Mathematical Framework for Field Reconstruction from Incomplete Measurements in Complex 3D Environments

Author

March 25, 2025

Abstract

This paper presents a mathematical framework for electromagnetic field reconstruction in complex 3D environments at millimeter-wave frequencies (28 GHz). We consider the problem of estimating current distributions on walls given only magnitude measurements on a 2D rectangular plane within the environment. The framework addresses challenges including phase recovery, sensitivity to geometrical misalignments, and the effects of affine transformations on field distributions. We derive expressions that relate field measurements to source current distributions through Green's functions and examine how geometric perturbations affect reconstruction accuracy.

1 Introduction

Electromagnetic field reconstruction from incomplete measurements represents a significant challenge in computational electromagnetics, particularly at millimeter-wave frequencies. In this work, we consider the following scenario:

- A 3D point cloud representation of a room environment
- A 2D rectangular measurement plane positioned within this environment
- Measurements of field magnitude (but not phase) on this plane at 28 GHz

- A digital twin of the environment with potential geometric discrepancies

The goal is to estimate the current distribution on the walls and use this distribution to predict fields throughout the environment. This inverse problem is complicated by several factors: the loss of phase information, high sensitivity to geometric misalignments at millimeter wavelengths, and uncertainties in the digital twin model.

We propose a mathematical framework that accounts for these challenges and investigates how affine transformations of the 3D environment affect the 2D field distribution on the measurement plane.

2 Problem Formulation

2.1 Notation and Coordinate Systems

Let us denote the 3D point cloud of the environment as $\mathcal{P} = \{(x_i, y_i, z_i) \in \mathbb{R}^3 : i = 1, 2, \ldots, N_p\}$, where N_p is the number of points.

The 2D measurement plane is defined as a rectangle $\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}], z = z_0\}$, where z_0 is the height of the plane.

We discretize the walls of the environment into N_s surface elements, with the current distribution represented as $\mathbf{J}(\mathbf{r}') = \sum_{j=1}^{N_s} \mathbf{J}_j \delta(\mathbf{r}' - \mathbf{r}'_j)$, where $\mathbf{r}'_j$ is the position of the j-th surface element.

2.2 Forward Problem: Field due to Current Distribution

The electric field at a point $\mathbf{r}$ on the measurement plane due to the current distribution on the walls is given by:

$$\mathbf{E}(\mathbf{r}) = \int_S \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') \, d\mathbf{r}' \tag{1}$$

where $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is the dyadic Green's function representing the field at $\mathbf{r}$ due to a point source at $\mathbf{r}'$.

In discretized form, this becomes:

$$\mathbf{E}(\mathbf{r}_i) = \sum_{j=1}^{N_s} \bar{\mathbf{G}}(\mathbf{r}_i, \mathbf{r}'_j) \cdot \mathbf{J}_j \Delta S_j \tag{2}$$

where $\mathbf{r}_i$ represents the i-th measurement point on the plane, and ΔS_j is the area of the j-th surface element.

2.3 Inverse Problem: Estimating Current Distribution

In our problem, we measure only the magnitude of the electric field at points on the measurement plane:

$$M_i = |\mathbf{E}(\mathbf{r}_i)| \quad \text{for } i = 1, 2, \ldots, N_m \tag{3}$$

where N_m is the number of measurement points.

The inverse problem involves estimating the current distribution $\mathbf{J}_j$ for $j = 1, 2, \ldots, N_s$ given the measurements M_i. This is an ill-posed problem, especially without phase information, requiring regularization techniques for stable solutions.

3 Green's Function for Field Propagation

The dyadic Green's function in free space at 28 GHz is given by:

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right) \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{4}$$

where $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength (approximately 10.7 mm at 28 GHz), and $\bar{\mathbf{I}}$ is the identity dyad.

For practical applications in complex environments, we can use the method of images or numerical techniques to account for multiple reflections.

4 Effects of Affine Transformations

4.1 Transformation of the 3D Point Cloud

Consider an affine transformation $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$\mathcal{T}(\mathbf{r}) = \mathbf{A}\mathbf{r} + \mathbf{b} \tag{5}$$

where $\mathbf{A} \in \mathbb{R}^{3\times3}$ is a non-singular matrix and $\mathbf{b} \in \mathbb{R}^3$ is a translation vector.

When this transformation is applied to the point cloud, each point $\mathbf{r} = (x, y, z)$ transforms to $\mathbf{r}_T = \mathcal{T}(\mathbf{r})$.

4.2 Impact on Field Distribution

The key insight is to understand how this transformation affects the field distribution on the measurement plane. Let $\mathbf{E}(\mathbf{r})$ be the original field and $\mathbf{E}_T(\mathbf{r})$ be the field in the transformed environment.

