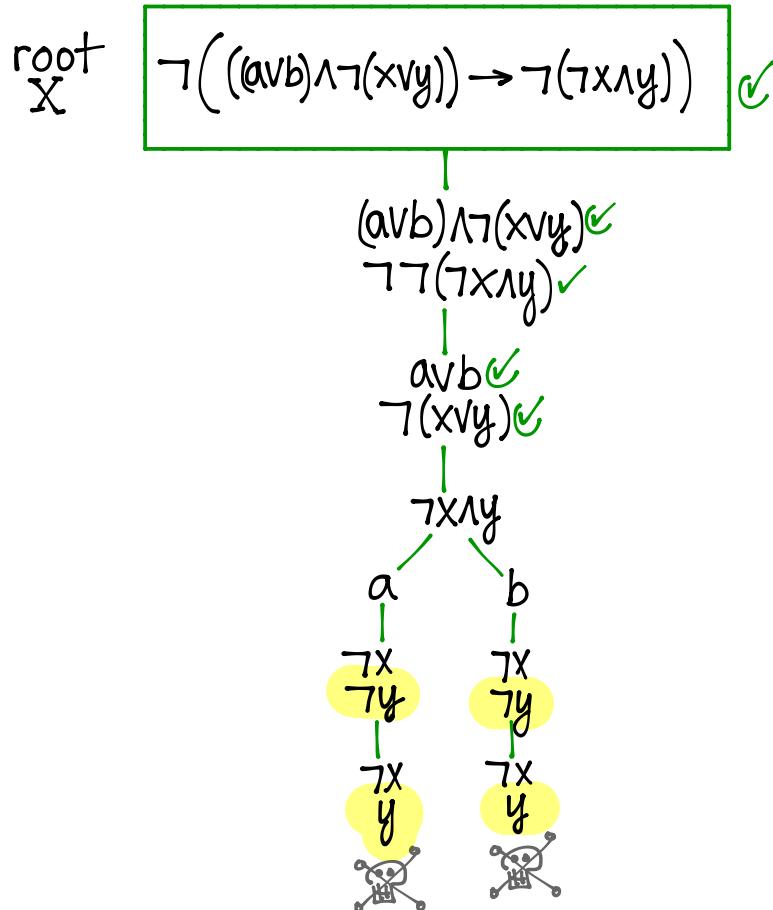


DGD 3**Q1. TRUTH TREES**

Use a truth tree to determine whether or not each of the following compound propositions is a contradiction. If it is not a contradiction, give all counterexamples. If it is a contradiction, explain how you know based on the tree.

i. $X : \neg((a \vee b) \wedge \neg(x \vee y)) \rightarrow \neg(\neg x \wedge y)$

