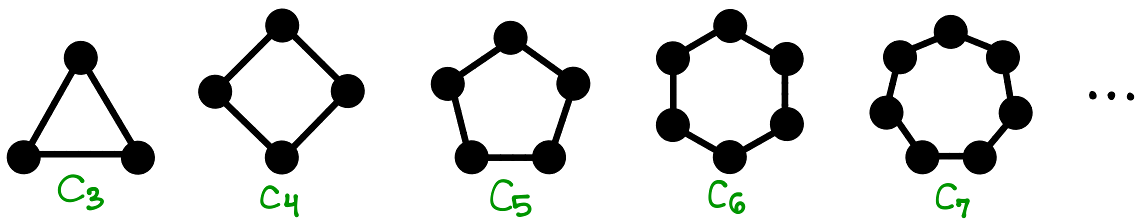
22. Bipartite Graphs & Proper Colourings

degree sequence	$(\deg_G(v_1), \ldots, \deg_G(v_n))$
Handshaking Theorem	every graph has an even number of vertices of odd degree
degree sequence	existence of a (simple) graph of given degree sequence
complete graph on n vertices: K_n	

SPECIAL FAMILIES OF SIMPLE GRAPHS

Cycles:

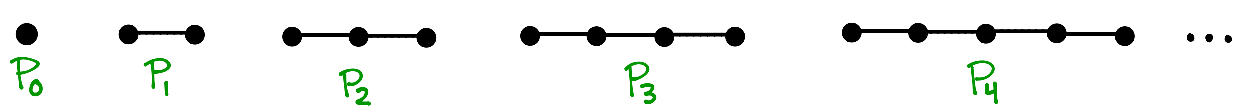
- for $n \geq 3$, the cycle of length n is denoted C_n
- $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ $\quad E(C_n) = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1\}$



- degree sequence of C_n: $(2, 2, \ldots, 2)$
- $|V(C_n)| = n$
- $|E(C_n)| = n$
- if n is odd, then C_n is an odd cycle
- if n is even, then C_n is an even cycle

Paths:

- for $n \geq 0$, the path of length n is denoted P_n
- $V(P_n) = \{u_0, u_1, \ldots, u_n\}$ $\quad E(P_n) = \{u_0u_1, u_1u_2, \ldots, u_{n-1}u_n\}$



- for $n \geq 1$, degree sequence of P_n: $(1, 2, \ldots, 2, 1)$
- $|V(P_n)| = n+1$
- $|E(P_n)| = n$

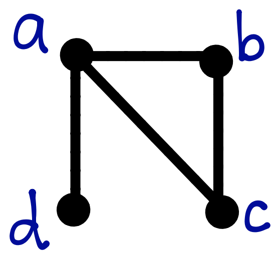
* These notes are solely for the personal use of students registered in MAT1348.

Let H and G be graphs.

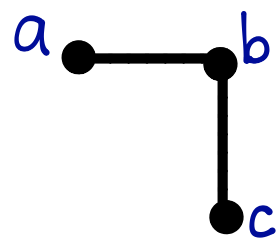
H is called a **subgraph** of G, denoted $H \subseteq G$, if both $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Example 22.1.

G



H



$V(G)=\{a,b,c,d\}$

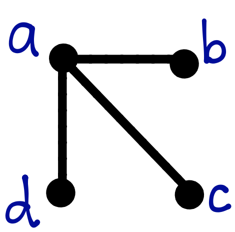
$V(H)=\{a,b,c\}$

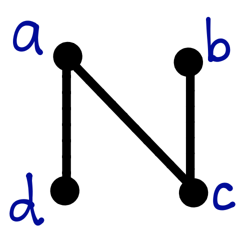
$E(G)=\{\{a,b\},\{a,c\},\{a,d\},\{b,c\}\}$

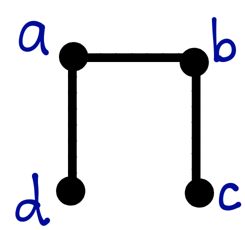
$E(H)=\{\{a,b\},\{b,c\}\}$

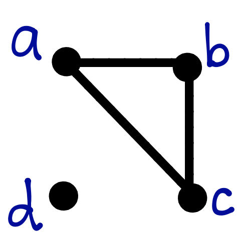
Since $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$, H is a subgraph of G $(H\subseteq G)$

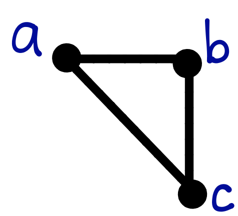
List all subgraphs of G with exactly 3 edges:





