
1. Propositional Logic

WHY MATHEMATICS IS AMAZING!

- Mathematics is a form of Art. The masterpieces of math are elegant arguments (proofs) that tell us some fragment of truth.
- Unlike other sciences where results are most often obtained from experimental data and observation, mathematical results are “discovered” when someone is able to prove that something must be true.
- For a proof to be correct, the underlying steps must be a valid logical argument and the language we use must be well-defined and exact.
- It is useful to know when something is true! Many scientific results are based on underlying mathematical models → physics, chemistry, biology, engineering...
- In this course, we will study DISCRETE MATH — the part of mathematics devoted to the study of “discrete” objects (consisting of distinct “whole” parts or unconnected elements).

Ex. DNA sequences = fundamental building blocks of living things (RNA)
 or = finite strings made of 4 symbols from the set {T(U), C, A, G}

translation: RNA ex. —AGC[AUC]GCAUCGA—

- From DNA to RNA to building proteins from amino acids encoded by 3-letter “words” called codons
- Logic itself is especially important in Computer Science; the language of computers is Boolean algebra. Using logic, we are able to translate our intentions so that a machine made of electrical circuits does our bidding.

THE ROUGH PLAN FOR MAT1348

- We will begin by studying Propositional Logic.
- We will work our way towards writing and understanding correct Mathematical Proofs.
- We will study the basics of Set Theory, Functions, and Relations (especially Equivalence Relations).
- We will learn basic methods of Counting, and Proof by Induction — these are fundamental tools for math and computer science.
- We will end with an introduction to Graph Theory, which is an essential tool for modelling many discrete structures.

Warnings!

- This course is very fast-paced.
- You will be given **many** new and precise definitions and techniques to use.
- Having precise (lawyer-like?) language is very important — when we communicate (in the form of written mathematical proofs, computer programs, and otherwise), we want to be certain we understand each other exactly.

* These notes are solely for the personal use of students registered in MAT1348.

BASIC BUILDING BLOCKS OF LOGIC

- A **proposition** is a declarative sentence that is either **true** or **false** but not both.
 - If a proposition is true, then its **truth value** is **TRUE**, denoted **T**.
 - If a proposition is false, then its **truth value** is **FALSE**, denoted **F**.
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Example 1.1. For each of the following sentences, determine whether or not it is a proposition. If it is a proposition, what is its truth value? If it is not a proposition, explain.

- (a) Ottawa is the capital of Canada. ← a proposition ✓ It's T.
- (b) $1 + 2 = 3$. ← a proposition ✓ It's T.
- (c) $2 + 2 = 3$. ← a proposition ✓ It's F.
- (d) $1 + x = 3$. ← not a proposition unless we know what x is.
- (e) There exists an integer x such that $1 + x = 3$. ← a proposition ✓ It's T.
- (f) For all integers x , the equation $1 + x = 3$ is true. ← a proposition ✓ It's F.
- (g) Eat your vegetables. ← not a proposition since it's not a declarative statement
- (h) What time is it? ← not a proposition since it's not a declarative statement
- (i) → All books in the library are yellow.
[a proposition ✓ It's F (at uOttawa library)]
- (j) → At least 2 books in the library are yellow.
[a proposition ✓ It's T (at uOttawa library)]

Propositional variables (a.k.a atoms) and compound propositions.

- For short, we will use variables to represent propositions.
- The variables we use to represent propositions are called **propositional variables** or “**atoms**” for short.

Ex. p : “Tigers are furry animals.”

Ex. q : “Elephants are wise.”

- A **compound proposition** is formed by combining one or more atoms using **logical connectives**.

LOGICAL CONNECTIVES

Negation. Let p be a proposition.

- The negation of p , denoted $\neg p$, read "not p ", is the proposition "It is not the case that p ."
- The truth value of $\neg p$ is the opposite of the truth value of p .

Truth table:	
p	$\neg p$
T	F
F	T

Conjunction. Let p and q be propositions.

- The conjunction of p and q , denoted $p \wedge q$, read " p and q ", is the proposition " p and q "
- $p \wedge q$ is T when both p and q are T; otherwise, $p \wedge q$ is F.

Truth table:		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction. Let p and q be propositions.

- The disjunction of p and q , denoted $p \vee q$, read " p or q ", is the proposition " p or q "
 - $p \vee q$ is F when both p and q are F; otherwise, $p \vee q$ is T.
- \Rightarrow "or" is the inclusive or i.e. "and/or"

Truth table:		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or. Let p and q be propositions.

- The exclusive or of p and q , denoted $p \oplus q$, read " p x-OR q ", is the proposition " p or q but not both"
- $p \oplus q$ is T when exactly one of p and q is T and the other is F; otherwise, $p \oplus q$ is F.

Truth Table:		
p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statement / Implication. Let p and q be propositions.

