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Mixtures and Allegations

by Rishabh Bafna

CSA102 Mathematics-1

Mean price refers to the average price of a mixture when two or more ingredients with different prices are combined. It represents the overall cost per unit of the mixture.

In the context of **Mixture and Allegation** problems, the mean price is the price at which you want to mix different quantities. It lies between the prices of the cheaper and dearer items.

Formula for Mean Price:

If you mix two items, one costing C_1 per unit and another costing C_2 per unit, the mean price is typically the price you want for the mixture.

- **Cheaper Quantity**: The quantity of the item with a lower price.
- **Dearer Quantity**: The quantity of the item with a higher price.

To calculate the mean price in these problems, we use the following formula:

$$\frac{\text{Cheaper Price}}{\text{Dearer Price}} = \frac{\text{Dearer Price - Mean Price}}{\text{Mean Price - Cheaper Price}}$$

Where:

- **Mean Price** is the target price of the mixture.

Explanation of Terms:

- **Cheaper Price**: Cost per unit of the cheaper item (let's call this C_1).
- **Dearer Price**: Cost per unit of the more expensive (dearer) item (let's call this C_2).
- **Mean Price**: The average cost per unit of the mixture (let's call this M).
- **Quantity of Cheaper**: The amount of the cheaper item in the mixture (let's call this Q_1).
- **Quantity of Dearer**: The amount of the dearer item in the mixture (let's call this Q_2).

Proof of the Allegation Formula

1. **Total Cost of the Mixture**:

   - The total cost of the mixture is the combined cost of the cheaper and dearer items.
   - Cost of the cheaper item = $Q_1 \times C_1$
   - Cost of the dearer item = $Q_2 \times C_2$

   So, the total cost is:

   $$\text{Total Cost} = (Q_1 \times C_1) + (Q_2 \times C_2)$$

2. **Mean Price of the Mixture**: The mean price (M) is the price per unit of the mixture, which can be found by dividing the total cost by the total quantity of the mixture:

   $$M = \frac{\text{Total Cost}}{\text{Total Quantity}}$$

   So:

   $$M = \frac{(Q_1 \times C_1) + (Q_2 \times C_2)}{Q_1 + Q_2}$$

3. **Rearranging the Equation**: Now, rearrange the equation to isolate the quantities on one side.

   First, cross-multiply to eliminate the denominator:

   $$M \times (Q_1 + Q_2) = (Q_1 \times C_1) + (Q_2 \times C_2)$$

   Expanding the left side:

   $$M \times Q_1 + M \times Q_2 = Q_1 \times C_1 + Q_2 \times C_2$$

4. **Isolate Terms**: Now, move the terms involving Q_1 and Q_2 to one side:

   $$Q_2 \times C_2 - Q_2 \times M = Q_1 \times M - Q_1 \times C_1$$

   Factor the terms involving Q_1 and Q_2:

   $$Q_2 \times (C_2 - M) = Q_1 \times (M - C_1)$$

5. **Solve for the Ratio**: Finally, divide both sides by Q_1 and Q_2 to isolate the ratio of the quantities:

   $$\frac{Q_1}{Q_2} = \frac{C_2 - M}{M - C_1}$$

Conclusion:

This gives us the desired allegation formula:

$$\frac{\text{Quantity of Cheaper}}{\text{Quantity of Dearer}} = \frac{\text{Dearer Price} - \text{Mean Price}}{\text{Mean Price} - \text{Cheaper Price}}$$

This formula helps calculate the ratio in which two items with different prices (or concentrations) should be mixed to achieve a desired mean price or concentration.

Problem: A merchant has two varieties of rice, one costing ₹60/kg and the other ₹80/kg. In what ratio should he mix the two varieties to get a mixture worth ₹70/kg?

60 80

70

$80 - 70 = \underline{\underline{10}}$ $70 - 60 = \underline{\underline{10}}$

$\Rightarrow \frac{10}{10} = \frac{1}{1}$

₹60/kg rice

₹80/kg rice

$\rightarrow \underline{\underline{1 : 1}}$ ratio

Problem: A merchant has two varieties of rice, one costing ₹60/kg and the other ₹80/kg. In what ratio should he mix the two varieties to get a mixture worth ₹70/kg?

Solution:

- Cheaper Price = ₹60
- Dearer Price = ₹80
- Mean Price = ₹70

Using the allegation formula:

$$\frac{\text{Cheaper Quantity}}{\text{Dearer Quantity}} = \frac{80 - 70}{70 - 60} = \frac{10}{10} = 1 : 1$$

Thus, the two varieties of rice should be mixed in the ratio 1:1.

Problem: A chemist has two solutions of acid, one containing 30% acid and the other containing 70% acid. How much of each should be mixed to obtain 40 liters of a solution that contains 50% acid?

30 70

50

$70-50=\underline{\underline{20}}$ $50-30=\underline{\underline{20}}$ $\Rightarrow \frac{20}{20} = \frac{1}{1}$

20L solution containing 30% acid + 20L solution containing 70% acid = 40L solution containing 50% acid

$\therefore \frac{1}{2} \times 40 = 20\,L$

Problem: A chemist has two solutions of acid, one containing 30% acid and the other containing 70% acid. How much of each should be mixed to obtain 40 liters of a solution that contains 50% acid?

Solution:

- Solution A (Cheaper) = 30%
- Solution B (Dearer) = 70%
- Desired concentration = 50%

Using the allegation formula:

$$\frac{\text{Cheaper Solution}}{\text{Dearer Solution}} = \frac{70-50}{50-30} = \frac{20}{20} = 1:1$$

Thus, the two solutions should be mixed in the ratio 1:1.

Total mixture = 40 liters. Therefore, each solution should be 20 liters.

Same question ; Take mean as solution containing 60% acid.

30   70

60

$70 - 60 = \underline{\underline{10}}$ $60 - 30 = \underline{\underline{30}}$

$$\Rightarrow \frac{10}{30} = \frac{1}{3}$$

Cheaper quantity

Dearer quantity

Note: 60 is closer to 70 as compared to 30.
So, 70% acid solution is more than 30% acid solution in the mixture.

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Problem: A 40-liter solution contains 25% salt. If 8 liters of the solution are replaced with pure water, what will be the new concentration of salt in the solution?

40L solution: 30L Water(W), 10L Salt (S)

$-8L$

$\frac{30}{40}\times 8 = 6L$ water ↓

$\frac{10}{40}\times 8 = 2L$ salt ↓

24L W, 8L S $\xrightarrow{+8L\,W}$ 32L W, 8L S

Concentration of salt in the new mixture $= \frac{8}{(32+8)} \times 100\% = \underline{20\%}$ ✓

Problem: A 40-liter solution contains 25% salt. If 8 liters of the solution are replaced with pure water, what will be the new concentration of salt in the solution?

Solution: Initial amount of salt = $40 \times \frac{25}{100} = 10$ liters. Amount of salt removed = $8 \times \frac{25}{100} = 2$ liters. After removal, the salt left = $10 - 2 = 8$ liters.

Total volume after replacement = 40 liters (since the solution is replaced with water).

New concentration = $\frac{8}{40} \times 100 = 20\%$.

