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COE 311K: Engineering Computation

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Problem 1:

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ -1/3 & -4/11 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & 22/3 & -14/3 \\ 0 & 0 & 40/11 \end{bmatrix}$

$A = LU$

$\vec{b} = \begin{bmatrix} -10 \\ 44 \\ -26 \end{bmatrix}$ Let $LU\vec{x} = \vec{b}$ $(A\vec{x} = \vec{b})$ & $U\vec{x} = \vec{d}$ so $L\vec{d} = \vec{b}$

$L\vec{d} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ -1/3 & -4/11 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 44 \\ -26 \end{bmatrix} \quad \text{where } \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$d_1 = -10$

$2/3\, d_1 + d_2 = 44 \Rightarrow d_2 = 152/3$

$-1/3\, d_1 - 4/11\, d_2 + d_3 = -26 \Rightarrow d_3 = -120/11$

$U\vec{x} = \vec{d}$

$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & 22/3 & -14/3 \\ 0 & 0 & 40/11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 152/3 \\ -120/11 \end{bmatrix} \quad \text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$40/11\ x_3 = -120/11$; $x_3 = -3$

$22/3\ x_2 - 14/3\ x_3 = 152/3$; $x_2 = 5$

$3x_1 - 2x_2 + x_3 = -10$; $x_1 = 1$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}}$$

Problem 2:

2. Short method :

$$A = \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix} = LU \quad , \quad L = \begin{bmatrix} 1 & 0 & 0 \\ L_{31} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

for all elements below / on subdiagonal,

$$L_{i,n} = \frac{A_{i,n}}{\underbrace{A_{n,n}}_{\text{pivot}}} \quad i > n \; \& \; i < \text{max rows.}$$

First pivot / step.

$$A_{n,n} = 8 \quad \text{where } n = 1$$

$$L_{2,1} = \frac{A_{2,1}}{A_{1,1}} = \frac{3}{8}$$

$$L_{31} = \frac{A_{3,1}}{A_{1,1}} = \frac{2}{8} = \frac{1}{4}$$

U is row reduced Echelon form of A:

$$U := \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix} \xrightarrow[R_3 - \frac{1}{4}R_1 \to R_3]{R_2 - \frac{3}{8}R_1 \to R_2} \begin{bmatrix} 8 & 2 & 1 \\ 0 & 25/4 & 13/8 \\ 0 & 5/2 & 35/4 \end{bmatrix}$$

Now, use $\begin{bmatrix} 8 & 2 & 1 \\ 0 & 25/4 & 13/8 \\ 0 & 5/2 & 35/4 \end{bmatrix}$ to determine L_{32}

$L_{32} = \dfrac{A_{3,2}}{A_{2,2}}$ where A is

$= \dfrac{5/2}{25/4} = \dfrac{5 \cdot 4}{2 \cdot 25} = \dfrac{2}{5}$

So, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3/8 & 1 & 0 \\ 1/4 & 2/5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 0.25 & 0.4 & 1 \end{bmatrix}$

reduce A one more step:

$$\begin{bmatrix} 8 & 2 & 1 \\ 0 & 25/4 & 13/8 \\ 0 & 5/2 & 35/4 \end{bmatrix} \xrightarrow{R_3 - \frac{2}{5}R_2 \to R_3} \begin{bmatrix} 8 & 2 & 1 \\ 0 & 25/4 & 13/8 \\ 0 & 0 & 81/10 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix} = U$$

So, $L = \begin{bmatrix} 1 & 0 & 0 \\ .375 & 1 & 0 \\ .25 & .4 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix}$

$$LU = \begin{bmatrix} 1\times8+0+0 & 2+0+0 & 1+0+0 \\ .375\times8 & .375\times2+6.25 & .375+1.625 \\ .25\times8 & .25\times2+.4\times6.25 & .25\times1+.4\times1.625+8.1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix} = A$$

b. $\det(A) = \det(U)$
$\hookrightarrow \det(L) = 1$

$\det(A)$: (regular method, without LU results)

$$\det(A) \left| \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix} \right| = 8(63-6) - 2(27-4) + (9-14) = 405$$

$$\det(U) = \left| \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix} \right| = 8 \times 6.25 \times 8.1 = 405$$

determinant is same, as expected

a. Regular Method

$$L_1 \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 2.5 & 8.75 \end{bmatrix}$$

$\hookrightarrow$ first two step towards row reducing A

$L_1 \; A = U_1$

$\downarrow$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3/8 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 - \frac{3}{8} R_1$

$R_3 = R_3 - \frac{1}{4} R_1$

$L_2 \quad U_1 = U_2 = U$ ↳ now reduced A.

$$L_2 \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 2.5 & 8.75 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix}$$

↳ $R_3 = R_3 - \frac{2}{5} R_2$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -.4 & 1 \end{bmatrix}$$

