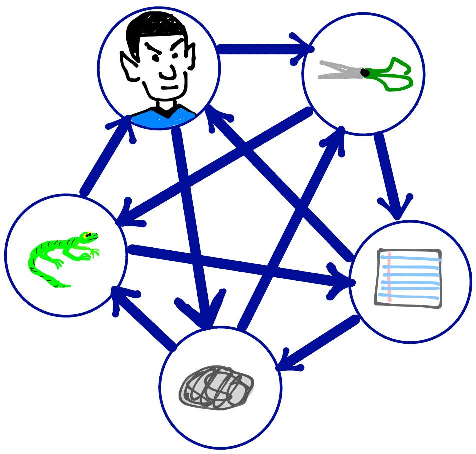
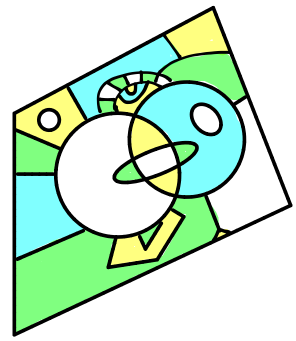
21. Introduction to Graph Theory

Graph Theory Examples (before we talk about the formal definitions)

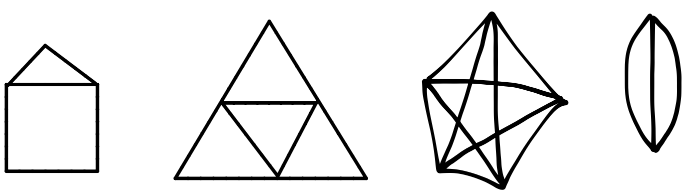
Ex. A generalization of the game "rock/paper/scissors" from 3 options to 5... why wouldn't 4 options work just as well?



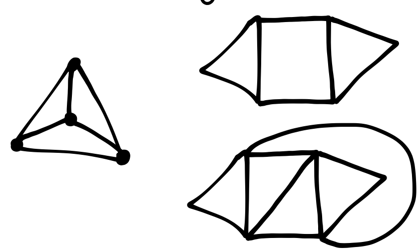
Ex Colour the regions/countries in any map so that no two countries that share a border are the same colour. What is the maximum number of colours you will ever need?



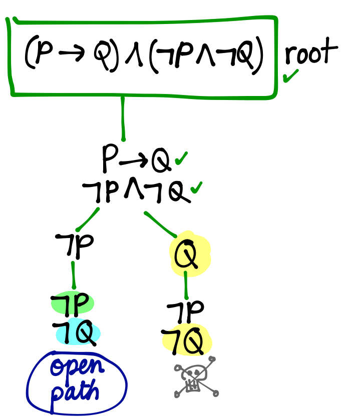
Ex. Draw these shapes without retracing any lines or lifting your pencil. Can your tracing start and end at the same corner?

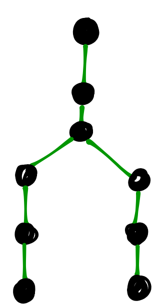


Ex Can you draw any of these shapes without retracing any lines or lifting your pencil?



Ex. Truth trees are special types of graphs





* These notes are solely for the personal use of students registered in MAT1348.

A **graph** G is an ordered pair $\big(V(G), E(G)\big)$, where

- $V(G)$ is a nonempty set whose elements are called vertices.

  $V(G)$ is called the vertex set of G

- $E(G)$ is a set whose elements are called edges.

  $E(G)$ is called the edge set of G ($E(G)$ can be the empty set)

- $V(G)$ and $E(G)$ are related to each other by a function

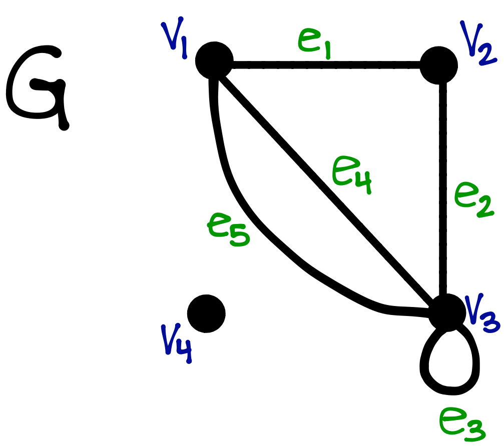
$$\psi_G : E(G) \longrightarrow \{\{u,v\} : u, v \in V(G)\}$$

ψ_G is called the incidence function of G

for each edge $e \in E(G)$, $\psi_G(e) = \{\text{set of endpoints of } e\}$

Remark 21.1. Although we are using the word "graph", a graph, as defined above, is not the same as the graph of a function.

