# 13. Functions

Recall:

- A function $f : A \to B$ is called **injective** or "1-1" if

for all $a_1, a_2 \in A$, the implication $(f(a_1) = f(a_2)) \longrightarrow (a_1 = a_2)$ is true.

- A function $f : A \to B$ is called **surjective** or "onto" if

for all $b \in B$, there is at least one $a \in A$ such that $f(a) = b$ ie $f^{-1}(b) \neq \emptyset$.

**The properties "injective" and "surjective" are independent properties. Any combination of these two properties is possible.**

Consider the following functions from $\mathbb{Z}$ to $\mathbb{Z}$

$id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$
$id_{\mathbb{Z}}(k) = k$

$id_{\mathbb{Z}}$ is both injective and surjective

$s : \mathbb{Z} \to \mathbb{Z}$
$s(k) = k + 1$

s is both injective and surjective

* Note the "floor function"

$\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$

Ex $\lfloor 1.8 \rfloor = 1$ Ex $\lfloor -5 \rfloor = -5$

Ex $\lfloor -1.8 \rfloor = -2$ Ex $\lfloor 1.999 \rfloor = 1$

$g : \mathbb{Z} \to \mathbb{Z}$
$g(k) = \lfloor k/2 \rfloor$

g is surjective but not injective

$h : \mathbb{Z} \to \mathbb{Z}$
$h(k) = k^3$

h is injective but not surjective

$f : \mathbb{Z} \to \mathbb{Z}$
$f(k) = 55$

f is neither injective nor surjective

Exercise give proofs or counterexamples for each of the above functions and properties

* These notes are solely for the personal use of students registered in MAT1348.

A function $f : A \to B$ is called a **bijection** if

f is both injective and surjective.

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**Example 13.1.** Let $g : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^- \times \mathbb{R}^+$ be the function defined as follows:

$$g(x, y) = \left(\frac{x}{y}, 3x\right)$$

Prove that $g$ is a bijection. → we must prove that g is both injective and surjective

Recall: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$

[injective].

Let $(a,b), (c,d) \in \mathbb{R}^+ \times \mathbb{R}^-$ be arbitrary elements of g's domain.

Assume $g(a,b) = g(c,d)$. (goal: prove $(a,b) = (c,d)$ )

Then $\left(\frac{a}{b}, 3a\right) = \left(\frac{c}{d}, 3c\right)$ (by def of g)

So $\frac{a}{b} = \frac{c}{d}$ and $\underbrace{3a = 3c}_{\therefore\, a = c}$ (two ordered pairs are equal if and only if their first coordinates are equal and their second coordinates are equal)

Consequently, $\frac{c}{d} = \frac{a}{b}$, thus $\frac{a}{b} = \frac{a}{d} \implies ad = ab$

$\implies ad - ab = 0$

$\implies a(d-b) = 0$

$\cancel{a=0}$ or $d = b$

since $a \in \mathbb{R}^+$ we know $a \neq 0$. $\therefore\, d = b$

So we proved $(g(a,b) = g(c,d)) \to (a = c \text{ and } b = d)$

ie $(g(a,b) = g(c,d)) \to ((a,b) = (c,d))$ $\therefore$ g is injective.

[Surjective]. Let $(r,s) \in \mathbb{R}^- \times \mathbb{R}^+$ be an arbitrary element of g's codomain.

(goal: prove that there exists some $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^-$ such that $g(x,y) = (r,s)$ )

We want $g(x,y) = (r,s)$. We'll figure out what $(x,y)$ should be based on g's rule.

$\Rightarrow$ we need $(x,y) \in$ g's domain such that $\left(\frac{x}{y}, 3x\right) = (r,s)$

$\Rightarrow$ we need $\frac{x}{y} = r$ and $3x = s$

$\therefore$ we need $x = \frac{s}{3}$ and $y = \frac{x}{r} = \frac{s/3}{r} = \frac{s}{3r}$

Now, let's verify that the $(x,y)$ we need is actually in g's domain:

$x = \frac{s}{3}$, $s \in \mathbb{R}^+$ $\therefore \frac{s}{3} \in \mathbb{R}^+$ ✓    $y = \frac{s}{3r}$, $s \in \mathbb{R}^+, r \in \mathbb{R}^-$ $\therefore \frac{s}{3r} \in \mathbb{R}^-$ ✓

Now, let's verify that $g(x,y) = (r,s)$:

$g(x,y) = g\left(\frac{s}{3}, \frac{s}{3r}\right) = \left(\frac{s/3}{s/3r}, 3\left(\frac{s}{3}\right)\right) = (r,s)$ ✓    $\therefore$ g is surjective.

Since g is both injective and surjective, g is indeed a bijection.