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ .375 & 1 & 0 \\ .25 & 0 & 1 \end{bmatrix}$$

$$L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .4 & 1 \end{bmatrix}$$

$L = L_1$ & L_2 clumped together,
where values below the diagonal & ones are merged.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ .375 & 1 & 0 \\ .25 & .4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix}$$

verification that $LU = A$ is done in the initial LU factorization method

Problem 3:

3. $$\begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

$$A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

Shortcut method.

$$L_{i,n} = \frac{A_{i,n}}{A_{n,n}} \quad i > n \text{ (values below diagonal of } L)$$

$A_{n,n}$ → pivot element

First pivot. $A_{1,1} = 10$.

$$L_{2,1} = \frac{A_{2,1}}{A_{1,1}} = \frac{-3}{10}$$

$$L_{3,1} = \frac{A_{3,1}}{A_{1,1}} = \frac{1}{10}$$

Row Reduce A by 1 step.

$$\underset{A}{\begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}} \xrightarrow[R_3 - \frac{1}{10}R_1 \to R_3]{R_2 + \frac{3}{10}R_1 \to R_2} \underset{A'}{\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix}}$$

use for next L value.

2^{nd} pivot : $A'_{2,2} = -5.4$

$$L_{3,2} = \frac{A'_{3,2}}{A'_{2,2}} = \frac{0.8}{-5.4} = -0.\overline{148} = -4/27$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & -4/27 & 1 \end{bmatrix}$$

U is a completely row reduced A or A'

$$A' = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix} \xrightarrow{R_3 + \frac{4}{27}R_2 \to R_3} \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.35 \end{bmatrix}$$

$\to \frac{289}{54}$

$$U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.35 \end{bmatrix}$$

Using regular LU decomposition method:

$L_1 A = A_1$ where A_1 is first step of reduction on A

$$L_1 \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix}$$

$\hookrightarrow R_2 = R_2 + \frac{3}{10} R_1$

$\hookrightarrow R_3 = R_3 - \frac{1}{10} R_1$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3/10 & 1 & 0 \\ -1/10 & 0 & 1 \end{bmatrix}$$

$L_2 A_1 = A_2$ $\hookrightarrow$ now reduced A_1 (1 more reduction step).

$A_2 = U$

$$L_2 \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.35 \end{bmatrix}$$

$\hookrightarrow 289/54$

$\hookrightarrow R_3 = R_3 + \frac{4}{27} R_2$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4/27 & 1 \end{bmatrix}$$

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & 0 & 1 \end{bmatrix}$$

$$L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4/27 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.35 \end{bmatrix}$$

L is L_1^{-1} & L_2^{-1} "clubbed" together:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & -4/27 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 289/54 \end{bmatrix}$$

$$LU\vec{x} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix} \text{ where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problem 4:

4. Solve for $\vec{x}$ where $A\vec{x} = \vec{b}$.

a $\vec{A} = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & -4/27 & 1 \end{bmatrix} \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.35 \end{bmatrix}$

$\frac{289}{54}$

$A\vec{x} = LU\vec{x} = \vec{b}$

let $U\vec{x} = \vec{d}$; $L\vec{d} = \vec{b}$ $\qquad b = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$

1. Solve $L\vec{d} = \vec{b}$ for $\vec{d}$

2. Solve $U\vec{x} = d$ for $\vec{x}$.

1.

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & -4/27 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix} \quad \text{let } \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$d_1 = 27$

$-\frac{3}{10} d_1 + d_2 = -61.5 \Rightarrow d_2 = -267/5 \approx -53.4$

$\frac{1}{10} d_1 - \frac{4}{27} d_2 + d_3 = -21.5 \Rightarrow d_3 = -289/9 \approx -32.1$

$d = \begin{bmatrix} 27 \\ -53.4 \\ -32.1 \end{bmatrix}$

2. $U\vec{x} = \vec{d}$

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 289/54 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -53.4 \\ -32.1 \end{bmatrix}$$

$x_3 = -6;$

$-5.4 x_2 + 1.7 x_3 = -53.4 \Rightarrow x_2 = 8$

$10 x_1 + 2 x_2 - x_3 = 27 \Rightarrow x_1 = 0.5$

$$\boxed{\vec{x} = \begin{bmatrix} 0.5 \\ 8 \\ -6 \end{bmatrix}}$$

b. $LU\vec{x} = \vec{b}$; let $U\vec{x} = \vec{d}$

1. $L\vec{d} = \vec{b}$ $\quad\quad \vec{b} = \begin{bmatrix} 12 \\ 18 \\ -6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & -4/27 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\vec{d}} = \begin{bmatrix} 12 \\ 18 \\ -6 \end{bmatrix}$$

$d_1 = 12$

$-3/10\, d_1 + d_2 = 18 \Longrightarrow d_2 = 21.6$

$1/10\, d_1 - 4/27\, d_2 + d_3 = -6 \Longrightarrow d_3 = -4$

$\vec{d} = \begin{bmatrix} 12 \\ 21.6 \\ -4 \end{bmatrix}$

2. $U\vec{x} = \vec{d}$

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 289/54 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 12 \\ 21.6 \\ -4 \end{bmatrix}$$