Example 21.2. Let G be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$, whose incidence function ψ_G is defined as follows:



$$\psi_G(e_1) = \{ v_1, v_2 \}$$

$$\psi_G(e_2) = \{ v_2, v_3 \}$$

e_3 is a loop $\rightarrow$ $\psi_G(e_3) = \{ v_3 \}$

e_4 and e_5 are parallel edges

$$\psi_G(e_4) = \{ v_1, v_3 \}$$

$$\psi_G(e_5) = \{ v_1, v_3 \}$$

◦ An edge $e \in E(G)$ is called a **loop** if

$\psi_G(e) = \{u\}$ for some vertex $u \in V(G)$ (ie the endpoints of a loop coincide)

◦ An edge $e \in E(G)$ is called a **link** if

$\psi_G(e) = \{u, v\}$ for two distinct vertices $u, v \in V(G)$.

◦ Distinct edges e_1 and e_2 are called **parallel edges** if

$\psi_G(e_1) = \psi_G(e_2)$ (ie e_1 and e_2 have the same endpoints)

◦ A graph G is called a **simple graph** if

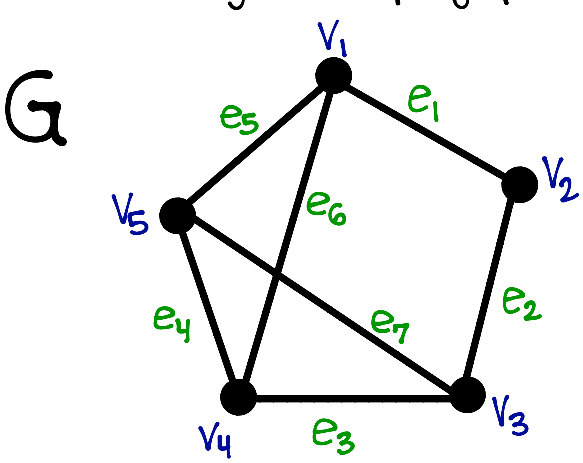
G has no loops and no parallel edges.

Observations:

◇ If G is a graph with no parallel edges, then we can think of an edge e interchangeably with its endpoints (since there is at most one edge joining any set of endpoints).

◇ If G has parallel edges, then we need the incidence function ψ_G to keep track of which edge we are talking about when we consider two endpoints.

Ex. A drawing of a simple graph G:



ex. $\psi_G(e_1) = \{v_1, v_2\}$ and e_1 is the only edge with endpoints $\{v_1, v_2\}$

we can think of e_1 as $e_1 = \{v_1, v_2\}$

Similarly, $e_7 = \{v_3, v_5\}$, etc...

- G has no loops ✓
- G has no parallel edges ✓

$\therefore$ G is a simple graph.

More Graph Terminology and Conventions

Let $G = (V, E)$ be a graph.

- vertices $u, v \in V(G)$ are called **adjacent** or **neighbours** if $\psi_G(e) = \{u, v\}$ for some edge $e \in E(G)$.

  Notation for adjacency: $u \sim v$ means u and v are adjacent

- an edge e is said to be **incident with** its endpoints

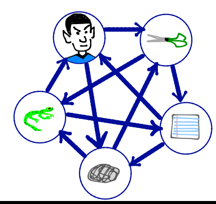
- if G has no parallel edges, then, for short, we will abuse notation and write $e = uv$ instead of $\psi_G(e) = \{u, v\}$.

  Notes: • If $\psi_G(e) = \{u,v\}$, then $e = uv$ and $e = vu$ are equivalent (order of endpoints is irrelevant in an undirected graph).

• we are often even lazier and we just write $uv \in E(G)$ (ie refer to an edge by its endpoints alone... forget about its name 'e'...)

• for a loop incident with vertex u, we will write $uu \in E(G)$.

* there is also the notion of a directed graph where order of endpoints of an "arc" $u \longrightarrow v$ is taken into consideration (but we'll focus on "undirected" graphs)



Degrees and Degree Sequences

Let G be a graph.

The **degree** of a vertex $u \in V(G)$, denoted $\deg_G(u)$, is

$\deg_G(u) =$ # of edges incident with u, where each loop incident with u is counted twice (ie each loop incident with u contributes 2 to $\deg_G(u)$).

