9. Basics of Set Theory

Important Proof Strategies:
☐ Direct Proof ☐ Indirect Proof ☐ Proof by Contradiction
☐ Proof by Cases ☐ Proof of Equivalence

SETS AND SET CONCEPTS

A **set** is a well-defined unordered collection of objects called **elements**.

Ex. $\underset{\text{name of set}}{A} = \{\underbrace{1, 2, a, \text{Ottawa}, \heartsuit}_{\text{list of elements contained in set.}}\}$

Ex. {people in this room}

Notation for set membership: $1 \in A$ means "1 is an element of A" (greek letter epsilon)

$3 \notin A$ means "3 is not an element of A"

Two sets are **equal** if they contain the same elements (regardless of the order and multiplicity).

Ex. $S = \{a, b, c\}$ $T = \{b, c, a\}$ $U = \{a, a, b, c\}$ (U contains 3 distinct elements, namely a, b, and c)

Note. $S = T = U$.

A set either contains some element or it does not.
↳ order elements are listed does not affect "is an element of"
↳ repeating an element more than once does not affect "is an element of"

Describing a Set.

by listing its elements between braces {} (using ellipses (. . .) if necessary)	**by using set-builder notation**
$A = \{a, e, i, o, u\}$	$A = \{\ell : \ell \text{ is a vowel of the English alphabet}\}$
$B = \{3, 6, 9, \ldots, 36\}$	$B = \{3n : n \in \mathbb{Z},\ 1 \leq n \leq 12\}$
$C = \{3, 4, 5, 6, \ldots\}$	$C = \{n : n \in \mathbb{Z},\ n \geq 3\}$
$D = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$	$D = \{n : n \text{ is an even integer}\}$

Important Sets of Numbers

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (The set of natural numbers)

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ (the set of integers)

$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, q \neq 0 \right\}$ (the set of rational numbers)

$\mathbb{R} = \{r : r \text{ is a real number}\}$ (the set of real numbers)

$\mathbb{Z}^+ = \{n \in \mathbb{Z} : n > 0\}$ (the set of positive integers)

$\mathbb{Z}^- = \{n \in \mathbb{Z} : n < 0\}$ (the set of negative integers)

Similarly, $\mathbb{Q}^+, \mathbb{Q}^-, \mathbb{R}^+, \mathbb{R}^- \ldots$

The Empty Set. $\emptyset$

The empty set, denoted $\emptyset$, is the set with no elements. i.e. $\emptyset = \{\ \}$

Note. $\{\emptyset\}$ is not the empty set because it does contain an element
(its one element happens to be a set, the empty set in fact. Regardless of what the element is, $\{\emptyset\}$ does contain an element)

The Universal Set. $\mathcal{U}$

The universal set, denoted $\mathcal{U}$, is the set of all objects under consideration.

Subsets.

Let A and B be sets.
Then A is said to be a **subset** of B (written $A \subseteq B$) if every element of A is also an element of B.

i.e. for all $x \in \mathcal{U}$, the implication $(x \in A) \to (x \in B)$ is true. exercise

Write the negation of this definition.
ie What is the definition for "A is not a subset of B" ?

Ex. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Ex. $A = \{a, b, c\}$ $B = \{a, c\}$ $C = \{a, \{b\}, c\}$

$B \subseteq A$

$B \subseteq C$

"is not a subset of"

$A \nsubseteq B$ because $b \in A$ but $b \notin B$

$A \nsubseteq C$ because $b \in A$ but $b \notin C$

$C \nsubseteq A$ because $\{b\} \in C$ but $\{b\} \notin A$

$A \neq C$ because A and C do not contain the same elements

just a letter of the English alphabet → b vs. $\{b\}$ ← a set containing one element, namely b.

Theorem 9.1. Let S be any set. Then

1. $S \subseteq S$
2. $\emptyset \subseteq S$

Theorem 9.2. Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Note. To prove $A = B$, we must prove two things: 1. $A \subseteq B$ and 2. $B \subseteq A$.

Proper Subsets.

