Maps, Sets and Fractals – A Topics Tutorial

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Overview

This project is deliberately open-ended, however this sheet specifies some initial numerical investigations and some suggested areas of research that might help you to get started.

You may use any language(s) you like to perform these experiments, but you may like to take the opportunity to learn a new environment or language that is appropriate to the task.

Keep in mind that the outputs of this project will be a project website and a 15-20 minute presentation on your work, to be presented at the end of the Spring Term. You are free to choose the format of your presentation, but it should support and build on your project website.

Important: citation and referencing

Please do not copy material from Wikipedia or other resources uncited. We will be checking both your website and presentation and you will get into trouble if you plagiarise[1]. By all means use these on-line resources to aid your understanding (some pages are better than others) and provide you with links into other references, but when you are forming mathematical or descriptive explanations always try to use your own words. If in doubt cite the source – this applies to pictures as well as text and equations. Books, papers and articles tend to carry more weight than crowd-sourced material as it will tend to have been through a more formal process of peer-review. You will get more credit for pictures that you have generated yourself (so make it clear that you have) than ones you have imported with citation (although sometimes this is necessary). If we have doubts we may ask you to show us your code generating a particular picture. Ensure that you have written your own core algorithm code, although you can use supporting libraries. You are advised to use BibTeX [1] for your referencing and you should take a look at a service such as `mendeley.com` to help you maintain your bibliography across a group of people.

The Initial Investigation

All around us, in nature, art and science, there are repeating patterns that are intricate, beautiful and sometimes deeply surprising. However, the mathematics behind such patterns, known as fractals, can arise from deceptively simple equations.

This project requires an investigation into the non-linear dynamics of discrete maps, leading to the exploration of a fractal landscape. This is meant to be fun, but there is some very serious maths underpinning the field that you are now in a good position to be able to start to appreciate.

[1]The first instance of plagiarism during an Imperial degree can result in a written warning; the second, no matter how small, can result in exclusion – so it really is not worth it.

The Verhulst process

The Verhulst process is a one dimensional discrete map that was designed to model bounded population growth. It has several forms but can be written as:

$$x_{k+1} = x_k + r x_k (1 - x_k) \qquad \text{for } k \geq 0 \qquad (1)$$

where $x_k \geq 0$. It is discrete as it evolves for discrete values of k; and 1 dimensional as there is only a single evolving quantity x_k. r is a real, positive, fixed parameter of the system and its value has a dramatic effect on the dynamics of the process.

Investigate what happens to x_k for suitably large k (with an initial value of $x_0 = 0.1$) for different values of r, $0 \leq r < 2.57$. You should find that for some range of r the process converges on a fixed point (a value of x_k such that $x_k = x_{k+1}$). Whereas, for other values of r, the process eventually oscillates between 2 or more fixed-points.

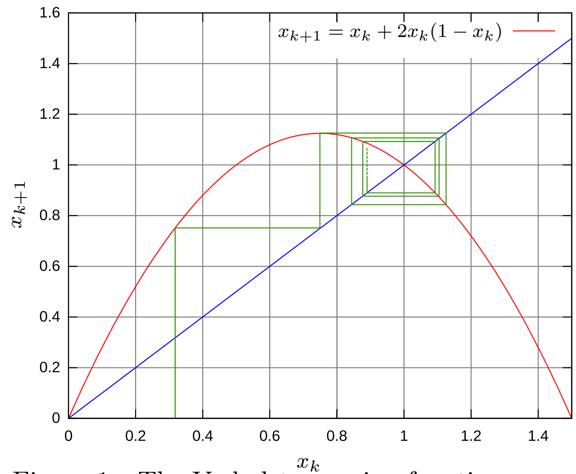

Figure 1: The Verhulst mapping function: $f(x) = x + rx(1 - x)$ for $r = 2$ and the line $y = x$. Successive iterations of the map reflect in the $y = x$ line to give the classic spider diagram (in green).

1. Plot the fixed point(s) of the Verhulst process against r for $0 \leq r < 2.57$ and find the first few r-values that capture the moment of so-called *period doubling* [2] in the number of fixed points[2].
2. Is there a relationship between the values of r that see a period-doubling in x_k?
3. Plot the fixed point(s) and then 200 or so consecutive points of x_k for large k of the process for $1 \leq r < 3$. What happens when $r > 2.57$?
4. Are the above features the same if the initial value x_0 is varied?
5. Show that this map displays fractal properties.

Research Idea: Other non-linear maps

As part of the research work in this project, you could attempt similar investigations for some other standard non-linear maps, such as:

1. The logistic map: $x_{k+1} = r x_k (1 - x_k)$, $k \geq 0$
2. The trigonometric map: $x_{k+1} = r \sin x_k$, $k \geq 0$

You may wish to investigate common features between these maps and the Verhulst map. You might want to see if these maps also display fractal properties.

Additionally, there are several more esoteric maps, such as the Gaussian map or the Tent map. You might like to discuss chaotic behaviour (not necessarily formally) with respect to these one dimensional processes and show how it can be applied to one of the maps in outline for a particular range of parameters. You could try to devise your own chaotic map. You could also construct your own investigation of two-dimensional chaotic maps, such as the Hénon map and the Gingerbreadman Map!

[2]We refer to this sequence later as the period-doubling cascade.

