Brief Review for Midterm 2

!Warning! This is not "everything" you need to know, but it's a good starting point for studying.

DEFINITIONS & TERMINOLOGY TO KNOW AND UNDERSTAND

- ☐ Proof Strategies:
  - ☐ Direct Proof ☐ Indirect Proof
  - ☐ Proof by Contradiction ☐ Proof by Cases
  - ☐ Proof of Equivalence ☐ Vacuous Proof ☐ Trivial Proof
- ☐ Mathematical definitions (for proof-writing or defining relations, etc.):
  - ☐ even / odd
  - ☐ divides remainder
  - ☐ rational / irrational number ☐ inequality
- ☐ Basic Concepts of Set Theory:
  - ☐ set
  - ☐ element
  - ☐ set equality ☐ subset ☐proper subset
  - ☐ ways to describe a set: list notation set-builder notation
  - ☐ empty set universal set
  - ☐ cardinality
  - ☐ the power set of a set
  - ☐ Cartesian product of 2 (or more) sets
  - ☐ disjoint sets
- ☐ Set Operations:
  - ☐ union
  - ☐ intersection
  - ☐ complement
  - ☐ set difference
  - ☐ symmetric difference
- ☐ ways to verify set identities:
  - ☐ membership table
  - ☐ rigorous proof
  - ☐ using identities from the Table of Important Set Identities
- ☐ Basic Concepts of Functions:
  - ☐ domain codomain
  - ☐ image of an element image of the domain image of a subset of the domain
  - ☐ preimage
  - ☐ injective (1-1) function
  - ☐ surjective (onto) function
  - ☐ bijection
  - ☐ inverse function (invertible)
  - ☐ identity function
  - ☐ composition of functions

- ☐ Basic Concepts of Relations:
  - ☐ a (binary) relation from A to B
  - ☐ a relation on a set A
- ☐ properties of a relation on a set:
  - ☐ reflexive
  - ☐ symmetric
  - ☐ antisymmetric
  - ☐ transitive
- ☐ Equivalence Relations:
  - ☐ reflexive, symmetric, and transitive
  - ☐ equivalence class of an equivalence relation

NOTATION TO KNOW AND UNDERSTAND

T, F	$\neg$	$\wedge$	$\vee$
$\equiv$	$\oplus$	$\rightarrow$	$\leftrightarrow$
$\mathbb{N}$	$\mathbb{Z}$ $\mathbb{Z}^+$ $\mathbb{Z}^-$	$\mathbb{Q}$ $\mathbb{Q}^+$ $\mathbb{Q}^-$	$\mathbb{R}$ $\mathbb{R}^+$ $\mathbb{R}^-$
$a \in A$	$a \notin A$	$A \subseteq B$	$A \nsubseteq B$
$\emptyset$	$\mathcal{U}$	$A \subset B$	$A = B$
$\mathcal{P}(A)$	$A \times B$	$A_1 \times \cdots \times A_n$	A^n
$\lvert A \rvert$	$A \cup B$	$A \cap B$	$\overline{A}$
$A - B$	$A \oplus B$		
$f : A \rightarrow B$	$f^{-1} : B \rightarrow A$	$\mathrm{id}_A : A \rightarrow A$	$g \circ f$
$f(a)$	$f(A)$	$f(S)$	$f^{-1}(b)$
$\mathcal{R}$		$[a]_{\mathcal{R}}$	
$(a, b) \in \mathcal{R}$	$(a, b) \notin \mathcal{R}$		
$a \mathcal{R} b$	$a \not\mathcal{R} b$		
$d \lvert n$	$d \nmid n$	$x = y$	$x \neq y$
$a \leq b$	$a \nleq b$	$P \equiv Q$	$P \not\equiv Q$
			etc...