Example: Suppose a 40-liter solution contains 25% salt. If 8 liters of the solution are replaced by water and this process is repeated twice, find the final concentration of salt.

40L solution: 30L Water(W), 10L Salt(S)

$-8L$

$\frac{30}{40} \times 8 = 6L$ water ↓

$\frac{10}{40} \times 8 = 2L$ salt ↓

24L W, 8L S $\xrightarrow{+8L\ W}$ 32L W, 8L S

32L	W
8L	S

$\xrightarrow{-8L}$

25.6L	W
6.4L	S

$\xrightarrow{+8L\,W}$

33.6L	W
6.4L	S

$$\frac{32}{(32+8)} * 8 \qquad \frac{8}{(32+8)} * 8$$

$$= \frac{32}{40} \times 8 \qquad = \frac{8}{40} \times 8$$

$$= \underline{\underline{6.4L}}\ W\downarrow \qquad = \underline{\underline{1.6L}}\ S\downarrow$$

Final concentration of salt

$$= \frac{6.4}{(33.6+6.4)} * 100 = \frac{6.4}{40} * 100$$

$$= \underline{\underline{16\%}}$$

Example: Suppose a 40-liter solution contains 25% salt. If 8 liters of the solution are replaced by water and this process is repeated twice, find the final concentration of salt.

Step-by-Step Solution:

- **Step 1: Calculate Initial Quantity of Salt**
  The initial quantity of salt = $40 \times \frac{25}{100} = 10$ liters.
- **Step 2: Apply the Replacement Process Once**
  - After removing 8 liters of the solution, the remaining volume of the solution is $40 - 8 = 32$ liters.
  - The quantity of salt removed with the 8 liters = $8 \times \frac{25}{100} = 2$ liters.
  - The remaining salt after the first replacement = $10 - 2 = 8$ liters.
- **Step 3: Recalculate the New Concentration of Salt**
  After replacing the 8 liters with pure water, the total volume of the solution is restored to 40 liters. Therefore, the new concentration of salt is:

$$\frac{8}{40} \times 100 = 20\%$$

- **Step 4: Apply the Replacement Process Again**
  - For the second replacement, we again remove 8 liters. The amount of salt removed is $8 \times \frac{20}{100} = 1.6$ liters.
  - The remaining salt after the second replacement = $8 - 1.6 = 6.4$ liters.
- **Step 5: Final Concentration** After the second replacement, the final concentration of salt in the solution is:

$$\frac{6.4}{40} \times 100 = 16\%$$

Conclusion: After two replacements, the concentration of salt in the solution has reduced to 16%.

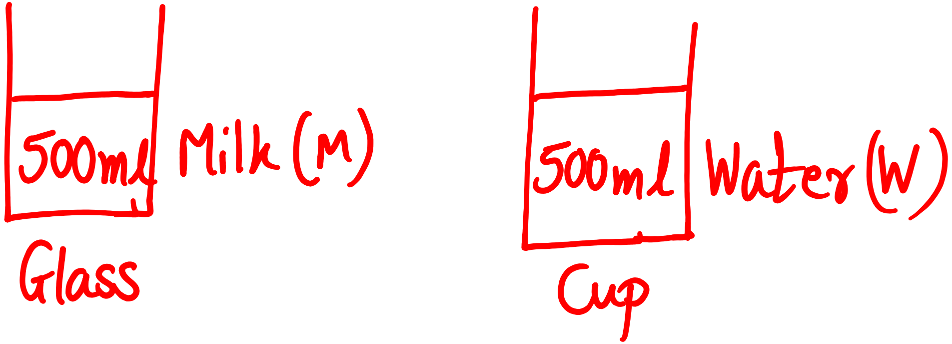
Q. 500ml milk in glass

500ml water in cup

150ml milk from glass is poured into the cup

Later, 150ml of mixture is poured from cup to the glass

What is the ratio of water quantity in glass to the milk quantity in cup?



150ml poured from glass to cup

Glass: 350ml M

Cup: 150ml M, 500ml W

150ml removed from cup

$-150ml$

$$\frac{150}{650} \times 150 = 34.6ml \text{ M} \downarrow$$

$$\frac{500}{650} \times 150 = 115.4ml \text{ W} \downarrow$$

Glass: 350ml M

Cup: 115.4 ml M, 384.6 ml W

+150ml

150ml removed from cup is added to the glass

Glass: 115.4ml W, 384.6ml M

Cup: 115.4ml M, 384.6ml W

Finally, $\frac{\text{Water quantity in glass}}{\text{Milk quantity in cup}} = \frac{115.4ml}{115.4ml} = \frac{1}{1}$

$\rightarrow$ 1:1

Problem:

A 100-liter mixture contains 40% alcohol. If 20 liters of the mixture is replaced with pure water, what will be the new percentage of alcohol in the mixture?

100L

60L	Water (W)
40L	Alcohol (A)

$\xrightarrow{-20L}$

$\frac{6\cancel{0}}{1\cancel{0}\cancel{0}} \times 2\cancel{0} = 12L$ W $\downarrow$

$\frac{4\cancel{0}}{1\cancel{0}\cancel{0}} \times 2\cancel{0} = 8L$ A $\downarrow$

48L	W
32L	A

$\xrightarrow{+20L\ W}$

68L	W
32L	A

$$\frac{32}{(68+32)} * 100 = \frac{32}{100} * 100$$

$$= 32\%$$ ✓

Problem:

A 100-liter mixture contains 40% alcohol. If 20 liters of the mixture is replaced with pure water, what will be the new percentage of alcohol in the mixture?

Solution:

- **Step 1: Calculate the Initial Quantity of Alcohol** Initial alcohol in the mixture = $100 \times \frac{40}{100} = 40$ liters.
- **Step 2: Calculate the Alcohol Removed** When 20 liters of the mixture is removed, the amount of alcohol removed = $20 \times \frac{40}{100} = 8$ liters. Remaining alcohol = $40 - 8 = 32$ liters.
- **Step 3: Calculate the New Concentration** After replacing the 20 liters with water, the total volume is restored to 100 liters. Thus, the new concentration of alcohol is:

$$\frac{32}{100} \times 100 = 32\%$$

Thus, after the replacement, the concentration of alcohol in the mixture is 32%.

