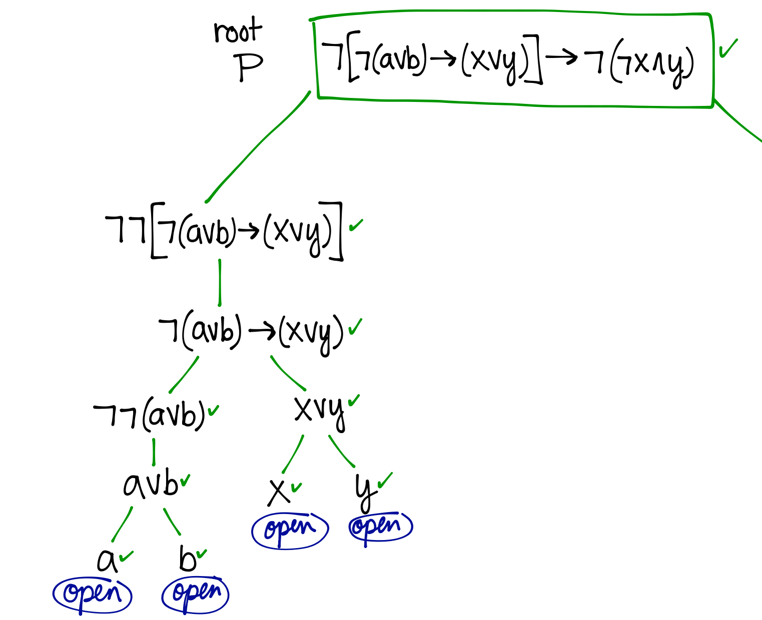
DGD 4

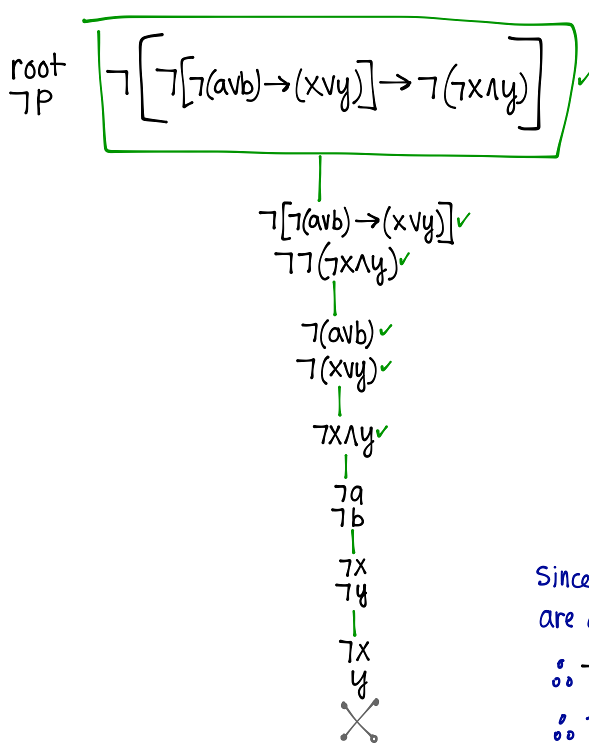
Q1. Truth Trees

Use appropriate truth tree(s) to determine whether the proposition P, defined as follows, is a contradiction, tautology, or contingency:

$$P: \quad \neg\big(\neg(a \vee b) \to (x \vee y)\big) \to \neg(\neg x \wedge y)$$



given that there is already at least one complete open path, we know the proposition at the root (P) can be True. (which means we don't need to complete the rest of the tree)



Since all paths (only one path in this case) are dead, the root ¬P can never be T.

∴ ¬P is a contradiction

∴ P is a tautology

Q2. VALID ARGUMENT

Q2a. Translate the following argument into propositional logic. Then use an appropriate **truth tree** to determine whether it's valid or not. If it's valid, explain how you know this based on your tree. If it's invalid, give all counterexamples.

If it is hungry, then the bear eats berries or the bear eats trout. The bear eats berries only if it does not see trout. Whenever the bear sees trout and is hungry, it eats trout. The bear eats berries. Therefore, the bear is not hungry.

