11. Set Identities

☐ **building new sets out of old:**

☐ power set of a set A: $\mathcal{P}(A)$ $\quad |\mathcal{P}(A)| = 2^{|A|}$

☐ Cartesian product of sets A and B $\quad A \times B$ $\quad |A \times B| = |A||B|$

☐ generalization of Cartesian product $\quad A_1 \times A_2 \times \cdots \times A_n$

☐ **set operations:**

union $\quad A \cup B$ $\qquad$ intersection $\quad A \cap B$ $\qquad$ complement $\quad \overline{A}$

set difference $\quad A - B$ $\qquad$ symmetric difference $\quad A \oplus B$

☐ **verifying set identities:**

using a membership table $\qquad$ with a rigorous proof

USING THE TABLE OF SET IDENTITIES

Example 11.1. Let A, B, and C be subsets of the universal set $\mathcal{U}$.

Using set identities, prove that $\quad \overline{(B \cup C) - A} = (\overline{C} \cap \overline{B}) \cup A$

$$LS = \overline{(B \cup C) - A}$$

$$= \overline{(B \cup C) \cap \overline{A}} \quad \text{(Difference Law)}$$

$$= \overline{(B \cup C)} \cup \overline{\overline{A}} \quad \text{(De Morgan's Law)}$$

$$= (\overline{B} \cap \overline{C}) \cup \overline{\overline{A}} \quad \text{(De Morgan's Law)}$$

$$= (\overline{B} \cap \overline{C}) \cup A \quad \text{((Double) Complementation Law)}$$

$$= (\overline{C} \cap \overline{B}) \cup A \quad \text{(Commutative Law)}$$

$$= RS \qquad \therefore LS = RS$$

* These notes are solely for the personal use of students registered in MAT1348.

Table of Important Set Identities

1. 2.	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
3. 4.	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination Laws
5. 6.	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8. 9.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
10. 11.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
12. 13.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
14. 15.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
16. 17.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
18. 19.	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	Complement Laws
20.	$A - B = A \cap \overline{B}$	Difference Law
21. 22.	$A \oplus B = (A - B) \cup (B - A)$ $A \oplus B = (A \cup B) - (A \cap B)$	Symmetric Difference Laws

Example 11.2. Prove the following theorem:

Theorem 11.2. Let A and B be subsets of the universal set.

$$\text{Then} \quad \underbrace{\overline{A} \subseteq \overline{B}}_{P} \quad \text{if and only if} \quad \underbrace{B \subseteq A}_{Q}.$$

Note: P says "for all $x \in \mathcal{U}$, $(x \in \overline{A}) \to (x \in \overline{B})$"

Equivalently (contrapos.): $(x \notin \overline{B}) \to (x \notin \overline{A})$ ☆☆☆

Q says "for all $x \in \mathcal{U}$, $(x \in B) \to (x \in A)$"

Equivalently (contrapos.): $(x \notin A) \to (x \notin B)$ ✧✧✧

Proof of Theorem 11.2 (using a proof of equivalence)

($\Longrightarrow$) We will prove $P \to Q$ with a direct proof.

Assume P is True. ie Assume $\overline{A} \subseteq \overline{B}$. (goal is to prove Q is True, ie $B \subseteq A$).

Let $x \in \mathcal{U}$.

Assume $x \in B$. Then $x \notin \overline{B}$ since $x \in B$.

$\Rightarrow x \notin \overline{A}$ since we assumed $\overline{A} \subseteq \overline{B}$. ☆☆☆

$\Rightarrow x \in A$

Thus, we proved $(x \in B) \to (x \in A)$ $\therefore$ $B \subseteq A$ (ie Q is True).

Overall, we proved $(\overline{A} \subseteq \overline{B}) \to (B \subseteq A)$

($\Longleftarrow$) We will prove $Q \to P$ with a direct proof.

Assume Q is True. ie Assume $B \subseteq A$ (goal is to prove P is True, ie $\overline{A} \subseteq \overline{B}$).

Let $x \in \mathcal{U}$.

Assume $x \in \overline{A}$. Then $x \notin A$ since $x \in \overline{A}$

$\Rightarrow x \notin B$ since we assumed $B \subseteq A$. ✧✧✧

$\Rightarrow x \in \overline{B}$

Thus, we proved $(x \in \overline{A}) \to (x \in \overline{B})$ $\therefore$ $\overline{A} \subseteq \overline{B}$ (ie P is True).

Overall, we proved $(B \subseteq A) \to (\overline{A} \subseteq \overline{B})$

We proved $P \to Q$ and $Q \to P$. $\therefore$ we proved $P \leftrightarrow Q$ is true. ■

Exercise 11.3. Give concrete examples of two sets A and B such that $\overline{A} \not\subseteq \overline{B}$.

Let $\mathcal{U} = \{1,2,3,4,5\}$ and let A and B be the following two sets:

$$A = \{1,2,3\} \qquad B = \{3,4,5\}$$

Then $\overline{A} = \{4,5\} \qquad \overline{B} = \{1,2\}$

and so $\overline{A} \not\subseteq \overline{B}$.

Why doesn't this example contradict Theorem 11.2?

Theorem 11.2 does not say that $\overline{A} \subseteq \overline{B}$ for all sets A, B.

Theorem 11.2 says $\overline{A} \subseteq \overline{B}$ if and only if $B \subseteq A$.

It means: ① If $\overline{A} \subseteq \overline{B}$, then $B \subseteq A$ and ② If $B \subseteq A$, then $\overline{A} \subseteq \overline{B}$

In our example, B was <u>not</u> a subset of A so the premise of ② was not fulfilled.

STUDY GUIDE

Basic terms and concepts of Set Theory:

☐ set	☐ element	☐ subset	☐ proper subset	☐ equality	☐ cardinality
S	$x \in S$	$T \subseteq S$	$T \subset S$	$S = T$	$\|S\|$

Some important sets:

☐ empty set	☐ universal set	☐ naturals	☐ integers	☐ rationals	☐ reals
$\emptyset$	$\mathcal{U}$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
			$\mathbb{Z}^-$ $\mathbb{Z}^+$	$\mathbb{Q}^-$ $\mathbb{Q}^+$	$\mathbb{R}^-$ $\mathbb{R}^+$

Building new sets from old:

☐ power set of S — $\mathcal{P}(S)$

☐ Cartesian product of two (or more) sets — $S \times T$ $\quad S_1 \times S_2 \times \cdots \times S_t$

Set Operations:

☐ union	☐ intersection	☐ complement	☐ difference	☐ symmetric difference
$S \cup T$	$S \cap T$	$\overline{S}$	$S - T$	$S \oplus T$

Set identities:

☐ verify using membership tables ☐ verify using a rigorous proof

☐ prove other identities using the laws from the Table of Important Set Identities

Exercises	Sup.Ex. §4 # 1, 2, 3, 4, 5, 6, 9, 11 Rosen §2.2 # 14, 15, 17, 19, 29, 35