DGD 7

Q1. PROPERTIES OF FUNCTIONS: INJECTIVE & SURJECTIVE

For each of the following functions, determine whether it is injective and/or surjective. If it is, give a proof; otherwise, provide a concrete numerical counterexample. Let f, g, and h be three functions defined as follows:

$$f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \qquad g : \mathbb{Q}^+ \times \mathbb{Q}^+ \to \mathbb{Q}^+ \qquad h : \mathbb{Z} \to \mathbb{N}$$
$$f(x) = (x, 5 - x) \qquad g(r, s) = rs \qquad h(n) = 2n^2 + 1$$

f is injective.

proof. Let $a, b \in \mathbb{Z}$. Assume $f(a) = f(b)$. Then $(a, 5-a) = (b, 5-b)$
$\Rightarrow a = b$ and $5 - a = 5 - b$
$\Rightarrow a = b$ ▨

f is not surjective:

counterexample: $(0,0) \in \mathbb{Z} \times \mathbb{Z}$ but there is no $x \in \mathbb{Z}$ such that $f(x) = (0,0)$ because $(x, 5-x) = 0 \Leftrightarrow x = 0$ and $x = 5$ ↯

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g is not injective.

counterexample: $(3,2)$ and $(1.5, 4) \in \mathbb{Q}^+ \times \mathbb{Q}^+$, $(3,2) \neq (1.5, 4)$
but $g(3,2) = 3 \cdot 2 = 6 = 1.5 \cdot 4 = g(1.5, 4)$

g is surjective.

proof Let $q \in \mathbb{Q}^+$. Then $q = 1 \cdot q = g(1, q)$.
Hence any element q of the codomain of g is the image of $(1, q)$. ▨

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h is not injective.

counterexample: -5 and $5 \in \mathbb{Z}$. $-5 \neq 5$ but $h(-5) = h(5)$.

h is not surjective.

counterexample: $2 \in \mathbb{N}$ but there is no integer $n \in \mathbb{Z}$ such that $2 = 2n^2 + 1$

Q2. INVERSE FUNCTIONS

Prove that the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $g(m, n) = (1 - n, m + 5)$ is a bijection. Determine the expression for its inverse g^{-1} and verify that $g \circ g^{-1} = g^{-1} \circ g = \mathrm{id}_{\mathbb{Z} \times \mathbb{Z}}$.

[injective]

Let $(m_1, n_1), (m_2, n_2) \in \mathbb{Z} \times \mathbb{Z}$
(arbitrary elements of g's domain).

Assume $g(m_1, n_1) = g(m_2, n_2)$.
(goal is to prove $(m_1, n_1) = (m_2, n_2)$)

Then $(1 - n_1, m_1 + 5) = (1 - n_2, m_2 + 5)$
$\Rightarrow 1 - n_1 = 1 - n_2$ and $m_1 + 5 = m_2 + 5$
$\Rightarrow n_1 = n_2$ and $m_1 = m_2$
$\Rightarrow (m_1, n_1) = (m_2, n_2)$
$\therefore g$ is injective (1-1).

[surjective]

Let $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ (arbitrary elements of g's codomain)

*goal is to find $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ (domain) such that

$$g(m, n) = (p, q)$$

Well, $g(m, n) = (p, q) \Leftrightarrow (1 - n, m + 5) = (p, q)$
(we want to reverse-engineer what (m, n) should equal in order to get $g(m, n) = (p, q)$).

$\Rightarrow 1 - n = p$ and $m + 5 = q$
$\Rightarrow n = 1 - p$ and $m = q - 5$

So $(m, n) = (q - 5, 1 - p)$ works and

since $q \in \mathbb{Z}$, so is $q - 5$
since $p \in \mathbb{Z}$, so is $1 - p$
$\therefore (q - 5, 1 - p) \in \mathbb{Z} \times \mathbb{Z}$.
and $g(q - 5, 1 - p) = (p, q)$

$\therefore g$ is surjective (onto)

Since g is both injective and surjective, g is a bijection!

