10. Set Operations & Set Identities

Basics of Set Theory:

- ☐ set element when two sets are equal
- ☐ describing a set:
  - set-builder notation
  - list notation (order / multiplicity do not affect an element's membership in a set)
- ☐ when two sets are equal subset proper subset
- ☐ empty set $\emptyset$ universal set $\mathcal{U}$
- ☐ cardinality of a finite set S: $|S|$ power set of a set S: $\mathcal{P}(S)$

CARTESIAN PRODUCT

Let A and B be sets.
The **Cartesian product "A cross B"**, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

i.e. $A \times B = \{(a,b) : a \in A, b \in B\}$

Ex. The Cartesian plane: $\mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

Ex. $A = \{a, b, c\}$ $B = \{1, 2\}$

$A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Ex. $R = \{A, 2, \ldots, 9, 10, J, Q, K\}$ $S = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$

$R \times S = \{(A,\heartsuit), (A,\diamondsuit), (A,\clubsuit), (A,\spadesuit), (2,\heartsuit), \ldots, (K,\diamondsuit), (K,\clubsuit), (K,\spadesuit)\}$

$|R \times S| = 13 \times 4 = 52$ Ex $(8, \diamondsuit) \in R \times S$

$S \times R = \{(\heartsuit, A), (\heartsuit, 2), (\heartsuit, 3), \ldots, (\spadesuit, Q), (\spadesuit, K)\}$ $|S \times R| = 4 \times 13 = 52.$

Ex $(8, \diamondsuit) \notin S \times R$ although $(\diamondsuit, 8) \in S \times R.$

Note Cartesian product is not commutative in general ($A \times B \neq B \times A$)

* These notes are solely for the personal use of students registered in MAT1348.

Cardinality of $A \times B$.

Theorem 10.1. Let A and B be sets. Then $|A \times B| = |A||B|$.

Proof. By definition, $A \times B = \{(a, b) : a \in A,\ b \in B\}$

There are $|A|$ choices for the 1st coordinate

There are $|B|$ choices for the 2nd coordinate.

$\therefore$ total # elements in $A \times B$ is $|A| \cdot |B|$ $\blacksquare$

Cartesian Product is not Associative. meaning $(A \times B) \times C \neq A \times (B \times C)$

because $(A \times B) \times C = \{((a,b), c) : (a,b) \in A \times B,\ c \in C\}$

while $A \times (B \times C) = \{(a, (b,c)) : a \in A,\ (b,c) \in B \times C\}$

So, omitting brackets would not make sense for Cartesian product the way it does make sense for other operations like multiplication; however, we do have a generalization of Cartesian product...

Generalization of Cartesian Product to More Than Two Sets.

Let $A_1, A_2, \ldots, A_n$ be sets.
The **Cartesian product "A_1 cross A_2 cross ... cross A_n"**, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set...

$$A_1 \times A_2 \times \cdots \times A_n = \{\underbrace{(a_1, a_2, \ldots, a_n)}_{\text{elements are ordered } n\text{-tuples}} : a_i \in A_i \text{ for } 1 \le i \le n\}$$

elements are ordered n-tuples (n-dimensional vectors)

Notation for Cartesian Product of A with itself n times:

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = A^n$$

Ex. $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$

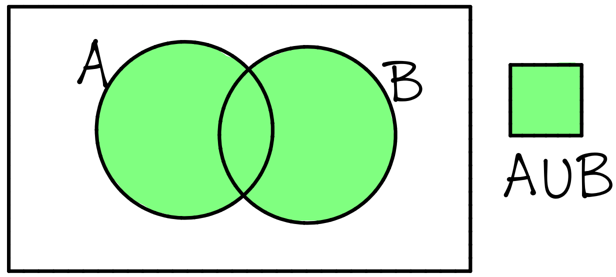
Sets Operations

Venn diagrams are a graphical method for depicting sets and set operations.

Union.