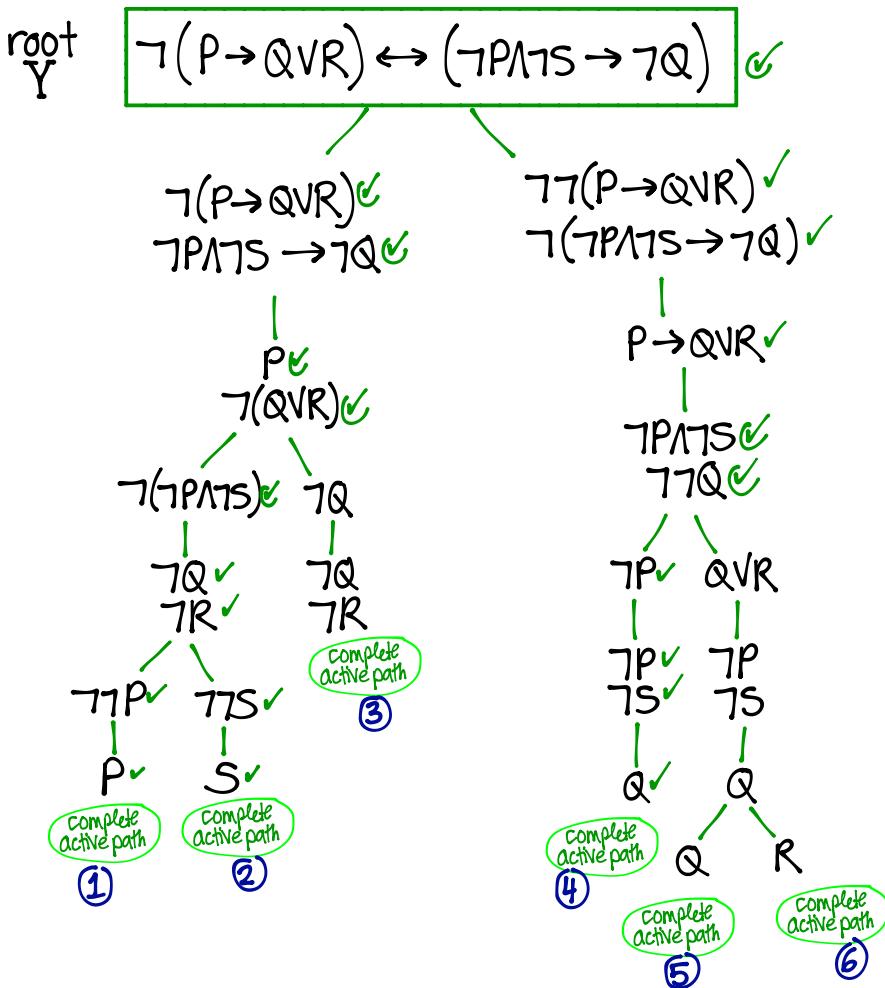


In the fully grown tree with root X , all paths are dead/inactive.
 \therefore the root X is a contradiction.

Based on your tree, give a DNF for X :

F

$$\text{ii. } Y: \neg(P \rightarrow (Q \vee R)) \leftrightarrow ((\neg P \wedge \neg S) \rightarrow \neg Q)$$



In the fully grown tree with root Y , there exists at least 1 complete path. \therefore the root Y is not a contradiction.

counterexamples:

$$\textcircled{1} \quad P=T, Q=F, R=F$$

$$\textcircled{4} \quad P=F, Q=T, S=F$$

$$\textcircled{2} \quad P=T, Q=F, R=F, S=T$$

$$\textcircled{5} \quad P=F, Q=T, S=F$$

$$\textcircled{3} \quad P=T, Q=F, R=F$$

$$\textcircled{6} \quad P=F, Q=T, R=T, S=F$$

Each of these truth assignments make the root Y true (which certifies that Y is not a contradiction)

Based on your tree, give a DNF for Y :

$$(P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R \wedge S) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg S) \vee (\neg P \wedge Q \wedge S) \vee (\neg P \wedge \neg Q \wedge R \wedge S)$$

① ② ③ ④ ⑤ ⑥

Note this DNF for Y has some redundancies, but it's still a DNF...

Q2. CONSISTENT SET OF PROPOSITIONS

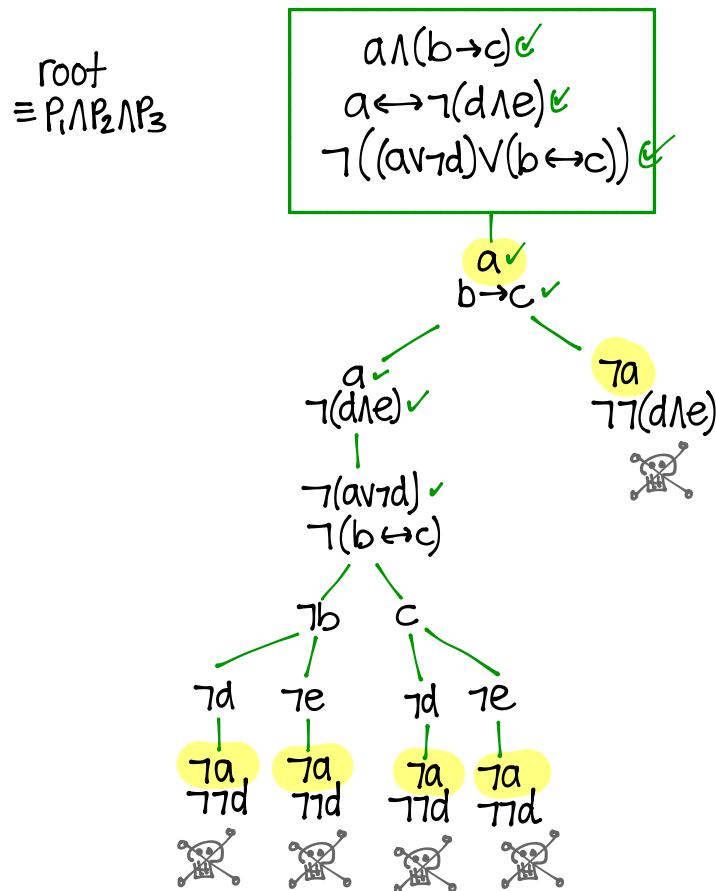
- A set $\{P_1, P_2, \dots, P_n\}$ of compound propositions is called **consistent** if there exists at least one truth assignment that makes all the propositions P_1, \dots, P_n true at the same time.
- If $\{P_1, P_2, \dots, P_n\}$ is not consistent, then, for each possible truth assignment, at least one of the propositions P_i is false; in this case, the set $\{P_1, P_2, \dots, P_n\}$ is called **inconsistent**.

