15. Relations, Equivalence Relations & Equivalence Classes

Let A and B be sets.

- ★ A **relation from** A **to** B is a subset $\mathcal{R}$ of $A \times B$.
- ★ A **relation on** A is a relation from A to itself, i.e., a subset of $A \times A$.
- ★ Given a relation $\mathcal{R}$ from A to B, we write $a \mathrel{\mathcal{R}} b$ if and only if $(a, b) \in \mathcal{R}$.

  $a \mathrel{\mathcal{R}} b$ means "a is related to b by the relation $\mathcal{R}$".

  $a \not\mathrel{\mathcal{R}} b$ means "a is **not** related to b by the relation $\mathcal{R}$".

NUMBER OF RELATIONS

Theorem 15.1. Let A and B be finite sets.

Then

- there are $\boxed{2^{|A|\cdot|B|}}$ relations from A to B, and
- there are $\boxed{2^{|A|^2}}$ relations on A.

proof. $\mathcal{R}$ is a relation from A to B $\Leftrightarrow$ $\mathcal{R}$ is a subset of A×B

$$\therefore \begin{pmatrix}\#\text{ relations}\\ \text{from } A \text{ to } B\end{pmatrix} = \begin{pmatrix}\#\text{ subsets}\\ \text{of } A\times B\end{pmatrix} = |\mathcal{P}(A\times B)| = 2^{|A\times B|} = 2^{|A|\cdot|B|}$$

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Example 15.2. Let $A = \{1, 2, 3\}$ and let $B = \{x, y\}$

How many relations from A to B are there?

There are $2^{|A|\cdot|B|} = 2^{(3)(2)} = 2^6 = 64$ relations from A to B

How many relations on A are there?

There are $2^{|A|^2} = 2^{3^2} = 2^9 = 512$ relations on A.

* These notes are solely for the personal use of students registered in MAT1348.

Example 15.3. Let $\mathcal{R}_3$ be a relation on $\mathbb{Z}$ defined by $\mathcal{R}_3 = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m|n\}$

Recall "m divides n" $\Leftrightarrow$ $n = km$ for some $k \in \mathbb{Z}$

[transitive] We must prove $x\mathcal{R}_3 y \wedge y\mathcal{R}_3 z \to x\mathcal{R}_3 z$

ie $[(x|y) \wedge (y|z)] \to [x|z]$

Let $x,y,z \in \mathbb{Z}$.

Assume $x|y$ and $y|z$ (goal: prove $x|z$)

Then $y = kx$ and $z = \ell y$ for some integers $k,\ell \in \mathbb{Z}$ (def of divides)

Consequently $z = \ell(kx) = (\ell k)x = jx$ where $j = \ell k$ $\therefore j \in \mathbb{Z}$

$\therefore x|z$ (def of divides)

We proved $x|y$ and $y|z \to x|z$ $\therefore \mathcal{R}_3$ ("divides") is transitive

⋄ $\mathcal{R}_3$ is reflexive and transitive, but not symmetric nor antisymmetric. (verify this!)

Example 15.4. Let $\mathcal{R}_4$ be a relation on $\mathbb{Z}^+$ defined by $\mathcal{R}_4 = \{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : m|n\}$

(same rule as $\mathcal{R}_3$ but different set)

[antisymmetric] We must prove $(x\mathcal{R}_4 y \wedge y\mathcal{R}_4 x) \to (x = y)$

ie $[(x|y) \wedge (y|x)] \to [x = y]$

Let $x,y \in \mathbb{Z}^+$

Assume $x|y$ and $y|x$ (goal: prove $x = y$)

Then $y = kx$ and $x = \ell y$ for some integers $k,\ell \in \mathbb{Z}$ (def. of divides)

Note: since $x,y \in \mathbb{Z}^+$, it follows that $k,\ell \in \mathbb{Z}^+$.

Thus $y = k(\ell y) \Rightarrow y - k\ell y = 0$

$\Rightarrow y(1 - k\ell) = 0$

$\Rightarrow \cancel{y = 0}$ or $k\ell = 1$

reject because $y \in \mathbb{Z}^+$

$\therefore k = \ell = 1$ because the only integer factors of 1 are ± 1 but since $k,\ell \in \mathbb{Z}^+$, the only option is $k = \ell = 1$.

$\therefore y = kx \Rightarrow y = 1 \cdot x = x$ $\therefore y = x$

We proved $x|y$ and $y|x \to x = y$ $\therefore \mathcal{R}_4$ is antisymmetric.

⋄ As a relation on $\mathbb{Z}^+$ (instead of on $\mathbb{Z}$), $\mathcal{R}_4$ is reflexive, antisymmetric, and transitive, but not symmetric. (verify this!)

