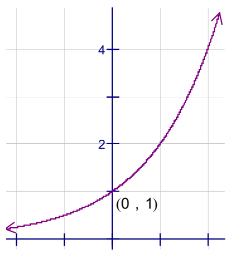
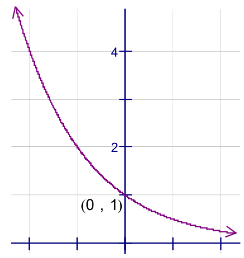
I. 1.4, Exponential Functions, p. 45

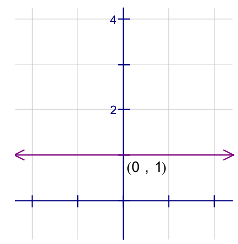
A. Laws of exponents 1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \dfrac{a^x}{a^y}$ 3. $\left(a^x\right)^y = a^{xy}$ 4. $\left(ab\right)^x = a^x b^x$

B. Three basic types of exponential functions

1. $y = a^x,\ a > 1$ 2. $y = a^x,\ 0 < a < 1$ 3. $y = 1^x$

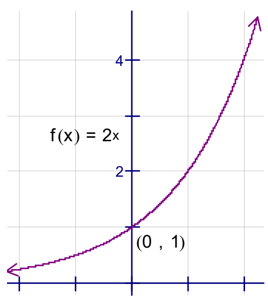


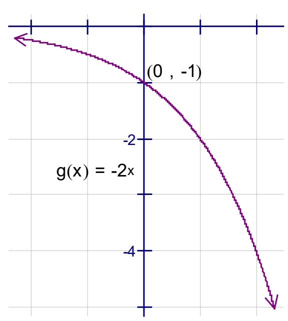


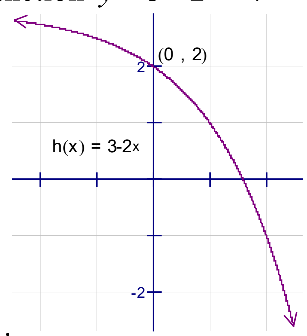


C. Example 1, p. 47

1. Find the domain, range, and sketch the graph of the function $y = 3 - 2^x \Rightarrow$







Domain: $\mathbb{R}$

Range: $\left(-\infty\ ,\ 3\right)$

D. Half-life: Time required for half of a given quantity to disintegrate

E. The slope of $y = e^x$ at $\left(0\ ,\ 1\right)$ is exactly 1, $e \approx 2.718281828\ldots$ (Andrew Jackson squared)

II. 1.5, Inverse Functions and Logarithms, p. 55

A. $1-1$ Functions

1. $1-1$ functions pass the horizontal line test: x^3 is $1-1$, x^2 is not $1-1$

2. Decreasing and increasing functions are $1-1$

B. Inverse functions

1. Inverse function: $y = f\left(x\right) \Leftrightarrow x = f^{-1}\left(y\right)$ The superscript is not an exponent.

2. Finding an inverse function

a. Algebraically: switch x and y in the equation and then solve for y

b. Geometrically: Reflect the graph of the function across the line $y = x$

C. Example 4, p. 58

1. Find the inverse function of $f\left(x\right) = x^3 + 2 \Rightarrow$

2. $y = x^3 + 2 \Rightarrow x = y^3 + 2\ ,\ y^3 = x - 2\ ,\ y = \sqrt[3]{x-2} \Rightarrow \boxed{f^{-1}\left(x\right) = \sqrt[3]{x-2}}$

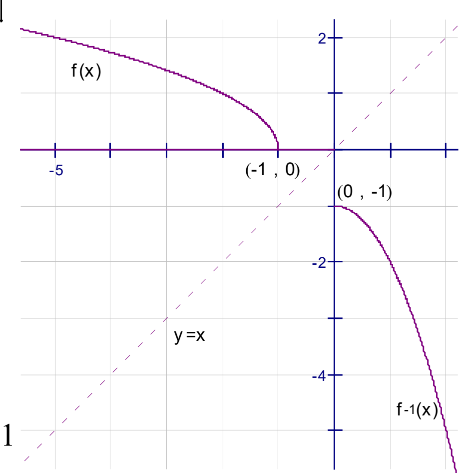
D. Example 5, p. 59

1. Sketch the graphs of $f\left(x\right) = \sqrt{-1-x}$ and its inverse $\Rightarrow$

2. $y = \sqrt{-1-x}\ ,\ y = \sqrt{-\left(1+x\right)}\ ,\ y^2 = -\left(1+x\right)\ ,\ x+1 = -y^2 \Rightarrow$

3. Upper-half of the parabola opening to the left with vertex $\left(-1\ ,\ 0\right)$

4. Right-half of the parabola opening down with vertex $\left(0\ ,\ -1\right)$



E. Logarithmic functions

1. $\log_a x = y \Leftrightarrow a^y = x$ (base a)

2. Logarithmic functions are inverses of exponential functions

3. Natural logarithm is the logarithm with base e: $\log_e x = \ln x \Rightarrow \ln e = 1$

F. Properties and laws of logarithms

1. $\log_a a^x = x$ 2. $a^{\log_a x} = x$ 3. $\log_a xy = \log_a x + \log_a y$ 4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

5. $\log_a x^r = r \log_a x$ 6. Change-of-Base Formula: $\log_a x = \dfrac{\ln x}{\ln a} = \dfrac{\log_b x}{\log_b a}$

G. Example 6, p. 60

1. $\log_2 80 - \log_2 5 \Rightarrow \log_2 \dfrac{80}{5} = \log_2 16 = \boxed{4}$

H. Extra example #1

1. $\log_3 81 = \boxed{4}$ 2. $\log_{25} 5 = \boxed{\dfrac{1}{2}}$ 3. $\log_{10} .001 = \boxed{-3}$ 4. $\log_4 2 + \log_4 32 \Rightarrow \log_4 64 = \boxed{3}$