Let A and B be sets.
Then A is said to be a **proper subset** of B (written $A \subset B$) if $A \subseteq B$ and $A \neq B$.

Ex. $\mathbb{N} \subset \mathbb{Z}$ because $\mathbb{N} \subseteq \mathbb{Z}$ but $\mathbb{N} \neq \mathbb{Z}$.

"is not a proper subset of"

Ex. Let S be a set. Then $S \not\subset S$. Although $S \subseteq S$ is True, $S = S$ is also true.
$\therefore$ S is not a proper subset of itself.

Cardinality.

If a set A has exactly n distinct elements (for some $n \in \mathbb{N}$), then A is called **finite** and the **cardinality** of A is n (its size). The **cardinality** of a set A is denoted $|A|$.

(what about cardinality of infinite sets ? we'll talk about it later...)

Ex. $A = \{a, b, c\}$ $|A| = 3$ "the cardinality of A is 3" because A contains 3 distinct elements.

Ex. $B = \{a, a, b\}$ $|B| = 2$ "the cardinality of B is 2" because B contains 2 distinct elements.

Ex. $C = \{3n : n \in \mathbb{Z}, 1 \leq n \leq 12\}$ $|C| = 12$

Ex $\emptyset$ $|\emptyset| = 0$

Ex $\{\emptyset\}$ $|\{\emptyset\}| = 1$

Ex. $D = \{a, \{a\}, \{a, \{a\}\}\}$ $|D| = 3$

these are the 3 elements of D
(one element of D is just a letter, each of the other 2 elements of D happen to be sets)

All Subsets of a Finite Set.

Ex. List all subsets of $A = \{a, b, c\}$ in increasing order of cardinality.

0-element subsets: $\emptyset$

1-element subsets: $\{a\}$, $\{b\}$, and $\{c\}$

2-element subsets: $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$

A has 7 proper subsets

3-element subsets: $\{a, b, c\}$ (ie $A \subseteq A$)

A has 8 subsets

The Power Set. $\mathcal{P}(A)$

Let A be a set.
The **power set of** A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Ex. $A=\{a,b,c\}$

$$\mathcal{P}(A)=\{\emptyset, \{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}, A\}$$

In particular, $\emptyset \subseteq A$. Thus $\emptyset \in \mathcal{P}(A)$. Similarly, $A \subseteq A$. Thus $A \in \mathcal{P}(A)$.

Observe also that $A \notin \mathcal{P}(A)$ because, for example, $a \in A$, but $a \notin \mathcal{P}(A)$.

Note. $|A|=3$ and $|\mathcal{P}(A)|=2^3=8$.

Ex $\mathcal{P}(\emptyset)=\{\emptyset\}$ Note. $|\emptyset|=0$ and $|\mathcal{P}(\emptyset)|=2^0=\underline{1}$

Theorem 9.3. Let A be a set. If $|A|=n$, then $|\mathcal{P}(A)|=2^n$. *(to be proved later)*

Well-defined Sets and Russell's Paradox.

Note. In the definition of set, we stated "well-defined"...
here is an example of something that seems like a set but is not actually well-defined:

Let S be the set of all sets that do not contain themselves as elements.

i.e. $S=\{x : x \notin x\}$

S might seem okay but S is not well-defined!

- Suppose $S \in S$. Then $S \notin S$ by definition of S ⚡
- Suppose $S \notin S$. Then $S \in S$ by definition of S ⚡

Russell's Paradox.
There is a village in which a barber shaves all those villagers and only those who do not shave themselves. Who shaves the barber?

Study Guide

Important terms and concepts:

- ◇ set element list notation set-builder notation
- ◇ when two sets are equal
- ◇ empty set $\emptyset$ universal set $\mathcal{U}$
- ◇ subset proper subset cardinality power set of S $\mathcal{P}(S)$

Exercises	Sup.Ex. §4 # 2, 3, 10 Rosen §2.1 # 1, 2, 5, 6, 7, 8, 9, 11, 19, 21, 23, 27, 31