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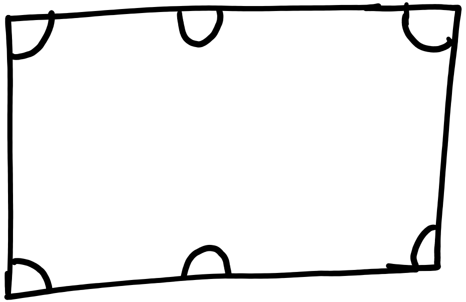
Billiard Problems

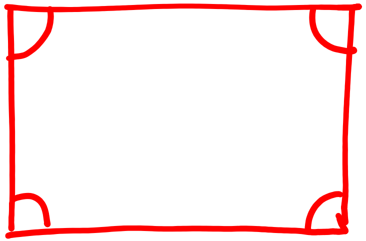
by Rishabh Bafna

CSA102 Mathematics-1



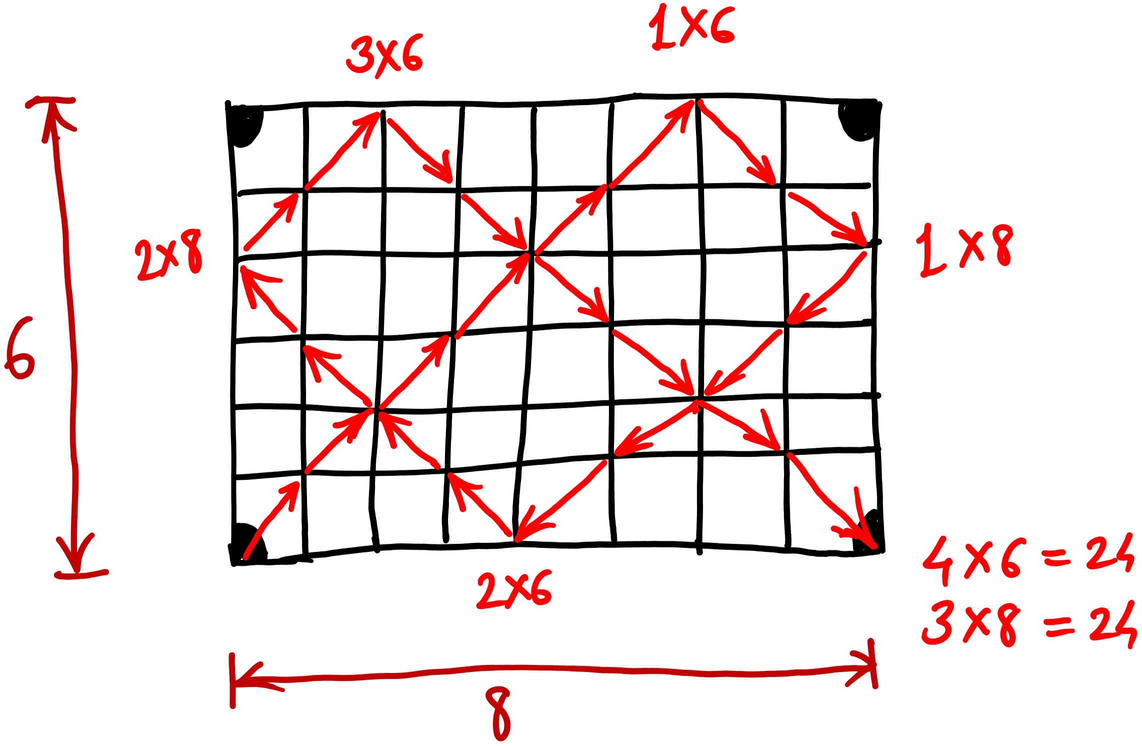
Actual Billiard Table

6 pockets (holes)



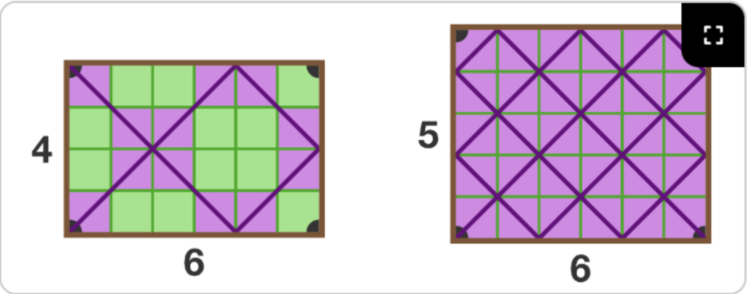
Modified Billiard Table

4 pockets (holes)

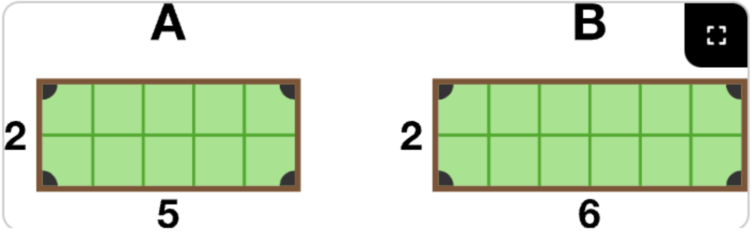


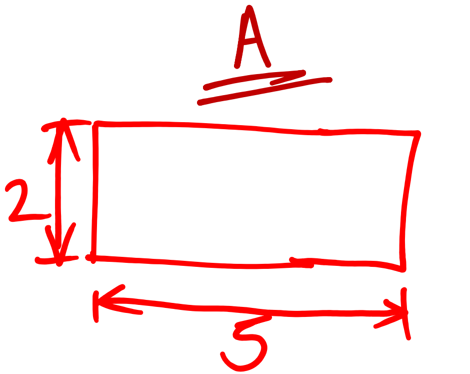
Starting from the bottom left corner, the ball will pass through LCM(6,8) number of squares before reaching the pocket.

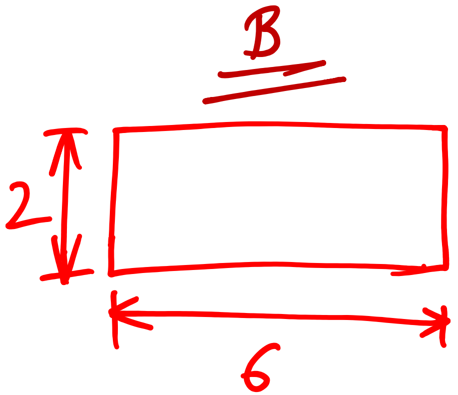
For example, on the 4×6 table, the ball passes through 12 squares, while on the 5×6 table it passes through all 30:



Now find out on which of these tables will the ball pass through more squares.







$LCM(2,5) = \underline{\underline{10}}$

$\Downarrow$

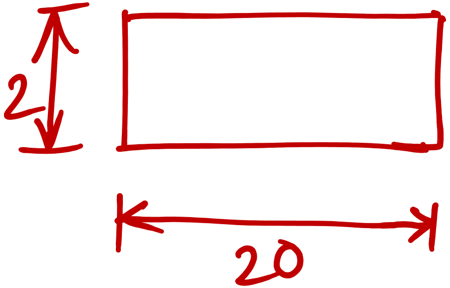
Ball will pass through 10 squares.

$LCM(2,6) = \underline{\underline{6}}$

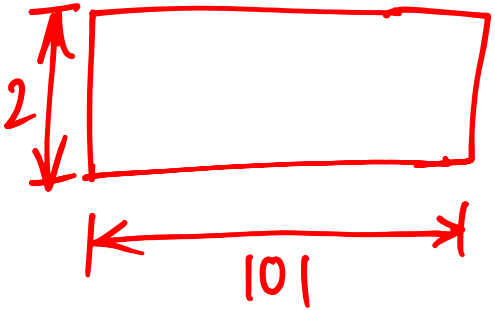
$\Downarrow$

Ball will pass through 6 squares.

Q. How many squares will the ball pass through on a 2×20 table?



$\rightarrow$ $LCM(2, 20) = \underline{\underline{20}}$



$\rightarrow$ $\text{LCM}(2, 101) = \underline{\underline{202}}$

So, the ball will pass through 202 squares.

