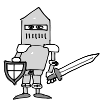
DGD 2

Q1. The Island of Knights & Knaves

Strolling along on The Island of Knights & Knaves, we meet A, B and C, each of whom is a knight or a knave. Note: knaves always lie and knights always speak the truth. Two people are said to be of the same type if they are both knights or both knaves. A and B make the following statements:



A says: "B is a knave." B says: "A and C are of the same type."

What (if anything) can you conclude?

Solution As an inhabitant of The Island of Knights and Knaves, either A is a Knight or a Knave.

Case 1. A is a Knight ⇒ "B is a Knave" must be true.
⇒ B's statement "A and C are of the same type" must be false.
⇒ A and C must be of different types
⇒ C is a Knave.
∴ Case 1 is possible (A Knight, B Knave, C Knave)

Case 2. A is a Knave. ⇒ "B is a Knave" must be false. ⇒ B must be a Knight.
⇒ B's statement "A and C are of the same type" must be true.
⇒ A and C must be of the same type.
⇒ C is a Knave.
∴ Case 2 is also possible (A Knave, B Knight, C Knave)

- Although we cannot determine the types of A and B, we do know that C must be a Knave in both possible cases.
- We also know that A and B must be opposite types of each other.

Truth Table Method

1. Define I-am-Knight atoms for each inhabitant:
   a: "A is a Knight."
   b: "B is a Knight."
   c: "C is a Knight."

2. Translate all statements:
   A says: "$\neg b$"
   B says: "$a \leftrightarrow c$"
   There are other logically equivalent ways to translate B's statement

3. Truth Table

↯ means "contradiction"

a	b	c	A says $\neg b$	B says $a \leftrightarrow c$	
T	T	T	F	T	← in this row, A is a Knight, but A's statement is F ↯
T	T	F	F	F	← in this row, A is a Knight, but A's statement is F ↯
T	F	T	T	T	← in this row, B is a Knave, but B's statement is T ↯
T	F	F	T	F	← this row is possible on The Island.
F	T	T	F	F	← in this row, B is a Knight, but B's statement is F ↯
F	T	F	F	T	← this row is possible on The Island.
F	F	T	T	F	← in this row, A is a Knave, but A's statement is T ↯
F	F	F	T	T	← in this row, A is a Knave, but A's statement is T ↯

compare these columns

compare these columns

Conclusion: In both possible cases, C is a Knave

Q2. Counterexamples

i. Suppose two compound propositions, say X and Y, are **not** logically equivalent. Describe what a counterexample would be.

A counterexample would be a truth assignment that certifies that $X \leftrightarrow Y$ is not a tautology.

Ex. Is $P \vee Q \to R \equiv R \to P \wedge Q$? If not, provide *all* counterexamples.

P	Q	R	$P\vee Q$	$P\vee Q \to R$ (X)	$P\wedge Q$	$R\to P\wedge Q$ (Y)	$X \leftrightarrow Y$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	F
T	F	T	T	T	F	F	F
T	F	F	T	F	F	T	F
F	T	T	T	T	F	F	F
F	T	F	T	F	F	T	F
F	F	T	F	T	F	F	F
F	F	F	F	T	F	T	T

X is not logically equivalent to Y

counterexamples:

① $P=T, Q=T, R=F$
② $P=T, Q=F, R=T$
③ $P=T, Q=F, R=F$
④ $P=F, Q=T, R=T$
⑤ $P=F, Q=T, R=F$
⑥ $P=F, Q=F, R=T$

for each of these truth assignments, $X\leftrightarrow Y$ is F $\therefore$ $X \not\equiv Y$

ii. Suppose a compound proposition X is **not** a tautology. Describe what a counterexample would be.

A counterexample would be a truth assignment for which the truth value of X is F.

Ex. Is $\neg(\neg P \wedge \neg Q) \leftrightarrow P \wedge Q$ a tautology ? If not, provide *all* counterexamples.

P	Q	$\neg P\wedge\neg Q$	$\neg(\neg P\wedge \neg Q)$	$P\wedge Q$	$\neg(\neg P\wedge Q)\leftrightarrow P\wedge Q$ (X)
T	T	F	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

X is not a tautology.
counterexamples:

① $P=T, Q=F$
② $P=F, Q=T$

for each of these truth assignments, the truth value of X is F
$\therefore$ X is not a tautology.

iii. Suppose a compound proposition X is **not** a contradiction. Describe what a counterexample would be.

A counterexample would be a truth assignment for which the truth value of X is T.

Ex. Is $P \to (Q \vee \neg P)$ a contradiction ? If not, provide *all* counterexamples.

P	Q	$Q\vee\neg P$	$P\to(Q\vee\neg P)$ (X)
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

X is not a contradiction
counterexamples:
① $P=T, Q=T$
② $P=F, Q=T$
③ $P=F, Q=F$

for each of these truth assignments, the truth value of X is T
$\therefore$ X is not a contradiction.