I. Example 7, p. 61

1. $\ln x = 5 \Rightarrow x = \boxed{e^5}$

J. Example 8, p. 61

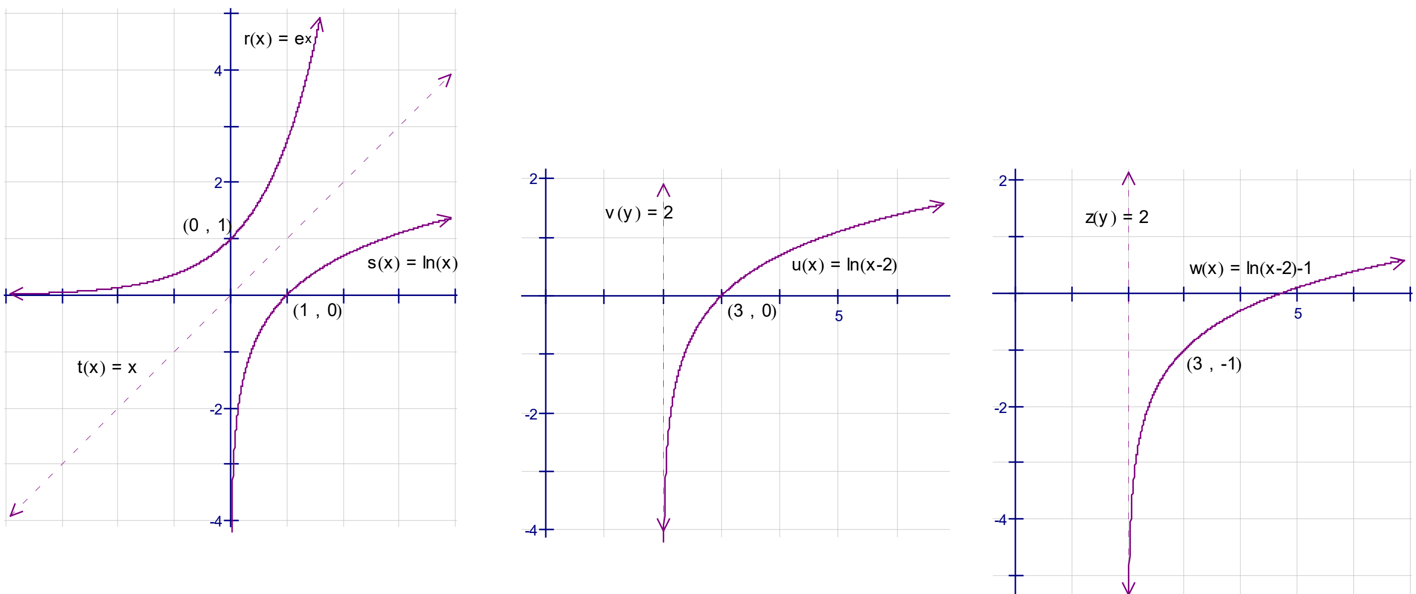
1. $e^{5-3x} = 10 \Rightarrow 5 - 3x = \ln 10$, $5 - \ln 10 = 3x$, $x = \boxed{\dfrac{5 - \ln 10}{3}}$

K. Example 9, p. 61

1. Express $\ln a + \dfrac{1}{2}\ln b$ as a single logarithm $\Rightarrow \ln a + \dfrac{1}{2}\ln b = \ln a + \ln b^{\frac{1}{2}} = \boxed{\ln ab^{\frac{1}{2}}} = \ln a\sqrt{b}$

L. Example 11, p. 62

1. Sketch the graph of $y = \ln(x-2) - 1 \Rightarrow$

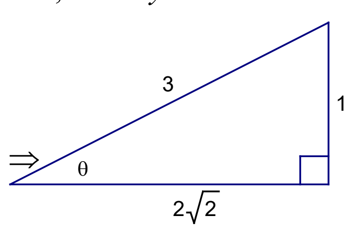


M. Inverse trigonometric functions

1. $y = \sin^{-1} x$, $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ 2. $y = \cos^{-1} x$, $0 \le y \le \pi$ 3. $y = \tan^{-1} x$, $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

N. Example 12, p. 64

1. a. $\sin^{-1}\left(\dfrac{1}{2}\right) = \boxed{\dfrac{\pi}{6}}$ b. $\tan\left(\arcsin\dfrac{1}{3}\right) \Rightarrow$

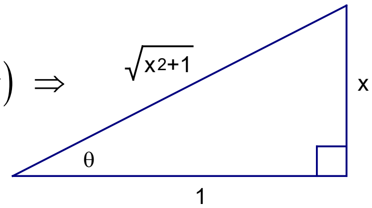


$\tan\theta = \dfrac{1}{2\sqrt{2}} = \boxed{\dfrac{\sqrt{2}}{4}}$

O. Extra example #2

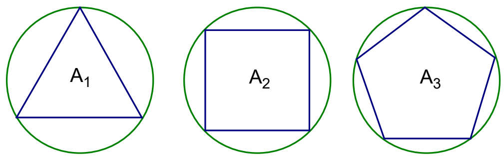
1. a. $\sin(\sin^{-1} .6) = \boxed{.6}$ b. $\sin^{-1}\left(\sin\dfrac{\pi}{12}\right) = \boxed{\dfrac{\pi}{12}}$ c. $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \boxed{\dfrac{\pi}{3}}$

P. Example 13, p. 65

1. $\cos\left(\tan^{-1} x\right) \Rightarrow$ 

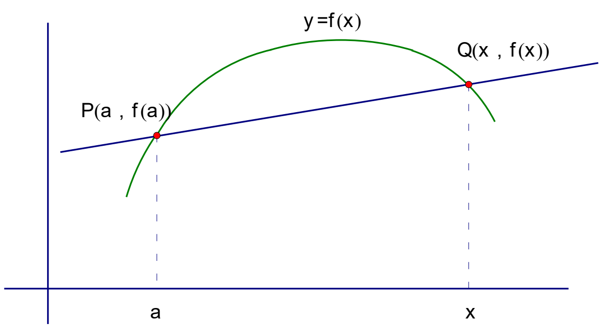
$\cos\theta = \boxed{\dfrac{1}{\sqrt{x^2+1}}}$

III. Calculus Preview, p. 1

A. Area problem 

$\cdots$ $A = \lim\limits_{n \to \infty} A_n \rightarrow$ integral calculus

B. Tangent problem



$$m_{\overline{PQ}} = \frac{f(x)-f(a)}{x-a}$$

$$m_{\text{tangent}} = \lim_{Q \to P} m_{\overline{PQ}}$$

$$m_{\text{tangent}} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \rightarrow \text{differential calculus}$$

C. Limits also applied to velocity problem; developed by Fermat, Wallis, Barrow, Newton, and Leibniz

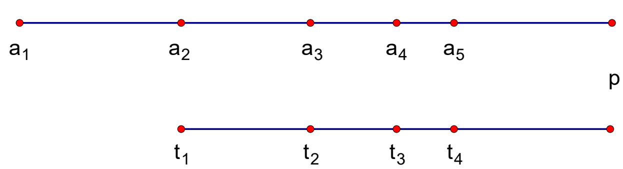
D. Limit of a sequence

1. Zeno of Elea $\rightarrow$ Zeno's paradoxes

a. 2^{nd} Paradox: Achilles and the tortoise

b. Achilles

Tortoise



$\leftarrow$ Achilles overtakes the tortoise.

c. Both sequences have the same limit.

E. Sum of a series

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$

2. Infinitely large positive value: ∞ , unbounded growth in the positive direction

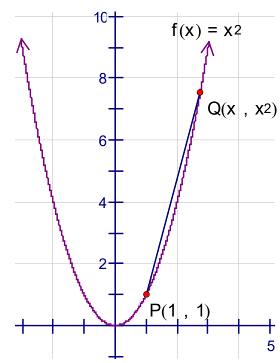
3. Infinitely large negative value: $-\infty$, unbounded growth in the negative direction

4. Infinitesimally small value: # that is practically zero, either negative or positive; represented by 0^- or 0^+

Chapter 2: Limits and Derivatives

IV. 2.1, Tangent and Velocity Problems, p. 78

A. Tangent problem



B. Example 1, p. 78

1. Find the equation of a tangent line to $y = x^2$ at the point $P(1\ ,\ 1)$ $\Rightarrow$

2. $m_{\overline{PQ}} = \frac{x^2 - 1}{x - 1}$

3. For $Q(1.5\ ,\ 1.5^2)$ $\rightarrow$ $Q(1.5\ ,\ 2.25)$, $m_{\overline{PQ}} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{.5} = 1.25(2) = 2.5$

x	1.5	0.5	1.1	0.9	1.01	0.99	1.001	0.999
$m_{\overline{PQ}}$	2.5	1.5	2.1	1.9	2.01	1.99	2.001	1.999

4. The slope approaches 2 . $\rightarrow$ The slope of the tangent line equals the limit of the secant slopes.

5. $\lim\limits_{Q \to P} m_{\overline{PQ}} = m_{\text{tangent}}$, $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

6. $\therefore$ The equation of the tangent line is $\boxed{y-1=2(x-1)}$ <u>OR</u> $y=2x-1$.

