Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

DISCRETE MATHEMATICS FOR COMPUTING — INSTRUCTOR: ELIZABETH MALTAIS

MAT1348C — Practice Final Exam — for practice only

- ◇ Clearly write your name and student number on this exam, and **sign it** below to confirm that you will read and follow these **instructions**:
- ◇ This is a 3-hour **closed-book** practice final exam. **No notes.** **No calculators.**
- ◇ This exam consists of 22 questions on 18 pages (including this cover page). The total number of pretend points possible is 60 points.
- ◇ Questions 1–6 are **multiple-choice**. In each question, you must select the correct response. You do not need to justify your answers.
- ◇ Questions 7–10 are **true-or-false**. Circle the correct response. You do not need to justify your answers.
- ◇ Questions 11–17 are **short-answer**. Write the final answer in the appropriate answer box. Whenever indicated, you must **briefly justify your answers in order to receive full (pretend) marks**.
- ◇ Questions 18–22 are **long-answer**. To receive full (pretend) marks, your solution/proof must be complete, correct, and show all relevant details.
- ◇ Read all questions carefully and be sure to follow the instructions for the individual problems. You may ask for clarification.
- ◇ You must use **proper mathematical notation and terminology**.
- ◇ For rough work or additional space, you may use Page 18 or the backs of pages. **Do not use any of your own scrap paper.**

SOLUTIONS

FAMILY NAME:	STUDENT NUMBER:
FIRST NAME:	†SIGNATURE:

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada

585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

Table of Points for marking purposes. *Do not write in this table.*

	multiple-choice	true-or-false	short-answer	long-answer	TOTAL
max points possible	12	9	15	24	60 points
points obtained					

MULTIPLE-CHOICE QUESTIONS.

Write your choice in the answer box. No justification is needed.

Q1. Let $A = \{1, 2\}$ and let $B = \{s, t, u, v, w\}$. How many **injective** functions are there from A to B?

A. 20 **B.** 25 **C.** 32

D. 0 **E.** 10 **F.** None of the previous answers.

$\underbrace{f(1)}_{\text{5 choices}} \cdot \underbrace{f(2)}_{\text{4 choices}}$

Answer: $\boxed{A}$ **[2 points]**

(mod 5) ≡ 4, 3, 0, 1, 4, 0, 1, 2, 0

Q2. Let $\mathcal{R}$ be the equivalence relation on the set $A = \{-6, -2, 0, 1, 4, 5, 6, 7, 15\}$ defined by

$$x\mathcal{R}y \iff \Big(x \equiv y \pmod 5 \quad \textbf{or} \quad x \equiv -y \pmod 5 \Big)$$

Which one (if any) of the following statements is **true**?

~~A.~~ $\{-6, 6\}$ is an equivalence class with respect to $\mathcal{R}$. F. $[-6]_R = \{-6, 1, 4, 6\}$

~~B.~~ $[-2]_{\mathcal{R}} = [6]_{\mathcal{R}}$. F. $-2 \not\equiv 6 \pmod 5$ and $-2 \not\equiv -6 \pmod 5$ $\therefore -2 \not\mathcal{R}\, 6$

not equiv. classes w.r.t. $\mathcal{R}$.

~~C.~~ $\mathcal{P} = \{\{-6, -2\}, \{0\}, \{1, 4, 5, 6, 7, 15\}\}$ is the partition of A into equivalence classes.

D. $\mathcal{P} = \{\{-6, 1, 4, 6\}, \{-2, 7\}, \{0, 5, 15\}\}$ is the partition of A into equivalence classes. T

E. $4\mathcal{R}7$. F. $4 \not\equiv 7 \pmod 5$ and $4 \not\equiv -7 \pmod 5$ $\therefore 4 \not\mathcal{R}\, 7$.

F. None of the above.

Answer: $\boxed{D}$ **[2 points]**

Q3. One of the following statements is **false**. Which one?