COMPREHENSION QUESTIONS

- **Proof Strategies**
  - How do you give a Direct Proof of a conditional statement like $P \to Q$? What do you assume? What must you prove?
  - How do you give an Indirect Proof of a conditional statement like $P \to Q$? What do you assume? What must you prove?
  - How do you give a Proof by Contradiction of a (possibly compound) proposition X? What do you assume? What must you prove?
  - How do you give a Proof of Equivalence? What must you prove?
- **Set Theory**
  - What are the elements of a set A? What are the subsets of a set A ? How do you distinguish an element from a subset?
  - When is one set a subset of another? How do you prove $A \subseteq B$?
  - What are the elements of... the power set of a set? the Cartesian product of two (or more) sets?
  - What is the cardinality of... a set? a Cartesian product of sets? a power set of a set?
  - What elements belong to... the union of two sets? the intersection of two sets? the complement of a set?
  - What elements belong to... the difference of two sets? the symmetric difference of two sets?
  - How do you verify a set identity using a membership table? How many rows does a membership table require?
  - How do you rigorously prove two sets are equal?
  - How do you use the laws in the Table of Important Set Identities to prove another set identity?
- **Functions**
  - What is the domain of a function? What is the codomain of a function? What do you expect when a function $f : A \to B$ has domain A and codomain B (i.e. what does f "do" with elements of A or B) ?
  - Given a function $f : A \to B$, what is the image of an element $a \in A$?
    What is image of the domain A ?
  - Given a function $f : A \to B$, what is the preimage of an element $b \in B$?
    Can $f^{-1}(b)$ be the empty set ? What would this mean about f ?
  - When is a function $f : A \to B$ called injective (1-1) ?
    How do you prove that $f : A \to B$ is injective?
    If $f : A \to B$ is **not** injective, then what does a counterexample consist of ?
  - When is a function $f : A \to B$ called surjective (onto) ?
    How do you prove that $f : A \to B$ is surjective?
    If $f : A \to B$ is **not** surjective, then what does a counterexample consist of ?

- When is a function $f : A \to B$ called a bijection?
  How do you prove that $f : A \to B$ is a bijection?
- If f and g are functions, when is the composition $f \circ g$ defined (i.e. make sense) ?
  What would be the domain and codomain of $f \circ g$ if it exists?
  Do both $g \circ f$ and $f \circ g$ necessarily exist if one or the other exists?
- When is a function $f : A \to B$ called invertible?
  How do you verify that two functions are inverses of each other?
- What does the existence of an injective/surjective function from A to B tell you about $|A|$ and $|B|$?

- **Relations**
  - What is a binary relation from A to B ? How many binary relations are there from A to B ?
  - What is a relation on a set A? How does it differ from a relation from A to another set B?
  - What does it mean for a relation $\mathcal{R}$ on a set A to be reflexive?
    How do you prove that $\mathcal{R}$ is reflexive?
    If $\mathcal{R}$ is not reflexive, then what does a counterexample consist of?
  - What does it mean for a relation $\mathcal{R}$ on a set A to be symmetric?
    How do you prove that $\mathcal{R}$ is symmetric?
    If $\mathcal{R}$ is not symmetric, then what does a counterexample consist of?
  - What does it mean for a relation $\mathcal{R}$ on a set A to be antisymmetric?
    How do you prove that $\mathcal{R}$ is antisymmetric?
    If $\mathcal{R}$ is not antisymmetric, then what does a counterexample consist of?
  - What does it mean for a relation $\mathcal{R}$ on a set A to be transitive?
    How do you prove that $\mathcal{R}$ is transitive?
    If $\mathcal{R}$ is not transitive, then what does a counterexample consist of?

- **Equivalence Relations**
  - How do you prove that $\mathcal{R}$ is an equivalence relation? What 3 properties must an equivalence relation possess?
  - If $\mathcal{R}$ is an equivalence relation on A, then what is the equivalence class $[a]_{\mathcal{R}}$ of an element $a \in A$?

EXTRA PRACTICE FOR MIDTERM 2 — SEE BRIGHTSPACE

Here is a list of specific questions from old tests/midterms:

Practice tests (old tests) *(Solutions are posted on Brightspace.)*

2017 Test 3 all questions

2017 Test 4 Questions 1, 2ab, 3, 4, 5, 6, 8, 9

2016 Midterm 2 Questions 3, 4, 5, 6, 7, 8, 9, 10

2015 Midterm Questions 1, 5, 9, 10, 11

2014 Midterm Questions 2, 4, 5, 10

2013 Midterm Questions 5, 7, 9, 10

You can also use these Practice Homework exercises to study! *(no solutions will be provided for old HW)*

Practice Homework 4 Question 3

Practice Homework 5 Questions 1, 2, 3

Practice Homework 6 Questions 1, 2, 3

Practice Homework 7 Question 1

Don't forget to practice even more by

- ☐ re-doing the DGD exercise lists
- ☐ working through the suggested exercises (at end of each of the LEC notes) from the Supplemental Exercise Lists and the Rosen textbook
- ☐ re-doing some of the examples that were done in class LEC notes
- ☐ reviewing your Homework Assignment solutions