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Engineering Computation: COE 311K

08 February 2024

Homework 1

Note: all code blocks are prefaced by the following import statements:

```
from matplotlib import pyplot as plt
from math import sqrt, tanh, factorial, pi
import numpy as np
from typing import List
```

Problem 1:

Function `exact_velocity()` found in HW1.py.

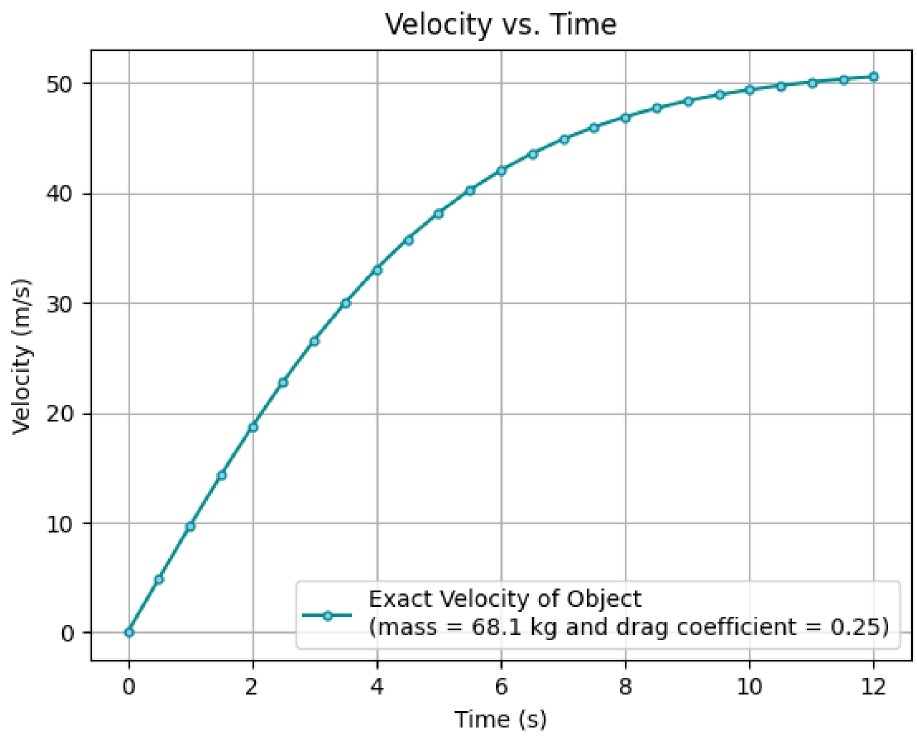
Problem 2:

Code block calling `exact_velocity()` and plotting velocity versus time:

```
# problem 1
t = np.linspace(0,12,25)
c_d = .25
m = 68.1
g = 9.81
v_exact = exact_velocity(c_d,m,t,g)

plt.figure(0)
plt.plot(t, v_exact,
            color = "darkcyan",
            markerfacecolor = "lightskyblue",
            marker = "o", markersize = 3.5,
            label = "Exact Velocity")
plt.title("Velocity vs. Time")
plt.xlabel("Time (s)") # assumed seconds
plt.ylabel("Velocity (m/s)") # assumed meters/second
plt.legend([f"Exact Velocity of Object\n(mass = {m} kg and drag coefficient = {c_d})"])
plt.grid()
plt.show()
plt.savefig("HW1P2.png")
```

