we can understand what the state of $\mathcal{U}$ is on its own terms without the need to appeal to any other extrinsic principle such as measurement. Just as we don't puzzle over how to interpret what a sphere is in terms of an extrinsic principle, we don't need to puzzle over how to interpret the states of $\mathcal{U}$. We can think of the mathematical formalism $|\Psi(t)\rangle = \sum_i c_i\,|\xi_i(t)\rangle\,|E_i(t)\rangle$ describing the state of $\mathcal{U}$ as being somewhat akin to the equation $x^2 + y^2 + z^2 = 1$ describing a sphere. Although we might be tempted to interpret $|\Psi(t)\rangle$ as describing the probability of outcomes, we are not obliged to do so, since these probabilities can instead be understood to be grounded in the symmetries the system possesses rather than in terms of the frequency of how many measurement outcomes are likely to occur. For instance, when we see a coin and judge that it will come up heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$, we intuit this by looking at the symmetry of the coin rather than tossing the coin millions of times and counting how often it comes up heads and how often it comes up tails.

As for the decomposition $|\Psi(t)\rangle = \sum_i c_i |\xi_i(t)\rangle |E_i(t)\rangle$ in terms of the $|\xi_i(t)\rangle |E_i(t)\rangle$ basis states, decoherence theory gives us a natural account of why we should choose this basis rather than any other. When $\mathcal{U}$ is in the state $|\Psi(t_0)\rangle = \Big(\sum_i c_i |\xi_i(t_0)\rangle\Big) |E_r(t)\rangle$, we can think of this state as describing one world, W say. But once $t > t_0 + \delta$ so that $\langle E_i(t)|E_j(t)\rangle \approx 0$ for $i \neq j$, we can think of each $|\xi_i(t)\rangle |E_i(t)\rangle$-component as a different world W_i. Thus, for $t > t_0 + \delta$, we say the world W has **branched** into as many-worlds W_i for which the c_i are non-zero.

But why should we think that there are literally many worlds? Well, from an ontological point of view, one might very well think that there is really only one world and that this world is described by $|\Psi(t)\rangle$; it would be a rather weird world since the entanglement between $\mathcal{V}$ and $\mathcal{E}$ would mean there wouldn't be any absolute matters of fact describing $\mathcal{V}$. But it might not be a bad thing to say that the "many" in the many-worlds interpretation is really just a figure of speech that we shouldn't take too literally, since a common objection to the many-worlds interpretation is that it is ontologically extravagant and that we should appeal to Occam's Razor. However, if we just say that there is actually only one world described by $|\Psi(t)\rangle$ then this "many"-worlds interpretation is actually rather parsimonious from an ontological point of view.

But if by literal, we mean descriptive rather than ontological, it does seem rather natural to say that there are literally many worlds. For although we might initially suspect that the worlds W_i and W_j are not well-defined given the fact that $\langle E_i(t)|E_j(t)\rangle$ is very small but not zero for $i \neq j$, we can nevertheless expect $\langle \xi_i(t)|\xi_j(t)\rangle$ to be identically zero for $i \neq j$, just as we can expect $\langle \hat{\mathbf{a}}+|\hat{\mathbf{a}}-\rangle$ to be identically zero.[71] Thus, if we define $|W_i(t)\rangle = |\xi_i(t)\rangle\, |E_i(t)\rangle$, then the $\langle W_i(t)|W_j(t)\rangle$ will be identically zero for $i \neq j$.

Still, the supposition that $|\Psi(t)\rangle = \sum_i c_i\, |W_i(t)\rangle$ with $\langle W_i|W_j\rangle = 0$ for $i \neq j$ is not of itself sufficient justification for describing the state $|\Psi(t)\rangle$ as a composition of mutually exclusive world descriptions given by the $|W_i(t)\rangle$. After all, the fact that

$$\begin{aligned}|\text{Cat Alive}\rangle =& \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)\Big)\\ &+ \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)\Big).\end{aligned}$$

does not incline us to think of the state $|\text{Cat Alive}\rangle$ as being composed of the mutually exclusive cat states $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle + |\text{Cat Dead}\rangle)$ and $\frac{1}{\sqrt{2}}(|\text{Cat Alive}\rangle - |\text{Cat Dead}\rangle)$.

The key justification for describing the state $|\Psi(t)\rangle$ as a composition of the mutually exclusive $|W_i(t)\rangle$-states is the fact that the states $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ decohere for $i \neq j$, that is, the off-diagonal entries $|\xi_i(t)\rangle\langle\xi_j(t)|$ of the reduced density matrix $\hat{\rho}_{\mathcal{V}}(t)$ will tend to zero, and as we saw in section 1.10, it will follow that quantum interference effects between $|\xi_i(t)\rangle$ and $|\xi_j(t)\rangle$ will then tend to zero. Thus, when it comes to observables defined on $\mathcal{V}$, using equations (1.19) and (1.26), we can calculate the expectation value of an observable $\hat{O}_{\mathcal{V}}$ as a weighted sum of expectation values for each of the states $|\xi_i(t)\rangle$:

$$\langle \hat{O}_{\mathcal{U}}\rangle_{\Psi(t)} \approx \sum_i |c_i|^2\, \langle \hat{O}_{\mathcal{V}}\rangle_{\xi_i(t)}\,. \tag{1.27}$$

{manyapprox}

[71] It is also reasonable to suppose that in situations such as the double-slit experiment described on page 49 that $\langle\psi_1(t)|\psi_2(t)\rangle$ is identically zero. This is because $\langle\psi_1(t)|\psi_2(t)\rangle$ when t is the time at which the particle is going through the slit, and this will remain zero because of a property of the time evolution operator known as unitarity.

The fact that (1.27) is only an approximation suggests that the time at which branching occurs is not well-defined. All that we can do is choose a time sufficiently large δ so that for $t > t_0 + \delta$, the approximation (1.27) meets our desired level of accuracy.

Despite this vagueness on when branching occurs, we can still form a natural and well-defined notion of worlds according to the following definition: rigorousworld a set $\{W_i : i\}$ is the set of worlds for a universe $\mathcal{U} = \mathcal{V} + \mathcal{E}$ when

1. W_i is a description of $\mathcal{U}$ given by $|W_i(t)\rangle = |\xi_i(t)\rangle\, |E_i(t)\rangle$,
2. $\mathcal{U}$ is in the state $|\Psi(t)\rangle = \sum_i c_i\, |W_i(t)\rangle$ with all $c_i \neq 0$.
3. $\langle \xi_i(t) | \xi_j(t) \rangle = 0$ for $i \neq j$,
4. $\langle E_i(t) | E_j(t) \rangle \to 0$ as $t \to \infty$ for $i \neq j$ and the convergence is such that for any observable $\hat{O}_{\mathcal{V}}$ defined on $\mathcal{V}$, $\langle \hat{O}_{\mathcal{U}} \rangle_{\Psi(t)} \to \sum_i |c_i|^2 \langle \hat{O}_{\mathcal{V}} \rangle_{\xi_i(t)}$.

Note that according to this definition, the description $|\Psi(t)\rangle$ is rather trivially a world – we just take the environment $\mathcal{E}$ to be empty so that there would be only one $|E_i(t)\rangle$ which would be the vacuum state. So there is at least one world according to this definition. There is a question of whether there could be more than one world and this would depend on whether we could really have a non-trivial decomposition $\mathcal{U} = \mathcal{V} + \mathcal{E}$, for the supposition that there is such a decomposition requires that it is possible to distinguish $\mathcal{V}$ and $\mathcal{E}$, but this might not in fact be possible. For instance, if the ultimate fate of the universe was that it would collapse into a singularity, then there would come a point at which it wouldn't be possible to make a distinction between $\mathcal{V}$

and $\mathcal{E}$. But despite this possible concern, the above definition makes it seem plausible that there could be many well-defined worlds W_i.[72]

When we look at a particular $|\xi_i(t)\rangle$ it will look like it is describing a fairly classical world with scientists performing their measurements and agreeing about what they measure. And as long as the $|\xi_i(t)\rangle$-states remain pointer states with respect to $|E_i(t)\rangle$ no branching will occur. But typically, a $|\xi_i(t)\rangle$-state will not indefinitely remain a pointer state with respect to $|E_i(t)\rangle$. We can think of how this happens with the Stern-Gerlach experiment. For if one Stern-Gerlach apparatus has its magnetic field orientated in the $\hat{\mathbf{a}}$-direction, then $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ will be pointer states for a silver atom in the vicinity of this apparatus. But if the same silver atom then travels onward to another Stern-Gerlach apparatus with its magnetic field now orientated in the $\hat{\mathbf{b}}$-direction, $|\hat{\mathbf{a}}+\rangle$ and $|\hat{\mathbf{a}}-\rangle$ will no longer be pointer states with respect to their environment, and so branching will occur. But this is not necessarily a problem for the definition of many-worlds given above on page 61, for when a $|\xi_i(t)\rangle$-state does not indefinitely remain a pointer state with respect to $|E_i(t)\rangle$, we can just rewrite $|\xi_i(t)\rangle$ as a sum of pointer states $|\xi_{ij}(t)\rangle$ and $|E_i(t)\rangle$ as a sum of their respective environments $|E_{ij}(t)\rangle$, and then $|E_i(t)\rangle$ will be like a ready state for the $|E_{ij}(t)\rangle$. Assuming we can do this so that the $|\xi_i j(t)\rangle$ are orthogonal to the $|\xi_{i'j'}(t)\rangle$ when $i' \neq i$ or $j' \neq j$, then we would still be able to have well-defined worlds according to the definition given above.

[72] Recall that the purpose of this chapter is only to show why physicists might find the many-worlds interpretation of quantum physics attractive. It is not the purpose of this chapter to show that the many-worlds interpretation is without any problems.

manyworldsinterpretation2

1.13 Evaluating the Many-Worlds Interpretation

Given the above account of the many-worlds hypothesis of quantum physics, it does seem understandable why physicists would find it so attractive. Although we can't specify an exact moment at which branching occurs, the idea of branching and of there being many worlds itself is not particularly mysterious. This can all be explained in terms of the dynamics of the system and the environment, and decoherence theory allows us to understand why the interference effects that are the hallmark of quantum physics generally disappear on the macroscopic level.

There are other advantages of the many-worlds hypothesis besides these which we need not discuss here.[73] But for all the advantages of the many-worlds hypothesis, there is one fundamental problem, and that is its patent absurdity. It seems that we should be able to say whether a cat is alive or dead without having to say what state the rest of the universe is in. However, the many-worlds hypothesis suggests that for any subsystem of the universe, we will in general only be able to say what state it is in with respect to the state of the rest of the universe. For example, if the state $\mathcal{S}$ is the system constituting a cat-wise configuration of particles and $\mathcal{E}$ is the rest of the universe, then given that the composite system $\mathcal{U} = \mathcal{S} + \mathcal{E}$ is described by the state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\big(\,|\text{Cat Alive}\rangle_{\mathcal{S}}\,|E_{\text{Cat Alive}}\rangle_{\mathcal{E}} + |\text{Cat Dead}\rangle_{\mathcal{S}}\,|E_{\text{Cat Dead}}\rangle_{\mathcal{E}}\,\big),$$

then we are in no position to make an absolute matter of fact claim about the system $\mathcal{S}$ and say the cat is dead or the cat is alive. Rather we have to say with respect to the environment described by $|E_{\text{Cat Alive}}\rangle_{\mathcal{E}}$, the cat is alive, and with respect to the

[73] More details can be found in Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* and E. Joos et al., *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer Berlin Heidelberg, 2013).
cbx@17

environment $|E_{\text{Cat Dead}}\rangle_{\mathcal{E}}$, the cat is dead. According to the many-worlds hypothesis, the branching into multiple worlds doesn't just occur in rare instances, such as in Schrödinger's cat type experiments. On the contrary, branching is supposed to be happening all the time. Now anyone who has a common sense understanding of science would say that science enables us to understand absolute matters of fact about subsystems of the universe and the principles that govern them. But if there really are no such matters of fact, then we have to abandon this common sense understanding of science. Intuitively, it also seems obvious that I can know I am alive without needing to know the state of the rest of the universe, but the many-worlds hypothesis does not allow me to make this absolute matter of fact claim. So from a common sense point of view, the many-worlds hypothesis really is absurd.

Of course some hypotheses may initially seem absurd, but once the hypothesis has been fully explained, it can appear far more plausible. For instance, time dilation in special relativity might initially sound absurd to some people, but once one has a better grasp of special relativity and is open to the possibility that systems moving close to the speed of light with respect to ourselves might have properties rather different to systems that move with much slower speeds, then special relativity doesn't seem absurd at all.

However, the many-worlds hypothesis as presented here is different in this regard since it is not hypothesizing about some extreme situation. It is hypothesizing about ordinary situations. And one can have a fairly good understanding of the many-worlds hypothesis and still find it absurd. Some people choose to embrace the absurdity and reject common sense. But throughout much of human history, when a hypothesis has entailed an absurd conclusion, reasonable people have usually thought it better to reject the hypothesis rather than embrace the absurdity.

But in rejecting a hypothesis as absurd, it doesn't mean that absolutely everything in the hypothesis needs to be rejected, for a hypothesis might be formulated in terms of sub-hypotheses, some of which might be very plausible, in which case something of the original hypothesis might be salvageable. In the case of the many-worlds hypothesis, I believe it does have something that is salvageable, namely decoherence theory. In the next chapter I will consider Adrian Kent's one-world interpretation of quantum physics in which the basic ideas of decoherence theory remain intact.

Chapter 2

Evaluating Kent's Solution to the Measurement Problem

In the previous chapter, we saw how a denial of the Copenhagen interpretation of quantum mechanics and a denial of hidden variables leads fairly naturally to a so-called Many-Worlds interpretation of quantum mechanics. However, the Many-Worlds interpretation seems to be radically opposed to the view that we can make common sense truth claims about the physical world. A strategy among some philosophers of physics who do not wish to endorse the Many-Worlds interpretation is therefore to reexamine the assumptions that lead to Bell's Inequality. One of these assumptions will have to be discarded since Bell's Inequality is experimentally violated. The false assumption that is used to prove Bell's Inequality is sometimes referred to as **the culprit**. We therefore need to identify the culprit, that is, we need to decide which assumption we should discard while keeping in mind that we wish to maintain a theory that is compatible with the experimental findings of quantum physics and special relativity.

Shimony noticed that there are two key assumptions in the proof of Bell's Inequality that might be identified as the culprit. He refers to one assumption as Outcome Independence (OI), and to the other assumption as Parameter Independence (PI).[1] Shimony argued that if we only denied OI, then the proof of Bell's Inequality would fail to go through. Yet by continuing to assume PI, there is a sense in which special

[1] See A. Shimony, "Events and processes in the quantum world," in *Search for a Naturalistic World View: Volume II: natural science and metaphysics* (1986; Cambridge: Cambridge University Press, 1993), 146–147.

relativity is not obviously violated. Shimony therefore thought that denying OI and assuming PI was sufficient to ensure peaceful coexistence between quantum theory and special relativity. In other words, Shimony thought OI was the culprit.

Butterfield,[2] however, argues that although PI is a necessary assumption if there is to be peaceful coexistence between quantum theory and special relativity, PI together with the denial of OI independence is not sufficient to ensure peaceful coexistence. Butterfield thinks this because Shimony fails to offer an account of what an outcome is. As mentioned in the previous chapter,[3] the important problem of outcomes was highlighted by d'Espagnat. It is this omission on Shimony's part that motivates Butterfield to explore whether Kent's interpretation of quantum physics provides what is lacking in Shimony's account.

In this chapter, I will present and evaluate Kent's interpretation of quantum physics. In order to provide such an evaluation, it will be necessary to show that the predictions of Kent's theory do not contradict the predictions of quantum theory that have been experimentally validated. We also need to show that Kent's theory also possesses the symmetries (known as Lorentz invariance) that belong to a theory that is consistent with Einstein's theory of special relativity. D'Espagnat's problem of outcomes also needs to be addressed, so we will need to consider Kent's interpretation in the light of decoherence theory and how Kent succeeds in giving us a convincing account of what an outcome is.

---

[2]See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem."

[3]See section 1.11.

Shimony's question of identifying the culprit will also take up several sections. For instance, we need to explain what is meant by OI and PI and how these two notions relate to one another. Also, in recent years, Leegwater et al.[4] have proved an important theorem, the so-called 'Collbeck-Renner Theorem' regarding PI which says that if a theory satisfies PI together with what is called a 'no conspiracy' criterion, then this theory is reducible to standard quantum theory without any hidden variables. This suggests that if Kent's interpretation is to offer a satisfactory solution to the measurement problem (that is a solution that addresses d'Espagnat's problem of outcomes in a way that ensures peaceful coexistence between special relativity and quantum physics), then it can't be a hidden-variables theory. We thus need to explain how Kent's model can be an interpretation of quantum reality without it being a hidden-variables theory.

We will first turn our attention to the notions of OI and PI.

[4]See Gijs Leegwater, "An impossibility theorem for parameter independent hidden-variable theories," *Studies in History and Philosophy of Modern Physics* 54 (2016): 18–34, Roger Colbeck and Renato Renner, "No extension of quantum theory can have improved predictive power," *Nature communications* (England) 2, no. 1 (2011): 411–411, Roger Colbeck and Renato Renner, "The completeness of quantum theory for predicting measurement outcomes," 2012, K Landsman, "On the Colbeck–Renner theorem," *Journal of mathematical physics* (United States) 56, no. 12 (2015): 122103, and Klaas Landsman, *Foundations of Quantum Theory : From Classical Concepts to Operator Algebras (Volume 188.0)*, vol. 188, Fundamental Theories of Physics (Cham: Springer Open, 2017).

2.1 Outcome Independence versus Parameter Independence

To explain Shimony's[5] notion of Outcome Independence and Parameter Independence, we suppose we have an experimental setup similar to the experimental setup described in section 1.3 on Bell's Inequality. Thus, we suppose there are two particles labeled q_A, and q_B, and that a measurement can be made on particle q_A at one location (e.g. Alice's laboratory), and a measurement can be made on particle q_B at some other location (e.g. Bob's laboratory). Alice can make a choice of one of n measurements to be made. These are labeled $a_1, \dots, a_n$. For example, a_1 might be a measurement of q_A's spin along the z-axis, whereas a_2 might be the measurement of q_A's spin along an axis that is at a $45°$ angle to the z-axis etc. We use the variable x to denote Alice's choice so that $x = a_i$ for some $i \in \{1, \dots, n\}$. If Alice chooses to make measurement a_i (i.e. $x = a_i$), the measurement outcome is labeled A_i, and this outcome can take values $+1$ or -1. For example, Alice could use the convention in which $+1$ corresponds to a spin up outcome, and -1 corresponds to a spin down outcome. We will use the variable X to denote the measurement outcome Alice obtains, so for example, if Alice chooses to make the a_1 measurement so that $x = a_1$ and obtains the outcome $A_1 = 1$, then $X = 1$. Similarly, we use the notation b_i, y, and B_i, Y to correspond to the measurement choices and measurement outcomes for Bob.

We now suppose that there is a complete state $\lambda \in \Lambda$ describing both q_A and q_B that is independent of Alice and Bob's measurement choices, but that encodes all other features that would influence the corresponding measurement outcomes. Here, the domain Λ of all such complete states will depend on how the two particles are

[5]See Shimony, "Events and processes in the quantum world," 146–147 and Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 7–9.

prepared and the model we are assuming. We also assume that q_A and q_B are initially coupled together in such a way that Alice and Bob would always get opposite results when they made their measurements in the same direction. For instance, for $n = 3$, we might assume a model in which

{bellLambda}

$$\Lambda = \big\{(A_1, A_2, A_3, B_1, B_2, B_3) : A_1,\ A_2,\ A_3 = \pm 1,\ B_i = -A_i\big\}. \tag{2.1}$$

In this case, $\lambda \in \Lambda$ would fully determine Alice and Bob's measurement outcomes. This would be like the model described in the proof of Bell's Inequality with all the states of Λ being described in table 1.1 of section 1.3.

However, in general, we don't insist on such determinism. Rather, we suppose that given a complete state $\lambda \in \Lambda$, and given that Alice makes a measurement choice x and Bob makes a measurement choice y, then there will be a probability $P_{\lambda,x,y}(X, Y)$ representing the probability Alice gets outcome X and Bob gets outcome Y. It is only in deterministic models that $P_{\lambda,x,y}(X, Y)$ will only have values restricted to either 0 or 1. In non-deterministic models, there will have to be some situations when $P_{\lambda,x,y}(X, Y)$ has a value strictly between 0 and 1. For example, if our model was standard quantum mechanics, we could take λ to be the Bell state (1.4). Then it follows from equation (1.5) that as long as Alice's and Bob's measurement choices x and y are in the same direction, then $P_{\lambda,x,y}(1, -1) = 1/2$. Incidentally, we also note that equation (1.5) implies the domain Λ consists of a single state:

{quantumLambda}

$$\Lambda = \left\{ \frac{1}{\sqrt{2}}\big(|\hat{\mathbf{a}}+\rangle_A \ |\hat{\mathbf{a}}-\rangle_B - |\hat{\mathbf{a}}-\rangle_A \ |\hat{\mathbf{a}}+\rangle_B \big) \right\}. \tag{2.2}$$

In both models (2.1) and (2.2), we see that if we define

{PIone}

$$P_{A,\lambda,x,y}(X) = P_{\lambda,x,y}(X,1) + P_{\lambda,x,y}(X,-1), \tag{2.3}$$

{PItwo}

$$P_{B,\lambda,x,y}(Y) = P_{\lambda,x,y}(1,Y) + P_{\lambda,x,y}(-1,Y), \tag{2.4}$$

then $P_{A,\lambda,x,y}(X)$ is independent of Bob's choice of measurement y, and $P_{B,\lambda,x,y}(Y)$ is independent of Alice's choice of measurement x.[6] In other models, however, it's possible that such independence does not hold. So to distinguish between such possibilities, we say a model satisfies **Parameter Independence** (PI) PIdef if and only if $P_{A,\lambda,x,y}(X)$ is independent of y, and $P_{B,\lambda,x,y}(Y)$ is independent of x. In other words, PI holds if and only if (2.3) and (2.4) hold for all λ, x, y, X, and Y.

One model in which PI fails to hold is the **pilot wave interpretation** of quantum mechanics. In this interpretation, it is assumed that at any instant of time t, the particles q_A and q_B will have definite positions $\mathbf{x}_A$ and $\mathbf{x}_B$ and definite momenta $\mathbf{p}_A$

[6]To see that this is true for model (2.1), it is obvious that $P_{A,\lambda,x,y}(X) = 1$ or 0 regardless of what y is. As for model (2.2), it is straightforward to show onehalf that $P_{A,\lambda,x,y}(X) = 1/2$ and $P_{B,\lambda,x,y}(Y) = 1/2$ for any X, Y. E.g. for $x = \hat{\mathbf{a}}$ and $y = \hat{\mathbf{b}}$, by (1.5), we can assume the two particles are in the state

$$|\zeta\rangle = \frac{1}{\sqrt{2}}\big(\,|\hat{\mathbf{b}}+\rangle_A\,|\hat{\mathbf{b}}-\rangle_B - |\hat{\mathbf{b}}-\rangle_A\,|\hat{\mathbf{b}}+\rangle_B\,\big).$$

Since the inner product on the composite system is given by $\langle\xi'|\xi\rangle = \langle\psi'|\psi\rangle_A\,\langle\chi'|\chi\rangle_B$ for $|\xi\rangle = |\psi\rangle_A\,|\chi\rangle_B$ and $|\xi'\rangle = |\psi'\rangle_A\,|\chi'\rangle_B$, it follows that

$${}_A\langle\hat{\mathbf{a}}+|\;{}_B\langle\hat{\mathbf{b}}\pm|\zeta\rangle = \mp\frac{1}{\sqrt{2}}\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}\mp\rangle_A\,.$$

Therefore, by the Born Rule (see page 9)

$$P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) + P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-) = \frac{1}{2}\big|\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}-\rangle_A\big|^2 + \frac{1}{2}\big|\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}+\rangle_A\big|^2.$$

But since

$$|\hat{\mathbf{a}}+\rangle_A = \langle\hat{\mathbf{b}}+|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{b}}+\rangle_A + \langle\hat{\mathbf{b}}-|\hat{\mathbf{a}}+\rangle_A\,|\hat{\mathbf{b}}-\rangle_A$$

it follows that

$$|\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}+\rangle_A|^2 + |\,\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}-\rangle_A|^2 = 1.$$

Therefore,

$$P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{b}}}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}+) + P_{\lambda,x,y}(\hat{\mathbf{a}}+,\hat{\mathbf{b}}-) = \frac{1}{2}.$$

and $\mathbf{p}_B$ respectively. But in addition to the positions and momenta of the particles, it is also assumed that there is a so-called **pilot wave**

$$\psi(\mathbf{x}_A, \mathbf{x}_B, t) = r(\mathbf{x}_A, \mathbf{x}_B, t)e^{iS(\mathbf{x}_A, \mathbf{x}_B, t)} \tag{2.5}$$

where $r(\mathbf{x}_A, \mathbf{x}_B, t) > 0$ is the magnitude of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$, and the real-valued function $S(\mathbf{x}_A, \mathbf{x}_B, t)$ is the complex phase of $\psi(\mathbf{x}_A, \mathbf{x}_B, t)$. The time evolution of the pilot wave is deterministically governed by the Schrödinger equation, and the phase $S(\mathbf{x}_A, \mathbf{x}_B, t)$ relates the positions $\mathbf{x}_A$ and $\mathbf{x}_B$ to the momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ via the gradient of S:

$$\mathbf{p}_A = \boldsymbol{\nabla}_A S(\mathbf{x}_A, \mathbf{x}_B), \qquad \mathbf{p}_B = \boldsymbol{\nabla}_B S(\mathbf{x}_A, \mathbf{x}_B). \tag{2.6}$$

In other words, if we fix $\mathbf{x}_B$ and consider S to be just a function of $\mathbf{x}_A$, then the momentum $\mathbf{p}_A$ is in the direction and has the magnitude of the steepest ascent of S considered as a function of $\mathbf{x}_A$. The momentum $\mathbf{p}_B$ is determined similarly.

