COE 347 HW4 17 February 25

1. $u'' = f(x) = 4\pi^2 \cos(4\pi x)$ 1

$x \in [0,1]$

$u''(x_i)h^2 = u_{i+1} - 2u_i + u_{i-1} + Ch^4 u^{(4)}(x_i) + \text{h.o.t.}$ (1)

$C = ?$

$u^{(3)} = -16\pi^3 \sin(4\pi x)$

$u^{(4)} = -64\pi^4 \cos(4\pi x)$

$u_{i+1} = u(x_i + h) = u_i + h u_i' + \frac{h^2}{2} u_i'' + \frac{h^3}{6} u_i^{(3)} + \dots h^n/n!\, u_i^{(n)}$

$u_{i-1} = u(x_i - h) = u_i - h u_i' + \frac{h^2}{2} u_i'' - \frac{h^3}{6} u_i^{(3)} + \dots \frac{(-1)^n h^n}{n!} u_i^{(n)}$

$u_{i+1} + u_{i-1} = 2\left[u_i + \frac{h^2}{2} u_i'' + \frac{h^4}{4!} u_i^{(4)} \dots \frac{h^{2n} u_i^{(2n)}}{(2n)!} \dots\right]$

$\underbrace{u_{i+1} + u_{i-1} - 2u_i}\ = h^2 u_i'' + \underbrace{\frac{h^4}{12} u_i^{(4)} + \dots \frac{h^{2n}}{(2n)!} u_i^{(2n)} \cdot 2 \dots}_{\text{H.O.T}}\Big]$ (2)

$\rightarrow u_i'' h^2 = u_{i+1} - 2u_i + u_{i-1} - \left[\frac{h^4}{12} u_i^{(4)} + \dots \quad \text{H.O.T}\right]$ (3)

$\rightarrow$ compare (3) with (1) $\rightarrow$

$\boxed{C = -1/12}$

p2_p3

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In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

def solve_bvp(N=10):
    L = 1.0
    h = L / (N + 1)
    x = np.linspace(0, 1, N + 2)

    f = lambda x: (4*np.pi)**2 * np.cos(4*np.pi*x)

    u = np.zeros(N + 2)
    u[0]  = 0
    u[-1] = 2

    A = np.zeros((N, N))
    b = np.zeros(N)

    for i in range(N):
        A[i, i] = -2.0
        if i > 0:
            A[i, i-1] = 1.0
        if i < N-1:
            A[i, i+1] = 1.0

        b[i] = h**2 * f(x[i+1])

    # Inject boundary conditions
    b[0]  -= u[0]
    b[-1] -= u[-1]

    u[1:-1] = np.linalg.solve(A, b)
    return x, u


def exact_solution(x):
    """
    Exact solution to u''(x) = (4π)² cos(4πx) with u(0)=0, u(1)=2

    The general solution is: u(x) = -cos(4πx) + Ax + B
    Applying boundary conditions:
    u(0) = -1 + B = 0 → B = 1
    u(1) = -cos(4π) + A + 1 = -1 + A + 1 = A = 2 → A = 2

    So exact solution: u(x) = -cos(4πx) + 2x + 1
    """
    return -np.cos(4*np.pi*x) + 2*x + 1
```

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```
def main(N=10, analytic=True, print_results=True, plot=True):
    # Solve with N=10

    # N = 10
    x, u_numerical = solve_bvp(N)

    # Calculate exact solution
    u_exact = exact_solution(x)

    # Print results
    if print_results:
        print(f"Solution with N={N} interior points (h=
{1/(N+1):.4f}):")
        print("-" * 60)
        print(f"{'x':>10} {'Numerical':>15} {'Exact
(Analytically)':>15} {'Error':>15}")
        print("-" * 60)
        for i in range(1, N+1):  # i from 1 to 10
            error = abs(u_numerical[i] - u_exact[i])
            print(f"{i:3d} {x[i]:10.6f} {u_numerical[i]:18.12f}
{u_exact[i]:18.12f} {error:15.6e}")
        print("=" * 60)


    try:
        # Load reference solution (N=10000)
        ref_data = np.loadtxt('solutionA_N10000.dat')
        if ref_data is not None:
            x_ref = ref_data[:, 0]
            u_ref = ref_data[:, 1]

            # Interpolate reference solution to our grid
            from scipy.interpolate import interp1d
            interp_func = interp1d(x_ref, u_ref, kind='linear')
            u_ref_interp = interp_func(x)

            # Calculate error compared to reference
            error_ref = np.max(np.abs(u_numerical - u_ref_interp))
            print(f"\nMaximum error compared to N=10000 reference:
{error_ref:.6e}")
                # Plot results
            if plot:
                plt.figure(figsize=(10, 6))
                assert x.shape == u_numerical.shape, "x and
u_numerical must have the same shape"
                plt.scatter(x, u_numerical, marker='o', color='b',
s=12, label=f'Numerical (N={N})')
```

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```
                plt.scatter(x_ref, u_ref, marker='*', color='g', s=12,
alpha = .5, label='Exact/Reference (N=10000)', linewidth=2)
                if analytic:
                    plt.scatter(x, u_exact, marker='x', color='r',
s=6, label='Analytic solution', linewidth=2)

                plt.xlabel('x', fontsize=12)
                plt.ylabel('u(x)', fontsize=12)
                plt.title('Solution of $u\'\'(x) = (4\\pi)^2
\\cos(4\\pi x)$ with $u(0)=0, u(1)=2$ (provided data)', fontsize=14)
                plt.legend(fontsize=12)

    except:
        print("\nNote: Reference file 'solutionA_N10000.dat' not found
for comparison.")
        print("Using exact solution for validation instead.")

        for i in range(len(x)):
            error = abs(u_numerical[i] - u_exact[i])
            print(f"{x[i]:10.4f} {u_numerical[i]:15.6f}
{u_exact[i]:15.6f} {error:15.6f}")

        # Calculate and print maximum error
        max_error = np.max(np.abs(u_numerical - u_exact))
        print("\n" + "=" * 60)
        print(f"Maximum error: {max_error:.6e}")
        print("=" * 60)

        # Plot results
        plt.figure(figsize=(10, 6))
        plt.scatter(x, u_numerical, 'bo-', label=f'Numerical (N={N})',
markersize=6)
        plt.plot(x, u_exact, 'r-', label='Exact Analytic Solution',
linewidth=2)
        plt.xlabel('x', fontsize=12)
        plt.ylabel('u(x)', fontsize=12)
        plt.title('Solution of $u\'\'(x) = (4\\pi)^2 \\cos(4\\pi x)$
with $u(0)=0, u(1)=2$ (analytical solution)', fontsize=14)
        plt.legend(fontsize=12)
        plt.grid(True, alpha=0.3)
        plt.tight_layout()
        plt.show()
    return x, u_numerical, u_ref, u_exact


if __name__ == "__main__":
    main(N=10)
```

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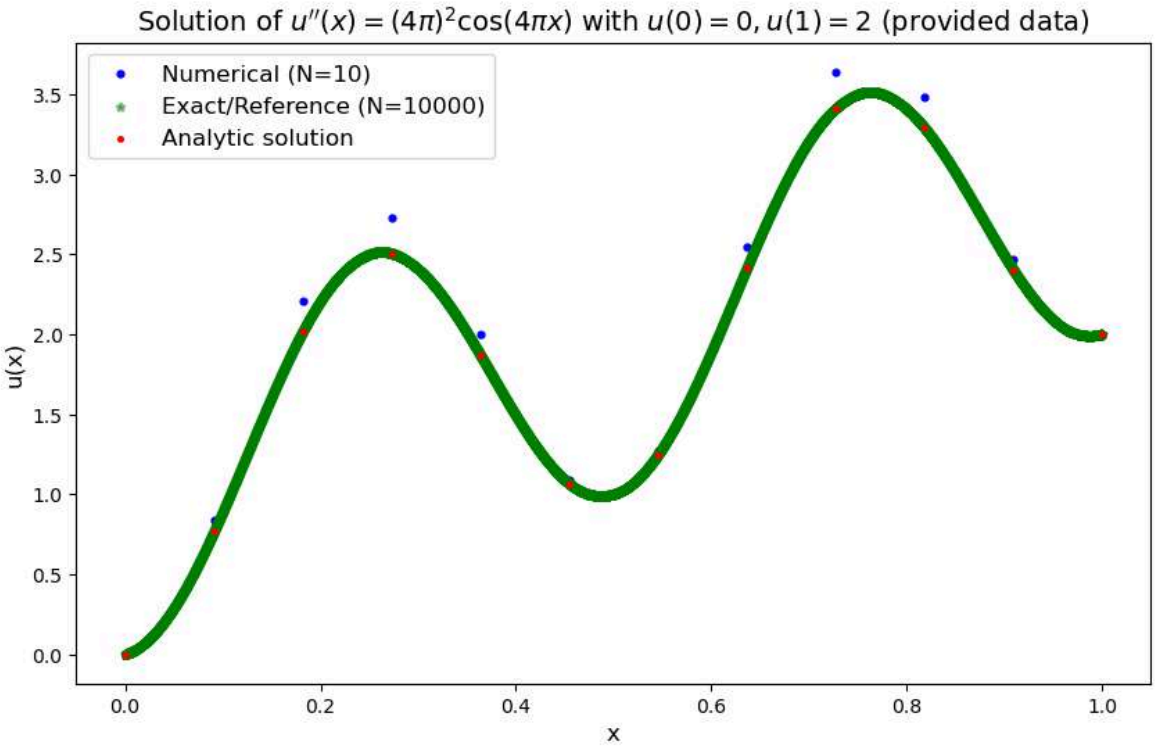
```
Solution with N=10 interior points (h=0.0909):
------------------------------------------------------------
         x       Numerical Exact (Analytically)           Error
------------------------------------------------------------
  1   0.090909     0.834354009989     0.766403168816     6.795084e-02
  2   0.181818     2.210854379067     2.018497097582     1.923573e-01
  3   0.272727     2.732714565420     2.504947519069     2.277670e-01
  4   0.363636     2.002367667450     1.869587565546     1.327801e-01
  5   0.454545     1.086289707772     1.067837376260     1.845233e-02
  6   0.545455     1.268107889591     1.249655558078     1.845233e-02
  7   0.636364     2.547822212905     2.415042111001     1.327801e-01
  8   0.727273     3.641805474511     3.414038428160     2.277670e-01
  9   0.818182     3.483581651794     3.291224370309     1.923573e-01
 10   0.909091     2.470717646353     2.402766805180     6.795084e-02
============================================================