$x_3 = \frac{-216}{289} \approx -.747$

$-5.4x_2 + 1.7x_3 = 21.6 \Longrightarrow x_2 = \frac{-72}{17} \approx -4.235$

$10x_1 + 2x_2 - x_3 = 12 \Longrightarrow x_1 = \frac{570}{289} \approx 1.97$

$$\boxed{\vec{x} \approx \begin{bmatrix} 1.9723 \\ -4.2353 \\ -.7474 \end{bmatrix}}$$

Problem 5:

5. $2x_1 - 6x_2 - x_3 = -38$

$-3x_1 - x_2 + 7x_3 = -34$

$-8x_1 + x_2 - 2x_3 = -40$

$$\begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} -38 \\ -34 \\ -40 \end{bmatrix}$$

let $A = \begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix}$,

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -38 \\ -34 \\ -40 \end{bmatrix}$

then $A\vec{x} = \vec{b}$.

To use LU factorization on A: $(PA = LU)$
(with partial pivoting)

$|-8| > |2|$ so use pivoting:

let $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ so $P_1 A = \begin{bmatrix} -8 & 1 & -2 \\ -3 & -1 & 7 \\ 2 & -6 & -1 \end{bmatrix}$

let $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3/8 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix}$ so $L_1 P_1 A = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -11/8 & 31/4 \\ 0 & -23/4 & -3/2 \end{bmatrix}$

note to get element l_{21} & l_{31} of matrix L_1:

$$l_{21} = \frac{-[P_1 A]_{2,1}}{[P_1 A]_{1,1} \longrightarrow \text{pivot element}}$$

$$l_{31} = \frac{-[P_1 A]_{3,1}}{[P_1 A]_{1,1} \longrightarrow \text{pivot element}}$$

$|-23/4| > |-11/8|$ so use pivoting again:

let $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ so $P_2 L_1 P_1 A = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -23/4 & -3/2 \\ 0 & -11/8 & 31/4 \end{bmatrix}$

let $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.239 & 1 \end{bmatrix}$ so $L_2 P_2 L_1 P_1 A = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.11 \end{bmatrix}$

$\hookrightarrow 11/46$ $\qquad \hookrightarrow \frac{373}{46}$

$U = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.11 \end{bmatrix}$ if $L_2 P_2 L_1 P_1 A = U$;

$L_2^{-1} L_2 P_2 L_1 P_1 A = L_2^{-1} U$; $P_2 L_1 P_1 A = L_2^{-1} U$

$P_2^{-1} P_2 L_1 P_1 A = P_2^{-1} L_2^{-1} U$; $L_1 P_1 A = P_2^{-1} L_2^{-1} U$

$L_1^{-1} L_1 P_1 A = L_1^{-1} P_2^{-1} L_2^{-1} U$; $P_1 A = L_1^{-1} P_2^{-1} L_2^{-1} U$

$P_1 A = L_1^{-1} P_2^{-1} L_2^{-1} U$

$\underbrace{P_2 P_1}_{} A = P_2 \underbrace{L_1^{-1} P_2^{-1} L_2^{-1}}_{} U$

upper triangular matrix, final permutation matrix, P

lower triangular matrix, final L

$$P_2 \cdot P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P.$$

$L_1^{-1} = L_1$ with non-zero values below the diagonal multiplied by -1

$L_2^{-1} = L_2$ with non-zero values below the diagonal multiplied by -1

$$\left.\begin{aligned} P_1^{-1} &= P_1^T \\ P_2^{-1} &= P_2^T \end{aligned}\right\} \text{transpose}$$

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3/8 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} ; \quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.239 & 1 \end{bmatrix}$$

→ 11/46

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} ; \quad P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2 L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3/8 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 0 & 1 \\ 3/8 & 1 & 0 \end{bmatrix}$$

$$(P_2 L_1^{-1}) P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 0 & 1 \\ 3/8 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 3/8 & 0 & 1 \end{bmatrix}$$

$$(P_2 L_1^{-1} P_2^{-1}) L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 3/8 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11/46 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 3/8 & 11/46 & 1 \end{bmatrix} = L$$

So, $PA = LU$ is :

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 3/8 & 11/46 & 1 \end{bmatrix} \begin{bmatrix} -8 & 1 & -2 \\ 0 & -23/4 & -3/2 \\ 0 & 0 & 373/46 \end{bmatrix}$$

if $A\vec{x}=\vec{b}$ & $PA=LU$,

$PA\vec{x}=P\vec{b}$. then $LU\vec{x}=P\vec{b}$

$$P\vec{b}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}-38\\-34\\-40\end{bmatrix}=\begin{bmatrix}-40\\-38\\-34\end{bmatrix}$$

let $U\vec{x}=\vec{d}$ so $L\vec{d}=P\vec{b}$; solve for $\vec{d}$

$$\begin{bmatrix}1&0&0\\-1/4&1&0\\3/8&11/46&1\end{bmatrix}\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}=\begin{bmatrix}-40\\-38\\-34\end{bmatrix} \text{ where } \vec{d}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$$

use forward substitution:

$d_1=-40$,

$-1/4\, d_1+d_2=-38 \Rightarrow d_2=-48$

$3/8\, d_1+11/46\, d_2+d_3=-34 \Rightarrow d_3=-\frac{173}{23}$

$$\vec{d}=\begin{bmatrix}-40\\-48\\-173/23\end{bmatrix}$$

Now, solve $U\vec{x}=\vec{d}$ for $\vec{x}$:

$$\begin{bmatrix}-8&1&-2\\0&-23/4&-3/2\\0&0&373/46\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-40\\-48\\-173/23\end{bmatrix} \text{ where } \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

use back substitution:

$x_3=-346/373 \approx -0.9276$

$-23/4\, x_2-3/2\, x_3=-48 \Rightarrow x_2=3204/373$
≈ 8.5898

$-8x_1 + x_2 - 2x_3 = -40 \Rightarrow x_1 = 2352/373$
$\approx 6.3056.$

Thus, $\boxed{\vec{x} \approx \begin{bmatrix} 6.3056 \\ 8.5898 \\ -0.9276 \end{bmatrix}}$

Problem 6:

Found in HW2.py.

Problem 7:

Found in HW2.py

Problem 8:

8. $10x_1 + 2x_2 - x_3 = 27$
$-3x_1 - 6x_2 + 2x_3 = -61.5$
$x_1 + x_2 + 5x_3 = -21.5$

let $A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$

so $A\vec{x} = \vec{b}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$AA^{-1} = I$:

$[A \mid I] = \left[\begin{array}{ccc|ccc} 10 & 2 & -1 & 1 & 0 & 0 \\ -3 & -6 & 2 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$ $R_2 + \frac{3}{10}R_1 \to R_2$, $R_3 - \frac{1}{10}R_1 \to R_3$

$\to \left[\begin{array}{ccc|ccc} 10 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5.4 & 1.7 & .3 & 1 & 0 \\ 0 & .8 & 5.1 & -.1 & 0 & 1 \end{array}\right]$ $R_3 + \frac{4}{27}R_2 \to R_3$

$\to \left[\begin{array}{ccc|ccc} 10 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5.4 & 1.7 & .3 & 1 & 0 \\ 0 & 0 & 289/54 & -1/18 & 4/27 & 1 \end{array}\right]$ $R_3 \times \frac{54}{289} \to R_3$

$\to \left[\begin{array}{ccc|ccc} 10 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5.4 & 1.7 & .3 & 1 & 0 \\ 0 & 0 & 1 & -3/289 & 8/289 & 54/289 \end{array}\right]$ $R_2 - 1.7R_3 \to R_2$, $R_3 + R_1 \to R_1$

$\to \left[\begin{array}{ccc|ccc} 10 & 2 & 0 & 286/289 & 8/289 & 54/289 \\ 0 & -5.4 & 0 & 27/85 & 81/85 & -27/85 \\ 0 & 0 & 1 & -3/289 & 8/289 & 54/289 \end{array}\right]$ $R_2/-5.4 \to R_2$, $R_1/2 \to R_1$

$$\rightarrow \left[\begin{array}{ccc|ccc} 5 & 1 & 0 & 143/289 & 4/289 & 27/289 \\ 0 & 1 & 0 & -1/17 & -3/17 & 1/17 \\ 0 & 0 & 1 & -3/289 & 8/289 & 54/289 \end{array}\right]$$

$$R_1 - R_2 \to R_1 \quad \longrightarrow \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 160/289 & 56/289 & 10/289 \\ 0 & 1 & 0 & -1/17 & -3/17 & 1/17 \\ 0 & 0 & 1 & -3/289 & 8/289 & 54/289 \end{array}\right]$$

$$R_1/5 \to R_1 \quad \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 32/289 & 11/289 & 2/289 \\ 0 & 1 & 0 & -1/17 & -3/17 & 1/17 \\ 0 & 0 & 1 & -3/289 & 8/289 & 54/289 \end{array}\right]$$

Now, we have row reduced $[A|I]$ into $[I|A^{-1}]$.

$$\text{So, } A^{-1} = \begin{bmatrix} 32/289 & 11/289 & 2/289 \\ -1/17 & -3/17 & 1/17 \\ -3/289 & 8/289 & 54/289 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.11073 & 0.03806 & 0.00692 \\ -0.05882 & -0.17647 & 0.05882 \\ -0.01038 & 0.02768 & 0.18685 \end{bmatrix}$$