In reality, we don't know the exact positions of all the particles, but based on what we know about an experimental setup, we can average over our uncertainty and recover exactly the same predictions that quantum mechanics would make.[7] So for instance, our knowledge of the experimental setup above should enable us to know both that q_A and q_B are contained within a region V, and also for us to work out the probability $p(V_i, V_j)$ that particle q_A will be in a region V_i, and q_B will be in a region V_j, where the V_i are small non-overlapping regions such that $V = \bigcup_i V_i$. If we are interested in some physical quantity $O(\mathbf{x}_A, \mathbf{x}_B)$ that depends on the positions $\mathbf{x}_A$ and $\mathbf{x}_B$ of the two particles, then when the regions V_i are sufficiently small so that $O(\mathbf{x}_i, \mathbf{x}_j)$ varies

[7]See David Bohm, "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I," *Physical review* 85, no. 2 (1952): 166–179 and David Bohm, "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II," *Physical review* 85, no. 2 (1952): 180–193.

negligibly for any $\mathbf{x}_i \in V_i$ and $\mathbf{x}_j \in V_j$, the average value

{bohmconsistency}

$$\langle O \rangle = \sum_{i,j} p(V_i, V_j) O(\mathbf{x}_i, \mathbf{x}_j) \tag{2.7}$$

calculated in the pilot wave model turns out to be the same as the expectation value for O predicted by standard quantum mechanics.[8]

To see why PI fails to hold in the pilot wave model, we first note that since the pilot wave model makes the same predictions as quantum mechanics when averaged over all the hidden variables, the violation of Bell's inequality (1.3) implies there must be some hidden variable λ and choices of measurement directions $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$ such that

{bellinequality2}

$$P_{\lambda,\hat{\mathbf{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) > P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+) + P_{\lambda,\hat{\mathbf{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+). \tag{2.8}$$

Since physics in the pilot wave model is deterministic, probabilities must be either 0 or 1. Therefore, the only way (2.8) can be satisfied is for

{PDproof1}

$$P_{\lambda,\hat{\mathbf{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = 1, \tag{2.9}$$

{PDproof2}

$$P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+) = 0, \tag{2.10}$$

{PDproof3}

$$P_{\lambda,\hat{\mathbf{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+) = 0. \tag{2.11}$$

We suppose that PI holds, and we will try to arrive at a contradiction. If both Alice and Bob make their measurement in the $\hat{\boldsymbol{c}}$-direction, there are two possibilities: either Alice measures q_A to be in the state $\hat{\mathbf{c}}-$ and Bob measures q_B to be in the state $\hat{\mathbf{c}}+$, or Alice measures q_A to be in the state $\hat{\mathbf{c}}+$ and Bob measures q_B to be in the state $\hat{\mathbf{c}}-$. So expressed in terms of probabilities, these two possibilities are equivalent to

---

[8] In this explanation, I've refrained from using measure theory, but basically this explanation is saying that we when we construct a measure μ on $V \times V$ based on our knowledge of the experimental setup, $\int_{V\times V} O(\mathbf{x}_i, \mathbf{x}_j)\mathrm{d}\mu$ will be the same as the expectation value for O predicted by standard quantum physics.

either

{PDproofcase1}

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+)=1 \qquad \text{and} \qquad P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)=0. \tag{2.12}$$

or

{PDproofcase2}

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)=1 \qquad \text{and} \qquad P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+)=0 \tag{2.13}$$

Let's first consider case (2.12). Note that

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}-)=0. \tag{2.14}$$

Therefore, since we are assuming PI,

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}-)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}-;\hat{\mathbf{c}}-)=0. \tag{2.15}$$

In particular,

{PDproof4}

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}-)=0. \tag{2.16}$$

But by (2.9), we know that

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}-)=1, \tag{2.17}$$

so using this together with PI, we must have

{PDproof5}

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}-)=1. \tag{2.18}$$

But by (2.10) and (2.16)

{PDproof6}

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}-)=0. \tag{2.19}$$

Since (2.18) contradicts (2.19), the assumption (2.12) must be false if PI is to hold.

So we now consider the alternative case when (2.13) holds. We will again see that this assumption leads to a contradiction. First note that

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}-)=0. \tag{2.20}$$

By PI

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}-)=0. \tag{2.21}$$

In particular,

{PDproof8}

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}+)=0. \tag{2.22}$$

But by (2.9), we know that

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}-;\hat{\mathbf{b}}+)=1, \tag{2.23}$$

so using this together with PI, we must have

{PDproof7}

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}+)=1. \tag{2.24}$$

But by (2.11) and (2.22)

{PDproof9}

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}+)=0. \tag{2.25}$$

Since (2.24) contradicts (2.25), the assumption (2.13) must also be false if PI is to hold. So we can only conclude that PI fails to hold in the pilot wave model. But we can conclude even more than that: any deterministic hidden variable model that gives the same predictions as quantum mechanics when averaged over the hidden variables must violate PI. PIdeterminism

Now the violation of PI in the pilot wave model does not sit easily with Einstein's theory of relativity, for according to Einstein's theory, it should be impossible to send signals faster than the speed of light. However, if PI is violated, then if Alice happened to know what λ was for each run of the experiment, and if Bob made the same measurement, then because the distribution of Alice's outcomes will depend on Bob's choice of measurement, with enough runs of the experiment, Alice should be able to work out what measurement Bob is making. And this should be possible even if Alice and Bob are separated by many light years. So it seems faster than light communication would be possible. The only thing preventing such communication would be Alice's lack of knowledge of λ.

But although a PI violation can account for the violation of Bell's Inequality, this is not the only possible culprit to consider. Another assumption of Bell's Inequality that might be violated is **Outcome Independence** (OI). Outcome independence is the assumption

{OI}

$$P_{\lambda,x,y}(X,Y) = P_{A,\lambda,x,y}(X) \cdot P_{B,\lambda,x,y}(Y), \tag{2.26}$$

We can see that if OI holds in any model which gives the same predictions as standard quantum theory when averaged over the hidden variables, then PI must be violated in such a model. For if both PI and OI hold, then for any measurement choices $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and hidden variable λ, we have

$$\begin{aligned}
P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+) &= P_{A,\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+)\cdot P_{B,\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{a}}}(\hat{\mathbf{c}}+) \\
&= \Big(P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}-)\Big)\cdot\Big(P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}-;\hat{\mathbf{c}}+)\Big) \\
&= \Big(P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}-)\Big)\cdot\Big(\cancelto{0}{P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}+)}+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+)\Big) \\
&= \Big(P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}-)\Big)\cdot P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+) \\
&\geq P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)\cdot P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+)
\end{aligned}$$

{OIPI1} (2.27)

Similarly, we have

$$\begin{aligned}
P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+) &= P_{A,\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+)\cdot P_{B,\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{b}}+) \\
&= \Big(P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}-)\Big)\cdot\Big(P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}+)\Big) \\
&= \Big(\cancelto{0}{P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}+)}+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)\Big)\cdot\Big(P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}-;\hat{\mathbf{b}}+)\Big) \\
&= P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)\cdot\Big(P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)+P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}-;\hat{\mathbf{b}}+)\Big) \\
&\geq P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)\cdot P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+).
\end{aligned}$$

{OIPI2} (2.28)

But since the hidden variable λ is assumed to be independent of Alice and Bob's measurement, and since Alice and Bob will always get opposite results when they make the same choice of measurement, it follows that

{OIPI3}

$$P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}+;\hat{\mathbf{c}}-)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{c}}}(\hat{\mathbf{c}}-;\hat{\mathbf{c}}+)=1 \tag{2.29}$$

Therefore, putting (2.27), (2.28), and (2.29) together, we have

$$P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)+P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)\geq P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+). \tag{2.30}$$

We have thus proved that OI and PI implies Bell's Inequality (2.8). But since Bell's Inequality does not hold in reality, it follows that if OI is always true, then PI must be violated. OIPIproofend OIdet

In the case of deterministic models, OI necessarily holds. To see why, we first note that for deterministic models, either $P_{\lambda,x,y}(X,Y) = 1$ or $P_{\lambda,x,y}(X,Y) = 0$. When $P_{\lambda,x,y}(X,Y) = 1$, then by (2.3), $P_{A,\lambda,x,y}(X) = 1$, and by (2.4), $P_{B,\lambda,x,y}(Y) = 1$, so (2.26) is seen to hold in this case. On the other hand, when $P_{\lambda,x,y}(X,Y) = 0$, if $P_{A,\lambda,x,y}(X) = 1$, then by (2.3), $P_{\lambda,x,y}(X,-Y) = 1$ so that by (2.4), $P_{B,\lambda,x,y}(Y) = 0$ in which case (2.26) holds. And similarly, if $P_{B,\lambda,x,y}(Y) = 1$, by (2.4), $P_{\lambda,x,y}(-X,Y) = 1$ so that by (2.3), $P_{A,\lambda,x,y}(X) = 0$, so again (2.26) holds. And (2.26) obviously holds when $P_{A,\lambda,x,y}(X) = P_{B,\lambda,x,y}(Y) = 0$. It therefore follows that OI holds in any deterministic model.

When it comes to standard quantum mechanics, however, OI fails to hold. For instance, if $x = y = \hat{\mathbf{a}}$, then $P_{\lambda,\hat{\mathbf{a}},\hat{\mathbf{a}}}(\hat{\mathbf{a}}+,\hat{\mathbf{a}}+) = 0$, but $P_{A,\lambda,\hat{\mathbf{a}},\hat{\mathbf{a}}}(\hat{\mathbf{a}}+) = P_{B,\lambda,\hat{\mathbf{a}},\hat{\mathbf{a}}}(\hat{\mathbf{a}}+) = 1/2$.[9] Hence, OI fails. Nevertheless, as long as PI holds, the failure of OI does not enable Bob to send messages to Alice faster than light because Bob only has control over the measurement he makes. Assuming Bob's mental states have no effect on the measurement outcome, there is nothing he can do to influence his outcome, so although Alice will be able to work out Bob's measurement outcome if she already happens to know which choice of measurement he has made, she will not be able to work out which measurement Bob makes (or even whether Bob has made a measurement at all) by measuring the

[9]See footnote 6.

outcome of her particle. For Shimony[10] this inability to send super-luminal messages between Alice and Bob when PI holds and OI is violated was deemed sufficient for the theories of standard quantum physics and special relativity to peacefully coexist.

However, Butterfield is not satisfied with Shimony's solution to peaceful coexistence.[11] Firstly, he notes that proofs of non-super-luminal signaling[12] make no assumptions about spacetime locations. One would have thought that any proof that super-luminal signalling between two points is impossible would have to show that a signal cannot be transmitted from one point to the other in less time than the time it takes light to travel between the two points. But if nothing is said about the location of these two points or what is so special about the speed of light compared to the speed of any other particle, then there does not seem to be enough information in the premises to draw the desired conclusion that super-luminal signaling is impossible in quantum physics.

Secondly, Butterfield notes that Shimony thinks peaceful coexistence of quantum physics and special relativity is guaranteed by the denial of OI and the acceptance of PI, but OI itself depends on the (often) rather vague notion of what an outcome really is. For instance, in the Many-Worlds interpretation described in sections 1.12 and 1.13, it is not clear that there are any outcomes at all. Rather, there is just a universal wave function that tells us the probability of certain outcomes, if there were

---

[10]See Shimony, "Events and processes in the quantum world," 146–147.

[11]See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," p. 12.

[12]e.g. see cbx@31 Michael Redhead, *Incompleteness, nonlocality, and realism : a prolegomenon to the philosophy of quantum mechanics* (Oxford : New York: Clarendon Press ; Oxford University Press, 1987), p. 113–116; cbx@32 David Bohm and B. J Hiley, *The undivided universe : an ontological interpretation of quantum theory* (London: Routledge, 1993), p. 139–140

such things as outcomes – it doesn't tell us that there really are any outcomes. But as we saw in the previous chapter, the Many-Worlds interpretation does flow rather naturally from the postulates of standard quantum theory.

Still, the notion of what an outcome is doesn't have to be vague. In the pilot wave interpretation of quantum physics, it is very clear what an outcome of an experiment is since all the particles have definite positions and momenta. Because of this, the pointers and displays of measuring devices which are made up of particles will have definite readouts which will correspond to the definite positions of particles being measured (assuming the measurement device is working properly). So unlike the Many-Worlds interpretation, measurements in the pilot wave interpretation have definite outcomes, and hence there is only a single world in the pilot wave interpretation of quantum physics. But as we've just seen, the problem with the pilot wave interpretation is the violation of PI.

Thus, a satisfactory account of the peaceful coexistence of quantum physics and special relativity requires an interpretation of quantum physics in which not only PI holds, but also an interpretation of quantum physics that has special relativity built into it (thus satisfying Butterfield's first objection), and in which we can make sense of what it means to be an outcome (thus satisfying Butterfield's second objection). To fully address Butterfield's first objection would require quantum field theory, and this would be beyond the scope of this dissertation. But a more modest aspiration that would go some way to address Butterfield's first objection would be to insist on an interpretation of quantum physics that has a property known as Lorentz invariance.

This provides a motivation for the consideration of Kent's interpretation of quantum physics that has this property of Lorentz invariance.

2.2 A description of Kent's Interpretation of Quantum Physics

In this section I will provide an account of Kent's interpretation of quantum physics focusing on the ideas Kent presents in his 2014 paper.[13] This section is primarily descriptive. We'll wait until the next section to consider how Kent's interpretation addresses the issues Butterfield raises.

Kent's interpretation of quantum physics has some similarities in common with the pilot wave interpretation. Firstly, there is no wave-function collapse in Kent's interpretation. Secondly, some additional values beyond standard quantum theory (i.e. in addition to the quantum wave function) are included in Kent's interpretation. And thirdly, Kent's interpretation is a one-world interpretation of quantum physics. I'll consider these three features of Kent's interpretation in some detail as I describe his theory. I'll then present an account of his toy model that provides a simple example of how the ideas of his theory fit together.

2.2.1 The No-collapse Feature of Kent's Interpretation

We first consider the no-collapse feature of Kent's interpretation. This is a feature that belongs both to the Many-World's interpretation and to the pilot wave interpretation. In all three interpretations, the wave-function deterministically evolves according to the Schrödinger equation. The Schrödinger equation itself describes how a quantum state evolves over time. The precise formula for the Schrödinger equation need not concern us here, but all we need to know is that the Schrödinger equation determines a so-called **unitary operator** $U(t', t)$. What this means is that if a system is in a

[13] cbx@33 Adrian Kent, "Lorentzian Quantum Reality: Postulates and Toy Models," 2014, https://doi.org/10.1098/rsta.2014.0241, eprint: arXiv:1411.2957.

state $|\psi\rangle$ at time t, then it will be in the state $|\psi'\rangle = U(t', t)\,|\psi\rangle$ at time t'. A unitary operator U has the property that if $|\psi'\rangle = U\,|\psi\rangle$ and $|\chi'\rangle = U\,|\chi\rangle$, then

{unitarycond}

$$\langle\chi'|\psi'\rangle = \langle\chi|\psi\rangle\,. \tag{2.31}$$

[14]

Under the Copenhagen interpretation, the state will evolve unitarily for the most part, but there will typically be a non-unitary change in the state whenever there is a measurement. To see why this is, we recall the situation described on page 27 where a state $|\psi\rangle$ is the sum of orthonormal eigenstates $|\psi_i\rangle$ of some observable:

$$|\psi\rangle = \sum_{i=1}^{N} \alpha_i\,|\psi_i\rangle\,. \tag{1.9 revisited}$$

If $|\psi_i'\rangle = U(t, t_0)\,|\psi_i\rangle$ and $|\psi'\rangle = U\,|\psi\rangle$, then the unitary condition (2.31) implies that $\langle\psi_i'|\psi'\rangle = \alpha_i$. But if on measurement at time t, the system collapses to the $\left|\psi_j'\right\rangle$ state for $j \neq i$ so that $|\psi'\rangle = \left|\psi_j'\right\rangle$, we will have $\langle\psi_i'|\psi'\rangle = 0$. So in the Copenhagen interpretation, the unitary condition (2.31) will fail if $\alpha_i \neq 0$.

However, in non-collapse models such as the pilot wave interpretation, the Many-Worlds interpretation, and Kent's interpretation, the wave function always evolves unitarily.

2.2.2 The Additional Values of Kent's Interpretation

additional

Secondly, like the pilot wave interpretation, some additional values beyond standard quantum theory (i.e. in addition to the quantum wave function) are included in Kent's interpretation. In the pilot wave interpretation, these additional values are the positions and momenta of all the particles, whereas in Kent's interpretation, the

[14] A unitary operator also has the property that it is invertible: there is an operator U^{-1} such that UU^{-1} and $U^{-1}U$ are the identity operator I, i.e. $U^{-1}U\,|\psi\rangle = UU^{-1}\,|\psi\rangle = |\psi\rangle$ for any state $|\psi\rangle$.

additional values specify the mass-energy density on a three-dimensional distant future hypersurface in spacetime. We refer to this hypersurface as S.

To understand the nature of this three-dimensional hyperspace S, we recall that in special relativity, there is no such thing as absolute time. So for instance, two spacetime locations might seem to be simultaneous from one frame of reference, but another person travelling at a different velocity would judge the same two spacetime locations to be non-simultaneous. But it is not the case that for any two spacetime locations we can always find a frame of reference in which the two spacetime locations are simultaneous. Sometimes this is not possible. But we refer to spacetime locations that could be simultaneous in some frame of references as being **spacelike-separated**. For example, the two spacetime locations O and A in figure 2.1 are spacelike-separated.

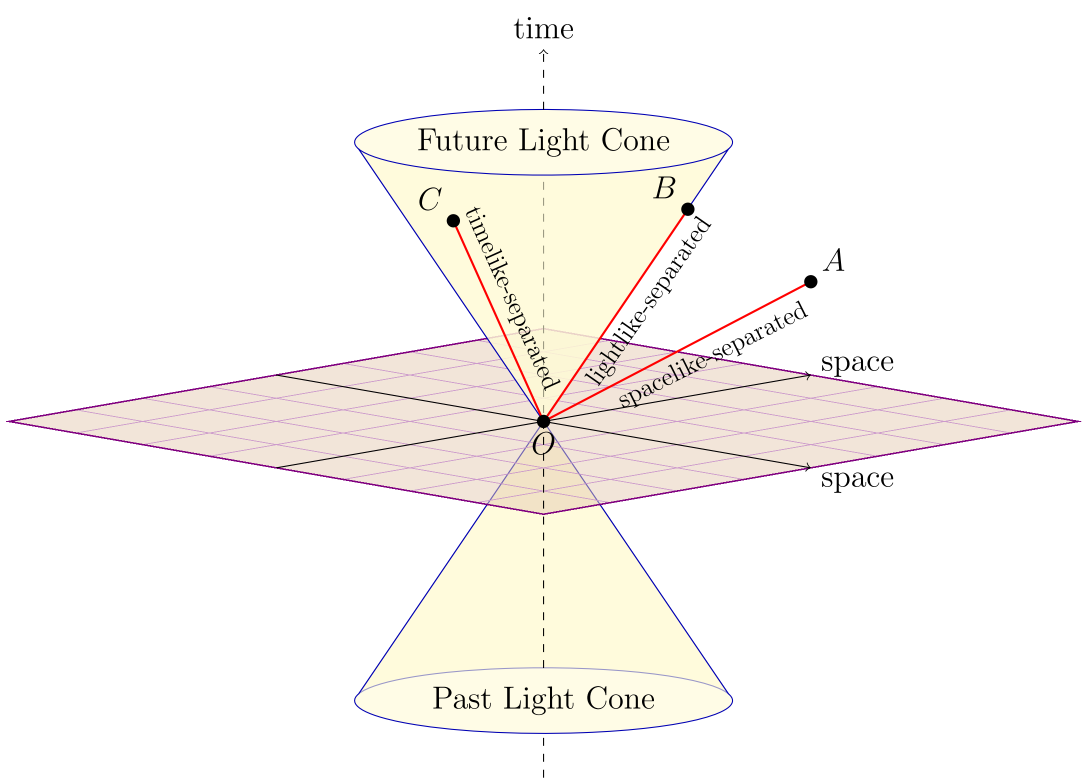


Figure 2.1: The meaning of spacelike, timelike and lightlike-separation when there are two space dimensions and one time dimension.

`cone`

There are also spacetime locations in spacetime that could be connected by a beam of light such as the two spacetime locations O and B in figure 2.1. Such spacetime locations are referred to as being **lightlike-separated**. For any given spacetime location, the spacetime locations that are lightlike-separated from it form two cones called the future light cone and the past light cone as shown in figure 2.1. Because light appears to travel at the same speed no matter what frame of reference one uses, the light cone of a spacetime location remains invariant when one change from one reference frame to another. In other words, if another spacetime location lies on the light cone of a spacetime location in one frame of reference, then it must lie on the light cone of this spacetime location in every frame of reference.

Figure 2.1 also depicts two spacetime locations O and C that are **timelike-separated**. Such spacetime locations lie within the light cones of each other, and when two spacetime locations are timelike-separated, it is always possible to choose a frame of reference in which the two spacetime locations are located at the same point in space, but with one spacetime location occuring after the other depending on which spacetime location is in the future light cone of the other.

Now a three-dimensional hypersurface S in spacetime is a three-dimensional surface in which all the spacetime locations of S are spacelike-separated. Kent assumes that this hypersurface S is in the distant future of an expanding universe so that nearly all the particles that can decay have already done so, and that all the particles that are not bound together are very far from each other so that the probability of any particle collisions is very small. In other words, all the interesting physics in the universe has played its course before S.

At every point of $x \in S$, there is a quantity $T_S(x)$ called the **mass-energy density**.[15] The important thing to note about $T_S(x)$ is that it does not depend on which frame of reference one is in.[16] This property is in contrast to many physical properties that do depend on which frame of reference one is in. For example, the kinetic energy of an object will depend on the calculated velocity of the object, and this velocity will in turn depend on the frame of reference in which this calculation is done.

