# DGD 11 — last DGD

**Q1.** INDUCTION Let $\varphi$ denote the **golden ratio**, that is, $\varphi = \frac{1+\sqrt{5}}{2}$.

Use **Mathematical Induction** to prove that the following formula is true for all integers $n \geq 1$:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

★ Define the proposition $P(n)$ for each integer $n \geq 1$ as follows:

$P(n)$: " $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$. "

★★ B.I. $n_0 = 1$ $P(1)$ says " $\varphi^2 = \varphi^1 + \varphi^0$ " ie $\varphi^2 = \varphi + 1$.

Let's check that $P(1)$ is true:

$LS = \varphi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}$ $RS = \varphi + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{1+\sqrt{5}}{2} + \frac{2}{2} = \frac{3+\sqrt{5}}{2}$

Since LS=RS, we proved that $P(1)$ is indeed true.

★★★ I.S. Let $k$ be an integer such that $k \geq n_0 = 1$.

We must prove $P(k) \longrightarrow P(k+1)$.

★★★★ I.H. Assume $P(k)$ is true for some $k \geq n_0 = 1$.

ie Assume $\varphi^{k+1} = \varphi^k + \varphi^{k-1}$ ] I.H.

(goal: prove $P(k+1)$ follows from $P(k)$, ie prove $\varphi^{k+1+1} = \varphi^{k+1} + \varphi^{k+1-1}$

ie prove $\varphi^{k+2} = \varphi^{k+1} + \varphi^k$

Well, LS of $P(k+1) = \varphi^{k+2}$

$= \varphi(\underbrace{\varphi^{k+1}})$

$= \varphi[\varphi^k + \varphi^{k-1}]$ using the I.H.: $\varphi^{k+1} = \varphi^k + \varphi^{k-1}$

$= \varphi^{k+1} + \varphi^k$ (by multiplying)

$=$ RS of statement for $P(k+1)$ $\therefore P(k+1)$ is true and so we've proved that $P(k) \rightarrow P(k+1)$.

★★★★★ Conclusion Since we proved $P(1)$ is true and we proved $P(k) \rightarrow P(k+1)$ for any $k \geq 1$, it follows from Mathematical Induction that $P(n)$ is true for all $n \geq 1$.

**Q2.** INDUCTION Use **Mathematical Induction** to prove that 8 divides $m^2 - 1$ for all **odd** integers $m \geq 1$. *(hint: think of odd numbers in the form $2n+1$, starting from $n = 0$, and define your proposition $P(n)$ corresponding to the $n$-th odd number)*

★ For each integer $n \geq 0$, define the proposition $P(n)$ as follows:

$P(n)$: " 8 divides $(2n+1)^2 - 1$."

ie $P(n)$ says "8 divides the nth positive odd integer."

★★ B.I. $n_0 = 0$ (ie $2(0)+1 = 1$ is the 1st positive odd number)

$P(0)$ says " 8 divides $(2(0)+1)^2 - 1$"

ie $P(0)$: " 8 divides $1^2 - 1$ "

ie $P(0)$: " 8 divides 0"

Since $0 = (0)(8)$ and $0 \in \mathbb{Z}$, it is indeed true that $8 \mid 0$.

∴ $P(0)$ is true.

★★★ I.S Let $k \geq n_0 = 0$ and prove $P(k) \rightarrow P(k+1)$.

★★★★ I.H. Assume $P(k)$ is true for some $k \geq n_0 = 0$. (goal: prove $P(k+1)$ follows from $P(k)$)

ie assume 8 divides $(2k+1)^2 - 1$ ie. prove 8 divides $(2(k+1)+1)^2 - 1$.

Thus, $4k^2 + 4k + 1 - 1 = 8\ell$ for some $\ell \in \mathbb{Z}$ (by def of divides)

$\Rightarrow$ $4k^2 + 4k = 8\ell$ for some $\ell \in \mathbb{Z}$ ] I.H.

Let's prove $P(k+1)$ is true using our I.H.

First, what does $P(k+1)$ say? $P(k+1)$: " $(2(k+1)+1)^2 - 1$ is divisible by 8"

$$
\begin{aligned}
\text{Well, } (2(k+1)+1)^2 - 1 &= (2k+3)^2 - 1 \\
&= 4k^2 + 12k + 9 - 1 \\
&= 4k^2 + 12k + 8 \\
&= \underbrace{4k^2 + 4k} + 8k + 8 \\
&= 8\ell + 8k + 8 \quad \text{using the I.H. } 4k^2 + 4k = 8\ell \\
&= 8(\ell + k + 1) \\
&= 8j \quad \text{for } j = \ell + k + 1. \text{ Since } \ell, k, 1 \in \mathbb{Z}, \text{ so is } j \in \mathbb{Z}.
\end{aligned}
$$

∴ 8 divides $(2(k+1)+1)^2 - 1$ ∴ $P(k+1)$ is true and so we proved that $P(k) \rightarrow P(k+1)$.

