DGD 8

Q1. A RELATION ON THE POWERSET OF A SET

Let $\mathcal{U}$ be a nonempty universal set. Let $\mathcal{R}$ be a relation on the set $\mathcal{P}(\mathcal{U})$ defined by

$$(A, B) \in \mathcal{R} \iff A \subseteq B$$

Determine whether $\mathcal{R}$ is reflexive, symmetric, antisymmetric, or transitive (for each property, give a proof or counterexample to support your claim).

Is $\mathcal{R}$ an equivalence relation?

Is $\mathcal{R}$ reflexive? Yes! For all $S \in \mathcal{P}(\mathcal{U})$, it is true that $S \subseteq S$

$\Rightarrow (S,S) \in \mathcal{R}$ (by the rule for $\mathcal{R}$)

Thus, $(S,S) \in \mathcal{R}$ for all $S \in \mathcal{P}(\mathcal{U})$ ie $\mathcal{R}$ is reflexive.

Is $\mathcal{R}$ symmetric? No. Since $\mathcal{U}$ is non-empty, we know that $\mathcal{U} \neq \emptyset$.

Moreover, $\emptyset \in \mathcal{P}(\mathcal{U})$ and $\mathcal{U} \in \mathcal{P}(\mathcal{U})$ and $\emptyset \subseteq \mathcal{U}$ is true while $\mathcal{U} \subseteq \emptyset$ is false

$\Rightarrow \underbrace{(\emptyset, \mathcal{U}) \in \mathcal{R} \quad \text{but } (\mathcal{U}, \emptyset) \notin \mathcal{R}}_{\text{counterexample}}$

From the counterexample, we see that it is not the case that for all $A, B \in \mathcal{P}(\mathcal{U})$, $\big((A,B) \in \mathcal{R}\big) \to \big((B,A) \in \mathcal{R}\big)$ ie $\mathcal{R}$ is not symmetric

Is $\mathcal{R}$ antisymmetric?

Yes! Let $A, B \in \mathcal{P}(\mathcal{U})$.

Assume $(A,B) \in \mathcal{R}$ and $(B,A) \in \mathcal{R}$.

Then $\underbrace{A \subseteq B \quad \text{and} \quad B \subseteq A}$.

$\therefore A = B$.

Thus, for all $A, B \in \mathcal{P}(\mathcal{U})$, we proved $\big((A,B) \in \mathcal{R} \text{ and } (B,A) \in \mathcal{R}\big) \to (A = B)$ ie $\mathcal{R}$ is antisymmetric

Is $\mathcal{R}$ transitive?

Yes. Let $A, B, C \in \mathcal{P}(\mathcal{U})$.

Assume $(A,B) \in \mathcal{R}$ and $(B,C) \in \mathcal{R}$.

Then $A \subseteq B$ and $B \subseteq C$. $\therefore (x \in A) \to (x \in B)$ and $(x \in B) \to (x \in C)$ ie $A \subseteq C$

$\Rightarrow (A,C) \in \mathcal{R}$ (by the rule for $\mathcal{R}$)

Thus, for any $A, B, C \in \mathcal{P}(\mathcal{U})$, we proved $\big((A,B) \in \mathcal{R} \text{ and } (B,C) \in \mathcal{R}\big) \to \big((A,C) \in \mathcal{R}\big)$ ie $\mathcal{R}$ is transitive

Is $\mathcal{R}$ an equivalence relation?

No. To be an equivalence relation, $\mathcal{R}$ would need to be reflexive, symmetric and transitive. Since $\mathcal{R}$ is not symmetric, $\mathcal{R}$ is not an equivalence relation on $\mathcal{P}(\mathcal{U})$.

Note: the relation $\mathcal{R}$ is defined on the set $\mathbb{R}^2$ which means the elements that $\mathcal{R}$ relates are themselves pairs

Q2. AN EQUIVALENCE RELATION ON $\mathbb{R}^2$

Define a binary relation on the set $\mathbb{R}^2$ (recall that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$):

$$\Big(\underbrace{(a, b)}_{\in\mathbb{R}^2},\ \underbrace{(c, d)}_{\in\mathbb{R}^2}\Big) \in \mathcal{R} \quad\Longleftrightarrow\quad \underbrace{a^2 + b^2 = c^2 + d^2}_{\text{the rule for } \mathcal{R}}$$

Q2i. Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}^2$.

