DGD 9

Q1. PARTITIONS OF A SET INTO EQUIVALENCE CLASSES

Let $A = \{-4, -3, -1, 0, 2, 5, 6, 7, 8, 15, 23\}$.

i. Determine the partition of A into equivalence classes of the equivalence relation $\equiv \pmod 7$.

	-4	-3	-1	0	2	5	6	7	8	15	23
remainder (mod7)	3	4	6	0	2	5	6	0	1	1	2

partition of A into equivalence classes (mod7) $\mathcal{P} = \{\{-4\}, \{-3\}, \{-1, 6\}, \{0, 7\}, \{2, 23\}, \{5\}, \{8, 15\}\}$

ii. Determine the partition of A into equivalence classes of the equivalence relation $\mathcal{S}$ given by the rule for all $x, y \in A$, $x \,\mathcal{S}\, y$ if and only if $x + 3y$ is even.

$[-4]_{\mathcal{S}} = \{-4, 0, 2, 6, 8\}$ $[-3]_{\mathcal{S}} = \{-3, -1, 5, 7, 15, 23\}$

partition of A into equivalence classes of $\mathcal{S}$ $\mathcal{P} = \{\{-4, 0, 2, 6, 8\}, \{-3, -1, 5, 7, 15, 23\}\}$

iii. Verify that the relation $\mathcal{T}$ given below is an equivalence relation on A.

$$\mathcal{T} = \Big\{(-4,-4), (-3,-3), (-1,-1), (0,0), (0,2), (0,5), (2,0), (2,2), (2,5), (5,0), (5,2), (5,5), (6,6), (6,7), (7,6), (7,7), (8,8), (15,15), (23,23)\Big\}$$

Give the equivalence classes $[-4]_{\mathcal{T}}$ and $[5]_{\mathcal{T}}$.

$[-4]_{\mathcal{T}} = \{-4\}$ $[5]_{\mathcal{T}} = \{5, 0, 2\}$

$[-3]_{\mathcal{T}} = \{-3\}$ $[-1]_{\mathcal{T}} = \{-1\}$ $[6]_{\mathcal{T}} = \{6, 7\}$ $[8]_{\mathcal{T}} = \{8\}$ $[15]_{\mathcal{T}} = \{15\}$ $[23]_{\mathcal{T}} = \{23\}$

Determine the partition of A into equivalence classes of $\mathcal{T}$.

$\mathcal{P} = \{\{-4\}, \{-3\}, \{-1\}, \{0, 2, 5\}, \{6, 7\}, \{8\}, \{15\}, \{23\}\}$

Q2. ANOTHER EXAMPLE OF AN EQUIVALENCE RELATION ON $\mathbb{Z}$

Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ given by the following rule:

for all $x, y \in \mathbb{Z}$, $\quad x \,\mathcal{R}\, y \quad$ if and only if $\quad x \equiv y \pmod 7$ or $x \equiv -y \pmod 7$.

i. Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

[reflexive] Let $x \in \mathbb{Z}$. Then $x \equiv x \pmod 7$ (since $\equiv \pmod 7$ is reflexive)

$\therefore x \mathcal{R} x$ (by the rule for $\mathcal{R}$)

We proved $(x \in \mathbb{Z}) \to (x \mathcal{R} x)$ $\therefore \mathcal{R}$ is reflexive.

[symmetric]. Let $x, y \in \mathbb{Z}$

Assume $x \mathcal{R} y$ (goal: prove $y \mathcal{R} x$)

Then $x \equiv y \pmod 7$ or $x \equiv -y \pmod 7$ (by $\mathcal{R}$'s rule)

Case 1. If $x \equiv y \pmod 7$, then $y \equiv x \pmod 7$ (since $\equiv \pmod 7$ is symmetric).

Case 2. If $x \equiv -y \pmod 7$, then $7 \mid (x - (-y))$ (by Theorem 16.2)

$\Rightarrow x + y = 7k$ for some $k \in \mathbb{Z}$

$\Rightarrow y + x = 7k$ $\therefore 7 \mid y - (-x)$

$\therefore y \equiv -x \pmod 7$ (by Theorem 16.2)

$\therefore y \mathcal{R} x$ (by $\mathcal{R}$'s rule)

In both cases, we proved $(x \mathcal{R} y) \to (y \mathcal{R} x)$ $\therefore \mathcal{R}$ is symmetric.

[transitive] Let $x, y, z \in \mathbb{Z}$

Assume $x \mathcal{R} y$ and $y \mathcal{R} z$. (goal: prove $x \mathcal{R} z$)

Then $[x \equiv y \pmod 7$ or $x \equiv -y \pmod 7]$ and $[y \equiv z \pmod 7$ or $y \equiv -z \pmod 7]$

(by $\mathcal{R}$'s rule)

Case 1 Assume $x \equiv y \pmod 7$ and $y \equiv z \pmod 7$.

Then $x \equiv z \pmod 7$ (since $\equiv \pmod 7$ is transitive)

$\therefore x\mathcal{R}z$ in Case 1.

Case 2 Assume $x \equiv y \pmod 7$ and $y \equiv -z \pmod 7$.

Then $x \equiv -z \pmod 7$ (since $\equiv \pmod 7$ is transitive)

$\therefore x\mathcal{R}z$ in Case 2.

Case 3 Assume $x \equiv -y \pmod 7$ and $y \equiv z \pmod 7$.

Then $7 \mid (x-(-y))$ and $7 \mid (y-z)$ (by Theorem 16.2)

Thus $x+y = 7k$ and $y-z = 7\ell$ for some integers $k, \ell \in \mathbb{Z}$.

$\therefore x-(-z) = 7k-y-(7\ell-y) = 7k-7\ell$

$\Rightarrow 7 \mid (x-(-z))$ $\therefore x \equiv -z \pmod 7$ (by Theorem 16.2)

$\therefore x\mathcal{R}z$ in Case 3.

Case 4 Assume $x \equiv -y \pmod 7$ and $y \equiv -z \pmod 7$.

Then $7 \mid (x-(-y))$ and $7 \mid (y-(-z))$ (by Theorem 16.2)

Thus $x+y = 7k$ and $y+z = 7\ell$ for some integers $k, \ell \in \mathbb{Z}$.

$\therefore x-z = 7k-y-(7\ell-y) = 7k-7\ell$

$\Rightarrow 7 \mid (x-z)$ $\therefore x \equiv z \pmod 7$ (by Theorem 16.2)

$\therefore x\mathcal{R}z$ in Case 4.

We proved $(x\mathcal{R}y \wedge y\mathcal{R}z) \rightarrow (x\mathcal{R}z)$ in all four cases. $\therefore \mathcal{R}$ is transitive.

Since $\mathcal{R}$ is reflexive, symmetric, and transitive, it is an equivalence relation.

ii. Determine the partition of A (from Q1) into equivalence classes of $\mathcal{R}$.

	−4	−3	−1	0	2	5	6	7	8	15	23
remainder (mod7)	3	4	6	0	2	5	6	0	1	1	2
negative's	4	3	1	0	−2	−5	−6	−7	−8	−15	−23
remainder (mod7)	4	3	1	0	5	2	1	0	6	6	5

partition of A into equivalence classes of $\mathcal{R}$ $\mathcal{P} = \{\{-4,-3\}, \{-1,6,8,15\}, \{0,7\}, \{2,23,5\}\}$

★ Review previous DGD exercises that were missed and Assignment 2 solutions if time.