- ◇ if $\deg_G(u) = 0$, then u is called **isolated.**
- ◇ if $\deg_G(u) = 1$, then u is called a **leaf** or **pendant vertex**.

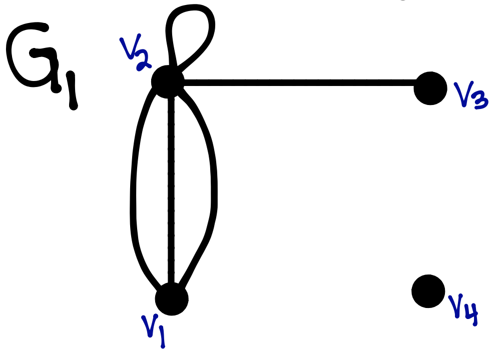
Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$.

The **degree sequence** of G is the sequence:

$$\left(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n)\right)$$

Note. The entries of a degree sequence may be listed in any order, but most often, we list the degrees in a non-decreasing order.

Example 21.3. What is the degree sequence of each of the following graphs?



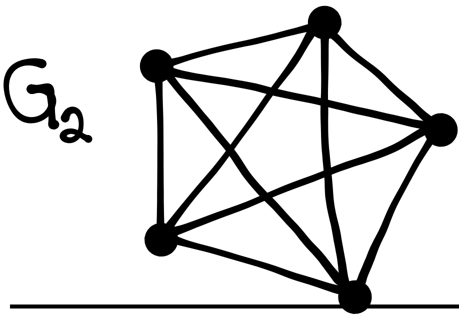
$\deg_{G_1}(v_1) = 3$

$\deg_{G_1}(v_2) = 6$ ← notice how the loop adds 2 to degree of v_2.

$\deg_{G_1}(v_3) = 1$ ← v_3 is a leaf

$\deg_{G_1}(v_4) = 0$ ← v_4 is isolated

degree sequence of G_1: $(3,6,1,0)$



degree sequence of G_2: $(4,4,4,4,4)$

The Handshaking Theorem

Theorem 21.4. (The Handshaking Theorem) Let G be any graph. Then

$$\boxed{\sum_{u \in V(G)} \deg_G(u) = 2|E(G)|}$$

Proof of The Handshaking Theorem:

Let e be any edge of G. Then there are 2 cases:

<u>Case 1</u> • if $e = uv$ is a link, then e contributes 1 to $\deg_G(u)$ and e contributes 1 to $\deg_G(v)$.

<u>Case 2</u> • if $e = uu$ is a loop, then e contributes 2 to $\deg_G(u)$.

In both possible cases e contributes 2 in total to the sum $\sum_{u \in V(G)} \deg_G(u)$

$$\therefore \sum_{u \in V(G)} \deg_G(u) = 2|E(G)|.$$

Exercise 21.5. Verify that the equation of The Handshaking Theorem is true for each of the undirected graphs in these notes.

Example 21.6. For each of the following sequences, determine whether there exists

i. a graph with that sequence as its degree sequence

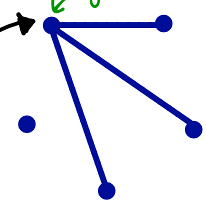
ii. a **simple** graph (no loops, no parallel edges) with that sequence as its degree sequence

If you claim such a graph exists, draw an example with the desired degree sequence. Otherwise, explain why no such simple graph exists.

i. $(3, 3, 3, 3, 3)$ (# vertices) = 5

degree = 3

(1.) without loss of generality we can give this vertex 3 neighbours

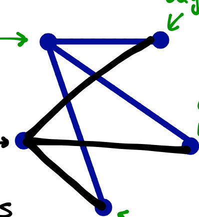


Now 2 vertices have deg. 3

deg. 2

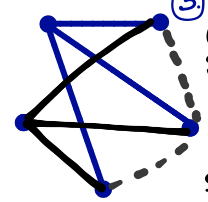
deg. 2

(2.) Now, this vertex needs 3 neighbours but it cannot be adjacent to the 1st vertex without making its degree 4.



deg 2

(3.) We need to add more edges so the last 3 vertices get 3 neighbours each but we seem to be stuck...



Wait! How many edges should this graph have!?

(Handshaking Theorem) $(\#\text{edges}) = \frac{1}{2}\sum_{u\in V} \deg(u) = \frac{1}{2}(3+3+3+3+3) = 7.5$ ⚡

There is no graph (simple or not) with this degree sequence, since it contradicts the Handshaking Theorem.