6, 8, 12, 16, 18

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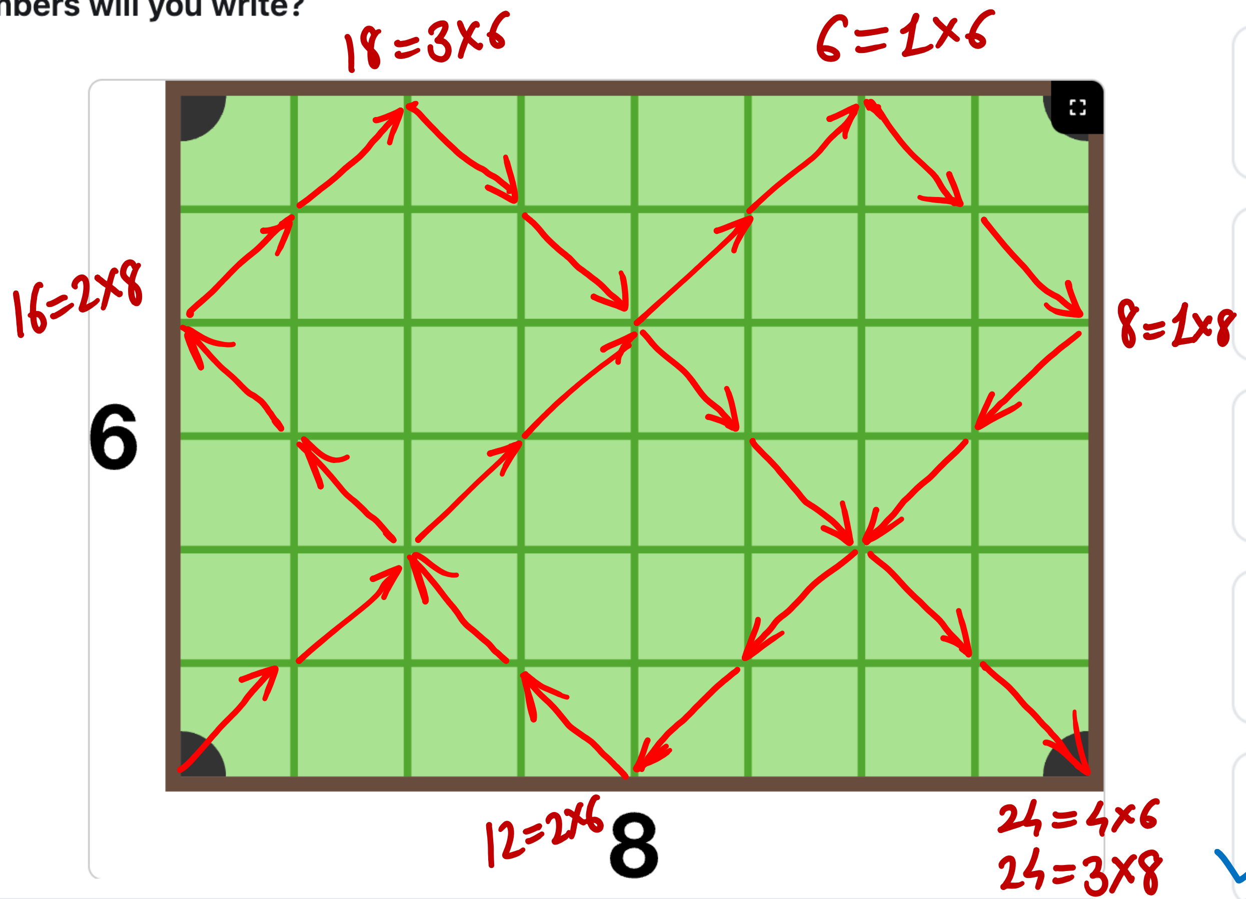
Imagine you shoot a billiard ball at a 45° angle from the bottom-left corner of this 6×8 table, and you write down the number of squares the ball has passed through each time it reaches one of the sides. Which of these numbers will you write?



A 7

B 4

C 15

D 20

E 16

Neither multiple of 6 nor multiple of 8

Multiple of 8

For this 12×21 table, we will eventually write the number 63 when the ball reaches the ____________.

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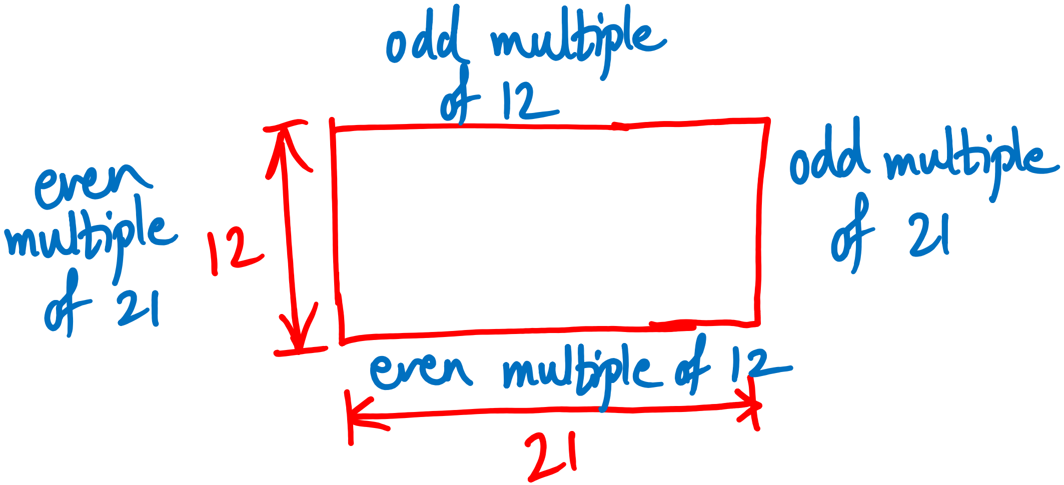