Example 15.5. Let $\mathcal{R}_5$ be a relation on $\mathbb{Q}$ defined by

$$q \,\mathcal{R}_5\, r \quad \iff \quad q < r$$

Is $\mathcal{R}_5$ reflexive?

No. Counterexample: $5.98 \in \mathbb{Q}$ but "$5.98 < 5.98$" is false. $\therefore (5.98, 5.98) \notin \mathcal{R}_5$

Thus, it is not the case that $(q,q) \in \mathcal{R}_5$ for all $q \in \mathbb{Q}$. $\therefore \mathcal{R}_5$ is not reflexive.

Is $\mathcal{R}_5$ antisymmetric?

Yes! proof Let $x,y \in \mathbb{Q}$. Then there are 3 cases for x and y:

Case 1 Assume $x=y$. Then $x<y$ is False, thus $[(x<y)\wedge(y<x)] \to [x=y]$ is vacuously true.

Case 2 Assume $x<y$. Then $y<x$ is False, thus $[(x<y)\wedge(y<x)] \to [x=y]$ is vacuously true.

Case 1 Assume $x>y$. Then $x<y$ is False, thus $[(x<y)\wedge(y<x)] \to [x=y]$ is vacuously true.

$\therefore$ for all $x,y \in \mathbb{Q}$, the implication $[(x<y)\wedge(y<x)] \to [x=y]$ is true. $\therefore \mathcal{R}_5$ is antisymmetric.

◇ $\mathcal{R}_5$ is transitive and antisymmetric, but not reflexive, nor symmetric. (verify this!)

Example 15.6. Let $\mathcal{R}_6$ be a relation on $\mathbb{Z}$ defined by

$$(r,s) \in \mathcal{R}_6 \quad \iff \quad s = 2r+1$$

Is $\mathcal{R}_6$ symmetric?

No. Counterexample: $0,1 \in \mathbb{Z}$

$1 = 2(0)+1 \implies (0,1) \in \mathcal{R}_6$

$0 \neq 2(1)+1 \implies (1,0) \notin \mathcal{R}_6$

Thus, it is not the case that $(r,s) \in \mathcal{R}_6 \to (s,r) \in \mathcal{R}_6$ for all $r,s \in \mathbb{Z}$

$\therefore \mathcal{R}_6$ is not symmetric.

◇ $\mathcal{R}_6$ is antisymmetric, but not reflexive, nor symmetric, nor transitive. (verify this!)

Example 15.7. Let $\mathcal{R}_7$ be a relation on $A = \{1,2,3,4\}$ defined by

$$\mathcal{R}_7 = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$$

Is $\mathcal{R}_7$ reflexive?

Yes! proof: $(1,1),(2,2),(3,3),(4,4) \in \mathcal{R}_7$ $\therefore (a \in A) \to (a \mathcal{R}_7 a)$ is true

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Is $\mathcal{R}_7$ antisymmetric?

No. Counterexample:

$2,3 \in A$ and $2 \neq 3$ but both $(2,3) \in \mathcal{R}_7$ and $(3,2) \in \mathcal{R}_7$

Thus, it is not the case that $(a \neq b) \to ((a,b) \notin \mathcal{R}_7$ or $(b,a) \notin \mathcal{R}_7)$ for all $a,b \in A$. $\therefore \mathcal{R}_7$ is not antisymmetric.

◇ $\mathcal{R}_7$ is reflexive, symmetric, and transitive, but not antisymmetric. (verify this!)

In particular, $\mathcal{R}_7$ is an equivalence relation on $\{1,2,3,4\}$

Note the set $\rightarrow$ $\mathcal{R}_8$ relates pairs of ordered pairs to each other by this rule

Example 15.8. Let $\mathcal{R}_8$ be a relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$\big((n_1, n_2), (m_1, m_2)\big) \in \mathcal{R}_8 \iff n_2 = m_1$$

Is $\mathcal{R}_8$ transitive?

No. Counterexample: $(1,5), (5,0), (0,9) \in \mathbb{N}\times\mathbb{N}$

$(1,5)\,\mathcal{R}_8\,(5,0)$ and $(5,0)\,\mathcal{R}_8\,(0,9)$

but $(1,5) \not\mathcal{R}_8 (0,9)$ $\therefore$ $\mathcal{R}_8$ is not transitive.

◇ $\mathcal{R}_8$ is not reflexive, nor symmetric, nor antisymmetric, nor transitive. (verify this!)