7. Alternative to limit: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x+1)(x-1)}{x-1} = \lim\limits_{x\to 1} (x+1) = \underline{2}$

C. Velocity problem

D. Example 3, p. 80 $\rightarrow$ read

1. Ball is dropped $450\ m$ above ground from the CN Tower in Toronto. Find its velocity $5\ sec$ later. $\Rightarrow$

2. Use Galileo's Law for a falling object accelerated by gravity: $s(t)=4.9t^2$

3. Average velocity $= \dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

4. For example, for the time interval from $5\ sec$ to $5.1\ sec$, the average velocity is $\dfrac{s(5.1)-s(5)}{5.1-5}$

$$= \frac{4.9(5.1)^2-4.9(5)^2}{.1} = \frac{4.9\left[(5.1)^2-(5)^2\right]}{.1} = 4.9(26.01-25)(10) = 49(1.01) = \underline{49.49\ {}^{m}/_{sec}}$$

time	5.1	5.05	5.01	5.001
average velocity	49.49	49.245	49.049	49.0049

5. $\therefore$ The instantaneous velocity after $5\ sec$ is the value approached in the table which is $\boxed{49\ {}^{m}/_{sec}}$.

E. Figure 6, p. 81: close connection between tangent problem and velocity problem

1. Observe that the slope of the secant line equals average velocity.

2. Observe that the slope of the tangent line equals instantaneous velocity.

V. 2.2, Limit of a Function, p. 83

A. Figure 1, p. 83: graph of $x^2 - x + 2$

1. Graph and table
2. $\lim_{x \to 2} \left(x^2 - x + 2\right) = 4$

B. Definition: $\lim_{x \to a} f(x) = L$

1. "The limit of $f(x)$, as x approaches a, equals L"
2. This is true if $f(x)$ is arbitrarily close to L as x is sufficiently close to a on either side, $x \neq a$

C. Figure 2, p. 84: three different situations for a limit of L to exist

1. Three graphs
2. $\lim_{x \to a} f(x) = L$ in all three cases

D. Example 1, p. 84

1. Predict the value of $\lim_{x \to 1} \dfrac{x-1}{x^2 - 1} \Rightarrow \lim_{x \to 1} \dfrac{x-1}{x^2 - 1} = \boxed{.5}$
   a. Figure 3 and Figure 4, p. 84: two different functions can have the same limit at a point
   b. Tables and graphs
2. Look at the piecewise function $g(x) = \begin{cases} \dfrac{x-1}{x^2 - 1} & x \neq 1 \\ 2 & x = 1 \end{cases}$
3. This function and the previous one both have limits of $\dfrac{1}{2}$ as $x \to 1$ from both sides of 1

E. Example 2, p. 85

1. Find $\lim_{t \to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2} \Rightarrow \lim_{t \to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2} = \boxed{\dfrac{1}{6}}$
   a. Tables, p. 85: numerical values in tables from a calculator may be misleading and not agree
   b. A calculator gives <u>false</u> values for extremely small values of t !
   c. Figure 5, p. 85: graphs of $\dfrac{\sqrt{t^2+9} - 3}{t^2}$ are inaccurate due to subtraction rounding errors

F. Example 3, p. 86

1. Find the value of $\lim_{x \to 0} \dfrac{\sin x}{x} \Rightarrow \lim_{x \to 0} \dfrac{\sin x}{x} = \boxed{1}$
   a. Table, p. 86: numerical evidence that the limit exists as the values appear to approach 1
   b. Figure 6, p. 86: graphical evidence from the graph of $\dfrac{\sin x}{x}$ that the limit exists and equals 1

G. Example 4, p. 86

1. Investigate $\lim_{x \to 0} \sin\dfrac{\pi}{x} \Rightarrow \lim_{x \to 0} \sin\dfrac{\pi}{x} \Rightarrow \boxed{\text{Does not exist}}$ or DNE
   a. Figure 7, p. 86: graph of $\sin\dfrac{\pi}{x}$ oscillates infinitely often near 0
   b. The values of $\sin\dfrac{\pi}{x}$ oscillate between -1 and 1

H. Example 5, p. 87

1. Find $\lim_{x \to 0} \left(x^3 + \dfrac{\cos 5x}{10{,}000}\right) \Rightarrow \lim_{x \to 0} \left(x^3 + \dfrac{\cos 5x}{10{,}000}\right) = \boxed{.0001}$

a. Tables, p. 87: demonstrate a pitfall of trying to guess a limit as numerical values approach $\frac{1}{10,000}$

b. Even though the limit may appear to be zero, it is not.

I. One-sided limits

1. If $t \to a^-$, then a is approached from the left, meaning from below.
2. If $t \to a^+$, then a is approached from the right, meaning from above.

J. Example 6, p. 87

1. The Heaviside function is $H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \Rightarrow$

a. $\lim\limits_{t \to 0^-} H(t) = 0$

b. $\lim\limits_{t \to 0^+} H(t) = 1$

c. These limits are not equal. $\therefore \lim\limits_{t \to 0} H(t) \Rightarrow \boxed{\text{DNE}}$

d. Figure 8, p. 87: graph of the Heaviside function

K. Definition of one-sided limits

1. $\lim\limits_{x \to a^-} f(x) = L$ is the left-hand limit of $f(x)$ as x approaches a
2. $\lim\limits_{x \to a^+} f(x) = L$ is the right-hand limit of $f(x)$ as x approaches a
3. A limit exists if and only if, iff, the left– and right–hand limits exist and are equal
4. $\Rightarrow \lim\limits_{x \to a} f(x) = L$ iff $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$

L. Example 7, p. 88

1. Use the graph in Figure 10, p. 88 to find the following limits

a. $\lim\limits_{x \to 2^-} g(x) \Rightarrow \lim\limits_{x \to 2^-} g(x) = \boxed{3}$

b. $\lim\limits_{x \to 2^+} g(x) \Rightarrow \lim\limits_{x \to 2^+} g(x) = \boxed{1}$

c. $\lim\limits_{x \to 2} g(x) \Rightarrow \lim\limits_{x \to 2} g(x) \Rightarrow \boxed{\text{DNE}}$ since $3 \neq 1$

d. $\lim\limits_{x \to 5^-} g(x) \Rightarrow \lim\limits_{x \to 5^-} g(x) = \boxed{2}$

e. $\lim\limits_{x \to 5^+} g(x) \Rightarrow \lim\limits_{x \to 5^+} g(x) = \boxed{2}$

f. $\lim\limits_{x \to 5} g(x) \Rightarrow \lim\limits_{x \to 5} g(x) = \boxed{2}$ since $2 = 2$

2. Observe: $g(5) \neq 2$

M. Infinite limits

N. Example 8, p. 89

1. Find $\lim\limits_{x \to 0} \frac{1}{x^2}$ if it exists $\Rightarrow f(x) = \frac{1}{x^2}$ becomes arbitrarily large as x becomes infinitesimally small, positive or negative. $\therefore$ The limit does not exist — DNE — since it does not approach a finite number.