## Cardinalities of Infinite Sets

**Note.** If $A$ and $B$ are finite sets and $f : A \to B$ is a bijection, then $|A| = |B|$.

For infinite sets, the way we compare their cardinality is through bijections. We define the notion of equality of cardinalities of infinite sets as follows:

$$|A| = |B| \quad \text{if and only if} \quad \text{there exists a bijection from } A \text{ to } B.$$

An infinite set $S$ is called **countable** if $|S| = |\mathbb{N}|$.

$\boxed{\begin{array}{ccc} (\text{injective}) & \wedge & (\text{surjective}) \\ \downarrow & & \downarrow \\ (|A| \leq |B|) & \wedge & (|B| \leq |A|) \end{array}}$

Ex. $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2|n| - 1 & \text{if } n < 0 \end{cases}$ is 1-1 and onto (verify this!)

$$f \; \begin{array}{c|cccccccccc} \mathbb{Z} & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \end{array}$$

$\therefore$ there is a bijection from $\mathbb{Z}$ to $\mathbb{N}$

$\therefore |\mathbb{Z}| = |\mathbb{N}|$

**Fact.** There is no bijection from $\mathbb{R}$ to $\mathbb{N}$. Therefore, the set of real numbers is called **uncountable.**

**The identity function.**

Let $A$ be any set.

The **identity function on** $A$, denoted $\mathrm{id}_A$, is the function $\mathrm{id}_A : A \to A$ defined by

$$\mathrm{id}_A(x) = x \quad \text{for all } x \in A.$$

In particular, $\mathrm{id}_A$ is a bijection from the set $A$ to itself.

## Compositions of Functions

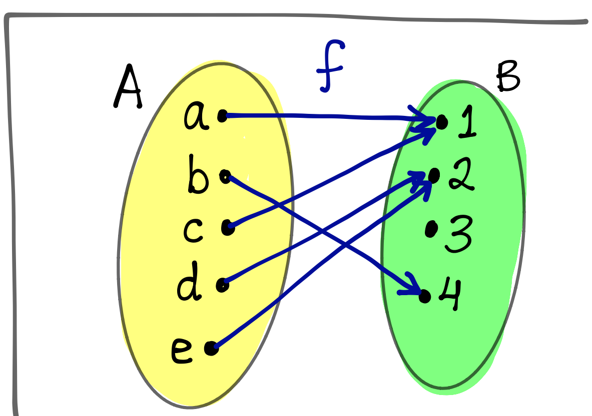
Let $f : A \to B$ and let $g : B \to C$ be functions.

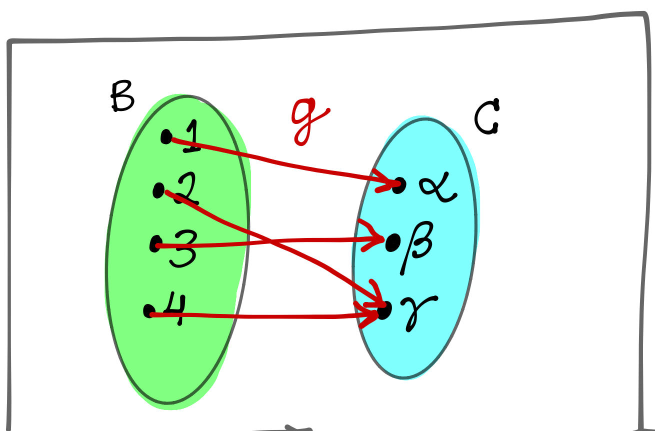
The **composition** $g$ of $f$, denoted $g \circ f$, is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

**Example 13.2.** Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$, and $C = \{\alpha, \beta, \gamma\}$.

Let $f : A \to B$ and $g : B \to C$ be functions defined as follows:



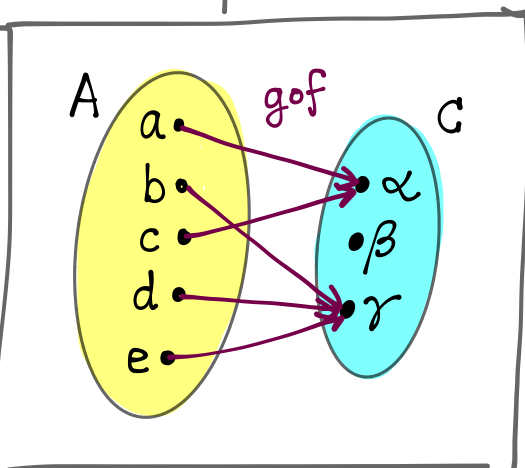

$g \circ f(a) = g(f(a)) = g(1) = \alpha$

$g \circ f(b) = g(f(b)) = g(4) = \gamma$

$g \circ f(c) = g(f(c)) = g(1) = \alpha$

$g \circ f(d) = g(f(d)) = g(2) = \gamma$

$g \circ f(e) = g(f(e)) = g(2) = \gamma$




$f : A \to B$ $\quad$ $g : B \to C$

$g \circ f : A \to C$

**Question.** In Example 13.2, does $f \circ g$ make sense? If so, what is $f \circ g$?

$f\circ g(x) = f(g(x))$ so $x$ needs to be in $g$'s domain $B$.

however, $g(x) \in C$ ($g$'s codomain) and $C \not\subseteq A$ ($f$'s domain)

$\therefore$ $f\circ g$ is not defined.

