2. Logical Equivalence, Translation between English and Propositional Logic

Lec 1 Mini Review.

✓ proposition ✓ truth value ✓ compound proposition

✓ logical connectives:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

Precedence of Logical Operators. Unless specified otherwise by parentheses, the order of precedence of logical connectives is

1. $\neg$ 2. $\wedge$ 3. $\vee$ 4. $\to$ 5. $\leftrightarrow$ 6. $\oplus$

ex. $\neg p \vee q \to q \wedge r$ means $((\neg p) \vee q) \to (q \wedge r)$

LOGICAL EQUIVALENCE

- Two propositions p and q are called **logically equivalent** if the biconditional statement $p \leftrightarrow q$ is a tautology.
- **Notation:** if p and q are logically equivalent, then we will write

$$p \equiv q.$$

- **Note:** $\equiv$ is not a logical connective; it's just short for "is logically equivalent to".

Example 2.1. Using a truth table and an explanation, prove that $(p \to q) \equiv (\neg p \vee q)$.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the biconditional statement $(p \to q) \leftrightarrow (\neg p \vee q)$ is T for all truth assignments, it's a tautology. By definition, this means that $p \to q$ is logically equivalent to $\neg p \vee q$ $\therefore p \to q \equiv \neg p \vee q$

* These notes are solely for the personal use of students registered in MAT1348.

Conditional Statement, Contrapositive, and Converse For a conditional statement $P \to Q$, there are two other conditional statements which are related to $P \to Q$ in important ways:

- $\neg Q \to \neg P$ is called the **contrapositive** of $P \to Q$.
- $Q \to P$ is called the **converse** of $P \to Q$.

Example 2.2. Use a truth table to determine whether a conditional statement $P \to Q$ is logically equivalent to its contrapositive, namely $\neg Q \to \neg P$.

P	Q	① $P \to Q$	$\neg Q$	$\neg P$	② $\neg Q \to \neg P$	① $\leftrightarrow$ ②
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is a tautology (true for all of its truth assignments) it means that $P \to Q \equiv \neg Q \to \neg P$ ✓

Ex. P: You can go to the party Q: You finish your homework.

$\boxed{P \to Q:}$ You can go to the party only if you finish your homework.

is logically equivalent to

$\boxed{\neg Q \to \neg P:}$ If you do not finish your homework, then you cannot go to the party.

Example 2.3. Using a truth table, prove that $P \to Q$ is **not** logically equivalent to its converse $Q \to P$. Give all counterexamples, that is, list all truth assignments for which $(P \to Q) \leftrightarrow (Q \to P)$ is **F**.

P	Q	① $P \to Q$	② $Q \to P$	① $\leftrightarrow$ ②
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Since $(P \to Q) \leftrightarrow (Q \to P)$ is <u>not</u> a tautology, $P \to Q \not\equiv Q \to P$

Counterexamples (truth assignments that certify that $(P \to Q) \leftrightarrow (Q \to P)$ is <u>not</u> a tautology)

- when $P = T$, $Q = F$, $(P \to Q) \leftrightarrow (Q \to P)$ is F.
- when $P = T$, $Q = F$, $(P \to Q) \leftrightarrow (Q \to P)$ is F.

Ex. $\boxed{P \to Q:}$ You can go to the party only if you finish your homework.

is <u>not</u> logically equivalent to

$\boxed{Q \to P}$ You finish your homework only if you can go to the party.

Example 2.4. Are all tautologies logically equivalent to each other? Explain.

- Suppose P is a tautology, where P is some compound proposition.
- Then, for all truth assignments (of whatever atoms P consists of), the truth value of P is T (by def. of tautology)
- It now follows that the biconditional statement $P \leftrightarrow T$ is of the form $T \leftrightarrow T$ for all truth assignments. $\therefore$ $P \leftrightarrow T$ is a tautology.

$\Rightarrow P \equiv T$.

So any tautology is logically equivalent to T

$\therefore$ all tautologies are logically equivalent to each other.

Translation between English and Propositional Logic

For the examples in this section, let us define the following propositional variables:

b : "A bear eats berries."
f : "A bear eats a fish."
r : "A bear is near the river."
s : "A bear sees a fish."

* assume these refer to one particular bear.

Negation.

Common translations in English

$\neg p$: "It is not the case that p."

Ex. $\neg r$: "It is not the case that a bear is near the river."
"A bear is not near the river."

Ex. $\neg s$: "A bear does not see a fish"

Ex "A bear does not eat berries." translation to propositional logic: $\neg b$

Ex "It is not the case that a bear does not eat berries." $\neg(\neg b)$

Conjunction.

Common translations in English

$p \wedge q$ — "p and q." / "p but q."