Find g^{-1}.

$$
\begin{aligned}
g^{-1}(a,b)=(m,n) &\iff g(m,n)=(a,b)\\
&\iff (1-n,\ m+5)=(a,b)\\
&\iff 1-n=a \quad\text{and}\quad m+5=b\\
&\iff n=1-a \quad\text{and}\quad m=b-5\\
&\iff (m,n)=(b-5,\ 1-a)
\end{aligned}
$$

$\therefore\ g^{-1}(a,b)=(b-5,1-a)$

Verify compositions:

$$
\begin{aligned}
g\circ g^{-1}(a,b)&=g(g^{-1}(a,b))\\
&=g(b-5,1-a)\\
&=\big(1-(1-a),\ (b-5)+5\big)\\
&=(a,b)
\end{aligned}
$$

$\therefore$ for all $(a,b)\in\mathbb{Z}\times\mathbb{Z}$, we have $g\circ g^{-1}(a,b)=(a,b)$ so $g\circ g^{-1}=id_{\mathbb{Z}\times\mathbb{Z}}$

$$
\begin{aligned}
g^{-1}\circ g(a,b)&=g^{-1}(g(a,b))\\
&=g^{-1}(1-b,a+5)\\
&=\big((a+5)-5,\ 1-(1-b)\big)\\
&=(a,b)
\end{aligned}
$$

$\therefore$ for all $(a,b)\in\mathbb{Z}\times\mathbb{Z}$, we have $g^{-1}\circ g(a,b)=(a,b)$ so $g^{-1}\circ g=id_{\mathbb{Z}\times\mathbb{Z}}$

Q3. PROPERTIES OF FUNCTIONS

i. Give an example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is neither injective nor surjective.

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(r,s) = (|r|, |s|)$

f is neither injective nor surjective.

ii. Give an example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is injective but not surjective.

Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(r,s) = \begin{cases} (r+1, s+1) & \text{if } r \geqslant 0 \text{ and } s \geqslant 0 \\ (r+1, s-1) & \text{if } r \geqslant 0 \text{ and } s < 0 \\ (r-1, s+1) & \text{if } r < 0 \text{ and } s \geqslant 0 \\ (r-1, s-1) & \text{if } r < 0 \text{ and } s < 0 \end{cases}$$

g is injective but not surjective

iii. Give an example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is surjective but not injective.

Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(r,s) = (r^3 - r, s^3 - s)$

h is surjective but h is not injective

iv. Give an example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is a bijection but is **not** the identity function $\mathrm{id}_{\mathbb{R}^2}$. Compute an expression for the inverse of your function.

Define $p : \mathbb{R}^2 \to \mathbb{R}^2$ by $p(r,s) = (r+s, r-s)$ $\quad$ p is a bijection

$p^{-1}(t,u) = (r,s) \iff p(r,s) = (t,u)$

$\iff (r+s, r-s) = (t,u)$ $\quad \therefore r = \frac{t+u}{2}$ and $s = \frac{t-u}{2}$

$\therefore p^{-1}(t,u) = \left(\frac{t+u}{2}, \frac{t-u}{2}\right)$

Check: $p^{-1}(p(r,s)) = p^{-1}(r+s, r-s) = \left(\frac{r+s+r-s}{2}, \frac{r+s-(r-s)}{2}\right) = (r,s)$

Check: $p(p^{-1}(t,u)) = p\left(\frac{t+u}{2}, \frac{t-u}{2}\right) = \left(\frac{t+u}{2} + \frac{t-u}{2}, \frac{t+u}{2} - \left(\frac{t-u}{2}\right)\right) = (t,u)$.

Q4. COMPOSITIONS OF FUNCTIONS

For each of the following statements, prove it (if it is true) or give a counterexample (to show that the statement can be false). Let $g : A \to B$ and $h : B \to C$ be functions.

i. If h and $h \circ g$ are injective (1-1), then g is injective (1-1).

True. proof. Assume h and $h\circ g$ are injective. (goal is to prove g must be injective)

To prove g is injective (the goal), we must prove

$$\underbrace{(g(a_1)=g(a_2))}\to(a_1=a_2) \text{ for all } a_1,a_2\in A \text{ (g's domain).}$$

Let $a_1,a_2\in A$ and assume $\boxed{g(a_1)=g(a_2)}$.

