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Project #2

(1) $u_t - u_{xx} = f(x,t) \quad (x,t) \in (0,1)\times(0,1), \quad f(x,t) = (\pi^2 - 1)e^{-t}\sin \pi x$

BC (initial & dirichlet):

$u(x,0) = \sin(\pi x)$

$u(0,t) = u(1,t) = 0$

let the test function be $v(x)$

1: Multiply strong form eq by test function & integrate over domain (by parts method):

$$\int_0^1 u_t v - u_{xx} v \, dx = \int_0^1 f(x,t) v \, dx$$

$$\int_0^1 u_t v \, dx dt - \int_0^1 u_{xx} v \, dx = \int_0^1 f(x,t) v \, dx$$

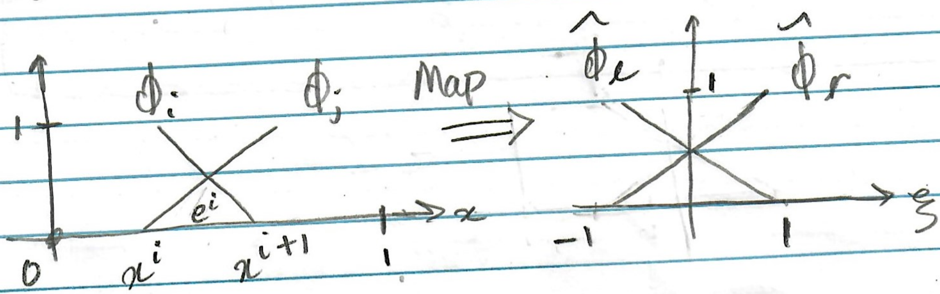
$$\int_0^1 u_t v \, dx - \underbrace{\left[u_x v\right]_{x=0}^{x=1}}_{=0} + \int_0^1 u_x v_x \, dx = \int_0^1 f(x,t) v \, dx$$

$$\boxed{\begin{gathered}\int_0^1 u_t v \, dx + \int_0^1 u_x v_x \, dx = \int_0^1 f(x,t) v \, dx \\ \text{or:} \\ \int_0^1 (u_t v + u_x v_x) \, dx = \int_0^1 f(x,t) v \, dx\end{gathered}}$$

we map each element e^i (of space Ω (0,1)) to $\hat{\Omega}$ (-1,1)

$$\int_{e^i} \phi_i \phi_j \, dx = \int_{\hat{\Omega}} \hat{\phi}_l \hat{\phi}_r \, |\det(J(e^i))| \, d\xi$$

$$\int_{e^i} \phi_i' \phi_j' \, dx = \int_{\hat{\Omega}} \hat{\phi}_l' \hat{\phi}_r' \, |\det J(e^i)| \, d\xi$$



$\hat{\phi}_l = \frac{1-\xi}{2}$ $\hat{\phi}_l' = \frac{-1}{2}$

$\hat{\phi}_r = \frac{1+\xi}{2}$ $\hat{\phi}_r' = \frac{1}{2}$

Assume linear map ξ:

$\xi(x) = \alpha(x - x_i) + \beta$

$\xi(x = x_{i+1}) = \alpha(x_{i+1} - x_i) + \beta = 1$ (1)

$\xi(x = x_i) = \alpha(x_i - x_i) + \beta = -1$ (2)

(2): $\beta = -1$

(1) $\alpha(x_{i+1} - x_i) - 1 = 1$; $\alpha(x_{i+1} - x_i) = 2$

$\alpha = \frac{2}{x_{i+1} - x_i}$ we assume constant node spacing (same element size, so, let $x_{i+1} - x_i = h$)

$\alpha = \frac{2}{h}$

$\xi(x) = \frac{2}{h}(x - x_i) - 1$

$\frac{d\xi}{dx} = \frac{2}{h} \Rightarrow \frac{dx}{d\xi} = \frac{h}{2}$

Forward Euler method.

