

14. Relations

Let A and B be sets. A (binary) relation from A to B is a subset of $A \times B$.

Let A be a set. A (binary) relation on a set A is a relation from A to itself.
ie a subset of $A \times A$

Example 14.1. Let $A = \{1, 2, 3\}$ and let $B = \{x, y\}$

$$R_1 = \{(1, x), (1, y), (3, y)\} \quad R_1 \subseteq A \times B \therefore R_1 \text{ is a relation from } A \text{ to } B$$

$$R_2 = \{(x, x), (y, x)\} \quad R_2 \subseteq B \times B \therefore R_2 \text{ is a relation on } B$$

$$R_3 = \{(1, 2), (2, 1), (2, 3), (3, 2)\} \quad R_3 \subseteq A \times A \therefore R_3 \text{ is a relation on } A$$

$$R_4 = \{(x, 1), (y, 2)\} \quad R_4 \subseteq B \times A \therefore R_4 \text{ is a relation from } B \text{ to } A$$

Notation for Relations

- since $(1, x) \in R_1$, it means "1 is related to x by R_1 "
- since $(2, x) \notin R_1$, it means "2 is not related to x by R_1 "

Special Relation Notation:

- for short, we will write $1 R_1 x$ for "1 is related to x by R_1 "
and $2 \not R_1 x$ for "2 is not related to x by R_1 "

Representing Relations: a finite list of pairs of related elements VS. a rule to relate elements

For a small finite relation, we can simply write it as a list (set) of ordered related pairs of elements:

$$\text{ex } R_3 = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$$

Alternatively, some relations may be described by a rule that tells us exactly when/how two elements are related:

ex for all $u, v \in A = \{1, 2, 3\}$, $u R_3 v$ if and only if uv is odd.

EXAMPLES OF RELATIONS YOU'VE ALREADY ENCOUNTERED

Example 14.2. Friends and Family

Ex for all people A,B, $A \mathcal{R} B$ if and only if A and B share a common ancestor.

Ex for all Facebook users X,Y, $X \mathcal{R} Y$ if and only if X and Y are "friends"

Example 14.3. Graphs

The graph of a function $f: A \rightarrow B$ is the set of coordinate pairs $\mathcal{G} = \{(a, f(a)) : a \in A\}$

$\mathcal{G} \subseteq A \times B \therefore \mathcal{G}$ is a relation from A to B.

rule for \mathcal{G} : for all $a \in A, b \in B$, $a \mathcal{G} b$ if and only if $f(a) = b$

Ex the graph of the equation $x^2 + y^2 = 1$ is a subset of $\mathbb{R}^2 \therefore$ it's a relation on \mathbb{R}

rule: for all $x, y \in \mathbb{R}$, $x \mathcal{R} y$ if and only if $x^2 + y^2 = 1$

Example 14.4. Equality

$=$ is a relation on \mathbb{R}

Let \mathcal{E} be the relation on the set of real numbers defined by the rule

For all $x, y \in \mathbb{R}$, $x \mathcal{E} y$ if and only if $x = y$

Ex $1 \mathcal{E} 1$ because $1 = 1$ so $(1, 1) \in \mathcal{E}$

Ex $1 \not\mathcal{E} \pi$ because $1 \neq \pi$ so $(1, \pi) \notin \mathcal{E}$

Example 14.5. Inequality

\leq is a relation on \mathbb{R}

$<$ is a relation on \mathbb{R}

Ex 2 is related to 2 because $2 \leq 2$

Ex 2 is related to 2.1 because $2 < 2.1$

2 is related to 1000.1 because $2 \leq 1000.1$

2 is not related to 2 because $2 < 2$ is false.

2 is not related to 0 because $2 \not\leq 0$

Example 14.6. Divides

$|$ ("divides") is a relation on \mathbb{Z}

forall $m, n \in \mathbb{Z}$, $m | n$ if and only if $n = km$ for some integer k .

Ex $5 | 100$ $100 \not| 5$ $3 | 99$

← these are not fractions! Never forget:
"divides" is a relation on \mathbb{Z} , not the arithmetic operation of division!

Example 14.7. Logical Equivalence

\equiv is a relation on the set of all compound propositions

rule: for all propositions P, Q , $P \equiv Q$ if and only if $P \leftrightarrow Q$ is a tautology

PROPERTIES OF RELATIONS ON A SET

[Reflexive] A relation \mathcal{R} on a set A is called **reflexive** if

the implication $(x \in A) \rightarrow (x, x) \in \mathcal{R}$ is true.

Equivalently, $(x \in A) \rightarrow (x \mathcal{R} x)$

[Symmetric] A relation \mathcal{R} on a set A is called **symmetric** if for all $x, y \in A$,

the implication $((x, y) \in \mathcal{R}) \rightarrow ((y, x) \in \mathcal{R})$ is true.

Equivalently, $(x \mathcal{R} y) \rightarrow (y \mathcal{R} x)$

[Antisymmetric] A relation \mathcal{R} on a set A is called **antisymmetric** if

for all $x, y \in A$, the implication $((x, y) \in \mathcal{R} \text{ and } (y, x) \in \mathcal{R}) \rightarrow (x = y)$ is true.