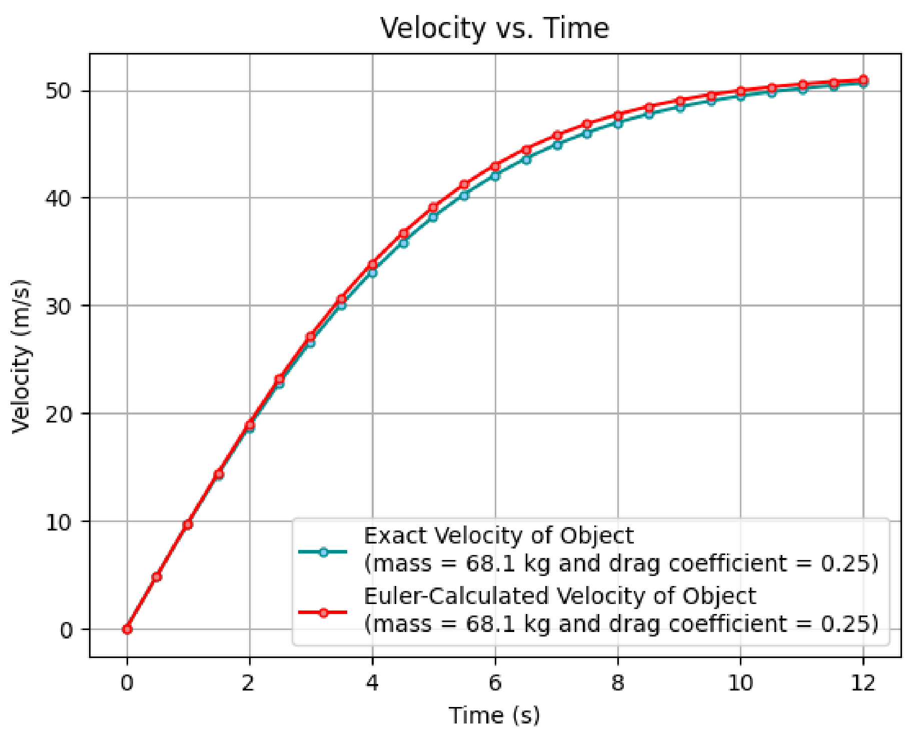
Plot on the next page.



Problem 3:

Function `forward_Euler_velocity()` found in HW1.py.

Problem 4:



The difference between the exact velocity and velocity calculated by the Euler approximation is most obvious at high velocities, before the object achieves terminal velocity. The exact velocity has $\frac{\partial v}{\partial t} \sim v(t)^2$ while the Euler approximation suggests $\frac{\partial v}{\partial t} \sim v(t)$. At low velocities, although $\frac{\partial v}{\partial t}$ is high, the $v(t)$ is low so the difference between $v(t)$ and $v(t)^2$ is not noticeable, leading to less error. At higher velocities, the difference $v(t)$ and $v(t)^2$ is quite high, resulting in larger errors. Near terminal velocity, $\frac{\partial v}{\partial t}$ is close to zero, so the error between the exact velocity and the Euler approximation is less noticeable.

Code block:

```
# problem 4
t = np.linspace(0,12,25)
c_d = .25
m = 68.1
g = 9.81
v_exact = exact_velocity(c_d,m,t,g)
v_euler = forward_Euler_velocity(c_d,m,t,g)

plt.figure(1)
plt.plot(t, v_exact,
         color = "darkcyan",
         markerfacecolor = "lightskyblue",
         marker = "o", markersize = 3.5,
         label = "Exact Velocity")
plt.plot(t, v_euler,
          color = "red",
          markerfacecolor = "lightcoral",
          marker = "o", markersize = 3.5,
          label = "Euler-Calculuated Velocity")
plt.title("Velocity vs. Time")
plt.xlabel("Time (s)") # assumed to be in seconds
plt.ylabel("Velocity (m/s)") # assumed to be in meters per second
plt.legend([f"Exact Velocity of Object \
            \n (mass = {m} kg and drag coefficient = {c_d})", \
            f"Euler-Calculated Velocity of Object\
            \n(mass = {m} kg and drag coefficient = {c_d})"])
plt.grid()
plt.show()
plt.savefig("HW1P4.png")
```

Problem 5:

Code block calculating and plotting root mean square error of Euler approximation:

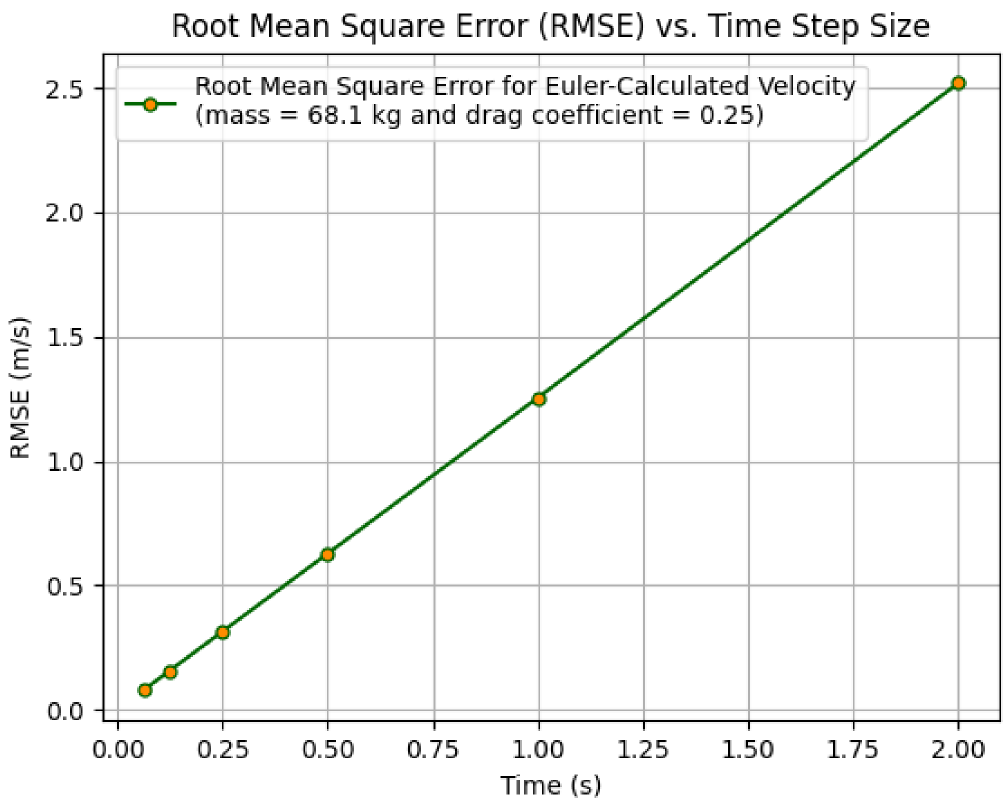
```python
def root_mean_square_error(maxt:float, dt:float, c_d:float, m:float, g:float) -> float:
    """
    Calculates the root mean square error of the velocity at any given time point.
    Args:
        float maxt: the maximum time to be calculated.
        float dt: the step size or delta time interval between velocity re-calculations.
        float c_d: drag coefficient
        float m: mass of the object
        float g: gravitational constant
    Returns:
        float rmse: the root mean square error of the velocities of the data sample. The data sample is defined\
        as all the times between 0 and maxt with the delta t as dt.
    """
    tsteps = int(maxt/dt) + 1 # the total number of time instances where to calculate velocity
    t = np.linspace(0,maxt, tsteps) # each time spot
    # the difference between the exact and approximated velocities at the times listed in t
    dv = np.array(exact_velocity(c_d, m, t, g) - forward_Euler_velocity(c_d, m, t, g))
    rmse = sqrt(np.sum(dv**2)/tsteps) # compute the quadrature of dv over the number of time steps
    return rmse
```

```python
# problem 5
c_d = .25
m = 68.1
g = 9.81
rmse = []
step_sizes = [.0625,.125,.25,.5,1,2]
max_time = 12
for dt in step_sizes:
    rmse+=[root_mean_square_error(max_time,dt,c_d,m,g)]
print(f"RMSE values from t = (0,12) seconds with step sizes of {step_sizes}:\n{rmse}")
plt.figure(2)
plt.plot(step_sizes, rmse,
            color = "darkgreen",
            markerfacecolor = "darkorange",
            marker = "o", markersize = 5,
            label = "Exact Velocity")
plt.title("Root Mean Square Error (RMSE) vs. Time Step Size")
plt.xlabel("Time (s)") # assumed to be in seconds
plt.ylabel("RMSE (m/s)") # assumed to be in meters per second
plt.legend([f"Root Mean Square Error for Euler-Calculated Velocity \
             \n(mass = {m} kg and drag coefficient = {c_d})"])
plt.grid()
plt.show()
plt.savefig("HW1P5.png")
```

Code output:

```
RMSE values from t = (0,12) seconds with step sizes of [0.0625, 0.125, 0.25, 0.5, 1, 2] seconds:
[0.07852760414273907, 0.15701845765627015, 0.3139061565534722, 0.6274243308775233, 1.254480667474395, 2.5199229449235085]
```

Plot on the next page.



