

# hw01

Exported 1/23/2026, 12:08:18 AM

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# 1.1

ASE379L HW1 23 January 2026

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$$\Omega = \{\emptyset, B, R, G\}$$

$$\begin{aligned} F &= \{\emptyset, \{\emptyset\}, \{\emptyset, B\}, \{\emptyset, R\}, \{\emptyset, G\}, \\ &\quad \{B\}, \{R\}, \{G\}, \{B, R\}, \{B, G\}, \{R, G\}, \{B, R, G\}\} \end{aligned}$$

$$P(\emptyset) = 0$$

$$P(B) = P(R) = P(G) = 1/3$$

$$P(\{\emptyset, B\}) = P(\{\emptyset, R\}) = P(\{\emptyset, G\}) = 2/3$$

$$P(\Omega) = P(\{\emptyset, B, R, G\}) = 1$$

# 1.2

1.2  $0+e=0$  where 0, e represent odd, even for the 1<sup>st</sup>/2<sup>nd</sup> dice roll  
 $e+0=0$  1<sup>st</sup> 2<sup>nd</sup> roll 1, 2, 3, 4

$$\Omega = \{ \underbrace{12, 14, 32, 34}_{0+e}, \underbrace{21, 23, 41, 43}_{0+0} \}$$

4 outcomes 4 outcomes  
 $a_1, a_2, a_3, a_4$  as  $a_5, a_6, a_7, a_8$  let these be the 8 specific outcomes.

$$\begin{aligned} F = & \{ \emptyset, \underbrace{\{a_1\}}, \underbrace{\{a_2\}}, \dots, \underbrace{\{a_8\}}, \underbrace{\{a_1, a_2\}}, \underbrace{\{a_1, a_3\}}, \dots, \underbrace{\{a_1, a_8\}}, \\ & \underbrace{\{a_2, a_3\}}, \underbrace{\{a_2, a_4\}}, \dots, \underbrace{\{a_2, a_8\}}, \underbrace{\{a_3, a_4\}}, \underbrace{\{a_3, a_5\}}, \dots, \underbrace{\{a_3, a_8\}} \\ & \underbrace{\{a_4, a_5\}}, \dots, \underbrace{\{a_4, a_8\}}, \underbrace{\{a_5, a_6\}}, \dots, \underbrace{\{a_5, a_8\}}, \underbrace{\{a_6, a_7\}}, \dots, \underbrace{\{a_6, a_8\}} \\ & \underbrace{\{a_7, a_8\}}, \text{ all } 3 a_i, a_j, a_k \text{ where } i, j, k \in \{1, 8\} \text{ &} \\ & i \neq j \neq k \} \end{aligned}$$

$8C_3$

$8C_4$  sets of 4 terms

$8C_5$  sets of 5

$8C_6$  sets of 6

$8C_7$  sets of 7

$$\begin{array}{rcl} & & \Omega \\ & \downarrow & \} \\ +8C_8 & \text{terms} & = 1 \end{array}$$

Q,

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$$Q = \{ \emptyset \} \cup \{ a_i : i=1 \dots 8 \} \cup \{ a_i, a_j : i < j, j=1 \dots 8 \}$$

$$\cup \{ a_i, a_j, a_k : i < j < k, k=1 \dots 8 \} \cup \dots$$

↓Φ

$$\text{Total terms: } 1 + 8c_1 + 8c_2 + 8c_3 \dots + 8c_8 = 1 + \sum_{i=1}^8 8c_i \\ = \sum_{i=0}^8 8c_i = 2^8 = 256 \text{ terms}$$

$$P(\emptyset) = 0$$

commonly written  $2^n$ 

$$P(\Omega) = 1$$

# of events

$$P(\{a_i : i=1 \dots 8\}) = 1/8$$

$$\binom{8}{1} = 8$$

$$P(\{a_i, a_j : i < j, j=1 \dots 8\}) = 2/8 = 1/4$$

$$\binom{8}{2} = 28$$

$$P(\{a_i, a_j, a_k : i < j < k, k=1 \dots 8\}) = 3/8$$

$$\binom{8}{3} = 56$$

$$P(\underbrace{\text{4 terms}}_{\text{4 terms}}) = 4/8 = 1/2$$

$$\binom{8}{4} = 70$$

$$P(\underbrace{\text{5 terms}}_{\text{5 terms}})$$

$$\binom{8}{5} = 56$$

$$6/8 = 3/4$$

$$\binom{8}{6} = 28$$

$$P(\underbrace{\{a_i, a_j, a_k, a_l : i < j < k < l, l=1 \dots 8\}}_{7 \text{ terms}}) = 7/8$$

$$\binom{8}{7} = 8$$

## 2.1

$$\mu_x = \frac{1}{\sigma_x^2} = 4 \quad \text{so } \mu_x = 4$$

$$X \sim N(0, 4) \quad Y = \sin(X)$$

$$2.1 \quad M = E[Y] = \int_{-\infty}^{\infty} \sin(x) f(x) dx = \int_{-\infty}^{\infty} \sin(x) \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx$$