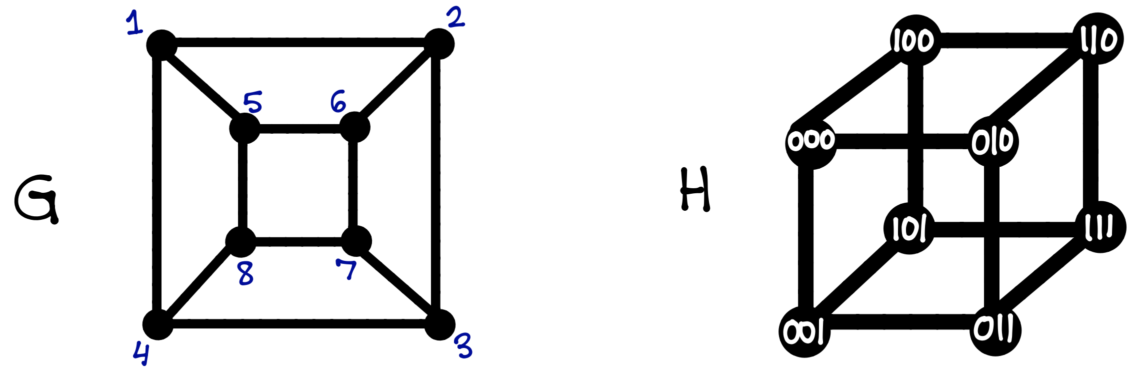




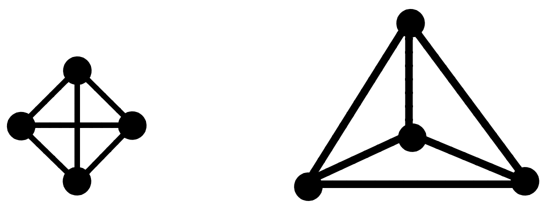


Note these two subgraphs of G are not the same (one has 4 vertices while the other has only 3 vertices).

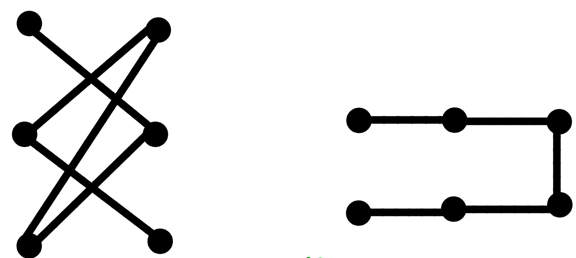
Graph Isomorphism



G and H are "essentially the same" (same structure of vertices/edges but different vertex labels and drawing perspective).



2 different drawings of K_4



2 different drawings of P_5

Let G and H be simple graphs.

An **isomorphism** from G to H is a bijection $f : V(G) \to V(H)$ such that, for all $u, v \in V(G)$,

$$\left(\{u,v\} \in E(G)\right) \longleftrightarrow \left(\{f(u),f(v)\} \in E(H)\right)$$

Graphs G and H are called **isomorphic** if there exists an isomorphism from G to H.

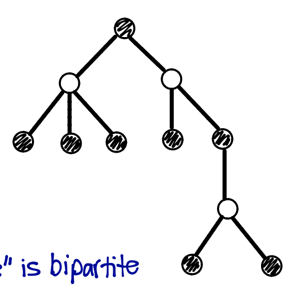
!! THE END OF WHAT WAS COVERED IN CLASS JULY 23 !!

Bipartite Graphs

Note: if "bipartite" graphs or "colourings" of graphs were to appear on the exam, then it would only be in the form of a bonus point.

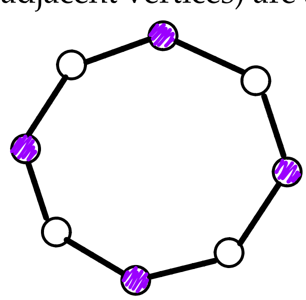
A graph G is called **bipartite** or **2-colourable** if we can colour the vertices of G using 2 colours so that no two neighbours (pair of adjacent vertices) are assigned the same colour.

Ex



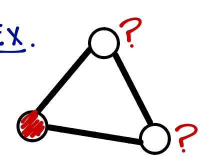
this "tree" is bipartite

Ex



C_8 is bipartite

Ex.