Problem 6:

1. As a computational engineer in general, what is error and why is it important to us?
   Error is the difference between a computed or estimated value and the true value. For computational engineers, error measurement is crucial to determine the validity of a model; a model with high error cannot represent real-world scenarios. Comparing errors between models (the error between each model and the true value) can also determine which model is better.
2. In your own words, what is roundoff error and why should we care about it?
   Roundoff error is error due to the computer's finite precision; computers cannot represent irrational numbers to infinite precision so the difference between the true value and the rounded value (that is stored in the computer) is the roundoff error.
3. In your own words, what is truncation error and why should we care about it?
   Truncation error is associated with converting mathematical operations to discrete operations that can be performed by computers. For example, derivatives or integrals are numerically approximated; exponential functions like e^x or functions $\sin(x)$ like may be approximated by a finite set of an infinite series.

Problem 7:

7 Order of Truncation error = ?

$$\frac{\partial^2 f(x)}{\partial x^2} \simeq \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \frac{f''''(x)h^4}{4!} + \frac{f^{(5)}(\xi)h^5}{5!}$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2!} - \frac{f'''(x)h^3}{3!} + \frac{f''''(x)h^4}{4!} - \frac{f^{(5)}(\xi)h^5}{5!}$$

$$f(x+h) + f(x-h) = 2\left[f(x) + \frac{f''(x)h^2}{2!} + \frac{f''''(x)h^4}{4!}\right]$$

$$f(x+h) + f(x-h) - 2f(x) = 2\left[\frac{f''(x)h^2}{2!} + \frac{f''''(x)h^4}{4!}\right]$$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = 2\left[\frac{f''(x)}{2!} + \frac{f''''(x)}{4!}h^2\right]$$

$$\underbrace{\left|\text{Error}_{\text{truncation}}\right|}_{\varepsilon_T} = \left|\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{\partial^2 f(x)}{\partial x^2}\right|$$

$$\varepsilon_T = \left|\left(f''(x) + \frac{f''''(x)h^2}{12}\right) - f''(x)\right|$$

$$= \frac{f^4(x)h^2}{12}$$

h^2 : $O(h^2)$: 2^{nd} Order of Truncation Error

Problem 8:

8.2 $[A] = \begin{bmatrix} 4 & 7 \\ 1 & 2 \\ 5 & 6 \end{bmatrix}$ $[B] = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}$ $[C] = \begin{Bmatrix} 3 \\ 6 \\ 1 \end{Bmatrix}$

$[D] = \begin{bmatrix} 9 & 4 & 3 & -6 \\ 2 & -1 & 7 & 5 \end{bmatrix}$ $[E] = \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}$

$[F] = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix}$ $[G] = [7 \ 6 \ 4]$

(a) dimensions:

$[A] : (3,2)$

$[B] : (3,3)$

$[C] : (3,1)$

$[D] : (2,4)$

$[E] : (3,3)$

$[F] : (2,3)$

$[G] : (1,3)$

(b) square matrices: [B], [E]

column matrices: [C]

row matrices: [G]

(c) value of elements:

$a_{12} = 7$

$b_{23} = 7$

d_{32} = does not exist

$e_{22} = 2$

$f_{12} = 0$

$g_{12} = 6$

(d) 1: $[E]+[B]=\begin{bmatrix}1&5&8\\7&2&3\\4&0&6\end{bmatrix}+\begin{bmatrix}4&3&7\\1&2&7\\2&0&4\end{bmatrix}=\begin{bmatrix}5&8&15\\8&4&10\\6&0&10\end{bmatrix}$

2: $[A]+[E]$ = not computable

$size([A]) \doteq (3,2) \neq size([B]) = (3,3)$

3: $[B]-[E]=\begin{bmatrix}4&3&7\\1&2&7\\2&0&4\end{bmatrix}-\begin{bmatrix}1&5&8\\7&2&3\\4&0&6\end{bmatrix}=\begin{bmatrix}3&-2&-1\\-6&0&4\\-2&0&-2\end{bmatrix}$

4: $7\times[B]=7\cdot\begin{bmatrix}4&3&7\\1&2&7\\2&0&4\end{bmatrix}=\begin{bmatrix}28&21&49\\7&14&49\\14&0&28\end{bmatrix}$

5: $[C]^T=\begin{Bmatrix}3\\6\\1\end{Bmatrix}^T=\begin{bmatrix}3&6&1\end{bmatrix}$

6: $[E]\times[B]=\begin{bmatrix}1&5&8\\7&2&3\\4&0&6\end{bmatrix}\times\begin{bmatrix}4&3&7\\1&2&7\\2&0&4\end{bmatrix}$

