19. Proof by Induction & The Pigeonhole Principle

proof by induction:

1. Define $P(n)$
2. **B.I.** Prove $P(n_0)$
3. **I.S.** Let $k \geq n_o$. Prove $P(k) \to P(k+1)$.
   4. **I.H.** Assume $P(k)$ is true (goal: prove $P(k+1)$ follows from I.H.)
5. **conclusion**

PROOF BY INDUCTION

Example 19.1. Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of numbers defined according to the following **recurrence relation:**

$$a_0 := 1$$
$$\text{for each integer } n \geq 1, \quad a_n = 5(a_{n-1})^2$$

Using the **recurrence relation**, compute the values for a_1, a_2, a_3, and a_4.

$a_0 := 1$

$a_1 = 5(a_0)^2 = 5(1)^2 = 5$

$a_2 = 5(a_1)^2 = 5(5)^2 = 5^3$

$a_3 = 5(a_2)^2 = 5(5^3)^2 = 5^7$

$a_4 = 5(a_3)^2 = 5(5^7)^2 = 5^{15}$

$a_5 = 5(a_4)^2 = 5(5^{15})^2 = 5^{31}$

What is the **general solution** to this recurrence relation? That is, what does a_n equal as a function of n? Prove this solution using a **Proof by Induction**

In general, it looks like $\boxed{a_n = 5^{2^n - 1}}$ Let's prove this!

1. For each integer $n \geq 0$, let $P(n)$ denote the following proposition:

$$P(n): \text{“} a_n = 5^{2^n - 1} \text{”}$$

2. B.I. $\underline{n_0 = 0}$ $P(0)$ says "$a_0 = 5^{2^0 - 1}$"

According to the recurrence relation, $a_0 = 1$ and the RS of $P(0)$ is equal to $5^{2^0 - 1} = 5^{1-1} = 5^0 = 1$. Thus, $P(0)$ is true.

3. I.S. Let $k \geq 0$. We must prove $P(k) \to P(k+1)$.
   4. I.H. Assume $P(k)$ is true. That is, assume $a_k = 5^{2^k - 1}$
   (goal: prove $P(k+1)$ follows, that is, prove $a_{k+1} = 5^{2^{k+1} - 1}$)

---
* These notes are solely for the personal use of students registered in MAT1348.

$$
\begin{aligned}
\text{LS of } P(k+1) &= a_{k+1} \\
&= 5(a_k)^2 && \text{(by the recurrence relation)} \\
&= 5\left[5^{2^k-1}\right]^2 && \text{(by the I.H.!)} \\
&= 5\cdot 5^{(2^k-1)(2)} && \text{(by laws of exponents)} \\
&= 5\cdot 5^{2\cdot 2^k-2} && \text{"} \\
&= 5\cdot 5^{2^{k+1}-2} && \text{"} \\
&= 5^1\cdot 5^{2^{k+1}-2} && \text{"} \\
&= 5^{2^{k+1}-2+1} && \text{"} \\
&= 5^{2^{k+1}-1} \\
&= \text{RS of } P(k+1) && \therefore P(k+1) \text{ does follow from } P(k)!
\end{aligned}
$$

5. Conclusion:

Since $P(1)$ is true and since we proved $P(k) \to P(k+1)$ for any $k \geq 1$, it follows from the principle of Mathematical Induction that $P(n)$ is true for all integers $n \geq 1$.

EXAMPLES OF THE PIGEONHOLE PRINCIPLE

Suppose we have a standard deck of 52 playing cards (with jokers removed).