A. $A \cap (B \cup \overline{C}) = (A \cap B) \cup (A \cap \overline{C})$ ✓

B. $\overline{A} \cap (B \cup C) = A \cup (\overline{B \cup C})$ X

C. $A \cup (\overline{B \cap C}) = \overline{\overline{A} \cap (B \cap C)}$ ✓

D. $(\overline{A} \cup \overline{B}) \cap C = (\overline{A} \cap C) \cup (\overline{B} \cap C)$ ✓

E. $(A \cup B) \cup \overline{C} = \overline{(\overline{A \cup B}) \cap C}$ ✓

Answer: $\boxed{B}$ **[2 points]**

Q4. How many strings of length 6 with symbols from $\{a, b, c, d, e\}$ start with '*ace*' or end with '*ee*'? (*this is inclusive or*)

A. 755 **B.** 36 **C.** 750 **D.** 35 **E.** 745 **F.** 34

$\underline{a}\,\underline{c}\,\underline{e}\,\underset{5}{\_}\,\underset{5}{\_}\,\underset{5}{\_}$ $5 \cdot 5 \cdot 5$ 5^3 start with 'ace'

$\underset{5}{\_}\,\underset{5}{\_}\,\underset{5}{\_}\,\underset{5}{\_}\,\underline{e}\,\underline{e}$ $5 \cdot 5 \cdot 5 \cdot 5$ 5^4 end with 'ee'

$\underline{a}\,\underline{c}\,\underline{e}\,\underset{5}{\_}\,\underline{e}\,\underline{e}$ 5^1 start with 'ace' and end with 'ee'

PIE: $5^3 + 5^4 - 5^1 = 125 + 625 - 5 = 745$ start with 'ace' <u>or</u> end with 'ee'.

Answer: $\boxed{E}$ **[2 points]**

Q5. Let G be a graph with 7 vertices and 10 edges. If every vertex of G has degree 2 or 3, then how many vertices of each degree does G have?

A. G has 0 vertices of degree 2 and 7 vertices of degree 3.

B. G has 1 vertex of degree 2 and 6 vertices of degree 3.

C. G has 2 vertices of degree 2 and 5 vertices of degree 3.

D. G has 3 vertices of degree 2 and 3 vertices of degree 3.

E. G has 4 vertices of degree 2 and 3 vertices of degree 3.

F. No such graph exists.

Let a = # vertices of degree 2 and b = # vertices of degree 3.

Then $a+b$ = total # vertices $=7$ and $2(\#\text{ of edges}) = \sum_{u \in V} \deg(u) = a(2)+b(3)$.

$a+b=7$

$2a+3b=20 \Rightarrow 2(7-b)+3b=20 \Rightarrow b=6 \Rightarrow a=1$

Answer: $\boxed{B}$ **[2 points]**

Q6. How many binary strings of length 7 contain at least 5 zeros?

A. 21 **B.** 32 **C.** 96 **D.** 4 **E.** 49 **F.** 29

G. None of the above.

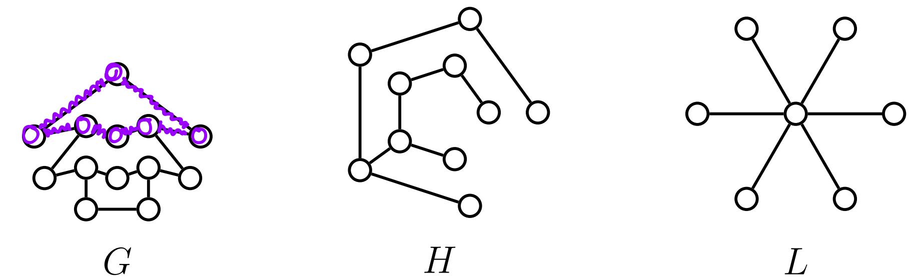
$$\binom{7}{5}+\binom{7}{6}+\binom{7}{7} = \frac{7!}{5!2!} + \frac{7!}{6!1!} + \frac{7!}{7!0!} = 21+7+1=29$$

Answer: $\boxed{F}$ **[2 points]**

TRUE-OR-FALSE QUESTIONS.

Circle the correct responses. No justification is needed.

Q7. Consider the following three graphs: **[1.5 points]**



Circle the correct responses.

G is a tree. **T** **F**

H is a forest. **T** **F**

L is a tree. **T** **F**

Q8. Consider the following set S: **[3 points]**

$$S = \{\emptyset, 1, \{1\}\} \qquad \text{and} \qquad T = \{1, \{\emptyset, 1\}\}$$

Circle the correct responses.

