p2

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In [2]:

```
import numpy as np
import matplotlib.pyplot as plt

# Construct the N x N tridiagonal matrix T_N
def T_N(N):
    A = -2 * np.eye(N)
    for i in range(N - 1):
        A[i, i + 1] = 1
        A[i + 1, i] = 1
    return A

# Analytical eigenvalues
def analytical_eigenvalues(N):
    i = np.arange(1, N + 1)
    return -2 * (1 - np.cos(np.pi * i / (N + 1)))

N = 10
A = T_N(N)

eig_num = np.sort(np.linalg.eigvals(A))
eig_ana = np.sort(analytical_eigenvalues(N))

print("Numerical eigenvalues (N=10):")
print(eig_num)

print("\nAnalytical eigenvalues (N=10):")
print(eig_ana)

print("\nMaximum error:")
print(np.max(np.abs(eig_num - eig_ana)))

# Plot eigenvalues for N = 10
plt.figure()
plt.plot(eig_num, 'o-', color = 'red')
plt.xlabel("Index i")
plt.ylabel("Eigenvalue")
plt.title("Eigenvalues of T_N for N = 10")
plt.grid(True)
plt.show()
```

```
Numerical eigenvalues (N=10):
[-3.91898595 -3.68250707 -3.30972147 -2.83083003 -2.28462968
-1.71537032
 -1.16916997 -0.69027853 -0.31749293 -0.08101405]

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Maximum error:
7.105427357601002e-15
```
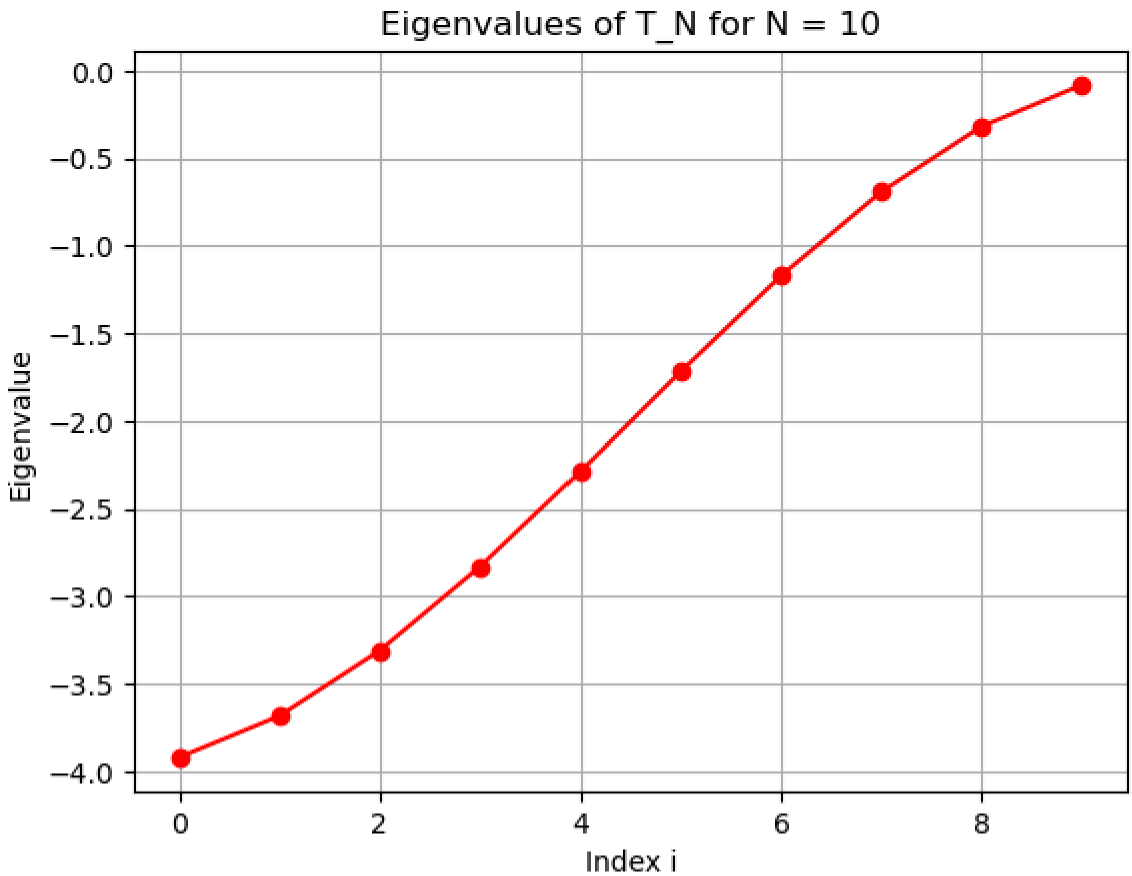
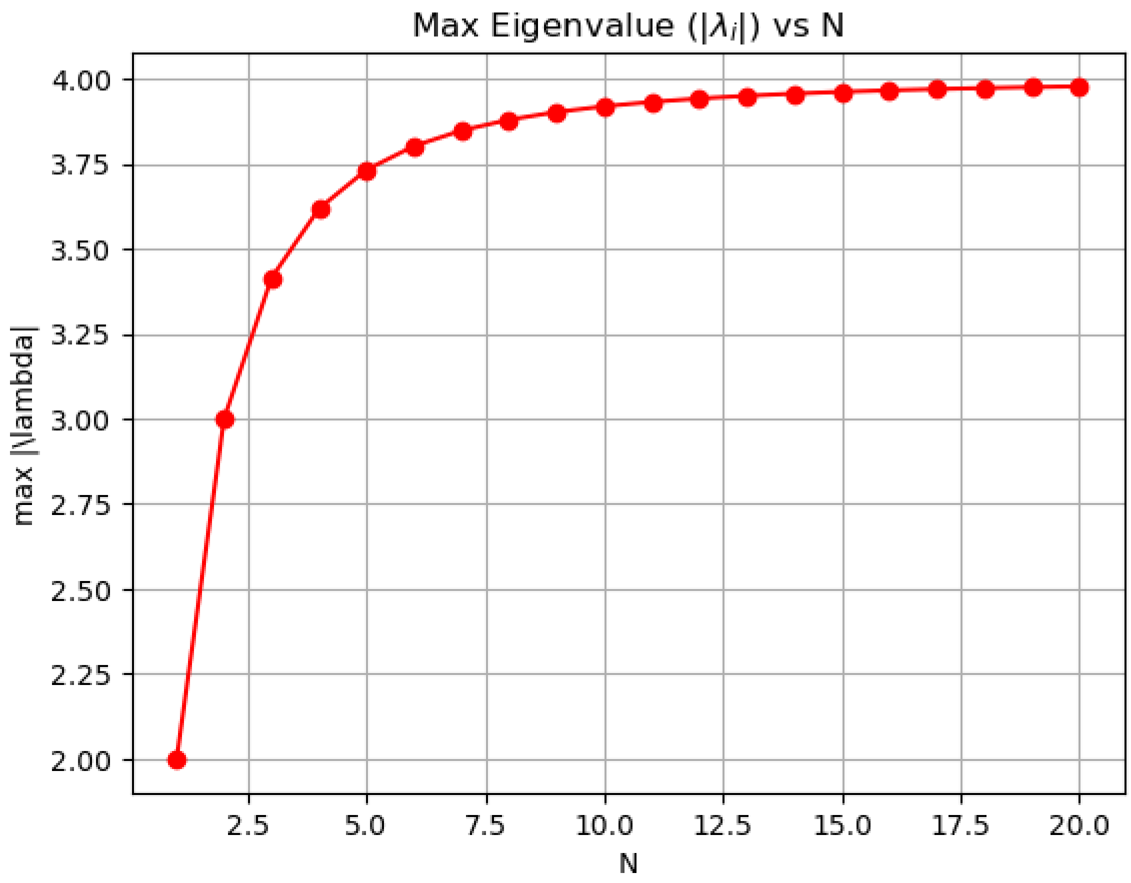