let $v = \phi$ (test functions are equal to basis functions)

with Continuous Galerkin Forward Euler Method (CG-FEM):

$$\int_0^1 \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \phi_i(x)\,dx + \int_0^1 \frac{\partial u}{\partial x}\frac{\partial \phi_i}{\partial x}\,dx$$
$$- \int_0^1 f(x,t)\,\phi_i(x) = 0.$$

let superscript indicate time index (n) while subscript indicates spatial index:

$$\frac{1}{\Delta t}\int_0^1 \left(\sum_{j=1}^{n} u_j^{n+1}\phi_j\right)\phi_i\,dx - \frac{1}{\Delta t}\int_0^1\left(\sum_{j=1}^{n} u^n \phi_j\right)\phi_i\,dx + \int_0^1 \frac{\partial}{\partial x}\left(\sum_{j=1}^{n} u_j^n \phi_j\right)\frac{\partial \phi_i}{\partial x}\,dx$$
$$- \int_0^1 f(x,t)\,\phi_i\,dx = 0$$

$$\frac{1}{\Delta t}\int_0^1 \sum_{j=1}^{n} u_j^{n+1}\phi_i\phi_j\,dx - \frac{1}{\Delta t}\int_0^1 \sum_{j=1}^{n} u_j^n \phi_i \phi_j\,dx + \int_0^1 \sum_{j=1}^{n} u_j^n \frac{\partial}{\partial x}\phi_j \frac{\partial \phi_i}{\partial x}\,dx$$
$$- \int_0^1 f(x,t)\,\phi_i\,dx = 0$$

$$\frac{1}{\Delta t}\sum_{j=1}^{n} u_j^{n+1}\int_0^1 \phi_i\phi_j\,dx - \frac{1}{\Delta t}\sum_{j=1}^{n} u_j^n \int_0^1 \phi_i\phi_j + \sum_{j=1}^{n} u_j^n \int_0^1 \phi_i'\phi_j'\,dx$$
$$= \int_0^1 f(x,t)\,\phi_i\,dx$$

let $\int_0^1 \phi_i\phi_j\,dx = M_{ij}$ of matrix M.

$\int_0^1 \phi_i'\phi_j'\,dx = K_{ij}$ of matrix K

also let $\int_0^1 f(x,t)\,\phi_i(x)\,dx = \vec{F}_i$ of vector $\vec{F}$

Then:

$$\frac{1}{\Delta t} M \vec{u}^{\,n+1} - \frac{1}{\Delta t} M \vec{u}^{\,n} + K \vec{u}^{\,n} = \vec{F}^{\,n}$$

$$\vec{u}^{\,n+1} = \vec{u}^{\,n} - \Delta t\, M^{-1} K \vec{u}^{\,n} + \Delta t\, M^{-1} \vec{F}^{\,n}$$

we apply directilet boundary

conditions on our $K\vec{u} = \vec{F}$ system:

Boundary $u(x=0)$

$$\begin{bmatrix} 1 & 0 & 0 & \\ 0 & k_{22} & k_{23} & \\ 0 & k_{32} & k_{33} & \ddots \\ \vdots & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \hat{u}_1 \\ F_2 - k_{21}\hat{u}_1 \\ F_3 - k_{31}\hat{u}_1 \\ \vdots \end{bmatrix}$$

Boundary $u(x=1)$

Backward Euler Method

all are $u(x, t+\Delta t)$ or $u(x,t)$ for simplicity, x is omitted.