Maximum error compared to N=10000 reference: 2.277670e-01
```



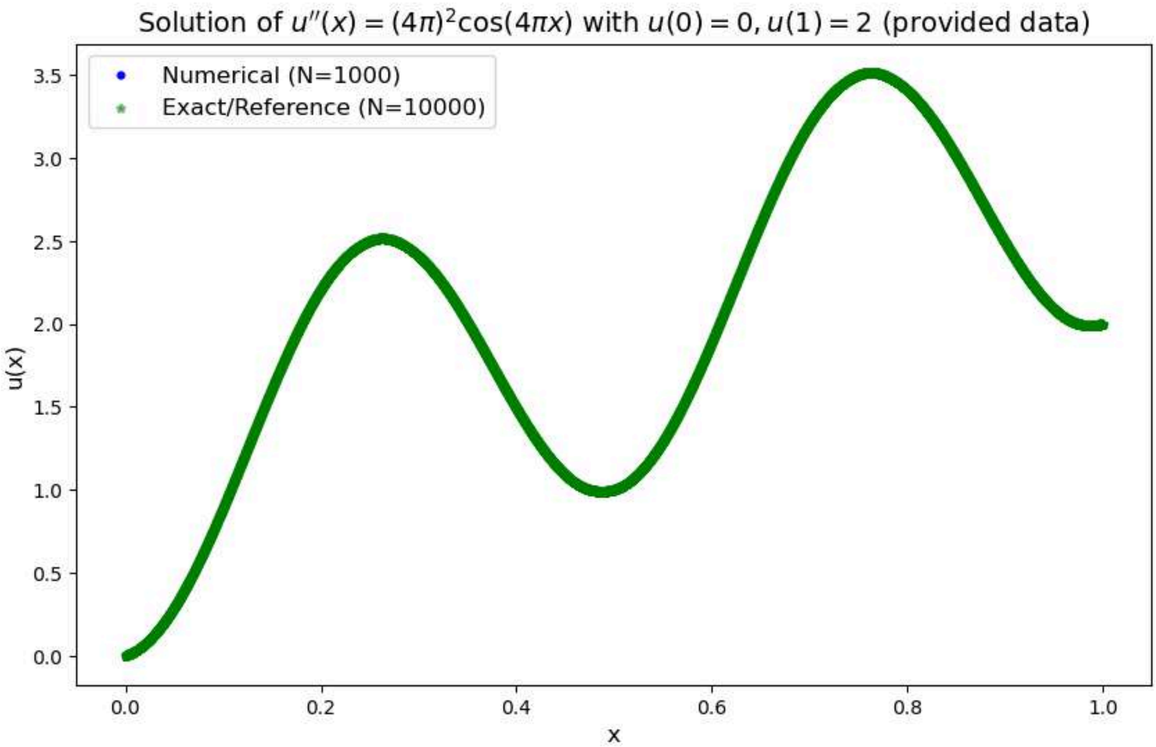
3

For confirmation that the numerical approximation converges to the true solution/the provided data, we can refine the mesh then compute the errors:

In [ ]:

```
x, u_numerical, u_ref, u_exact = main(1000, analytic=False,
print_results=False)
```

```
Maximum error compared to N=10000 reference: 2.615429e-05
```



In [3]:

```
from scipy.interpolate import CubicSpline
from scipy.optimize import curve_fit
import warnings; warnings.filterwarnings('ignore')



L = 1.
E = lambda u_num, u_ref: np.linalg.norm(u_num - u_ref)
e = lambda u_num, u_ref: np.linalg.norm(u_num - u_ref) / len(u_num)
N_values = [5, 10, 20, 40, 80, 160, 320, 640, 1280]
results = {
    'N': [],
    'h': [],
    'E': [],
    'e': [],
}

print(f"{'N':>6} {'h':>12} {'E (Frobenius)':>18} {'e (scaled)':>18}")

for N in N_values:
    x, u_numerical, u_ref, u_exact = main(N, analytic=False,
print_results=False, plot=False)
    x_ref = np.linspace(0, 1, 10000 + 2)
    uinterior_ref = CubicSpline( x_ref, u_ref)(x[1:-1])

    Error, error, = E(u_numerical[1:-1], uinterior_ref),
e(u_numerical[1:-1], uinterior_ref)
    h = L / (N + 1)

    results['N'].append(N)
    results['h'].append(h)
    results['E'].append(Error)
    results['e'].append(error)
    print(f"{N:6d} {h:12.6e}"
          f"{Error:18.6e} {error:18.6e}")

print("=" * 80)

for key in results:
    results[key] = np.array(results[key])

log_h = np.log(results['h'])
log_E = np.log(results['E'])
log_e = np.log(results['e'])



# For E (Frobenius norm)
```

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```
coeffs_E = np.polyfit(log_h, log_E, 1)
α_E = coeffs_E[0]
C_E = np.exp(coeffs_E[1])

# For e (scaled Frobenius norm)
coeffs_e = np.polyfit(log_h, log_e, 1)
α_e = coeffs_e[0]
C_e = np.exp(coeffs_e[1])

print(f"E (Frobenius) fit: E = {C_E:.6f} * h^{α_E:.4f}")
print(f"e (scaled) fit:    e = {C_e:.6f} * h^{α_e:.4f}")



# Create log-log plot
fig, ax = plt.subplots(figsize=(14, 10))

ax.loglog(1/results['h'], results['E'], 'bo', linewidth=2, markersize=8, label='Data E')
ax.loglog(1/results['h'], results['e'], 'bo', linewidth=2, markersize=8, label='Data e')

# Plot fitted curve
h_fit = np.logspace(np.log10(min(results['h'])), np.log10(max(results['h'])), 100)
E_fit = C_E * (h_fit**α_E)
ax.loglog(1/h_fit, E_fit, 'r--', linewidth=2, label=f'E with Fit: α={α_E:.3f}')

e_fit = C_e * (h_fit**α_e)
ax.loglog(1/h_fit, e_fit, 'r--', linewidth=2, label=f'e with Fit: α={α_e:.3f}')

ax.set_xlabel('1/h', fontsize=12)
ax.set_ylabel('E (Frobenius norm)', fontsize=12)
ax.set_title('Frobenius Error vs 1/h', fontsize=14)
ax.grid(True, alpha=0.3)
ax.legend(fontsize=12)



plt.tight_layout()
plt.savefig('error_convergence.png', dpi=150, bbox_inches='tight')
plt.show()

# Print convergence rates
print("\n" + "=" * 80)
print("CONVERGENCE RATE ANALYSIS:")
print("=" * 80)
print(f"Theoretical expectation: 2nd-order scheme → α ≈ 2")
```

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```
print(f"Frobenius norm (E): α = {α_E:.4f} (expectation: 2.0)")
print(f"Scaled norm (e):    α = {α_e:.4f} (expectation: 2.0)")
```

```
     N            h      E (Frobenius)         e (scaled)

Maximum error compared to N=10000 reference: 6.932453e-01
     5 1.666667e-01      1.386490e+00       2.772981e-01

Maximum error compared to N=10000 reference: 2.277670e-01
    10 9.090909e-02      4.721594e-01       4.721594e-02

Maximum error compared to N=10000 reference: 6.042456e-02
    20 4.761905e-02      1.705179e-01       8.525896e-03

Maximum error compared to N=10000 reference: 1.570735e-02
    40 2.439024e-02      6.167992e-02       1.541998e-03

Maximum error compared to N=10000 reference: 4.014668e-03
    80 1.234568e-02      2.213354e-02       2.766693e-04

Maximum error compared to N=10000 reference: 1.015461e-03
   160 6.211180e-03      7.889783e-03       4.931114e-05

Maximum error compared to N=10000 reference: 2.553685e-04
   320 3.115265e-03      2.799704e-03       8.749076e-06

