

DGD 10

Q1. COUNTING: BINARY STRINGS i. How many **binary strings of length 9** are there?

Procedure:

Build a binary string
of length 9

Task T_i : choose i th entry

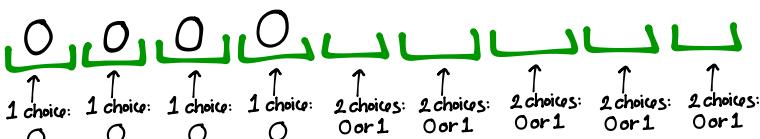


There are $2^9 = 512$ binary strings of length 9.

ii. How many **binary strings of length 9** are there that begin with 4 zeros?

Procedure:

Build a binary string
of length 9 that begins
with four 0's.



Task T_i : choose i th entry

There are $1^4 \cdot 2^5 = 32$ such strings.

iii. How many **binary strings of length 9** are there that contain exactly 4 zeros?

Task T_1 : choose 4 of the 9 entries in the string to place the zeros

$$\text{(in one of } C(9,4) = \binom{9}{4} = \frac{9!}{4!5!} = \frac{39,876}{120} = 126 \text{ ways)}$$

Task T_2 : fill the other 5 entries with ones (in 1 way)

\therefore there are $\binom{9}{4} \cdot (1) = 126$ such binary strings.

iv. How many **binary strings of length 9** are there that contain at most 4 zeros?

We will break this up into 5 separate cases:

Case 0: String contains no zeros (hence 9 ones)

There are $\binom{9}{0} = 1$ binary strings of length 9 with no zeros

Case 1: string contains exactly 1 zero (hence 8 ones)

There are $\binom{9}{1} = 9$ binary strings of length 9 with exactly 1 zero

Case 2: string contains exactly 2 zeros (hence 7 ones)

There are $\binom{9}{2} = 36$ binary strings of length 9 with exactly 2 zeros

Case 3: string contains exactly 3 zeros (hence 6 ones)

There are $\binom{9}{3} = 84$ binary strings of length 9 with exactly 3 zeros

Case 4: string contains exactly 4 zeros (hence 5 ones)

There are $\binom{9}{4} = 126$ binary strings of length 9 with exactly 4 zeros

\therefore there are $1 + 9 + 36 + 84 + 126 = 256$ binary strings with at most 4 zeros.

Q2. COUNTING: RELATIONS AND FUNCTIONS

i. How many relations are there on the set $A = \{1, 2, 3\}$?

Recall that a relation R on A is simply a subset of $A \times A$

In this example, $|A|=3 \Rightarrow |A \times A|=9$

Procedure: build a subset of a 9-element set $A \times A = \{(1,1), (1,2), \dots, (3,3)\}$

Task T_i : either put the i th element of $A \times A$ in the relation or not (2 ways)

There are $2^{|A \times A|} = 2^9 = 512$ relations on A .

ii. How many relations on the set $A = \{1, 2, 3\}$ are reflexive?

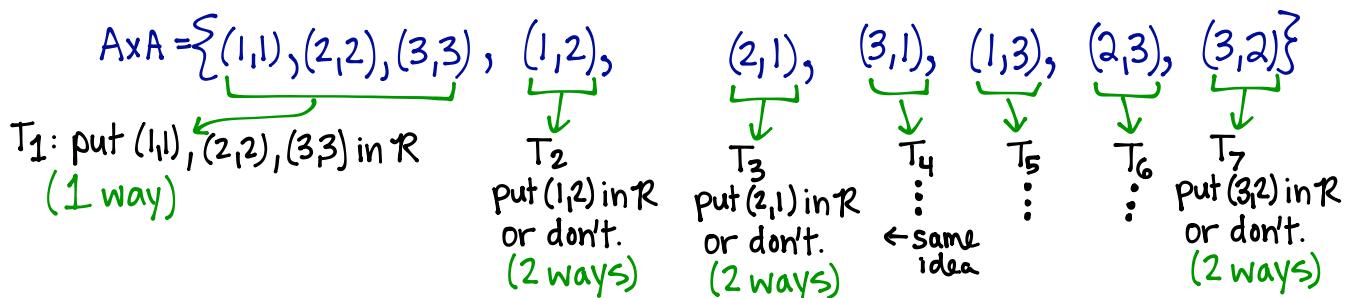
Recall that a relation R on A is reflexive if, for all $a \in A$, $(a, a) \in R$.