$$\int_0^1 \frac{u(t+\Delta t) - u(t)}{\Delta t}\,\phi_i(x)\,dx + \int_0^1 \frac{\partial u(t+\Delta t)}{\partial x}\,\frac{\partial \phi_i(x)}{\partial x}\,dx$$

$$= \int_0^1 f(x, t+\Delta t)\,\phi_i(x)\,dx$$

$$\frac{1}{\Delta t}\int_0^1 u(t+\Delta t)\,\phi_i(x)\,dx - \frac{1}{\Delta t}\int_0^1 u(t)\,\phi_i(x)\,dx$$

$$+\int_0^1 \frac{\partial}{\partial x} u(t+\Delta t)\,\frac{\partial \phi_i(x)}{\partial x}\,dx = \int_0^1 f(x,t+\Delta t)\,\phi_i(x)\,dx$$

$$\frac{1}{\Delta t}\int_0^1 \sum_{j=1}^{n} u_j^{n+1}\phi_j(x)\,\phi_i(x)\,dx - \frac{1}{\Delta t}\int_0^1 \sum_{j=1}^{n} u_j^{n}\,\phi_j(x)\,\phi_i(x)\,dx$$

$$+\int_0^1 \frac{\partial}{\partial x}\left(\sum_{j=1}^{n} u_j^{n+1}\,\phi_j(x)\right)\frac{\partial}{\partial x}\,\phi_i(x)\,dx$$

$$=\int_0^1 f(x,t+\Delta t)\,\phi_i(x)\,dx$$

$$\frac{1}{\Delta t}\sum_{j=1}^{n} u_j^{n+1}\int_0^1 \phi_i\phi_j\,dx - \frac{1}{\Delta t}\sum_{j=1}^{n} u_j^{n}\int_0^1 \phi_i\phi_j\,dx$$

$$+\int_0^1 \sum_{j=1}^{n} u_j^{n+1}\,\frac{\partial}{\partial x}\phi_i\,\frac{\partial}{\partial x}\phi_j\,dx = \int_0^1 f(x,t+\Delta t)\,\phi_i(x)\,dx$$

$$\frac{1}{\Delta t}\sum_{j=1}^{n} u_j^{n+1}\int_0^1 \phi_i\phi_j\,dx - \frac{1}{\Delta t}\sum_{j=1}^{n} u_j^{n}\int_0^1 \phi_i\,\phi_j\,dx$$

$$+\sum_{j=1}^{n} u_j^{n+1}\int_0^1 \phi_i'\,\phi_j'\,dx = \int_0^1 f(x,t+\Delta t)\,\phi_i(x)\,dx$$

$$\frac{1}{\Delta t} M\vec{u}^{n+1} - \frac{1}{\Delta t} M\vec{u}^{n} + K\vec{u}^{n+1} = \vec{F}$$

$$M\vec{u}^{n+1} - M\vec{u}^{n} + K\Delta t\,\vec{u}^{n+1} = F\Delta t$$

$$\vec{u}^{n+1} = \underbrace{(M+K\Delta t)^{-1}}_{\text{matrix } B}\underbrace{(M\vec{u}^{n}+F\Delta t)}_{\text{vector } b}$$

(in code)

at specific element

$$K_{ij} = \int_{e^i} \phi_i' \phi_j' \, dx \qquad , i \neq j$$

Map to parent element

$$= \int_{-1}^{1} \left(\frac{\partial \hat{\phi}_\ell'}{\partial \xi} \cdot \frac{d\xi}{dx} \right) \left(\frac{\partial \hat{\phi}_r'}{\partial \xi} \cdot \frac{d\xi}{dx} \right) \frac{dx}{d\xi} \, d\xi$$

$$= \int_{-1}^{1} \left(\frac{-1}{2} \cdot \frac{2}{h} \right) \left(\frac{1}{2} \cdot \frac{2}{h} \right) \cdot \frac{h}{2} \, d\xi$$

$$= \frac{-1}{2h} \int_{-1}^{1} d\xi = \frac{-1}{2h} \left[\xi \right]_{-1}^{1} = \frac{-1}{h}$$

$$M_{ij} = \int_{e^i} \phi_i \phi_j \, dx \qquad , i \neq j$$

Map to parent element

$$= \int_{-1}^{1} \hat{\phi}_\ell \, \hat{\phi}_r \, \det(J(e^i)) \cdot d\xi$$

$$= \int_{-1}^{1} \left(\frac{1-\xi}{2} \right) \left(\frac{1+\xi}{2} \right) \cdot \frac{dx}{d\xi} \cdot d\xi$$