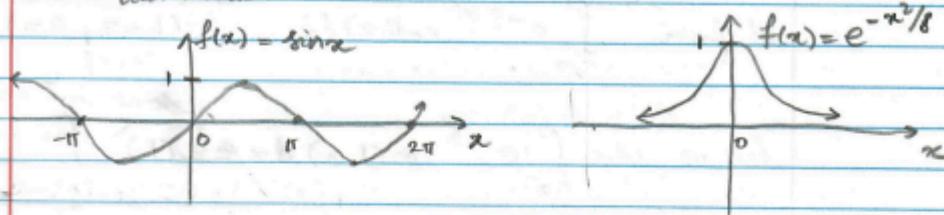
pdf  $f(x)$   
of  $X$

$$= \int_{-\infty}^{\infty} (\sin(x)) \frac{e^{-x^2/8}}{2\sqrt{2\pi}} dx$$

$$\mu = \int_{-\infty}^{\infty} (\sin(x)) \frac{e^{-x^2/8}}{2\sqrt{2\pi}} dx = 0$$

even function

odd function



We know the product of an odd & even function results in an odd function.

Also, any odd function integrated from  $-\infty$  to  $\infty$  (symmetric) is 0.

$$\text{So, } \mu = 0 \cdot \frac{1}{2\sqrt{2\pi}} = 0.$$

$$\text{Var}[Y] = E[Y^2] - E[Y]^2 = E[Y^2] - \mu^2 = E[Y^2]$$

$$E[Y^2] = \int_{-\infty}^{\infty} (\sin(x))^2 \frac{1}{2\sqrt{2\pi}} e^{-x^2/8} dx$$

even even

$$\sin^2 x = 1 - \cos 2x \text{ so,}$$

$$E[Y^2] = \left[ \int_{-\infty}^{\infty} e^{-x^2/8} dx - \int_{-\infty}^{\infty} (\cos 2x) e^{-x^2/8} dx \right] \frac{1}{4\sqrt{2\pi}}$$

$$= \left[ \underbrace{\left( \text{integral of gaussian} + 2\sqrt{2\pi} \right)}_{\text{of mean 0, variance 2}} - \int_{-\infty}^{\infty} (\cos 2x) e^{-x^2/8} dx \right] \frac{1}{4\sqrt{2\pi}}$$

$$\mathbb{E}[\gamma^2] = \frac{1}{2} - \int_{-\infty}^{\infty} (\cos 2x) e^{-x^2/8} dx + \frac{1}{4\sqrt{2\pi}} \quad 4/9$$

of form  $\int_{-\infty}^{\infty} e^{-ax^2} \cos(bx) dx \quad (b=2, a=1/8)$

let us solve  $\int_{-\infty}^{\infty} e^{-ax^2} \cos(bx) dx = I(b)$  some integral function

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a^2}} e^{-(x^2)/2a^2} dx \cdot \sqrt{\frac{\pi}{a}}$$

$$\text{where } a' = \frac{1}{\sqrt{2a}}$$

$$= \sqrt{\pi/a}$$

$$I'(b) = \int_{-\infty}^{\infty} -xe^{-ax^2} \sin(bx) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2a} \frac{\partial}{\partial x} (e^{-ax^2}) \sin bx dx$$

$$\text{let } u = \sin bx \quad du = b \cos bx dx \\ dv = \frac{1}{2a} \frac{\partial}{\partial x} (e^{-ax^2}) dx \quad v = \frac{1}{2a} e^{-ax^2}$$

$$I'(b) = \frac{1}{2a} \sin bx e^{-ax^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2a} e^{-ax^2} b \cos bx dx$$

$$= 0 - \frac{b}{2a} I(b)$$

$$\gamma'(b) = -\frac{b}{2a} I(b)$$

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$$\frac{dI}{db} = \frac{-b}{2a} I ; \quad \frac{dI}{I} = \frac{-b}{2a} db$$