Maximum error compared to N=10000 reference: 6.397205e-05
   640 1.560062e-03      9.890482e-04       1.545388e-06

Maximum error compared to N=10000 reference: 1.597234e-05
  1280 7.806401e-04      3.457620e-04       2.701266e-07
======================================================================
==========
E (Frobenius) fit: E = 19.084012 * h^1.5306
e (scaled) fit:    e = 22.446809 * h^2.5578
```

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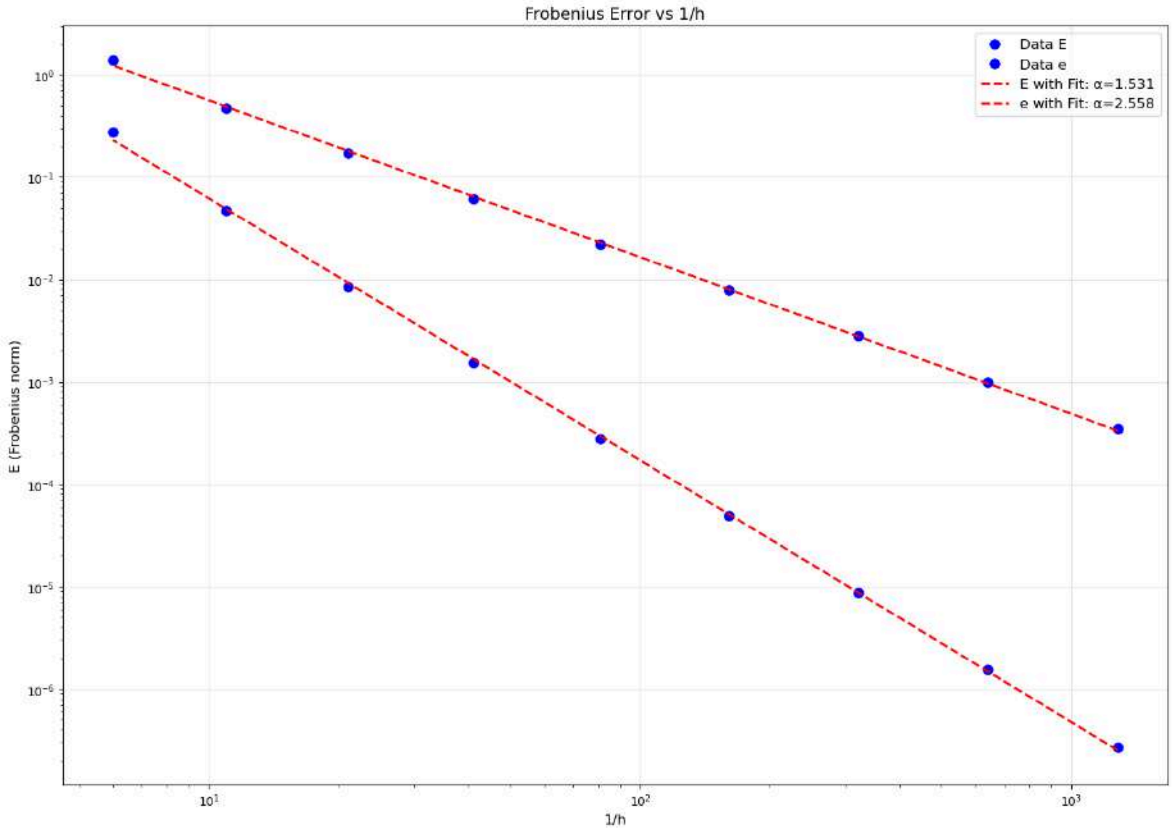


```
================================================================================
CONVERGENCE RATE ANALYSIS:
================================================================================
Theoretical expectation: 2nd-order scheme → α ≈ 2
Frobenius norm (E): α = 1.5306 (expectation: 2.0)
Scaled norm (e):    α = 2.5578 (expectation: 2.0)
```

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Approximately, E converges at order 1.5 and e converges at order 2.5. Theoretically, E should converge at 1.5, while e is expected to be of order 1 greater than the order of E: e is proportional to 1/(number of interior points on mesh) which is proportional to the step size h.

For proof/explanation on expected vs theoretical orders of convergence:

Assume the pointwise discretization error satisfies

$$u_h(x_i) - u(x_i) \mid \ = O(h^2) .$$

Then the discrete ℓ^2 error is

$$E = \left(\sum_{i=1}^{N} O(h^4) \right)^{1/2} = O!\left(\sqrt{N}, h^2 \right) = O(h^{3/2}),$$

since

$$N \sim \frac{1}{h}.$$

Next, define the scaled error

$$e = \frac{1}{N}, \|u_h - u\|_{\ell^2} .$$

Because

$$\frac{1}{N} \sim h,$$

we obtain

$$e = h, E .$$

Therefore,

$$e = O!\left(h^{3/2+1} \right) = O!\left(h^{5/2} \right) .$$

Hence, the observed convergence rate of order 2.5 for e is as theoretically expeccted.

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4. $u'(x_i)h = au_i + bu_{i+1} + cu_{i+2} + O(h^3)$ (1) 2

$$u_{i+1} = u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i^{(3)} + \dots O(h^4) \quad (2)$$

$$u_{i+2} = u_i + 2hu_i' + \frac{(2h)^2}{2}u_i'' + \frac{(2h)^3}{6}u_i^{(3)} + O(h^4)$$

$$= u_i + 2hu_i' + 2h^2u_i'' + \frac{4}{3}h^3u_i^{(3)} + O(h^4) \quad (3)$$

in equation 1, there are no second order terms (with u_i''), so, some combination of (2), (3), and u_i must eliminate 2nd order terms. u_i has no u_i'' so, some combination of (2), (3) eliminates u_i'':

$$(4u_{i+1} - u_{i+2}) = 4u_i + 4hu_i' + 2h^2u_i'' + 2/3\, h^3 u_i^{(3)} + \dots O(h^4)$$
$$- u_i - 2hu_i' - 2h^2u_i'' + 4/3\, h^3 u_i^{(3)} + O(h^4)$$

$$= 3u_i + 2hu_i' + 2h^3 u_i^{(3)} + O(h^4)$$

So, we have:

$$u'(x_i)h = d(\overset{b=4d}{4u_{i+1}} \overset{c=-d}{- u_{i+2}}) + au_i + O(h^3)$$

$$= d(3u_i + 2hu_i' + 2h^3u_i^{(3)}) + au_i + O(h^3)$$

$$= 3du_i + 2dhu_i' + 2dh^3u_i^{(3)} + O(h^3)$$
$$+ au_i$$

Since we have (1) $u_i'h$ on the LHS, $2d = 1$ and $d = 1/2$.

We also have (0) u_i on the LHS, so $3d + a = 0$.

$$\Rightarrow a = -3d = -3/2$$

Therefore: $b = 4d = 2$, $c = -d = -1/2$

$$u'(x_i)h = \underbrace{-\tfrac{3}{2}}_{a} u_i + \underbrace{2}_{b} u_{i+1} \underbrace{-1/2}_{c} u_{i+2} + O(h^3) \quad (4)$$

$$\boxed{\begin{aligned} a &= -3/2 \\ b &= 2 \\ c &= -1/2 \end{aligned}}$$

To recheck:

RHS of eq (4):

$$-\frac{3}{2}u_i + 2\left[u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{3!}u^{(3)} + O(h^4)\right]$$
$$-1/2\left[u_i + 2hu_i' + \frac{4h^2}{2}u_i'' + \frac{8h^3}{3!}u^{(3)} + O(h^4)\right]$$

$$= \left(-3/2 + 2 - 1/2\right)u_i$$
$$+ (2h - h)\,u_i'$$
$$+ \left(2h^2/2 - 4h^2/4\right)u_i''$$
$$+ O(h^3)$$

$$= (0)u_i + hu_i' + (0)h^2u_i'' + O(h^3) = hu_i' = u'(x_i)\,h = \text{RHS} \checkmark$$

p5_p6

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5

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In [7]:

```
import numpy as np
import matplotlib.pyplot as plt
import numpy as np


def solve_bvp(N=10, fd_order=2):
    L = 1.0
    h = L / (N + 1)
    x = np.linspace(0, 1, N + 2)

    f = lambda x: (4*np.pi)**2 * np.cos(4*np.pi*x)

    u = np.zeros(N + 2)

    u[-1] = 2

    A = np.zeros((N+1, N+1))
    b = np.zeros(N+1)

    for i in range(N+1):
        A[i, i] = -2.0
        if i > 0:
            A[i, i-1] = 1.0
        if i < N:
            A[i, i+1] = 1.0

        b[i] = h**2 * f(x[i])
    if fd_order == 2:
        b0  = 20*h
        A[0,0] = -3
        A[0,1] = 4
        A[0,2]=-1
    elif fd_order == 1:
        b0 = 10*h
        A[0,0] = -1
        A[0,1] = 1