2. Figure 11, p. 89: graph of $\frac{1}{x^2}$

O. Use the notation $\lim_{x \to 0} \frac{1}{x^2} = \infty$ to indicate the type of behavior exhibited.

1. This does not imply that the limit exists.

2. This simply expresses the particular way in which the limit fails to exist.

3. Generally, $\lim_{x \to 0} \frac{1}{x^2} = \infty$ indicates $f(x)$ increases without bound.

4. Definition: $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a.

5. Figure 12, p. 89: illustration of $f(x) \to \infty$ as $x \to a$.

P. Similar definition for $\lim_{x \to a} f(x) = -\infty$

3. Figures 13 and 14, p. 90: illustrations of two-sided and one-sided infinite limits

Q. Definition: the line $x = a$ is a vertical asymptote of $y = f(x)$ if at least one of the following is true:

1. $\lim_{x \to a} f(x) = \infty$
2. $\lim_{x \to a^-} f(x) = \infty$
3. $\lim_{x \to a^+} f(x) = \infty$
4. $\lim_{x \to a} f(x) = -\infty$
5. $\lim_{x \to a^-} f(x) = -\infty$
6. $\lim_{x \to a^+} f(x) = -\infty$

R. Example 9, p. 91
1. Find the following limits:
   a. $\lim_{x \to 3^+} \frac{2x}{x-3} \Rightarrow \lim_{x \to 3^+} \frac{2x}{x-3} \to \infty \Rightarrow \boxed{\text{DNE}}$
   b. $\lim_{x \to 3^-} \frac{2x}{x-3} \Rightarrow \lim_{x \to 3^-} \frac{2x}{x-3} \to -\infty \Rightarrow \boxed{\text{DNE}}$
2. Figure 15, p. 91: graph of $\frac{2x}{x-3}$
   a. $\Rightarrow x = 3$ is a vertical asymptote

S. Example 10, p. 91

1. Find the vertical asymptotes of $f(x) = \tan x \Rightarrow$

   a. $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x \to \infty$

   b. $\lim\limits_{x \to \frac{\pi}{2}^+} \tan x \to -\infty$

   c. $\therefore$ Vertical asymptotes occur when $x = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$

2. Figure 16, p. 91: graph of $\tan x$

T. Figure 17, p. 91: graph of $\ln x$

1. The natural log function has the line $x = 0$, i.e. the $y-$axis, as a vertical asymptote.
2. Observe: $\lim\limits_{x \to 0^+} \ln x \to -\infty$

VI. 2.3, Calculating Limits and Limit Laws, p. 95

A. Laws of limits: Suppose c is a constant and that the two limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ both exist.

1. $\lim\limits_{x \to a} \left[f(x) + g(x)\right] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ Sum Law
2. $\lim\limits_{x \to a} \left[f(x) - g(x)\right] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$ Difference Law
3. $\lim\limits_{x \to a} \left[c f(x)\right] = c \lim\limits_{x \to a} f(x)$ Constant Multiple Law
4. $\lim\limits_{x \to a} \left[f(x) g(x)\right] = \lim\limits_{x \to a} f(x) \bullet \lim\limits_{x \to a} g(x)$ Product Law
5. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ if $\lim\limits_{x \to a} g(x) \neq 0$ Quotient Law
6. $\lim\limits_{x \to a} \left[f(x)\right]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$, $n \in \mathbb{N}$ Power Law
7. $\lim\limits_{x \to a} c = c$
8. $\lim\limits_{x \to a} x = a$
9. $\lim\limits_{x \to a} x^n = a^n$, $n \in \mathbb{N}$
10. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, $n \in \mathbb{N}$ Assume that $a > 0$ if n is even.
11. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$, $n \in \mathbb{N}$ Assume that $\lim\limits_{x \to a} f(x) > 0$ if n is even. Root Law

B. Example 1, p. 95

1. Figure 1, p. 95: use graphs of f and g to calculate limits
2. Evaluate the following limits.

   a. $\lim\limits_{x \to -2} \left[f(x) + 5g(x)\right] \Rightarrow \lim\limits_{x \to -2} \left[f(x) + 5g(x)\right] = 1 + 5(-1) = 1 - 5 = \boxed{-4}$

   b. $\lim\limits_{x \to 1} \left[f(x) g(x)\right] \Rightarrow$ Observe that $\lim\limits_{x \to 1^-} g(x) = -2$ and $\lim\limits_{x \to 1^+} g(x) = -1$.

   These two limits are not equal, $-2 \neq -1$, so $\lim\limits_{x \to 1} g(x) \Rightarrow \underline{\text{DNE}}$.

   Even though $\lim\limits_{x \to 1} f(x) = 2$, the $\lim\limits_{x \to 1} \left[f(x) g(x)\right] \Rightarrow \boxed{\text{DNE}}$

   c. $\lim\limits_{x \to 2} \dfrac{f(x)}{g(x)} \Rightarrow$ Observe that $\lim\limits_{x \to 2} f(x) \approx 1.4$ and $\lim\limits_{x \to 2} g(x) = 0$.

   Since $\lim\limits_{x \to 2} \dfrac{f(x)}{g(x)} \to \dfrac{1.4}{0^{\pm}} \to \pm\infty \Rightarrow \boxed{\text{DNE}}$

C. Five common techniques for evaluating limits:
1. Direct substitution — with continuous functions
2. Dividing out common factors — cancellation
3. Rationalization
4. Finding a common denominator
5. Expanding a binomial

D. Example 2, p. 97
1. Evaluate the following limits
a. $\lim_{x \to 5} \left(2x^2 - 3x + 4\right) \Rightarrow \lim_{x \to 5} \left(2x^2 - 3x + 4\right) = 2(5)^2 - 3 \cdot 5 + 4 = 2 \cdot 25 - 15 + 4 = 50 - 11 = \boxed{39}$
b. $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x} \Rightarrow \lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x} = \dfrac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$
$= \dfrac{-8 + 2 \cdot 4 - 1}{5 + 6} = \dfrac{-8 + 8 - 1}{11} = \boxed{-\dfrac{1}{11}}$
c. Extra example: $\lim_{x \to 1} \left[\sqrt[5]{x^2 - x} + \left(x^3 + x\right)^9\right] \Rightarrow \lim_{x \to 1} \left[\sqrt[5]{x^2 - x} + \left(x^3 + x\right)^9\right] = \left[\sqrt[5]{1^2 - 1} + \left(1^3 + 1\right)^9\right]$
$= \left[\sqrt[5]{1 - 1} + (1 + 1)^9\right] = \left(\sqrt[5]{0} + 2^9\right) = 0 + 512 = \boxed{512}$

E. Example 3, p. 98
1. Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} \Rightarrow \lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim_{x \to 1} \dfrac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1} (x + 1) = \boxed{2}$

F. Example 4, p. 98
1. Find $\lim_{x \to 1} g(x)$, $g(x) = \begin{cases} x + 1, & x \neq 1 \\ \pi, & x = 1 \end{cases} \Rightarrow \lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = (1 + 1) = \boxed{2}$

G. Figure 2, p. 98: graphs of $\dfrac{x^2 - 1}{x - 1}$ and $g(x) = \begin{cases} x + 1, & x \neq 1 \\ \pi, & x = 1 \end{cases}$
1. The previous two examples are identical except at $x = 1$, but they result in the same limit.