Equivalently, a set $\{P_1, P_2, \dots, P_n\}$ is...

- ▷ **consistent** if the conjunction $P_1 \wedge \dots \wedge P_n$ is *not* a contradiction.
- ▷ **inconsistent** if the conjunction $P_1 \wedge \dots \wedge P_n$ is a contradiction.

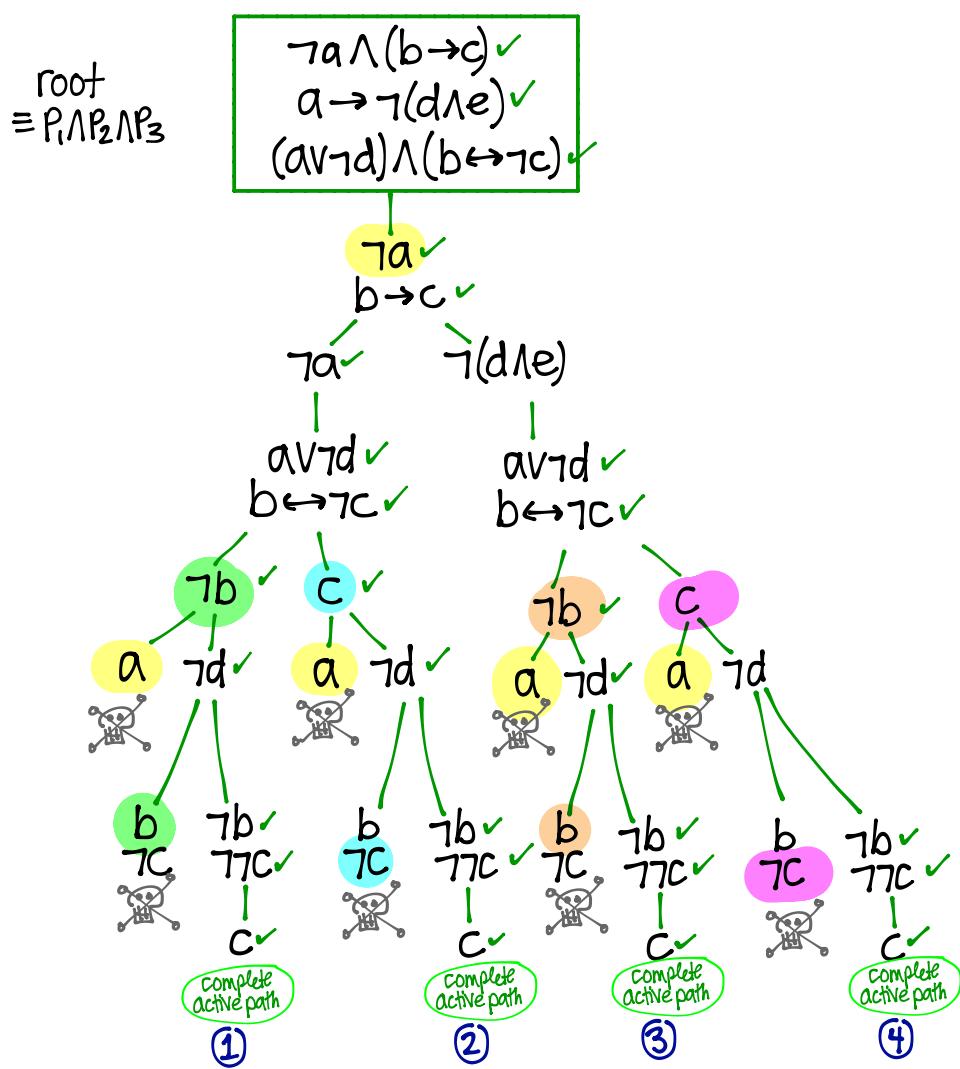
Using an appropriate truth tree, determine whether each of the following sets of propositions is consistent. If you claim the set is consistent, give all truth assignments that certify the set is consistent. If you claim the set is inconsistent, explain based on your tree.