Exercise 15.9. Give an example of a relation on the set $A = \{1, 2, 3\}$

- that is both symmetric and antisymmetric:

$\mathcal{R} = \{(1,1)\}$ is symmetric and antisymmetric (there are other possible answers)

- that is neither symmetric nor antisymmetric:

$\mathcal{S} = \{(1,2),(2,1),(2,3)\}$ is neither symmetric, nor antisymmetric

(there are other possible answers)

Exercise 15.10. There are 16 relations on the set $A = \{1, 2\}$. Here they all are:

$\mathcal{R}_1 = \emptyset$	$\mathcal{R}_9 = \{(1,2),(2,1)\}$
$\mathcal{R}_2 = \{(1,1)\}$	$\mathcal{R}_{10} = \{(1,2),(2,2)\}$
$\mathcal{R}_3 = \{(1,2)\}$	$\mathcal{R}_{11} = \{(2,1),(2,2)\}$
$\mathcal{R}_4 = \{(2,1)\}$	$\mathcal{R}_{12} = \{(1,1),(1,2),(2,1)\}$
$\mathcal{R}_5 = \{(2,2)\}$	$\mathcal{R}_{13} = \{(1,1),(1,2),(2,2)\}$
$\mathcal{R}_6 = \{(1,1),(1,2)\}$	$\mathcal{R}_{14} = \{(1,1),(2,1),(2,2)\}$
$\mathcal{R}_7 = \{(1,1),(2,1)\}$	$\mathcal{R}_{15} = \{(1,2),(2,1),(2,2)\}$
$\mathcal{R}_8 = \{(1,1),(2,2)\}$	$\mathcal{R}_{16} = \{(1,1),(1,2),(2,1),(2,2)\}$

◇ Which of these relations is reflexive? $\mathcal{R}_i$ for all $i \in \{8,13,14,16\}$

◇ Which of these relations is symmetric? $\mathcal{R}_i$ for all $i \in \{1,2,5,8,9,12,15,16\}$

◇ Which of these relations is antisymmetric? $\mathcal{R}_i$ for all $i \in \{1,2,3,4,5,6,7,8,10,11,13,14\}$

◇ Which of these relations is transitive? $\mathcal{R}_i$ for all $i \in \{1,2,3,4,5,6,7,8,10,11,13,14,16\}$

◇ Which of these relations is an equivalence relation on A? $\mathcal{R}_i$ for all $i \in \{8,16\}$

Example 15.11. Logical Equivalence:

Let $\mathscr{A}$ be the set of all compound propositions. Logical equivalence $\equiv$ is a relation on $\mathscr{A}$ given by the rule:

for all $P, Q \in \mathscr{A}$, $\quad P \equiv Q \quad$ if and only if $\quad P \leftrightarrow Q$ is a tautology.

Prove that $\equiv$ is an equivalence relation on $\mathscr{A}$.

We must prove that $\equiv$ is reflexive, symmetric and transitive.

[reflexive] Let $P \in \mathscr{A}$. Then $P \leftrightarrow P \equiv (P \wedge P) \vee (\neg P \wedge \neg P)$ (biconditional law)

$\equiv P \vee \neg P$ (idempotent law x2)

$\equiv T$ (negation law)

$\therefore$ $P \leftrightarrow P$ is a tautology $\therefore$ $P \equiv P$.

We proved $(P \in \mathscr{A}) \rightarrow (P \equiv P)$ $\therefore$ $\equiv$ is reflexive.

[symmetric] Let $P, Q \in \mathscr{A}$.

Assume $P \equiv Q$ (goal: prove $Q \equiv P$)

Then $T \equiv P \leftrightarrow Q$ (def of $\equiv$)

$\Rightarrow T \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$ (biconditional law)

$\Rightarrow T \equiv (Q \wedge P) \vee (\neg Q \vee \neg P)$ (commutative law x2)

$\Rightarrow T \equiv Q \leftrightarrow P$ (biconditional law)

$\therefore$ $Q \leftrightarrow P$ is a tautology $\therefore$ $Q \equiv P$.

We proved $(P \equiv Q) \rightarrow (Q \equiv P)$ $\therefore$ $\equiv$ is symmetric.

[transitive] Let $P, Q, R \in \mathscr{A}$.

Assume $P \equiv Q$ and $Q \equiv R$. (goal: prove $P \equiv R$)

Then $P \leftrightarrow Q$ and $Q \leftrightarrow R$ are both tautologies (def of $\equiv$)

We will prove $P \leftrightarrow R$ is a tautology by proving $P \rightarrow R$ and $R \rightarrow P$ are true.

($\Rightarrow$) Assume P is T. Then Q is T since $P \leftrightarrow Q \equiv T$.

