4. Logical Equivalences

Example 4.1. Using a truth table, verify each of the following laws from The Table of Logical Equivalences:

- ◇ $p \wedge \mathbf{T} \equiv p$ (Identity Law)
- ◇ $p \vee \mathbf{F} \equiv p$ (Identity Law)
- ◇ $p \vee \mathbf{T} \equiv \mathbf{T}$ (Domination Law)
- ◇ $p \wedge \mathbf{F} \equiv \mathbf{F}$ (Domination Law)
- ◇ $p \vee \neg p \equiv \mathbf{T}$ (Negation Law)
- ◇ $p \wedge \neg p \equiv \mathbf{F}$ (Negation Law)

Identity Laws:

p	$p \wedge T$	$p \wedge T \leftrightarrow p$
T	T	T
F	F	T

$p \wedge T \equiv p$ because $p \wedge T \leftrightarrow p$ is a tautology.

p	$p \vee F$	$p \vee T \leftrightarrow p$
T	T	T
F	F	T

$p \vee T \equiv p$ because $p \vee T \leftrightarrow p$ is a tautology.

Domination Laws:

p	$p \vee T$	$p \vee T \leftrightarrow T$
T	T	T
F	T	T

$p \vee T \equiv T$ because $p \vee T \leftrightarrow T$ is a tautology.

p	$p \wedge F$	$p \wedge F \leftrightarrow F$
T	F	T
F	F	T

$p \wedge F \equiv F$ because $p \wedge F \leftrightarrow F$ is a tautology.

Negation Laws:

p	$p \vee \neg p$	$p \vee \neg p \leftrightarrow T$
T	T	T
F	T	T

$p \vee \neg p \equiv T$ because $p \vee \neg p \leftrightarrow T$ is a tautology.

p	$p \wedge \neg p$	$p \wedge \neg p \leftrightarrow F$
T	F	T
F	F	T

$p \wedge F \equiv F$ because $p \wedge F \leftrightarrow F$ is a tautology.

* These notes are solely for the personal use of students registered in MAT1348.

THE TABLE OF LOGICAL EQUIVALENCES

$P \to Q$ is T when P is F or Q is T	1.	$P \to Q \equiv \neg P \vee Q$	Implication Law
2 ways to think about $\leftrightarrow$	2.	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	Biconditional Laws
	3.	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	
out of $P/\neg P$ exactly one is T and other is F.	4.	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
	5.	$P \wedge \neg P \equiv \mathbf{F}$	
	6.	$P \vee \mathbf{F} \equiv P$	Identity Laws
	7.	$P \wedge \mathbf{T} \equiv P$	
	8.	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
	9.	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
	10.	$P \vee P \equiv P$	Idempotent Laws
	11.	$P \wedge P \equiv P$	
	12.	$\neg\neg P \equiv P$	Double Negation Law
	13.	$P \vee Q \equiv Q \vee P$	Commutative Laws
	14.	$P \wedge Q \equiv Q \wedge P$	
	15.	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
	16.	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
a bit like factoring	17.	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
	18.	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
a way to switch $\wedge/\vee$ with $\neg\neg$	19.	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
	20.	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	

Example 4.2. Prove $(x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \equiv x \vee y$

$(x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \equiv [x \wedge (y \vee \neg y)] \vee (\neg x \wedge y)$ (distributive law)

$\equiv [x \wedge T] \vee (\neg x \wedge y)$ (negation law)

$\equiv (x) \vee (\neg x \wedge y)$ (identity law)

$\equiv (x \vee \neg x) \wedge (x \vee y)$ (distributive law)

$\equiv T \wedge (x \vee y)$ (negation law)

$\equiv x \vee y$ (identity law)

$\therefore (x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \equiv x \vee y$

Example 4.3. Prove that $(a \wedge \neg b) \wedge (\neg a \vee b)$ is a contradiction <u>Note</u> contradiction $\equiv F$

$(a \wedge \neg b) \wedge (\neg a \vee b) \equiv a \wedge (\neg b \wedge (\neg a \vee b))$ (associative law)

$\equiv a \wedge [(\neg b \wedge \neg a) \vee (\neg b \wedge b)]$ (distributive law)

$\equiv a \wedge [(\neg b \wedge \neg a) \vee F]$ (negation law)

$\equiv a \wedge [\neg b \wedge \neg a]$ (identity law)

$\equiv a \wedge (\neg a \wedge \neg b)$ (commutative law)

$\equiv (a \wedge \neg a) \wedge \neg b$ (associative law)

$\equiv F \wedge \neg b$ (negation law)

$\equiv F$ (domination law)

$\therefore (a \wedge \neg b) \wedge (\neg a \vee b) \equiv F$ so $(a \wedge \neg b) \wedge (\neg a \vee b)$ is a contradiction.