Ex. "$\underbrace{\text{A bear sees a fish}}_{s}$, $\underbrace{\text{but}}_{\wedge}$ $\underbrace{\text{she does not eat a fish}}_{\neg f}$." $\quad s \wedge \neg f$

Disjunction.

Common translations in English

$p \vee q$ — "p or q." / "p unless q."

* "unless" is not always used as an inclusive or in English but this will be the accepted convention for 1348.

Ex. $\underbrace{\text{A bear eats berries}}_{b}$ $\underbrace{\text{unless}}_{\vee}$ $\underbrace{\text{she sees a fish}}_{s}$. $\quad b \vee s$

Ex. $\underbrace{\text{A bear eats berries}}_{b}$ $\underbrace{\text{or}}_{\vee}$ fish. $\quad b \vee f$

f

Exclusive Or.

Common translations in English

$p \oplus q$ — "Either p or q." / "p or q but not both."

* "either...or" is not always used as an exclusive or in English but this will be the accepted convention for 1348.

Ex. "Either $\underbrace{\text{a bear eats fish}}_{f}$ $\underbrace{\text{or}}_{\oplus}$ $\underbrace{\text{she eats berries}}_{b}$." $\quad f \oplus b$

Ex $r \oplus$: "A bear is near the river or she sees a fish, but not both."

* important note on translation: you are only translating between English and propositional logic. The propositions themselves do not need to make sense with respect to the "real world". Do not translate based on context... it is irrelevant to consider how a real bear would behave... Just translate the given propositions.

Conditional Statement $p \to q$

- p is the premise or hypothesis
- q is the conclusion or consequence.

Important: $p \to q$ is not logically equivalent to its converse $q \to p$

∴ when translating conditional statements you must be careful not to mix up premise and conclusion.

* in the following chart, each phrase is an English translation of $\boxed{p \to q}$

$p \to q$ — Common translations in English

"if p, then q." "if p, q."	"q if p."
"p implies q."	"q is implied by p." "q follows from p."
"p is a sufficient condition for q."	"A sufficient condition for q is p."
"A necessary condition for p is q."	"q is a necessary condition for p."
"p only if q." "whenever it is the case that p, then q." "when p, q."	"q if p" "q whenever p." "q when p."

Ex. "$\underbrace{\text{A bear eats berries}}_{b}$ $\underset{\to}{\text{only if}}$ $\underbrace{\text{she is not near the river}}_{\neg r}$." $b \to \neg r$

Ex "If $\underbrace{\text{a bear sees a fish}}_{s}$, $\underset{\to}{\text{then}}$ $\underbrace{\text{she eats a fish}}_{f}$." $s \to f$

Ex "A bear eats a fish whenever she's near the river." $r \to f$

$\underbrace{\text{A bear eats a fish}}_{f}$ whenever $\leftarrow$ $\underbrace{\text{she's near the river}}_{r}$

Ex. "A bear does not eat berries when she sees a fish." $s \to \neg b$

$\underbrace{\text{A bear does not eat berries}}_{\neg b}$ when $\leftarrow$ $\underbrace{\text{she sees a fish}}_{s}$

Ex. "A sufficient condition for a bear to eat a fish is that she sees a fish." $s \to f$

A sufficient condition for $\underbrace{\text{a bear to eat a fish}}_{f}$ is that $\underbrace{\text{she sees a fish}}_{s}$

Ex. "A necessary condition for a bear to eat a fish is that she sees a fish." $f \to s$

A necessary condition for $\underbrace{\text{a bear to eat a fish}}_{f}$ is that $\underbrace{\text{she sees a fish}}_{s}$

Biconditional Statement $p \leftrightarrow q$ * Exercise show that $p \leftrightarrow q \equiv (q \to p) \wedge (p \to q)$

So we have translations: 1. (p if q) and (p only if q)

2. (p is necessary for q) and (p is sufficient for q)

Thus, a biconditional statement is logically equivalent to the conjunction of a conditional statement and its converse.

$p \leftrightarrow q$

Common translations in English

"p if and only if q."

"if p, then q, and conversely."

"p is necessary and sufficient for q."

Ex. "A bear eats berries if and only if she does not see a fish." $b \leftrightarrow \neg s$

$\underbrace{\text{A bear eats berries}}_{b}$ if and only if $\leftrightarrow$ $\underbrace{\text{she does not see a fish}}_{\neg s}$

Study Guide

Important terms and concepts:

logical equivalence	translation between English and propositional logic
truth assignment	consistent vs. inconsistent set of propositions

Rosen Exercises	1.3 # 17–33, 1.1 # 23, 1.2 # 1, 5, 9, 11