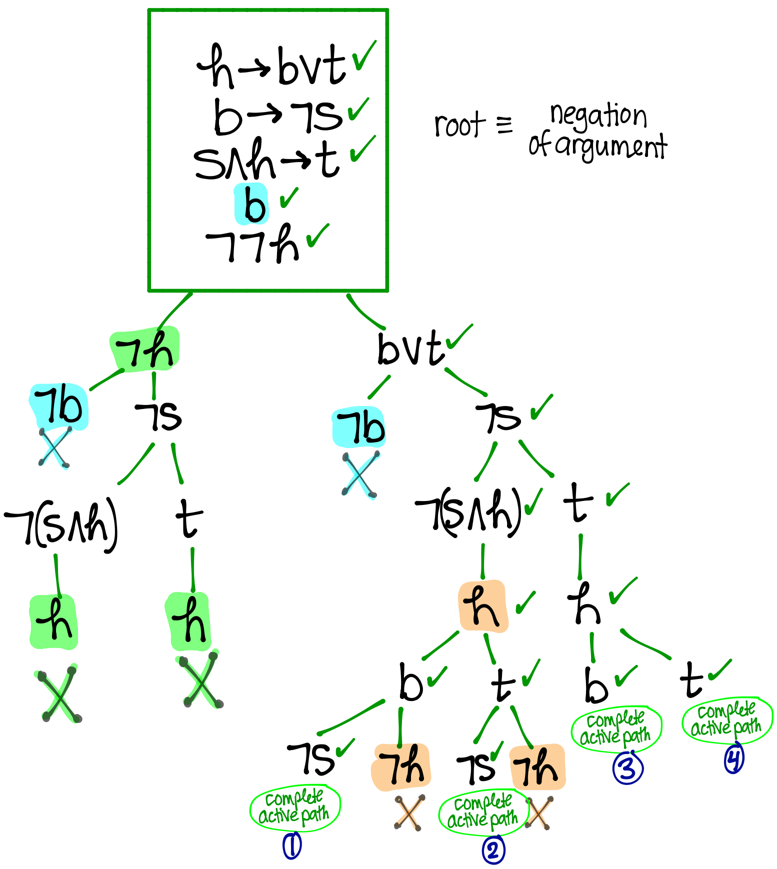
Use the propositional variables:

h: The bear is hungry.
b: The bear eats berries.
t: The bear eats trout.
s: The bear sees trout.

translation of argument:

P_1: $h \rightarrow b \vee t$
P_2: $b \rightarrow \neg s$
P_3: $s \wedge h \rightarrow t$
P_4: b
$\therefore$ C: $\neg h$

root ≡ negation of argument



The argument's negation is not a contradiction.

$\therefore$ the argument itself is not a tautology, hence it's an invalid argument.

Counterexamples:

① b=T, h=T, s=F
② b=T, h=T, s=F, t=T
③ b=T, h=T, s=F, t=T
④ b=T, h=T, s=F, t=T

For each of the above truth assignments, all premises are true but the conclusion is False.

Q2b. Now, use a truth table to verify your answer from **Q2b.**. Explain your conclusion based on the table. If the argument is invalid, give all counterexamples. Consider the connections/differences between your solution with the truth-tree approach compared with the truth-table approach.

	h	b	t	s	P_1: $h \to (b \vee t)$	P_2: $b \to \neg s$	P_3: $(s \wedge h) \to t$	P_4: b	C: $\neg h$	
	T	T	T	T	T	F	T	T	F	
(i)	T	T	T	F	T	T	T	T	F	← counterexample (all premises are T but conclusion is F)
	T	T	F	T	T	F	F	T	F	
(ii)	T	T	F	F	T	T	T	T	F	← counterexample (all premises are T but conclusion is F)
	T	F	T	T	T	T	T	F	F	
	T	F	T	F	T	T	T	F	F	
	T	F	F	T	F	T	F	F	F	
	T	F	F	F	F	T	T	F	F	
	F	T	T	T	T	F	T	T	T	
	F	T	T	F	T	T	T	T	T	
	F	T	F	T	T	F	T	T	T	
	F	T	F	F	T	T	T	T	T	
	F	F	T	T	T	T	T	F	T	
	F	F	T	F	T	T	T	F	T	
	F	F	F	T	T	T	T	F	T	
	F	F	F	F	T	T	T	F	T	

There are two distinct counterexamples:

(i) when h=T, b=T, t=T, s=F, the argument is false

(ii) when h=T, b=T, t=F, s=F, the argument is false

Note that the complete active path ① in the truth tree already told us both of the above counterexample since ① implied that the argument is False when ① b=T, h=T, s=F ← because this complete open path did not include a literal for the variable t, it means t can be either T or F while b=T, h=T, s=F, and the argument will be false regardless of the truth value of t. The other 3 complete active path counterexamples did not give us any "new" information.

Observations:

- In a truth table, each distinct row corresponds to a distinct truth assignment
- In a truth tree, distinct complete active paths may actually correspond to the same truth assignment

Q3. PROOF STRATEGIES — DIRECT PROOF

Definition. Let d and n be positive integers. We say that d **divides** n if there exists an integer k such that $n = kd$. It means the **remainder** of n divided by d is 0.

Note. If d does not divide n, then there exist positive integers k and r such that $n = kd + r$, and $0 < r < d$. We call r the **remainder** of n divided by d. Indeed, if d divides n, then $n = kd + 0$ for some integer k, i.e. the remainder is $r = 0$.

For example, 2 divides -100 because $-100 = (-50)(2)$ and -50 is an integer.

For example, 3 does not divide 22 because $22 = (7)(3) + 1$ (so the remainder of 22 divided by 3 is not 0).

Give a **direct proof** of each of the following theorems:

Theorem 1. Let n be an integer. If $\underbrace{3 \text{ divides } (n+1)}_{P}$, then $\underbrace{3 \text{ divides } (n^3+n+2)}_{Q}$.