Example 4.4. Find a DNF for $(p \to q) \vee (\neg(p \vee q) \wedge r)$

$(p \to q) \vee (\neg(p \vee q) \wedge r) \equiv (\neg p \vee q) \vee (\neg(p \vee q) \wedge r)$ (implication law)

$\equiv \neg p \vee q \vee (\neg(p \vee q) \wedge r)$ (associative law)

$\equiv \neg p \vee q \vee ((\neg p \wedge \neg q) \wedge r)$ (De Morgan's law)

$\equiv \underbrace{\neg p}_{\text{conjunctive clause } \checkmark} \vee \underbrace{q}_{\text{conjunctive clause } \checkmark} \vee (\underbrace{\neg p \wedge \neg q \wedge r}_{\text{conjunctive clause } \checkmark})$ (associative law)

$\therefore (p \to q) \vee (\neg(p \vee q) \wedge r) \equiv \neg p \vee q \vee (\neg p \wedge \neg q \wedge r)$ and this is in DNF since it's a disjunction of conjunctive clauses.

Example 4.5. Find a compound proposition that is logically equivalent to $X \wedge Y$ that uses only the logical connectives $\to$ and $\neg$.

$X \wedge Y \equiv \neg\neg(X \wedge Y)$ (double negation law)

$\equiv \neg(\neg X \vee \neg Y)$ (De Morgan's law)

$\equiv \neg(X \to \neg Y)$ (implication law)

Thus, we found a proposition $\neg(X \to \neg Y)$ such that $X \wedge Y \equiv \neg(X \to \neg Y)$ and $\neg(X \to \neg Y)$ uses only the logical connectives $\to$ and $\neg$.

Example 4.6. Find a compound proposition that is logically equivalent to $p \to (q \vee r)$ that uses only the logical connectives $\neg$ and $\wedge$.

$$
\begin{aligned}
p \to (q \vee r) &\equiv \neg p \vee (q \vee r) && \text{(implication law)}\\
&\equiv \neg\neg[\neg p \vee (q \vee r)] && \text{(double negation)}\\
&\equiv \neg[\neg\neg p \wedge \neg(q \vee r)] && \text{(De Morgan's law)}\\
&\equiv \neg[p \wedge \neg(q \vee r)] && \text{(double negation law)}\\
&\equiv \neg[p \wedge (\neg q \wedge \neg r)] && \text{(De Morgan's law)}
\end{aligned}
$$

Thus, $p \to (q \vee r) \equiv \neg[p \wedge (\neg q \wedge \neg r)]$ and $\neg[p \wedge (\neg q \wedge \neg r)]$ uses only the logical connectives $\neg$ and $\wedge$.

*A collection of logical connectives is called <u>functionally complete</u> if every compound proposition is logically equivalent to a compound proposition involving only these logical connectives.

Show that $\{\neg, \to\}$ is functionally complete.

Study Guide

Important terms and concepts:

- ◇ The Island of Knights and Knaves — via reasoning or with the truth table method
- ◇ DNF (atoms, literals, conjunctive clauses) — how to find DNF from a truth table
- ◇ The Laws from the Table of Logical Equivalences

Exercises	Sup.Ex. §1 # 7c (using the Laws)
	Sup.Ex. §2 # 1, 2, 3, 4, 5, 7, 8, 11, 15
	Rosen §1.3 # 7, 11, and using the Laws: # 22, 23, 24, 25, 26, 27, 28, 29, 30
	**For each of the following, find a compound proposition that uses only the connectives $\neg$ and $\to$ i. $p \vee q$ ii. $p \wedge q$ iii. $p \oplus q$ iv. $p \leftrightarrow q$
	Rosen §1.3 optional: # 44, 45, 46, 47, 48, 49, 50, 51, 52