Now according to the Tomonaga-Schwinger formulation of relativistic quantum physics,[17] for any hypersurface S, there is a Hilbert space H_S of states describing S. One of the properties of H_S is that for any spacetime location $x \in S$, there is an observable $\hat{T}_S(x)$ acting on H_S such that if $|\Psi\rangle \in H_S$ is an eigenstate of $\hat{T}_S(x)$ with eigenvalue $\tau(x)$, then $|\Psi\rangle$ corresponds to a state of S in which the energy-density at x is $\tau(x)$. This can be done in such a way that $\hat{T}_S(x)$ only depends on x rather than on the hypersurface S that contains x. Furthermore, if x and y are any two spacetime locations of S, then the observables $\hat{T}_S(x)$ and $\hat{T}_S(y)$ commute. In other words,

$$\hat{T}_S(x)\hat{T}_S(y) = \hat{T}_S(y)\hat{T}_S(x).$$

The commutativity of all the $\hat{T}_S(x)$ for $x \in S$ means that if $|\Psi\rangle$ is an eigenstate of $\hat{T}_S(x)$, then for any $y \in S$, $\hat{T}_S(y)\,|\Psi\rangle$ is also an eigenstate of $\hat{T}_S(x)$ with the same

[15] The definition of $T_S(x)$ will be discussed in section 2.2.2.

[16] The reason for why this is will be discussed in section 2.2.2.

[17] See Julian Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation," *Physical review* 74, no. 10 (1948): 1439–1461; S. Tomonaga, "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields," *Progress of theoretical physics* (Tokyo) 1, no. 2 (1946): 27–42

eigenvalue as $|\Psi\rangle$. The invariance of the $\hat{T}_S(x)$-eigenspace under the action of $\hat{T}_S(y)$ means that we can create simultaneous eigenstates for both $\hat{T}_S(x)$ and $\hat{T}_S(y)$, albeit with different eigenvalues. But because x and y are arbitrary points of S, this means that we can express any state H_S as a superposition of simultaneous $\hat{T}_S$-eigenstates of the form $\left|\Psi^{(i)}\right\rangle$ where $\hat{T}_S(x)\left|\Psi^{(i)}\right\rangle = \tau_S^{(i)}(x)\left|\Psi^{(i)}\right\rangle$ for all $x \in S$, where $\tau_S^{(i)}(x) \geq 0$ is a possible energy-density measurement over the whole of S.[18] simultaneous

The additional values beyond standard quantum theory that are included in Kent's interpretation are given by one of these possible outcomes for an energy-density measurement over the whole of S. We will denote this outcome as $\tau_S(x)$, and we will let $|\Psi\rangle$ be the state such that $\hat{T}_S(x)|\Psi\rangle = \tau_S(x)|\Psi\rangle$ for all $x \in S$. But although we speak of the measurement of $T_S(x)$ over S as being $\tau_S(x)$, this is only a notional measurement. Thus, we speak of the measurement of $T_S(x)$ on S only to mean that $T_S(x)$ has a determinate value on S despite the quantum state of S given by Schrödinger's equation in general being in a superposition of simultaneous $\hat{T}_S(x)$-eigenstate for every $x \in S$. How this determination of $T_S(x)$ comes about is up to one's philosophical preferences. For example, one could suppose that it was simply by divine fiat that this determination of $T_S(x)$ came about.[19]

Nevertheless, the particular density $\tau_S(x)$ which is found to describe S can't be absolutely anything. Rather, we suppose there is a much earlier hypersurface S_0 which is described by a state $|\Psi_0\rangle$ belonging to a Hilbert space H_{S_0} as shown in figure 2.2.

---

[18] We will gloss over glosssim the details of how to make this rigorous for continuous variables x and continuous indices i. It is sufficient to approximate the continuous variables and indices as discrete variables and indices when thinking about the simultaneous $\hat{T}_S(x)$-eigenspaces, and one can choose the granularity of this approximation to achieve whatever level of accuracy one desires.

[19] I will discuss my philosophical preference in the final chapter.

It is assumed that all physics that we wish to describe takes place between these two hypersurfaces S_0 and S. In figure 2.2, we therefore let y depicts a generic spacetime location that we wish to describe.

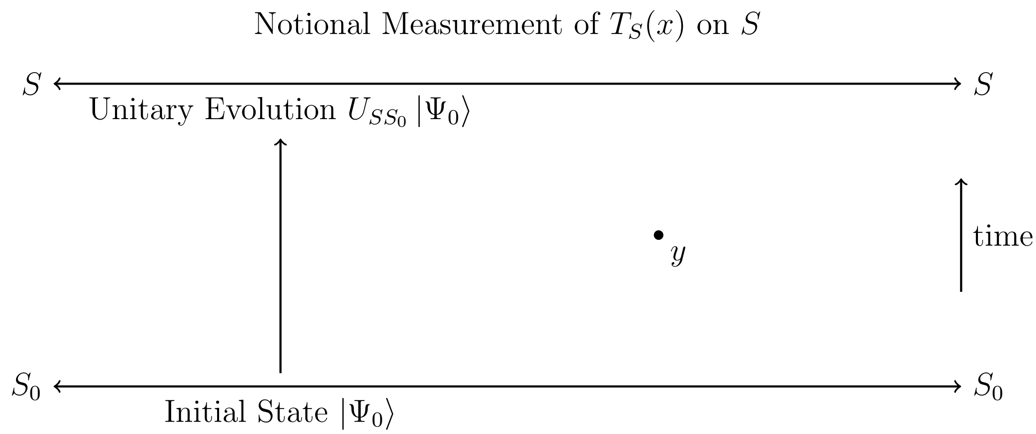


Figure 2.2: A notional measurement of $T_S(x)$ is made for all $x \in S$. The simultaneous $\hat{T}_S$-eigenstate $|\Psi\rangle$ with $\hat{T}_S(x)\,|\Psi\rangle = \tau_S(x)\,|\Psi\rangle$ is selected with probability $|\langle\Psi|U_{SS_0}|\Psi_0\rangle|^2$. The values $\tau_S(x)$ obtained for $T_S(x)$ are then used to calculate the physical properties at the spacetime location y.

S1

According to Schwinger,[20] there is a unitary operator SchwingerUnitaryOP U_{SS_0} that maps states in H_{S_0} such as $|\Psi_0\rangle$ to states in H_S. Then the probability $P(\Psi|\Psi_0)$ that S will be found to be in the state $|\Psi\rangle$ with mass-energy density $\tau_S(x)$ given that S_0 was initially in the state $|\Psi_0\rangle$ will be given by the Born rule (see page 9):

{bornpsi}

$$P(\Psi|\Psi_0) = |\langle\Psi|U_{SS_0}|\Psi_0\rangle|^2. \tag{2.32}$$

It's possible that there might be several different states of H_S that have the same mass-energy density $\tau_S(x)$, but one of these states is still realized with probability given by equation (2.32). But it is $\tau_S(x)$ rather than one of the eigenstates with mass-energy density $\tau_S(x)$ that constitute the additional values that Kent adds to standard quantum theory.

---

[20]Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation," p.1459

2.2.3 The One-World Feature of Kent's Interpretation

The third similarity Kent's interpretation shares with the pilot wave interpretation is that it is a one-world interpretation of quantum physics. It will be helpful to contrast this with the Many-Worlds interpretation.

Unlike the Many-Worlds interpretation, there are no superpositions of living and dead cats in Kent's interpretation. Recall that in the Many-Worlds interpretation, Schrödinger will still only observes his cat to be either dead or alive, and not both dead and alive, but Schrödinger himself goes into a superposition of observing his cat to be alive and his cat to be dead. In the Many-Worlds interpretation, there is thus a difference between observing something to be so, and something actually being so: the observation is of a particular outcome, but the reality is a superposition of different outcomes.

To capture this distinction between observation and reality, Bell speaks of **beables**. Bell introduces the term beable when speculating on what would be a more satisfactory physical theory than quantum physics currently has to offer.[21] Bell says that such a theory should be able to say of a system not only that such and such is observed to be so, but that such and such be so. In other words, a more satisfactory theory would be a theory of beables rather than a theory of observables. On the macroscopic level, these beables should be the underlying reality that gives rise to all the familiar things in the world around us, things like cats, laboratories, procedures, and so on. For example proponents of the pilot wave interpretation believe that the beables are all

cbx@37
[21]See J. S. Bell, "Subject and Object," in *Speakable and unspeakable in quantum mechanics*, 2nd ed. (Cambridge: Cambridge University Press, 2004), 40–44

the particles with their precise position and momentum. But whatever these beables are, it is because of them that a scientist can observe a physical system to be in such and such a state. Thus, observables are ontologically dependent on beables.

Now the beables in Kent's one world interpretation are expressed in terms of a physical quantity called the **stress-energy tensor** $T^{\mu\nu}(y)$. stressenergy For any spacetime location y, the stress-energy tensor $T^{\mu\nu}(y)$ is an array of 16 values corresponding to each combination of $\mu, \nu = 0, 1, 2,$ or 3. The value $T^{00}(y)$ is the energy density at y divided by c^2,[22] whereas the other values of $T^{\mu\nu}(y)$ indicate how much energy and momentum flow across different surfaces in the neighborhood of y.

It was mentioned in the previous section that for any spacetime location $x \in S$, there is an observable $\hat{T}_S(x)$ acting on H_S. It turns out that for any $\mu, \nu = 0, 1, 2,$ or 3, there is also an observable $\hat{T}^{\mu\nu}(x)$ acting on H_S, such that if $|\Psi\rangle \in H_S$ is a simultaneous eigenstate of $\hat{T}^{\mu\nu}(x)$ with eigenvalue $\tau^{\mu\nu}(x)$ for all $x \in S$, then $|\Psi\rangle$ corresponds to a state of S in which $T^{\mu\nu}(x)$ is $\tau^{\mu\nu}(x)$ for all $x \in S$.[23] Moreover, the observable $\hat{T}_S(x)$ is expressible in terms of the $\hat{T}^{\mu\nu}(x)$-observables.[24]

Now the beables in Kent's interpretation are defined at each spacetime location y that occurs after S_0 and before S. For such a spacetime location y, the beables will be determinate values of the stress-energy tensor $T^{\mu\nu}(y)$, but calculated from the

---

[22] This is not to be confused with the mass-energy density $T_S(x)$ defined for x on a hypersurface S. As will be shown in section 2.3.1, all 16 elements of $T^{\mu\nu}(x)$ will typically be needed to calculate $T_S(x)$.

[23] Note however, that such a simultaneous eigenstate is only for a fixed choice of μ and ν, since in general, $\hat{T}^{\mu\nu}(x)$ and $\hat{T}^{\mu'\nu'}(x)$ will not commute for $\mu \neq \mu'$ or $\nu \neq \nu'$.

[24] See section 2.3.1 for an explanation for why this is so.

expectation of the observable $\hat{T}^{\mu\nu}(y)$ conditional on the energy-density on S being given by $\tau_S(x)$ for all x outside the light cone of y.

To explain what this means in more detail, recall the definition of expectation in equation (1.7) and the expectation formula (1.8) for an observable. If the beable in question was simply the expectation of $\hat{T}^{\mu\nu}(y)$ without conditioning on the value of the energy-density on S, then the $T^{\mu\nu}(y)$-beable would just be $\langle\Psi'|\hat{T}^{\mu\nu}(y)|\Psi'\rangle$ where $|\Psi'\rangle = U_{S'S_0}|\Psi_0\rangle$ for any hypersurface S' that goes through y.[25] However, such a beable would give a description of reality that was very different from what we observe – for instance, in a Schrödinger cat-like experiment, there would be energy-densities corresponding to both the cat being alive and the cat being dead in the same world. To overcome this defect, information about the mass-energy density on S is required, specifically the values of $\tau_S(x)$ for $x \in S^1(y)$ where $S^1(y)$ is defined to consist of all the spacetime locations of S outside the light cone of y as depicted in figure 2.3.

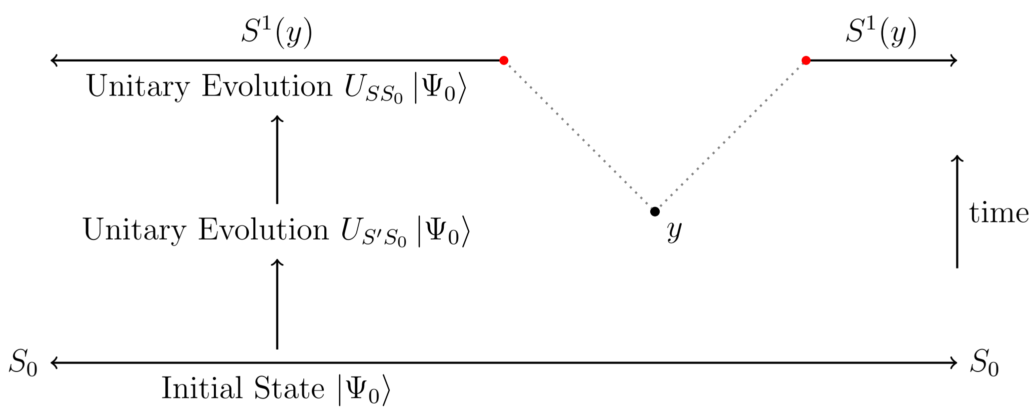


Figure 2.3: The set $S^1(y)$ consists of all the spacetime locations of S outside the light cone of y. The $T^{\mu\nu}(y)$-beables are calculated using the initial state $|\Psi_0\rangle$ together with the values of $\tau_S(x)$ for $x \in S^1(y)$.

S2

[25]This can be done such that $\langle\Psi'|\hat{T}^{\mu\nu}(y)|\Psi'\rangle$ does not depend on the hypersurface S' other than the fact that it contains y. For more details see Schwinger, "Quantum Electrodynamics. I. A Covariant Formulation."

The conditional expectation that we need to calculate depends on the notion of **conditional probability**. In probability theory, the conditional probability $P(q|r)$ that a statement q is true given that a statement r is true is given by the formula

{conditionalprobability}
$$P(q|r) = \frac{P(q\&r)}{P(r)}. \tag{2.33}$$

If we now define $q(\kappa)$ to be the statement that some quantity K takes the value κ, then the **conditional expectation** of K given r will be given by the formula

{conditionalexpectation}
$$\langle K\rangle_r \stackrel{\text{def}}{=} \sum_\kappa \frac{P(q(\kappa), r)\kappa}{P(r)} \tag{2.34}$$

where the summation is over all the possible values κ that K can take.

Kentbeable

If we let $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ stand for Kent's $T^{\mu\nu}(y)$-beable, then this can be calculated from (2.34) by taking r to be the statement that $T_S(x) = \tau_S(x)$ for all $x \in S^1(y)$, and $q(\tau)$ to be the statement that the universe is found to be in a quantum eigenstate of the observable $\hat{T}^{\mu\nu}(y)$ with eigenvalue τ. It is these $T^{\mu\nu}(y)$-beables that give a one-world picture of reality in Kent's interpretation.

toysection

2.2.4 Kent's toy example

To get a feel for how all the elements of Kent's interpretation fit together, it is helpful to consider Kent's toy model example that he discusses in his 2014 paper.[26] In his toy model, Kent considers a system in one spatial dimension which is the superposition of two localized states $\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$ where ψ_1^{sys} is localized at spatial location z_1, ψ_2^{sys} is localized at spatial location z_2, and $|c_1|^2 + |c_2|^2 = 1$. According to the Copenhagen interpretation, a measurement on this system would collapse the wave function of ψ_0^{sys} to the wave function of ψ_1^{sys} with probability $|c_1|^2$, and to the wave

[26]See Kent, "Lorentzian Quantum Reality: Postulates and Toy Models," p.3–4.

function of ψ_2^{sys} with probability $|c_2|^2$. The purpose of Kent's toy model is to show that within his interpretation, there is something analogous to wave function collapse. In order for this "collapse" to happen, one needs to consider how the system interacts with light. Thus, Kent supposes that a photon (which is modelled as a point particle) comes in from the left, and as it interacts with the two states ψ_1^{sys} and ψ_2^{sys}, the photon enters into a superposition of states, corresponding to whether the photon reflects off the localized ψ_1^{sys}-state at time t_1 or the localized ψ_2^{sys}-state at time t_2. The photon in superposition then travels to the left and eventually reaches the one dimensional hypersurface S at locations γ_1 and γ_2 as shown in figure 2.4.

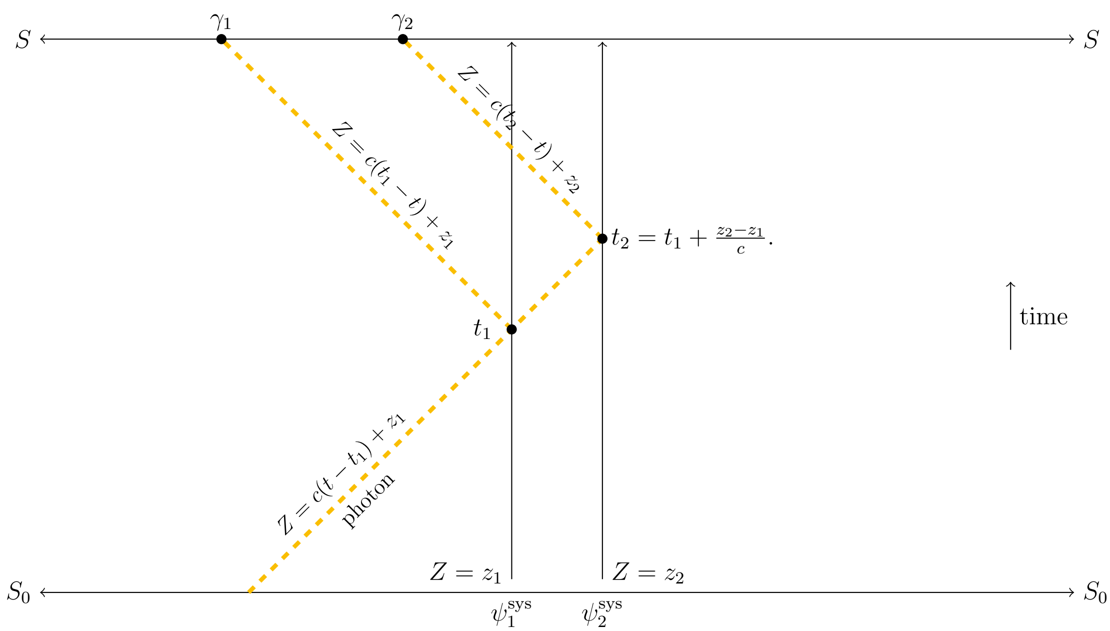
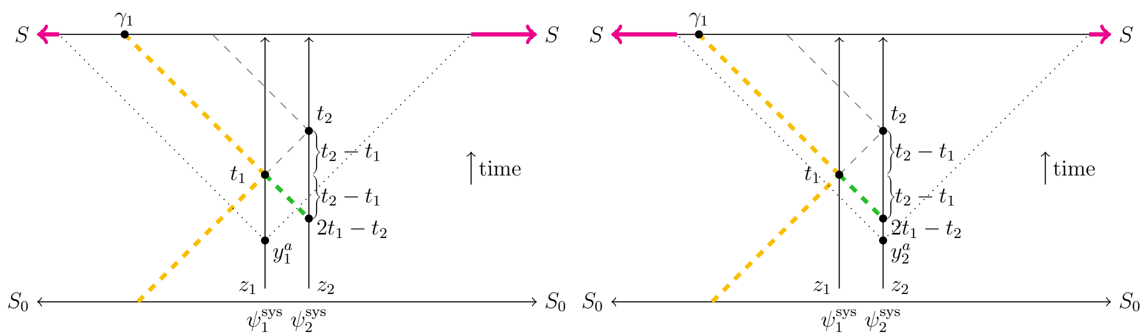


Figure 2.4: Kent's toy model

TM1

We now suppose that when the mass-energy density S is "measured", the energy of the photon is found to be at γ_1 rather than at γ_2. We then consider the mass-density at early spacetime locations $y_1^a = (z_1, t_a)$ and $y_2^a = (z_2, t_a)$ as show in figure 2.5 (a) and (b).



(a) (b)

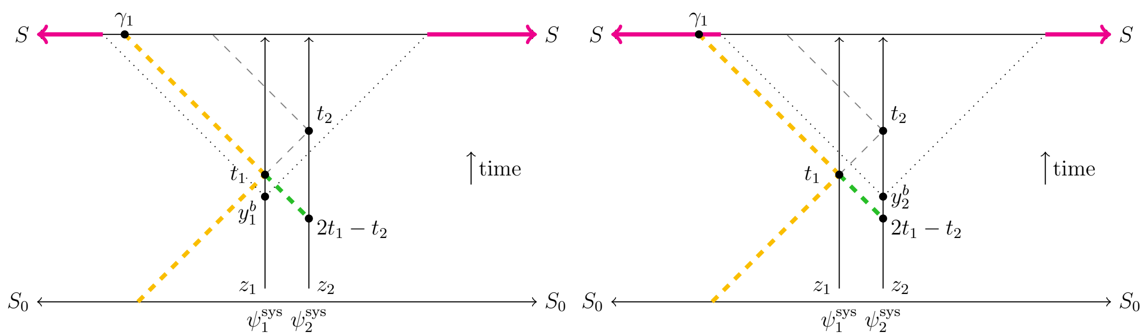
Figure 2.5: (a) highlights the part of S used to calculate the energy density at y_1^a whose time is less than $2t_1 - t_2$. (b) highlights the part of S used to calculate the energy density at y_2^a whose time is less than $2t_1 - t_2$.

TM2

By early, we mean that $t_a < 2t_1 - t_2$. This will mean that the possible detection locations γ_1 and γ_2 will be outside the forward light cones of y_1^a and y_2^a. Hence, $S^1(y_1^a) \cap S$ and $S^1(y_2^a) \cap S$ contain no additional information beyond standard quantum theory by which we could calculate the conditional expectation values of the energy at y_1^a and y_2^a. Hence, according to Kent's interpretation, the total energy at time t_a will be divided between the two spatial locations with a proportion of $|c_1|^2$ at z_1 and a proportion of $|c_2|^2$ at z_2.

However, the situation is different for two spacetime locations $y_1^b = (z_1, t_b)$ and $y_2^b = (z_2, t_b)$ with t_b slightly after $2t_1 - t_2$ as depicted in figure 2.6.

In this situation, when we consider the location y_1^b, there is no additional information in $S^1(y_1^b) \cap S$ beyond standard quantum theory, so there will be a proportion of $|c_1|^2$ of the total initial energy of the system at y_1^b. But at location y_2^b, the information in $S^1(y_2^b) \cap S$ shows that the photon has reflected from the localized ψ_1^{sys}-state, and so this additional information tells us that after time t_b, there is no energy localized at z_2 since from the perspective of y_2^b, the energy is known to be localized at z_1. So it



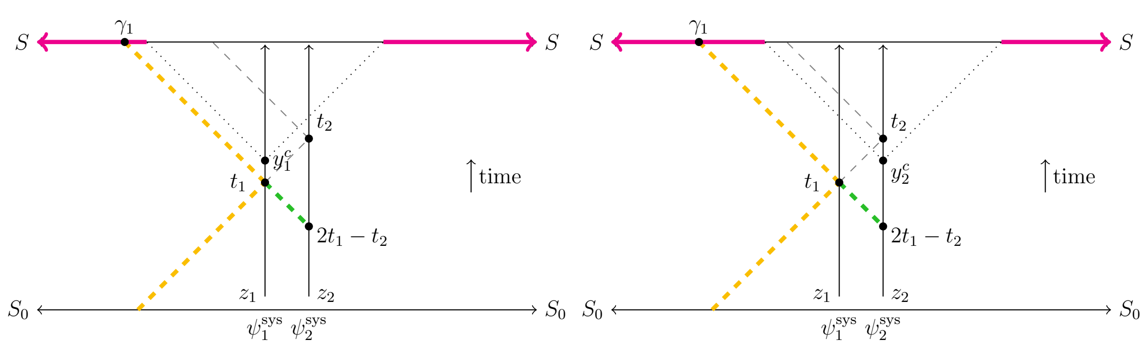
(a) (b)

Figure 2.6: (a) highlights the part of S used to calculate the energy density at y_1^b whose time is greater than $2t_1 - t_2$. (b) highlights the part of S used to calculate the energy density at y_2^b whose time is greater than $2t_1 - t_2$.