A Top

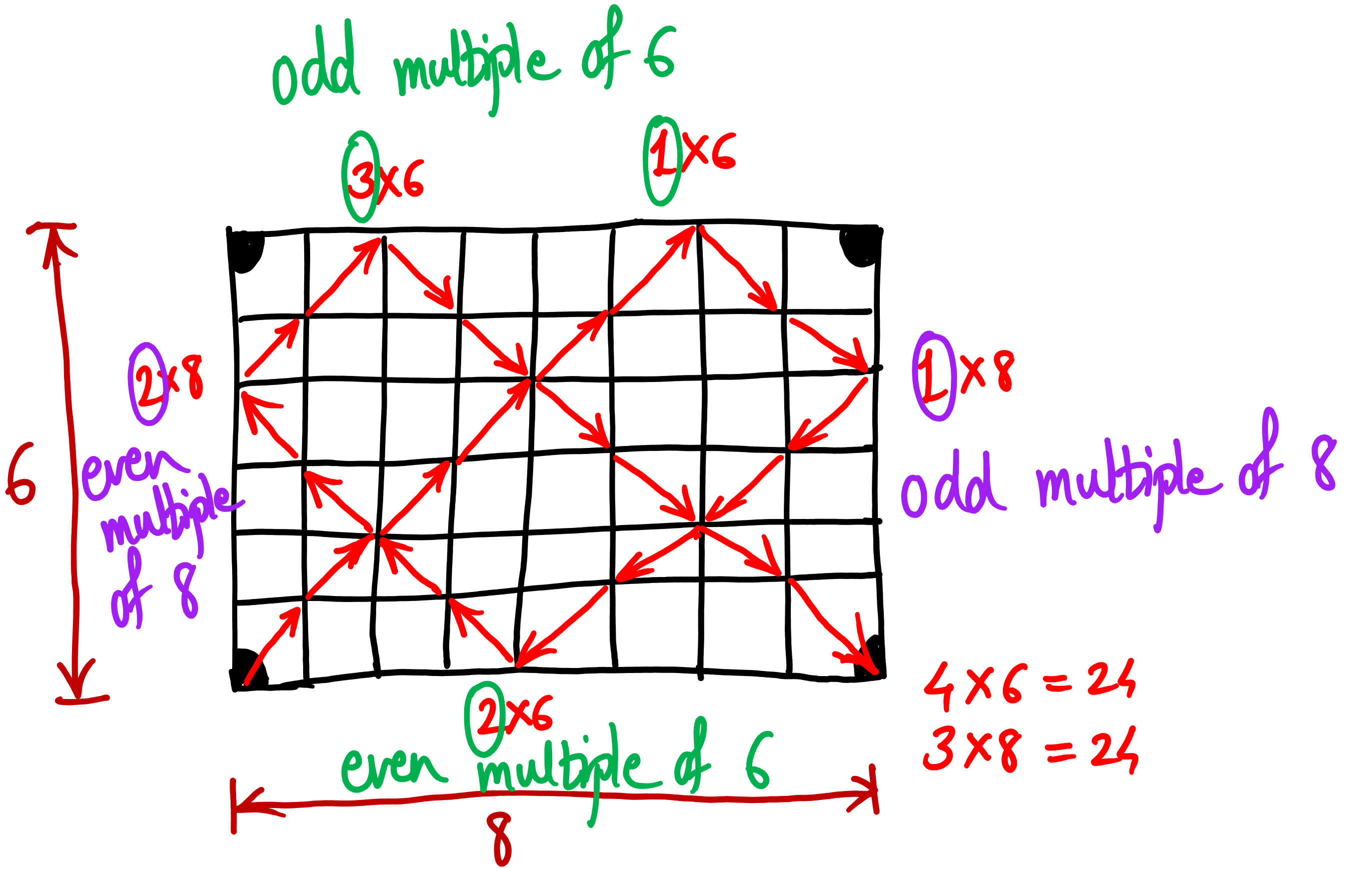
B Bottom

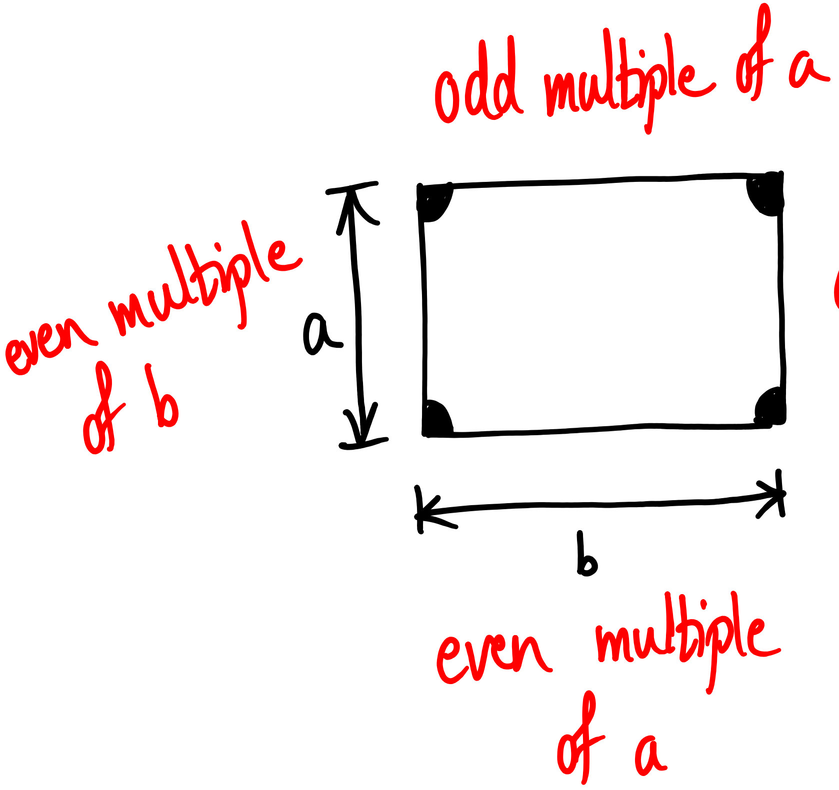
C Right Side

D Left Side

63 is odd multiple of 21.

$3 \times 21 = 63$


odd multiple of 6
3x6
1x6
2x8
6
even multiple of 8
1x8
odd multiple of 8
4x6 = 24
3x8 = 24
2x6
even multiple of 6
8



Logic: If the ball covers the number of squares equivalent to the odd multiple of b, then the ball reaches right side of the table.

How many squares will the ball pass through on a 36×45 table?

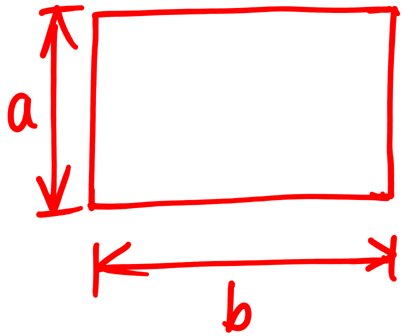
A $2^2 \times 3^2 \times 5^1 = 180$

B $2^3 \times 3^2 \times 5^1 = 360$

C $2^1 \times 3^4 \times 5^1 = 810$

D $2^2 \times 3^4 \times 5^1 = 1620$

$LCM(36, 45) = 180$



→ Ball covers LCM (a,b) number of squares.

→ Total number of squares = a * b

→ If a ball pass through every grid square, then the following equation must be satisfied:

Total number of squares = Number of squares covered

$$\therefore \boxed{a * b = LCM(a,b)}$$

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Total number of squares = Number of squares covered

$$\therefore \quad a * b = LCM(a,b)$$

$$\therefore \quad \cancel{LCM(a,b)} * GCD(a,b) = \cancel{LCM(a,b)}$$

$$\therefore \quad \boxed{GCD(a,b) = 1}$$

If the ball pass through every grid square in an a x b size table, then 'a' & 'b' must be relatively prime.

For which of these tables will the ball pass through every grid square?

A. 15 x 24 → $GCD(15,24) = 3$ ✗

B. 16 x 25 → $GCD(16,25) = 1$ ✓

C. 17 x 26 → $GCD(17,26) = 1$ ✓

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A Only A

B Only B

C Only C

D Both A and B

E Both B and C ✓

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What fraction of the squares will the ball pass through on an a×b table?