H. Example 5, p. 99
1. Evaluate $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h} \Rightarrow$
a. Observe that $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h} \to \dfrac{0}{0}$ which is an indeterminate form
b. Compare that to limits approaching values such as $\dfrac{0}{5} = 0$ and $\dfrac{5}{0} \to \pm\infty \Rightarrow$ DNE
c. $\therefore \lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} \dfrac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \dfrac{6h + h^2}{h}$
$= \lim_{h \to 0} \dfrac{h(6 + h)}{h} = \lim_{h \to 0} (6 + h) = (6 + 0) = \boxed{6}$ <u>OR</u>
d. Factor as the difference of squares: $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} \dfrac{(3 + h)^2 - 3^2}{h}$
$= \lim_{h \to 0} \dfrac{(3 + h - 3)(3 + h + 3)}{h} = \lim_{h \to 0} \dfrac{h(6 + h)}{h} = \cdots = \underline{6}$

I. Example 6, p. 99
1. Find $\lim_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2} \Rightarrow$

a. Observe that $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}\ -3}{t^2} \to \dfrac{0}{0}$ which is an indeterminate form

b. Need to rationalize the numerator

c. $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}\ -3}{t^2} = \lim\limits_{t \to 0} \dfrac{\left(\sqrt{t^2+9}\ -3\right)\left(\sqrt{t^2+9}\ +3\right)}{t^2\left(\sqrt{t^2+9}\ +3\right)} = \lim\limits_{t \to 0} \dfrac{\left(t^2+9\right)+3\sqrt{t^2+9}-3\sqrt{t^2+9}-9}{t^2\left(\sqrt{t^2+9}\ +3\right)}$

$= \lim\limits_{t \to 0} \dfrac{t^2+9-9}{t^2\left(\sqrt{t^2+9}\ +3\right)} = \lim\limits_{t \to 0} \dfrac{t^2}{t^2\left(\sqrt{t^2+9}\ +3\right)} = \lim\limits_{t \to 0} \dfrac{1}{\sqrt{t^2+9}\ +3} = \dfrac{1}{\sqrt{0^2+9}\ +3}$

$= \dfrac{1}{\sqrt{0+9}\ +3} = \dfrac{1}{\sqrt{9}\ +3} = \dfrac{1}{3+3} = \boxed{\dfrac{1}{6}}$

J. Theorem: $\lim\limits_{x \to a} f(x) = L$ iff, if and only if, $\lim\limits_{x \to a^+} f(x) = L = \lim\limits_{x \to a^-} f(x)$

a. This means two-sided limits exist only when both one-sided limits exist and are equal.

K. Example 7, p. 100

1. Show $\lim\limits_{x \to 0} |x| = 0 \Rightarrow$

a. Recall $|x| = \begin{cases} x\ , & x \geq 0 \\ -x\ , & x < 0 \end{cases}$

b. For $x \geq 0$, $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} x = 0$

c. For $x < 0$, $\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} -x = 0$

d. $\therefore \lim\limits_{x \to 0} |x| = 0$ QED!

2. Figure 3, p. 100: graph of $|x|$

L. Example 8, p. 100

1. Prove $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist $\Rightarrow$

a. $\lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{x \to 0^+} \dfrac{x}{x} = \lim\limits_{x \to 0^+} 1 = \underline{1}$

b. $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} \dfrac{-x}{x} = \lim\limits_{x \to 0^-} -1 = \underline{-1}$

c. $\therefore \lim\limits_{x \to 0} \dfrac{|x|}{x}$ $\boxed{\text{DNE}}$ since the right– and left–hand limits are different

2. Figure 4, p. 100: graph of $\dfrac{|x|}{x}$

M. Example 9, p. 100

1. Does $\lim\limits_{x \to 4} f(x)$ exist for $f(x) = \begin{cases} \sqrt{x-4}\ , & x > 4 \\ 8-2x\ , & x < 4 \end{cases}$? $\Rightarrow$

a. $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = \sqrt{0} = \underline{0}$

b. $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (8-2x) = 8-2\bullet 4 = 8-8 = \underline{0}$

c. $\therefore \lim\limits_{x \to 4} f(x) = 0$, and $\boxed{\text{yes, the limit exists}}$ since the right– and left–hand limits are equal.

2. Figure 5, p. 100: graph of $f(x) = \begin{cases} \sqrt{x-4} , & x > 4 \\ 8-2x , & x < 4 \end{cases}$

N. Example 10, p. 101

1. The <u>greatest integer function</u> $[\![x]\!]$ or $[x]$ represents the largest integer $\le x$.
2. This is also known as — aka — the <u>floor function</u> $\lfloor x \rfloor$ or <u>rounding down function</u>.
3. For example, $[\![4.8]\!]=4$, $[\![4]\!]=4$, $[\![\pi]\!]=3$, $[\![\sqrt{2}]\!]=1$, $\left[\!\!\left[-\frac{1}{2}\right]\!\!\right] = -1$
4. Show $\lim\limits_{x \to 3} [\![x]\!]$ DNE $\Rightarrow$
   a. Since $[\![x]\!]=3$ for $3 \le x < 4$, $\lim\limits_{x \to 3^+} [\![x]\!] = \lim\limits_{x \to 3^+} 3 = 3$.
   b. Since $[\![x]\!]=2$ for $2 \le x < 3$, $\lim\limits_{x \to 3^-} [\![x]\!] = \lim\limits_{x \to 3^-} 2 = 2$.
   c. $\therefore \lim\limits_{x \to 3} [\![x]\!]$ $\boxed{\text{DNE}}$ since the right– and left–hand limits are different, $3 \ne 2$.
5. Figure 6, p. 101: graph of $[\![x]\!]$ which is a type of step function

O. Theorem: If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as $x \to a$, then $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$.

P. Sandwich — Squeeze — Pinching Theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim\limits_{x \to a} g(x) = L$. This means all three functions have the same limit L .

1. Figure 7, p. 101: illustration of the Sandwich Theorem

Q. Example 11, p. 101

1. Show $\lim\limits_{x \to 0} x^2 \sin\frac{1}{x} = 0 \Rightarrow$ Observe that the limits may not be multiplied since $\lim\limits_{x \to 0} \sin\frac{1}{x}$ DNE
   a. Also, $-1 \le \sin\frac{1}{x} \le 1 \therefore -x^2 \le x^2 \sin\frac{1}{x} \le x^2$
   b. Since $\lim\limits_{x \to 0} x^2 = 0$ and $\lim\limits_{x \to 0} \left(-x^2\right) = 0 \Rightarrow \lim\limits_{x \to 0} x^2 \sin\frac{1}{x} = \boxed{0}$ by the Sandwich Theorem.
2. Figure 8, p. 102: illustration of the Sandwich Theorem applied to the example

VII. 2.4, Precise Definition of a Limit, p. 104

A. Concept of neighborhood

B. Delta-epsilon proof

1. Figure 1, p. 106: illustration of a delta-epsilon proof
2. Given ε , find δ for a neighborhood around a $(=3)$

VIII. 2.5, Continuity, p. 114

A. Definition: a function f is <u>continuous</u> at a if $\lim\limits_{x \to a} f(x) = f(a)$. This implies three requirements:

1. $f(a)$ is defined which means a is in the domain of f

2. $\lim\limits_{x \to a} f(x)$ exists

3. $\lim\limits_{x \to a} f(x) = f(a)$

B. Most physical phenomena are continuous — have no breaks. Otherwise, f is discontinuous at a which means f has a discontinuity at a.