[reflexive] Let $(x,y)\in\mathbb{R}^2$. Then $x^2+y^2=x^2+y^2$ is true. $\therefore \big((x,y),(x,y)\big)\in\mathcal{R}$

We proved $\big((x,y),(x,y)\big)\in\mathcal{R}$ for all $(x,y)\in\mathbb{R}^2$.

$\therefore \mathcal{R}$ is a reflexive relation on $\mathbb{R}^2$.

[symmetric]. Let $(p,q),(r,s)\in\mathbb{R}^2$.

Assume $\big((p,q),(r,s)\big)\in\mathcal{R}$. Then $p^2+q^2=r^2+s^2$ by the rule for $\mathcal{R}$.

$\Rightarrow r^2+s^2=p^2+q^2$ is true.

$\Rightarrow \big((r,s),(p,q)\big)\in\mathcal{R}$ (since the rule for $\mathcal{R}$ is satisfied).

We proved $\Big(\big((p,q),(r,s)\big)\in\mathcal{R}\Big)\rightarrow\Big(\big((r,s),(p,q)\big)\in\mathcal{R}\Big)$

$\therefore \mathcal{R}$ is a symmetric relation on $\mathbb{R}^2$

[transitive] Let $(p,q),(r,s),(t,u)\in\mathbb{R}^2$.

Assume $\big((p,q),(r,s)\big)\in\mathcal{R}$ and $\big((r,s),(t,u)\big)\in\mathcal{R}$.

Then $p^2+q^2=r^2+s^2$ and $r^2+s^2=t^2+u^2$ (by the rule for $\mathcal{R}$)

$\therefore p^2+q^2=t^2+u^2$

$\Rightarrow \big((p,q),(t,u)\big)\in\mathcal{R}$ (since the rule for $\mathcal{R}$ is satisfied).

We proved $\Big(\big((p,q),(r,s)\big)\in\mathcal{R} \text{ and } \big((r,s),(t,u)\big)\in\mathcal{R}\Big)\rightarrow\Big(\big((p,q),(t,u)\big)\in\mathcal{R}\Big)$

$\therefore \mathcal{R}$ is a transitive relation on $\mathbb{R}^2$.

Since $\mathcal{R}$ is reflexive, symmetric and transitive, $\mathcal{R}$ is an equivalence relation on $\mathbb{R}^2$.

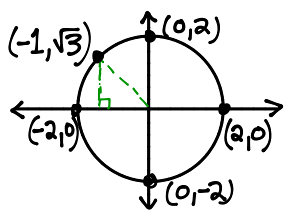
Q2ii. **Recall:** If $\mathcal{R}$ is an equivalence relation on a set A, then, for each element $a \in A$, we define the **equivalence class of** a, denoted $[a]_{\mathcal{R}}$, as follows: $[a]_{\mathcal{R}} = \{x \in A : a \mathrel{\mathcal{R}} x\}$

For the equivalence relation $\mathcal{R}$ (proved in **Q2i**), determine the equivalence class of the element $(0, 2) \in \mathbb{R}^2$, that is, describe the equivalence class $[(0, 2)]_{\mathcal{R}}$ using set-builder notation.

$$\begin{aligned}[(0,2)]_{\mathcal{R}} &= \{(x,y) \in \mathbb{R}^2 : ((0,2),(x,y)) \in \mathcal{R}\} \\ &= \{(x,y) \in \mathbb{R}^2 : 0^2 + 2^2 = x^2 + y^2\} \\ &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}\end{aligned}$$

Do you see a geometric interpretation for $[(0, 2)]_{\mathcal{R}}$?

All $x, y \in \mathbb{R}$ such that $\underbrace{x^2+y^2=4}_{\text{eq}^{\text{n}} \text{ for (the boundary of) a circle of radius 2 centred at the origin.}}$



List three elements in the equivalence class $[(0, 2)]_{\mathcal{R}}$ other than $(0, 2)$.

There are infinitely many possible answers (uncountably many, in fact) but here are three:

$(-1,\sqrt{3}) \in [(0,2)]_{\mathcal{R}}$ $\quad (2,0) \in [(0,2)]_{\mathcal{R}}$ $\quad (0,-2) \in [(0,2)]_{\mathcal{R}}$

Q3. RELATIONS ON A FINITE SET AND THEIR PROPERTIES

Let $A = \{1, 2, 3, 4\}$. For each of the following relations on A, determine whether it is reflexive, symmetric, or transitive. In each case, if the relation is an equivalence relation, determine all of the distinct equivalence classes of the given relation.