A $\dfrac{\mathrm{lcm(a,\ b)}}{\mathrm{gcd(a,\ b)}}$

B $\dfrac{\mathrm{gcd(a,\ b)}}{\mathrm{lcm(a,\ b)}}$

C $\dfrac{1}{\mathrm{gcd(a,\ b)}}$

D $\dfrac{1}{\mathrm{lcm(a,\ b)}}$

→ Ball will pass through LCM (a,b) number of squares out of total a * b number of squares on an a × b sized table.

$$\frac{LCM(a,b)}{a*b} = \frac{LCM(a,b)}{LCM(a,b)*GCD(a,b)} = \frac{1}{GCD(a,b)}$$

A ball will pass through every square on an $a \times b$ table if and only if a and b are relatively prime.

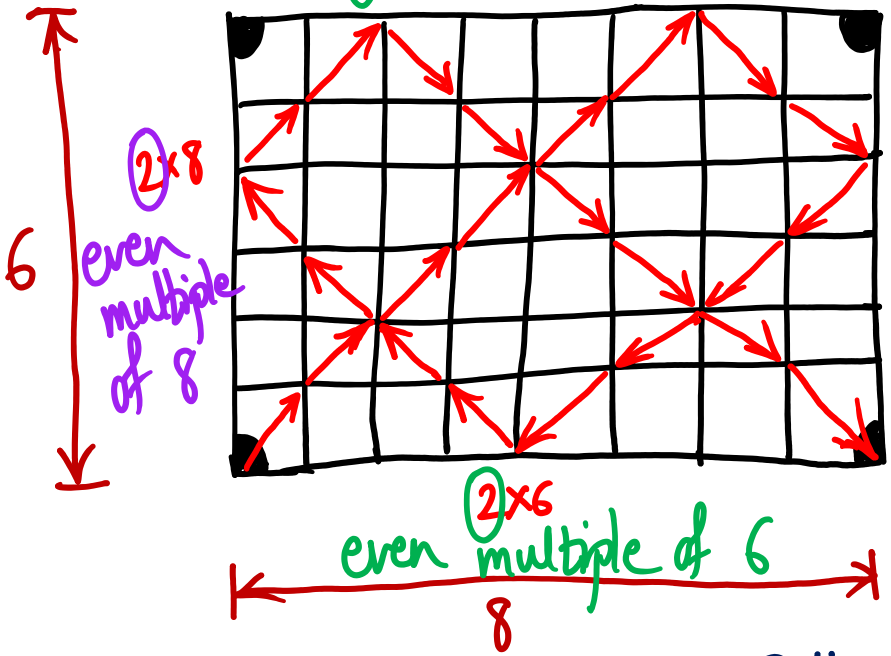
A True

B False

Ball will pass through every square if $\underbrace{GCD(a,b) = 1}_{\text{'a' \& 'b' are relatively prime.}}$

odd multiple of 6

3x6 1x6



1x8
odd multiple of 8

$4 \times 6 = 24$

$3 \times 8 = 24$

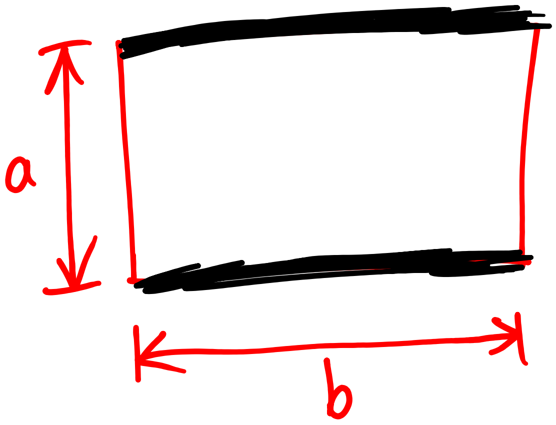
Ball hits Top or Bottom surface 3 times &

$4 = 3+1$

Ball hits Left or Right surface 2 times

& $3 = 2+1$

Top-Bottom



Consider that the ball hits Top or Bottom 'x' times.

$$(x+1) * a = LCM(a,b)$$

$$\therefore \boxed{x = \frac{LCM(a,b)}{a} - 1}$$

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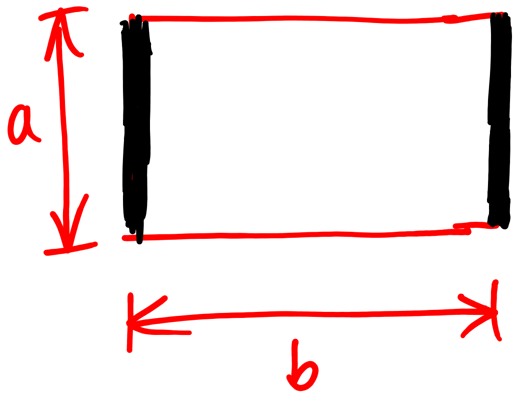
Left-Right

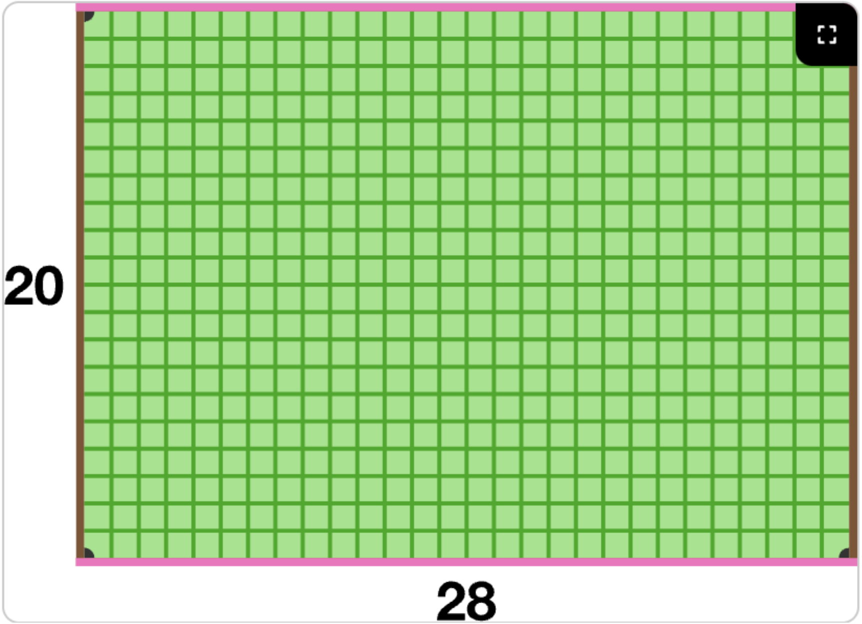


Consider that the ball hits Left or Right 'y' times.

$$(y+1) * b = LCM(a,b)$$

$$\therefore \boxed{y = \frac{LCM(a,b)}{b} - 1}$$

How many times will the ball bounce off the **top** or **bottom** of a 20×28 table?