i. $\left\{ \underbrace{a \wedge (b \rightarrow c)}_{P_1}, \underbrace{a \leftrightarrow \neg(d \wedge e)}_{P_2}, \underbrace{\neg((a \vee \neg d) \vee (b \leftrightarrow c))}_{P_3} \right\}$



All paths are dead/inactive \therefore the root (which corresponds to $P_1 \wedge P_2 \wedge P_3$) is a contradiction. \therefore the set $\{P_1, P_2, P_3\}$ is inconsistent (no truth assignment makes all 3 propositions P_1, P_2, P_3 simultaneously true).

ii. $\left\{ \underbrace{\neg a \wedge (b \rightarrow c)}_{P_1}, \underbrace{a \rightarrow \neg(d \wedge e)}_{P_2}, \underbrace{(a \vee \neg d) \wedge (b \leftrightarrow \neg c)}_{P_3} \right\}$



There exists at least one complete active path.

\therefore the root (which corresponds to $P_1 \wedge P_2 \wedge P_3$) is not a contradiction.

\therefore the set $\{P_1, P_2, P_3\}$ is consistent

Each of the following truth assignments makes all 3 propositions P_1, P_2, P_3 simultaneously true:

① $a=F, b=F, c=T, d=F$

② $a=F, b=F, c=T, d=F$

③ $a=F, b=F, c=T, d=F$

④ $a=F, b=F, c=T, d=F$

In fact, all 4 complete active paths happen to give the same truth assignment (for other sets, your tree might give more than one distinct truth assignment).

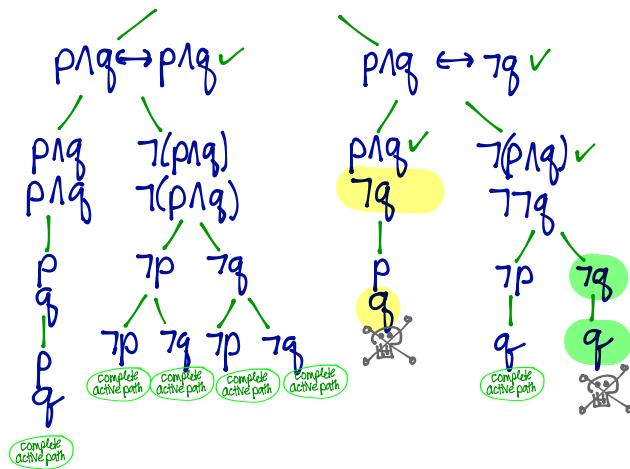
Q3. POSSIBLE MISCONCEPTIONS ABOUT TRUTH TREES

Give examples of compound propositions X and Y , consisting of the propositional variables p and q , such that

- X is a tautology and the fully grown truth tree with root X contains **at least one** dead/inactive path.

There is more than one correct answer, but here is one example:

$$X: ((p \wedge q) \leftrightarrow (p \wedge q)) \vee (\neg(p \wedge q) \leftrightarrow \neg(p \wedge q)) \checkmark$$



X is a tautology

*Verify this with a truth table!

Nevertheless, the tree with root X does have at least one dead path

- Y is a contingency and the fully grown truth tree with root X contains **no** dead/inactive paths.

There is more than one correct answer, but here is one example:

$$Y: p \vee q$$

Y is a contingency

*Verify this with a truth table!

Nevertheless, the tree with root Y has no dead paths.

- iii. Suppose all paths in a fully grown truth tree with root R are complete active paths. Does this mean that the root is a tautology? Explain.

No. In Q3ii) we saw that a contingency can have a tree with all paths being complete active paths.

Each complete active path gives a truth assignment that makes the root of the tree true, but does not guarantee that the root is true for all truth assignments.

- iv. Suppose all paths in a fully grown truth tree with root R are dead/inactive paths. Does this guarantee that the root is a contradiction? Explain.

Yes.

When all paths are dead, it means there is no truth assignment that makes the root true, hence, for all truth assignments, the root's truth value must be F . \therefore in this situation, we know the root must be a contradiction.

- iv. Suppose a fully grown truth tree with root R contains at least one complete active path and at least one dead/inactive path. Does this guarantee that the root is a contingency? Explain.

No.

At least one complete active path guarantees the root can be true (hence is not a contradiction); however, even a tautology can have a tree with dead paths (such as the proposition X given in Q3i). \therefore the presence of at least one dead path does not imply the root can be false.