    # Inject boundary conditions
    b[0]  = b0
    b[-1] -= u[-1]

    u[0:-1] = np.linalg.solve(A, b)

    return x, u


def exact_solution(x):
```

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```
    """
    Exact solution to u''(x) = (4π)² cos(4πx) with u'(0)=1, u(1)=2
    u'= 4π sin(4πx) + A
    The general solution is: u(x) = -cos(4πx) + Ax + B
    Applying boundary conditions:
    u'(0) = 0 + A = 10 → A = 10
    u(1) = -cos(4π) + 10 + B = -1 + 10 + B = 9+ B = 2 → B = -7

    So exact solution: u(x) = -cos(4πx) + 1x + 2
    """
    return -np.cos(4*np.pi*x) + 10*x - 7

def main(N=10, analytic=True, print_results=True, plot=True, size =
None, fd_order=2):
    # Solve with N=10

    # N = 10
    x, u_numerical = solve_bvp(N, fd_order = fd_order)

    # Calculate exact solution
    u_exact = exact_solution(x)

    # Print results
    if print_results:
        print(f"Solution with N={N} interior points (h=
{1/(N+1):.4f}):")
        print("-" * 60)
        print(f"{'x':>10} {'Numerical':>15} {'Exact
(Analytically)':>15} {'Error':>15}")
        print("-" * 60)
        for i in range(1, N+1):  # i from 1 to 10
            error = abs(u_numerical[i] - u_exact[i])
            print(f"{i:3d} {x[i]:10.6f} {u_numerical[i]:18.12f}
{u_exact[i]:18.12f} {error:15.6e}")
        print("=" * 60)



    try:
        # Load reference solution (N=10000)
        ref_data = np.loadtxt('solutionB_N10000.dat')
        if ref_data is not None:
            x_ref = ref_data[:, 0]
            u_ref = ref_data[:, 1]

            # Interpolate reference solution to our grid
            from scipy.interpolate import interp1d
            interp_func = interp1d(x_ref, u_ref, kind='linear')
            u_ref_interp = interp_func(x)
```

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```
            # Calculate error compared to reference
            error_ref = np.max(np.abs(u_numerical - u_ref_interp))
            print(f"\nMaximum error compared to N=10000 reference:
{error_ref:.6e}")
                # Plot results
            if plot:
                plt.figure(figsize=(10, 6))
                assert x.shape == u_numerical.shape, "x and
u_numerical must have the same shape"
                if size:
                    ms = size
                else:
                    ms = 12
                plt.scatter(x, u_numerical, marker='o', color='b',
s=ms, label=f'Numerical (N={N})')

                plt.scatter(x_ref, u_ref, marker='*', color='g', s=12,
alpha = .5, label='Exact/Reference (N=10000)', linewidth=2)
                if analytic:
                    plt.scatter(x, u_exact, marker='x', color='r',
s=6, label='Analytic solution', linewidth=2)

                plt.xlabel('x', fontsize=12)
                plt.ylabel('u(x)', fontsize=12)
                plt.title('Solution of $u\'\'(x) = (4\\pi)^2
\\cos(4\\pi x)$ with $u\'(0)=10, u(1)=2$ (provided data)',
fontsize=14)
                plt.legend(fontsize=12)

    except:
        print("\nNote: Reference file 'solutionA_N10000.dat' not found
for comparison.")
        print("Using exact solution for validation instead.")

        for i in range(len(x)):
            error = abs(u_numerical[i] - u_exact[i])
            print(f"{x[i]:10.4f} {u_numerical[i]:15.6f}
{u_exact[i]:15.6f} {error:15.6f}")

        # Calculate and print maximum error
        max_error = np.max(np.abs(u_numerical - u_exact))
        print("\n" + "=" * 60)
        print(f"Maximum error: {max_error:.6e}")
        print("=" * 60)

        # Plot results
        plt.figure(figsize=(10, 6))
```

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```
        plt.scatter(x, u_numerical, 'bo-', label=f'Numerical (N={N})',
markersize=6)
        plt.plot(x, u_exact, 'r-', label='Exact Analytic Solution',
linewidth=2)
        plt.xlabel('x', fontsize=12)
        plt.ylabel('u(x)', fontsize=12)
        plt.title('Solution of $u\'\'(x) = (4\\pi)^2 \\cos(4\\pi x)$
with $u\'\(0)=10, u(1)=2$ (analytical solution)', fontsize=14)
        plt.legend(fontsize=12)
        plt.grid(True, alpha=0.3)
        plt.tight_layout()
        plt.show()
    return x, u_numerical, u_ref, u_exact


if __name__ == "__main__":
    main(N=10)
```

```
Solution with N=10 interior points (h=0.0909):
------------------------------------------------------------
         x       Numerical Exact (Analytically)          Error
------------------------------------------------------------
  1   0.090909    -2.623746776465    -6.506324103911    3.882577e+00
  2   0.181818    -0.901436328743    -4.526957447873    3.625521e+00
  3   0.272727    -0.033766063743    -3.313234299113    3.279468e+00
  4   0.363636    -0.418302883068    -3.221321525363    2.803019e+00
  5   0.454545    -0.988570764100    -3.295798987377    2.307228e+00
  6   0.545455    -0.460942503637    -2.386708078286    1.925766e+00
  7   0.636364     1.164581898323    -0.494048798090    1.658631e+00
  8   0.727273     2.604375238575     1.232220246342    1.372155e+00
  9   0.818182     2.791961494503     1.836678915763    9.552826e-01
 10   0.909091     2.124907567708     1.675494077907    4.494135e-01
============================================================

Maximum error compared to N=10000 reference: 4.196089e+00
```

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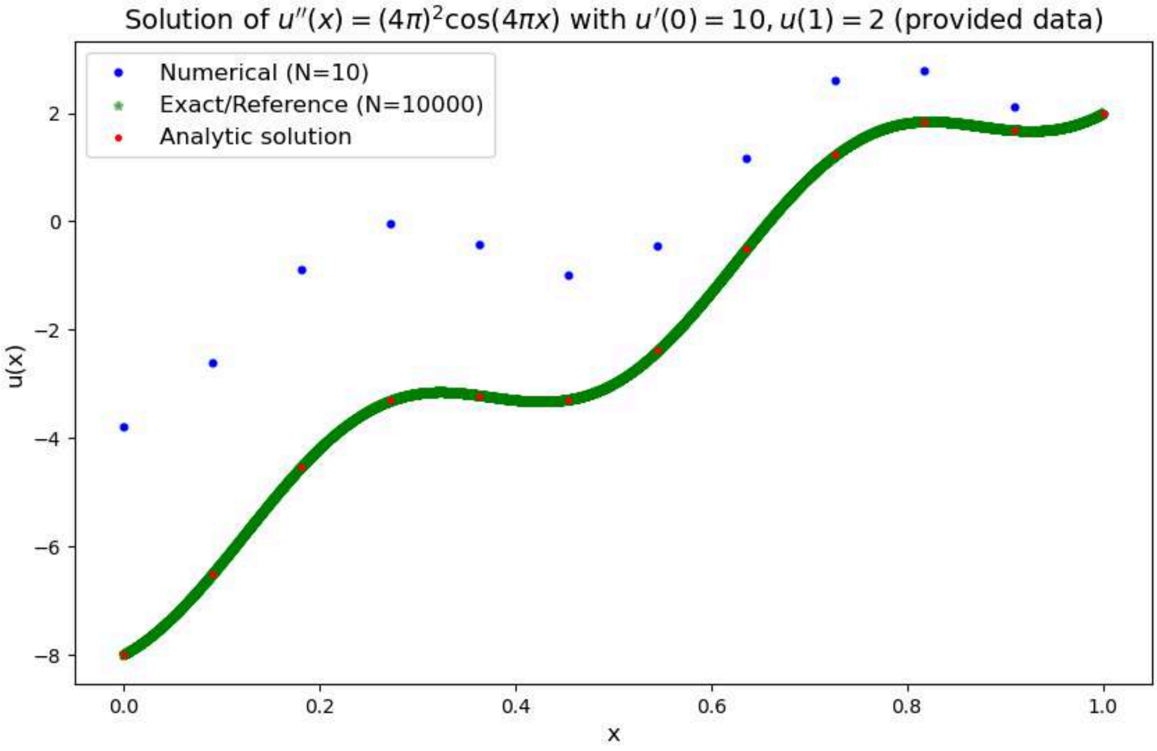
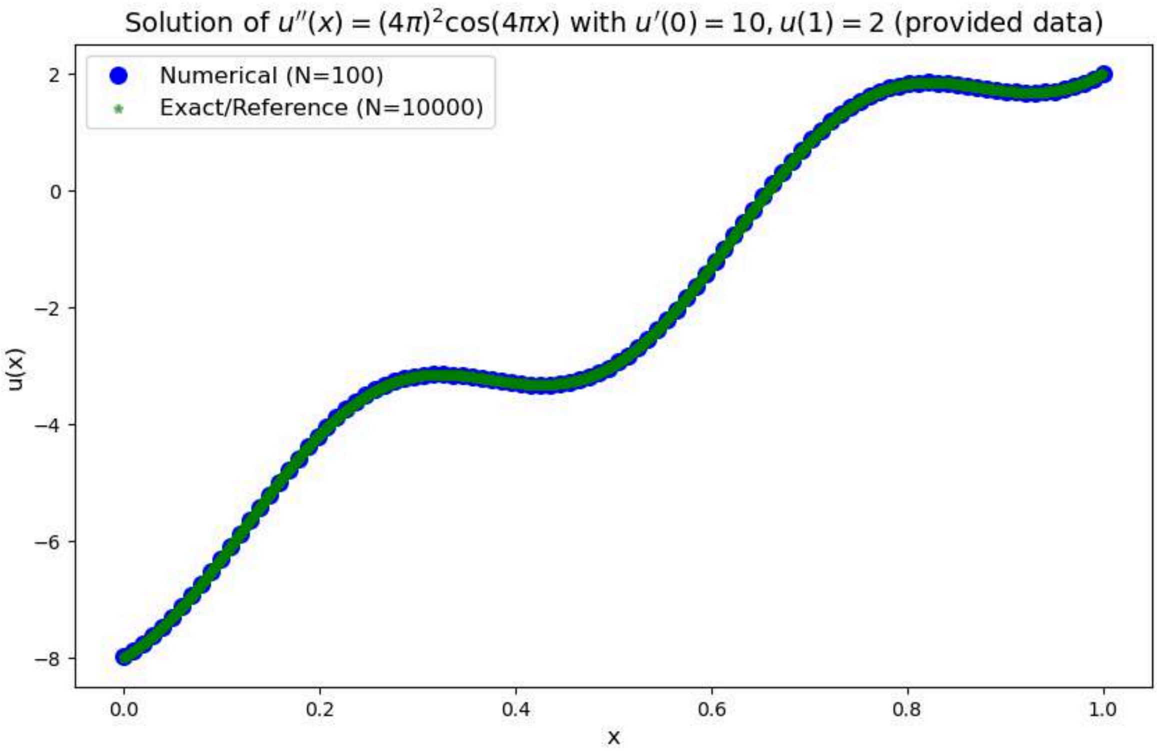