Since we have a max error between the analytical (via equation) and computed eigenvalues of order 1e-15, the true eigenvalues are the analytical eigenvalues of $\lambda_i = -2(1 - \cos(\frac{\pi i}{N+1}))$.

In [6]:

```
Ns = np.arange(1, 21)
max_abs_lambda = []

for N in Ns:
    eigs = np.linalg.eigvals(T_N(N))
    max_abs_lambda.append(np.max(np.abs(eigs)))

plt.figure()
plt.plot(Ns, max_abs_lambda, 'o-', color = 'red')
plt.xlabel("N")
plt.ylabel(f"max |\lambda|")
plt.title(r"Max Eigenvalue ($|\lambda_{i}|)$ vs N")
plt.grid(True)
plt.show()
```



The theoretical eigenvalues are

$$\lambda_i = -2\left(1 - \cos\left(\frac{\pi i}{N+1}\right)\right), \qquad i = 1, \dots, N.$$

To understand the behavior of $\max\ |\lambda_i|$, note that the cosine function $\cos(\theta)$ is strictly decreasing on the interval $\theta \in [0, \pi]$. Therefore, the function

$$p(i) = 1 - \cos\left(\frac{\pi i}{N+1}\right)$$

is strictly increasing as i increases from 1 to N.

Since i takes only discrete integer values, we do not need to differentiate with respect to i. Instead, monotonicity of the cosine function directly implies that $p(i)$ is maximized at the largest index, $i = N$. Hence, the eigenvalue with the largest magnitude is

$$\lambda_{\max} = \lambda_N = -2\left(1 - \cos\left(\frac{\pi N}{N+1}\right)\right).$$

Taking the limit as $N \to \infty$, we observe that

$$\frac{\pi N}{N+1} \to \pi, \qquad \cos\left(\frac{\pi N}{N+1}\right) \to \cos(\pi) = -1.$$

Substituting into the expression for λ_N yields

$$\lambda_N \to -2(1 - (-1)) = -4.$$

Therefore,

$$\max_{1 \le i \le N} |\lambda_i| = |\lambda_N| \to 4 \quad \text{as } N \to \infty.$$

When N is small, the maximum absolute eigenvalue is strictly less than 4. As N increases, the quantity $\frac{\pi N}{N+1}$ moves closer to π, causing $\cos\left(\frac{\pi N}{N+1}\right)$ to decrease and $|\lambda_N|$ to increase.

Thus, the numerical plot of $\max\ |\lambda|$ versus N increases monotonically and approaches 4 as N increases.