For small perturbations, we can express the transformed field as:

$$\mathbf{E}_T(\mathbf{r}) \approx \mathbf{E}(\mathbf{r}) + \delta\mathbf{E}(\mathbf{r}) \tag{6}$$

where $\delta\mathbf{E}(\mathbf{r})$ is the perturbation in the field.

4.2.1 First-Order Approximation

For small perturbations where $\mathbf{A} = \mathbf{I} + \delta\mathbf{A}$ and $\mathbf{b} = \delta\mathbf{b}$, the field perturbation can be approximated to first order as:

$$\delta\mathbf{E}(\mathbf{r}) \approx -\sum_{j=1}^{N_s} \left[(\delta\mathbf{A}\mathbf{r}'_j + \delta\mathbf{b}) \cdot \nabla_{\mathbf{r}'} \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j)\right] \cdot \mathbf{J}_j \Delta S_j \tag{7}$$

This expression gives us insight into how small geometric perturbations affect the field distribution.

4.2.2 Phase Sensitivity Analysis

Of particular importance is understanding how these transformations affect the phase of the field, since phase information is highly sensitive to geometric changes, especially at millimeter wavelengths.

The phase perturbation $\delta\phi(\mathbf{r})$ can be approximated as:

$$\delta\phi(\mathbf{r}) \approx -k\hat{\mathbf{R}} \cdot (\delta\mathbf{A}\mathbf{r}'_j + \delta\mathbf{b}) \tag{8}$$

where $\hat{\mathbf{R}} = (\mathbf{r} - \mathbf{r}'_j)/|\mathbf{r} - \mathbf{r}'_j|$ is the unit vector pointing from the source to the observation point.

At 28 GHz, $k \approx 587$ rad/m, which means that a displacement of just 1 mm can cause a phase shift of approximately 0.587 radians or 33.6 degrees.

5 Phase Recovery from Magnitude-Only Measurements

Since our measurements provide only the magnitude $|\mathbf{E}(\mathbf{r}_i)|$, we need methods to recover or estimate the phase information. Several approaches are possible:

5.1 Optimization-Based Phase Retrieval

We can formulate the phase retrieval problem as an optimization:

$$\min_{\mathbf{J}} \sum_{i=1}^{N_m} (|\mathbf{E}(\mathbf{r}_i)| - M_i)^2 + \lambda \mathcal{R}(\mathbf{J}) \tag{9}$$

where $\mathcal{R}(\mathbf{J})$ is a regularization term (e.g., L_1 or L_2 norm) and λ is a regularization parameter.

5.2 Iterative Phase Retrieval Algorithms

Iterative algorithms such as the Gerchberg-Saxton algorithm can be adapted to our problem:

Algorithm 1 Modified Gerchberg-Saxton for Current Estimation

1: Initialize $\mathbf{J}^{(0)}$
2: **for** $k = 0, 1, 2, \ldots$ **do**
3: Compute $\mathbf{E}^{(k)}(\mathbf{r}_i) = \sum_{j=1}^{N_s} \bar{\mathbf{G}}(\mathbf{r}_i, \mathbf{r}'_j) \cdot \mathbf{J}_j^{(k)} \Delta S_j$
4: Replace magnitude: $\hat{\mathbf{E}}^{(k)}(\mathbf{r}_i) = M_i \frac{\mathbf{E}^{(k)}(\mathbf{r}_i)}{|\mathbf{E}^{(k)}(\mathbf{r}_i)|}$
5: Update $\mathbf{J}^{(k+1)}$ by inverting the forward problem with $\hat{\mathbf{E}}^{(k)}(\mathbf{r}_i)$
6: **end for**

6 Compensation for Geometric Misalignments

To address the issue of misalignments between the digital twin and the actual environment, we propose a two-step approach:

6.1 Estimating the Affine Transformation

We formulate the problem of finding the optimal affine transformation as:

$$(\mathbf{A}^*, \mathbf{b}^*) = \arg\min_{\mathbf{A},\mathbf{b}} \sum_{i=1}^{N_m} (|\mathbf{E}_{\mathbf{A},\mathbf{b}}(\mathbf{r}_i)| - M_i)^2 \tag{10}$$

where $\mathbf{E}_{\mathbf{A},\mathbf{b}}(\mathbf{r}_i)$ is the field computed after applying the transformation $(\mathbf{A}, \mathbf{b})$ to the digital twin.

6.2 Joint Optimization

Alternatively, we can perform joint optimization of the current distribution and the geometric transformation:

$$(\mathbf{J}^*, \mathbf{A}^*, \mathbf{b}^*) = \arg\min_{\mathbf{J},\mathbf{A},\mathbf{b}} \sum_{i=1}^{N_m} \left(|\mathbf{E}_{\mathbf{J},\mathbf{A},\mathbf{b}}(\mathbf{r}_i)| - M_i\right)^2 + \lambda \mathcal{R}(\mathbf{J}) \tag{11}$$

This is a challenging non-convex optimization problem that may require advanced techniques such as alternating minimization or genetic algorithms.