- The conditional statement "if p , then q ", is denoted $p \rightarrow q$, read " p implies q ", is the proposition "If p , then q ".

p is the premise $\xrightarrow{p \rightarrow q}$ q is the conclusion

- $p \rightarrow q$ is F when p is T and q is F ; otherwise, $p \rightarrow q$ is T .

- $p \rightarrow q$ is like a promise or bargain. If p is T , then q is promised to be T . The only way a promise is broken is when p is T but q is F .

Note: $[F \rightarrow (\text{anything})]$ is T .
 $[(\text{anything}) \rightarrow T]$ is T .

Truth Table:		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional Statement. Let p and q be propositions.

- The biconditional statement " p if and only if q " is denoted $p \leftrightarrow q$, read " p if and only if q ".

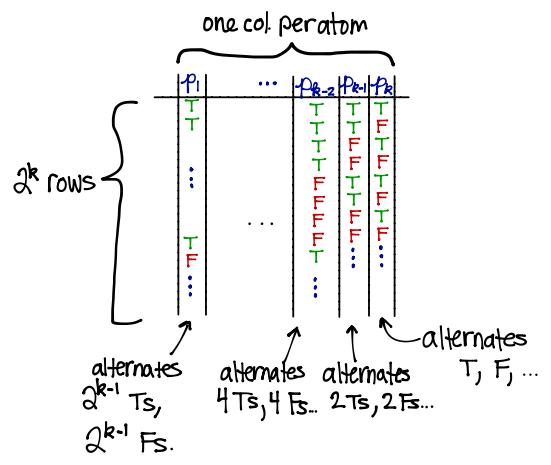
- $p \leftrightarrow q$ is T when p and q have the same truth value (both T or both F); otherwise, $p \leftrightarrow q$ is F .

Truth Table:		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TRUTH TABLES

When making a truth table for a compound proposition consisting of atoms p_1, p_2, \dots, p_k , we will always use the following convention:

- 2^k rows for k atoms p_1, \dots, p_k
- truth values always T s and F s (not 0 s/ 1 s)
- Each atom has its own column.
- For rightmost atom's column, we alternate T, F, T, F, \dots from top to bottom of column
- Moving one column to the left, the next atom's column alternates twice as many T s and twice as many F s as the column on its immediate right



Example 1.2. Write a truth table for each of the following compound propositions:

- $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
- $(p \oplus q) \rightarrow \neg q$
- $(p \rightarrow q) \wedge (q \rightarrow p)$
- $\neg(\neg p \vee q)$
- $(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$
- $q \leftrightarrow (p \leftrightarrow r)$

a)	$\neg p$	q	$p \rightarrow q$	$\neg p \vee q$	$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
	T	T	T	T	T
	T	F	F	F	T
	F	T	T	T	T
	F	F	T	T	T

b)	$\neg p$	q	$p \oplus q$	$\neg q$	$(p \oplus q) \rightarrow \neg q$
	T	T	F	F	T
	T	F	T	T	T
	F	T	T	F	F
	F	F	F	T	T

c)	$\neg p$	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
	T	T	T	T	T
	T	F	F	T	F
	F	T	T	F	F
	F	F	T	T	T

d)	$\neg p$	q	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$
	T	T	F	T	F
	T	F	F	F	T
	F	T	T	T	F
	F	F	T	T	F

e)	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg p \vee \neg q$	$(\neg p \wedge \neg q) \leftrightarrow (\neg p \vee \neg q)$
	T	F	F	T	F
	F	T	F	T	F

f)

p	q	r	$p \leftrightarrow r$	$q \leftrightarrow (p \leftrightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	T
F	T	T	F	F
F	T	F	T	T
F	F	T	F	T
F	F	F	T	F

TAUTOLOGIES, CONTRADICTIONS, AND CONTINGENCIES

- A compound proposition that is always true (in all rows of its truth table) is called a **tautology**.
- A compound proposition that is always false (in all rows of its truth table) is called a **contradiction**.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example 1.3. For each compound proposition given in Example 1.2, determine whether it is a tautology, contradiction, or contingency.

a) $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is always T \therefore tautology

b) $(p \oplus q) \rightarrow \neg q$ is sometimes T, sometimes F \therefore contingency

c) $(p \rightarrow q) \wedge (q \rightarrow p)$ is sometimes T, sometimes F \therefore contingency

d) $\neg(\neg p \vee q)$ is sometimes T, sometimes F \therefore contingency

e) $(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$ is always F \therefore contradiction

f) $q \leftrightarrow (p \leftrightarrow r)$ is sometimes T, sometimes F \therefore contingency

STUDY GUIDE

- ◊ proposition propositional variable / atom compound proposition
- ◊ truth value truth table (how many rows?)
- ◊ logical connectives: negation \neg disjunction (or) \vee conjunction (and) \wedge
implication \rightarrow biconditional (if and only if) \leftrightarrow exclusive or (XOR) \oplus
- ◊ tautology contradiction contingency