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**Note.** In order for the composition $f \circ g$ to be defined, we need the image of the domain of $g$ (a subset of the codomain of $g$) to be a subset of the domain of $f$.

Informally, for $f\circ g$ to make sense, we need $g$ to "give" $f$ elements that are in the domain of $f$.

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**Example 13.3.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x - 7$ and $g(x) = x^2$.

Find $g \circ f$.

f's domain ; g's codomain

$g\circ f : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} g\circ f(x) &= g(f(x)) \\ &= g(5x-7) \\ &= (5x-7)^2 \\ &= 25x^2 - 70x + 49 \end{aligned}$$

Find $f \circ g$.

g's domain ; f's codomain

$f\circ g : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} f\circ g(x) &= f(g(x)) \\ &= f(x^2) \\ &= 5(x^2) + 7 \\ &= 5x^2 + 7 \end{aligned}$$

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**Note.** In general, $f \circ g \neq g \circ f$, even if both compositions are defined.

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## Inverse Functions

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Let $f: A \to B$ be a function.

The <u>inverse</u> of $f$ (if it exists) is the function $f^{-1}: B \to A$ such that

$$\boxed{f^{-1}\circ f = id_A \quad \text{and} \quad f\circ f^{-1} = id_B}$$

Equivalently,

the <u>inverse</u> of $f$ (if it exists) is the function $f^{-1}: B \to A$ such that

for all $a \in A$, $b \in B$, $\boxed{f^{-1}(b) = a \text{ if and only if } f(a) = b.}$

**Example 13.4.** For the function $g : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^- \times \mathbb{R}^+$, defined by $g(x, y) = \left(\frac{x}{y}, 3x\right)$, verify that $g$'s inverse $g^{-1} : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^-$ is given by the rule

$$g^{-1}(r, s) = \left(\frac{s}{3}, \frac{s}{3r}\right)$$

Let $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^-$.

$$\begin{aligned} \text{Then } (g^{-1} \circ g)(x,y) &= g^{-1}(g(x,y)) \\ &= g^{-1}\Big(\underbrace{\tfrac{x}{y}}_{r}, \underbrace{3x}_{s}\Big) \\ &= \left(\frac{3x}{3}, \frac{3x}{3(x/y)}\right) \\ &= (x,y) \end{aligned}$$

$\therefore\ g^{-1} \circ g = \mathrm{id}_{\mathbb{R}^+ \times \mathbb{R}^-}$

Let $(r,s) \in \mathbb{R}^- \times \mathbb{R}^+$.

$$\begin{aligned} \text{Then } (g \circ g^{-1})(r,s) &= g(g^{-1}(r,s)) \\ &= g\Big(\underbrace{\tfrac{s}{3}}_{x}, \underbrace{\tfrac{s}{3r}}_{y}\Big) \\ &= \left(\frac{s/3}{s/3r}, 3\left(\frac{s}{3}\right)\right) \\ &= (r,s) \end{aligned}$$

$\therefore\ g \circ g^{-1} = \mathrm{id}_{\mathbb{R}^- \times \mathbb{R}^+}$

## Some facts about inverse functions

- Not every function has an inverse.
- If a function $f: A \to B$ has an inverse, then we call $f$ invertible.
- If $f$ is invertible, then its inverse is unique meaning there is one and only one function from $B$ to $A$ whose compositions with $f$ give the respective identity functions.
- Theorem Let $f: A \to B$ be a function. Then $f$ is invertible if and only if $f$ is a bijection.

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STUDY GUIDE

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**Important terms and concepts:**

☐ bijection
injective & surjective

☐ identity function
for all $x \in A$, $\mathrm{id}_A(x) = x$

☐ composition
$(f \circ g)(x) = f(g(x))$

☐ inverse of $g : A \to B$
$g^{-1} \circ g = \mathrm{id}_A$
$g \circ g^{-1} = \mathrm{id}_B$

Exercises

Sup.Ex. §5 # 1c, 2, 8, 11
Rosen §2.3 # 33, 34, 35, 36, 39, 69

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