TM3

is as though the information of $S^1(y_2^b) \cap S$ has determined that we are in a world in which there is an energy density of zero at y_2^b, and there are no other worlds in which the energy density at y_2^b is non-zero since all worlds have to be consistent with the notional measurement made on S. So for a short time the total energy of the system is reduced by a factor of $|c_1|^2$.

However, as shown in figure 2.7, for times t_c greater than t_1, the total energy of the system is once again restored to the initial energy the system had when in the state ψ_0^{sys}.



(a) (b)

Figure 2.7: (a) highlights the part of S used to calculate the energy density at y_1^c whose time is greater than t_1. (b) highlights the part of S used to calculate the energy density at y_2^c whose time is greater than t_1.

TM4

In this situation, there is now information in $S^1(y_1^c) \cap S$ that determines that the photon reflected off the localized ψ_1^{sys}-state. This means that when the conditional expectation of the energy density of y_1^c is calculated, the extra information in $S^1(y_1^c) \cap S$ determines that all the energy of the system is located at location z_1 for times t_c greater than t_1, and the energy is equal to the initial energy of the system so that energy is conserved.

2.3 Evaluating Kent's Interpretation

In this section I will consider how well Kent's interpretation allows for peaceful coexistence between standard quantum physics and special relativity. I'll begin by showing that Kent's interpretation is consistent with standard quantum theory. I'll then show that Kent's interpretation is Lorentz invariant. The problems of outcomes will be addressed in a subsection that considers how Kent's interpretation ties in with decoherence theory and d'Espagnat's objection about improper mixtures. I'll then consider PI in Kent's interpretation and the consistency of Kent's interpretation with Colbeck and Renner's theorem. Finally, I will raise some issues about the nature of Kent's beables.

2.3.1 Consistency of Kent's interpretation with Standard Quantum Physics*

LorentzInvarianceSection

If we are to take Kent's interpretation seriously, it had better not contradict empirical observations. Standard quantum physics is a firmly established scientific theory, and so far, it has not been contradicted by any experimental observations. Thus, standard quantum physics is empirically adequate in its domain of applicability. Thus, if we can show that Kent's interpretation is consistent with standard quantum physics, then it too will be empirically adequate to the same degree.

In order to show that Kent's interpretation is consistent with standard quantum theory and does not contradict it, we will need to express $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ in terms of the observable $\hat{T}^{\mu\nu}(y)$ and the initial state $|\Psi_0\rangle$. To find such an expression, we would ideally like to find a hypersurface S' that contains both $S^1(y)$ and y. Then we could

*As mentioned in the introduction on page 4, sections marked with an asterisk may be challenging to readers who do not have a mathematics or physics background.

consider how the observables $\hat{T}_S(x)$ and $\hat{T}^{\mu\nu}(y)$ act on the state $|\Psi'\rangle = U_{S'S_0}|\Psi_0\rangle$ and use this action to determine the probabilities $P(q(\tau), r)$ and $P(r)$ needed to define the conditional probability $P(q(\kappa)|r)$ as defined in (2.33). However, since by definition, a hypersurface must be continuous with any two locations on it being spacelike-separated, it is going to be impossible to find a hypersurface S' with the desired properties of containing both $S^1(y)$ and y. Nevertheless, what we can do is find a sequence of hypersurfaces $S_n(y)$ such that siydef $S_n(y) \subset S_{n'}(y)$ for $n < n'$, and such that for any $x \in S^1(y)$, there exists n such that $x \in S_n(y)$. An example of one such $S_n(y)$ is shown in figure 2.8. When there is no ambiguity, we will drop the y and write S_n instead of $S_n(y)$.

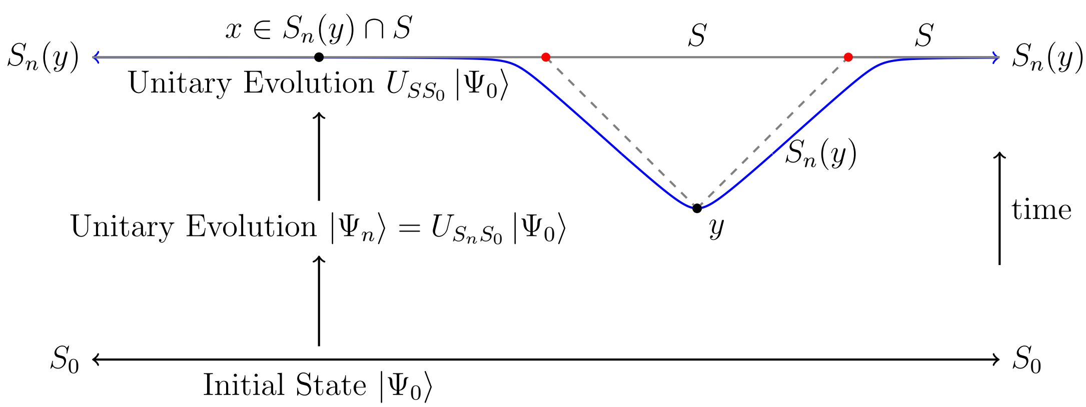


Figure 2.8: $S_n \stackrel{\text{def}}{=} S_n(y)$ is a hypersurface containing y and all of $S^1(y)$ in the limit as $n \to \infty$.

S3

Now if r_n is the statement that $T_S(x) = \tau_S(x)$ for all $x \in S_n(y) \cap S$, then so long as $\tau_S(x)$ is chosen by the Born Rule so that $P(r) \neq 0$, it will follow that

$$P(q(\tau)|r) = \lim_{n\to\infty} P(q(\tau)|r_n). \tag{2.35}$$

Therefore, from the definition of the beable $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ given on page 92 together with the definition of conditional expectation given in equation (2.34), we have

{beable1}
$$\langle T^{\mu\nu}(y)\rangle_{\tau_S} = \lim_{n\to\infty} \sum_{\tau} \frac{P(q(\tau), r_n)\tau}{P(r_n)}. \tag{2.36}$$

To calculate $P(q(\tau)|r_n)$, we note that since S_n is a hypersurface, there will exist a unitary operator $U_{S_n S_0}$ which maps the Hilbert space of states H_{S_0} describing S_0 to the Hilbert space of states H_{S_n} HSidef describing S_n in accord with how the states of H_{S_0} evolve over time. Now let $H_{S_n,\tau_S} \subset H_{S_n}$ be the subspace of states $|\xi\rangle$ for which $\hat{T}_S(x)\,|\xi\rangle = \tau_S(x)\,|\xi\rangle$ for all $x \in S_n \cap S$, and let $\{|\xi_1\rangle, |\xi_2\rangle, \ldots\}$ be an orthonormal basis of H_{S_n,τ_S}. Given that the initial state of the world is $|\Psi_0\rangle$, the probability $P(r_n)$ of "measuring" the value of $T_S(x)$ on $S_n \cap S$ to be $\tau_S(x)$ will be

{Pri}
$$P(r_n) = \sum_j |\langle \xi_j|\Psi_n\rangle|^2, \tag{2.37}$$

where $|\Psi_n\rangle = U_{S_n S_0}\,|\Psi_0\rangle$, and this probability will be independent of the particular orthonormal basis $\{|\xi_j\rangle : j\}$ of H_{S_n,τ_S}.[27] If we define

{tauprojection}
$$\pi_n = \sum_j |\xi_j\rangle\langle\xi_j|\,, \tag{2.38}$$

then it is easy to see that

{Prn}
$$P(r_n) = \langle \Psi_n|\pi_n|\Psi_n\rangle\,. \tag{2.39}$$

We also see that π_n is Hermitian (i.e. has real eigenvalues) and that $\pi_n\pi_n = \pi_n$. Any Hermitian operator π with $\pi^2 = \pi$ is called a **projection**. We thus see that π_n is a projection.

---

[27] To see why this is, we note that we can extend the orthonormal set $\{|\xi_1\rangle, |\xi_2\rangle, \ldots\}$ to an orthonormal basis $\{|\xi_1\rangle, |\xi_2\rangle, \ldots\} \cup \{|\zeta_1\rangle, |\zeta_2\rangle, \ldots\}$ of H_{S_n} which consists entirely of $\hat{T}_S$-eigenstates. We can think of each of the states of this orthonormal basis as the possible measurement outcomes when making the notional measurement of $T_S(x)$ on $S_n \cap S$. By the Born rule, it therefore follows that $P(r_n) = \sum_j |\langle\xi_j|\Psi_n\rangle|^2$. But to see that this probability is independent of the particular basis, we can uniquely write $|\Psi_n\rangle$ as a sum $|\Psi_n\rangle = |\xi\rangle + |\zeta\rangle$ where $|\xi\rangle$ belongs to the span of $\{|\xi_j\rangle : j\}$ and $|\zeta\rangle$

Turning to the calculation of $P(q(\tau), r_n)$, note that for the Tomonaga-Schwinger formulation of relativistic quantum physics, the operators $\hat{T}_S(x)$ and $\hat{T}^{\mu\nu}(y)$ for fixed μ, ν commute when x and y are spacelike-separated. It therefore follows that we can express any state of H_{S_n} as a superposition of simultaneous eigenstates of $\hat{T}^{\mu\nu}(y)$ and $\hat{T}_S(x)$ for $x \in S_n \cap S$.[28] For a particular choice of μ, ν, we can then form an orthonormal basis $\{|\eta_j\rangle : j\}$ of H_{S_n} consisting of simultaneous $\hat{T}^{\mu\nu}(y)$, $\hat{T}_S(x)$-eigenstates so that $\hat{T}^{\mu\nu}(y)\,|\eta_j\rangle = \tau^{(j)}\,|\eta_j\rangle$ and $\hat{T}_S(x)\,|\eta_j\rangle = \tau_S^{(j)}(x)\,|\eta_j\rangle$ for $x \in S_n \cap S$, where $\tau^{(j)}$ and $\tau_S^{(j)}(x)$ are the corresponding eigenstates. If we define $\pi_{n,\tau} = \sum_j |\chi_{j,\tau}\rangle\langle\chi_{j,\tau}|$ where $\{|\chi_{j,\tau}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ such that $\hat{T}^{\mu\nu}(y)\,|\chi_{j,\tau}\rangle = \tau\,|\chi_{j,\tau}\rangle$ and $\hat{T}_S(x)\,|\chi_{j,\tau}\rangle = \tau_S(x)\,|\chi_{j,\tau}\rangle$ for all $x \in S_n \cap S$, then

{pqtauri}

$$P(q(\tau), r_n) = \sum_j |\langle\chi_{j,\tau}|\Psi_n\rangle|^2 = \langle\Psi_n|\pi_{n,\tau}|\Psi_n\rangle\,.^{29} \tag{2.40}$$

But if we define $\pi_\tau = \sum_j |\eta_{j,\tau}\rangle\langle\eta_{j,\tau}|$ where $\{|\eta_{j,\tau}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ with $\hat{T}^{\mu\nu}(y)\,|\eta_{j,\tau}\rangle = \tau\,|\eta_{j,\tau}\rangle$, then we also have $\pi_{n,\tau} = \pi_n\pi_\tau$.[30] Hence,

{pqtauri2}

$$P(q(\tau), r_n) = \langle\Psi_n|\pi_n\pi_\tau|\Psi_n\rangle\,. \tag{2.41}$$

But clearly $\hat{T}^{\mu\nu}(y) = \sum_\tau \tau\pi_\tau$. Therefore, combining (2.36), (2.40), and (2.41), we have

{kentconsistency0}

$$\langle T^{\mu\nu}(y)\rangle_{\tau_S} = \lim_{n\to\infty}\frac{\sum_\tau \langle\Psi_n|\pi_n\pi_\tau|\Psi_n\rangle\,\tau}{\langle\Psi_n|\pi_n|\Psi_n\rangle} = \lim_{n\to\infty}\frac{\langle\Psi_n|\pi_n\hat{T}^{\mu\nu}(y)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle}. \tag{2.42}$$

---

belongs to the span of $\{|\zeta_j\rangle : j\}$. Then since $|\xi\rangle = \sum_j \langle\xi_j|\Psi_n\rangle\,|\xi_j\rangle$, it follows that

$$\langle\xi|\xi\rangle = \sum_j |\langle\xi_j|\Psi_n\rangle|^2 = P(r_n).$$

Therefore, since $\langle\xi|\xi\rangle$ is independent of the particular basis chosen of H_{S_n,τ_S}, then so is $P(r_n)$. priproof

[28] We make the same approximation as mentioned on page 87 in footnote 18.

[29] The proof of this is very similar to the proof given in footnote 27.

[30] To see why this is, we first show that $\pi_n = \sum_j |h_{n,j}\rangle\langle h_{n,j}|$ where $\{|h_{n,j}\rangle : j\}$ is the subset of $\{|\eta_j\rangle : j\}$ for which $|h_{n,j}\rangle \in H_{S_n,\tau_S}$. Note that $\pi_n\,|h_{n,j}\rangle = |h_{n,j}\rangle$ since $\{|\xi_j\rangle : j\}$ is a basis for H_{S_n,τ_S} and $|h_{n,j}\rangle \in H_{S_n,\tau_S}$. Therefore, $\pi_n\pi_{n,h} = \pi_{n,h}$ where $\pi_{n,h} = \sum_j |h_{n,j}\rangle\langle h_{n,j}|$. But $\pi_{n,h}\,|\xi_j\rangle = |\xi_j\rangle$ since $\{|h_{n,j}\rangle : j\}$ is a basis for H_{S_n,τ_S} and $|\xi_j\rangle \in H_{S_n,\tau_S}$. Therefore, $\pi_{n,h}\pi_n = \pi_n$. But $\pi_{n,h}\pi_n = \pi_n\pi_{n,h}$ since π_n and $\pi_{n,h}$ are Hermitian. Hence, $\pi_n = \pi_{n,h}$. Now the summands of $\pi_n\pi_\tau$ are only going to

We are now in a position to show that Kent's theory is consistent with standard quantum theory. First let us consider what we need to show.

In the pilot wave interpretation, its consistency with standard quantum theory requires that if one averages the expectation values of an observable over the hidden variables (i.e. the positions and the momenta of all the particles) then one obtains the expectation value of the observable given by standard quantum theory as indicated in equation (2.7).

Now in Kent's interpretation, the hidden variables on which his beables $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ depend are the values $\tau_S(x)$ of $T_S(x)$ for $x \in S^1(y) \cap S$. The operator π_n in equation (2.42) in the limit as $n \to \infty$ encapsulates this hidden information. To remind ourselves of π_n's dependency on τ_S restricted to $S_n \cap S$, we will now write $\pi_n(\tau_{S_n \cap S})$ for π_n where $\tau_{S_n \cap S}$ is the function τ_S restricted to $S_n \cap S$. Likewise, we will write $r_n(\tau_{S_n \cap S})$ for r_n, the statement that $T_S(x) = \tau_S(x)$ for all $x \in S_n(y) \cap S$. If we let j index all possible functions $\tau^{(j)}_{S_n \cap S}$ taking real values on $S_n \cap S$, then the analogue of (2.7) requires us to show that

$$\langle \hat{T}^{\mu\nu}(y)\rangle = \lim_{n\to\infty} \sum_j P\big(r_n(\tau^{(j)}_{S_n \cap S})\big) \, \langle T^{\mu\nu}(y)\rangle_{\tau^{(j)}_{S_n \cap S}} \quad \text{\{kentconsistency\}} \tag{2.43}$$

for all y lying between S_0 and S, where the left-hand side of (2.43) is just the expectation value of $\hat{T}^{\mu\nu}(y)$ predicted by standard quantum mechanics. Equation (2.43) is sufficient to establish consistency with standard quantum theory because ultimately, all observables are going to be reducible to expressions dependent on $\hat{T}^{\mu\nu}(y)$, since once we know what to expect for $\hat{T}^{\mu\nu}(y)$, we will know what to expect

---

consist of those $|\eta_j\rangle\langle\eta_j|$ for which $\hat{T}^{\mu\nu}(y)\,|\eta_j\rangle = \tau\,|\eta_j\rangle$ and for which $\hat{T}_S(x)\,|\eta_j\rangle = \tau_S(x)\,|\eta_j\rangle$ for all $x \in S_n \cap S$, and these are just the $|\chi_{j,\tau}\rangle\langle\chi_{j,\tau}|$ which are the summands of $\pi_{n,\tau}$. Hence, $\pi_n \pi_\tau = \pi_{n,\tau}$.

for the energy and momentum densities for all measuring apparatus readouts etc. and hence what to expect for all measurement outcomes. But from (2.39) and (2.42), we have

{kentconsistency1}

$$\lim_{n\to\infty}\sum_j P\big(r_n(\tau^{(j)}_{S_n\cap S})\big)\,\langle T^{\mu\nu}(y)\rangle_{\tau^{(j)}_{S_n\cap S}} = \lim_{n\to\infty}\sum_j \langle\Psi_n|\pi_n(\tau^{(j)}_{S_n\cap S})T^{\mu\nu}(y)|\Psi_n\rangle \tag{2.44}$$

Since there is an orthonormal basis $\{|\eta_j\rangle : j\}$ of H_{S_n} consisting of simultaneous $\hat{T}_S(x)$-eigenstates so that $\hat{T}_S(x)\,|\eta_j\rangle = \tau^{(j)}_{S_n\cap S}(x)\,|\eta_j\rangle$ for all $x \in S_n \cap S$, it follows that $\sum_j \pi_n(\tau^{(j)}_{S_n\cap S}) = I$. Therefore, equation (2.43) follows from (2.44) which is what we were aiming to show for standard quantum consistency to hold.

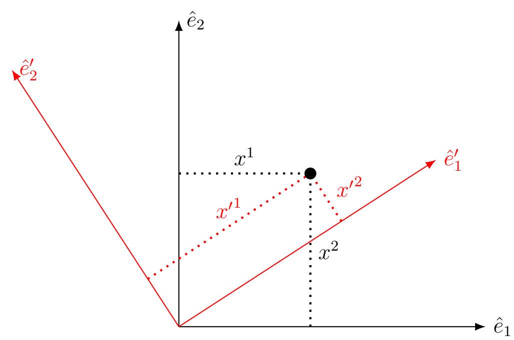
2.3.2 Kent's interpretation and Lorentz Invariance

LorentzInvariance

In order to explain what it means for Kent's interpretation to be Lorentz invariant, we first need to explain how spacetime coordinates look to different observers.

A spacetime location is represented by a four-tuple (x^0, x^1, x^2, x^3) where $(x^i)_{n=1}^3$ are spatial coordinates, and where $x^0 = ct$ with c being equal to the speed of light and t being the time. If $(1, 0, 0, 0)$ corresponds to the spacetime location $\hat{e}_0$, and $(0, 1, 0, 0)$ corresponds to the spacetime location $\hat{e}_1$, etc., then we can express any other spacetime location as a sum $\sum_{\mu=0}^3 x^\mu \hat{e}_\mu$. We will use the so-called Einstein convention of dropping the summation sign and implicitly assuming that there is a summation whenever an upper index and a lower index are the same so that we can write $x^\mu \hat{e}_\mu$ instead of $\sum_{\mu=0}^3 x^\mu \hat{e}_\mu$.

Now suppose an observer O expresses spacetime locations in terms of $\{\hat{e}_\mu : \mu = 0, \ldots, 3\}$ and hence uses the coordinates (x^0, x^1, x^2, x^3) to describe various spacetime locations. For another observer O', it may be more natural to express spacetime locations in terms of a different set of spacetime locations $\{\hat{e}'_\mu : \mu = 0, \ldots, 3\}$ so that the location described by O as (x^0, x^1, x^2, x^3) would be described by O' as (x'^0, x'^1, x'^2, x'^3) where $x'^\mu \hat{e}'_\mu = x^\mu \hat{e}_\mu$. For instance if O and O' are moving with respect to each other, they may both want to use coordinates in which their own spatial coordinates are fixed and in which the spatial coordinates of the other observer are changing. As another example, figure 2.9 shows how the (x^1, x^2)-coordinates transform under a spatial rotation.



(a)

Figure 2.9: Shows how a location (marked as •) can be expressed either in coordinates (x^1, x^2) with respect to the basis $\{\hat{e}_1, \hat{e}_2\}$ or in coordinates (x'^1, x'^2) with respect to the basis $\{\hat{e}'_1, \hat{e}'_2\}$.

`rotfigure`

Now the key fact about all observers is that they must always observe light in a vacuum to have a constant speed c. Thus, for a photon that goes through the spacetime locations $(0, 0, 0, 0)$ and (x^0, x^1, x^2, x^3) in the coordinates of O, we must have $(x^0, x^1, x^2, x^3) = (ct, tv^1, tv^2, tv^3)$ where

$$\sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2} = c.$$

But if $(0, 0, 0, 0)$ and (x^0, x^1, x^2, x^3) corresponds to $(0, 0, 0, 0)$ and (x'^0, x'^1, x'^2, x'^3) respectively in the coordinates of another observer O', then we must also have

$(x'^0, x'^1, x'^2, x'^3) = (ct', t'v'^1, t'v'^2, t'v'^3)$ where

$$\sqrt{(v'^1)^2 + (v'^2)^2 + (v'^3)^2} = c.$$

In either case, we must have

{invariant}

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = 0. \tag{2.45}$$

If we define $\eta_{00} = 1$, $\eta_{ni} = -1$ for $i = 1, 2, 3$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, then using the Einstein summation convention as well as the convention of lowering indices so that we define $x_\mu \stackrel{\text{def}}{=} \eta_{\mu\nu} x^\nu$, then (2.45) is equivalent to

$$x_\mu x^\mu = x'_\mu x'^\mu = 0.$$

Thus, for any coordinate transformation $x \to x'$ such that $x_\mu x^\mu = x'_\mu x'^\mu$, if the speed of light is c in the x-coordinates, then the speed of light is also guaranteed to be c in the x'-coordinates. A **Lorentz Transformation** Λ is any coordinate transformation of the form $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x_\mu x^\mu = x'_\mu x'^\mu$. Since a Lorentz transformation must satisfy

$$x_\mu x^\mu = \eta_{\mu\rho} \Lambda^\rho{}_\sigma x^\sigma \Lambda^\mu{}_\nu x^\nu$$

for all x, it follows that

{lorentztrans}

$$\Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu}. \tag{2.46}$$ [31]

Having considered how the coordinates of a spacetime location viewed by one observer relate to the coordinates of the same spacetime location viewed by a different observer, we can now consider how physical quantities viewed by different observers relate to each

[31] To see why this is, note that if $x_\mu x^\mu = x'_\mu x'^\mu$ for all x, then for any other spacetime location y, we have $(x+y)_\mu (x+y)^\mu = (x'+y')_\mu (x'+y')^\mu$. If we expand this out and cancel $x_\mu x^\mu$ with $x'_\mu x'^\mu$ and cancel $y_\mu y^\mu$ with $y'_\mu y'^\mu$, and using the fact that $y_\mu x^\mu = x_\mu y^\mu$, etc. we find that $x_\mu y^\mu = x'_\mu y'^\mu$ for all x and y. Hence,

$$\eta_{\nu\mu} x^\mu y^\nu = x_\mu y^\mu = \eta_{\sigma\rho} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu x^\mu y^\nu.$$

other. The simplest kind of physical quantity is called a **scalar**. A scalar defined at a particular spacetime location has the same value no matter what frame of reference an observer uses. One example of a scalar is an object's **rest mass** which is the mass an object would have if it had no velocity. There is still a transformation rule for scalars since the spacetime location at which the scalar is measured is usually expressed in terms of an observer's coordinate system, and the coordinates of such a location will differ for different observers. Thus, if $\phi(x) \stackrel{\text{def}}{=} \phi(x^0, x^1, x^2, x^3)$ is the value of a scalar defined at the spacetime location (x^0, x^1, x^2, x^3) as described by an observer O, then another observer O' using a different set of coordinate (x'^0, x'^1, x'^2, x'^3) to describe the location (x^0, x^1, x^2, x^3) will describe this same scalar as $\phi'(x') \stackrel{\text{def}}{=} \phi'(x'^0, x'^1, x'^2, x'^3)$ where $\phi'(x') = \phi(x)$. Since ϕ' is just a function of the four numbers x'^0, x'^1, x'^2, and x'^3, we can rename these numbers x^0, x^1, x^2, and x^3, and then

{lorentzscalar}

$$\phi'(x) = \phi(\Lambda^{-1}x) \tag{2.47}$$

where Λ^{-1} is the inverse Lorentz transformation that takes the coordinates $x' = (x'^0, x'^1, x'^2, x'^3)$ of a location to the coordinates $x = (x^0, x^1, x^2, x^3)$ describing that location. Thus, equation (2.47) shows us how a scalar transforms under a Lorentz transformation Λ.