For confirmation that the numerical approximation converges to the true solution/the provided data, we can refine the mesh then compute the errors:x,

In [8]:

```
x, u_numerical, u_ref, u_exact = main(100, analytic=False,
print_results=False, size = 70)
```

```
Maximum error compared to N=10000 reference: 7.204568e-03
```



6

In [13]:

```
from scipy.interpolate import CubicSpline
from scipy.optimize import curve_fit
import warnings; warnings.filterwarnings('ignore')

def fit_and_plot_convergence(fd_order=2,  N_values = [5, 10, 20, 40, 
80, 160, 320, 640, 1280]):
    L = 1.
    E = lambda u_num, u_ref: np.linalg.norm(u_num - u_ref)
    e = lambda u_num, u_ref: np.linalg.norm(u_num - u_ref) / 
len(u_num)

    results = {
        'N': [],
        'h': [],
        'E': [],
        'e': [],
    }

    print(f"{'N':>6} {'h':>12} {'E (Frobenius)':>18} {'e 
(scaled)':>18}")

    for N in N_values:
        x, u_numerical, u_ref, u_exact = main(N, analytic=False, 
print_results=False, plot=False, fd_order=fd_order)
        x_ref = np.linspace(0, 1, 10000 + 2)
        uinterior_ref = CubicSpline( x_ref, u_ref)(x[1:-1])

        Error, error, = E(u_numerical[1:-1], uinterior_ref), 
e(u_numerical[1:-1], uinterior_ref)
        h = L / (N + 1)

        results['N'].append(N)
        results['h'].append(h)
        results['E'].append(Error)
        results['e'].append(error)
        print(f"{N:6d} {h:12.6e}"
                f"{Error:18.6e} {error:18.6e}")

    print("=" * 80)

    for key in results:
        results[key] = np.array(results[key])

    log_h = np.log(results['h'])
    log_E = np.log(results['E'])
    log_e = np.log(results['e'])
```

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```
    # For E (Frobenius norm)
    coeffs_E = np.polyfit(log_h, log_E, 1)
    α_E = coeffs_E[0]
    C_E = np.exp(coeffs_E[1])

    # For e (scaled Frobenius norm)
    coeffs_e = np.polyfit(log_h, log_e, 1)
    α_e = coeffs_e[0]
    C_e = np.exp(coeffs_e[1])

    print(f"E (Frobenius) fit: E = {C_E:.6f} * h^{α_E:.4f}")
    print(f"e (scaled) fit:    e = {C_e:.6f} * h^{α_e:.4f}")



    # Create log-log plot
    fig, ax = plt.subplots(figsize=(14, 10))

    ax.loglog(1/results['h'], results['E'], 'bo', linewidth=2,
markersize=8, label='Data E')
    ax.loglog(1/results['h'], results['e'], 'bo', linewidth=2,
markersize=8, label='Data e')

    # Plot fitted curve
    h_fit = np.logspace(np.log10(min(results['h'])),
np.log10(max(results['h'])), 100)
    E_fit = C_E * (h_fit**α_E)
    ax.loglog(1/h_fit, E_fit, 'r--', linewidth=2, label=f'E with Fit:
α={α_E:.3f}')

    e_fit = C_e * (h_fit**α_e)
    ax.loglog(1/h_fit, e_fit, 'r--', linewidth=2, label=f'e with Fit:
α={α_e:.3f}')

    ax.set_xlabel('1/h', fontsize=12)
    ax.set_ylabel('E (Frobenius norm)', fontsize=12)
    ax.set_title('Frobenius Error vs 1/h', fontsize=14)
    ax.grid(True, alpha=0.3)
    ax.legend(fontsize=12)



    plt.tight_layout()
    plt.savefig('error_convergence.png', dpi=150, bbox_inches='tight')
    plt.show()

    # Print convergence rates
    print("\n" + "=" * 80)
```

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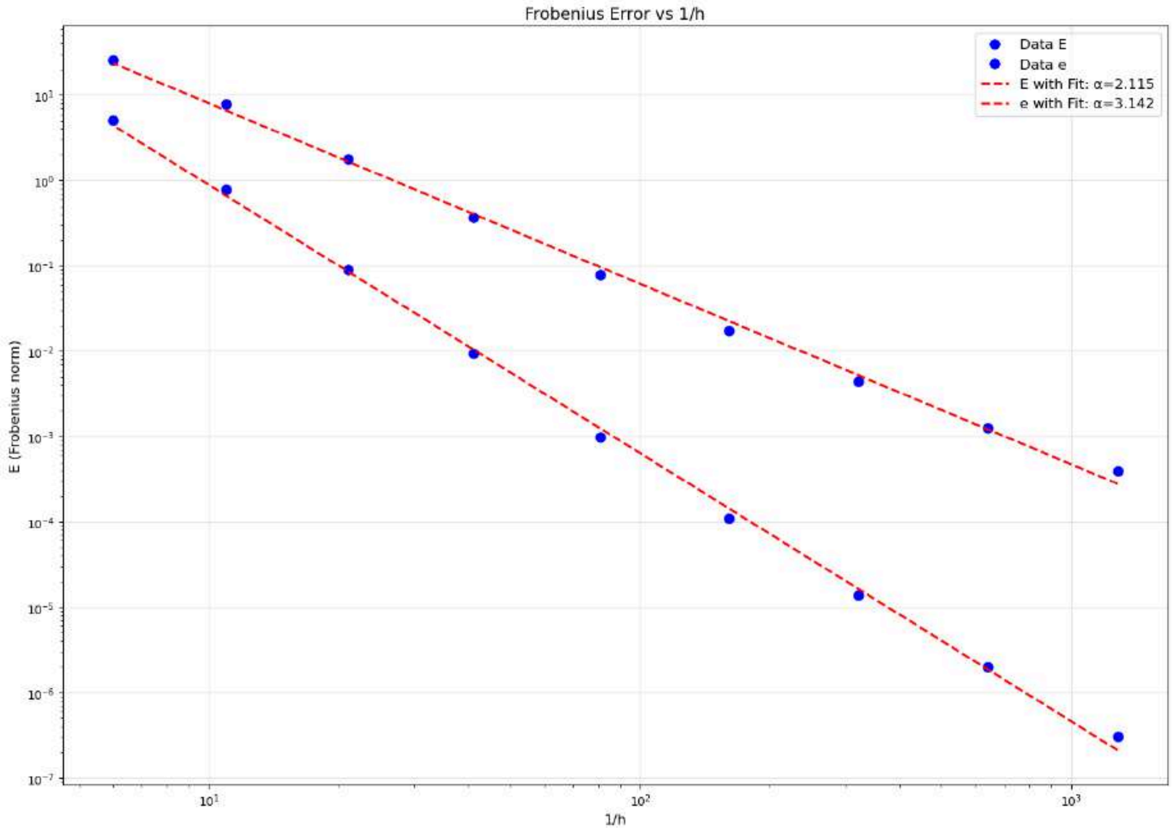
```
    print("CONVERGENCE RATE ANALYSIS:")
    print("=" * 80)
    print(f"Frobenius norm (E): α = {α_E:.4f})")
    print(f"Scaled norm (e):    α = {α_e:.4f}")
    return results, E_fit, e_fit, α_E, α_e
results_2, E_fit_2, e_fit_2, α_E2, α_e2=
fit_and_plot_convergence(fd_order=2)
```