$\mathcal{R}_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$\mathcal{R}_1$ is reflexive since $(a,a) \in \mathcal{R}_1$ for all $a \in A = \{1,2,3,4\}$.

$\mathcal{R}_1$ is symmetric since $((a,b) \in \mathcal{R}_1) \rightarrow [(b,a) \in \mathcal{R}_1]$ for all $a \in A = \{1,2,3,4\}$.

$\mathcal{R}_1$ is transitive since $[(a,b) \in \mathcal{R}_1 \text{ and } (b,c) \in \mathcal{R}_1] \rightarrow [(a,c) \in \mathcal{R}_1]$ for all $a,b,c \in A = \{1,2,3,4\}$

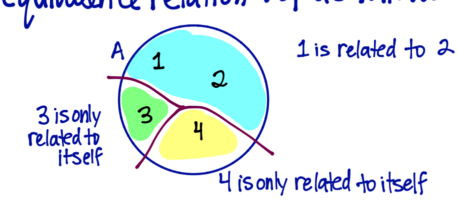
Since $\mathcal{R}_1$ is reflexive, symmetric, and transitive, $\mathcal{R}_1$ is an equivalence relation on A.

$[1]_{\mathcal{R}_1} = \{1,2\}$

$[3]_{\mathcal{R}_1} = \{3\}$

$[4]_{\mathcal{R}_1} = \{4\}$

The set A can be partitioned by the equivalence relation $\mathcal{R}_1$ as follows:



$\mathcal{R}_2 = \{(1,1),(2,2),(3,3),(4,4)\}$

$\mathcal{R}_2$ is reflexive since $(a,a) \in \mathcal{R}_2$ for all $a \in A = \{1,2,3,4\}$.

$\mathcal{R}_2$ is symmetric since $\left((a,b) \in \mathcal{R}_2\right) \rightarrow \left[(b,a) \in \mathcal{R}_2\right]$ for all $a \in A = \{1,2,3,4\}$.

$\mathcal{R}_2$ is transitive since $\left[(a,b) \in \mathcal{R}_2 \text{ and } (b,c) \in \mathcal{R}_2\right] \rightarrow \left[(a,c) \in \mathcal{R}_2\right]$ for all $a,b,c \in A = \{1,2,3,4\}$

Since $\mathcal{R}_2$ is reflexive, symmetric, and transitive, $\mathcal{R}_2$ is an equivalence relation on A.

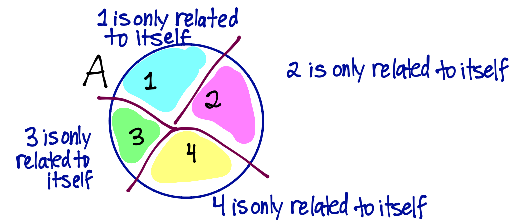
$[1]_{R_2} = \{1\}$

$[2]_{R_2} = \{2\}$

$[3]_{R_2} = \{3\}$

$[4]_{R_2} = \{4\}$

The set A can be partitioned by the equivalence relation $\mathcal{R}_2$ as follows:



$\mathcal{R}_3 = \{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$

$\mathcal{R}_3$ is <u>not</u> reflexive since $1 \in A$ but $(1,1) \notin \mathcal{R}_3$.

$\mathcal{R}_3$ is <u>not</u> symmetric since $(1,4) \in \mathcal{R}_3$ but $(4,1) \notin \mathcal{R}_3$.

$\mathcal{R}_3$ is <u>not</u> transitive since $(2,3) \in \mathcal{R}_3$ and $(3,1) \in \mathcal{R}_3$ but $(2,1) \notin \mathcal{R}_3$

* in each case, one counterexample was provided, but there might be other counterexamples as well.

$\mathcal{R}_3$ is <u>not</u> an equivalence relation on A.

$\mathcal{R}_4 = \emptyset$

$\mathcal{R}_4$ is <u>not</u> reflexive since $3 \in A$, but $(3,3) \notin \mathcal{R}_4$.