Many physical quantities, however, are not scalars and so will look different to different observers. For instance, the energy of an object has a kinetic component that depends on the velocity the object has relative to an observer. However, it turns out that if

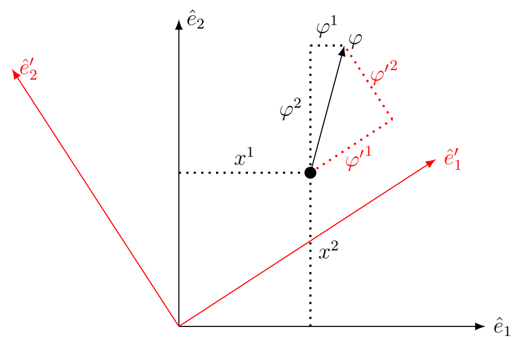
---

Since we can choose x such that $x^\mu = 1$ and $x^\alpha = 0$ for $\alpha \neq \mu$, and can choose y such that $y^\nu = 1$ and $y^\beta = 0$ for $\beta \neq \nu$. Then we get

$$\eta_{\mu\nu} = \eta_{\sigma\rho}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu,$$

and hence the result follows.

an observer O considers an object's energy E together with its three components of momentum p^1, p^2, and p^3 (in the directions $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ respectively) to form the four-tuple $p \stackrel{\text{def}}{=} (E/c, p^1, p^2, p^3)$ known as the object's **four-momentum**, then p transforms in the same way as spacetime coordinates transform between different observers. In other words, a different observer O' whose coordinates are given by $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ would observe the object's four-momentum to be $p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$.[32] More generally, any list of four physical quantities $(\varphi^0, \varphi^1, \varphi^2, \varphi^3)$ that transforms as $\varphi \to \varphi'$ with $\varphi'^{\mu} = \Lambda^{\mu}{}_{\nu} \varphi^{\nu}$ is called a **four-vector**. Figure 2.10 shows how (two of) the components of a four-vector



(b)

Figure 2.10: Shows how a four-vector φ (of which only two components are shown) defined at a spacetime location (indicated by $\bullet$) can be expressed either as (φ^1, φ^2) with respect to the basis $\{\hat{e}_1, \hat{e}_2\}$ or as (φ'^1, φ'^2) with respect to the basis $\{\hat{e}'_1, \hat{e}'_2\}$. rotfigure2

[32] In order for p to transform in this way, we have to redefine what we mean by energy and momentum. In classical mechanics, the momentum of an object is the product of the object's mass and its velocity. In the context of special relativity, however, the four-momentum of an object is defined to be the product of its rest mass m_0 and its **four-velocity** where the four velocity of an object is a four-tuple (u^0, u^1, u^2, u^3) with $u_\mu u^\mu = c^2$ such that the object's velocity (in the classical sense) is the vector $(c\frac{u^1}{u^0}, c\frac{u^2}{u^0}, c\frac{u^3}{u^0})$. The motivation for this definition can be seen by considering an object whose classical velocity is $\mathbf{v} = (v^1, v^2, v^3)$ that goes through $(0,0,0,0)$. It will have a spacetime trajectory $x(t) = (ct, v^1 t, v^2 t, v^3 t)$. u is just the four-vector proportional to $x(1)$ with $u_\mu u^\mu = c^2$. We can easily work out the four-velocity u of an object whose classical velocity is $\mathbf{v}$. For we must have $u^i = \frac{v^i u^0}{c}$, for $i = 1$ to 3. Therefore, since $u_\mu u^\mu = c^2$, we must have $(u^0)^2\big(1 - \frac{v^2}{c^2}\big) = c^2$ where $v = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}$. Thus, if we define $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, then $u^0 = c\gamma$ and $u^i = \gamma v^i$ for $i = 1$ to 3, and hence the four-velocity of the object must be $u = \gamma(c, v^1, v^2, v^3)$. From this, we see that the object's four-momentum will be $\gamma m_0(c, v^1, v^2, v^3)$. If the object's velocity is very small compared to the speed of light, then $\gamma \approx 1 + \frac{v^2}{2c^2}$, and hence the object's four-momentum $(E/c, p^1, p^2, p^3)$ will be approximately $(m_0 c + \frac{1}{2} m_0 v^2/c, m_0 v^1, m_0 v^2, m_0 v^3)$. Therefore, (p^1, p^2, p^3) is approximately equal to the classical momentum. However, the energy is now $E = m_0 c^2 + \frac{1}{2} m_0 v^2$. Thus, in addition to the kinetic energy term $\frac{1}{2} m_0 v^2$, there is a rest mass energy $m_0 c^2$. If we define the **relativistic mass** $m = \gamma m_0$, then we obtain Einstein's famous formula $E = mc^2$.

φ at a particular location will differ for different observers under a spatial rotation of the coordinates. A four-vector $\varphi^\mu(x)$ defined at every spacetime location x is called a **four-vector field**. If O observes this vector-field $\varphi^\mu(x)$, and O' is another observer whose coordinates are related to the coordinates O via the Lorentz transformation Λ, then O' will describe this vector-field as $\varphi'^\mu(x') \stackrel{\text{def}}{=} \Lambda^\mu{}_\nu \varphi^\nu(x)$. Hence, under the Lorentz transformation Λ, a vector field $\varphi^\mu(x)$ transforms as $\varphi^\mu(x) \to \varphi'^\mu(x')$ where

{lorentzvector}

$$\varphi'^\mu(x) = \Lambda^\mu{}_\nu \varphi^\nu(\Lambda^{-1}x). \tag{2.48}$$

From a four-vector φ^μ, we can also define the so-called **four-covector**:

{covector}

$$\varphi_\mu \stackrel{\text{def}}{=} \eta_{\mu\nu}\varphi^\nu. \tag{2.49}$$

To see how four-covectors transform under a Lorentz transformation Λ, it will be helpful to define

{colambda}

$$\Lambda_\mu{}^\nu \stackrel{\text{def}}{=} \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma \tag{2.50}$$

where $\eta^{\nu\sigma} = \eta_{\nu\sigma}$. If we also define the **Kronecker-delta** δ^ν_μ such that $\delta^\nu_\mu = 1$ when $\mu = \nu$ and $\delta^\nu_\mu = 0$ otherwise, then using the fact that $\eta_{\mu\rho}\eta^{\nu\rho} = \delta^\nu_\mu$ together with equation (2.46), we have

{lambdainverse}

$$\Lambda^\rho{}_\mu \Lambda_\rho{}^\nu = \delta^\nu_\mu. \tag{2.51}$$

Since by definition, the inverse of Λ^{-1} satisfies $(\Lambda^{-1})^\nu{}_\rho \Lambda^\rho{}_\mu = \delta^\nu_\mu$, we have $(\Lambda^{-1})^\nu{}_\rho = \Lambda_\rho{}^\nu$. From (2.48), (2.49), and (2.50), we therefore see that under a Lorentz transformation Λ, a four-covector field $\varphi_\mu(x)$ transforms as $\varphi_\mu(x) \to \varphi'_\mu(x')$ where

{lorentzcovector}

$$\varphi'_\mu(x) = \Lambda_\mu{}^\nu \varphi_\nu(\Lambda^{-1}x) \tag{2.52}$$

Besides scalars, four-vectors, and four-covectors, we also need to consider physical quantities called rank-two tensors. The stress-energy tensor $T^{\mu\nu}$ mentioned on page 90 is an example of a rank-two tensor. The defining property of a rank-two tensor field $\varphi^{\mu\nu}(x)$ is that under a Lorentz transformation Λ, it transforms as $\varphi^{\mu\nu}(x) \to \varphi'^{\mu\nu}(x')$ where

{lorentztensor}

$$\varphi'^{\mu\nu}(x) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\varphi^{\rho\sigma}(\Lambda^{-1}x). \tag{2.53}$$

On page 86, we introduced the mass-energy density $T_S(x)$ on a hypersurface S. As explained in section 2.2.2, the values of $T_S(x)$ are the additional values that Kent uses to supplement standard quantum theory. It was mentioned in passing that $T_S(x)$ does not depend on which frame of reference one is in. In other words, $T_S(x)$ is a scalar. I will now explain why this is so.

We first need to consider the precise definition of $T_S(x)$. At each spacetime location on the hypersurface S which an observer O describes as having coordinates $x = (x^\mu)_{\mu=0}^3$, we define $\eta^\mu(x)$ to be the future-directed unit four-vector at x that is orthogonal to S. In other words, $\eta^0(x) > 0$, $\eta_\mu(x)\eta^\mu(x) = 1$, and if $y \in S$ is very close to x, then $\frac{(x-y)_\mu \eta^\mu(x)}{\sqrt{(x-y)_\nu (x-y)^\nu}} \approx 0$. $T_S(x)$ is then given by the formula

{TSdef}

$$T_S(x) = T^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x). \tag{2.54}$$

For example, if S was the hypersurface consisting of all spacetime locations $x = (0, x^1, x^2, x^3)$, then $\big(\eta^0(x), \eta^1(x), \eta^3(x), \eta^3(x)\big) = (1, 0, 0, 0)$, and hence $T_S(x) = T^{00}(x)$ which is the density of relativistic mass at x, i.e. the energy density at x divided by c^2.

To see why $T_S(x)$ is a scalar, suppose that Λ is a Lorentz transformation such that ${\Lambda^0}_\mu \eta^\mu > 0$ for any future-directed unit four-vector vector η^μ. We refer to a Λ with this property as an **orthochronous** Lorentz transformation. Also, suppose that O and O' are two observers such that spacetime locations that observer O describes as having coordinates $x = (x^\mu)_{\mu=0}^3$ are described by O' as having coordinates $x' = ({\Lambda^\mu}_\nu x^\nu)_{\mu=0}^3$. Then since ${x'}_\mu {y'}^\mu = x_\mu y^\mu$, it follows that the future-directed unit four-vector orthogonal to S at x which O describes as $\eta^\mu(x)$ will be described by O' as ${\eta'}^\mu(x') = {\Lambda^\mu}_\nu \eta^\nu(x)$. Thus, for any location in S that O' describes as having coordinates x' with corresponding future-directed S-orthogonal unit four-vector ${\eta'}^\mu(x')$, O' can construct a function $T'_S(x')$ with

{TSprimedef}

$$T'_S(x') = T'^{\mu\nu}(x')\eta'_\mu(x')\eta'_\nu(x'). \tag{2.55}$$

Then using (2.52) and (2.53) on the right-hand side of (2.55), we have

{invariantTS1}

$$\begin{aligned}
T'_S(x') &= {\Lambda^\mu}_\rho {\Lambda^\nu}_\sigma T^{\rho\sigma}(x) {\Lambda_\mu}^\alpha \eta_\alpha(x) {\Lambda_\nu}^\beta \eta_\beta(x) \\
&= {\Lambda^\mu}_\rho {\Lambda_\mu}^\alpha {\Lambda^\nu}_\sigma {\Lambda_\nu}^\beta T^{\rho\sigma}(x)\eta_\alpha(x)\eta_\beta(x) \\
&= \delta^\alpha_\rho \delta^\beta_\sigma T^{\rho\sigma}(x)\eta_\alpha(x)\eta_\beta(x) \\
&= T^{\alpha\beta}(x)\eta_\alpha(x)\eta_\beta(x) \\
&= T_S(x)
\end{aligned} \tag{2.56}$$

where on the third line we have used (2.51), and on the last line we have used (2.54). To obtain (2.56), we assumed that Λ is orthochronous, but if Λ is non-orthochronous, we would need to take the negations of ${\eta'}^\mu(x')$ to get the future-directed S-orthogonal unit four-vector. But clearly this will not affect the equality in (2.56), so (2.56) holds for all Lorentz transformations, whether they are orthochronous or non-orthochronous. We thus see that $T_S(x)$ is a scalar.

Let us now consider the Hilbert space H_{S_n} as defined on page 99 for a hypersurface S_n. Given that $\hat{T}^{\mu\nu}(x)$ is the observable whose eigenstates with eigenvalues τ are the states of S_n for which an observer O observes the stress-energy tensor $T^{\mu\nu}(x)$ to take the value τ at x, it follows from (2.54) that

{TShat}

$$\hat{T}_S(x) \stackrel{\text{def}}{=} \hat{T}^{\mu\nu}(x)\eta_\mu(x)\eta_\nu(x) \tag{2.57}$$

will be the observable whose eigenstates with eigenvalues $\tau_S(x)$ are the states of S_n for which an observer O observes $T_S(x)$ to take the value $\tau_S(x)$ at x. Two observers O and O' will typically assign different physical states to S_n based on their frame of reference. E.g. if O and O' are traveling at different speeds, they will attribute different energy levels and momenta to the spacetime locations of S_n. For the Lorentz transformation that relates the coordinates of O' to the coordinates of O, i.e. $x' = \Lambda x$, there will then be a unitary operator $U(\Lambda) : H_{S_n} \to H_{S_n}$ such that if O observes S_n to be in the state $|\psi_n\rangle \in H_{S_n}$, then O' will observe S_n to be in the state $U(\Lambda)\,|\psi_n\rangle$. But in order for $U(\Lambda)\,|\psi_n\rangle$ to be meaningful, we need to specify how the Hermitian operators that act on $U(\Lambda)\,|\psi_n\rangle$ correspond to the physical quantities that O' observes. So we specify that the Hermitian operator

{Thatprime}

$$\hat{T}'^{\mu\nu}(x') = \hat{T}^{\mu\nu}(x') \tag{2.58}$$

will be the observable whose eigenstates with eigenvalues τ' are the states of S_n for which O' observes the stress-energy tensor $T'^{\mu\nu}(x')$ to take the value τ' at x'. Since $T^{\mu\nu}(x)$ transforms according to (2.53), it will follow that

{TUrelation}

$$U(\Lambda)^{-1}\hat{T}^{\mu\nu}(x)U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu_\sigma \hat{T}^{\rho\sigma}(\Lambda^{-1}x). \tag{2.59}$$

We also insist that $U(\Lambda)$ is unitary because this means that if O calculates the probability S_n transitions from state $|\psi_n\rangle$ to state $|\chi_n\rangle$, then O' would calculate the same probability for the corresponding transition from the state $|\psi'_n\rangle = U(\Lambda)\,|\psi_n\rangle$ to the state $|\chi'_n\rangle = U(\Lambda)\,|\chi_n\rangle$.[33]

Now to say that Kent's model is Lorentz invariant, is to say that (2.42) defines a rank-two tensor, for then this quantity and the quantities on which it depends will transform in the way that physical quantities should transform under a Lorentz transformation. Thus, in order to show that Kent's model is Lorentz invariant, we need to show that if $\{|\xi_j\rangle : j\}$ is an orthonormal basis of the Hilbert space of states H_{S_n,τ_S} for which O observes $T_S(x)$ to be $\tau_S(x)$ for all $x \in S_n(y) \cap S$, and if $\{|\xi'_j\rangle : j\}$ is an orthonormal basis of the Hilbert space of states H_{S_n,τ'_S} for which O' observes $T'_S(x')$ to be $\tau'_S(x')$ for all $x' \in S_n(y') \cap S$, then

$$\lim_{n\to\infty} \frac{\langle \Psi'_n|\pi'_n \hat{T}^{\mu\nu}(y')|\Psi'_n\rangle}{\langle \Psi'_n|\pi'_n|\Psi'_n\rangle} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} \lim_{n\to\infty} \frac{\langle \Psi_n|\pi_n \hat{T}^{\rho\sigma}(y)|\Psi_n\rangle}{\langle \Psi_n|\pi_n|\Psi_n\rangle} \quad \texttt{\{kentlorentz\}} \tag{2.60}$$

where $\pi_n = \sum_j |\xi_j\rangle\langle\xi_j|$, $\pi'_n = \sum_j |\xi'_j\rangle\langle\xi'_j|$, and $|\Psi'_n\rangle = U(\Lambda)\,|\Psi_n\rangle$.

To see why (2.60) holds, we first recall that π'_n will be independent of which orthonormal basis we choose for H_{S_n,τ'_S}.[34] Therefore, if we can show that $\{|\xi'_j\rangle \stackrel{\text{def}}{=} U(\Lambda)\,|\xi_j\rangle : j\}$ is an orthonormal basis of H_{S_n,τ'_S}, it will follow that $\pi'_n = U(\Lambda)\pi_n U(\Lambda)^{-1}$.

That the elements of $\{U(\Lambda)\,|\xi_j\rangle : j\}$ are orthonormal follows from the unitarity of $U(\Lambda)$ together with the orthonormality of $\{|\xi_j\rangle : j\}$. Since $\hat{T}^{\mu\nu}(x')$ is the observable whose eigenstates with eigenvalue τ' are the states of $S_n(y')$ for which O' observes

[33] This follows from (2.31) which implies $|\langle\chi'_n|\psi'_n\rangle|^2 = |\langle\chi_n|\psi_n\rangle|^2$, together with the Born rule given on page 9.

[34] We showed this was the case for π_n in footnote 27 on page 100.

the stress-energy tensor $T'^{\mu\nu}(x')$ to take the value τ' at x', it follows from (2.55) and (2.58) that

$$\hat{T}'_S(x') \stackrel{\text{def}}{=} \eta'_\mu(x')\eta'_\nu(x')\hat{T}^{\mu\nu}(x') \tag{2.61}$$

{TShatprime}

will be the observable whose eigenstates with eigenvalue τ'_S are the states of $S_n(y')$ for which O' observes $T'_S(x')$ to take the value τ'_S at x', where as usual, $\eta'^\mu(x')$ is the unit four-vector orthogonal to $S_n(y')$ at x'. Now if $x' \in S_n(y') \cap S$ in the coordinates of O', then $x = \Lambda^{-1}x' \in S_n(y) \cap S$ in the coordinates of O. Using the same calculation as in (2.56) together with (2.52), we have

$$\begin{aligned} \hat{T}_S(x) &\stackrel{\text{def}}{=} \eta_\mu(x)\eta_\nu(x)\hat{T}^{\mu\nu}(x) \\ &= \eta'_\mu(x')\eta'_\nu(x')\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\hat{T}^{\rho\sigma}(x). \end{aligned} \tag{2.62}$$

{TSLambda}

By (2.59) we have

$$U(\Lambda)^{-1}\hat{T}^{\mu\nu}(x') = \Lambda^\mu{}_\rho\Lambda^\nu_\sigma\hat{T}^{\rho\sigma}(x)U(\Lambda)^{-1}, \tag{2.63}$$

so using this with (2.62) and (2.61), we have

$$\begin{aligned} \hat{T}_S(x)U(\Lambda)^{-1} &= \eta'_\mu(x')\eta'_\nu(x')\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\hat{T}^{\rho\sigma}(x)U(\Lambda)^{-1} \\ &= U(\Lambda)^{-1}\eta'_\mu(x')\eta'_\nu(x')\hat{T}^{\mu\nu}(x') \\ &= U(\Lambda)^{-1}\hat{T}'_S(x'). \end{aligned} \tag{2.64}$$

{TSU}

Now suppose that $|\xi'\rangle$ is a state for which O' observes $T'_S(x')$ to be $\tau'_S(x')$ for all $x' \in S_n(y') \cap S$. Then $\hat{T}'_S(x')\,|\xi'\rangle = \tau'_S(x')$, and so by (2.64),

$$\begin{aligned} \hat{T}_S(x)U(\Lambda)^{-1}\,|\xi'\rangle &= U(\Lambda)^{-1}\hat{T}'_S(x')\,|\xi'\rangle \\ &= \tau'_S(x')U(\Lambda)^{-1}\,|\xi'\rangle \\ &= \tau_S(x)U(\Lambda)^{-1}\,|\xi'\rangle \end{aligned} \tag{2.65}$$

{TSUxi}

where on the last line we have used the fact that $T_S(x)$ is a scalar. Therefore, $U(\Lambda)^{-1}|\xi'\rangle$ can be expressed as a linear combination of the basis elements $\{|\xi_j\rangle : j\}$ of H_{S_n,τ_S}, and hence $|\xi'\rangle$ can be expressed as a linear combination of $\{U(\Lambda)|\xi_j\rangle : j\}$. From (2.64) we also see that $U(\Lambda)\hat{T}_S(x) = \hat{T}'_S(x')U(\Lambda)$, so

$$\hat{T}'_S(x')U(\Lambda)|\xi_j\rangle = U(\Lambda)\hat{T}_S(x)|\xi_j\rangle = \tau_S(x)U(\Lambda)|\xi_j\rangle = \tau'_S(x')U(\Lambda)|\xi_j\rangle$$

for all $x' \in S_n(y') \cap S$. Therefore, $U(\Lambda)|\xi_j\rangle \in H_{S_n,\tau'_S}$. Since $\{|\xi'_j\rangle \overset{\text{def}}{=} U(\Lambda)|\xi_j\rangle : j\}$ is a spanning orthonormal subset of H_{S_n,τ'_S}, it must therefore be an orthonormal basis of H_{S_n,τ'_S}. From this it follows that $\pi'_n = U(\Lambda)\pi_n U(\Lambda)^{-1}$. Therefore,

$$\begin{aligned}
\frac{\langle\Psi'_n|\pi'_n\hat{T}^{\mu\nu}(y')|\Psi'_n\rangle}{\langle\Psi'_n|\pi'_n|\Psi'_n\rangle} &= \frac{\langle\Psi_n|U(\Lambda)^{-1}U(\Lambda)\pi_n U(\Lambda)^{-1}\hat{T}^{\mu\nu}(y')U(\Lambda)|\Psi_n\rangle}{\langle\Psi_n|U(\Lambda)^{-1}U(\Lambda)\pi_n U(\Lambda)^{-1}U(\Lambda)|\Psi_n\rangle} \\
&= \frac{\langle\Psi_n|\pi_n U(\Lambda)^{-1}\hat{T}^{\mu\nu}(y')U(\Lambda)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle} \qquad \texttt{\{kentlorentz2\}} \quad (2.66) \\
&= \frac{\langle\Psi_n|\pi_n\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}_{\sigma}\hat{T}^{\rho\sigma}(y)|\Psi_n\rangle}{\langle\Psi_n|\pi_n|\Psi_n\rangle}
\end{aligned}$$

where on the last line we have used (2.59). Thus, equation (2.60) holds, and hence Kent's model is Lorentz invariant.

Note that in this proof of Lorentz invariance, we don't need to take the limit of S_n as $n \to \infty$. That is, we could remove the $\lim_{n\to\infty}$ from equation (2.42) and consider a particular S_n, and the corresponding $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ would still be a rank-two tensor. Butterfield tells us that Kent's interpretation is Lorentz invariant because his algorithm respects the light cone structure of y.[35] However, this statement could be slightly misleading because we don't need to consider the subset $S^1(y) \subset S$ of locations outside the light cone of y in order to obtain a Lorentz invariant model. Doing the calculation on any Tomonaga-Schwinger hypersurface is sufficient to guarantee Lorentz invariance

[35]See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 30.

since any such hypersurface (e.g. S_n) is not altered at all by a Lorentz transformation – only its coordinate description changes under a Lorentz transformation, and so the additional information of the scalar $\tau_S(x)$ on $S_n \cap S$ is Lorentz invariant. The only reason we need to consider the limit $\lim_{n\to\infty} S_n$ and hence $S^1(y) = \lim_{n\to\infty} S_n \cap S$ is that it is only in the limit that we use all the available information in $\tau_S(x)$ to calculate $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$.

2.3.3 Kent's Interpretation and Decoherence Theory*

In section 1.11 we saw that decoherence theory by itself does not offer a solution to the problem of outcomes. In this section, we consider how the additional information in Kent's interpretation is sufficient to address this problem. We will explain this by again considering Kent's toy model discussed in section 2.2.4.

We thus suppose that a system is in a superposition $\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$ of two local states ψ_1^{sys} and ψ_2^{sys} where $|c_1|^2 + |c_2|^2 = 1$, and that there is a photon coming in from the left that interacts with the system. We also suppose that y_1 is a spacetime location with spatial location z_1 between the two hypersurfaces S_0 and S, and we consider a hypersurface $S_n = S_n(y_1)$ in a sequence of hypersurfaces that each contain y_1 as described on page 98.