$|S \times T| = 9.$ **T** **F**

$\{\{1\}\} \subseteq S$ **T** **F**

$|S \cap T| = 2$ **T** **F**

$(\{1\}, \{1\}) \in T \times S$ **T** **F**

$|S \cup T| = 4$ **T** **F**

$|\mathcal{P}(T)| = 4$ **T** **F**

Q9. Consider the following relation on the set $\mathbb{Z}$: **[2 points]**

$$x\mathcal{R}y \quad \Longleftrightarrow \quad x(1+y) \quad \text{is even.}$$

What properties does the relation $\mathcal{R}$ possess? *Circle the correct responses.*

$\mathcal{R}$ is reflexive. $x(1+x)$ is even for all $x \in \mathbb{Z}$. **T** **F**

$\mathcal{R}$ is symmetric. No. counterexample: $2\mathcal{R}1$ but $1 \not\mathcal{R} 2$ **T** **F**

$\mathcal{R}$ is antisymmetric. No. counterexample: $1\mathcal{R}3$ and $3\mathcal{R}1$ but $1 \neq 3$ **T** **F**

$\mathcal{R}$ is transitive. Yes. Assume $x\mathcal{R}y$ and $y\mathcal{R}z$. **T** **F**

Case 1: If x is even, then $x(1+z)$ will be even $\Rightarrow x\mathcal{R}z$.

Case 2: If x is odd, then y must be odd since $x(1+y)$ is even.
Since y is odd, z must also be odd since $y(1+z)$ is even $\therefore$ $1+z$ is even so $x(1+z)$ is even $\therefore x\mathcal{R}z$

Q10. Consider the following propositions with atomic variables x and y: **[2.5 points]**

$P_1 : \quad x \wedge \neg y$

$P_2 : \quad \neg x \vee \neg y$

$P_3 : \quad x \to y$

$C : \quad y \to x$

x	y	P_1	P_2	P_3	C
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	T	T	F
F	F	F	T	T	T

Circle the correct responses.

The set $\{P_1, P_2, P_3\}$ is a consistent set of propositions. **T** **F**

The set $\{P_1, P_2\}$ is a consistent set of propositions. **T** **F**

The argument $(P_1 \wedge P_2 \wedge P_3) \to C$ is a valid argument. **T** **F**

The compound proposition $P_1 \wedge P_2 \to C$ is a tautology. **T** **F**

Compound propositions P_1 and $\neg P_3$ are logically equivalent. **T** **F**

Short-Answer Questions.

Write your final answer in the answer box.
Wherever indicated, you must **briefly justify your answers** to receive full marks.

Q11. Let a, b. and c be propositional variables.

Give a **disjunctive normal form (DNF)** for the following compound proposition:

$$P: \quad (a \leftrightarrow b) \wedge (b \vee \neg c)$$

DNF for P:

$(a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge \neg c)$

Justification: **[1.5 points]**

a	b	c	$a \leftrightarrow b$	$b \vee \neg c$	P: $(a \leftrightarrow b) \wedge (b \vee \neg c)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	F	F
F	F	F	T	T	T

conjunctive clauses

$a \wedge b \wedge c$

$a \wedge b \wedge \neg c$

$\neg a \wedge \neg b \wedge \neg c$

Q12. **Fully evaluate** the following expression:

$\binom{9}{6} + \binom{9}{7} =$ 120

Justification: **[1.5 points]**

$$\binom{9}{6} + \binom{9}{7} = \binom{10}{7}$$

$$= \frac{10!}{7!\,3!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4} \cdot \cancel{7!}}{\cancel{7!}\,\cancel{3} \cdot \cancel{2} \cdot 1} = 10 \cdot 3 \cdot 4 = 120$$

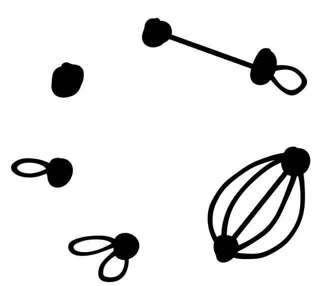
Q13. Consider the following sequence:

7 vertices

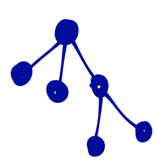
$$(0, 1, 2, 3, 4, 5, 5)$$ $\frac{1}{2}(0+1+2+3+4+5+5) = 10$ edges

(a) Does there exist a graph with the above degree sequence? *Circle:* **YES** **NO**

If so, draw an example of such a graph; otherwise, briefly explain why no such graph exists.