```
     N            h      E (Frobenius)         e (scaled)

Maximum error compared to N=10000 reference: 1.973921e+01
     5 1.666667e-01      2.553305e+01       5.106610e+00

Maximum error compared to N=10000 reference: 4.196089e+00
    10 9.090909e-02      7.849262e+00       7.849262e-01

Maximum error compared to N=10000 reference: 6.533160e-01
    20 4.761905e-02      1.795448e+00       8.977241e-02

Maximum error compared to N=10000 reference: 8.974813e-02
    40 2.439024e-02      3.726019e-01       9.315047e-03

Maximum error compared to N=10000 reference: 1.301626e-02
    80 1.234568e-02      7.762972e-02       9.703715e-04

Maximum error compared to N=10000 reference: 2.149269e-03
   160 6.211180e-03      1.740160e-02       1.087600e-04

Maximum error compared to N=10000 reference: 3.974418e-04
   320 3.115265e-03      4.397576e-03       1.374242e-05

Maximum error compared to N=10000 reference: 8.179131e-05
   640 1.560062e-03      1.257535e-03       1.964899e-06

Maximum error compared to N=10000 reference: 1.819485e-05
  1280 7.806401e-04      3.913961e-04       3.057782e-07
======================================================================
==========
E (Frobenius) fit: E = 1043.646475 * h^2.1152
e (scaled) fit:    e = 1227.547608 * h^3.1425
```




```
================================================================================
CONVERGENCE RATE ANALYSIS:
================================================================================
Frobenius norm (E): α = 2.1152)
Scaled norm (e):    α = 3.1425
```

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In [14]:

```
results_1, E_fit_1, e_fit_1, α_E1, α_e1 =
fit_and_plot_convergence(fd_order=1)
```

```
     N            h      E (Frobenius)         e (scaled)

Maximum error compared to N=10000 reference: 1.315947e+01
     5 1.666667e-01      1.740636e+01       3.481271e+00

Maximum error compared to N=10000 reference: 7.177894e+00
    10 9.090909e-02      1.316567e+01       1.316567e+00

Maximum error compared to N=10000 reference: 3.759849e+00
    20 4.761905e-02      9.717364e+00       4.858682e-01

Maximum error compared to N=10000 reference: 1.925776e+00
    40 2.439024e-02      7.033490e+00       1.758373e-01

Maximum error compared to N=10000 reference: 9.747757e-01
    80 1.234568e-02      5.033981e+00       6.292476e-02

Maximum error compared to N=10000 reference: 4.904151e-01
   160 6.211180e-03      3.581529e+00       2.238456e-02

Maximum error compared to N=10000 reference: 2.459714e-01
   320 3.115265e-03      2.540388e+00       7.938712e-03

Maximum error compared to N=10000 reference: 1.231776e-01
   640 1.560062e-03      1.799122e+00       2.811129e-03

Maximum error compared to N=10000 reference: 6.163687e-02
  1280 7.806401e-04      1.273162e+00       9.946577e-04
======================================================================
==========
E (Frobenius) fit: E = 42.687632 * h^0.4891
e (scaled) fit:    e = 50.209627 * h^1.5164
```

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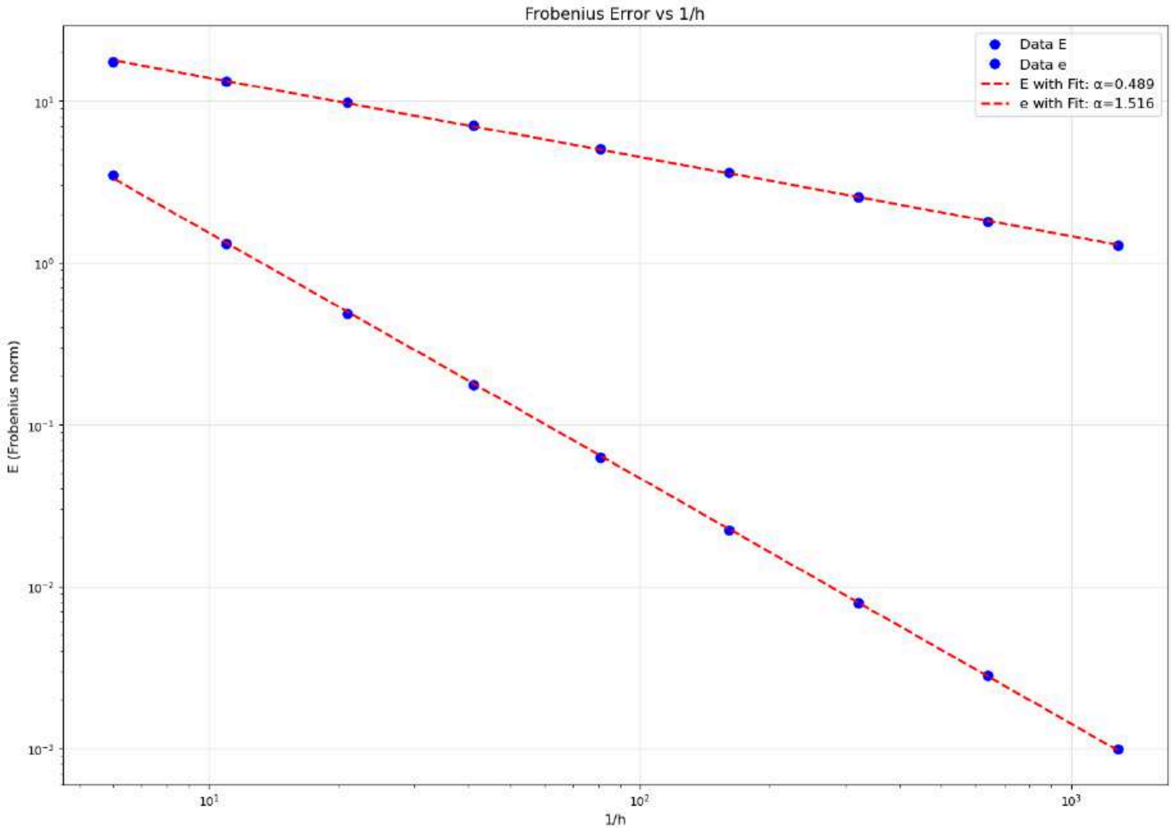


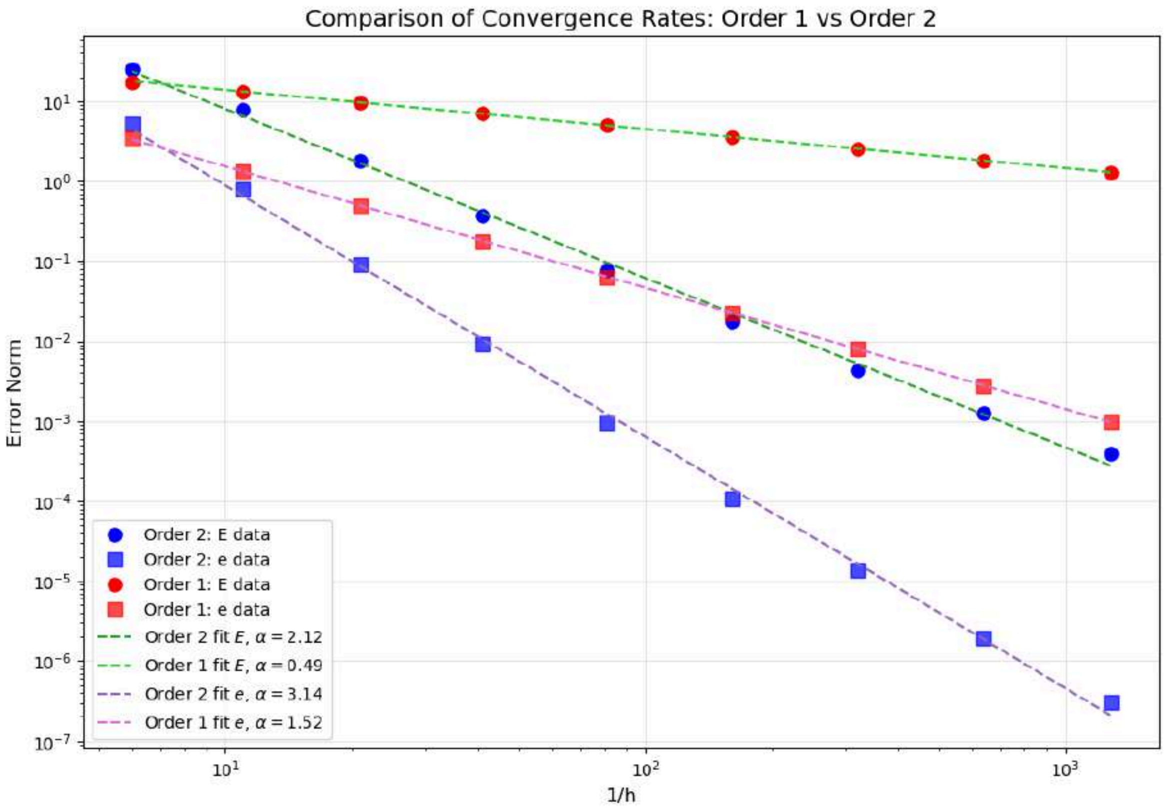