In order to obtain a sufficiently simple description of the state $|\Psi_n\rangle \in H_{S_n}$ of S_n for which we can use the formula (2.42) to calculate Kent's beable, we will use a coarse-grained model so that S_n is treated as a mesh of tiny cells labeled by a sequence $(y_k)_{k=1}^{\infty}$. Thus, for each cell y_k there will be a Hilbert space H_k describing the state of that cell. We can think of each of these y_k as systems that can become entangled with one another, but we will assume that y_1 is entangled with only a finite number M of the other y_k. What this means is that the most general expression for $|\Psi_n\rangle$ will be of the form

$$|\Psi_n\rangle = \Big(\sum_j \sum_{n\in\mathbb{N}^M} c_{j,n}\,|\xi_{1,j}\rangle \prod_{l=1}^{M} |\xi_{k_l,n_l}\rangle\Big)\Xi. \tag{2.67}$$

{Sistate}

In this expression, $\{|\xi_{1,j}\rangle : j\}$ is an orthonormal basis of H_1, $\mathbb{N}^M$ means the set of all lists $(n_1,\ldots,n_M)$ with each $n_l \in \mathbb{N}$ where $\mathbb{N}$ is the set of positive integers greater than 0. The set of states $\{|\xi_{k_l,n_l}\rangle : n_l \in \mathbb{N}\}$ form an orthonormal basis of H_{k_l} for each k_l, and the k_l are all distinct from each other and from 1. Also, M is chosen to be as small as possible so that any common factors of $|\Psi_n\rangle$ belong to Ξ which is a sum of states of the form $\prod_l |\xi_{\kappa_l}\rangle$ where the states $|\xi_{\kappa_l}\rangle \in H_{\kappa_l}$ ranging over all the cells of S_n not included in the set $\{k_l : l = 1,\ldots,M\}$. We also assume that each summand $c_{j,n}\,|\xi_{1,j}\rangle \prod_{l=1}^{M} |\xi_{k_l,n_l}\rangle\,\Xi$ of Ψ_n contains a state in each H_k for every cell k of S_n. In other words if $k \neq 1$ and does not belong to the set $\{k_l : l = 1,\ldots,M\}$ then k belongs to the set $\{\kappa_l : l\}$. Also, we will give H_{S_n} an inner product so that if

$$|\Psi'_n\rangle = \Big(\sum_j \sum_{n\in\mathbb{N}^M} c'_{j,n}\,|\xi_{1,j}\rangle \prod_{l=1}^{N_j} |\xi_{k_l,n_l}\rangle\Big)\Xi',$$

then

$$\langle\Psi'_n|\Psi_n\rangle = \Big(\sum_j \sum_{n\in\mathbb{N}^M} \overline{c'_{j,n}}\,c_{j,n}\Big)\,\langle\Xi'|\Xi\rangle$$

where $\langle\Xi'|\Xi\rangle$ is defined in the obvious way. With this inner product, we will assume that $|\Psi_n\rangle$ is appropriately normalized so that $\langle\Psi_n|\Psi_n\rangle = 1$. If we also assume that $\langle\Xi|\Xi\rangle = 1$, it will follow that $\sum_j \sum_{n\in\mathbb{N}^M} |c_{j,n}|^2 = 1$.

We now consider several scenarios from Kent's toy model. In each scenario, we will use the decomposition (2.67) of $|\Psi_n\rangle$ to calculate the partial trace encapsulating all the information needed to calculate expectation values at different spacetime locations.

First, consider Figure 2.11 which depicts the hypersurface $S_n(y_1^a)$ for a spacetime location y_1^a that occurs before the photon has interacted with the system.

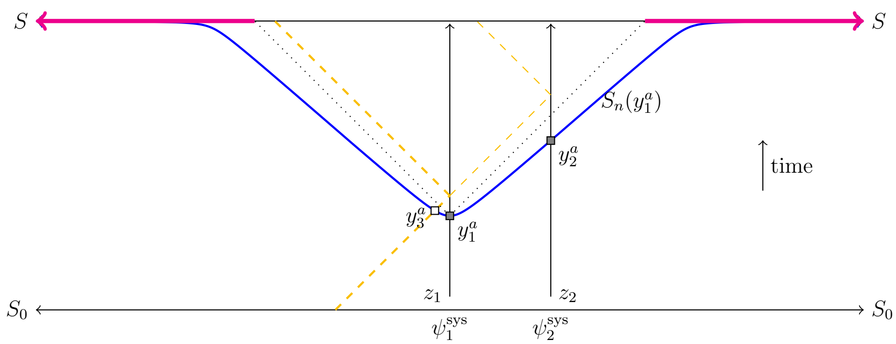


Figure 2.11: Depiction of a superposition of two local states at z_1 and z_2 before the photon has interacted with them. The gray squares indicate cells in $S^1(y_1^a)$ whose states are among the summands in (2.67) rather than in Ξ. The white square indicates a cell in $S_n(y_1^a)$ whose state is a factor in Ξ. `kentdeco1`

The gray squares correspond to the summands that appear in (2.67). If the system were in the ψ_1^{sys}-state, then the state describing $S_n(y_1^a)$ would have a factor $|\psi_1^{\text{sys}}\rangle \in H_1$ indicating that there is a non-zero mass at the y_1^a-cell, and there would also be a factor $|0_2\rangle \in H_2$ which we use to indicate that there is zero mass/energy at y_2^a. There is also an incoming photon at the y_3^a-cell, and so we use $|\gamma_3\rangle$ to indicate that there is a photon there. Thus, if the system were in the ψ_1^{sys}-state, we would write the state of $S_n(y_1^a)$ as $|\Psi_n\rangle = |\psi_1^{\text{sys}}\rangle\,|0_2\rangle\,|\gamma_3\rangle\,\Xi'$, where Ξ' describes the states of all the other cells of $S_n(y_1^a)$. In this very simple scenario, $\Xi' = \sum_{k\neq 1,2,3} |0_k\rangle$ indicating that there is zero mass/energy at all the other y_k.

Alternatively, if the system were in the state ψ_2^{sys}, then the state describing $S_n(y_1^a)$ would have a factor $|\psi_2^{\text{sys}}\rangle \in H_2$ indicating that there is a non-zero mass at the y_2^a-cell, and there would also be a factor $|0_1\rangle \in H_1$ which we use to indicate that there is zero mass at y_1^a, and again the y_3^a-cell would be in the $|\gamma_3\rangle$, and every other cell would be described by Ξ' just as if the system had been in the ψ_1^{sys}-state. Therefore, when the system is in the state ψ_2^{sys}, we would write the state of $S_n(y_1^a)$ as $|\Psi_n\rangle = |0_1\rangle\,|\psi_2^{\text{sys}}\rangle\,|\gamma_3\rangle\,\Xi'$.

Now since the system is actually in a supposition $\psi_0^{\text{sys}} = c_1\psi_1^{\text{sys}} + c_2\psi_2^{\text{sys}}$, the state of $S_n(y_1^a)$ will be

$$|\Psi_n\rangle = \big(c_1\,|\psi_1^{\text{sys}}\rangle\,|0_2\rangle + c_2\,|0_1\rangle\,|\psi_2^{\text{sys}}\rangle\,\big)\,|\gamma_3\rangle\,\Xi' = \big(c_1\,|\psi_1^{\text{sys}}\rangle\,|0_2\rangle + c_2\,|0_1\rangle\,|\psi_2^{\text{sys}}\rangle\,\big)\Xi$$

where we have absorbed the $|\gamma_3\rangle$-state into Ξ (i.e. $\Xi = |\gamma_3\rangle\,\Xi'$).

Now as it stands, the state $|\Psi_n\rangle$ describing $S_n(y_1^a)$ has a definite mass-energy density $\tau_S(x)$ for $x \in S_n(y_1^a) \cap S$, namely 0. Thus, if π_n is the operator featuring in (2.42) that corresponds to this definite mass-energy density, then $\pi_n\,|\Psi_n\rangle = |\Psi_n\rangle$. Therefore, equation (2.42) for Kent's beables tells us that

$$\langle T^{\mu\nu}(y_1^a)\rangle_{\tau_S} = \langle\Psi_n|\hat{T}^{\mu\nu}(y_1^a)|\Psi_n\rangle\,,$$

where we have also used the fact that $\langle\Psi_n|\Psi_n\rangle = 1$.

Now as we saw in section 1.8, if we are interested only in the expectation values of observables for a system $\mathcal{S}$ contained within a universe $\mathcal{U} = \mathcal{S}+\mathcal{E}$, then the information needed to do this can be encapsulated in the reduced density matrix for $\mathcal{S}$. Thus, if the universe is described by a state $|\Psi\rangle = \sum_j c_j\,|\psi_j\rangle_{\mathcal{S}}\,|E_j\rangle$ with corresponding density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi| \in M(H_{\mathcal{U}})$, then the reduced density matrix $\hat{\rho}_{\mathcal{S}} \in M(H_{\mathcal{S}})$ is the Hermitian operator acting on the state space $H_{\mathcal{S}}$ with the property that

$$\langle \hat{O}_{\mathcal{U}} \rangle_{\rho} = \mathrm{Tr}_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}} \hat{O}_{\mathcal{S}}) \qquad \text{(1.19 revisited)}$$

where $\hat{O}_{\mathcal{S}}$ is an observable on $H_{\mathcal{S}}$, and $\hat{O}_{\mathcal{U}}$ is the corresponding observable on $H_{\mathcal{U}}$. Furthermore, we also have

$$\hat{\rho}_{\mathcal{S}} = \sum_j |c_j|^2 |\psi_j\rangle\langle\psi_j| + \sum_{j \neq k} c_j \overline{c_k} \langle E_k|E_j\rangle |\psi_j\rangle\langle\psi_k| .^{36} \qquad \texttt{\{reduced2\}} \; (2.68)$$

We can thus apply this to the situation at hand by taking S_n to be our universe $\mathcal{U}$ and y_1^a to be the system $\mathcal{S}$, and $S_n \setminus \{y_1^a\}$ to be the environment $\mathcal{E}$. If we assume that $\langle 0_2|\psi_2^{\mathrm{sys}}\rangle = 0$, then by (2.68), the corresponding reduced density matrix $\hat{\rho}_{y_1^a}$ takes the form of an improper mixture

$$\hat{\rho}_{y_1^a} = |c_1|^2 |\psi_1^{\mathrm{sys}}\rangle\langle\psi_1^{\mathrm{sys}}| + |c_2|^2 |0_1\rangle\langle 0_1| . \qquad \texttt{\{kentred\}} \; (2.69)$$

Kent's beables at y_1^a will thus take the form

$$\begin{aligned} \langle T^{\mu\nu}(y_1^a)\rangle_{\tau_S} &= \mathrm{Tr}_{y_1^a}(\hat{\rho}_{y_1^a} \hat{T}^{\mu\nu}(y_1^a)) \\ &= |c_1|^2 \langle\psi_1^{\mathrm{sys}}|\hat{T}^{\mu\nu}(y_1^a)|\psi_1^{\mathrm{sys}}\rangle + |c_2|^2 \langle 0_1|\hat{T}^{\mu\nu}(y_1^a)|0_1\rangle . \end{aligned} \qquad \texttt{\{kentbe\}} \; (2.70)$$

Let us now consider Kent's beables at the spacetime location y_1^b depicted in figure 2.12.

The state of $S_n(y_1^b)$ will then be

$$|\Psi_n\rangle = \big(c_1 |\psi_1^{\mathrm{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle + c_2 |0_1\rangle |\psi_2^{\mathrm{sys}}\rangle |0_3\rangle |\gamma_4\rangle \big)\Xi$$

where the notation is analogous to that in the previous example. Since no photon detections are registered on $S_n(y_1^b) \cap S$, we again have $\pi_n |\Psi_n\rangle = |\Psi_n\rangle$ so that the

36 cf. (1.20)

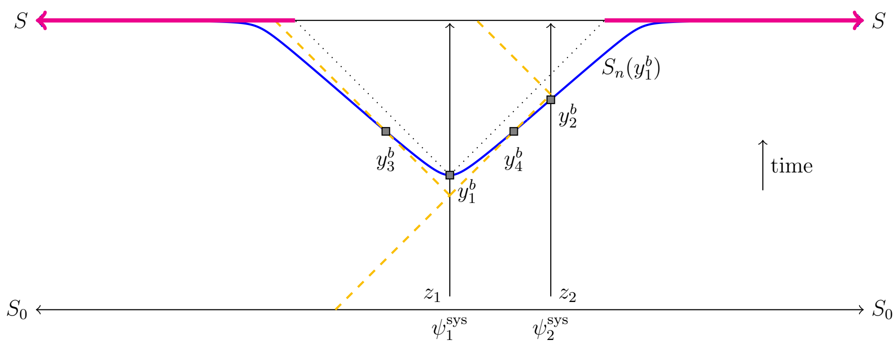


Figure 2.12: Depiction of a superposition of two local states at z_1 and z_2 with $S_n(y_1^b)$ being after the photon has interacted without the photon intersecting $S_n(y_1^b) \cap S$. The gray squares indicate cells in $S^1(y_1^b)$ whose states are among the summands in (2.67). kentdecoh2

reduced density matrix $\hat{\rho}_{y_1^b}$ will again be given by (2.69) with y_1^a replaced by y_1^b. However, in this case, Kent's beables $\langle T^{\mu\nu}(y_1^b)\rangle_{\tau_S}$ will not be given by (2.70) because in the limit as $n \to \infty$, the photon *will* be registered on $S_n(y_1^b) \cap S$.

To deal with the case when a photon is registered on $S_n(y_1^b) \cap S$, we consider a third example as depicted in figure 2.13.

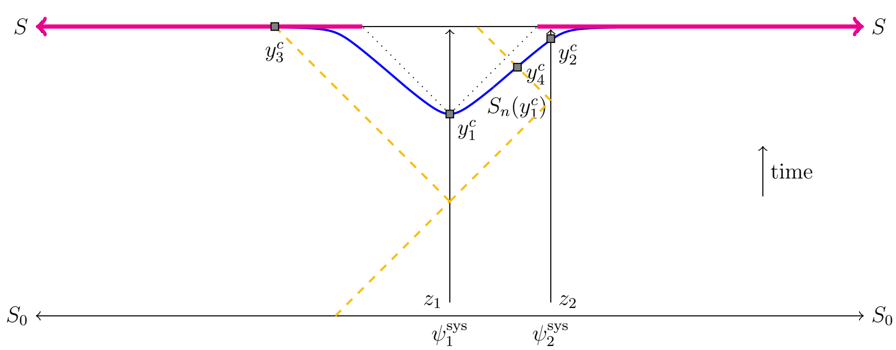


Figure 2.13: Depiction of a superposition of two local states at z_1 and z_2 with y_1^c sufficiently late that the photon intersects $S_n(y_1^c) \cap S$. The gray squares indicate cells in $S^1(y_1^c)$ whose states are among the summands in (2.67) kentdecoh3

In this case, the state of $S_n(y_1^c)$ will be

$$|\Psi_n\rangle = \big(c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle + c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |0_3\rangle |\gamma_4\rangle \big)\Xi$$

but now we have to consider the fact that the photon intersects $S_n(y_1^c) \cap S$. There are two possible (notional) measurement outcomes that can occur on $S_n(y_1^c) \cap S$: either $T_S = \tau_{S,1}$ where $\tau_{S,1}(y_3^c) \neq 0$, or $T_S = \tau_{S,2}$ where $\tau_{S,2}(y_3^c) = 0$.

The case $T_S = \tau_{S,1}$ indicates that there is a photon detection at y_3^c so that the local state at the y_3^c-cell is $|\gamma_3\rangle$. Therefore, if we write $\pi_{n,1}$ for the operator π_n, we have

$$\pi_{n,1} |\Psi_n\rangle = c_1 |\psi_1^{\text{sys}}\rangle |0_2\rangle |\gamma_3\rangle |0_4\rangle \, \Xi.$$

Therefore, $\langle \Psi_n | \pi_{n,1} \hat{T}^{\mu\nu}(y_1^c) | \Psi_n \rangle = |c_1|^2 \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^c) | \psi_1^{\text{sys}} \rangle$ and $\langle \Psi_n | \pi_{n,1} | \Psi_n \rangle = |c_1|^2$. Hence, by (2.42), Kent's beables at y_1^c will be

$$\langle T^{\mu\nu}(y_1^c) \rangle_{\tau_{S,1}} = \langle \psi_1^{\text{sys}} | \hat{T}^{\mu\nu}(y_1^c) | \psi_1^{\text{sys}} \rangle \, .$$

From this, it follows that the reduced density matrix at y_1^c will take the form of a pure state:

$$\hat{\rho}_{y_1^c} = |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}| \, . \tag{2.71}$$

On the other hand, for the case when $T_S = \tau_{S,2}$, this indicates that there is no photon detection at y_3^c, so that the local state at the y_3^c-cell will be $|0_3\rangle$. So if we now write $\pi_{n,2}$ for the operator π_n, we have

$$\pi_{n,2} |\Psi_n\rangle = c_2 |0_1\rangle |\psi_2^{\text{sys}}\rangle |0_3\rangle |\gamma_4\rangle \, \Xi.$$

Therefore, $\langle \Psi_n | \pi_{n,2} \hat{T}^{\mu\nu}(y_1^c) | \Psi_n \rangle = |c_2|^2 \langle 0_1 | \hat{T}^{\mu\nu}(y_1^c) | 0_1 \rangle$ and $\langle \Psi_n | \pi_{n,2} | \Psi_n \rangle = |c_2|^2$, and so by (2.42), Kent's beables at y_1^c will be

$$\langle T^{\mu\nu}(y_1^c) \rangle_{\tau_{S,2}} = \langle 0_1 | \hat{T}^{\mu\nu}(y_1^c) | 0_1 \rangle \, .$$

In this case, the reduced density matrix at y_1^c will be

$$\hat{\rho}_{y_1^c} = |0_1\rangle\langle 0_1| \, , \tag{2.72}$$

which is again a pure state.

In these examples we have therefore seen how the additional information concerning photon detection on $S_n(y_1) \cap S$ is able to determine whether the reduced density matrix at y_1 is a pure state or an improper mixture. Hence, Kent's interpretation offers an answer to d'Espagnat's problem of outcomes. As mentioned in section 1.11, d'Espagnat noticed that with decoherence theory alone, we are not entitled to give an ignorance interpretation to the reduced density matrix for a system that is an improper mixture, and thus we are not able to conclude from the reduced density matrix alone that an outcome has occurred. However, if the reduced density matrix of a system goes from being an improper mixture to a pure state of the form $|\psi\rangle\langle\psi|$ as it does when Kent's additional information is taken into account, then we can say that an outcome has occurred, namely the outcome of the system being in the state $|\psi\rangle$.

2.3.4 Butterfield's Analysis of Outcome Independence in Kent's interpretation

Let us now consider Kent's interpretation in the light of Shimony's notion of Outcome Independence (OI) as defined in section 2.1.

Butterfield[37] tries to answer the question of whether OI holds in Kent's interpretation by considering an example that builds on Kent's toy model. Butterfield's example is designed to capture the salient features of a Bell experiment where two spatially separated observers always observe opposite outcomes of some measurement. Following Kent, Butterfield thus considers a universe in one spatial dimensional. In this universe,

[37]See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 30–32

there are two entangled systems, a left-system and a right-system as depicted in figure 2.14.

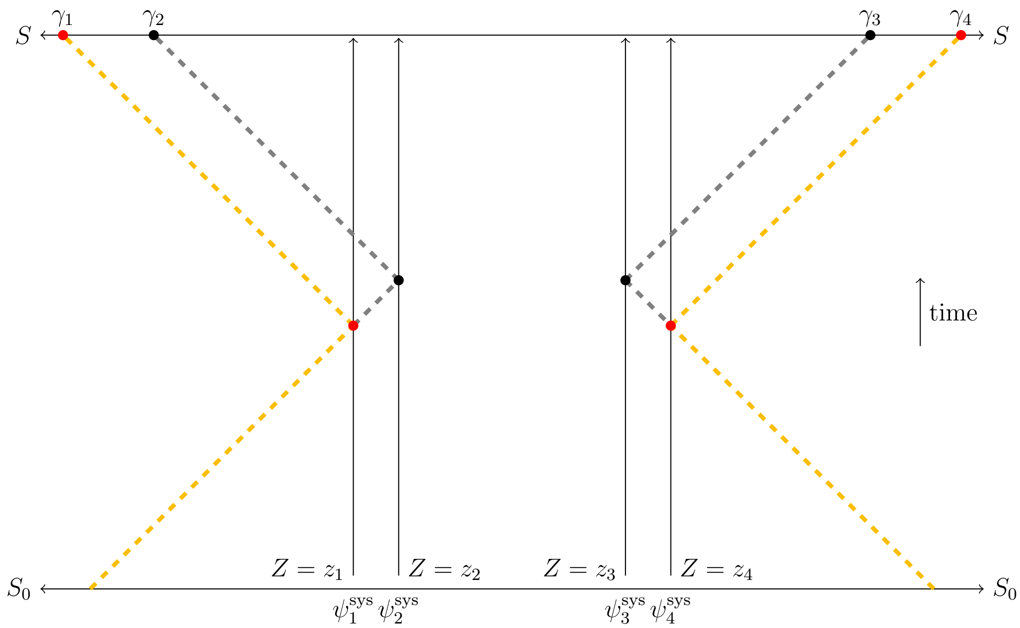


Figure 2.14: Butterfield's thought experiment for analyzing OI ButterfieldOI

Two locations z_1 and z_2 with $z_2 > z_1$ belong to a left-system, and there are two possible outcomes for a measurement on the left-system: either all the mass/energy of the left-system is localized at z_1 or all the mass/energy of the left-system is localized at z_2. These two possibilities are analogous to a spin up or a spin down measurement outcome in a Stern-Gerlach statement. Likewise, two locations z_3 and z_4 with $z_3 < z_4$ and $z_3 \gg z_2$ belong to a right-system, and again, there are two possible measurement outcomes: either all the mass/energy of the right-system is localized at z_3 or all the mass/energy of the right-system is localized at z_4.

The initial joint state of the two systems is $a\psi_1\psi_4 + b\psi_2\psi_3$. This means that the left-system will be found to be localized at z_1 with probability $|a|^2$, and at z_2 with probability $|b|^2$, and if the left-system is localized at z_1, the right system must be localized at z_4, whereas if the left-system is localized at z_2, then the right system must be localized at z_3.

Now Butterfield supposes that there are two photons, one coming in from the left that interacts with the left system, and one coming in from the right that interacts with the right system. As in Kent's toy model, there is a late time hypersurface S, on which the photons are "measured". Since the joint state of the two systems is in superposition, there will be two possible measurement outcomes for the two photons that arrive at S. Either the left-photon is measured at γ_1 and the right-photon is measured at γ_4, or the left-photon is measured at γ_2 and the right photon is measured at γ_3. Thus, if we suppose that the (notional) measurement for $T_S(x)$ yields an energy distribution $\tau_S(x)$ that is nonzero at γ_1 and γ_4, but is zero at γ_2 and γ_3, then we can say that the outcome of the measurement on the two systems is that the left system is localized at z_1 and the right system is localized at z_4. Moreover, the probability of this outcome is 1 given that the (notional) measurement of $T_S(x)$ on S is $\tau_S(x)$. In other words, this model is deterministic. But as we saw on page 78, if a model is deterministic, then OI must hold. This is the conclusion that Butterfield draws.

Now if Kent's interpretation is to be consistent with special relativity, OI being satisfied might initially seem concerning. Indeed, we saw on pages 76–78 that OI implies the negation of PI, and the negation of PI is not consistent with special relativity. However, there is one salient feature of a Bell experiment that is not captured in Butterfield's scenario, namely, in a Bell experiment, one can perform different measurements. PI and its negation only make sense when there are parameters that can be changed.