In this example, $|A|=3 \Rightarrow |A \times A|=9$

and to be reflexive, R needs to contain the three pairs $(1,1), (2,2)$, and $(3,3)$

Thus, there are $9-3=6$ other pairs $(a,b) \in A \times A$ for which each has 2 choices with respect to belonging to R : either $(a,b) \in R$ or $(a,b) \notin R$.

To be explicit (since this set A is reasonably small), here is $A \times A$:

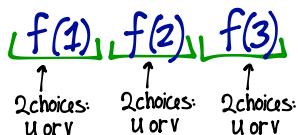


We can build a reflexive relation R on A using the procedure (T_1, T_2, \dots, T_7)

\therefore there are $(1)(2)(2)(2)(2)(2)(2) = 2^6 = 64$ reflexive relations on A

iii. How many functions are there from A to $B = \{u, v\}$?

Procedure: build a function $f: A \rightarrow B$ by choosing images for each element of A



There are $2^3 = 8$ functions from A to B .

Q3. COUNTING A 9-member fellowship is to be formed by selecting its members from among a group of 15 people consisting of 1 wizard, 5 hobbits, 5 elves, 2 men, and 2 dwarves.

i. How many different 9-member fellowships could be formed from these people?

• choose 9 out of 15 people (in one of $\binom{15}{9} = \frac{15!}{9!6!} = 5005$ ways)

∴ there are 5005 different 9-member fellowships that could be formed

ii. Suppose one member will be distinguished from the rest as a "ring-bearer". How many different 9-member fellowships with one ring-bearer can be formed from these people?

T₁: choose the ring-bearer (in one of $\binom{15}{1} = 15$ ways)

T₂: choose 8 other members from among the remaining 14 people
(in one of $\binom{14}{8} = 3003$ ways)

∴ there are $15 \cdot 3003 = 45045$ different 9-member fellowships that could be formed containing one ring-bearer.

iii. How many fellowships are there that include 1 wizard, 1 hobbit as the ring-bearer and 3 other hobbits, 1 elf, 2 men, and 1 dwarf?

T₁: choose 1 wizard (in $\binom{1}{1} = 1$ way)

T₂: choose one ring-bearer from among the 5 hobbits (in one of $\binom{5}{1} = 5$ ways)

T₃: choose 3 other hobbits from among the 4 remaining hobbits
(in one of $\binom{4}{3} = 4$ ways)

T₄: choose an elf (in one of $\binom{5}{1} = 5$ ways)

T₅: choose 2 men (in $\binom{2}{2} = 1$ way)

T₆: choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways).

∴ there are $1 \cdot 5 \cdot 4 \cdot 5 \cdot 1 \cdot 2 = 200$ such fellowships that could be formed.

- iv. After the fellowship has been selected in some way, its nine members will line up for a group photo before they depart on a perilous quest. How many ways can these 9 people be arranged for the photo?

There are $P(9,9) = 9! = 362\,880$ different arrangements (permutations) of the 9 members

- v. How many fellowships are there that include 1 hobbit as the ring-bearer and an equal number of dwarves and elves?

We will break this up into cases where #dwarves = #elves.

Case 1 0 elves & 0 dwarves means we need

9 members from the 1 wizard, 5 hobbits, and 2 men (not possible)

Case 2: 1 dwarf + 1 elf means we need

7 members from the 1 wizard, 5 hobbits, and 2 men

T_0 : choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1 : choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways)

T_2 : choose 1 elf (in one of $\binom{5}{1} = 5$ ways)

T_3 : choose 6 other members from among the 7 others

(1 wizard, 4 hobbits, 2 men)

(in one of $\binom{7}{6} = 7$ ways)

Case 3: 2 dwarves + 2 elves means we need

5 members from the 1 wizard, 5 hobbits, and 2 men

T_0 : choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1 : choose 2 dwarves (in $\binom{2}{2} = 1$ way)

T_2 : choose 2 elves (in one of $\binom{5}{2} = 10$ ways)

T_3 : choose 4 other members from among the 7 others

(1 wizard, 4 hobbits, 2 men)

(in one of $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ways)

\therefore there are $5 \cdot 2 \cdot 5 \cdot 7 + 5 \cdot 1 \cdot 10 \cdot 35 = 350 + 1750 = 2100$ ways

to form such a fellowship.