$$\ln I = \frac{-b^2}{4a} + c \quad \downarrow \text{const} \quad I(0) = \sqrt{\pi}/a$$

$$I = C \exp\left(\frac{-b^2}{4a}\right) \quad I(0) = C = \sqrt{\pi}/a$$

$$I(b) = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right)$$

$$\text{So, } \mathbb{E}[Y^2] = \frac{1}{2} - \frac{1}{4\sqrt{2\pi}} \left[ \sqrt{8\pi} \exp\left(-\frac{4^2}{4\cdot 1/2}\right) \right]$$

$$= \frac{1}{2} - \frac{1}{4\sqrt{2\pi}} \left[ \sqrt{8\pi} \exp(-8) \right]$$

$$= \frac{1}{2} - \frac{1}{2} \exp(-8) = \frac{1-e^{-8}}{2}$$

$$\text{So, } \boxed{\mu=0 \quad \sigma^2 = \frac{1-e^{-8}}{2} \approx 0.499832}$$

## 2.2

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

# =====
# 2.1: Solve using computer integration/simulation
# =====

# True analytical values (from theory)
mu_true = 0.0
sigma2_true = (1 - np.exp(-8)) / 2
print(f"Analytical values:")
print(f"  μ = E[Y] = {mu_true}")
print(f"  σ² = Var[Y] = {sigma2_true:.8f}")

# =====
# Method 1: Direct Monte Carlo simulation
# =====

np.random.seed(42)

# Use a large number of samples for accurate estimation
n_mc = int(1e7) # 10 million samples
print(f"\nMonte Carlo simulation with n = {n_mc:,} samples...")

# Generate X ~ N(0, 4) and Y = sin(X)
X = np.random.normal(loc=0, scale=2, size=n_mc) # scale=2 for
variance 4
Y = np.sin(X)

# Compute empirical mean and variance
mu_mc = np.mean(Y)
sigma2_mc = np.var(Y, ddof=1)

print(f"Monte Carlo results:")
print(f"  E[Y] ≈ {mu_mc:.8f} (error = {abs(mu_mc - mu_true):.8f})")
print(f"  Var[Y] ≈ {sigma2_mc:.8f} (error = {abs(sigma2_mc -
sigma2_true):.8f})")