Furthermore, in the proof that OI implies the negation of PI,[38] it is assumed that the choice of parameter is not determined by the hidden variable λ. If the choice of parameters did depend on λ, then for $\hat{a} \neq \hat{b}$, at least one of the probabilities $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{c}}}(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+)$, $P_{\lambda,\hat{\boldsymbol{c}},\hat{\boldsymbol{b}}}(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+)$ or $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)$ would not be well-defined.[39] Even though Butterfield is only considering OI in his thought experiment, a proper analysis of OI shouldn't be undertaken without considering an experiment with parameters (e.g. knob settings that correspond to measurement axes of a Stern-Gerlach experiment). This is because the determination of whether OI holds will depend on what one counts as being the hidden variable of a system, and we need the hidden variable of a system to be such that the notion of PI is well-defined. Otherwise, one's verdict on OI will be irrelevant to Shimony's analysis of why Bell's inequality fails to hold.

2.3.5 Hidden variables and the Colbeck-Renner theorem

Butterfield assumes that the hidden variables in Kent's interpretation consist in the outcome $\tau_S(x)$ of $T_S(x)$ over the whole of S, and so far, I haven't questioned this assumption. However, this assumption is going to cause difficulties in the context of Shimony's analysis. This is because in Kent's interpretation, the information in $\tau_S(x)$ over the whole of S clearly would determine which parameters are chosen in a Bell experiment, for this information would determine where a silver atom coming out of a Stern-Gerlach apparatus would be detected on a detection screen (as depicted in figure 1.4), and from the position of this detection, one could determine the orientation of the magnetic field used in the Stern-Gerlach experiment. So if we stipulated that

[38]The proof that determinism implies the negation of PI (on pages 73 to 75), also assumes that the choice of parameter is not determined by the hidden variable λ.

[39]For example, if we thought of $P_{\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}}(X;Y)$ as a conditional probability $P(X;Y|\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}})$ and the probability $P(\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}) = 0$, then according to the definition of conditional probability, $P(X;Y|\lambda,\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}) = \frac{0}{0}$.

$\lambda = \tau_S$ is the hidden variable of every system in Kent's interpretation, then Kent's interpretation wouldn't satisfy the preconditions necessary for defining OI and PI. This would make Kent's Interpretation radically different from the pilot wave interpretation where one can define OI and PI because the hidden variables, being the positions and momenta of the particles, are independent of the measurement choices. An unfortunate consequence of not being able to define OI and PI is that we wouldn't be able to evaluate Kent's interpretation in the light of Shimony's analysis of why Bell's inequality fails to hold.

But it is not obvious that we should stipulate that $\lambda = \tau_S$ is the hidden variable of every system in Kent's interpretation. Just because we give τ_S a single label λ, it doesn't follow that τ_S is a single piece of information. There is typically going to be a huge amount of information in τ_S, and for a given system $\mathcal{S}$, we should discern carefully what collection of information in τ_S should be stipulated as being the hidden variable λ of $\mathcal{S}$. The criteria on which we should make such a decision should at least include the following:

1. all the information of λ is about $\mathcal{S}$ so that a change in λ corresponds to a change in the system $\mathcal{S}$. hidden1

In the pilot wave interpretation, the positions and momenta of the particles that constitute a system would fulfil this criterion. On the other hand, all the information in τ_S of Kent's interpretation would not fulfil this criterion unless of course $\mathcal{S}$ was the whole universe.

Note, however, that we don't insist that a difference in $\mathcal{S}$ entails a difference in λ. This is because a hidden-variables theory is envisaged as augmenting standard quantum theory. So in the case when $\mathcal{S}$ is not entangled with any other system, there will be a quantum state describing $\mathcal{S}$, and this quantum state can be other than it is (indicating that $\mathcal{S}$ can be in a different physical state) whilst the hidden variable remains the same. We thus impose a second criterion for a hidden-variables theory: hidden3

2. If λ is the hidden variable of a system $\mathcal{S}$ and if $|\phi\rangle$ is the quantum state of $\mathcal{S}$ or of some composite system $\mathcal{U}$ that contains $\mathcal{S}$ as a subsystem, then it is possible for there to be a different quantum state $|\phi'\rangle$ of $\mathcal{S}$ (or $\mathcal{U}$) while the hidden variable λ remains unchanged, and it is possible for there to be a different hidden variable λ while $|\phi\rangle$ remains unchanged.

This criterion is satisfied in the pilot wave interpretation, since the quantum state is the pilot wave itself. The pilot could be other than it is without any of the positions and momenta of the particles changing, but changing the pilot wave would result in a physical change of the system since the pilot wave governs how the positions and the momenta of the particles subsequently evolve over time.

Another criterion for a collection of information λ to constitute the hidden variable of a system $\mathcal{S}$ is the following: hidden2

3. it should be possible to change the measurement parameters when measuring $\mathcal{S}$ without this having any affect on λ.

If this criterion doesn't hold, we cannot even begin to consider whether PI holds in a given theory. In the pilot wave interpretation, the positions and momenta of the particles that constitute a system would fulfil this criterion, whereas all the information in τ_S of Kent's interpretation would not. We used this criterion when showing that OI implies the negation of PI.

Closely related to criterion 3 is the following criterion: `hidden5`

4. If p_λ is the probability that a system $\mathcal{S}$ has hidden variable λ, then p_λ must be independent of any choice of measurement made on $\mathcal{S}$.

We are thus assuming there is a whole range of possibilities for the hidden variable λ, but because we don't know what the hidden variable λ is, we can only assign it a probability. Knowledge of the quantum state of the system may help us assign such a probability, but this probability cannot depend on the choice of any future measurement we might make on the system. Butterfield refers to criterion 4 as the 'no-conspiracy' assumption, though he adds that this is a rather unfair label since there wouldn't necessarily be anything conspiratorial if this assumption was violated.[40]

We should also state explicitly a fifth criterion: `hidden4`

5. Suppose $\mathcal{A}$ is any system that is entangled with $\mathcal{S}$, and that the quantum state of the composite system $\mathcal{S}+\mathcal{A}$ is $|\phi\rangle_{\mathcal{S}+\mathcal{A}}$. Then for any measurement $O_\mathcal{S}$ on $\mathcal{S}$ and $O_\mathcal{A}$ on $\mathcal{A}$, there is a probability $P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S}=o_\mathcal{S}, O_\mathcal{A}=o_\mathcal{A})$

[40] See Butterfield, "Peaceful Coexistence: Examining Kent's Relativistic Solution to the Quantum Measurement Problem," 34.

for the joint measurement of $O_{\mathcal{S}}$ and $O_{\mathcal{A}}$ on $\mathcal{S}+\mathcal{A}$ that is a function of λ despite λ only referring to the system $\mathcal{S}$.

In addition to these five criteria for a hidden variable λ of a system $\mathcal{S}$, it is also desirable for a hidden-variables theory to satisfy PI and empirical adequacy. We defined PI for a two-outcome measurement on page 71, but it is easy to generalize the definition of PI for measurements with more than two outcomes. Thus, using the notation of criterion 5 and letting $O'_{\mathcal{A}}$ denote a second choice of measurement on $\mathcal{A}$, PI states that

$$\sum_{o_{\mathcal{A}}} P_{\lambda}^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}}=o_{\mathcal{S}}, O_{\mathcal{A}}=o_{\mathcal{A}}) = \sum_{o'_{\mathcal{A}}} P_{\lambda}^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}}=o_{\mathcal{S}}, O'_{\mathcal{A}}=o'_{\mathcal{A}}) \tag{PI}$$

where the summations on both sides are over all the possible measurement outcomes of $O_{\mathcal{A}}$ and $O'_{\mathcal{A}}$ respectively.

As for the definition of **empirical adequacy** (EA), using the notation of criteria 4 and 5, this states that

{adeq}

$$\sum_{\lambda\in\Lambda} p_{\lambda} P_{\lambda}^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}}=o_{\mathcal{S}}, O_{\mathcal{A}}=o_{\mathcal{A}}) = P^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}}=o_{\mathcal{S}}, O_{\mathcal{A}}=o_{\mathcal{A}}) \tag{EA}$$

where Λ is the set of all hidden variables so that $\sum_{\lambda\in\Lambda} p_{\lambda} = 1$, and where

$$P^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_{\mathcal{S}}=o_{\mathcal{S}}, O_{\mathcal{A}}=o_{\mathcal{A}})$$

is the standard probability calculated using the Born rule with the eigenstates of the observables $\hat{O}_{\mathcal{S}}$ and $\hat{O}_{\mathcal{A}}$ and the quantum state $|\phi\rangle_{\mathcal{S}+\mathcal{A}}$. EA is essentially the same as equation (2.7). It also has some similarities with (2.43), though the main difference is the range of the summation – the index of the summands of (2.43) does not parametrize hidden variables that satisfy criteria 1 to 5 above.

Now it turns out that criteria 1 to 5 together with the conditions of PI and EA are very restrictive. In his 2016 paper, Leegwater proves a version of the Colbeck-Renner theorem.[41] Leegwater's version takes the following form: if one defines hidden variables according to criteria 1 to 5, then in any hidden-variables theory for which PI and EA hold, the hidden variables are redundant. In other words, in the notation of criterion 5,

{colbeckrenner}

$$P_\lambda^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S}, O_\mathcal{A} = o_\mathcal{A}) = P^{|\phi\rangle_{\mathcal{S}+\mathcal{A}}}(O_\mathcal{S} = o_\mathcal{S}, O_\mathcal{A} = o_\mathcal{A}) \tag{2.73}$$

for any measurement $O_\mathcal{S}$ on $\mathcal{S}$ and $O_\mathcal{A}$ on $\mathcal{A}$.[42]

Thus, the Colbeck-Renner theorem means that we cannot hope to make Kent's interpretation into a hidden-variables theory that satisfies PI and AE by simply defining more carefully what the hidden variables should be, for the information in Kent's interpretation is clearly non-redundant.

But nevertheless, it still seems that we should be able to make some kind of sense of PI and AE in Kent's interpretation and that we should be able to evaluate Kent's interpretation on the basis of whether these notions of PI and EA are true in this context. To achieve this aim, one strategy would be to relax one of the five criteria for a hidden variable. Since we still want to be able to make sense of PI and AE, a process of elimination suggests that the most obvious hidden variable criterion to drop would be criterion 2. In other words, instead of thinking of τ_S as an augmentation of standard quantum theory, we could instead think of τ_S as a rather elaborate way of

[41] See Leegwater, "An impossibility theorem for parameter independent hidden-variable theories."

[42] Strictly speaking, we should say that equation (2.73) holds for almost all λ, but we need not concern ourselves here with the details of measure theory that would be needed to make sense of this qualification.

stipulating the initial quantum states of experiments as well as the quantum states of measurement outcomes. The information of τ_S would then be non-redundant. Moreover, if we could appropriately partition the information in $\tau_S(x)$ on the basis of whether it determined the quantum state of the particle, or the quantum state of the apparatus, or the quantum state of the rest of the universe, we could then consider whether Kent's interpretation gave the same predictions as standard quantum theory. If it did, then PI and AE would hold in Kent's interpretation, since these both hold in standard quantum theory. And since Kent's interpretation is formulated in the Lorentz invariant setting of Schwinger and Tomonaga, this would mean that Kent's interpretation is a solution to the measurement problem!

2.3.6 Kent's interpretation and standard quantum theory*

In this section, I will show that Kent's interpretation does indeed give the same predictions as standard quantum physics in the case of an experimental apparatus $\mathcal{A}$ measuring the properties of a particle $\mathcal{S}$.

Let's assume that the apparatus has already interacted with many photons during its existence up until the time t_i. Likewise, let's assume the particle has interacted with many photons up until this time. Now suppose that the hypersurface S has energy density $\tau_S(x)$ indicating that some photons have been "measured" on S to be in a state $|\gamma_i^{(\mathcal{A})}\rangle$ which is correlated with the apparatus being in a state $|a\rangle$ shortly before time t_i and in the vicinity of spatial location z_1 as depicted in figure 2.15. Similarly, we suppose that $\tau_S(x)$ also indicates that some photons have been "measured" on S to be in a state $|\gamma_i^{(\mathcal{S})}\rangle$ which is correlated with the particle being in a state $|s\rangle$ also shortly before time t_i and in the vicinity of spatial location z_1. We assume that the time t_i is just before the particle enters the apparatus (given the measurent of $|\gamma_i^{(\mathcal{S})}\rangle$ and $|\gamma_i^{(\mathcal{A})}\rangle$ on S) and that no more photons are measured on S that have become entangled with particle or the apparatus until the particle emerges from the apparatus at time t_f. Then some more photons interact with the apparatus $\mathcal{A}$ and get entangled with it shortly before time t_m, and these photons are detected on S to be in a state $|\gamma_f''\rangle$, and this state is correlated with the apparatus now being in a state $|a_f\rangle$, and hence the particle being in state $|s_f\rangle$.

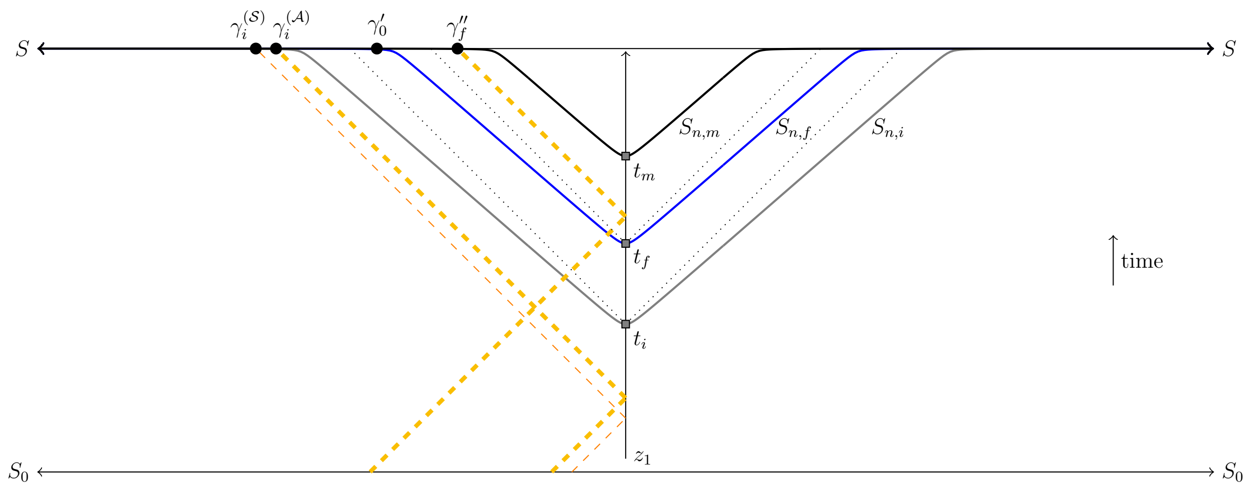


Figure 2.15: Depicts an experiment where the state of some photons $\gamma_i^{(\mathcal{S})}$ and $\gamma_i^{(\mathcal{A})}$ on the hypersurface S determines the initial conditions of an experimental setup of a particle $\mathcal{S}$ and apparatus $\mathcal{A}$ in the vicinity of the spacetime location (z_1, t_i). The state of the photons γ_f'' on the hypersurface S determines the final state of the apparatus after the particle has left it at time t_f so that the apparatus at time t_m displays a definite measurement outcome. It is assumed that no incoming photons have become entangled with the experiment after the $\gamma_i^{(\mathcal{S})}$ and $\gamma_i^{(\mathcal{A})}$ photons and before the γ_f'' photons have become entangled with the experiment.

`pisolution`

We aim to show that within Kent's interpretation, we can calculate the probability the particle emerges from the measuring apparatus $\mathcal{A}$ in state $|s_f\rangle$ given that it enters $\mathcal{A}$ in state $|s\rangle$, and that this probability is the same as if one ignored S and just applied the Born rule to $|s\rangle$ and $|s_f\rangle$.

In order to show this, let us choose a sequence of hypersurfaces $S_{n,i}$ which go through the spacetime location $y_i = (t_i, z_1)$ such that $\lim_{n\to\infty} S_{n,i} \cap S = S^1(y_i)$, where as usual, $S^1(y_i)$ consists of all the spacetime locations of S outside the light cone of y_i. Let us assume that n is sufficiently large so that the photons described by $|\gamma_i^{(\mathcal{S})}\rangle$ and $|\gamma_i^{(\mathcal{A})}\rangle$ belong to $S_{n,i}$. The hypersurface $S_{n,i}$ and the photons being reflected from the vicinity of z_1 just before time t_i are also depicted in figure 2.15.

Typically, the quantum state $|\Psi_{n,i}\rangle = U_{S_{n,i},S_0}|\Psi_0\rangle$ of the hypersurface $S_{n,i}$ (where $U_{S_{n,i},S_0}$ is the unitary operator relating the states of two hypersurfaces as discussed on page 88) will also include photon correlations with $\mathcal{S}$ and $\mathcal{A}$ corresponding to other possible "measurements" of $T_S(x)$ besides $\tau_S(x)$. So in general, we would expect the state of $S_{n,i}$ to be of the form

$$|\Psi_{n,i}\rangle = \sum_{j,k} c_{j,k}\, |\sigma_j\rangle\, |\alpha_k\rangle\; |\gamma_j^{(\mathcal{S})}\rangle\; |\gamma_k^{(\mathcal{A})}\rangle\,,$$

where $\{|\sigma_j\rangle : j\}$ is an orthonormal basis of states for the particle $\mathcal{S}$ with $|s\rangle \in \{|\sigma_j\rangle : j\}$, $\{|\alpha_k\rangle : k\}$ is an orthonormal basis of states describing the apparatus $\mathcal{A}$ with $|a\rangle \in \{|\alpha_k\rangle : k\}$, $\{\,|\gamma_j^{(\mathcal{S})}\rangle : j\}$ are normalized states of photons in $S_{n,i} \cap S$ that are entangled with the particle $\mathcal{S}$ such that $\langle\gamma_j^{(\mathcal{S})}|\gamma_{j'}^{(\mathcal{S})}\rangle \approx 0$ for $j \neq j'$, $\{\,|\gamma_k^{(\mathcal{A})}\rangle : k\}$ are normalized states of photons in $S_{n,i} \cap S$ that are entangled with the apparatus $\mathcal{A}$ such that $\langle\gamma_k^{(\mathcal{A})}|\gamma_{k'}^{(\mathcal{A})}\rangle \approx 0$ for $k \neq k'$, and for clarity, we have absorbed any other environmental information into the states $|\alpha_k\rangle$.

If we now define the projection $\pi_{n,i}$ corresponding to the "measurement outcome" $\tau_S(x)$ on $S_{n,i} \cap S$ as in equation (2.38), and if we also assume that the bases are indexed so that $|s\rangle = |\sigma_i\rangle$ and $|a\rangle = |\alpha_i\rangle$ then

{piphi}

$$\pi_{n,i}\,|\Psi_{n,i}\rangle \approx c\,|s\rangle\,|a\rangle\,\left|\gamma_i^{(\mathcal{S})}\right\rangle\,\left|\gamma_i^{(\mathcal{A})}\right\rangle \tag{2.74}$$

where $c = c_{i,i}$. For convenience, we will omit the reference to n and write S_i for $S_{n,i}$ and π_i for $\pi_{n,i}$. We will also write $|\Phi_i\rangle$ for the normalized state of $\pi_i\,|\Psi_i\rangle$ so that

{phii}

$$|\Phi_i\rangle \approx |s\rangle\,|a\rangle\,\left|\gamma_i^{(\mathcal{S})}\right\rangle\,\left|\gamma_i^{(\mathcal{A})}\right\rangle. \tag{2.75}$$

We now suppose that at time t_i, $|a\rangle$ is the ready state of the apparatus with pointer states $\{|a_j\rangle : j\}$ so that if $|s\rangle = \sum_i c_j\,|s_j\rangle$, then under Schrödinger evolution from time t_i to t_f,

$$|s\rangle\,|a\rangle \to \sum_j c_j\,|s_j\rangle\,|a_j\rangle\,.$$

We assume that before time t_f, no photons have had a chance to get entangled with $\mathcal{S} + \mathcal{A}$. It is only by time t_m that we assume a measurement of photons in state $\left|\gamma_f''\right\rangle$ on S outside the light cone of (t_m, z_1) is able to determine that the apparatus is in state $|a_f\rangle$ and hence that the particle is in state $|s_f\rangle$. Although the measurement outcome $\tau_S(x)$ on the whole of S determines with probability 1 that the apparatus and the particle will be in the states $|a_f\rangle$ and $|s_f\rangle$ respectively at time t_m, if we consider the probability $P(f|\,|\Phi_i\rangle)$ that this outcome occurs based just on the state $|\Phi_i\rangle$, then typically this probability is going to be less than 1.

To calculate this probability, we first consider the evolution of $|\Phi_i\rangle$ to $U_{S_f,S_i}\,|\Phi_i\rangle$. It's possible that there may be photons "measured" on S between the hypersurfaces S_i and S_f as indicated in figure 2.15 (i.e. on $(S \cap S_f) \setminus (S \cap S_i)$), but we are assuming that they do not get entangled with the different pointer states of the apparatus. In other words, $|\Phi_i\rangle$ will evolve to a state of the form

$$U_{S_f,S_i}|\Phi_i\rangle \approx \sum_j c_j|s_j\rangle|a_j\rangle\,|\gamma_i^{(\mathcal{S})}\rangle\,|\gamma_i^{(\mathcal{A})}\rangle\sum_k g_k|\gamma_k'\rangle\,, \tag{2.76}$$

{USfievolve1}

where $|\gamma_k'\rangle$ correspond to the possible measurements of $T_S(x)$ on $(S\cap S_f)\setminus(S\cap S_i)$, and $\sum_k |g_k|^2 = 1$.

But from time t_f to t_m, we assume that the apparatus does get entangled with photons which are measured on $S\cap S_m$. Thus, if $\{|\gamma_j''\rangle : j\}$ are the normalized states representing the possible measurements outcomes of these photons such that $\langle\gamma_j''|\gamma_k''\rangle \approx 0$ for $j\neq k$, then

$$U_{S_m,S_f}U_{S_f,S_i}|\Phi_i\rangle \approx \sum_j c_j|s_j\rangle|a_j\rangle\,|\gamma_i^{(\mathcal{S})}\rangle\,|\gamma_i^{(\mathcal{A})}\rangle\sum_k g_k|\gamma_k'\rangle\,|\gamma_j''\rangle\,.$$

Since we are assuming that at time t_m a measurement of photons on $S\cap S_m$ is able to determine that the apparatus is in state $|a_f\rangle$, this can only happen if $U_{S_m,S_f}U_{S_f,S_i}|\Phi_i\rangle$ is found to be in one of the states $|\Phi_{k,f}\rangle$ for some k where

$$|\Phi_{k,j}\rangle = |s_j\rangle|a_j\rangle\,|\gamma_i^{(\mathcal{S})}\rangle\,|\gamma_i^{(\mathcal{A})}\rangle|\gamma_k'\rangle\,|\gamma_j''\rangle\,.$$

By the Born Rule, the probability $|\Phi_i\rangle$ will be found to be in state $|\Phi_{k,j}\rangle$ will be

$$|\langle\Phi_{k,j}|\Phi_i\rangle| = |c_j|^2|g_k|^2.$$

Therefore,

$$P(f|\,|\Phi_i\rangle) = \sum_k |\langle\Phi_{k,f}|\Phi_i\rangle| = |c_f|^2 = |\langle s_f|s\rangle|^2.$$

Hence, the probability that a complete measurement of $T_S(x)$ on S will give a measurement outcome of the particle being in state $|s_f\rangle$ given the partial measurement of $T_S(x)$ on $S_i\cap S$ determines the particle to be initially in the state $|s\rangle$ will be the same as the standard Born rule probability $|\langle s_f|s\rangle|^2$ of $|s\rangle$ being found to be in state $|s_f\rangle$.

We can also recover this probability using Kent's conditional expectation. To do this, we recall that in standard quantum mechanics, if for some state $|\psi\rangle$ of a system we define the operator $[\psi] = |\psi\rangle\langle\psi|$, then when the system is in some initial state $|\chi\rangle$, the Born rule implies that $\langle\chi|[\psi]|\chi\rangle = P(\psi|\chi)$, where $P(\psi|\chi)$ is the probability that the system will be found to be in state $|\psi\rangle$ given that it was initially in state $|\chi\rangle$. But by (1.8), $\langle\chi|[\psi]|\chi\rangle$ is just the expectation $\langle\psi\rangle_\chi$ of $[\psi]$ when $[\psi]$ is treated as an observable.