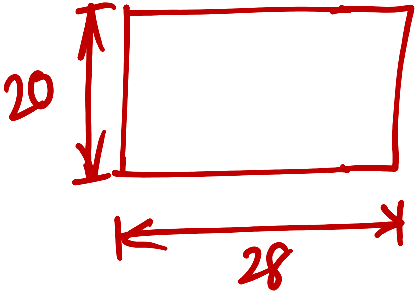


A 4

B 6

C 9

D 13



Top-Bottom

$$x = \frac{LCM(a,b)}{a} - 1$$

$$= \frac{LCM(20,28)}{20} - 1$$

$$= \frac{140}{20} - 1$$

$$= 7 - 1$$

$$= \underline{\underline{6}}$$

How many times will the ball bounce off the sides of a 39×45 table before it lands in a pocket?

NOTE: We will always have the ball start from the bottom-left corner.

- A 26
- B 39
- C 45
- D 82

Top-Bottom

$$x = \frac{LCM(39,45)}{39} - 1 = \frac{585}{39} - 1 = 15 - 1 = 14$$

Left-Right

$$y = \frac{LCM(39,45)}{45} - 1 = \frac{585}{45} - 1 = 13 - 1 = 12$$

$$\rightarrow x + y = 14 + 12 = 26$$

How many times will the ball bounce off the sides of a 100×101 table before it lands in a pocket

A	99
B	101
C	199
D	201

Top-Bottom

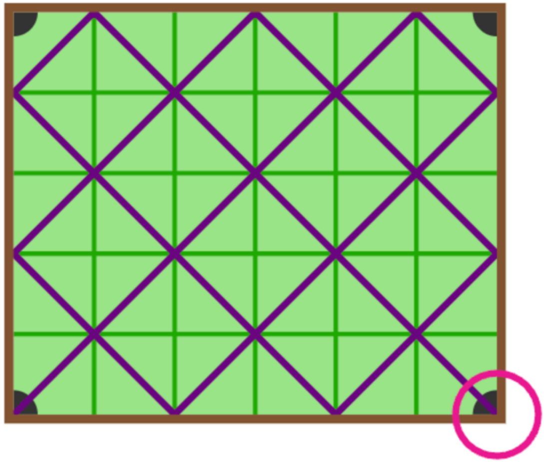
$$x = \frac{LCM(100,101)}{100} - 1 = \frac{10100}{100} - 1 = 101 - 1 = \underline{\underline{100}}$$

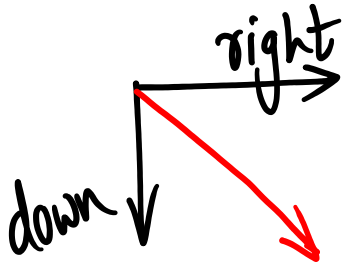
Left-Right

$$y = \frac{LCM(100,101)}{101} - 1 = \frac{10100}{101} - 1 = 100 - 1 = \underline{\underline{99}}$$

$\rightarrow x + y = 100 + 99 = \underline{\underline{199}}$

On a 5×6 table, the ball lands in the **bottom-right** pocket:





If a ball lands in the **bottom-right** pocket, what direction must it be going immediately **before** it reaches the pocket?

A Up and to the right

B Up and to the left

C Down and to the right

D Down and to the left

A ball on an $a \times b$ table has bounced off the sides of the table in this order:

Top, Right, Bottom, Top,

Left, Bottom, Right, Top

Which direction is the ball currently going? Which direction is the ball currently going?

After hitting top side, the ball must go in downward direction.

After hitting right side, the ball must go in left direction.

- A Up and to the right
- B Up and to the left
- C Down and to the right
- D Down and to the left
- E It's not possible to be certain without additional information.

On an a×b table, suppose a ball has bounced

- off the **top** or **bottom** of the table 88 times and
- off the **right** or **left** side of the table 55 times.

As a reminder, The ball **starts** going **up** and to the **right**. Find out the direction of the ball after the given bounces?

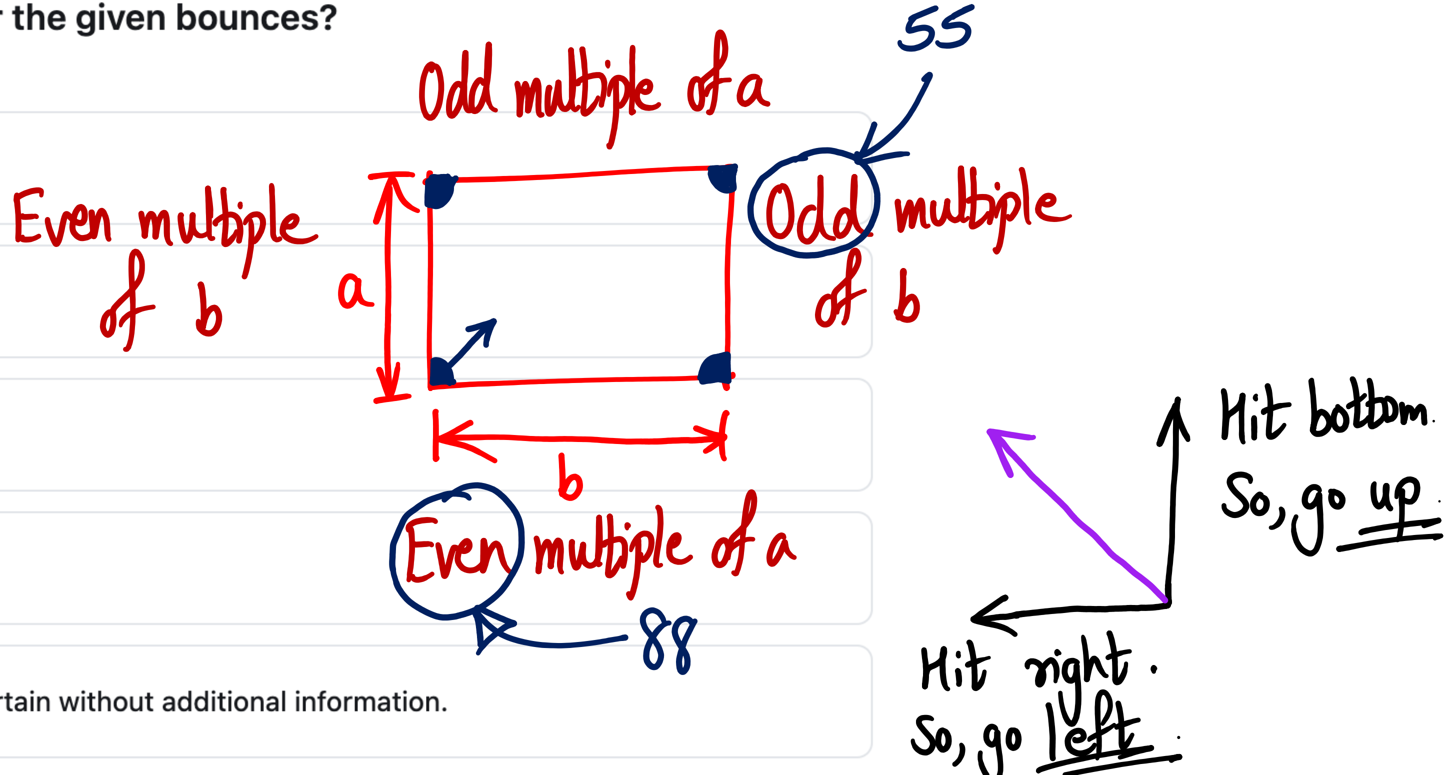
A Up and to the right

B Up and to the left ✓

C Down and to the right

D Down and to the left

E It's not possible to be certain without additional information.