# =====
print(f"\nVisualizing convergence of Monte Carlo estimates...")

# Compute cumulative means and variances
cumulative_means = np.cumsum(Y) / np.arange(1, n_mc + 1)
cumulative_sumsq = np.cumsum(Y**2)
cumulative_vars = (cumulative_sumsq / np.arange(1, n_mc + 1) -
cumulative_means**2)
```

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```
# Create figure with convergence plots
fig, axes = plt.subplots(2, 3, figsize=(15, 10))
fig.suptitle('Convergence of Monte Carlo Estimates for $Y = \sin(X)$, $X \sim N(0,4)$', fontweight='bold')

# Plot 1: Mean convergence (full range)
ax = axes[0, 0]
ax.plot(np.arange(1, n_mc + 1), cumulative_means, 'b-', alpha=0.7, linewidth=1)
ax.axhline(y=mu_true, color='r', linestyle='--', linewidth=2, label=f'True $\mu = {mu_true:.6f}$')
ax.set_xscale('log')
ax.set_xlabel('Number of samples')
ax.set_ylabel('Empirical mean')
ax.set_title('Convergence of E[Y] estimate')
ax.legend()
ax.grid(True, alpha=0.3)

# Plot 2: Variance convergence (full range)
ax = axes[1, 0]
ax.plot(np.arange(1, n_mc+1), cumulative_vars, 'g-', alpha=0.7, linewidth=1)
ax.axhline(y=sigma2_true, color='r', linestyle='--', linewidth=2, label=f'True $\sigma^2 = {sigma2_true:.6f}$')
ax.set_xscale('log')
ax.set_xlabel('Number of samples')
ax.set_ylabel('Empirical variance')
ax.set_title('Convergence of Var[Y] estimate')
ax.legend()
ax.grid(True, alpha=0.3)

# Plot 3: Error in mean (log-log)
ax = axes[0, 1]
sample_sizes = np.logspace(1, 5, 100).astype(int)
sample_sizes = np.unique(sample_sizes) # Remove duplicates
errors_mean = []

for n in sample_sizes:
    sample = Y[:n]
    errors_mean.append(abs(np.mean(sample) - mu_true))

ax.loglog(sample_sizes, errors_mean, 'bo-', markersize=4, linewidth=1)
ax.loglog(sample_sizes, 1/np.sqrt(sample_sizes), 'r--', linewidth=2, label='1/$\sqrt{n}$ reference')
ax.set_xlabel('Number of samples')
ax.set_ylabel('Empirical mean - True mean')  
The document was generated with Code-Format https://code-format.com/
```

```
ax.set_title('Error in mean estimate (log-log)')
ax.legend()
ax.grid(True, alpha=0.3)

# Plot 4: Error in variance (log-log)
ax = axes[1, 1]
errors_var = []

for n in sample_sizes:
    sample = Y[:n]
    errors_var.append(abs(np.var(sample, ddof=1) - sigma2_true))

ax.loglog(sample_sizes, errors_var, 'go-', markersize=4, linewidth=1)
ax.loglog(sample_sizes, 1/np.sqrt(sample_sizes), 'r--', linewidth=2,
label='1/sqrt(n) reference')
ax.set_xlabel('Number of samples')
ax.set_ylabel('|Empirical variance - True variance|')
ax.set_title('Error in variance estimate (log-log)')
ax.legend()
ax.grid(True, alpha=0.3)

# Plot 5: Distribution of Y
ax = axes[0, 2]
ax.hist(Y, bins=100, density=True, alpha=0.7, edgecolor='black',
linewidth=0.5)
ax.set_xlabel('Y = sin(X)')
ax.set_ylabel('Density')
ax.set_title(f'Distribution of Y (n={n_mc:,})')
ax.grid(True, alpha=0.3)

# Plot 6: Summary statistics
ax = axes[1, 2]
ax.axis('off')
summary_text = (
    f'Analytical values:\n'
    f'  μ = {mu_true:.8f}\n'
    f'  σ² = {sigma2_true:.8f}\n\n'
    f'Monte Carlo (n={n_mc:,}): \n'
    f'  μ ≈ {mu_mc:.8f}\n'
    f'  σ² ≈ {sigma2_mc:.8f}\n\n'
    f'Errors (Monte Carlo):\n'
    f'  |μ - μ_true| = {abs(mu_mc - mu_true):.2e}\n'
    f'  |σ² - σ²_true| = {abs(sigma2_mc - sigma2_true):.2e}'
)
ax.text(0.1, 0.95, summary_text, transform=ax.transAxes, fontsize=10,
        verticalalignment='top', bbox=dict(boxstyle='round',
        facecolor='gray', alpha=0.5))
```

```
plt.tight_layout()
plt.show()
```

Analytical values:

$$\mu = E[Y] = 0.0$$

$$\sigma^2 = \text{Var}[Y] = 0.49983227$$

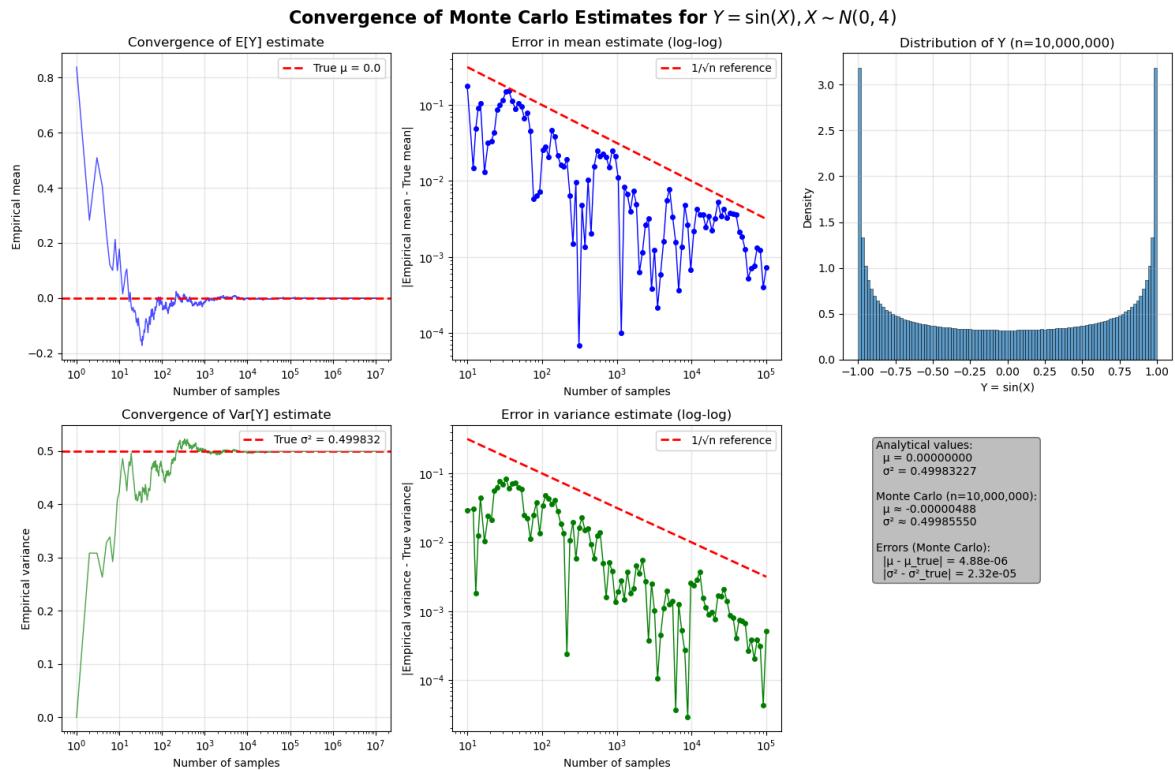
Monte Carlo simulation with  $n = 10,000,000$  samples...