$$= \int_{-1}^{1} \frac{1}{4} (1-\xi^2) \frac{h}{2} \, d\xi = \frac{h}{8} \left[\xi - \frac{\xi^3}{3} \right]_{-1}^{1}$$

$$= \frac{h}{8} \left[1 - \frac{1}{3} \right] \cdot 2 = \frac{h}{4} \left(\frac{2}{3} \right) = \frac{h}{6}$$

for $i=j$:

$$K_{ii} = \int_{e^i} \phi_i' \phi_i' \, dx$$

$$= \int_{-1}^{1} \left(\frac{\partial \hat{\phi_i}}{\partial \xi} \cdot \frac{d\xi}{dx} \right) \left(\frac{\partial \hat{\phi_i}}{\partial \xi} \frac{d\xi}{dx} \right) \frac{dx}{d\xi} \, d\xi$$

$$= \int_{-1}^{1} \left(\frac{-1}{2} \cdot \frac{2}{h} \right) \left(\frac{-1}{2} \frac{2}{h} \right) \cdot \frac{h}{2} \, d\xi$$

$$= \frac{1}{2h} \int_{-1}^{1} d\xi = \frac{1}{2h} \xi \Big]_{-1}^{1} = \frac{1}{h}$$

$$M_{ii} = \int_{e^i} \phi_i \phi_i \, dx$$

$$= \int_{-1}^{1} \hat{\phi_i} \hat{\phi_i} \det(J(e^i)) \, d\xi$$

$$= \int_{-1}^{1} \left(\frac{1-\xi}{2} \right) \left(\frac{1-\xi}{2} \right) \frac{dx}{d\xi} \, d\xi$$

$$= \int_{-1}^{1} \frac{1}{4} (1-\xi)^2 \cdot \frac{h}{2} \, d\xi = \frac{h}{8} \left[\frac{(1-\xi)^3}{3} \right]_{1}^{-1}$$

$$= \frac{h}{8} \cdot \frac{2^3}{3} = \frac{h}{3}$$

2 hat functions contribute to element $e^i = [x^i, x^{i+1}]$

$$F_i = \int_{e^i} f(x,t)\,\phi_i(x)\,dx + \int_{e^i} f(x,t)\,\phi_{i+1}(x)\,dx$$

map to parent element space:

$$= \int_{-1}^{1} \underbrace{\hat{f}(\xi,t)}_{\text{function } f \text{ transformed to } \xi \text{ space}} \phi_i(\xi)\,\det(J(e^i))\,d\xi$$

$$+ \int_{-1}^{1} \hat{f}(\xi,t)\,\phi_{i+1}(\xi)\,\underbrace{\det(J(e^i))}_{= dx/d\xi}\,d\xi$$

using gaussian quadrature $(n=2)$ to numerically integrate:

$$= \sum_{j=1}^{2} \hat{f}(\xi_j, t)\,\phi_i(\xi_j)\,\frac{dx}{d\xi}$$

quadrature points

$$+ \sum_{j=1}^{2} \hat{f}(\xi_j, t)\,\phi_{i+1}(\xi_j)\,\frac{dx}{d\xi}$$

per $\xi(x) = \frac{2}{h}(x - x_i) - 1$,

$$x(\xi) = \frac{h}{2}(\xi + 1) + x_i$$

$$= \sum_{j=1}^{2} f\left(\frac{h}{2}(\xi_j + 1) + x_i, t\right) \cdot \phi_i(\xi_j)\,\frac{dx}{d\xi}$$

$$+ \sum_{j=1}^{2} f\left(\frac{h}{2}(\xi_j + 1) + x_i, t\right)\,\phi_{i+1}(\xi_j)\,\frac{dx}{d\xi}$$