- Each card is of one of 2 colours: **red** or **black**
- Each card is of one of 13 ranks: **A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K**
- Each card is of one of 4 suits: ♡ ♢ ♣ ♠

A♡	**2♡**	**3♡**	**4♡**	**5♡**	**6♡**	**7♡**	**8♡**	**9♡**	**10♡**	**J♡**	**Q♡**	**K♡**
A♢	**2♢**	**3♢**	**4♢**	**5♢**	**6♢**	**7♢**	**8♢**	**9♢**	**10♢**	**J♢**	**Q♢**	**K♢**
A♣	**2♣**	**3♣**	**4♣**	**5♣**	**6♣**	**7♣**	**8♣**	**9♣**	**10♣**	**J♣**	**Q♣**	**K♣**
A♠	**2♠**	**3♠**	**4♠**	**5♠**	**6♠**	**7♠**	**8♠**	**9♠**	**10♠**	**J♠**	**Q♠**	**K♠**

Example 19.2. How many cards do I need to draw from the deck in order to guarantee that at least two cards are **black**?

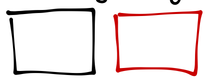
Worst-case scenario (27 is too few): We draw 1 black card and all 26 red cards before getting a 2nd black card.

28 cards is enough: Since there are only 26 red cards that could thwart us, 28 cards leaves us with at least $28-26=2$ cards that must both be black.

∴ we need to draw 28 cards in order to guarantee that at least 2 cards are black.

Example 19.3. How many cards do I need to draw from the deck in order to guarantee that at least two cards are the same colour?

<u>Worst-case scenario</u>: (2 is too few) — we draw 1 of each colour before getting a 2nd card of one of the 2 colours

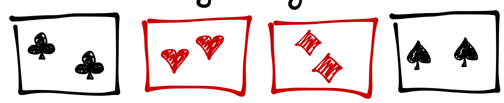


<u>3 cards is enough</u> — If we draw 3 cards, but there are only 2 possible colours, then we will end up with at least 2 that are the same colour.

∴ we need to draw $\underline{\underline{3}}$ cards in order to guarantee that at least 2 are the same colour.

Example 19.4. How many cards do I need to draw from a standard deck of 52 playing cards in order to guarantee that at least **three** cards are the same suit?

<u>Worst-case scenario</u>: (8 is too few) — we draw 2 of each suit before getting a 3rd card of one of the 4 suits



<u>9 cards is enough</u>: If we draw 9 cards, but there are only 4 possible suits, then we will end up with at least 3 that are the same suit.

Proof (by contradiction) that 9 cards is enough to guarantee at least 3 of the same suit.

Assume we drew 9 cards and did <u>not</u> get at least 3 of the same suit.

Then, of each suit, we drew at most 2 cards.

Now, $(\text{total \# cards drawn}) = \left(\begin{smallmatrix}\#\heartsuit\text{'s}\\ \text{drawn}\end{smallmatrix}\right) + \left(\begin{smallmatrix}\#\diamondsuit\text{'s}\\ \text{drawn}\end{smallmatrix}\right) + \left(\begin{smallmatrix}\#\clubsuit\text{'s}\\ \text{drawn}\end{smallmatrix}\right) + \left(\begin{smallmatrix}\#\spadesuit\text{'s}\\ \text{drawn}\end{smallmatrix}\right)$

$$\leq 2 + 2 + 2 + 2$$

But we drew a total of 9 cards. This means $9 \leq 2+2+2+2 = 8$ ⚡

Since $9 \leq 8$ is a contradiction, 9 cards really must have been enough ∎

THE CEILING FUNCTION (a function from $\mathbb{R}$ to $\mathbb{Z}$)

for $x \in \mathbb{R}$, the ceiling of x, denoted $\lceil x \rceil$, is defined as follows:

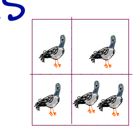
$$\lceil x \rceil = \min\{n \in \mathbb{Z} : x \leq n\}$$ "rounding up"

Ex. $\lceil 2 \rceil = 2$ Ex $\lceil 2.3 \rceil = 3$ Ex $\lceil -2.3 \rceil = -2$

THEOREM: THE GENERALIZED PIGEONHOLE PRINCIPLE (G.P.P.)