The <u>union</u> of sets A and B, denoted $A \cup B$, is the set

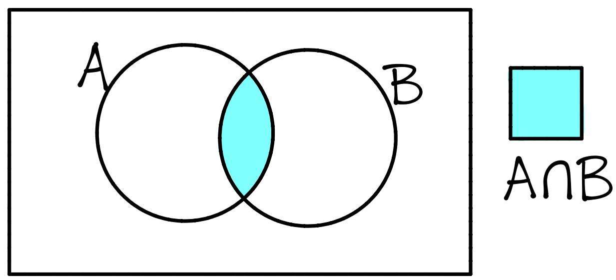
$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$



Intersection.

The <u>intersection</u> of sets A and B, denoted $A \cap B$, is the set

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$



Example 10.2. Let $A = \{1, 2, 3\}$ $\quad B = \{1, 3, 5, 7\}$ $\quad C = \{5, 7\}$.

$A \cup B = \{1, 2, 3, 5, 7\}$

$A \cap B = \{1, 3\}$

$A \cap C = \{\,\} = \emptyset$

$B \cap C = \{5, 7\} = C$

$B \cup C = \{1, 3, 5, 7\} = B$

Disjoint Sets.

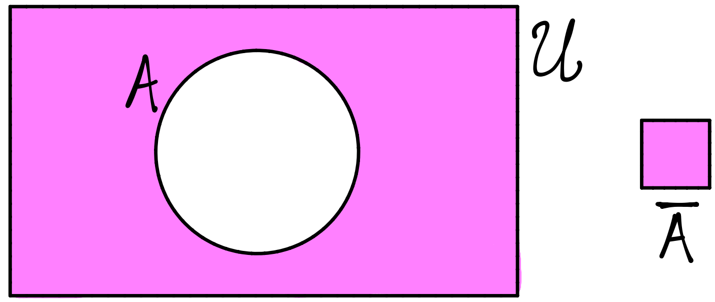
Sets A and B are called **disjoint** if $A \cap B = \emptyset$.

<u>Ex.</u> A and C are disjoint.
(from above example)

Complement.

The complement of a set A, denoted $\overline{A}$, is the set

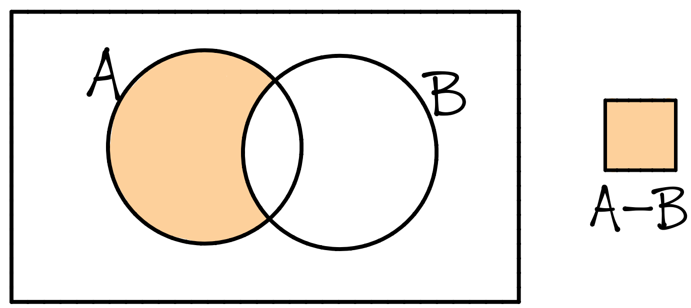
$$\overline{A} = \{x : (x \in \mathcal{U}) \wedge (x \notin A)\}$$



Difference.

The difference of sets A and B, denoted $A - B$, is the set

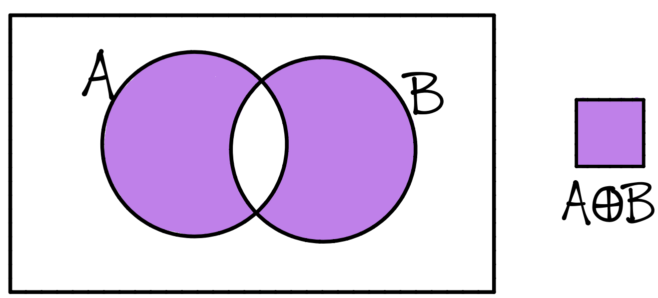
$$A - B = \{x : (x \in A) \wedge (x \notin B)\}$$



Symmetric Difference.

The symmetric difference of sets A and B, denoted $A \oplus B$, is the set

$$A \oplus B = \{x : (x \in A) \oplus (x \in B)\}$$



Set Identities

A **set identity** is an equation involving sets and set operations that is true *no matter what* particular sets we consider.

Example 10.3. For all sets A, B, C, the following equation is true:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

look familiar?
▶ distributive law!

Ways to verify set identities

How can we verify that a set identity will be true no matter what the sets A, B, and C are?

- ◇ Recall that two sets S and T are equal if

  for all $x \in \mathcal{U}$, the biconditional statement $(x \in S) \leftrightarrow (x \in T)$ is true.

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1. We can verify a set identity using a **membership table**.