$=\begin{bmatrix}4+5+16&3+10+0&7+35+32\\28+2+6&21+4+0&49+14+12\\16+0+12&12+0+0&28+0+24\end{bmatrix}$

$=\begin{bmatrix}25&13&74\\36&25&75\\28&12&52\end{bmatrix}$

7: $[B]\times[A]=\begin{bmatrix}4&3&7\\1&2&7\\2&0&4\end{bmatrix}\times\begin{bmatrix}4&7\\1&2\\5&6\end{bmatrix}$

$=\begin{bmatrix}16+3+35&28+6+42\\4+2+35&7+4+42\\8+0+20&14+0+24\end{bmatrix}$

$=\begin{bmatrix}54&76\\41&53\\28&38\end{bmatrix}$

8: $[D]^T = \begin{bmatrix} 9 & 4 & 3 & -6 \\ 2 & -1 & 7 & 5 \end{bmatrix}^T = \begin{bmatrix} 9 & 2 \\ 4 & -1 \\ 3 & 7 \\ -6 & 5 \end{bmatrix}$

9: $[A] \times [C] = \begin{bmatrix} 4 & 7 \\ 1 & 2 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$ = not computable

[A] size = (3,2); [C] size = (3,1)

for matrix multiplication, the number of columns of the first matrix (matrix [A], 2 columns) should be equal to the number of rows in the second matrix (matrix [B], 3 rows).

10: [I] x [B] · assuming identity is of a compatable size with matrix [B]. The identity matrix is a square matrix. For matrix multiplication, number of columns of identity matrix must equal number of rows of [B] matrix.

$[B] = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}$ size([B]) = (3,3)

rows[B] cl =

so, size([I]) = (3,3)

$[I] \times [B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4+0+0 & 3+0+0 & 7+0+0 \\ 0+1+0 & 0+2+0 & 0+7+0 \\ 0+0+2 & 0+0+0 & 0+0+4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix} = [B]$

11: $[E]^T \times [E]$

$$[E]^T = \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 & 4 \\ 5 & 2 & 0 \\ 8 & 3 & 6 \end{bmatrix}$$

$$[E]^T \times [E] = \begin{bmatrix} 1 & 7 & 4 \\ 5 & 2 & 0 \\ 8 & 3 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1+49+16 & 5+14+0 & 8+21+24 \\ 5+14+0 & 25+4+0 & 40+6+0 \\ 8+21+24 & 40+6+0 & 64+9+36 \end{bmatrix}$$

$$= \begin{bmatrix} 66 & 19 & 53 \\ 19 & 29 & 46 \\ 53 & 46 & 109 \end{bmatrix}$$

12: $[C]^T \times [C]$

$$[C]^T = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}^T = [3 \ 6 \ 1]$$

$$[C]^T \times [C] = [3 \ 6 \ 1] \times \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} = [9+36+1] = [46]$$

Problem 9:

9. (8.3)

$50 = 15x_3 - 7x_2$ eq 1

$4x_1 + 7x_3 + 30 = 3x_2$ eq 2

$5x_1 - 7x_3 = 42 - 9x_2 + 25x_1$ eq 3

$0x_1 - 7x_2 + 15x_3 = 50$ eq 1

$4x_1 - 3x_2 + 7x_3 = -30$ eq 2

$\underbrace{5x_1 - 25x_1}_{-20x_1} + 9x_2 - 7x_3 = 42$ eq 3

$$\begin{bmatrix} 0 & -7 & 15 \\ 4 & -3 & 7 \\ -20 & 9 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ -30 \\ 42 \end{bmatrix}$$

Problem 10:

10. (9.4) $[A] = \begin{bmatrix} 6 & -1 \\ 12 & 8 \\ -5 & 4 \end{bmatrix}$ $[B] = \begin{bmatrix} 4 & 0 \\ 0.5 & 2 \end{bmatrix}$ $[C] = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$

a) all combinations of matrices A, B, C:

$[A] \times [B]$	$(3,2) \times (2,2)$	$2 = 2$ ✓
$[B] \times [A]$	$(2,2) \times (3,2)$	$2 \neq 3$ ✗
$[A] \times [C]$	$(3,2) \times (2,2)$	$2 = 2$ ✓
$[C] \times [A]$	$(2,2) \times (3,2)$	$2 \neq 3$ ✗
$[B] \times [C]$	$(2,2) \times (2,2)$	$2 = 2$ ✓
$[C] \times [B]$	$(2,2) \times (2,2)$	$2 = 2$ ✓

size $[A]$: $(3,2)$
size $[B]$: $(2,2)$
size $[C]$: $(2,2)$

b) Note: number of columns in first matrix must equal number of rows in second matrix. Both of these are circled above.

a) Thus, the following matrices can be multiplied

1. $[A] \times [B] = \begin{bmatrix} 6 & -1 \\ 12 & 8 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 0 \\ .5 & 2 \end{bmatrix} = \begin{bmatrix} 24 - .5 & 0 - 2 \\ 48 + 4 & 0 + 16 \\ -20 + 2 & 0 + 8 \end{bmatrix}$

$= \begin{bmatrix} 23.5 & -2 \\ 52 & 16 \\ -18 & 8 \end{bmatrix}$

2. $[A] \times [C] = \begin{bmatrix} 6 & -1 \\ 12 & 8 \\ -5 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 12 - 3 & -12 - 1 \\ 24 + 24 & -24 + 8 \\ -10 + 12 & 10 + 4 \end{bmatrix}$

$= \begin{bmatrix} 9 & -13 \\ 48 & -16 \\ 2 & 14 \end{bmatrix}$

$$[B]\times[C] = \begin{bmatrix} 4 & 0 \\ .5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8+0 & -8+0 \\ 1+6 & -1+2 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ 7 & 1 \end{bmatrix}$$

$$[C]\times[B] = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 0 \\ .5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-1 & 0-4 \\ 12+.5 & 0+2 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 12.5 & 2 \end{bmatrix}$$

c) For two matrices, [A] & [B], when the matrices are multiplied, the number of columns of the first matrix must be equal to the number of rows of the second matrix. Let the size of [A] be (l, m) & the size of [B] be (n, o) where l, m, n, o are arbitrary integers. $[A]\times[B]$ is valid only if $m=n$, $[B]\times[A]$ is only valid if $o=l$. These two conditions ($m=n$ & $o=l$) are independent; if $[A]\times[B]$ is valid, this means nothing about $[B]\times[A]$.

For part a, note that $[A]\times[B]$ is valid but $[B]\times[A]$ is not. Similarily, $[A]\times[C]$ is valid but $[C]\times[A]$ is not. The order of multiplication determines if the product can be computed. Additionally, note that $[B]\times[C]$ & $[C]\times[B]$ are both valid, but their products are not equal.

The sizes of the 2 factors are flipped:
let [B] have size (l, m) & [C] have size (n, o). For $[B]\times[C]=[D]$ to be computable, $m=n$ & for $[C]\times[B]=[E]$ to be computable, $o=l$. Thus, [B], [C] have sizes $[l, m]$ & $[m, l]$.
$[D] \neq [C]$; matrix multiplication is not generally commutative.

Problem 11:

11.
(9.4)

$$\begin{aligned} -3x_2+7x_3&=4 \\ x_1+2x_2-x_3&=0 \\ 5x_1-2x_2&=3 \end{aligned} \Rightarrow \begin{aligned} 0x_1-3x_2+7x_3&=4 \\ x_1+2x_2-x_3&=0 \\ 5x_1-2x_2+0x_3&=3 \end{aligned}$$

$$\begin{bmatrix} 0 & -3 & 7 \\ 1 & 2 & -1 \\ 5 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

a)

$$\begin{vmatrix} 0 & -3 & 7 \\ 1 & 2 & -1 \\ 5 & -2 & 0 \end{vmatrix} = 0\left(2\cdot 0-(-1)\cdot(-2)\right)-(-3)\left(1\cdot 0-(-1)\cdot 5\right)+7\left(1\cdot(-2)-2\cdot 5\right)$$

$$= 0+3(0+5)+7(-2-10)$$

$$= 15+7(-12) = -69$$

11.