The Mandelbrot set

The Mandelbrot set, which many of you will have seen before, is a set of points in the complex plane. It is based around the complex, one dimensional map:

$$z_{k+1} = z_k^2 + c \qquad \text{for } k \geq 0 \tag{2}$$

where $z_k, c \in \mathbb{C}$ and $z_0 = 0$. The Mandelbrot set is defined to be the set of points $c \in \mathbb{C}$ for which Eq. (2) converges in the limit as $k \to \infty$. This definition can be refined into a computed test that states that a point c lies within the set if:

$$|z_k| \leq 2 \qquad \text{for all } k \geq 0 \tag{3}$$

In Fig. 2, the set of such c-values is shown by the central black area of the diagram.

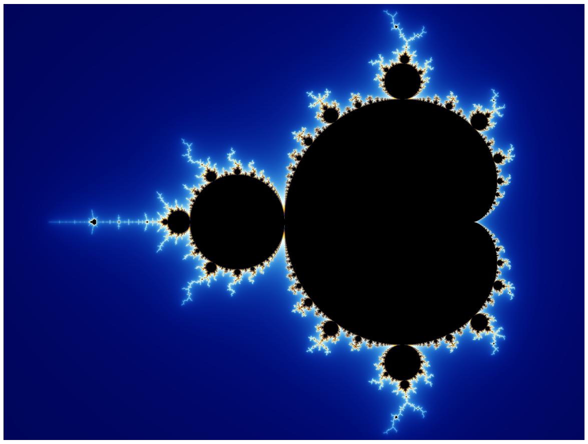

Figure 2: The famous beetle-like image of the Mandelbrot set from [3].

Attempt the following activities:

1. Reproduce Fig. 2 for $-1.2 \leq \mathrm{Im}\,(c) \leq 1.2$ and $-2.1 \leq \mathrm{Re}\,(c) \leq 0.7$ (by all means use your artistic input in selecting colours)[3]
2. Explain what the colours mean in your reproduction of the Mandelbrot set diagram
3. The Mandelbrot set is a compact set. Give a formal definition of a compact set.
4. Explain why, in Eq. (3), we restrict $|z_k| \leq 2$.
5. Demonstrate the relationship between the sequence of real points $c_k = a_k + 0i$ in the Mandelbrot set and the period-doubling cascade a_k of the logistic map.

Research Idea: Boll's Results and Julia Sets

As part of the research work in this project, you could verify numerically Boll's results (as is done in [4]) where, given that n_c is the least number of iterations of Eq. (2) required to have z_k exit the disc of radius 2, i.e. $n_c = \min(k : |z_k| > 2)$,

1. for $c = -0.75 + \epsilon i$ show that $\epsilon \times n_c \to \pi$ as $\epsilon \to 0$
2. for $c = 0.25 + \epsilon + 0i$ show that $\sqrt{\epsilon} \times n_c \to \pi$ as $\epsilon \to 0$

Choose values of ϵ other than 10^{-n}.

By considering the location of Boll's results relative to the Mandelbrot set, you might like to see if you can find other occurrences of π at other likely points around the Mandelbrot set.

For each c-point, there is an associated Julia set of z-points. You might like to research Julia sets and then plot them for a number of different values of c.

[3] Note that $\mathrm{Re}\,(c)$ and $\mathrm{Im}\,(c)$ refer to the real and imaginary parts of c, respectively.

Additionally, for the related complex map:

$$z_{k+1} = z_k^d + c \qquad \text{for } k \geq 0 \tag{4}$$

you could explore/plot the different types of set obtained as d becomes fractional $0 < d < 1$; as d goes negative $-2 < d < 0$; as d goes to higher powers $d \in \{3, 4, \ldots, 8\}$. Are there period-doubling cascades and Boll-style results to be found in these various more exotic sets?

Going Further

This tutorial should be used as a starting point for your wider project, but you will get some credit for attempting, documenting and presenting the initial investigations described above as an introduction to the subject area.

The bulk of the research credit for this project will be awarded for extra details that you and your group document/present. This extra material can take the form of one or more of the following:

- historical background and development of the mathematical concepts and structures
- further mathematical detail surrounding the concepts above (there is plenty)
- description of algorithmic and programming optimisations that your group has researched and implemented (you should include details of any memory or time savings as a result of using these optimisations and comparisons with some base case)

Several possible investigations are given above in the **Research Idea** sections, but these are only suggestions. You are at liberty to attempt your own variations on the above and we encourage you to be imaginative in how you pursue your investigations.

References

[1] A. Feder, "How to use BibTeX," 2006. `http://www.bibtex.org/Using/`.

[2] S. Lynch, *Dynamical Systems with Applications Using Mathematica*, ch. Nonlinear Discrete Dynamical Systems. No. 12, Birkhäuser, 2007. `http://www.phy.duke.edu/~hx3/physics/map.pdf`.

[3] W. Beyer, "The Mandelbrot set," 2006. `http://en.wikipedia.org/wiki/File:Mandel_zoom_00_mandelbrot_set.jpg`.

[4] A. Klebanoff, "π in the Mandelbrot set," *Fractals*, vol. 9, no. 4, pp. 393–402, 2001. `doi://10.1142/S0218348X01000828` or `http://www.doc.ic.ac.uk/~jb/teaching/jmc/pi-in-mandelbrot.pdf`.

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