C. Example 1, p. 115

1. Analyze Figure 2 for discontinuities $\Rightarrow f$ is discontinuous at $a = 1\ ,\ 3\ ,\ 5$

D. Example 2, p. 115

1. Find discontinuities of the following:

a. $f(x) = \dfrac{x^2 - x - 2}{x - 2} \Rightarrow \boxed{a = 2}$ $\qquad f(2)$ is undefined

b. $f(x) = \begin{cases} \dfrac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases} \Rightarrow \boxed{a = 0}$ $\qquad \lim\limits_{x \to 0} \dfrac{1}{x^2}$ DNE

c. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases} \Rightarrow \boxed{a = 2}$ $\qquad 1 = f(2) \neq \lim\limits_{x \to 2} f(x) = 3$

i. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x+1)}{x - 2} = \lim\limits_{x \to 2} (x+1) = 2 + 1 = \underline{3}$

d. $\lim\limits_{x \to a} f(x) = [\![x]\!] \Rightarrow \boxed{\text{discontinuities at all integers}}$ $\qquad \lim\limits_{x \to n} [\![x]\!]$ DNE , $n \in \mathbb{Z}$

E. Figure 3, p. 116: graphs of the four functions in the previous example

1. Graphical analysis of points of discontinuity

   a. Graphs a and c have point — removable — discontinuities.

   b. Graph b has an infinite discontinuity.

   c. Graph d has jump discontinuities.

   d. Infinite and jump discontinuities are non-removable.

F. Definition: A function is <u>continuous from the right</u> at a number a if $\lim_{x \to a^+} f(x) = f(a)$ and

a function is <u>continuous from the left</u> at a number a if $\lim_{x \to a^-} f(x) = f(a)$

G. Example 3, p. 116

1. At each integer n, $f(x) = [\![x]\!]$ is continuous from the right, but discontinuous from the left since

$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$ and $\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$

H. Definition: A function f is <u>continuous on an interval</u> if it is continuous at every number in the interval.

Continuity may mean right or left continuity at an endpoint of the interval.

I. Example 4, p. 117

1. Show $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1] \Rightarrow$

   a. $\lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) = 1 - \sqrt{\lim_{x \to a} \left(1 - x^2\right)} = 1 - \sqrt{1 - a^2} = f(a) \Rightarrow f$ is continuous on $(-1, 1)$

   b. $\lim_{x \to -1^+} f(x) = 1 = f(-1)$, $\lim_{x \to 1^-} f(x) = 1 = f(1)$

   c. Since f is continuous from the right and left at its endpoints, f is continuous on $[-1, 1]$

2. Figure 4, p. 117: graph of continuous semicircle

J. Theorem: If f and g are continuous at a , then so are

1. $f+g$ 2. $f-g$ 3. cf 4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

K. Theorem: The following functions are continuous on their domains:

1. Polynomials 2. Rational functions 3. Root functions 4. Trigonometric functions

5. Inverse trigonometric functions 6. Exponential functions 7. Logarithmic functions

L. Example 6, p. 120

1. Where is $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous? $\Rightarrow$

a. $x > 0$ and $x \neq \pm 1 \therefore$ continuous on $(0, 1) \cup (1, \infty)$

M. Theorem: $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right)$

N. Example 8, p. 121

1. Evaluate $\lim\limits_{x \to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right) \Rightarrow \lim\limits_{x \to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim\limits_{x \to 1} \dfrac{1-\sqrt{x}}{1-x}\right)$

$$= \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)}\right) = \arcsin\left(\lim_{x \to 1} \frac{1}{1+\sqrt{x}}\right) = \arcsin\frac{1}{2} = \boxed{\frac{\pi}{6}}$$

O. Theorem: $(f \circ g)(x)$ is continuous at a if g is continuous at a and if f is continuous at $g(a)$.

P. Example 9, p. 122

1. Find where the following functions are continuous.

a. $h(x) = \sin\left(x^2\right) \Rightarrow \boxed{\mathbb{R}}$

b. $F(x) = \ln(1+\cos x) \Rightarrow F$ has discontinuities at $\pm\pi$, $\pm 3\pi$, … when x is an odd multiple of π

$\therefore F$ is continuous on the intervals between these discontinuities

c. Figure 7, p. 122: graph of $\ln(1+\cos x)$

Q. Intermediate Value Theorem: Suppose f is continuous on $[a\ ,\ b]$, $f(a) < N < f(b)$, $\exists$ a number c in $(a\ ,\ b)$ such that $f(c) = N$.

1. Figure 8, p. 122: illustrations of the Intermediate Value Theorem — IVT, number c is and is not unique
2. Figure 9, p. 122: geometric interpretation of the IVT with a horizontal line

R. Example 10, p. 123

1. Show that a root of $4x^3 - 6x^2 + 3x - 2 = 0$ exists in $(1\ ,\ 2) \Rightarrow$ Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$.

   Observe f is continuous, $f(1) = 4 - 6 + 3 - 2 = -1 < 0$, $f(2) = 32 - 24 + 6 - 2 = 12 > 0$

   $\therefore$ By the Intermediate Value Theorem $\exists\ c$ between 1 and 2 such that $f(c) = 0$

   $\therefore$ The equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a root in the interval $(1\ ,\ 2)$. Quod Erat Demonstrandum!

2. Figures 10 and 11, p. 123: standard and zoomed in graph views showing the root of $4x^3 - 6x^2 + 3x - 2 = 0$

IX. 2.6, Limits at Infinity and Horizontal Asymptotes, p. 126

A. Definition: The limit $\lim_{x \to \infty} f(x) = L$ means values of $f(x)$ can be made arbitrarily close to L by making x sufficiently large; similar conclusion for $\lim_{x \to -\infty} f(x) = L$ meaning for large negative values of x .

1. Figure 1, p. 127: graph of $y = \dfrac{x^2 - 1}{x^2 + 1}$, with one horizontal asymptote since both limits at infinity equal 1
2. Figure 2, p. 127: three examples illustrating $\lim_{x \to \infty} f(x) = L$
3. Figure 3, p. 128: two examples illustrating $\lim_{x \to -\infty} f(x) = L$

B. Definition: The line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if

either $\lim\limits_{x \to \infty} f(x) = L$ or $\lim\limits_{x \to -\infty} f(x) = L$

C. A curve may have two different horizontal asymptotes, e.g. $y = \tan^{-1} x$

1. Figure 4, p. 128: graph of $\tan^{-1} x$

D. Example 1, p. 128

1. Analyze Figure 5, p. 128: find infinite limits, limits at infinity, and asymptotes of the given graph $\Rightarrow$

   a. $\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to 2^+} f(x) = \infty$, $\lim\limits_{x \to -1} f(x) = \infty$, $\lim\limits_{x \to \infty} f(x) = 4$, $\lim\limits_{x \to -\infty} f(x) = 2$

   b. $\therefore$ $\underline{x = -1 \text{ and } x = 2}$ are vertical asymptotes and $\underline{y = 4 \text{ and } y = 2}$ are horizontal asymptotes