```
================================================================================
CONVERGENCE RATE ANALYSIS:
================================================================================
Frobenius norm (E): α = 0.4891)
Scaled norm (e):    α = 1.5164
```

The document was converted with **Code-Format**: https://code-format.com/

In [15]:

```
fig, ax = plt.subplots(figsize=(10, 7))

# Plot order 2 results
ax.loglog(1/results_2['h'], results_2['E'], 'bo', linewidth=2,
markersize=8,
          label='Order 2: E data')
ax.loglog(1/results_2['h'], results_2['e'], 'bs', linewidth=2,
markersize=8,
          label='Order 2: e data', alpha=0.7)

# Plot order 1 results
ax.loglog(1/results_1['h'], results_1['E'], 'ro', linewidth=2,
markersize=8,
          label='Order 1: E data')
ax.loglog(1/results_1['h'], results_1['e'], 'rs', linewidth=2,
markersize=8,
          label='Order 1: e data', alpha=0.7)

h_fit = np.logspace(np.log10(min(results_1['h'])),
np.log10(max(results_1['h'])), 100)
ax.loglog(1/h_fit, E_fit_2, color='tab:green', linestyle='--',
linewidth=1.5,
        label=rf'Order 2 fit $E$, $\alpha={α_E2:.2f}$')

ax.loglog(1/h_fit, E_fit_1, color='limegreen', linestyle='--',
linewidth=1.5,
        label=rf'Order 1 fit $E$, $\alpha={α_E1:.2f}$')

ax.loglog(1/h_fit, e_fit_2, color='tab:purple', linestyle='--',
linewidth=1.5,
        label=rf'Order 2 fit $e$, $\alpha={α_e2:.2f}$')

ax.loglog(1/h_fit, e_fit_1, color='orchid', linestyle='--',
linewidth=1.5,
        label=rf'Order 1 fit $e$, $\alpha={α_e1:.2f}$')

ax.set_xlabel('1/h', fontsize=12)
ax.set_ylabel('Error Norm', fontsize=12)
ax.set_title('Comparison of Convergence Rates: Order 1 vs Order 2',
fontsize=14)
ax.grid(True, alpha=0.3)
ax.legend(fontsize=10)
plt.tight_layout()
plt.show()
```

The document was converted with **Code-Format**: https://code-format.com/



Check with smaller step sizes:

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In [16]:

```
results_2, E_fit_2, e_fit_2, α_E2, α_e2=
fit_and_plot_convergence(fd_order=2, N_values = [5*2**i for i in
range(5,10)])
results_1, E_fit_1, e_fit_1, α_E1, α_e1 =
fit_and_plot_convergence(fd_order=1,  N_values = [5*2**i for i in
range(5,10)])
fig, ax = plt.subplots(figsize=(10, 7))

# Plot order 2 results
ax.loglog(1/results_2['h'], results_2['E'], 'bo', linewidth=2,
markersize=8,
          label='Order 2: E data')
ax.loglog(1/results_2['h'], results_2['e'], 'bs', linewidth=2,
markersize=8,
          label='Order 2: e data', alpha=0.7)

# Plot order 1 results
ax.loglog(1/results_1['h'], results_1['E'], 'ro', linewidth=2,
markersize=8,
          label='Order 1: E data')
ax.loglog(1/results_1['h'], results_1['e'], 'rs', linewidth=2,
markersize=8,
          label='Order 1: e data', alpha=0.7)

h_fit = np.logspace(np.log10(min(results_1['h'])),
np.log10(max(results_1['h'])), 100)
ax.loglog(1/h_fit, E_fit_2, color='tab:green', linestyle='--',
linewidth=1.5,
        label=rf'Order 2 fit $E$, $\alpha={α_E2:.2f}$')

ax.loglog(1/h_fit, E_fit_1, color='limegreen', linestyle='--',
linewidth=1.5,
        label=rf'Order 1 fit $E$, $\alpha={α_E1:.2f}$')

ax.loglog(1/h_fit, e_fit_2, color='tab:purple', linestyle='--',
linewidth=1.5,
        label=rf'Order 2 fit $e$, $\alpha={α_e2:.2f}$')

ax.loglog(1/h_fit, e_fit_1, color='orchid', linestyle='--',
linewidth=1.5,
        label=rf'Order 1 fit $e$, $\alpha={α_e1:.2f}$')

ax.set_xlabel('1/h', fontsize=12)
ax.set_ylabel('Error Norm', fontsize=12)
ax.set_title('Comparison of Convergence Rates: Order 1 vs Order 2',
fontsize=14)
ax.grid(True, alpha=0.3)
```

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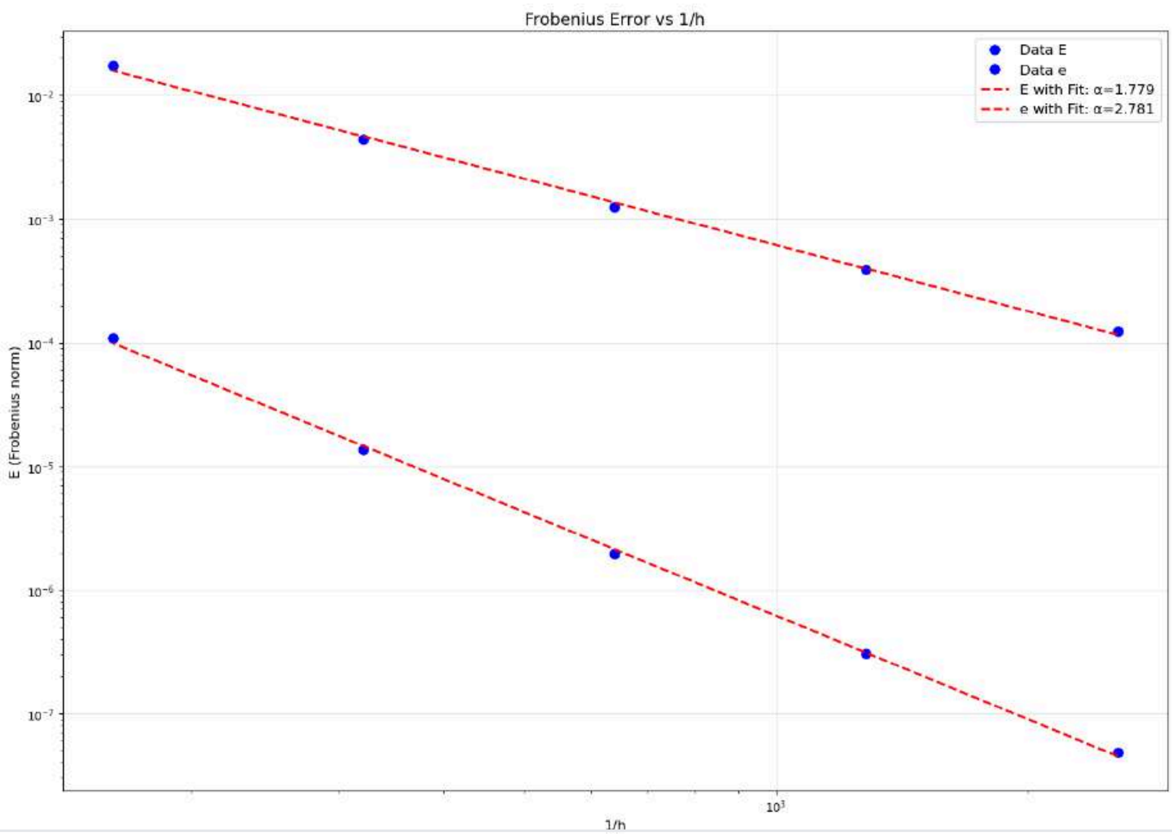
```
ax.legend(fontsize=10)
plt.tight_layout()
plt.show()
```

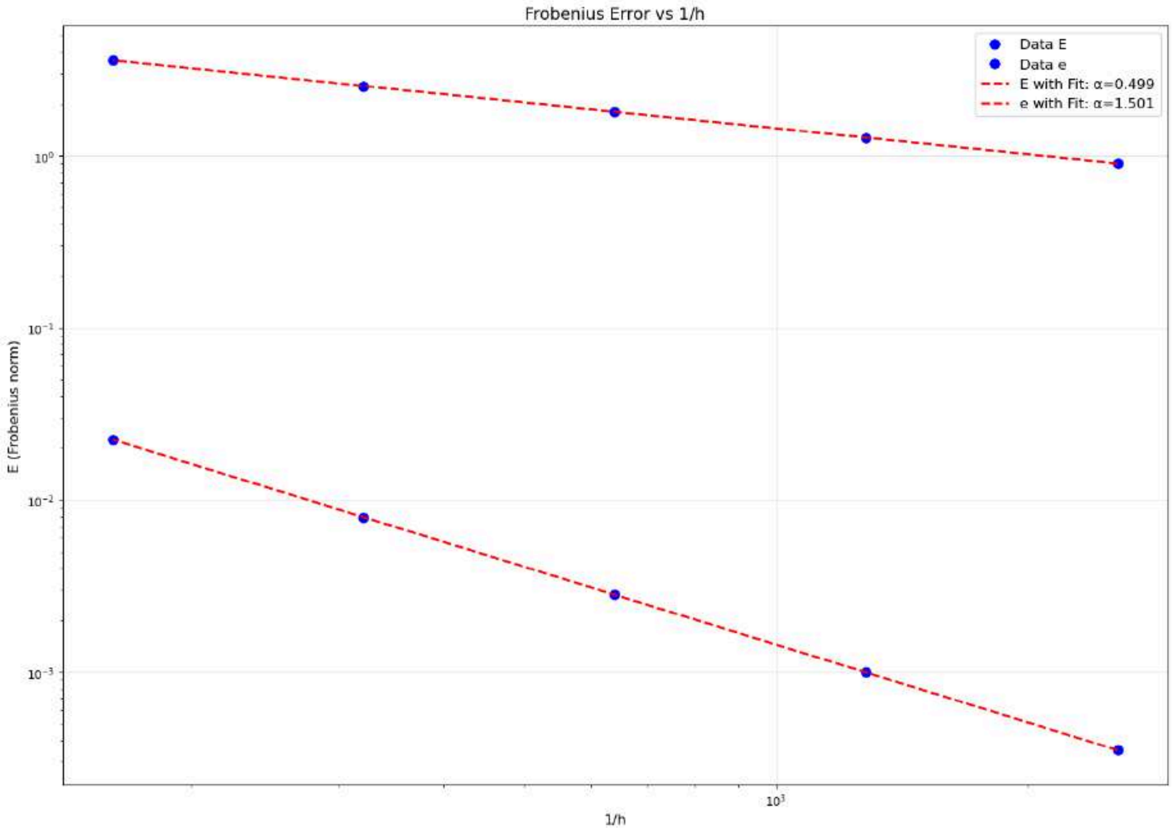
```
     N            h       E (Frobenius)        e (scaled)