7 Numerical Framework

For practical implementation, we propose the following numerical framework:

Algorithm 2 Field Reconstruction Framework

1: **Input:** Point cloud $\mathcal{P}$, measurement plane $\mathcal{M}$, field magnitudes $\{M_i\}$
2: **Output:** Estimated current distribution $\{\mathbf{J}_j\}$, geometric correction $(\mathbf{A}, \mathbf{b})$
3: Discretize the walls into surface elements
4: Initialize $\mathbf{A} = \mathbf{I}$, $\mathbf{b} = \mathbf{0}$
5: **for** $iter = 1, 2, \ldots, max\_iter$ **do**
6: Compute Green's functions $\bar{\mathbf{G}}(\mathbf{r}_i, \mathcal{T}(\mathbf{r}'_j))$
7: Estimate current distribution $\{\mathbf{J}_j\}$ using phase retrieval
8: Update $(\mathbf{A}, \mathbf{b})$ to improve alignment
9: Check convergence
10: **end for**
11: Use final $\{\mathbf{J}_j\}$ to predict fields at arbitrary locations

8 Challenges and Mitigation Strategies

8.1 High-Frequency Challenges at 28 GHz

At 28 GHz, the wavelength is approximately 10.7 mm, which presents several challenges:

- High sensitivity to geometric misalignments
- Increased computational complexity due to finer discretization requirements

- More significant multipath effects

Mitigation strategies include adaptive mesh refinement, multi-resolution analysis, and specialized high-frequency asymptotic techniques.

8.2 Phase Recovery Limitations

Phase retrieval from magnitude-only data is inherently ill-posed and often suffers from non-uniqueness. Strategies to address this include:

- Using spatial diversity in measurements
- Incorporating prior knowledge about the environment
- Employing multiple frequencies to leverage dispersion characteristics

8.3 Computational Efficiency

For large-scale problems, direct implementation of the proposed framework may be computationally intensive. Acceleration techniques include:

- Fast multipole methods (FMM) for efficient Green's function computations
- Model order reduction techniques
- GPU acceleration for parallelizable computations

9 Conclusion and Future Work

In this paper, we have derived a mathematical framework for electromagnetic field reconstruction in complex 3D environments at millimeter-wave frequencies. The framework addresses the challenges of phase recovery from magnitude-only measurements and compensates for geometric misalignments between the digital twin and the actual environment.

Future work includes:

- Experimental validation with controlled testbeds
- Extension to multi-frequency measurements to improve phase retrieval
- Integration of machine learning techniques for more robust reconstruction
- Development of real-time reconstruction algorithms for dynamic environments

A Derivation of the Sensitivity Expression

Here we provide a detailed derivation of the sensitivity expression for the field perturbation due to affine transformations.

Starting with the expression for the electric field:

$$\mathbf{E}(\mathbf{r}) = \sum_{j=1}^{N_s} \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j) \cdot \mathbf{J}_j \Delta S_j \tag{12}$$

After an affine transformation $\mathcal{T}(\mathbf{r}) = \mathbf{A}\mathbf{r}+\mathbf{b}$, the source position changes from $\mathbf{r}'_j$ to $\mathcal{T}(\mathbf{r}'_j) = \mathbf{A}\mathbf{r}'_j + \mathbf{b}$.

For small perturbations where $\mathbf{A} = \mathbf{I} + \delta\mathbf{A}$ and $\mathbf{b} = \delta\mathbf{b}$, we have:

$$\mathcal{T}(\mathbf{r}'_j) \approx \mathbf{r}'_j + \delta\mathbf{A}\mathbf{r}'_j + \delta\mathbf{b} \tag{13}$$

The perturbed field can then be expressed as:

$$\mathbf{E}_T(\mathbf{r}) = \sum_{j=1}^{N_s} \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j + \delta\mathbf{A}\mathbf{r}'_j + \delta\mathbf{b}) \cdot \mathbf{J}_j \Delta S_j \tag{14}$$

Using a first-order Taylor expansion of the Green's function:

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j + \delta\mathbf{r}'_j) \approx \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j) + \delta\mathbf{r}'_j \cdot \nabla_{\mathbf{r}'} \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j) \tag{15}$$

where $\delta\mathbf{r}'_j = \delta\mathbf{A}\mathbf{r}'_j + \delta\mathbf{b}$.

This leads to the perturbation in the field:

$$\delta\mathbf{E}(\mathbf{r}) = \mathbf{E}_T(\mathbf{r}) - \mathbf{E}(\mathbf{r}) \approx \sum_{j=1}^{N_s} \left[(\delta\mathbf{A}\mathbf{r}'_j + \delta\mathbf{b}) \cdot \nabla_{\mathbf{r}'} \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}'_j) \right] \cdot \mathbf{J}_j \Delta S_j \tag{16}$$