★★★★★ Conclusion Since we proved $P(0)$ is true and we proved $P(k) \rightarrow P(k+1)$ for any $k \geq 0$, it follows from Mathematical Induction that $P(n)$ is true for all $n \geq 0$.

**Q3.** BINOMIAL THEOREM What are the coefficients of

**i.** $x^4$

**ii.** $x^6$

**iii.** $x^{-1}$

in the expansion of $\left(\frac{3}{x^2} - x^3\right)^8$ ?

Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} \cdot b^i$

$$\Longrightarrow \quad \left(\frac{3}{x^2} - x^3\right)^8 = \sum_{i=0}^{8} \binom{8}{i} \left(\frac{3}{x^2}\right)^{8-i} (-x^3)^i$$

$$= \sum_{i=0}^{8} \binom{8}{i} \cdot 3^{8-i} (x^{-2})^{8-i} (-1)^i (x^3)^i$$

$$= \sum_{i=0}^{8} \binom{8}{i} 3^{8-i} x^{2i-16} \cdot (-1)^i \cdot x^{3i}$$

$$= \sum_{i=0}^{8} \binom{8}{i} 3^{8-i} \cdot (-1)^i x^{5i-16}$$

i. for coeff. of $x^4$ we need the index $i$ such that $4 = 5i - 16 \iff 20 = 5i$
$\iff i = 4$

$\therefore$ coeff. of $x^4$ is $\binom{8}{4} \cdot 3^4 (-1)^4 = \binom{8}{4} 3^4$ and $i \in \{0, 1, \ldots, 8\}$ ✓

ii. for coeff. of $x^2$ we need the index $i$ such that $2 = 5i - 16$
$\iff 18 = 5i$
$\iff i = \frac{18}{5} = 3.6$ ← no integer solution for $i$ in the range $0 \leq i \leq 8$

$\therefore$ coeff. of $x^2$ is $0$ (zero).

iii. for coeff. of $x^{-1}$ we need the index $i$ such that $-1 = 5i - 16 \iff 15 = 5i$
$\iff i = 3$

$\therefore$ coeff. of $x^{-1}$ is $\binom{8}{3} \cdot 3^5 (-1)^3 = -\binom{8}{3} \cdot 3^5$

**Q4.** PIGEONHOLE PRINCIPLE **i.** How many people do we need to guarantee that at least two among them were born on the same day of the week and in the same month (but possibly a different year)?

boxes = {day of week} x {month}
$k = 7 \times 12 = 84$

objects = people
N is to be determined

goal: we want at least $\boxed{r=2}$ objects (people) to be in the same box (have same day of week & month of birth)

$\therefore$ we need $N = k(r-1)+1$
$= (84)(2-1)+1$
$= 85$ people.

Why 85?
- 84 is too few in worst-case scenario when exactly 1 person was born on each of the 84 distinct day of week/month combos.
- 85 is enough because GPP guarantees at least one box will contain at least $\lceil \frac{N}{k} \rceil = \lceil \frac{85}{84} \rceil = 2$ objects.

**ii.** How many people do we need to guarantee that at least 5 among them were born on the same day of the week and in the same month (but possibly a different year)?

Same as in Q4i, but now we want at least $r=5$ people to have been born on same day of week and month.

$\therefore$ we need $N = k(r-1)+1$
$= (84)(5-1)+1$
$= (84)(4)+1$
$= 337$

Why 337?
- 336 is too few in worst-case scenario when exactly 4 people were born on each of the 84 distinct day of week/month combos.
- 337 is enough because GPP guarantees at least one box will contain at least $\lceil \frac{N}{k} \rceil = \lceil \frac{337}{84} \rceil = \lceil 4 + \frac{1}{84} \rceil = 5$ objects.

**Q5.** PIGEONHOLE PRINCIPLE On a test in a class with 30 students, a student called Ace made 12 mistakes, while every other student made fewer than 12 mistakes. Show that at least 3 students in this class made the same number of mistakes.

* Note, we know Ace is the only student to make 12 mistakes, so we will consider the 29 other students that each made $\leq 11$ mistakes.

boxes = {# of mistakes made}
$= \{0, 1, 2, \ldots, 11\}$
$\therefore k = 12$

objects = students that made $\leq 11$ mistakes
$N = 29$

By the GPP, at least one box contains at least $\lceil \frac{N}{k} \rceil = \lceil \frac{29}{12} \rceil = \lceil 2 + \frac{5}{12} \rceil = 3$ objects.

In other words, at least 3 students (objects) made the same # of mistakes (are in the same box).

★ If time, go over select solutions to 2017 Practice Final.