E. Example 2, p. 129

1. Find these limits

   a. $\lim\limits_{x \to \infty} \dfrac{1}{x} \Rightarrow \lim\limits_{x \to \infty} \dfrac{1}{x} = \boxed{0}$ since $\dfrac{1}{\infty} \to 0^+$

   b. $\lim\limits_{x \to -\infty} \dfrac{1}{x} \Rightarrow \lim\limits_{x \to -\infty} \dfrac{1}{x} = \boxed{0}$ since $\dfrac{1}{-\infty} \to 0^-$

2. Figure 6, p. 129: graph of $y = \dfrac{1}{x}$

   a. Equilateral hyperbola

   b. $y = 0$ is a horizontal asymptote

F. Theorem

1. If $r > 0$, $r \in \mathbb{Q}$, then $\lim\limits_{x \to \infty} \dfrac{1}{x^r} = 0$.

2. If $r > 0$, $r \in \mathbb{Q}$, and x^r is defined $\forall\ x$, then $\lim\limits_{x \to -\infty} \dfrac{1}{x^r} = 0$.

G. Example 3, p. 129

1. Evaluate $\lim_{x\to\infty} \dfrac{3x^2-x-2}{5x^2+4x+1} \Rightarrow \lim_{x\to\infty} \dfrac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty} \dfrac{3-\dfrac{1}{x}-\dfrac{2}{x^2}}{5+\dfrac{4}{x}+\dfrac{1}{x^2}} = \boxed{\dfrac{3}{5}}$

2. Technique: divide numerator and denominator by the largest power in the denominator

3. Highest degree polynomial will dominate in a rational function

H. Example 4, p. 130

1. Find the horizontal and vertical asymptotes of the graph of $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5} \Rightarrow$

   a. Recall $\sqrt{x^2}=x$, $x>0$. $\lim_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \dfrac{\sqrt{2x^2+1}\bullet\dfrac{1}{\sqrt{x^2}}}{(3x-5)\bullet\dfrac{1}{x}} = \lim_{x\to\infty} \dfrac{\sqrt{2+\dfrac{1}{x^2}}}{3-\dfrac{5}{x}} = \boxed{\dfrac{\sqrt{2}}{3}}$

   b. Also, $\sqrt{x^2}=-x$, $x<0$. $\lim_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \dfrac{\sqrt{2x^2+1}\bullet\dfrac{1}{\sqrt{x^2}}}{(3x-5)\bullet\dfrac{1}{-x}} = \lim_{x\to-\infty} \dfrac{\sqrt{2+\dfrac{1}{x^2}}}{-3+\dfrac{5}{x}} = \boxed{-\dfrac{\sqrt{2}}{3}}$

   c. $\lim_{x\to\frac{5}{3}^+} \dfrac{\sqrt{2x^2+1}}{3x-5} = \dfrac{\text{pos. \#}}{0^+} = \underline{\infty}$, $\lim_{x\to\frac{5}{3}^-} \dfrac{\sqrt{2x^2+1}}{3x-5} = \dfrac{\text{pos. \#}}{0^-} = \underline{-\infty}$

   d. $\therefore$ $\boxed{y=\pm\dfrac{\sqrt{2}}{3}}$ are horizontal asymptotes and $\boxed{x=\dfrac{5}{3}}$ is a vertical asymptote.

2. Figure 8, p. 131: graph of $y=\dfrac{\sqrt{2x^2+1}}{3x-5}$

3. Observe that the end behavior of $\dfrac{\sqrt{2x^2+1}}{3x-5}$ becomes $\dfrac{\sqrt{2x^2}}{3x} = \dfrac{x\sqrt{2}}{3x} = \dfrac{\sqrt{2}}{3}$

4. Observe that constants have negligible value when compared to unbounded values

I. Example 5, p. 131

1. Compute $\lim_{x \to \infty} \left(\sqrt{x^2+1}-x\right) \Rightarrow \lim_{x \to \infty} \left(\sqrt{x^2+1}-x\right) \bullet \dfrac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}\;+x\right)} = \lim_{x \to \infty} \dfrac{x^2+1-x^2}{\sqrt{x^2+1}\;+x}$

$= \lim_{x \to \infty} \dfrac{1}{\sqrt{x^2+1}\;+x} = \dfrac{1}{\infty} = \boxed{0}$

2. Figure 9, p. 131: graph of $y = \sqrt{x^2+1}-x$

3. Observe that the end behavior of $\sqrt{x^2+1}-x$ becomes $\sqrt{x^2}-x = x-x = 0$ for $x > 0$

J. Figure 10, p. 132: graph of $y = e^x$

1. $\lim_{x \to -\infty} e^x = 0$

K. Example 7, p. 132

1. Evaluate $\lim_{x \to 0^-} e^{\frac{1}{x}} \Rightarrow$ Let $t = \dfrac{1}{x}$, then $t \to -\infty$ as $x \to 0^-$. $\therefore \lim_{x \to 0^-} e^{\frac{1}{x}} = \lim_{t \to -\infty} e^t = \boxed{0}$

L. Example 8, p. 132

1. Evaluate $\lim_{x \to \infty} \sin x \Rightarrow \sin x$ oscillates between -1 and 1 infinitely often; $\therefore \lim_{x \to \infty} \sin x$ $\boxed{\text{DNE}}$

M. Infinite limits at infinity

N. Example 9, p. 133
1. Find these limits

a. $\lim_{x \to \infty} x^3 \Rightarrow \boxed{\infty}$ b. $\lim_{x \to -\infty} x^3 \Rightarrow \boxed{-\infty}$

b. Figure 11, p. 133: graph of $y = x^3$

O. Figure 12, p. 133: comparison of graphs shows e^x grows faster than x^3

1. Observe that exponential functions grow much faster than, or “dominate,” polynomials.

P. Example 10, p. 133

1. Find $\lim\limits_{x \to \infty} \left(x^2 - x\right) \Rightarrow \lim\limits_{x \to \infty} x(x-1) = \boxed{\infty}$

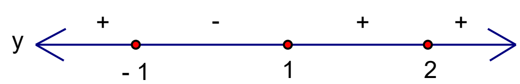
Q. Example 11, p. 133

1. Find $\lim\limits_{x \to \infty} \dfrac{x^2 + x}{3 - x} \Rightarrow \lim\limits_{x \to \infty} \dfrac{x+1}{\frac{3}{x} - 1} = \dfrac{\infty}{-1} = \boxed{-\infty}$

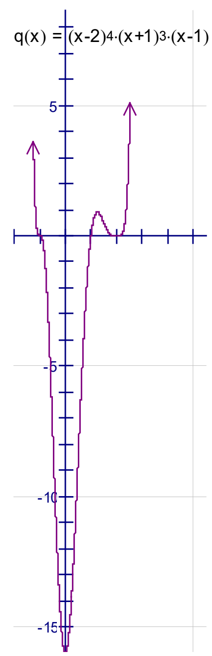
R. Example 12, p. 134

1. Sketch $y = (x-2)^4 (x+1)^3 (x-1)$ using intercepts and limits $\Rightarrow$

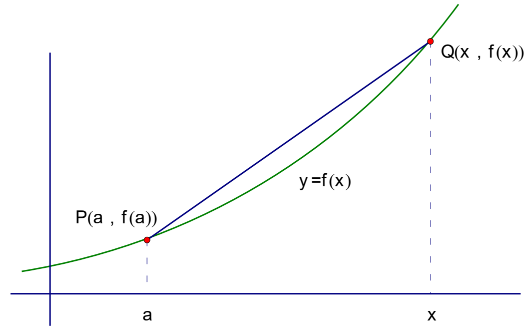
2. $\lim\limits_{x \to \pm\infty} y = \boxed{\infty}$; $y-$intercept: $(0\ ,\ -16)$