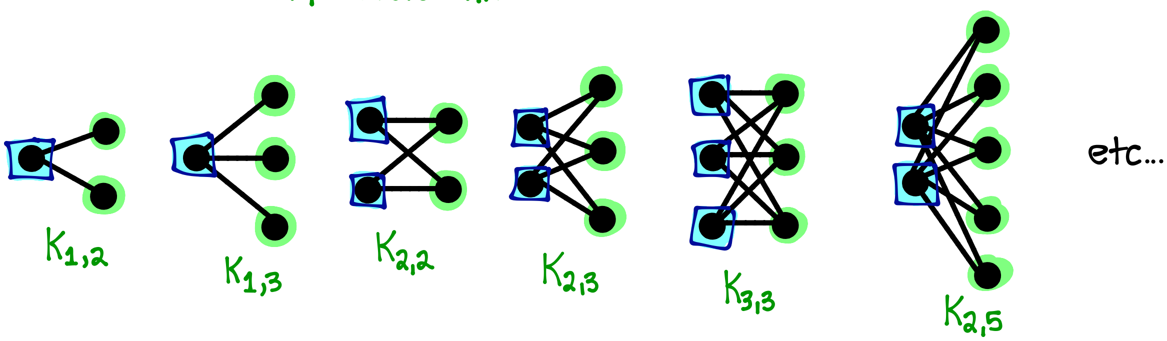
C_3 is not bipartite...

Ex all complete bipartite graphs are bipartite

- for $m,n \geq 1$, the complete bipartite graph $K_{m,n}$ has

- $V(K_{m,n}) = \{x_1,\dots,x_m\} \cup \{y_1,\dots,y_n\} = X \cup Y$ where $|X|=m$ and $|Y|=n$
  Moreover, $X \cap Y = \emptyset$ (so $\{X,Y\}$ is a partition of $V(K_{m,n})$ into 2 classes)

- $E(K_{m,n}) = \{x_i y_j : x_i \in X, y_j \in Y\}$
  ↖all possible links from X vertices to Y vertices



- $|V(K_{m,n})| = m+n$

- $|E(K_{m,n})| = m \cdot n$

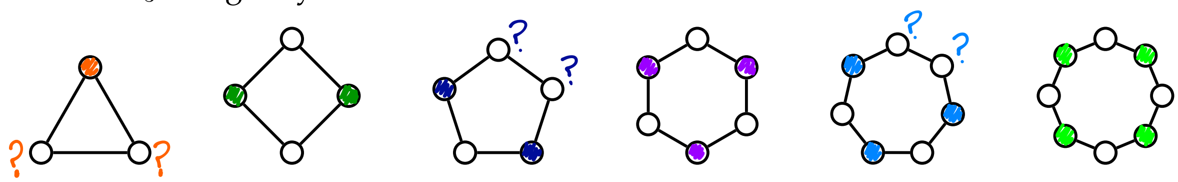
- for each $x_i \in X$, $\deg_{K_{m,n}}(x_i) = n$

- for each $y_i \in Y$, $\deg_{K_{m,n}}(y_i) = m$

- degree sequence of $K_{m,n}$

$$(\underbrace{n,n,\dots,n}_{\substack{m\text{ times}\\ \text{(degrees of}\\ X\text{ vertices)}}}, \underbrace{m,m,\dots,m}_{\substack{n\text{ times}\\ \text{(degrees of}\\ Y\text{ vertices)}}})$$

- ◇ A graph G is called **bipartite** or **2-colourable** if we can colour the vertices of G using 2 colours so that no two neighbours (pair of adjacent vertices) are assigned the same colour.
- ◇ So the cycle C_3 is **not** bipartite. No matter what we try, we cannot properly colour the vertices of C_3 using only 2 colours.



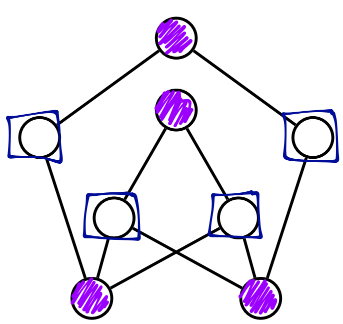
- ◇ In fact, no cycle of odd length is bipartite, whereas every cycle of even length is bipartite. (why? think about this!)
- ◇ Given that it is impossible to properly 2-colour the vertices of any cycle of odd length, it follows that all graphs which contain an odd cycle as a subgraph cannot be properly 2-coloured.
- ◇ In other words, in order for a graph G to be 2-colourable, it is **necessary that** G **contains no odd cycle as a subgraph**.
- ◇ What may be surprising is that the above necessary condition **is also sufficient.**

Theorem 22.2. Let G be a graph. Then

G is bipartite/2-colourable **if and only if** G has no odd-length cycle as a subgraph.

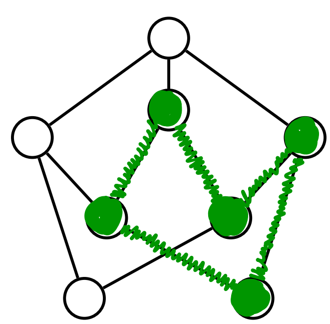
Thus, you can either properly 2-colour the vertices of a graph G, or else you are guaranteed to find at least one subgraph of G that is (isomorphic to) a cycle of odd length.

Example 22.3. Which of the following graphs are bipartite? Either give a proper 2-colouring or find an odd cycle to justify your answer.