Recheck that $A^{-1}A = I$:

$$\begin{bmatrix} 32/289 & 11/289 & 2/289 \\ -1/17 & -3/17 & 1/17 \\ -3/289 & 8/289 & 54/289 \end{bmatrix} \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} (320-33+2)/289 & (64-66+2)/289 & (-32+22+10)/289 \\ (-10+9+1)/17 & (-2+18+1)/17 & (1-6+5)/17 \\ (-30-24+54)/289 & (-6-48+54)/289 & (3+16+270)/289 \end{bmatrix}$$

$$= \begin{bmatrix} 289/289 & 0/289 & 0/289 \\ 0/17 & 17/17 & 0/17 \\ 0/289 & 0/289 & 289/289 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

To solve $A\vec{x} = \vec{b}$; apply A^{-1} to both sides of the equation:

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b} \quad \text{or} \quad \vec{x} = A^{-1}\vec{b}$$

$$A^{-1}\vec{b} = \begin{bmatrix} 32/289 & 11/289 & 2/289 \\ -1/17 & -3/17 & 1/17 \\ -3/289 & 8/289 & 54/289 \end{bmatrix} \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 8 \\ -6 \end{bmatrix} \quad \text{Thus, } \boxed{\vec{x} = \begin{bmatrix} 1/2 \\ 8 \\ -6 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 8 \\ -6 \end{bmatrix}}$$

Problem 9:

9. $-8x_1 + x_2 - 2x_3 = -20$

$2x_1 - 6x_2 - x_3 = -38$

$-3x_1 - x_2 + 7x_3 = -34$

Let $A = \begin{bmatrix} -8 & 1 & -2 \\ 2 & -6 & -1 \\ -3 & -1 & 7 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -28 \\ -38 \\ -34 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

such that $A\vec{x} = \vec{b}$.

To use LU decomposition without partial pivoting on A:

$A = LU$; where L is a lower triangular and U is an upper triangular matrix

for each element below the diagonal,

$$l_{ij} = \frac{A_{ij}}{A_{jj}}$$

$A_{jj} \longrightarrow$ pivot element

so, $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -3/8 & 0 & 1 \end{bmatrix}$

such that $L_1 A = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & -1.375 & 7.75 \end{bmatrix}$.

then let $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -11/46 & 1 \end{bmatrix}$

such that $L_2 L_1 A = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 373/46 \end{bmatrix} = U$

$\underbrace{L_2^{-1} L_2}_{I} L_1 A = L_2^{-1} U$ so $L_1 A = L_2^{-1} U$

$L_1^{-1} \underbrace{L_1}_{I} A = L_1^{-1} L_2^{-1} U$ so $A = \underbrace{L_1^{-1} L_2^{-1}}_{L} U$

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 3/8 & 0 & 1 \end{bmatrix} \quad \& \quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11/46 & 1 \end{bmatrix}$$

(all nonzero elements below the diagonal are negated).

$$L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 3/8 & 11/46 & 1 \end{bmatrix}$$

(all elements below diagonal are "clubbed" together).

$= L.$

So, $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.239 & 1 \end{bmatrix}$

and $U = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.109 \end{bmatrix}$

To find A^{-1}:

1. let $\underbrace{LU}_{A} \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ such that $\vec{x}_1$ is the first column of A^{-1} & $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the first column of the I matrix.

2. $LU\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

x_2, x_3 are 2^{nd} & 3^{rd} columns of A^{-1}

3. $LU\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

1. $LU\vec{x}_1 = \vec{b}_1$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

then let $U\vec{x}_1 = \vec{d}_1$ so $L\vec{d}_1 = \vec{b}_1$

$L\vec{d}_1 = \vec{b}_1$ where $\vec{d}_1 = \begin{bmatrix} d_{1a} \\ d_{2a} \\ d_{3a} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -.25 & 1 & 0 \\ 0.375 & 11/46 & 1 \end{bmatrix} \begin{bmatrix} d_{1a} \\ d_{2a} \\ d_{3a} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$d_{1a} = 1$

$-.25 d_{1a} + d_{2a} = 0$; $d_{2a} = 0.25$

$.375 d_{1a} + \frac{11}{46} d_{2a} + d_{3a} = 0$; $d_{3a} = -10/23$

$$\vec{d}_1 = \begin{bmatrix} 1 \\ 0.25 \\ -10/23 \end{bmatrix}$$

then, $U\vec{x}_1 = \vec{d}_1$: where $\vec{x}_1 = \begin{bmatrix} x_{1a} \\ x_{2a} \\ x_{3a} \end{bmatrix}$

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 373/46 \end{bmatrix} \begin{bmatrix} x_{1a} \\ x_{2a} \\ x_{3a} \end{bmatrix} = \begin{bmatrix} 1 \\ .25 \\ -10/23 \end{bmatrix}$$

$\frac{373}{46} x_{3a} = \frac{-10}{23}$; $x_{3a} = -20/373$

$-5.75 x_{2a} - 1.5 x_{3a} = .25$; $x_{2a} = -11/373$

$-8 x_{1a} + x_{2a} - 2x_{3a} = 1$; $x_{1a} = -43/373$.