(b) Draw an example of a **tree** whose degree sequence is $(1, 1, 1, 1, 3, 3)$.



No justification is needed. **[2 points]**

Q14. Determine the **coefficient** of $\dfrac{1}{x^8}$ in the expansion of $\left(3x^2 + \dfrac{5}{x^4}\right)^{10}$.

Your answer may include unevaluated factorials, binomial coefficients, powers, products, or sums.

Coefficient of x^{-8}: 0.

Justification: **[2.5 points]**

$$\left(3x^2+\frac{5}{x^4}\right)^{10} = \sum_{i=0}^{10}\binom{10}{i}(3x^2)^{10-i}\left(\frac{5}{x^4}\right)^{i} = \sum_{i=0}^{10}\binom{10}{i}3^{10-i}5^{i}\cdot(x^2)^{10-i}\cdot(x^{-4})^{i}$$

$$= \sum_{i=0}^{10}\binom{10}{i}3^{10-i}5^{i}\cdot x^{20-6i}$$

For x^{-8} we need $i\in\{0,1,\dots,10\}$ such that $-8=20-6i \iff 6i=28$

$i=\frac{28}{6}\notin\mathbb{Z}$

Since there is no solution for $i\in\{0,1,\dots,10\}$ the coef. is zero.

Q15. Give an example of a compound proposition P such that

- P consists of the propositional variables x, y, and z (all three variables must be used).
- P contains **only** the logical connectives $\neg$ and $\rightarrow$ and P must contain **both** these connectives.
- P is a **contradiction** (justification required below).

Make sure your proposition P contains only the variables x, y, and z, the logical connectives $\neg$ and $\rightarrow$ and appropriate parentheses.

$P = \neg[(y \rightarrow z) \rightarrow (x \rightarrow x)]$ (one possible answer)

Justification that P is a contradiction: **[2 points]**

$$(x \wedge \neg x) \wedge (y \rightarrow z) \equiv F \wedge (y \rightarrow z) \equiv F$$

$$\begin{aligned} (x \wedge \neg x) \wedge (y \rightarrow z) &\equiv \neg(x \rightarrow x) \wedge (y \rightarrow z) \\ &\equiv (y \rightarrow z) \wedge \neg(x \rightarrow x) \\ &\equiv \neg[(y \rightarrow z) \rightarrow (x \rightarrow x)] \end{aligned}$$

Q16. Let $f : \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ be a function defined by $f(x) = (x^3, x^4)$. **[3.5 points]**

Answer the following questions regarding the above function f.

To justify your answers, either give a proof or a concrete numerical counterexample.

Is f is injective? *Circle:* **YES** **NO**

Justification (proof or counterexample):

Proof. Let $x_1, x_2 \in \mathbb{Q}$ (the domain of f).

Assume $f(x_1) = f(x_2)$. Then $((x_1)^3, (x_1)^4) = ((x_2)^3, (x_2)^4)$

$\Rightarrow x_1^3 = x_2^3$ and $x_1^4 = x_2^4$

$\Rightarrow x_1 = x_2$ and $x_1 = \pm x_2$

$\therefore x_1 = x_2$

so f is injective.

Is f is surjective? *Circle:* **YES** **NO**

Justification (proof or counterexample):

Counterexample: $(0,1) \in \mathbb{Q} \times \mathbb{Q}$ (the codomain of f).

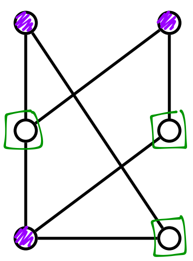
but $f^{-1}(0,1) = \emptyset$ because there is no $x \in \mathbb{Q}$ (the domain of f) such that $(x^3, x^4) = (0,1)$.

Is f is invertible? *Circle:* **YES** **NO**

No justification is needed for this part.

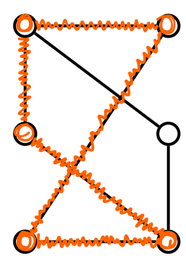
Q17. Which of the following four graphs are **bipartite**? **[2 points]**

For each graph, circle the correct response and justify your answer by either giving a proper 2-colouring of the graph, or by indicating an odd cycle in the graph.