In which direction will the ball be going after its **last bounce** on a 14×22 table?

Note: The ball **starts** going **up** and to the **right**.

A Up and to the right

B Up and to the left

C Down and to the right

D Down and to the left

Top-Bottom

$$x = \frac{LCM(14,22)}{14} - 1$$

$$= \frac{154}{14} - 1 = 11 - 1 = 10$$

Even

Hit bottom ⇒ Go up.

Left-Right

$$y = \frac{LCM(14,22)}{22} - 1$$

$$= \frac{154}{22} - 1 = 7 - 1 = 6$$

Even

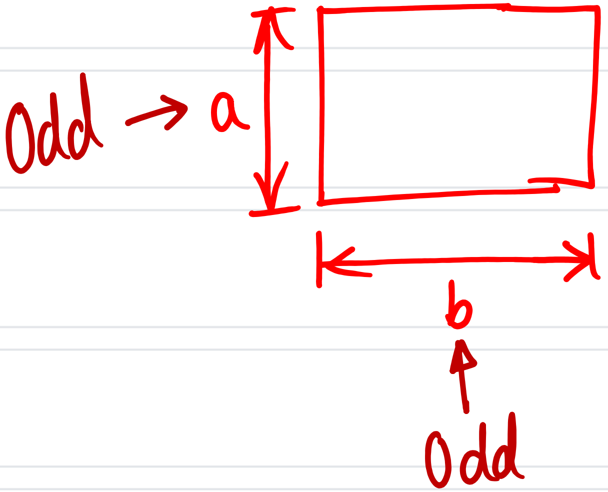
Hit left ⇒ Go right

up

right

If a and b are **both odd**, which pocket will the ball **land in** on an a×b table? Initially The ball **starts** going **up** and to the **right**.

- A Top Right
- B Bottom Right
- C Top Left
- D Bottom Left
- E It's not possible to be certain without additional information.



Top-Bottom

$$x = \frac{LCM(a,b)}{a} - 1 = \frac{LCM(Odd, Odd)}{Odd} - 1$$

$$= \frac{Odd}{Odd} - 1$$

$$= Odd - 1$$

$$= \underline{\underline{Even}} \Rightarrow \text{Hit bottom}$$

$$\Downarrow$$

Go up.

Left-Right

$$y = \frac{\mathrm{LCM}(a,b)}{b} - 1 = \frac{\mathrm{LCM}(\mathrm{Odd}, \mathrm{Odd})}{\mathrm{Odd}} - 1$$

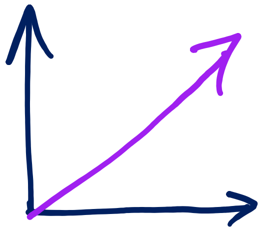
$$= \frac{\mathrm{Odd}}{\mathrm{Odd}} - 1$$

$$= \mathrm{Odd} - 1$$

$$= \underline{\underline{\mathrm{Even}}} \Rightarrow \text{Hit left}$$

$\Downarrow$

Go <u>right</u>



If a is **even** and b is **odd**, which pocket will the ball land in on an a×b table?

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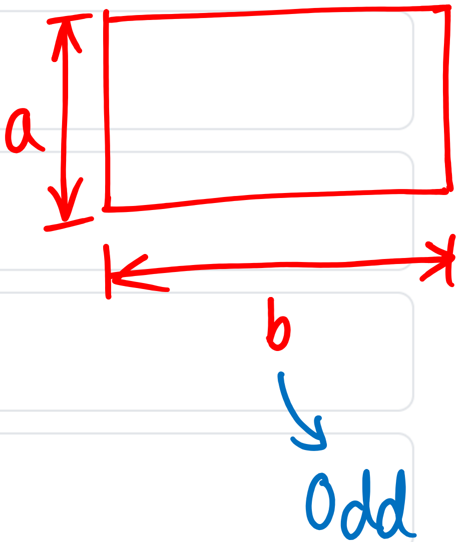
A Top right

B Bottom right

C Top left

D Bottom left

E It's not possible to be certain without additional information.



~~Top-Bottom~~

$$\boxed{x = \frac{LCM(a,b)}{a} - 1}$$

We know that $a * b = LCM(a,b) * GCD(a,b)$

$$\therefore \frac{LCM(a,b)}{a} = \frac{b}{GCD(a,b)}$$

$$\therefore \boxed{x = \frac{b}{GCD(a,b)} - 1}$$

both formulae are same

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Left-Right

$$\boxed{y = \frac{LCM(a,b)}{b} - 1}$$

We know that $a * b = LCM(a,b) * GCD(a,b)$

$$\therefore \frac{LCM(a,b)}{b} = \frac{a}{GCD(a,b)}$$

$$\therefore \boxed{y = \frac{a}{GCD(a,b)} - 1}$$

both formulae are same

Top-Bottom

$$x = \frac{b}{GCD(a,b)} - 1 = \frac{Odd}{GCD(Even, Odd)} - 1$$

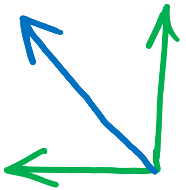
$$= \frac{Odd}{Odd} - 1$$

$$= Odd - 1$$

$$= \underline{\underline{Even}} \Rightarrow \text{Hit bottom}$$

$\Downarrow$

Go up



~~Left-Right~~

$$y = \frac{a}{GCD(a,b)} - 1 = \frac{Even}{GCD(Even, Odd)} - 1$$

$$= \frac{Even}{Odd} - 1$$

$$= Even - 1$$

$$= \underline{\underline{Odd}} \Rightarrow \text{Hit right}$$

$\Downarrow$

Go <u>left</u>

If a and b are both **even**, which pocket will the ball land in on an a×b table?

A Top right

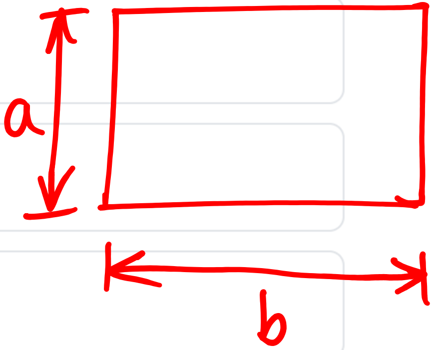
B Bottom right

C Top left

D Bottom left

E It's not possible to be certain without additional information.

Even



Even

$$GCD(a,b) = GCD(Even, Even) = Even$$

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Top-Bottom

$$x = \frac{b}{GCD(a,b)} - 1 = \frac{Even}{Even} - 1$$

May be Even or Odd.

⇓

We can't determine the pocket in which the ball will land.

⇑

Left-Right

$$y = \frac{a}{GCD(a,b)} - 1 = \frac{Even}{Even} - 1$$

May be Even or Odd.

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Thank You!