$P \longrightarrow Q$

Direct proof of Theorem 1. Let n be an integer.

Assume P is true. ie Assume 3 divides $n+1$. (goal: prove 3 divides n^3+n+2)

Then $n+1 = 3k$ for some integer k (by def. of divides)

Consequently, $n = 3k-1$

$$\begin{aligned} \text{So } n^3+n+2 &= (3k-1)^3 + (3k-1) + 2 \\ &= 27k^3 - 27k^2 + 9k - 1 + 3k - 1 + 2 \\ &= 27k^3 - 27k^2 + 12k \\ &= 3[9k^3 - 9k^2 + 4k] \\ &= 3j \quad \text{where } j = 9k^3 - 9k^2 + 4k \text{ so } j \text{ is an integer.} \end{aligned}$$

$\therefore$ n^3+n+2 is an integer multiple of 3 ie $3 \mid (n^3+n+2)$

so Q is true. $\therefore$ we proved $P \to Q$ is true ■

Theorem 2. Let M be an integer. If $\underbrace{10 \text{ divides } M}_{P}$, then $\underbrace{5 \text{ divides } \left(\frac{M^2}{4} + M\right)}_{Q}$.

$P \longrightarrow Q$

Direct proof of Theorem 2 Let M be an integer.

Assume P is true. ie Assume 10 divides M. (goal: prove 5 divides $\frac{M^2}{4} + M$)

Then $M = 10k$ for some integer k (by def. of divides)

$$\begin{aligned} \text{Consequently, } \frac{M^2}{4} + M &= \frac{(10k)^2}{4} + (10k) \\ &= \frac{100k^2}{4} + 10k \\ &= 25k^2 + 10k \\ &= 5(5k^2 + 2k) \\ &= 5\ell \quad \text{where } \ell = 5k^2 + 2k \text{ so } \ell \text{ is indeed an integer.} \end{aligned}$$

$\therefore$ $\frac{M^2}{4} + M$ is an integer multiple of 5 $\therefore$ 5 divides $\frac{M^2}{4} + M$

so Q is true. $\therefore$ we proved $P \to Q$ is true ■

Q4. PROOF STRATEGIES — INDIRECT PROOF

Prove the following theorem with an **indirect proof**:

Theorem 3. Let n be an integer. If $\underbrace{n^3+5 \text{ is odd}}_{P}$, then $\underbrace{n+2 \text{ is even}}_{Q}$.

$P \longrightarrow Q$

To prove $P \to Q$ with an indirect proof, we prove $\neg Q \to \neg P$ (the contrapositive $\neg Q \to \neg P$ is $\equiv P \to Q$)

$\neg Q$: $n+2$ is odd. $\neg P$: n^3+5 is even.

Indirect Proof of Theorem 3.

Let n be an integer.

Assume $\neg Q$ is True. ie Assume $n+2$ is odd. (goal is to prove $\neg P$: n^3+5 is even)

Then $n+2 = 2m+1$ for some integer m.

$\Rightarrow n = 2m+1-2$

$= 2m-1$

Consequently, $n^3+5 = (2m-1)^3+5$

$= \left((2m)^3 - 3(2m)^2 + 3(2m) - 1\right) + 5$

$= 8m^3 - 12m^2 + 6m + 4$

$= 2\left[4m^3 - 6m^2 + 3m + 2\right]$

$= 2j$ for $j = 4m^3 - 6m^2 + 3m + 2$. Thus $j \in \mathbb{Z}$

Thus, by def. n^3+5 is even. ie $\neg P$ is True.

Therefore, we proved $\neg Q \to \neg P$ is true.

$\equiv P \to Q$.

∎

Q5. PROOF STRATEGIES — PROOF BY CASES

Strolling along on The Island of Knights and Knaves, we meet an inhabitant called A (who is either a knight or a knave).

A says: "There is gold on The Island of Knights and Knaves if and only if I am a knight."

Is there gold on The Island of Knights and Knaves?

hint: break the problem down into two cases based on A's type. (proof by cases)

Either A is a Knight or A is a Knave

Case 1. Assume A is a Knight.

Then A's statement must be T since Knights never lie.

A says: $\underbrace{(\text{There is gold})}_{?} \longleftrightarrow \underbrace{(\text{A is a Knight})}_{\text{T in this case}}$

↳ to make A's statement T, (There is gold) must be T. (because (T↔T) is T).

∴ If A is a Knight, then there is gold on The Island.

Case 2. Assume A is a Knave.

Then (in this case), A's statement must be F since Knaves always lie

A says: $\underbrace{(\text{There is gold})}_{?} \longleftrightarrow \underbrace{(\text{A is a Knight})}_{\text{F in this case}}$

↳ to make A's statement F, (There is gold) must be T. (because (T↔F) is F)

∴ If A is a Knave, then there is gold on The Island.

In both possible cases, it follows that there is gold on The Island of K&K.

∴ there is gold on The Island.