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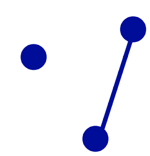
ii. $(0, 1, 1)$ (# vertices) = 3 $(\#\text{edges}) = \frac{1}{2}(0+1+1) = 1$ edge

Here is an example of a simple graph with degree sequence (0,1,1)



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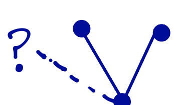
iii. $(1, 2, 3)$ (# vertices) = 3 $(\#\text{edges}) = \frac{1}{2}(1+2+3) = 3$



needs degree 3...

There is no simple graph on 3 vertices with this degree sequence because, in a simple graph on 3 vertices, any vertex has at most 2 distinct neighbours

On the other hand, there does exist a non-simple graph with this degree sequence...

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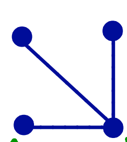
iv. $(1, 2, 2, 3)$ (# vertices) = 4 $(\#\text{edges}) = \frac{1}{2}(1+2+2+3) = 4$

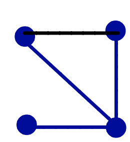
W.L.O.G.



a vertex of degree 1

a vertex of degree 3

Now we need one more edge and the other 2 vertices to have degree 2



✓

here is a simple graph with degree sequence (1,2,2,3)

Can you find a non-simple graph with this degree sequence?

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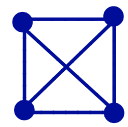
v. $(3, 3, 3, 3)$ $(\#\text{vertices}) = 4$ $\quad (\#\text{edges}) = \frac{1}{2}(3+3+3+3) = 6$

all vertices have degree 3 and the graph has 4 vertices so each vertex must be adjacent to each of the other 3 vertices



here is a simple graph with degree sequence $(3,3,3,3)$

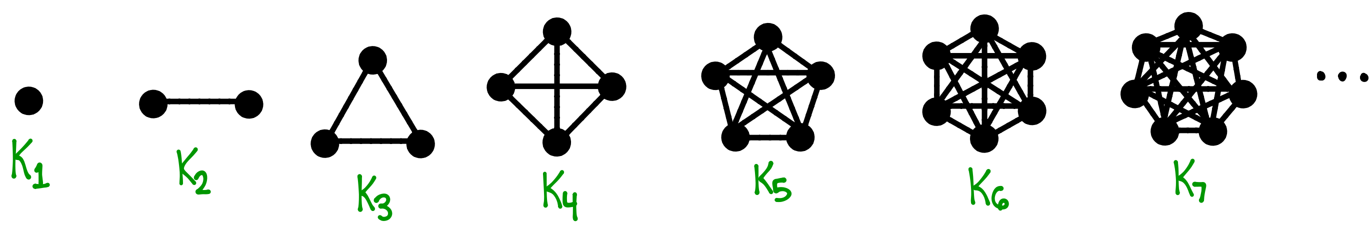
Can you find a non-simple graph with this degree sequence?

SPECIAL FAMILIES OF SIMPLE GRAPHS

Complete Graphs: • for $n \geq 1$, the <u>complete graph on n vertices</u> is denoted K_n

• $|V(K_n)| = n$ $\qquad E(K_n) = \{\{u,v\} : u,v \in V(K_n), u \neq v\}$ $\qquad$ What is $|E(K_n)|$?

all possible links



• degree sequence of K_n: $(n-1, n-1, \ldots, n-1)$

STUDY GUIDE

Important terms and concepts:

◇ graph G	digraph D
◇ vertex set $V(G)$ edge set $E(G)$	vertex set $V(D)$ arc set $A(D)$
◇ incidence function ψ_G	
◇ endpoints of an edge	initial and terminal vertices of an arc
◇ loop	
◇ parallel edges	
◇ simple graph	
◇ adjacent vertices neighbours $u \sim_G v$	
◇ degree $\deg_G(u)$	
◇ isolated vertex leaf (or pendant vertex)	
◇ degree sequence $(\deg_G(v_1), \ldots, \deg_G(v_n))$	
◇ **Handshaking Theorem**: $\sum_{v \in V(G)} \deg_G(v) = 2\lvert E(G)\rvert$	

Supp. Exercise List (on Brightspace)	§12 # 1, 2, 3, 4, 5
Graph Theory Notes (on Brightspace)	§1.4 # 1a, 2ab, 3, 5, 6
Graph Theory Notes (on Brightspace)	§2.5 # 1, 2, 3, 6, 9, 10