(c) $\begin{bmatrix} 0 & -3 & 7 \\ 1 & 2 & -1 \\ 5 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & -3 & 7 & 4 \\ 1 & 2 & -1 & 0 \\ 5 & -2 & 0 & 3 \end{array}\right] \xrightarrow{R_5 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 5 & -2 & 0 & 3 \\ 1 & 2 & -1 & 0 \\ 0 & -3 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{R_1/5 \to R_1} \left[\begin{array}{ccc|c} 1 & -.4 & 0 & .6 \\ 1 & 2 & -1 & 0 \\ 0 & -3 & 7 & 4 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -.4 & 0 & .6 \\ 0 & 2.4 & -1 & -.6 \\ 0 & -3 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -.4 & 0 & .6 \\ 0 & -3 & 7 & 4 \\ 0 & 2.4 & -1 & -.6 \end{array}\right]$$

$$\xrightarrow{R_2 \cdot 4/5 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -.4 & 0 & .6 \\ 0 & -3 & 7 & 4 \\ 0 & 0 & \frac{23}{5} & 2.6 \end{array}\right]$$

$\frac{23}{5} x_3 = 2.6 \quad \Rightarrow x_3 = 13/23 \approx 0.565$

$-3x_2 + 7x_3 = 4 \quad \Rightarrow -3x_2 + 3.96 = 4; \; x_2 = \frac{-1}{69} \approx -.0145$

$x_1 - .4x_2 = .6 \quad \Rightarrow x_1 + .006 = .6; \; x_1 = \frac{41}{69} \approx .594$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .594 \\ -.0145 \\ .565 \end{bmatrix}$

determinant of $\begin{bmatrix} 1 & -.4 & 0 \\ 0 & -3 & 7 \\ 0 & 0 & 4.6 \end{bmatrix} = 1 \times -3 \times 4.6$
$= -13.8$

Since in step 2, row 1 was divided by 5 (scaled by 1/5), the determinant was also scaled by 1/5.
Additionally, 2 rows were swapped due to partial pivoting; so the determinant was scaled by $(-1)^2 = +1$

Applying these steps to the calculated determinant of the reduced matrix in the reverse order brings the determinant to:

$5 \times 1 \times -13.8 = -69$, which is equal to the determinant of the non-reduced initial matrix.

(d) To substitute solutions back into original equations:

$$\left.\begin{array}{l} -3x_2 + 7x_3 = 4 \\ x_1 + 2x_2 - x_3 = 0 \\ 5x_1 - 2x_2 = 3 \end{array}\right\} \vec{x} = \begin{bmatrix} 41/69 \\ -1/69 \\ 13/23 \end{bmatrix}$$

$$-3\left(\frac{-1}{69}\right) + 7\left(\frac{13}{23}\right) = \frac{3}{69} + \frac{91}{23} = \frac{6348}{1587} = 4 \checkmark$$

$$\frac{41}{69} + 2\left(\frac{-1}{69}\right) - \frac{13}{23} = \frac{39}{69} - \frac{13}{23} = 0 \checkmark$$

$$5\left(\frac{41}{69}\right) - 2\left(\frac{-1}{69}\right) = \frac{205}{69} + \frac{2}{69} = \frac{207}{69} = 3 \checkmark$$

Problem 12:

12
(9.6)
$$10x_1 + 2x_2 - x_3 = 27 \quad \text{eq 1}$$
$$-3x_1 - 5x_2 + 2x_3 = -61.5 \quad \text{eq 2}$$
$$x_1 + x_2 + 6x_3 = -21.5 \quad \text{eq 3}$$

a)
$$\begin{bmatrix} 10 & 2 & -1 \\ -3 & -5 & 2 \\ 1 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

12
a)
$$\left[\begin{array}{ccc|c} 10 & 2 & -1 & 27 \\ -3 & -5 & 2 & -61.5 \\ 1 & 1 & 6 & -21.5 \end{array}\right] \xrightarrow{R_2 + \frac{3}{10}R_1 \to R_2} \left[\begin{array}{ccc|c} 10 & 2 & -1 & 27 \\ 0 & -4.4 & 1.7 & -53.4 \\ 1 & 1 & 6 & -21.5 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{R_1}{10} \to R_3} \left[\begin{array}{ccc|c} 10 & 2 & -1 & 27 \\ 0 & -4.4 & 1.7 & -53.4 \\ 0 & .8 & 6.1 & -24.2 \end{array}\right]$$

$$\xrightarrow{R_3 + \frac{2}{11}R_2 \to R_3} \left[\begin{array}{ccc|c} 10 & 2 & -1 & 27 \\ 0 & -4.4 & 1.7 & -53.4 \\ 0 & 0 & \frac{141}{22} & \frac{-373}{11} \end{array}\right]$$