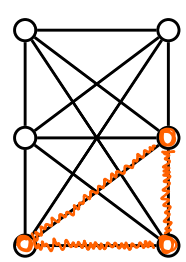
G

Is G bipartite? *Circle:* **YES** **NO**



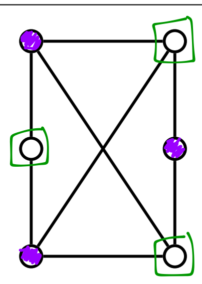
H

Is H bipartite? *Circle:* **YES** **NO**



K

Is K bipartite? *Circle:* **YES** **NO**



L

Is L bipartite? *Circle:* **YES** **NO**

Long-Answer Questions.

Detailed solutions are required.

Q18. Let m be a positive integer.

Give a **proof by contradiction** of the following statement:

Statement:

If $\underbrace{3m+1 \text{ marbles are placed into } m \text{ jars}}_{P}$, then $\underbrace{\text{at least one jar will contain at least 4 marbles}}_{Q}$.

Statement to be proved: $P \to Q$

Assume $\neg(P \to Q)$ is true. (goal: show that contradiction follows from this assumption)

Since $\neg(P \to Q) \equiv P \wedge \neg Q$ we are assuming

P: "$3m+1$ marbles are placed into m jars."

and $\neg Q$: "No jar contains more than 3 marbles."

$$\text{Then } 3m+1 = \begin{pmatrix}\text{total \# of}\\ \text{marbles placed}\\ \text{into jars}\end{pmatrix} \quad \text{(since we assumed } P \text{ is true)}$$

$$= \begin{pmatrix}\text{\# marbles}\\ \text{placed into}\\ \text{1st jar}\end{pmatrix} + \begin{pmatrix}\text{\# marbles}\\ \text{placed into}\\ \text{2nd jar}\end{pmatrix} + \cdots + \begin{pmatrix}\text{\# marbles}\\ \text{placed into}\\ m\text{th jar}\end{pmatrix}$$

$$\leq 3 + 3 + \cdots + 3 \quad \text{(since we assumed } \neg Q \text{ is true)}$$

$$= 3m$$

$\therefore 3m+1 \leq 3m$ ↯ this is a contradiction. $\therefore$ $P \to Q$ must be true. ■

[**5 points**]

Q19. Use **Mathematical Induction** to prove that

$$(2n)! \quad \text{is a multiple of} \quad 2^{n+1} \qquad\qquad \text{for all integers } n \geq 2.$$

Clearly state the proposition to be proved, Basis of Induction, Induction Hypothesis, and Induction Step. Clearly indicate where the Induction Hypothesis is used in your proof.

$P(n)$: " $(2n)!$ is a multiple of 2^{n+1}. "

BOI: $n_0 = 2$ $\quad P(2)$: " $(2(2))!$ is a multiple of 2^{2+1}. "

check: $(2(2))! = 4! = 4 \cdot 3 \cdot 2 = 3(8)$ thus $4!$ is a multiple of 2^3. So $P(2)$ is True.

I.H. Assume $P(k)$ is true for some $k \geq n_0 = 2$.

ie Assume $(2k)!$ is a multiple of 2^{k+1}

I.S. Prove that $P(k+1)$ follows from I.H. ie prove $P(k) \rightarrow P(k+1)$.

$P(k+1)$ says " $(2(k+1))!$ is a multiple of 2^{k+1+1} "

Let's show that $P(k+1)$ is true:

$$\begin{aligned}
(2(k+1))! &= (2k+2)! \\
&= (2k+2)(2k+1)\cdot(2k)! \\
&= (2k+2)(2k+1)\cdot[m \cdot 2^{k+1}] \text{ for some integer } m \quad \text{(using the IH here: } (2k)! \text{ is a multiple of } 2^{k+1}) \\
&= 2(k+1)\cdot(2k+1)\cdot m \cdot 2^{k+1} \\
&= (k+1)(2k+1)(m)\cdot 2^{k+2} \\
&= [(k+1)\cdot(2k+1)\cdot(m)]\cdot 2^{k+1+1} \\
&= \ell \cdot 2^{k+1+1} \text{ for } \ell = (k+1)(2k+1)(m). \text{ Thus } \ell \in \mathbb{Z}.
\end{aligned}$$

$\therefore (2(k+1))!$ is a multiple of 2^{k+1+1} Thus $P(k+1)$ is True.