2. $L U \vec{x_2} = \vec{b_2}$ where $\vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

then let $U \vec{x_2} = \vec{d_2}$ so $L \vec{d_2} = \vec{b_2}$

$L \vec{d_2} = \vec{b_2}$ where $\vec{d_2} = \begin{bmatrix} d_{1b} \\ d_{2b} \\ d_{3b} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -.25 & 1 & 0 \\ 0.375 & 11/46 & 1 \end{bmatrix} \begin{bmatrix} d_{1b} \\ d_{2b} \\ d_{3b} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$d_{1b} = 0$

$-.25\, d_{1b} + d_{2b} = 1; \quad d_{2b} = 1$

$.375\, d_{1b} + 11/46\, d_{2b} + d_{3b} = 0; \quad d_{3b} = -11/46$

$\vec{d_2} = \begin{bmatrix} 0 \\ 1 \\ -11/46 \end{bmatrix}$

then $U \vec{x_2} = \vec{d_2}$: where $\vec{x_2} = \begin{bmatrix} x_{1b} \\ x_{2b} \\ x_{3b} \end{bmatrix}$

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 373/46 \end{bmatrix} \begin{bmatrix} x_{1b} \\ x_{2b} \\ x_{3b} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -11/46 \end{bmatrix}$$

$373/46\, x_{3b} = -11/46; \quad x_{3b} = -11/373$

$-5.75\, x_{2b} - 1.5 x_{3b} = 1; \quad x_{2b} = -62/373$

$-8x_{1b} + x_{2b} - 2x_{3b} = 0; \quad x_{1b} = -5/373$

3. $LU\vec{x_3} = \vec{b_3}$ where $\vec{b_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Then let $U\vec{x_3} = \vec{d_3}$ so $L\vec{d_3} = \vec{b_3}$

$L\vec{d_3} = \vec{b_3}$ where $\vec{d_3} = \begin{bmatrix} d_{1c} \\ d_{2c} \\ d_{3c} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -.25 & 1 & 0 \\ .375 & 11/46 & 1 \end{bmatrix} \begin{bmatrix} d_{1c} \\ d_{2c} \\ d_{3c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$d_{1c} = 0$

$-.25\, d_{1c} + d_{2c} = 0; \quad d_{2c} = 0$

$-.375\, d_{1c} + 11/46\, d_{2c} + d_{3c} = 1; \quad d_{3c} = 1$

$$\vec{d_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then $U\vec{x_3} = \vec{d_3}$ where $\vec{x_3} = \begin{bmatrix} x_{1c} \\ x_{2c} \\ x_{3c} \end{bmatrix}$

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 373/46 \end{bmatrix} \begin{bmatrix} x_{1c} \\ x_{2c} \\ x_{3c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$373/46\, x_{3c} = 1; \quad x_{3c} = 46/373$

$-5.75\, x_{2c} - 1.5x_{3c} = 0; \quad x_{2c} = -12/373$

$-8x_{1c} + x_{2c} - 2x_{3c} = 0; \quad x_{1c} = -13/373$

$$A^{-1} = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \vec{x_3} \end{bmatrix} = \begin{bmatrix} x_{1a} & x_{1b} & x_{1c} \\ x_{2a} & x_{2b} & x_{2c} \\ x_{3a} & x_{3b} & x_{3c} \end{bmatrix}$$

$$= \begin{bmatrix} -43/373 & -5/373 & -13/373 \\ -11/373 & -62/373 & -12/373 \\ -20/373 & -11/373 & 46/373 \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.11528 & -0.01340 & -0.03485 \\ -0.02949 & -0.16622 & -0.03217 \\ -0.05362 & -0.02949 & 0.12332 \end{bmatrix}$$

To solve for $\vec{x}$:

$$A\vec{x} = \vec{b} \ ; \quad A^{-1}A\vec{x} = A^{-1}\vec{b} \ ; \quad \vec{x} = A^{-1}\vec{b}$$

$$A^{-1}\vec{b} = \begin{bmatrix} -43/373 & -5/373 & -13/373 \\ -11/373 & -62/373 & -12/373 \\ -20/373 & -11/373 & 46/373 \end{bmatrix} \begin{bmatrix} -20 \\ -38 \\ -34 \end{bmatrix}$$

$$= \begin{bmatrix} 1492/373 \\ 2984/373 \\ -746/373 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}$$

So, $\boxed{\vec{x} = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}}$

Problem 10:

10.

a. $A = \begin{bmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{bmatrix}$ size $(A) = m \times n$
$= 3 \times 3$.