Monte Carlo results:

$$E[Y] \approx -0.00000488 \text{ (error} = 0.00000488)$$

$$\text{Var}[Y] \approx 0.49985550 \text{ (error} = 0.00002323)$$

Visualizing convergence of Monte Carlo estimates...



The absolute errors of both the sample mean and the sample variance converge at an  $O(n^{-1/2})$  rate, as evidenced by the approximately  $-\frac{1}{2}$  slope on the log--log plot. By the central limit theorem, the standard deviation of the average of  $n$  independent and identically distributed (IID) samples scales as  $1 / \sqrt{n}$ . Consequently, for the sample mean  $\bar{X}_n$ ,

$$|\bar{X}_n - \mu| = O(n^{-1/2}).$$

A similar convergence rate holds for the sample variance  $s_n^2$ . Writing  $s_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ , we see that it is itself an average of transformed random variables. Although the terms  $(X_i - \bar{X}_n)^2$  are not independent due to the shared dependence on  $\bar{X}_n$ , this dependence becomes negligible asymptotically. By applying the central limit theorem to the sequence  $Z_i = (X_i - \mu)^2$ , which has finite variance under standard assumptions, we obtain

$$s_n^2 - \sigma^2 = O(n^{-1/2}).$$

Both the sample mean and the sample variance can therefore be viewed as sample averages of functions of the data:  $\bar{X}_n$  averages  $X_i$ , while  $s_n^2$  averages squared deviations. Any such sample mean of IID data with finite variance inherits the characteristic  $1 / \sqrt{n}$  scaling of its standard error. As a result, on a log--log plot, the absolute errors of both statistics display an approximate slope of  $-\frac{1}{2}$ , reflecting the fundamental  $n^{-1/2}$  convergence rate of Monte Carlo estimators.

### 3: Heat Equation

In [ ]:

```
from heat_rod import SteadyStateRod
import numpy as np

# --- Parameters from Example 3.5 ---
a = 0.95 # cm
b = 0.95 # cm
h = 0.002 # W/cm^2·C
k = 2.37 # W/cm·C
Phi = -18 # W/cm^2 (use negative for heat flux)
Tamb = 21 # C
L = 70
# --- Create rod instance ---
rod = SteadyStateRod(a, b, h, k, Phi, Tamb, L)

# --- Measurement locations ---
x0 = 10
dx = 4
x_vals = x0 + np.arange(15)*dx # 15 locations: 10, 14, 18, ..., 66

# --- Evaluate Ts and derivatives ---
Ts_vals = rod.Ts(x_vals)
dTs_dh_vals = rod.dTs_dh(x_vals)
dTs_dk_vals = rod.dTs_dk(x_vals)
dTs_dPhi_vals = rod.dTs_dPhi(x_vals)

# --- Print results like Table 3.2 ---
print(f"{'x (cm)':>6} | {'Ts (C)':>7} | {'dTs/dh (C^2 cm^2/W)':>10} |"
      f"{'dTs/dk (C^2 cm/W)':>10} | {'dTs/dPhi () b b':>10} ")
print("-"*55)
for i, x in enumerate(x_vals):
    print(f"{x:6.1f} | {Ts_vals[i]:7.4f} | {dTs_dh_vals[i]:10.4f} |"
          f"{dTs_dk_vals[i]:10.4f} | {dTs_dPhi_vals[i]:10.4f}")
```