Maximum error compared to N=10000 reference: 2.149269e-03
   160 6.211180e-03      1.740160e-02       1.087600e-04

Maximum error compared to N=10000 reference: 3.974418e-04
   320 3.115265e-03      4.397576e-03       1.374242e-05

Maximum error compared to N=10000 reference: 8.179131e-05
   640 1.560062e-03      1.257535e-03       1.964899e-06

Maximum error compared to N=10000 reference: 1.819485e-05
  1280 7.806401e-04      3.913961e-04       3.057782e-07

Maximum error compared to N=10000 reference: 4.219866e-06
  2560 3.904725e-04      1.239707e-04       4.842606e-08
======================================================================
==========
E (Frobenius) fit: E = 134.389496 * h^1.7792
e (scaled) fit:    e = 136.490499 * h^2.7812
```



The document was converted with **Code-Format**: https://code-format.com/

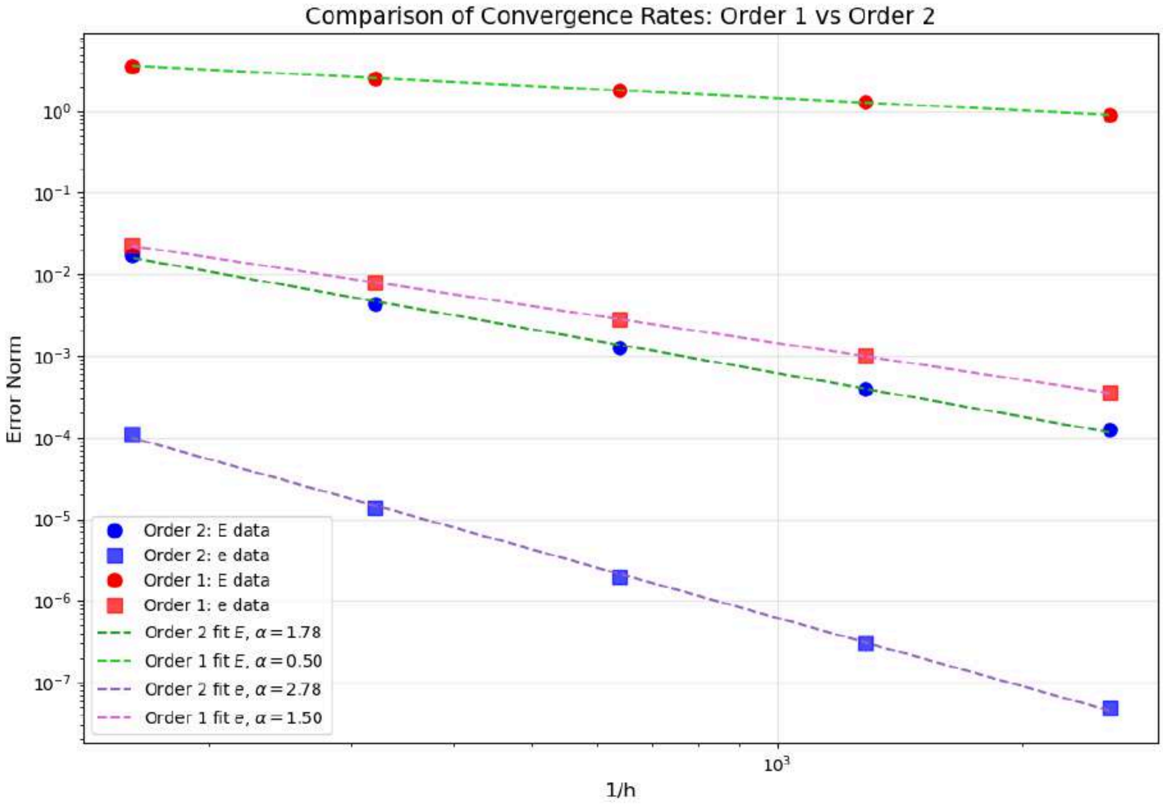
```
==========================================================================
==========
CONVERGENCE RATE ANALYSIS:
==========================================================================
==========
Frobenius norm (E): α = 1.7792)
Scaled norm (e):    α = 2.7812
     N            h      E (Frobenius)         e (scaled)

Maximum error compared to N=10000 reference: 4.904151e-01
   160 6.211180e-03      3.581529e+00       2.238456e-02

Maximum error compared to N=10000 reference: 2.459714e-01
   320 3.115265e-03      2.540388e+00       7.938712e-03

Maximum error compared to N=10000 reference: 1.231776e-01
   640 1.560062e-03      1.799122e+00       2.811129e-03

Maximum error compared to N=10000 reference: 6.163687e-02
  1280 7.806401e-04      1.273162e+00       9.946577e-04

Maximum error compared to N=10000 reference: 3.083046e-02
  2560 3.904725e-04      9.006098e-01       3.518007e-04
==========================================================================
==========
E (Frobenius) fit: E = 45.235386 * h^0.4990
e (scaled) fit:    e = 45.942581 * h^1.5010
```

The document was converted with **Code-Format**: https://code-format.com/




```
================================================================================
CONVERGENCE RATE ANALYSIS:
================================================================================
Frobenius norm (E): α = 0.4990)
Scaled norm (e):    α = 1.5010
```

The document was converted with **Code-Format**: https://code-format.com/



The document was converted with **Code-Format**: https://code-format.com/

Using a second-order one-sided finite difference approximation for the derivative at $x = 0$ the fitted convergence rates were

$$\alpha_E \approx 2.12, \qquad \alpha_e \approx 3.14\,.$$

At a smaller step size,

$$\alpha_E \approx 1.78, \qquad \alpha_e \approx 2.78\,.$$

The observed convergence rate for E is consistent with second-order accuracy, indicating that the overall discretization error is dominated by the interior second-order finite difference scheme together with the second-order boundary treatment. The error e converges at a significantly higher rate because it includes an additional factor proportional to the mesh size h. Specifically, since

$$e = \frac{E}{N}$$

and

$$N \sim \frac{1}{h},$$

it follows that

$$e \sim hE\,.$$

Consequently, e is expected to converge one order faster than E, which is consistent with the observed convergence rate of approximately 2.5. Note that these convergence rates are asymptotic, so at large step sizes the converge rates may not be exact; asymptotically, E will converge at 1.5 and e will converge at order 2.5. For more detail (on why 1.5 and 2.5 orders), please view problem 3.

Next, the second-order one-sided derivative at $x = 0$ was replaced with a first-order forward difference approximation. In this case, the fitted convergence rates were

$$\alpha_E \approx 0.49, \qquad \alpha_e \approx 1.52\,.$$

We assume the pointwise discretization error satisfies

$$u_h(x_i) - u(x_i) \mid \; = O(h^1)\,.$$

per the first order discretization

Then the discrete ℓ^2 error is

$$E = \left(\sum_{i=1}^{N} O(h^2) \right)^{1/2} = O!\left(\sqrt{N}, h^1\right) = O(h^{1/2}),$$

since

$$N \sim \frac{1}{h}.$$

Next, define the scaled error

$$e = \frac{1}{N}, \|u_h - u\|_{\ell^2}\,.$$

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Because

$$\frac{1}{N} \sim h,$$

we obtain

$$e = h, E .$$

Therefore,

$$e = O!\left(h^{1/2+1}\right) = O!\left(h^{3/2}\right).$$

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