Now in equation (2.42), we saw how to calculate the expectation value $\langle T^{\mu\nu}(y)\rangle_{\tau_S}$ of the observable $\hat{T}^{\mu\nu}(y)$ given the notional measurement τ_S on S outside the light cone of y. This suggests that the expectation value of any observable $\hat{O}$ defined at spacetime location (t_i, z_1) given the notional measurement τ_S on S outside the light cone of (t_i, z_1) is going to be

$$\langle\hat{O}\rangle_{\tau_S} = \frac{\langle\Psi_i|\pi_i\hat{O}|\Psi_i\rangle}{\langle\Psi_i|\pi_i|\Psi_i\rangle}.$$

By (2.74), $\langle\Psi_i|\pi_i|\Psi_i\rangle = |c|^2$, and so taking $\hat{O}$ to be $[s_f]$ we have

$$\langle[s_f]\rangle_{\tau_S} = \frac{|c|^2|\langle s_f|s\rangle|^2}{|c|^2} = |\langle s_f|s\rangle|^2.$$

Thus, Kent's conditional expectation $\langle[s_f]\rangle_{\tau_S}$ gives us the same probability $|\langle s_f|s\rangle|^2$ for a particle transitioning from state $|s\rangle$ to state $|s_f\rangle$ as in standard quantum mechanics.

Also note that we can typically expect the $|\gamma_i^{(\mathcal{S})}\rangle$-state to be independent of the $|\gamma_i^{(\mathcal{A})}\rangle$-state. Therefore, since $|\gamma_i^{(\mathcal{A})}\rangle$ will determine the measurement choice, and since

$|\gamma_i^{(\mathcal{S})}\rangle$ determines the initial state of the particle, we can expect the state of the particle to be independent of the measurement choice in Kent's interpretation. Thus, we can fulfil one of the necessary criteria (i.e. criterion 3) for PI to be a well-defined notion.

2.3.7 Kent's Interpretation and Parameter Independence

In addition to criterion 3 being satisfied, criterion 5 must also be true if PI is to be a well-defined notion. In the previous section, we saw how we can generalize Kent's beable $\langle \hat{T}^{\mu\nu}(y)\rangle_{\tau_S}$ to calculate conditional expectations $\langle \hat{O}\rangle_{\tau_S}$ for any observable $\hat{O}$ defined at a particular spacetime location (t_i, z_1). Calculating the probability for two measurements requires calculating the conditional expectation of an observable that depends on two spacetime locations. In order to do this, we need to make a further adaption to Kent's interpretation. In this section, we will describe this adaption and show that with it, Kent's interpretation allows us to calculate probabilities for Bell-type experiments, and that these probabilities are the same as in standard quantum physics. Since PI holds in standard quantum theory, a consequence of Kent's interpretation agreeing with standard quantum theory is that PI will also hold in Kent's interpretation.

So let's consider figure 2.16 which depicts a one-dimensional view of a Bell-type experiment. There is a left wing of the experiment located in the vicinity of z_L, and a right wing of the experiment located in the vicinity of z_R. Shortly before time t_i, photons interact with a Stern-Gerlach apparatus on the left wing and a Stern-Gerlach apparatus on the right wing, with the result that the photons being measured on a hypersurface $S_{n,i} \cap S$ to be in states $|\gamma_i^{(\mathcal{A}_L)}\rangle$ and $|\gamma_i^{(\mathcal{A}_R)}\rangle$ determine the measurement

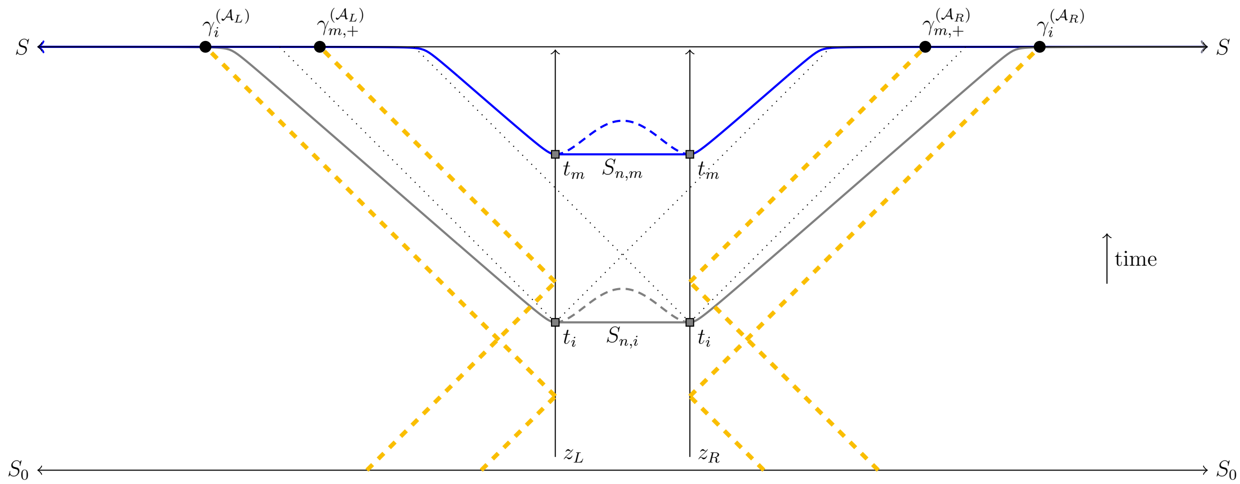


Figure 2.16: Depicts a Bell-type experiment where the state of some photons $\gamma_i^{(\mathcal{A}_L)}$ and $\gamma_i^{(\mathcal{A}_R)}$ on the hypersurface S determines the choice of measurement parameters of the left wing and right wing of the experiment respectively, and some photons $\gamma_{m,+}^{(\mathcal{A}_L)}$ and $\gamma_{m,+}^{(\mathcal{A}_R)}$ on the hypersurface S determine the measurement outcome of the experiment on the left wing and the right wing respectively. The dashed lines on the hypersurfaces $S_{n,m}$ and $S_{n,i}$ indicate other choices for the hypersurfaces, but they still lead to the same probability being calculated. `bellsolution`

parameters of the apparatuses on the left wing and the right wing of the experiment respectively.

We need to adapt Kent's sequences of hypersurfaces in order to proceed. Sequences of hypersurfaces $S_{n,i}$ are chosen so that they all contain the spacetime locations $y_L = (t_i, z_L)$ and $y_R = (t_i, z_R)$, and that in the limit, $\lim_{n\to\infty} S_{n,i}$ contains as much of $S^1(y_L)$ and $S^1(y_R)$ as possible, where as usual, $S^1(y)$ denotes the subset of S lying outside the light cone of y. Ultimately, this limit (unlike the limit of Kent's hypersurfaces) will not contain the whole of $S^1(y_L)$ or $S^1(y_R)$, but only serves to guarantee that we use as much of the information in S as possible in calculating the expectation values of observables at (t_i, z_L) and (t_i, z_R). There will be some degree of freedom in what we choose for the hypersurface between (t_i, z_L) and (t_i, z_R) as depicted by the dashed line in the figure. However, such freedom will have no effect on the probabilities calculated, because under the assumption that the hypersurface is very far into the future, there will be no choice of hypersurface in this region that would give us more information in S to condition on. Also, we recall that the stress-energy operators in the Tomonaga-Schwinger formulation of relativistic quantum physics are chosen so that they are invariant under perturbations of the hypersurface, so under the assumption that all physical observables will be ultimately expressible in terms of the stress-energy operators, the arbitrary choice of the hypersurfaces in regions that can't intersect with S will have no effect of the probabilities calculated.

On the hypersurface $S_{n,i}$, we assume that there are some photons "measured" on it to be in the states $|\gamma_i^{(\mathcal{A}_L)}\rangle$ and $|\gamma_i^{(\mathcal{A}_R)}\rangle$ that determine the choice of measurement axes for the left and right wings of the experiment respectively. We assume that the axis of orientation of the right wing Stern-Gerlach apparatus makes an angle θ with the axis of the left wing apparatus.

We also assume that there are two particles that together form a Bell-state

{bellstatePI}

$$\frac{1}{\sqrt{2}}\big(\left|\hat{\mathbf{s}}+\right\rangle_L \left|\hat{\mathbf{s}}-\right\rangle_R - \left|\hat{\mathbf{s}}-\right\rangle_L \left|\hat{\mathbf{s}}+\right\rangle_R\big). \tag{2.77}$$

We saw in footnote 17 on page 17 that a Bell state does not depend on the orientation of $\hat{\mathbf{s}}$, so without loss of generality, we can suppose that the $\left|\hat{\mathbf{s}}+\right\rangle_L$ and $\left|\hat{\mathbf{s}}-\right\rangle_L$ are pointer states for the apparatus on the left-wing of the experiment. This means there will be a ready state $\left|a\right\rangle_L$ as well as two states $\left|a+\right\rangle_L$ and $\left|a-\right\rangle_L$ of the left wing apparatus such that

$$\left|\hat{\mathbf{s}}\pm\right\rangle_L \left|a\right\rangle_L \to \left|\hat{\mathbf{s}}\pm\right\rangle_L \left|a\pm\right\rangle_L .$$

As for the right wing of the experiment, we let $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R$ and $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R$ be pointer states for the apparatus so that there is a ready state $|a\rangle_R$ as well as two states $|a_\theta+\rangle_R$ and $|a_\theta-\rangle_R$ of the right wing apparatus such that

$$|\hat{\boldsymbol{s}}_\theta\pm\rangle_R\,|a\rangle_R \to |\hat{\boldsymbol{s}}_\theta\pm\rangle_R\,|a_\theta\pm\rangle_R\,.$$

As in approximation (2.75), the detections of the photons on $S_{n,i}\cap S$ being in state $|\gamma_i^{(\mathcal{A}_L)}\rangle$ and $|\gamma_i^{(\mathcal{A}_R)}\rangle$ determine the two particles and the apparatuses on both wings of the experiment to be in the state

{bellstatePI2}

$$|\Phi_i\rangle \approx \frac{1}{\sqrt{2}}(\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\mathrm{s}}-\rangle_R - \;|\hat{\mathrm{s}}-\rangle_L\;|\hat{\mathrm{s}}+\rangle_R)\,|a\rangle_L\,|a\rangle_R\;|\gamma_i^{(\mathcal{A}_L)}\rangle\;|\gamma_i^{(\mathcal{A}_R)}\rangle\,. \tag{2.78}$$

As in equations (1.2a) and (1.2b), we have

$$|\hat{\mathrm{s}}+\rangle_R = \alpha_\theta\,|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R + \beta_\theta\,|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R\,,$$

$$|\hat{\mathrm{s}}-\rangle_R = \alpha_\theta\,|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R - \beta_\theta\,|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R\,,$$

where $\alpha_\theta = \cos(\theta/2)$, and $\beta_\theta = \sin(\theta/2)$. Substituting this into (2.78), we can express the state of the two particles as

{bellstatePI3}

$$\begin{aligned}|\Phi_{n,i}\rangle \approx \frac{1}{\sqrt{2}}(&\alpha_\theta\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R - \beta_\theta\,|\hat{\mathrm{s}}+\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R \\ &- \alpha_\theta\,|\hat{\mathrm{s}}-\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R - \beta_\theta\,|\hat{\mathrm{s}}-\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}-\rangle_R)\,|a\rangle_L\,|a\rangle_R\;|\gamma_i^{(\mathcal{A}_L)}\rangle\;|\gamma_i^{(\mathcal{A}_R)}\rangle\,.\end{aligned} \tag{2.79}$$

If we apply the unitary operator $U_{S_{n,m},S_{n,i}}$ to each of the terms of (2.79), we get

$$\begin{aligned}&U_{S_{n,m},S_{n,i}}\,|\hat{\mathrm{s}}\pm\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}\pm'\rangle_R\,|a\rangle_L\,|a\rangle_R\;|\gamma_i^{(\mathcal{A}_L)}\rangle\;|\gamma_i^{(\mathcal{A}_R)}\rangle \\ &\qquad = |\hat{\mathrm{s}}\pm\rangle_L\;|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}\pm'\rangle_R\,|a\pm\rangle_L\,|a_\theta\pm'\rangle_R\;|\gamma_{m,\pm}^{(\mathcal{A}_L)}\rangle\;|\gamma_{m,\pm'}^{(\mathcal{A}_R)}\rangle\end{aligned} \tag{2.80}$$

where $|\gamma_{m,\pm}^{(\mathcal{A}_L)}\rangle$ are the states of possible detections of photons on $S_{n,m}$ that would determine the left wing apparatus to be in the state $|a\pm\rangle_L$, and where $|\gamma_{m,\pm}^{(\mathcal{A}_R)}\rangle$ are the states of possible detections of photons on $S_{n,m}$ that would determine the right wing apparatus to be in the state $|a_\theta\pm'\rangle_R$. Using the Born rule, we therefore see that given a measurement of $T_S(x)$ determines the state of the hypersurface $S_{n,i}$ to be in state $|\Phi_{n,i}\rangle$, the probability that the hypersurface $S_{n,m}$ will be found to be in the state

$$|\hat{\mathbf{s}}+\rangle_L \; |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R \, |a+\rangle_L \, |a_\theta+\rangle_R \; |\gamma_{m,+}^{(\mathcal{A}_L)}\rangle \; |\gamma_{m,+}^{(\mathcal{A}_R)}\rangle$$

will be

$$\frac{1}{2}|\beta_\theta|^2 = \frac{1}{2}\sin^2(\theta/2)$$

From this it follows that the probability that the left wing particle will be in state $|\hat{\mathbf{s}}+\rangle_L$ and that the right wing particle will be in state $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R$ given the initial conditions will also be $\frac{1}{2}\sin^2(\theta/2)$. This is the same probability as that given by standard quantum theory on page 17.

Also note that if we define the observable $[\hat{\mathbf{s}}+]_L = |\hat{\mathbf{s}}+\rangle_{L\,L}\langle\hat{\mathbf{s}}+|$ that depends on spacetime location (t_1, z_L), and the observable $[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R = |\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_{R\,R}\langle\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+|$ that depends on spacetime location (t_1, z_R), then we can construct the observable $[\hat{\mathbf{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R$, and with the adapted sequence $S_{n,i}$ of hypersurfaces, we can calculate the conditional expectation

$$\langle[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R\rangle_{\tau_S} = \lim_{n\to\infty} \frac{\langle\Psi_{n,i}|\pi_{n,i}[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R|\Psi_{n,i}\rangle}{\langle\Psi_{n,i}|\pi_{n,i}|\Psi_{n,i}\rangle}.$$

With this adaption and the notional measurement of $T_S(x)$ on S described in this section, it is easy to see that

$$\langle[\hat{\boldsymbol{s}}+]_L[\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+]_R\rangle_{\tau_S} = \frac{1}{2}\sin^2(\theta/2)$$

which is the joint probability for finding the left wing particle in state $|\hat{\mathbf{s}}+\rangle_L$ and the right wing particle in state $|\hat{\boldsymbol{s}}_{\boldsymbol{\theta}}+\rangle_R$. Thus, we can adapt Kent's model so that criterion

5 of page 127 is satisfied and such that Kent's model gives the same probabilities as standard quantum physics. Hence, PI holds in Kent's interpretation.

2.3.8 An objection to Kent's beables

There is still the question of why Kent decides that the beables of his theory should take the form of expectation values. As an analogy, it seems a bit like saying a six sided dice can actually come up with a 3.5 since this is its expectation value. As an alternative to Kent's $\langle \hat{T}_{\mu,\nu}(y) \rangle_{\tau_S}$ beables, we could instead stipulate the reduced density matrices $\hat{\rho}_y$ as calculated in section **??** as beaare more fundamental beables than the conditional expectation values of $\hat{T}_{\mu,\nu}(y)$, and that the determinate value of T_S is more fundamental still. There are a few things I need to explain here such as what I mean fundamental, what I mean by saying a beable is a reduced density matrix, and how the beable of a reduced density matrix can give rise to Kent's expectation value beables.

By saying that the determinate value of T_S is more fundamental than the reduced density matrices, I mean that if we have a statement of the form "the beable at y is $\hat{\rho}_y$", then such a statement is reducible to statements about T_S. Such statements might be simple factual statements such as "there is photon is at x_1, and x_2 but not at x_3 or x_4" where the locations are indicated in figure 2.17.

With enough such factual statements, we would have enough information on T_S to say that there is a local state ψ_1^{sys} at spacetime location y_1 and hence conclude that the statement p="the beable at y_1 is $|\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}|$" is true. But we would very likely need quite a lot more information than knowledge of the fact that a photon is at

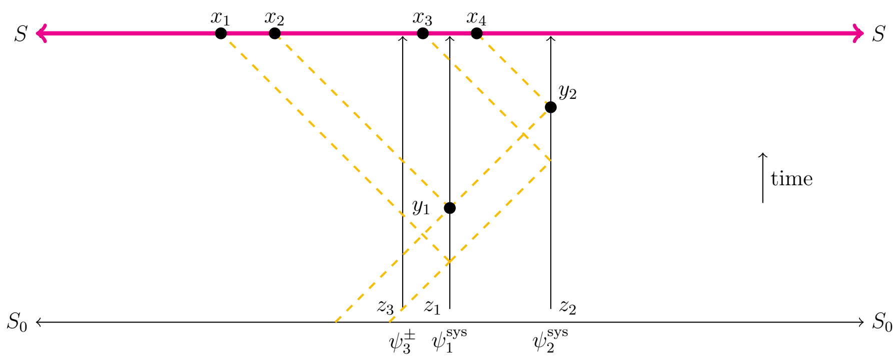


Figure 2.17: Depiction of a superposition of two local states at z_1 and z_2 with y_1^c sufficiently late that the photon intersects $S_n(y_1^c) \cap S$. With enough photon detections on S we can make statements about

`kentdecoh5`

x_1, and x_2, to conclude p since we would need to be able to work out something about time at which the incoming arrived in the vicinity of the system, and this would depend on physics of photon creation. But it seems plausible that from all the information in T_S one could make statements such as p that involve pure states. It is no more controversial than the assumption that we can draw valid conclusions about the physical world based on which cells in our retina are excited. What is controversial is my (and Kent's) suggestion that the information contained in T_S determines the state of physical reality on earlier hypersurfaces rather than the physical state of the earlier hypersurfaces determining the information contained in T_S. In the final chapter, I will aim to justify this suggestion and show that it is not quite as alien to common sense as it might first seem.

As for statements that involve improper mixtures, for example statements of the form q="the beable at y_1 is $|c_1|^2 \, |\psi_1^{\text{sys}}\rangle\langle\psi_1^{\text{sys}}| + |c_2|^2 \, |0_1\rangle\langle 0_1|$", we could take these to be expressible in terms of modal statement about such as "it's possible that a photon could have been detected at either x_1 or x_3 but not both" as depicted in figure 2.17. As in the case for pure states, we would also need other modal statements and declarative propositions about T_S in order to say enough about the times at which incoming photons would arrive at the z_1 and z_2. But with enough information on T_S it seems plausible that we could build up a picture of such photon interaction on earlier hypersurfaces. Also, by comparing the values of T_S over different region of S, we could detect similar configurations from which a super-intelligent being could conclude that these similar configurations correspond to some kind of activity such as a human being performing an experiment. But these multiple configurations would also have differences, some of which would correspond to different measurement outcomes. By surveying many of these configurations, the super-intelligent being could assign probabilities to these outcomes and hence calculate expectation values for observables and the reduced density matrices that would give rise to these expectation values and hence make statements like q which involve improper mixtures.

At this point it is worth reminding ourselves that Kent is not saying that an actual measurement of T_S on S is made, but only a notional measurement which is to say if a measurement were made on T_S it would have a determinate value τ_S, say. One could choose a hypersurface S' even later than S, but Kent supposes that the description of physical reality between S_0 and S is not going to be significantly different if the one used the notional measurements for $T_{S'}$ on S' rather than those on T_S.

So let's now consider how the understanding of beables in terms of reduced density matrices can give rise to Kent's beables as conditional expectation values of $T_{\mu,\nu}(y)$. In this section we have been approximating S_n as a coarse-grained model so that S_n

is treated as a mesh of tiny cells labeled. The state of these one of these tiny cells, y_k would not in general be pure eigenstates of $T_{\mu,\nu}(y_k)$

Bibliography

Bell, J. S. “Subject and Object.” In *Speakable and unspeakable in quantum mechanics*, 40–44.

Bohm, David. “A Suggested Interpretation of the Quantum Theory in Terms of ”Hidden” Variables. I.” *Physical review* 85, no. 2 (1952): 166–179.

———. “A Suggested Interpretation of the Quantum Theory in Terms of ”Hidden” Variables. II.” *Physical review* 85, no. 2 (1952): 180–193.

Bohm, David, and B. J Hiley. *The undivided universe : an ontological interpretation of quantum theory.* London: Routledge, 1993.

Butterfield, Jeremy. “Peaceful Coexistence: Examining Kent’s Relativistic Solution to the Quantum Measurement Problem,” 2017. eprint: arXiv:1710.07844.

Colbeck, Roger, and Renato Renner. “No extension of quantum theory can have improved predictive power.” *Nature communications* (England) 2, no. 1 (2011): 411–411.

———. “The completeness of quantum theory for predicting measurement outcomes,” 2012.

Einstein, Albert. *Albert Einstein, Philosopher Scientist.* Edited by P. A. Schilp. Evanston, Illinois: Library of Living Philosophers, 1949.

Espagnat, Bernard d’. *Conceptual foundations of quantum mechanics.* 2nd ed., completely rev., enl., reset. Mathematical physics monograph series ; 20. Reading, Mass.; London: W. A. Benjamin, 1976.

Joos, E., H.D. Zeh, C. Kiefer, D.J.W. Giulini, J. Kupsch, and I.O. Stamatescu. *Decoherence and the Appearance of a Classical World in Quantum Theory.* Springer Berlin Heidelberg, 2013.

Kent, Adrian. “Lorentzian Quantum Reality: Postulates and Toy Models,” 2014. https://doi.org/10.1098/rsta.2014.0241. eprint: arXiv:1411.2957.

Landsman, K. “On the Colbeck–Renner theorem.” *Journal of mathematical physics* (United States) 56, no. 12 (2015): 122103.

Landsman, Klaas. *Foundations of Quantum Theory : From Classical Concepts to Operator Algebras (Volume 188.0).* Vol. 188. Fundamental Theories of Physics. Cham: Springer Open, 2017.

Leegwater, Gijs. "An impossibility theorem for parameter independent hidden-variable theories." *Studies in History and Philosophy of Modern Physics* 54 (2016): 18–34.

Napolitano, Jim J., and J. J. Sakurai. *Modern Quantum Mechanics.* Pearson Education, 2013.

Pascazio, Saverio. "All you ever wanted to know about the quantum Zeno effect in 70 minutes." *44th Symposium on Mathematical Physics on New Developments in the Theory of Open Quantum Systems*, 2013. https://doi.org/10.1142/S1230161214400071. eprint: arXiv:1311.6645v1[quant-ph].

Redhead, Michael. *Incompleteness, nonlocality, and realism : a prolegomenon to the philosophy of quantum mechanics.* Oxford : New York: Clarendon Press ; Oxford University Press, 1987.

Rudin, Walter. *Functional Analysis.* Second Edition. McGraw-Hill, 1991.

Schlosshauer, Maximilian. *Decoherence and the Quantum-to-Classical Transition.* Berlin: Springer-Verlag, 2007.

Schwinger, Julian. "Quantum Electrodynamics. I. A Covariant Formulation." *Physical review* 74, no. 10 (1948): 1439–1461.

Shimony, A. "Events and processes in the quantum world." In *Search for a Naturalistic World View: Volume II: natural science and metaphysics*, 140–162.

———. *Search for a Naturalistic World View: Volume II: natural science and metaphysics.* Cambridge: Cambridge University Press, 1993.

Speakable and unspeakable in quantum mechanics. 2nd ed. Cambridge: Cambridge University Press, 2004.

Tomonaga, S. "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields." *Progress of theoretical physics* (Tokyo) 1, no. 2 (1946): 27–42.