Note: $h\circ g: A\to C$. $g:A\to B$ $h:B\to C$

Since $g:A\to B$, $g(a_1)\in B$ and $g(a_2)\in B$

So $g(a_1)$ $(=g(a_2))$ is an element of h's domain (B).

$\therefore h(g(a_1))=h(g(a_2))$ (since $g(a_1)=g(a_2)$, we are applying h to the same element of B)

$\Rightarrow h\circ g(a_1)=h\circ g(a_2)$ (by def. of composition)

$\Rightarrow a_1=a_2$ because $h\circ g$ is injective.

Thus $(g(a_1)=g(a_2))\to(a_1=a_2)$ $\therefore g$ is injective. (goal!)

ii. If h and $h \circ g$ are surjective (onto), then g is surjective (onto).

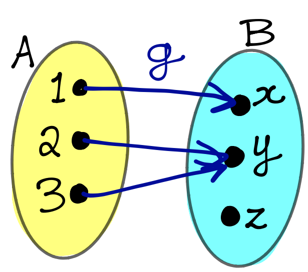
False Counterexample

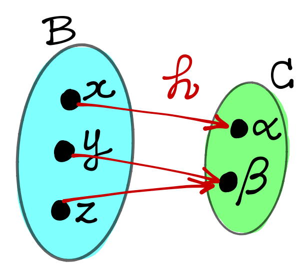
Sets:

$A=\{1,2,3\}$

$B=\{x,y,z\}$

$C=\{\alpha,\beta\}$



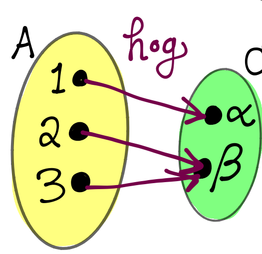


$g:A\to C$ is a Function

g is not surjective

$h:B\to C$ is a function

h is surjective.



$h\circ g:A\to C$ exists and is a function

$h\circ g$ is surjective.

$\therefore$ we have sets and functions that show the statement can be false

Q5. COUNTABLE INFINITE SETS

Let $\mathbf{S} = \{x \in \mathbb{N} : x = n^2 \text{ for some } n \in \mathbb{N}\}$ be the set of perfect squares.

i. Prove that $\mathbf{S} \subset \mathbb{N}$, that is, prove that $\mathbf{S}$ is a *proper* subset of $\mathbb{N}$.

Let $x \in S$. Then $x = n^2$ for some $n \in \mathbb{N}$ and so x is the product of a natural number with itself. $\therefore x \in \mathbb{N}$ so $S \subseteq \mathbb{N}$

To show that S is a proper subset of $\mathbb{N}$ we need only give an element $\mathbb{N}$ that is not an element of S. For example, $2 \in \mathbb{N}$ but $2 \notin S$ since there is no natural number x such that $x^2 = 2$. $\therefore S \subset \mathbb{N}$

ii. Prove that $|\mathbf{S}| = \mathbb{N}$ by giving a bijection $f : \mathbb{N} \to \mathbf{S}$. Justify that your function is a bijection.

Let $f: \mathbb{N} \longrightarrow S$ be the function defined by $f(n) = n^2$.

f is injective.

proof. Let $n_1, n_2 \in \mathbb{N}$ and assume $f(n_1) = f(n_2)$.
Then $(n_1)^2 = (n_2)^2 \Rightarrow n_1 = \pm n_2$ but natural #s are all ≥ 0, so $n_1 = n_2$.

f is surjective.

proof. Let $x \in S$. Then $x = n^2$ for some $n \in \mathbb{N}$ by def. of S.
$\Rightarrow x = f(n)$.
$\therefore f$ is surjective.

Since $f: \mathbb{N} \to S$ is both injective and surjective, f is a bijection.

Since there exists a bijection from $\mathbb{N}$ to S, we conclude that $|\mathbb{N}| = |S|$.