$$\|A\|_f = \sqrt{\sum_{\substack{i=1 \\ \downarrow \\ \text{rows}}}^{m} \sum_{\substack{j=1 \\ \downarrow \\ \text{columns}}}^{n} A_{ij}^2}$$

$$= \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij}^2}$$

$$= \sqrt{\sum_{j=1}^{3} A_{1j}^2 + \sum_{j=1}^{3} A_{2j}^2 + \sum_{j=1}^{3} A_{3j}^2}$$

$$= \sqrt{\left(8^2 + 2^2 + (-10)^2\right) + \left((-9)^2 + 1^2 + 3^2\right) + \left(15^2 + (-1)^2 + 6^2\right)}$$

$$= \boxed{\sqrt{521} \approx 22.825}$$

$$\|A\|_1 = \max_{\substack{1 \le j \le n \\ \text{columns}}} \sum_{i=1}^{m} |A_{ij}|$$

$$= \max_{1 \le j \le n} \Big((|8| + |-9| + |15|), (|2|, |1|, |-1|), (|-10|, |-3|, |6|) \Big) =$$

$= \max(32, 4, 19) = \boxed{32}$

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |A_{ij}|$$

$$= \max\big((|8| + |2| + |-10|), (|-9| + |1| + |3|), (|15| + |-1| + |6|) \big)$$

$$= \max(20, 13, 22) = \boxed{22}$$

b. $A = \begin{bmatrix} 2 & 7 \\ 3 & 4 \\ 6 & 5 \end{bmatrix}$ $\quad$ size $(A) = m \times n = 3 \times 2$

$$\|A\|_f = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (A_{ij})^2}$$

$\underset{\text{rows}}{\downarrow}$ $\underset{\text{cols}}{\downarrow}$

$$= \sqrt{\sum_{j=1}^{n} (A_{1j})^2 + \sum_{j=1}^{n} (A_{2j})^2 + \sum_{j=1}^{n} (A_{3j})^2}$$

$$= \sqrt{(2^2 + 7^2) + (3^2 + 4^2) + (6^2 + 5^2)}$$

$$= \boxed{\sqrt{139} \approx 11.7898}$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |A_{ij}|$$

$$= \max\left((|2|+|3|+|6|), (|7|+|4|+|5|) \right)$$

$$= \max(11, 16) = \boxed{16}$$

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |A_{ij}|$$

$$= \max\left((|2|+|7|), (|3|+|4|), (|6|+|5|) \right)$$

$$= \max(9, 7, 11)$$

$$= \boxed{11}$$

c. $A = \begin{bmatrix} 5 & 4 & 3 \\ 11 & 10 & 8 \end{bmatrix}$ $size(A) = m \times n = 2 \times 3$

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

$$= \sqrt{\sum_{j=1}^{n} A_{1j}^2 + \sum_{j=1}^{n} A_{2j}^2}$$

$$= \sqrt{(25+16+9) + (121+100+64)}$$

$$= \boxed{\sqrt{335} = 18.3030}$$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |A_{ij}|$$

$$= \max\left(\left(|5| + |11|\right), \left(|4| + |10|\right), \left(|3|, |8|\right)\right)$$

$$= \max(16, 14, 11) = \boxed{16}$$

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |A_{ij}|$$

$$= \max\left(\left(|5| + |4| + |3|\right), \left(|11| + |10| + |8|\right)\right)$$

$$= \max(12, 29) = \boxed{29}$$

Problem 11:

Found in HW2.py

Problem 12:

Found in HW2.py

Problem 13:

13. $$A = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax_0 = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 17 \\ 22 \end{bmatrix}$$

$$\max(Ax_0) = 22 = \lambda_{1,a}$$

$$\frac{Ax_0}{\lambda_{1,a}} = \begin{bmatrix} 10/11 \\ 17/22 \\ 1 \end{bmatrix} = x_{1,a}$$

will calculate error with next step.

$$Ax_{1,a} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix} \begin{bmatrix} 10/11 \\ 17/22 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 169/11 \\ 439/22 \end{bmatrix}$$

$$\max(Ax_{1,a}) = 439/22 = \lambda_{1,b}$$

$$\frac{Ax_{1,a}}{\lambda_{1,b}} = \begin{bmatrix} 396/439 \\ 338/439 \\ 1 \end{bmatrix} = x_{1,b}$$

$$error_1 = \left| \frac{\lambda_{1,b} - \lambda_{1,a}}{\lambda_{1,b}} \right| = \frac{439/22 - 22}{439/22} = \frac{45}{439}$$

$$Ax_{1,b} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix} \begin{bmatrix} 396/439 \\ 338/439 \\ 1 \end{bmatrix} = \begin{bmatrix} 7886/439 \\ 6715/439 \\ 8723/439 \end{bmatrix}$$

$$\max(Ax_{1,b}) = 8723/439 = \lambda_{1,c}$$

$$x_{1,c} = \frac{Ax_{1,b}}{\lambda_{1,c}} = \begin{bmatrix} 7886/8723 \\ 6715/8723 \\ 1 \end{bmatrix}$$

$$error_2 = \left| \frac{\lambda_{1,c} - \lambda_{1,b}}{\lambda_{1,c}} \right| = \left| \frac{8723/439 - 439/22}{8723/439} \right|$$

$$= \frac{815}{191906} \approx 0.0042468$$

highest eigenvalue $\approx \lambda_{1,c} = \boxed{8723/439 = 19.870}$

associated highest eigenvector $\approx x_{1,c} = \begin{bmatrix} 7886/8723 \\ 6715/8723 \\ 1 \end{bmatrix}$

$$\approx \boxed{\begin{bmatrix} 0.9040 \\ 0.7698 \\ 1 \end{bmatrix}}$$

Problem 14:

14.

$$A = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix} \quad \text{find } A^{-1} \text{ first}$$

$$[A|I] \Rightarrow [I|A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 2 & 8 & 10 & 1 & 0 & 0 \\ 8 & 4 & 5 & 0 & 1 & 0 \\ 10 & 5 & 7 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 2 & 8 & 10 & 1 & 0 & 0 \\ 0 & -28 & -35 & -4 & 1 & 0 \\ 0 & -35 & -43 & -5 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & .5 & 0 & 0 \\ 0 & 4 & 5 & 4/7 & -1/7 & 0 \\ 0 & -35 & -43 & -5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & .5 & 0 & 0 \\ 0 & 4 & 5 & 4/7 & -1/7 & 0 \\ 0 & 0 & .75 & 0 & -1.25 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/14 & 1/7 & 0 \\ 0 & 4 & 0 & 4/7 & 172/21 & -20/3 \\ 0 & 0 & 1 & 0 & -5/3 & 4/3 \end{array}\right] \rightarrow \left[\begin{array}{c|ccc} & -1/14 & 1/7 & 0 \\ I & 1/7 & 43/21 & -5/3 \\ & 0 & -5/3 & 4/3 \end{array}\right]$$

$$A^{-1} \approx \begin{bmatrix} -.0714 & .1428 & 0 \\ .1428 & 2.0476 & -1.667 \\ 0 & -1.667 & 1.33 \end{bmatrix}$$

Continued on next page.

let $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$A^{-1}x_0 = \begin{bmatrix} -1/14 & 1/7 & 0 \\ 1/7 & 43/21 & -5/3 \\ 0 & -5/3 & 4/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/14 \\ 11/21 \\ -1/3 \end{bmatrix}$$

normalize $A^{-1}x_0$ such that max value = 1:

let $\lambda_{1,a} = \frac{11}{21}$

$$\frac{1}{\lambda_{1,a}} A^{-1}x_0 = \begin{bmatrix} 3/22 \\ 1 \\ -7/11 \end{bmatrix} = x_{1,a}$$

$$A^{-1}x_{1,a} = \begin{bmatrix} -1/14 & 1/7 & 0 \\ 1/7 & 43/21 & -5/3 \\ 0 & -5/3 & 4/3 \end{bmatrix} \begin{bmatrix} 3/22 \\ 1 \\ -7/11 \end{bmatrix}$$

$$= \begin{bmatrix} 41/308 \\ 1445/462 \\ -83/33 \end{bmatrix}$$

normalize $A^{-1}x_{1,a}$:

let $\lambda_{1,b} = \frac{1445}{462}$

$$\boxed{\text{error}_1 = \left| \frac{\lambda_{1,b} - \lambda_{1,a}}{\lambda_{1,b}} \right| = 0.833}$$

$$\frac{1}{\lambda_{1,b}} A^{-1}x_{1,a} = \begin{bmatrix} 123/2890 \\ 1 \\ -1162/1445 \end{bmatrix} = x_{1,b}$$

$$A^{-1} x_{1,b} = \begin{bmatrix} -1/14 & 1/7 & 0 \\ 1/7 & 43/21 & -5/3 \\ 0 & -5/3 & 4/3 \end{bmatrix} \begin{bmatrix} 123/2890 \\ 1 \\ -1162/1445 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1398 \\ 3.3939 \\ -2.7388 \end{bmatrix}$$

normalize $A^{-1} x_{1,b}$:

let $\lambda_{1,c} = 3.3939$

$$\frac{1}{\lambda_{1,c}} A^{-1} x_{1,b} = \begin{bmatrix} 0.04119 \\ 1 \\ -.8069 \end{bmatrix} = x_{1,c}$$

$$\text{error}_2 = \left| \frac{\lambda_{1,c} - \lambda_{1,b}}{\lambda_{1,c}} \right| \simeq 0.078$$

$\lambda_{1,c}$ is approximated largest eigenvalue for A^{-1}. $\frac{1}{\lambda_{1,c}}$ is approximated smallest eigenvalue for A. $\frac{1}{\lambda_{1,c}} \simeq \boxed{0.2946}$

Problem 15:

Found in HW2.py

Bonus Part 1:

Found in HW2.py

Bonus Part 2:

Found in HW2.py