Theorem 19.5. (**The Generalized Pigeonhole Principle**) Let k and N be positive integers.
If N objects are placed into k boxes, then there is at least one box that contains at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

(P.P.) **Theorem 19.6.** (THE PIGEONHOLE PRINCIPLE) Let k be a positive integer.
If $k+1$ objects are placed into k boxes, then there is at least one box that contains at least 2 objects.



The Pigeon Version of Theorem:
If a flock of $k + 1$ pigeons flies into a set of k pigeonholes to roost, then at least one pigeonhole will contain more than one pigeon (because there are more pigeons than pigeonholes).

Proof by Contradiction of Theorem 19.6

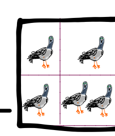
Let $k \in \mathbb{Z}^+$ (k is an integer and $k \geq 1$). Assume the Pigeonhole Principle is false.

Assume $k+1$ objects are placed into k boxes and assume that it is not the case that at least one box contains at least 2 objects.

Then each box contains at most 1 object.

Now,
$$\begin{aligned} k+1 &= \begin{pmatrix}\text{total \#}\\ \text{objects}\end{pmatrix} \\ &= \begin{pmatrix}\text{\# objects}\\ \text{in Box 1}\end{pmatrix} + \begin{pmatrix}\text{\# objects}\\ \text{in Box 2}\end{pmatrix} + \cdots + \begin{pmatrix}\text{\# objects}\\ \text{in Box } k\end{pmatrix} \\ &\leq 1 + 1 + \cdots + 1 \\ &= k \ \lightning \end{aligned}$$

Since our assumption leads to the contradiction $k+1 \leq k$, we were wrong.
$\therefore$ the Pigeonhole Principle is true



Lemma 19.7. Let $x \in \mathbb{R}$. Then $\lceil x \rceil < x + 1$.

Proof. Let $x \in \mathbb{R}$. We consider 2 cases: either x is an integer or not.

<u>Case 1</u>. Assume x is an integer. Then $\lceil x \rceil = x$ which is $< x+1$.

$\therefore$ in case 1, $\lceil x \rceil < x+1$ is true.

<u>Case 2</u>. Assume x is not an integer. Then we can write $x = n + \delta$ for some integer n and some real number δ such that $0 < \delta < 1$. That is, we can write x in decimal form as an integer plus whatever comes after the decimal point.

Then $\lceil x \rceil = \lceil n + \delta \rceil = n + 1 < n + 1 + \delta = x + 1$

$\therefore$ in case 2, $\lceil x \rceil < x+1$ is true.

In both possible cases, we showed that $\lceil x \rceil < x+1$ $\therefore$ $\lceil x \rceil < x+1$ is true for all $x \in \mathbb{R}$. $\blacksquare$

Proof of Theorem 19.5 (The Generalized Pigeonhole Principle).

We will prove GPP by contradiction.

GPP says: $\underbrace{\left(\text{placing } N \text{ objects into } k \text{ boxes}\right)}_{P} \rightarrow \underbrace{\left(\text{at least one box contains at least } \lceil N/k \rceil \text{ objects}\right)}_{Q.}$

Assume the negation of GPP is true. That is, assume $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$.

Assume we place N objects into k boxes <u>and</u> it is not the case that at least one box contains at least $\lceil N/k \rceil$ objects.

Then each box contains at most $\lceil N/k \rceil - 1$ objects.

$$\Rightarrow \begin{pmatrix}\text{total \#}\\ \text{of objects}\end{pmatrix} = \begin{pmatrix}\text{\# objects}\\ \text{in box 1}\end{pmatrix} + \begin{pmatrix}\text{\# objects}\\ \text{in box 2}\end{pmatrix} + \dots + \begin{pmatrix}\text{\# objects}\\ \text{in box } k\end{pmatrix}$$

$$\leq \lceil \tfrac{N}{k} \rceil - 1 + \lceil \tfrac{N}{k} \rceil - 1 + \dots + \lceil \tfrac{N}{k} \rceil - 1$$

$$= k\left(\lceil N/k \rceil - 1\right)$$

$$< k\left(\tfrac{N}{k} + 1 - 1\right) \quad \text{by Lemma 21.3}$$

$$= N$$ ↯ this is a contradiction since total # objects $= N$ and is $< N$.