| x (cm) | Ts (C)  | dTs/dh      | dTs/dk  | dTs/dPhi |
|--------|---------|-------------|---------|----------|
| 10.0   | 91.2698 | -28169.0259 | -5.8784 | -3.9039  |
| 14.0   | 76.3885 | -25543.2402 | -1.8152 | -3.0771  |
| 18.0   | 64.6710 | -22787.0997 | 0.8030  | -2.4262  |
| 22.0   | 55.4481 | -20079.4108 | 2.4096  | -1.9138  |
| 26.0   | 48.1929 | -17530.1660 | 3.3196  | -1.5107  |
| 30.0   | 42.4909 | -15201.9433 | 3.7607  | -1.1939  |
| 34.0   | 38.0165 | -13125.3738 | 3.8963  | -0.9454  |
| 38.0   | 34.5142 | -11310.2370 | 3.8423  | -0.7508  |
| 42.0   | 31.7837 | -9753.3536  | 3.6806  | -0.5991  |
| 46.0   | 29.6693 | -8444.1446  | 3.4679  | -0.4816  |
| 50.0   | 28.0500 | -7368.5026  | 3.2435  | -0.3917  |
| 54.0   | 26.8334 | -6511.4519  | 3.0335  | -0.3241  |
| 58.0   | 25.9501 | -5858.9472  | 2.8556  | -0.2750  |
| 62.0   | 25.3495 | -5399.0651  | 2.7210  | -0.2416  |
| 66.0   | 24.9973 | -5122.7715  | 2.6364  | -0.2221  |

# 3.1 & 3.2

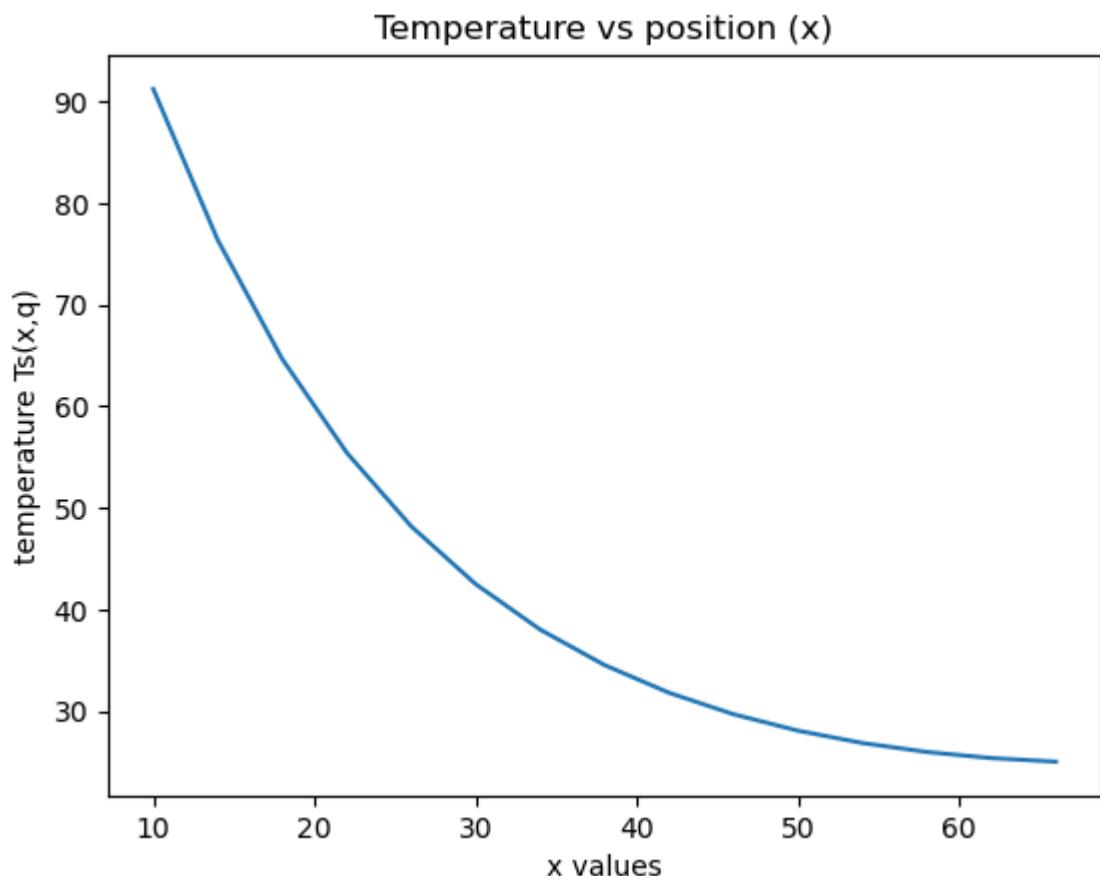
The values of  $T_s(x, q)$  at  $x_j$  and the respective partials at the same  $x_j$  are presented in the below table.