$\therefore$ R is T since $Q \leftrightarrow R \equiv T$. $\therefore$ $P \rightarrow R$ is true.

($\Leftarrow$) Assume R is T. Then Q is T since $Q \leftrightarrow R \equiv T$.

$\therefore$ P is T since $P \leftrightarrow Q \equiv T$. $\therefore$ $R \rightarrow P$ is true.

Since $P \rightarrow R$ and $R \rightarrow P$ are true, $P \leftrightarrow R$ is a tautology $\therefore$ $P \equiv R$.

We proved $(P \equiv Q) \wedge (Q \equiv R) \rightarrow (P \equiv R)$ $\therefore$ $\equiv$ is transitive.

Since $\equiv$ is reflexive, symmetric and transitive, it is an equivalence relation on $\mathscr{A}$

Given an equivalence relation $\mathcal{R}$ on A, for each element $a \in A$, we define **the equivalence class of a with respect to $\mathcal{R}$** as follows:

$$\boxed{[a]_{\mathcal{R}} = \{x \in A : a \mathcal{R} x\}}$$

$=$ set of all elements of A which are related to a by $\mathcal{R}$

Example 15.12. Let x and y be propositional variables, and let $A = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$, where the elements $P_i \in A$ are the following compound propositions:

$P_1: \quad x \to y$	$P_3: \quad \neg(\neg x \vee y)$	$P_5: \quad \neg(x \to y)$	$P_7: \quad x \wedge \neg y$
$P_2: \quad x \vee y$	$P_4: \quad \neg x \vee y$	$P_6: \quad x \oplus y$	$P_8: \quad \neg(x \leftrightarrow y)$

Let $\mathcal{R}$ be a relation on the set A defined by $\quad (P_i, P_j) \in \mathcal{R} \iff P_i \equiv P_j$

Note: Because $\equiv$ is an equivalence relation on the set of *all* compound propositions, it follows that $\mathcal{R}$ is an equivalence relation on A.

Compute the equivalence class for each element of A.

$[P_1]_{\mathcal{R}} = \{P_1, P_4\}$

$[P_2]_{\mathcal{R}} = \{P_2\}$

$[P_3]_{\mathcal{R}} = \{P_3, P_5, P_7\}$

$[P_4]_{\mathcal{R}} = \{P_1, P_4\}$

$[P_5]_{\mathcal{R}} = \{P_3, P_5, P_7\}$

$[P_6]_{\mathcal{R}} = \{P_6, P_8\}$

$[P_7]_{\mathcal{R}} = \{P_7, P_3, P_5\}$

$[P_8]_{\mathcal{R}} = \{P_8, P_6\}$

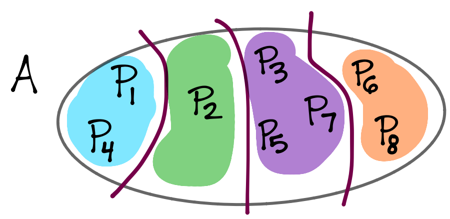
Some observations:

$[P_1]_{\mathcal{R}} = [P_4]_{\mathcal{R}} \qquad [P_3]_{\mathcal{R}} = [P_5]_{\mathcal{R}} = [P_7]_{\mathcal{R}} \qquad [P_6]_{\mathcal{R}} = [P_8]_{\mathcal{R}}$

$[P_1]_{\mathcal{R}} \cap [P_2]_{\mathcal{R}} = \emptyset \qquad [P_2]_{\mathcal{R}} \cap [P_3]_{\mathcal{R}} = \emptyset$

$[P_1]_{\mathcal{R}} \cap [P_3]_{\mathcal{R}} = \emptyset \qquad [P_2]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$

$[P_1]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset \qquad [P_3]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$



Note. Since $\mathcal{R}$ is an equivalence relation on A, the relation $\mathcal{R}$ gives us a way to "partition" the elements of A into disjoint "equivalence classes" of pairwise related elements.

STUDY GUIDE

relation on a set A $\mathcal{R} \subseteq A \times A$	**properties of a relation on a set:**	reflexive, symmetric antisymmetric, transitive
number of relations from A to B $= \lvert\mathcal{P}(A \times B)\rvert = 2^{\lvert A\rvert\lvert B\rvert}$	**equivalence relations:** **equivalence classes:**	reflexive, symmetric, & transitive $[a]_{\mathcal{R}} = \{x \in A : x \mathcal{R} a\}$
Exercises		Sup.Ex. §7 # 1a, 2, 3, 4a, 6a, 8a, 9, 10a, 11 Rosen §9.5 # 1, 3, 7, 11, 15, 25, 26, 29, 55