Q3. THE LAWS OF LOGICAL EQUIVALENCES

Using only the Laws in the Table of Logical Equivalences, naming the law used at each step, prove the following equivalence:

$$\underbrace{(\neg Q \wedge (P \to Q)) \to \neg P}_{LS} \equiv \underbrace{Q \vee \neg Q}_{RS}$$

$LS = (\neg Q \wedge (P \to Q)) \to \neg P$

$\equiv \neg(\neg Q \wedge (P \to Q)) \vee \neg P$ (implication law)

$\equiv \neg(\neg Q \wedge (\neg P \vee Q)) \vee \neg P$ (implication law)

$\equiv (\neg\neg Q \vee \neg(\neg P \vee Q)) \vee \neg P$ (De Morgan's law)

$\equiv (Q \vee \neg(\neg P \vee Q)) \vee \neg P$ (double negation law)

$\equiv (Q \vee (\neg\neg P \wedge \neg Q)) \vee \neg P$ (De Morgan's law)

$\equiv (Q \vee (P \wedge \neg Q)) \vee \neg P$ (double negation law)

$\equiv ((Q \vee P) \wedge (Q \vee \neg Q)) \vee \neg P$ (distributive law)

$\equiv ((Q \vee P) \wedge T) \vee \neg P$ (negation law)

$\equiv (Q \vee P) \vee \neg P$ (identity law)

$\equiv Q \vee (P \vee \neg P)$ (associative law)

$\equiv Q \vee T$ (negation law)

$\equiv T$ (domination law)

$\equiv Q \vee \neg Q$ (negation law)

$= RS$

$\therefore LS \equiv RS$

Q4. DNF

Using a truth table, find a DNF for each of the compound propositions X and Y, defined as follows:

i. X : $(a \vee \neg b) \oplus (b \wedge c)$

a	b	c	$a \vee \neg b$	$b \wedge c$	$(a \vee \neg b) \oplus (b \wedge c)$ = X	conjunctive clauses
T	T	T	T	T	F	
T	T	F	T	F	T	$a \wedge b \wedge \neg c$
T	F	T	T	F	T	$a \wedge \neg b \wedge c$
T	F	F	T	F	T	$a \wedge \neg b \wedge \neg c$
F	T	T	F	T	T	$\neg a \wedge b \wedge c$
F	T	F	F	F	F	
F	F	T	T	F	T	$\neg a \wedge \neg b \wedge c$
F	F	F	T	F	T	$\neg a \wedge \neg b \wedge \neg c$

DNF for X:

$$(a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$$

ii. Y : $\neg(A \rightarrow B) \leftrightarrow (\neg A \vee B)$

A	B	$\neg(A \rightarrow B)$	$\neg A \vee B$	$\neg(A \rightarrow B) \leftrightarrow (\neg A \vee B)$ = Y
T	T	F	T	F
T	F	T	F	F
F	T	F	T	F
F	F	F	T	F

← there is no truth assignment for which Y is true (ie Y is a contradiction).

Well, technically 'F' is an atom, technically an atom by itself is a conjuctive clause, and the disjunction of one conjuctive clause is in DNF.

∴ A DNF for Y is F.

Q5 EQUIVALENCES

i. Using the Laws of Logical Equivalences (naming the law used at each step), find a DNF for the compound proposition R, defined as follows:

$$R: \quad (a \vee \neg b) \rightarrow \neg(b \wedge c)$$

$R = (a \vee \neg b) \rightarrow \neg(b \wedge c)$

$\equiv \neg(a \vee \neg b) \vee \neg(b \wedge c)$ (implication law)

$\equiv (\neg a \wedge \neg\neg b) \vee \neg(b \wedge c)$ (De Morgan's law)

$\equiv (\neg a \wedge b) \vee \neg(b \wedge c)$ (double negation law)

$\equiv \underbrace{(\neg a \wedge b) \vee (\neg b \vee \neg c)}$ (De Morgan's law)

this is a DNF for R

Note: $(\neg a \wedge b) \vee (\neg b \vee \neg c) \equiv (\neg a \wedge b) \vee (\neg b) \vee (\neg c)$ by associativity of $\vee$.

ii. Using the Laws of Logical Equivalences (naming the law used at each step), find a formula for the compound proposition S that uses only the logical connectives $\neg$ and $\leftarrow$ and parentheses where appropriate.

$$S: \quad \neg(x \wedge y) \vee \neg(\neg y \vee \neg x)$$

$S = \neg(x \wedge y) \vee \neg(\neg y \vee \neg x)$

$\equiv (\neg x \vee \neg y) \vee \neg(\neg y \vee \neg x)$ (De Morgan's law)

$\equiv (x \rightarrow \neg y) \vee \neg(\neg y \vee \neg x)$ (implication law)

$\equiv (x \rightarrow \neg y) \vee \neg(y \rightarrow \neg x)$ (implication law)

$\equiv \neg(y \rightarrow \neg x) \vee (x \rightarrow \neg y)$ (commutative law)

$\equiv (y \rightarrow \neg x) \rightarrow (x \rightarrow \neg y)$ (implication law)