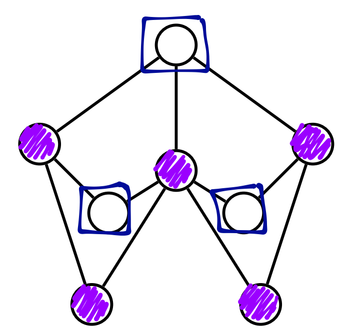
G_1

G_1 is bipartite.
(see the proper 2-colouring of the vertices of G_1)



G_2

G_2 is not bipartite
(see the odd cycle of length 5 in G_2)

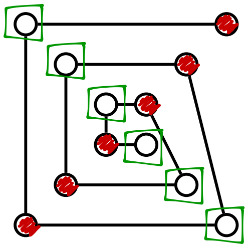


G_3

G_3 is bipartite.
(see the proper 2-colouring of the vertices of G_3)

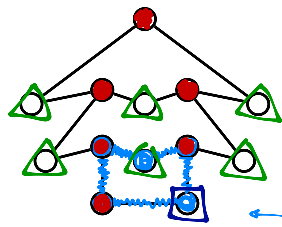
Example 22.4. Determine whether each of the following graphs is bipartite or not. Justify your answer. For each graph below, can you determine the **minimum number** of colours we would need in order to properly colour the vertices of the graph so that no pair of adjacent vertices are assigned the same colour? **For your interest:** for a given graph G, this minimum number of colours is called the **chromatic number of** G, and is denoted $\chi(G)$. The 4-colour Theorem is a theorem about the chromatic number of a special class of graphs called **planar graphs**. It says, if G is a planar graph, then $\chi(G) \leq 4$.

P_{11}



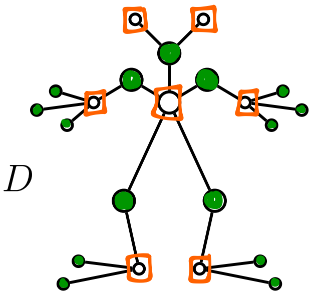
The path P_{11} is bipartite
$\chi(P_{11})=2$

G



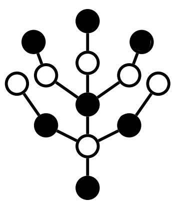
G contains an odd cycle C_5
$\Rightarrow$ not bipartite
$\chi(G)=3$

D



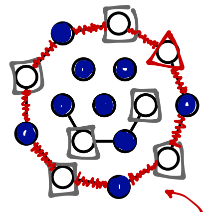
D is bipartite
$\chi(G)=2$

F



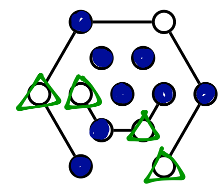
F is bipartite
$\chi(F)=2$

H



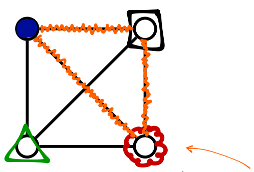
H contains an odd cycle C_9
$\Rightarrow$ not bipartite
$\chi(H)=3$

L



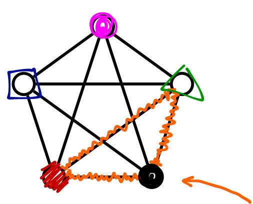
L is bipartite
$\chi(L)=2$

K_4



K_4 contains an odd cycle: C_3
$\Rightarrow$ not bipartite
$\chi(K_4)=4$

K_5

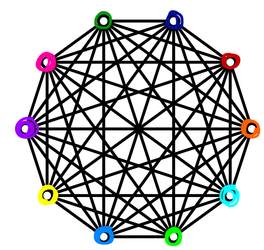


K_5 contains an odd cycle: C_3
$\Rightarrow$ not bipartite
$\chi(K_5)=5$

K_1

K_1 is bipartite
$\chi(K_1)=1$

K_{10}



not bipartite
$\chi(K_{10})=10$

Study Guide

Important graph theory terms and concepts:

- ◇ graph (vertex set, edge set) (loops, parallel edges, simple graph)
- ◇ degree sequence
  find a (simple/non-simple) graph of given degree sequence
  or, explain why no (simple/non-simple) graph could have that degree sequence
- ◇ Handshaking Theorem
- ◇ special families of graphs: K_n C_n P_n $K_{m,n}$
- ◇ subgraph graph isomorphism
- ◇ bipartite/2-colourable graphs
  either give a proper 2-colouring of the graph or find an odd cycle in the graph

Supp. Exercise List (on Brightspace)	§12 # 1, 2, 3, 4, 5, 10, 12
Graph Theory Notes (on Brightspace)	§2.5 # 4, 7, 8abd, 13