| x (cm) | Ts (C)  | dTs/dh      | dTs/dk  | dTs/dPhi |
|--------|---------|-------------|---------|----------|
| 10.0   | 91.2698 | -28169.0259 | -5.8784 | -3.9039  |
| 14.0   | 76.3885 | -25543.2402 | -1.8152 | -3.0771  |
| 18.0   | 64.6710 | -22787.0997 | 0.8030  | -2.4262  |
| 22.0   | 55.4481 | -20079.4108 | 2.4096  | -1.9138  |
| 26.0   | 48.1929 | -17530.1660 | 3.3196  | -1.5107  |
| 30.0   | 42.4909 | -15201.9433 | 3.7607  | -1.1939  |
| 34.0   | 38.0165 | -13125.3738 | 3.8963  | -0.9454  |
| 38.0   | 34.5142 | -11310.2370 | 3.8423  | -0.7508  |
| 42.0   | 31.7837 | -9753.3536  | 3.6806  | -0.5991  |
| 46.0   | 29.6693 | -8444.1446  | 3.4679  | -0.4816  |
| 50.0   | 28.0500 | -7368.5026  | 3.2435  | -0.3917  |
| 54.0   | 26.8334 | -6511.4519  | 3.0335  | -0.3241  |
| 58.0   | 25.9501 | -5858.9472  | 2.8556  | -0.2750  |
| 62.0   | 25.3495 | -5399.0651  | 2.7210  | -0.2416  |
| 66.0   | 24.9973 | -5122.7715  | 2.6364  | -0.2221  |

In [5]:

```
import matplotlib.pyplot as plt
plt.plot(x_vals,Ts_vals)
plt.xlabel("x values")
plt.ylabel("temperature Ts(x,q)")
plt.title("Temperature vs position (x)")
```

```
Text(0.5, 1.0, 'Temperature vs position (x)')
```



# 4 - 4.1

4.  $a=2$ 

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$$(4.1) \quad \text{Prove: } \mathbb{E}\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i \mathbb{E}(x_i)$$

where  $a_1, a_n$  are constants.

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^2 a_i x_i\right] &= \mathbb{E}[a_1 x_1 + a_2 x_2] \\ &= a_1 \mathbb{E}(x_1) + a_2 \mathbb{E}(x_2) \end{aligned}$$

or:

$$\begin{aligned} \mathbb{E}[a_1 x_1 + a_2 x_2] &= \iint (a_1 x_1 + a_2 x_2) f(x_1, x_2) dx_1 dx_2 \\ &= a_1 \iint x_1 f(x_1, x_2) dx_1 dx_2 + a_2 \iint x_2 f(x_1, x_2) dx_1 dx_2 \\ &= a_1 \int x_1 f(x_1) dx_1 + a_2 \int x_2 f(x_2) dx_2 \\ &= a_1 \mathbb{E}(x_1) + a_2 \mathbb{E}(x_2) \end{aligned}$$