Conclusion. Since $P(2)$ is True and since we proved $P(k) \rightarrow P(k+1)$ is true for any $k \geq 2$, it follows from the principle of Mathematical Induction that $P(n)$ is true for all integers $n \geq 2$.

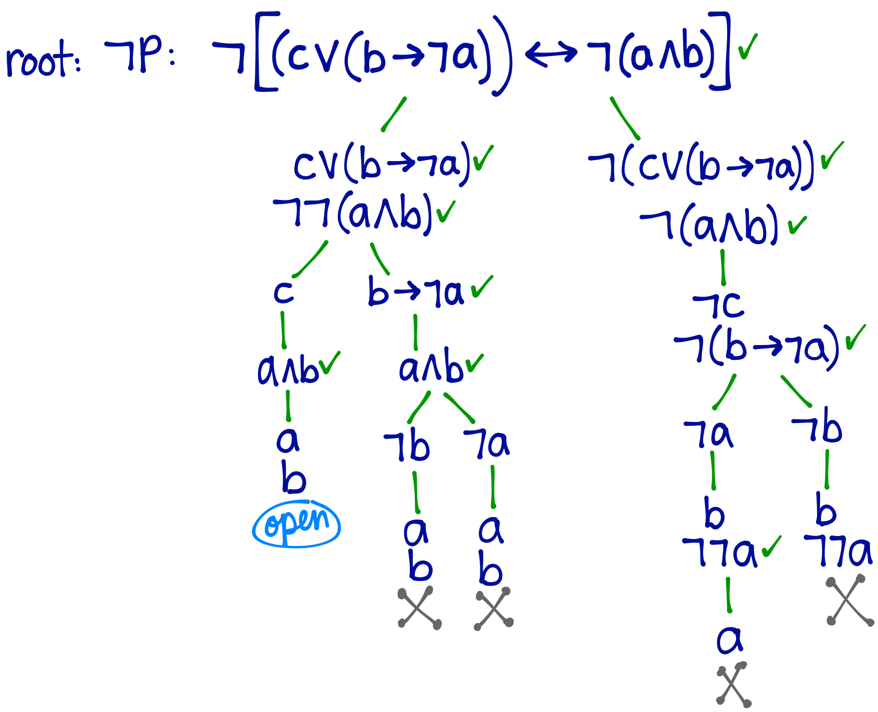
[5 points]

Q20. Use a **truth tree** to determine whether or not the proposition P below is a **tautology**. If you claim that P is not a tautology, give **all** counterexamples.

$$P: \quad \big(c \vee (b \to \neg a)\big) \leftrightarrow \neg(a \wedge b)$$

Clearly indicate the root of the tree. Use the branching rules precisely as taught in class. Do not use equivalences. Do not skip steps or combine branching rules. Make sure your truth tree is fully grown.

Complete truth tree:



Is P a tautology?	*Circle:*	**YES**	**NO**

If you circled **NO**, give **all counterexamples**:

$a = T$
$b = T$
$c = T$.

[**5 points**]

Q21. Define a binary relation $\mathcal{R}$ on the set $A = \{\text{binary strings of length 4}\}$ as follows:

For all strings $s, t \in A$,

$$s\mathcal{R}t \qquad \text{if and only if} \qquad s \text{ and } t \text{ have the same number of ones.}$$

(a) Prove that $\mathcal{R}$ is an equivalence relation.

[ref.] Let $s \in A$. Then s has the same # of ones as itself $\Rightarrow s\mathcal{R}s$.

$\therefore \mathcal{R}$ is reflexive.

[sym.] Let $s, t \in A$. Assume $s\mathcal{R}t$ (goal: show $t\mathcal{R}s$)

Then s and t have the same # of ones.

$\Rightarrow$ t and s have the same # of ones.

$\Rightarrow t\mathcal{R}s$ $\therefore \mathcal{R}$ is symmetric.

[trans.] Let $s, t, u \in A$. Assume $s\mathcal{R}t$ and $t\mathcal{R}u$ (goal: show $s\mathcal{R}u$)

Then s and t have the same # of ones, and t and u have the same # of ones.