$$\frac{141}{22}x_3 = \frac{-373}{11} \Rightarrow x_3 = \frac{-746}{141} \approx 5.29$$

$$-4.4x_2 + 1.7x_3 = -53.4 \Rightarrow x_2 = \frac{1423}{141} \approx 10.09$$

$$10x_1 + 2x_2 - x_3 = 27 \Rightarrow x_1 = \frac{43}{282} \approx 0.152$$

b) eq 1: $10x_1 + 2x_2 - x_3 = 27$
$$10\left(\frac{43}{282}\right) + 2\left(\frac{1423}{141}\right) - \left(\frac{-746}{141}\right) = 27 \checkmark$$

eq 2: $-3x_1 - 5x_2 + 2x_3 = -61.5$
$$-3\left(\frac{43}{282}\right) - 5\left(\frac{1423}{141}\right) + 2\left(-\frac{746}{141}\right) = -61.5 \checkmark$$

eq 3: $x_1 + x_2 + 6x_3 = -21.5$
$$\frac{43}{282} + \frac{1423}{141} + 6\left(-\frac{746}{141}\right) = -21.5 \checkmark$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 43/282 \\ 1423/141 \\ -746/141 \end{bmatrix} \approx \begin{bmatrix} 0.152 \\ 10.09 \\ -5.29 \end{bmatrix}$$

Problem 13:

13
(9.7)

$$2x_1 - 6x_2 - x_3 = -38$$
$$-3x_1 - x_2 + 7x_3 = -34$$
$$-8x_1 + x_2 - 2x_3 = -20$$

a) $$\begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -38 \\ -34 \\ -20 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -6 & -1 & -38 \\ -3 & -1 & 7 & -34 \\ -8 & 1 & -2 & -20 \end{array}\right] \xRightarrow{R_3 \to R_1} \left[\begin{array}{ccc|c} -8 & 1 & -2 & -20 \\ -3 & -1 & 7 & -34 \\ 2 & -6 & -1 & -38 \end{array}\right]$$

$$\xRightarrow{R_2 - \frac{3}{8}R_1 \to R_2} \left[\begin{array}{ccc|c} -8 & 1 & -2 & -20 \\ 0 & -11/8 & 31/4 & -53/2 \\ 2 & -6 & -1 & -38 \end{array}\right]$$

$$\xRightarrow{R_2 \to R_3} \left[\begin{array}{ccc|c} -8 & 1 & -2 & -20 \\ 2 & -6 & -1 & -38 \\ 0 & -11/8 & 31/4 & -53/2 \end{array}\right]$$

$$\xRightarrow{R_2 + \frac{1}{4}R_1 \to R_2} \left[\begin{array}{ccc|c} -8 & 1 & -2 & -20 \\ 0 & \frac{-23}{4} & \frac{-3}{2} & -43 \\ 0 & -\frac{11}{8} & 31/4 & -53/2 \end{array}\right]$$

$$\xRightarrow{R_3 - \frac{11}{46} R_2 \to R_3} \left[\begin{array}{ccc|c} -8 & 1 & -2 & -20 \\ 0 & \frac{-23}{4} & \frac{-3}{2} & -43 \\ 0 & 0 & \frac{373}{46} & \frac{-373}{23} \end{array}\right]$$

$$\frac{373}{46} x_3 = \frac{-373}{26} \quad \text{eq 1}$$

$$-\frac{23}{4} x_2 - \frac{3}{2} x_3 = -43 \quad \text{eq 2}$$

$-8x_1 + x_2 - 2x_3 = -20$ eq 3

$x_3 = -2$
$x_2 = 8$
$x_1 = 4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}$$

determinant $= -8 \cdot -\frac{23}{4} \cdot \frac{373}{46} = 373$

b) $2x_1 - 6x_2 - x_3 = -38$ eq 1

$2 \cdot 4 - 6 \cdot 8 - (-2) = -38$ ✓

$-3x_1 - x_2 + 7x_3 = -34$ eq 2

$-3 \cdot 4 - 8 + 7 \cdot (-2) = -34$ ✓

$-8x_1 + x_2 - 2x_3 = -20$ eq 3

$-8 \cdot 4 + 8 - 2(-2) = -20$ ✓

Bonus : Part 1 & Part 2.

Found in HW1.py.