Prove:

$$\text{Var}(a_1 x_1 + a_2 x_2) = a_1^2 \text{Var}(x_1) + a_2^2 \text{Var}(x_2) + 2a_1 a_2 \text{Cov}(x_1, x_2)$$

$$\text{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}[Y])^2] \quad \&$$

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

if  $\gamma = a_1 x_1 + a_2 x_2$ ,

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$$\mathbb{E}[\gamma] = a_1 \mathbb{E}[x_1] + a_2 \mathbb{E}[x_2] =$$

$$\gamma - \mathbb{E}[\gamma] = a_1 [x_1 - \mathbb{E}[x_1]] + a_2 [x_2 - \mathbb{E}[x_2]]$$

$$(\gamma - \mathbb{E}[\gamma])^2 = a_1^2 [x_1 - \mathbb{E}[x_1]]^2 + a_2^2 [x_2 - \mathbb{E}[x_2]]^2$$

$$+ 2a_1 a_2 [x_1 - \mathbb{E}[x_1]][x_2 - \mathbb{E}[x_2]]$$

$$\mathbb{E}[(\gamma - \mathbb{E}[\gamma])^2] = \mathbb{E}[a_1^2 [x_1 - \mathbb{E}[x_1]]^2] + \mathbb{E}[a_2^2 [x_2 - \mathbb{E}[x_2]]^2]$$

$$+ \mathbb{E}[2a_1 a_2 [x_1 - \mathbb{E}[x_1]][x_2 - \mathbb{E}[x_2]]]$$

$$= a_1^2 \mathbb{E}[(x_1 - \mathbb{E}[x_1])^2] + a_2^2 \mathbb{E}[(x_2 - \mathbb{E}[x_2])^2]$$

$$+ 2a_1 a_2 \mathbb{E}[(x_1 - \mathbb{E}[x_1])(x_2 - \mathbb{E}[x_2])]$$

$$= a_1^2 \text{Var}(x_1) + a_2^2 \text{Var}(x_2)$$

$$+ 2a_1 a_2 \text{Cov}(x_1, x_2)$$

## 5 - 4.2

5.

(4.2) PDF

$$X \sim U(a, b) \rightarrow f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases}$$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{x^2}{2} \Big|_a^b \frac{1}{b-a} \\ &= \frac{b^2 - a^2}{2} \frac{1}{b-a} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

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$$\begin{aligned} \mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_a^b x^2 \frac{1}{b-a} dx \\ &= \frac{x^3}{3} \Big|_a^b \frac{1}{b-a} = \frac{b^3 - a^3}{(b-a)} \frac{1}{3} \end{aligned}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{(b-a)} \frac{1}{3} = \frac{1}{3} (b^2 + ab + a^2)$$

$$\text{Var}(X) = \frac{1}{3} (b^2 + ab + a^2) - \left(\frac{a+b}{2}\right)^2$$

$$= (4b^2 + 4ab + 4a^2 - 3b^2 - 3a^2 - 6ab) \frac{1}{12}$$

$$= (b^2 - 2ab + a^2) \frac{1}{12} = \frac{(b-a)^2}{12}$$

## 6 - 4.4

6.  $\gamma: \text{RV}, c, d \in \mathbb{R}$

$$\mathbb{E}[c\gamma + d] = \int_{-\infty}^{\infty} (c\gamma + d) f_{\gamma}(y) dy$$

$$\text{since } \mathbb{E}[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

pdf of  $x$ .

$$\int_{-\infty}^{\infty} (c\gamma + d) f_{\gamma}(y) dy$$

$$= \int_{-\infty}^{\infty} cy f_{\gamma}(y) dy + \int_{-\infty}^{\infty} d f_{\gamma}(y) dy$$

$$= c \int_{-\infty}^{\infty} y f_{\gamma}(y) dy + d \int_{-\infty}^{\infty} f_{\gamma}(y) dy$$

$= c E[Y] + d$ , since  $\int_{-\infty}^{\infty} f_Y(y) dy$  is the integral of a PDF across the whole domain, it equals 1.

$$\text{var}(cY+d) = E[(cY+d)^2] - (E[cY+d])^2$$

$$E[(cY+d)^2] = E[c^2Y^2 + d^2 + 2cdY]$$

$$= E[c^2Y^2] + E[d^2] + 2cdE[Y]$$

$$= c^2 \int_{-\infty}^{\infty} y^2 f_Y(y) dy + d^2 + 2cd \int_{-\infty}^{\infty} y f_Y(y) dy$$

$$= c^2 E[Y^2] + d^2 + 2cd E[Y]$$

$$\text{so, } \text{var}(cY+d) = c^2 E[Y^2] + d^2 + 2cd E[Y]$$

$$- (c E[Y] + d)^2$$

$$= c^2 E[Y^2] + d^2 + 2cd E[Y]$$

$$- c^2 E[Y]^2 - d^2 - 2cd E[Y]$$

$$= c^2 (E[Y^2] - E[Y]^2)$$

$$= c^2 \text{var}(Y)$$