$\Rightarrow$ s and u have the same # of ones.

$\Rightarrow s\mathcal{R}u$ $\therefore \mathcal{R}$ is transitive.

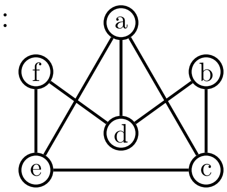
Since we proved that $\mathcal{R}$ is reflexive, symmetric, and transitive, it follows that $\mathcal{R}$ is an equivalence relation on A.

(b) Give the **equivalence class** of the string $s =$ '1010' with respect to the relation $\mathcal{R}$.

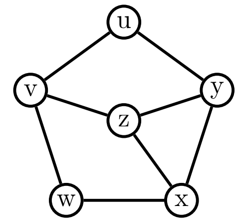
$[\text{'1010'}]_{\mathcal{R}} = \{$'1010', '1100', '1001', '0110', '0101', '0011'$\}$

[5 points]

Q22a. Consider the following two graphs: **[4 points]**



G



H

Are G and H isomorphic? *If so, give an isomorphism between them and verify that your function is an isomorphism; otherwise, clearly explain why they are not isomorphic.*

define $\varphi : V(G) \to V(H)$ as follows:

$\alpha \in V(G)$	a	b	c	d	e	f
$\varphi(\alpha) \in V(H)$	z	u	y	v	x	w

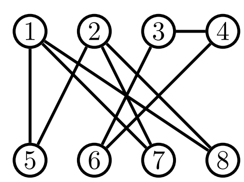
edges of G	corresponding edges of H
$\{a,d\}$	$\{\varphi(a),\varphi(d)\} = \{z,v\}$ ✓
$\{a,e\}$	$\{\varphi(a),\varphi(e)\} = \{z,x\}$ ✓
$\{a,c\}$	$\{\varphi(a),\varphi(c)\} = \{z,y\}$ ✓
$\{b,c\}$	$\{\varphi(b),\varphi(c)\} = \{u,y\}$ ✓
$\{b,d\}$	$\{\varphi(b),\varphi(d)\} = \{u,v\}$ ✓
$\{c,e\}$	$\{\varphi(c),\varphi(e)\} = \{y,x\}$ ✓
$\{d,f\}$	$\{\varphi(d),\varphi(f)\} = \{v,w\}$ ✓
$\{e,f\}$	$\{\varphi(e),\varphi(f)\} = \{x,w\}$ ✓

Since, for all vertices $\alpha, \beta \in V(G)$, we have

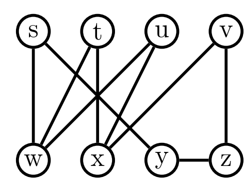
$$\{\alpha,\beta\} \in E(G) \iff \{\varphi(\alpha),\varphi(\beta)\} \in E(H)$$

and since $\varphi : V(G) \to V(H)$ is a bijection, it follows that φ is an isomorphism from G to H. $\therefore G \cong H$.

Q22b. Now consider these two graphs:



K



L

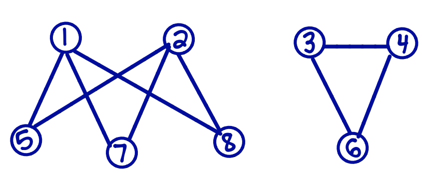
Are K and L isomorphic? *If so, give an isomorphism between them and verify that your function is an isomorphism; otherwise, clearly explain why they are not isomorphic.*

K and L are not isomorphic:

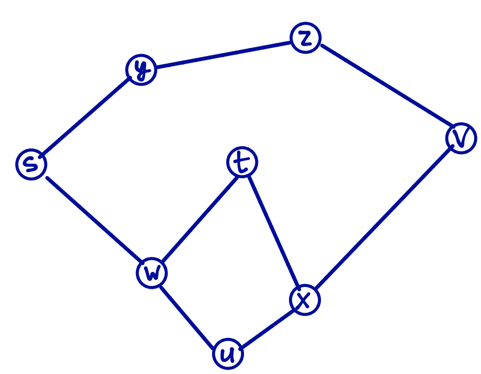
K is not a connected graph but L is connected.

redrawn:

K:



L:



Extra page for scrap work.

:-)