Equivalently, $(x \mathcal{R} y \text{ and } y \mathcal{R} x) \rightarrow (x = y)$

↗ this should remind you of $(x \leq y \text{ and } y \leq x) \rightarrow (x = y)$

contrapositive form: $(x \neq y) \rightarrow ((x, y) \notin \mathcal{R} \text{ or } (y, x) \notin \mathcal{R})$

[Transitive] A relation \mathcal{R} on a set A is called **transitive** if

for all $x, y, z \in A$,

the implication $((x, y) \in \mathcal{R} \text{ and } (y, z) \in \mathcal{R}) \rightarrow ((x, z) \in \mathcal{R})$ is true.

Equivalently, $(x \mathcal{R} y \text{ and } y \mathcal{R} z) \rightarrow (x \mathcal{R} z)$

↗ this should remind you of $(x \leq y \text{ and } y \leq z) \rightarrow (x \leq z)$

EQUIVALENCE RELATIONS

A relation \mathcal{R} on a set A is called **an equivalence relation** if

\mathcal{R} is reflexive, symmetric, and transitive.

EXAMPLES OF RELATIONS ON A SET AND THEIR PROPERTIES

Example 14.8. Let \mathcal{R}_2 be a relation on \mathbb{Z} defined by the rule

for all $a, b \in \mathbb{Z}$, $(a, b) \in \mathcal{R}_2$ if and only if $a + b$ is even

Examples $(0, 0) \in \mathcal{R}_2$ because $0+0=0$ is even

$(2, -23) \notin \mathcal{R}_2$ because $2+(-23)=-21$ is not even

$5 \mathcal{R}_2 -55$ (meaning $(5, -55) \in \mathcal{R}_2$) because $5+(-55)=-50$ is even

$2 \mathcal{R}_2 3$ (meaning $(2, 3) \notin \mathcal{R}_2$) because $2+3=5$ is not even

PROPERTIES

[reflexive] To prove \mathcal{R}_2 is reflexive, we must prove $(a \in \mathbb{Z}) \rightarrow (a \mathcal{R}_2 a)$

Let $a \in \mathbb{Z}$. Then $a+a=2a$ and $a \in \mathbb{Z}$

$\therefore a+a$ is even. (def of even)

$\therefore a \mathcal{R}_2 a$ (by the rule for \mathcal{R}_2) $\therefore \mathcal{R}_2$ is reflexive.

[symmetric] To prove \mathcal{R}_2 is symmetric, we must prove $a \mathcal{R}_2 b \rightarrow b \mathcal{R}_2 a$.

Let $a, b \in \mathbb{Z}$ be arbitrary elements of the set \mathbb{Z} .

Assume $a \mathcal{R}_2 b$. (goal is to prove $b \mathcal{R}_2 a$.)

Then $a+b$ is even (by the rule for \mathcal{R}_2)

$\Rightarrow b+a$ is even (since $a+b=b+a$)

$\Rightarrow b \mathcal{R}_2 a$ (by the rule for \mathcal{R}_2) $\therefore \mathcal{R}_2$ is symmetric

[antisymmetric] To prove \mathcal{R}_2 is antisymmetric, we must prove

$\lceil (a \mathcal{R}_2 b) \wedge (b \mathcal{R}_2 a) \rceil \rightarrow [a=b]$ for all $a, b \in \mathbb{Z}$

Wait! This is not true for all $a, b \in \mathbb{Z}$

Here is a counterexample:

$3, 7 \in \mathbb{Z}$

$3+7$ is even and $7+3$ is even

\Rightarrow both $(3, 7) \in \mathcal{R}_2$ and $(7, 3) \in \mathcal{R}_2$, but $3 \neq 7$. $\therefore \mathcal{R}_2$ is not antisymmetric

[transitive]

To prove R_2 is transitive, we must prove $[(aR_2b) \wedge (bR_2c)] \rightarrow [aR_2c]$

Let $a, b, c \in \mathbb{Z}$ be arbitrary elements of the set \mathbb{Z} .

Assume aR_2b and bR_2c . (goal is to prove aR_2c)

Then $a+b$ is even and $b+c$ is even. (by R_2 's rule)

$$\Rightarrow a+b = 2k \text{ for some } k \in \mathbb{Z}$$

$$\text{and } b+c = 2l \text{ for some } l \in \mathbb{Z}$$

(by def. of even).

$$\Rightarrow a = 2k - b \text{ and } c = 2l - b$$

$$\text{Thus, } a+c = (2k-b) + (2l-b)$$

$$= 2k + 2l - 2b$$

$$= 2(k+l-b)$$

$$= 2j \text{ where } j = k+l-b \quad \text{so } j \in \mathbb{Z}$$

$\therefore a+c$ is even.

$\therefore aR_2c$ (by R_2 's rule)

$\therefore R_2$ is transitive

[equivalence relation]

To prove R_2 is an equivalence relation on \mathbb{Z} , we must prove that

R_2 is reflexive, symmetric, and transitive \leftarrow we did prove these

✓ ✓ ✓

$\therefore R_2$ is an equivalence relation on \mathbb{Z} .

STUDY GUIDE

Important terms and concepts:

relation from A to B

$$R \subseteq A \times B$$

relation on A

$$R \subseteq A \times A$$

x is related to y by R

$$(x, y) \in R \quad x R y$$

important relations on \mathbb{Z}

$$= \leq < |$$

properties of relations:

reflexive

symmetric

antisymmetric

transitive

Exercises

Sup.Ex. §6 # 1, 2, 3, 4, 5, 6, 7, 9

Rosen §9.1 # 1, 3, 5, 6, 7, 8, 9, 10, 42, 43, 44, 45a