- Membership tables are similar to truth tables, but more like attendance sheets.
- If there are n sets involved in an identity, then the table will have 2^n rows.
- Each row corresponds to one possible "location" of an element $x \in \mathcal{U}$, *relative to* the sets in the identity.

Example 10.4. Using a membership table, prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (De Morgan)

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Since membership is the same for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ it follows that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

for 2 sets A and B, these 4 rows cover all possible cases for the membership of an arbitrary element x of the universal set relative to A and B.

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2. We can verify a set identity using what is called a **rigorous proof** (essentially a *proof of equivalence* of the definition of set equality).

▷▷▷ To prove $S = T$ with a **rigorous proof**, we must prove 2 things:

i. Let $x \in \mathcal{U}$ be an arbitrary element of the universal set.

   Then prove that the conditional statement $(x \in S) \to (x \in T)$ is true. (this proves $S \subseteq T$) (using a direct proof or another appropriate strategy).

ii. Let $x \in \mathcal{U}$ be an arbitrary element of the universal set.

   Then prove that the conditional statement $(x \in T) \to (x \in X)$ is true. (this proves $T \subseteq S$) (using a direct proof or another appropriate strategy).

If we prove i) $(x \in S) \to (x \in T)$ and ii) $(x \in T) \to (x \in S)$, then we have completed a proof of equivalence that

$$(x \in S) \longleftrightarrow (x \in T)$$

which is the definition of two sets S and T being equal.

Example 10.5. Use a rigorous proof to prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (De Morgan)

To prove $\overline{A\cap B} = \overline{A} \cup \overline{B}$ with a rigorous proof, we must prove

1. $\overline{A\cap B} \subseteq \overline{A} \cup \overline{B}$ and 2. $\overline{A} \cup \overline{B} \subseteq \overline{A\cap B}$

i. ($\subseteq$) Let $x \in \mathcal{U}$.

Assume $x \in \overline{A\cap B}$. By def. (of complement), this means that $x \notin A \cap B$.

$\therefore$ it is not the case that $(x\in A) \wedge (x \in B)$

$\therefore x \notin A$ or $x \notin B$

We are making use of De Morgan's Law $\neg\big((x\in A)\wedge(x\in B)\big) \equiv \neg(x\in A) \vee \neg(x \in B)$

By def. of complement, this means that

$x \in \overline{A}$ or $x \in \overline{B}$

By def. of union, this means that

$x \in \overline{A} \cup \overline{B}$

$\therefore$ We proved that $(x \in \overline{A\cap B}) \rightarrow (x \in \overline{A} \cup \overline{B})$ is True. Hence, $\overline{A\cap B} \subseteq \overline{A} \cup \overline{B}$.

ii. ($\supseteq$) Let $x \in \mathcal{U}$. Assume $x \in \overline{A} \cup \overline{B}$.

Then (by def. of $\cup$) $x \in \overline{A}$ or $x \in \overline{B}$

$\Rightarrow$ (by def. of $\overline{}$) $x \notin A$ or $x \notin B$

$\therefore x \notin A \cap B$ (not being an element of A or not being an element of B is sufficient to guarantee that x is not an element of $A \cap B$)

$\Rightarrow$ (by def. of $\overline{}$) $x \in \overline{A\cap B}$

$\therefore$ We proved that $(x \in \overline{A} \cup \overline{B}) \rightarrow (x \in \overline{A\cap B})$ is True. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A\cap B}$.

Since $\overline{A\cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A\cap B}$, we have proved that $\overline{A\cap B} = \overline{A} \cup \overline{B}$ $\blacksquare$

Study Guide

Important terms and concepts:

- ◇ Cartesian product of two (or more) sets
- ◇ **set operations**: union intersection complement difference symmetric difference
- ◇ **Set Identities**
- ◇ verifying set identities using: membership tables rigorous proofs

Exercises	Sup.Ex. §4 # 1, 2, 3, 5(using membership table), 6(using rigorous proof), 9, 11 Rosen §2.1 # 1, 2, 5, 6, 7, 8, 9, 11, 19, 21, 23, 27, 31, 32a Rosen §2.2 # 1, 3, 4, 5–13(using membership tables or rigorous proofs), 17, 19