3. Figure 13, p. 134: graph of $y = (x-2)^4 (x+1)^3 (x-1)$



X. 2.7, Tangents, Velocities, and Other Rates of Change, p. 140



A. Tangents

1. $m_{\overline{PQ}} = \dfrac{f(x)-f(a)}{x-a}$

2. Definition: A tangent line to the curve $y=f(x)$ at the point

$P(a\ ,\ f(a))$ has slope $\boxed{m = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}}$ if it exists

3. Figure 1, p. 141: shows secant slope approaching tangent slope

B. Example 1, p. 141

1. Find the equation of the tangent line to $y=x^2$ at $P(1\ ,\ 1) \Rightarrow$

2. $a=1$ so $m = \lim\limits_{x \to 1} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x \to 1} \dfrac{(x+1)(x-1)}{x-1} = \lim\limits_{x \to 1} (x+1) = 2$

$\therefore$ The equation of the tangent line is $\boxed{y-1=2(x-1)}$ <u>OR</u> $y=2x-1$.

3. Recall point-slope form of a line: $\underline{y-y_1=m(x-x_1)}$

C. Figure 2, p. 141: when zooming in, a curve becomes almost indistinguishable from its tangent line

1. Slope of the curve means slope of the tangent line.

D. Alternative definition for calculating slope of a tangent line

1. Figure 3, p. 142: illustration of alternative slope calculation

a. Let $x=a+h\ ,\ h=x-a \Rightarrow m_{\overline{PQ}} = \dfrac{f(x)-f(a)}{x-a} = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$

$\therefore \boxed{m_{\text{tangent}} = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}}$

E. Example 2, p. 142

1. Find the equation of the tangent line to the hyperbola $y = \dfrac{3}{x}$ at the point $(3\ ,\ 1) \Rightarrow$

2. $$m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3+h}-1}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3+h}-\dfrac{3+h}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{3-3-h}{(3+h)h} = \lim_{h \to 0} \frac{-h}{(3+h)h} = \lim_{h \to 0} \frac{-1}{3+h} = \boxed{-\frac{1}{3}}$$

$\therefore$ The equation of the tangent line is $\boxed{y-1 = -\frac{1}{3}(x-3)}$ <u>OR</u> $y = -\frac{1}{3}x+2$ <u>OR</u> $x+3y=6$

3. Figure 4, p. 142: the answer for the tangent line equation appears reasonable.

F. Extra example #1

1. Find the slopes of the tangent lines to the graph of $f(x)=\sqrt{x}$ at $(1\ ,\ 1)$, $(4\ ,\ 2)$, and $(9\ ,\ 3) \Rightarrow$

2. Find slope at the general point $\left(a\ ,\ \sqrt{a}\right)$ to be efficient.

3. $$m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h \to 0} \frac{\left(\sqrt{a+h}-\sqrt{a}\right)}{h} \cdot \frac{\left(\sqrt{a+h}+\sqrt{a}\right)}{\left(\sqrt{a+h}+\sqrt{a}\right)}$$

$$= \lim_{h \to 0} \frac{a+h-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{a+h}+\sqrt{a}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$$

4. $(1\ ,\ 1) \to a=1 \to \boxed{m=\frac{1}{2}}$; $(4\ ,\ 2) \to a=4 \to \boxed{m=\frac{1}{4}}$; $(9\ ,\ 3) \to a=9 \to \boxed{m=\frac{1}{6}}$

G. Velocities

1. The position function of an object describes motion.

    a. Figure 5, p. 142: illustration of change in position during a time interval

    b. Figure 6, p. 143: slope of secant line representing average velocity

2. Average velocity: $\dfrac{f(a+h)-f(a)}{h} = \dfrac{\text{displacement}}{\text{time}}$

3. Instantaneous velocity: $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h} = v(a)$

H. Example 3, p. 143

1. Ball is dropped $450\ m$ above ground from the CN Tower in Toronto.

    a. Find the velocity of the ball after $5\ sec$. $\Rightarrow$

        i. Recall: $s = f(t) = 4.9t^2$; $v(a) = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h \to 0} \dfrac{4.9(a+h)^2 - 4.9a^2}{h}$

        $= \lim\limits_{h \to 0} \dfrac{4.9a^2 + 9.8ah + 4.9h^2 - 4.9a^2}{h} = \lim\limits_{h \to 0} \dfrac{9.8ah + 4.9h^2}{h} = \lim\limits_{h \to 0} (9.8a + 4.9h) = \underline{9.8a}$

        ii. $\therefore$ Velocity after $5\ sec$ is $v(5) = 9.8(5) = \boxed{49\ {}^{m}\!/\!_{sec}}$

    b. How fast is the ball traveling when it strikes the ground?

        i. $s(t) = 450$, $4.9t^2 = 450$, $t^2 = \dfrac{450}{4.9}$, $t^2 \approx 91.83673469$, $t \approx 9.583148$ $(t > 0)$,

        $v(9.583148) \approx 9.8 \bullet 9.58 \approx \boxed{94\ {}^{m}\!/\!_{sec}}$

I. Rates of change: for $y = f(x)$

1. The change — increment — in x: $\Delta x = x_2 - x_1$

2. The change — increment — in y: $\Delta y = f(x_2) - f(x_1) = y_2 - y_1$

3. $\dfrac{\Delta y}{\Delta x}$ is the average rate of change of y with respect to — wrt — x

4. The instantaneous rate of change $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

J. Extra example #2
1. Suppose time and temperature data was recorded in Whitefish, Montana: temperature T and time x
   a. Find the average rate of change of temperature wrt time
      i. From noon to 3 PM
         A. $\dfrac{\Delta T}{\Delta x} = \dfrac{18.2 - 14.3}{15 - 12} = \dfrac{3.9}{3} = \boxed{1.3\ {}^{\circ}C/hr}$
      ii. From noon to 2 PM
         A. $\dfrac{\Delta T}{\Delta x} = \dfrac{17.3 - 14.3}{14 - 12} = \dfrac{3}{2} = \boxed{1.5\ {}^{\circ}C/hr}$
      iii. From noon to 1 PM
         A. $\dfrac{\Delta T}{\Delta x} = \dfrac{16.0 - 14.3}{13 - 12} = \dfrac{1.7}{1} = \boxed{1.7\ {}^{\circ}C/hr}$
   b. Estimate the instantaneous rate of change in temperature at noon
      i. Use a graph and the slope of the tangent line to estimate the instantaneous rate of change
      ii. For instance, $\dfrac{\text{rise}}{\text{run}} = \dfrac{10.3}{5.5} \approx \boxed{1.9\ {}^{\circ}C/hr}$

K. May interpret rates of change as slopes of tangent lines.