$\therefore$ the negation of GPP must be false since it leads to contradiction

$\therefore$ the GPP must be true $\blacksquare$

Let k and r be positive integers. Suppose some objects are to be placed into k boxes.

QUESTION:
What is the minimum number N of objects that we need to place into k boxes in order to guarantee that at least one box contains at least r objects?

ANSWER:
The minimum number N that is needed to guarantee that at least one box contains at least r objects is

$$N = k(r-1)+1$$

EXPLANATION:

By G.P.P., at least one box is guaranteed to contain at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

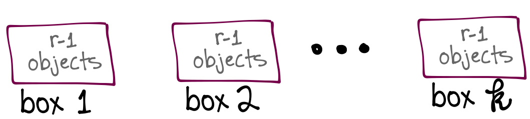
With $N=k(r-1)+1$, we get

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{k(r-1)+1}{k} \right\rceil = \left\lceil (r-1) + \frac{1}{k} \right\rceil = r \text{ objects}$$

$\therefore$ $N=k(r-1)+1$ is enough.

On the other hand, with only $k(r-1)$ objects, we could end up in the worst-case scenario:

each of the k boxes contains $r-1$ objects



(each box is just 1 object short of the goal).

$\therefore$ $k(r-1)$ objects is too few.

$\therefore$ $N=k(r-1)+1$ is exactly the minimum threshold to guarantee that at least one box contains at least r objects.

Example 19.8. How many people do we need to invite to a party in order to guarantee that at least three of our guests were born on the same month?

boxes = months

$k = 12$

objects = guests

N is to be determined

goal: we want at least $\boxed{r=3}$ objects (guests) to be in the same box (have same month of birth)

So we need $N = k(r-1)+1$

$= (12)(3-1)+1$

$= 25$ guests

Why 25?

↳ 24 is too few in the worst-case scenario when exactly two guests were born in each of the 12 different months.

↳ $N=25$ is enough since the GPP guarantees at least one box will contain at least $\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{25}{12} \right\rceil = 3$ objects

---

Example 19.9. How many people do we need in order to guarantee that at least 2 have the same two initials (first letter of first name and first letter of family name)?

boxes = all possible 2-letter initials

$k = 26 \cdot 26 = 26^2 = 676$

objects = people

N is to be determined

goal: we want at least $\boxed{r=2}$ objects (people) to be in the same box (have same 1st-name/last-name initials).

So we need $N = k(r-1)+1$

$= (676)(2-1)+1$

$= 676+1$

$= 677$ people

Why 677?

- 676 is too few in the worst-case scenario when we have exactly one person with each of the possible 676 2-letter initials.
- By the P.P., placing $k+1 = 677$ people into $k = 676$ boxes guarantees that at least one box will contain at least 2 people.

STUDY GUIDE

◇ **The Pigeonhole Principle** $\qquad k+1$ objects $\quad k$ boxes

worst-case scenario analysis

◇ the ceiling function $\quad \lceil x \rceil$

◇ **The Generalized Pigeonhole Principle** $\qquad N$ objects $\quad k$ boxes

at least one box contains at least $\lceil N/k \rceil$ objects

examples with the pigeonhold principle:	how many .... do we need in order to guarantee that at least .... have the same.... ? worst-case analysis
Exercises	Sup.Ex. §11 # 1, 2, 3, 4, 5, 7, 8, 9, 10, 14 Sup.Ex. §9 # 3, 4, 5, 6, 8, 12 Rosen §2.4 # 1, 3, 9 Rosen §5.1 # 3, 4, 5, 6, 7, 9, 11, 13, 15, 19, 21, 31, 33, 35