$\mathcal{R}_4$ is symmetric, since for all $a,b \in A$, the implication $\left(\underbrace{(a,b) \in \mathcal{R}_4}_{F}\right) \rightarrow \left(\underbrace{(b,a) \in \mathcal{R}_4}_{F}\right)$ is true.

$(F \rightarrow F) \equiv T.$

$\mathcal{R}_4$ is transitive since, for all $a,b,c \in A$, the implication

$\left(\underbrace{(a,b) \in \mathcal{R}_4 \text{ and } (b,c) \in \mathcal{R}_4}_{F}\right) \rightarrow \left(\underbrace{(a,c) \in \mathcal{R}_4}_{F}\right)$ is true

$(F \rightarrow F) \equiv T$

Since $\mathcal{R}_4$ is not reflexive, it is <u>not</u> an equivalence relation on A.

Q4. CONGRUENCE MOD m (AN EQUIVALENCE RELATION ON $\mathbb{Z}$)

Let $m \in \mathbb{Z}^+$. **Fact:** For each integer $x \in \mathbb{Z}$, there exist unique integers $q, r \in \mathbb{Z}$ such that

$$x = qm + r \quad \text{and} \quad 0 \le r < m$$

Given the "modulus" m and the integer x, the unique integer r in the above equation is called the "**remainder** of $x \pmod{m}$".

Let $\equiv \pmod{m}$ be the relation on $\mathbb{Z}$ defined by the following rule:

for all $x, y \in \mathbb{Z}$, $x \equiv y \pmod{m}$ if and only if x and y have the same remainder $\pmod{m}$

We say that x and y are **congruent** $\pmod{m}$ whenever $x \equiv y \pmod{m}$.

Q4i. Determine the remainder $\pmod 7$ of each element of the set

$$A = \{-15, -12, -7, -6, -4, -1, 0, 1, 2, 7, 8, 9, 10, 11, 12, 13, 14\}$$

remainder: 6 2 0 1 3 6 0 1 2 0 1 2 3 4 5 6 0

* remainder (mod 7) must be an integer r such that $0 \le r < 7$

$-15 = (-3)(7) + 6$	$-4 = (-1)(7) + 3$	$2 = (0)(7) + 2$	$10 = (1)(7) + 3$	
$-12 = (-2)(7) + 2$	$-1 = (-1)(7) + 6$	$7 = (1)(7) + 0$	$11 = (1)(7) + 4$	$14 = (2)(7) + 0$
$-7 = (-1)(7) + 0$	$0 = (0)(7) + 0$	$8 = (1)(7) + 1$	$12 = (1)(7) + 5$	
$-6 = (-1)(7) + 1$	$1 = (0)(7) + 1$	$9 = (1)(7) + 2$	$13 = (1)(7) + 6$	

Q4ii. Which elements of the set A (in Q4i) belong to the equivalence class $[-6]_{\equiv \pmod 7}$?

the remainder of $-6 \pmod 7$ is 1 so all elements in A with remainder $1 \pmod 7$ are in the equivalence class of -6. $\therefore [-6]_{\equiv (\text{mod } 7)} = \{-6, 1, 8\}$

Q4iii. Prove that $\equiv \pmod 7$ is an equivalence relation on A.

[reflexive]. Let $x \in A$. The remainder of $x \pmod 7$ equals the remainder of $x \pmod 7$ $\therefore x \equiv x \pmod 7$

We proved $(x \in A) \to (x \equiv x \pmod 7)$ $\therefore \equiv \pmod 7$ is reflexive.

[symmetric] Let $x, y \in A$. Assume $x \equiv y \pmod 7$. Then x and y have the same remainder (mod 7)

$\therefore$ y and x have the same remainder (mod 7)

$\therefore y \equiv x \pmod 7$

We proved $(x \equiv y \pmod 7) \to (y \equiv x \pmod 7)$ $\therefore \equiv \pmod 7$ is symmetric

[transitive] Let $x, y, z \in A$.

Assume $x \equiv y \pmod 7$ and $y \equiv z \pmod 7$.

Then x and y have the same remainder (mod 7) and y and z have the same remainder (mod 7)

$\therefore$ x and z have the same remainder (mod 7) $\therefore x \equiv z \pmod z$.

We proved $(x \equiv y \pmod 7 \wedge y \equiv z \pmod 7) \to (x \equiv z \pmod 7)$ $\therefore \equiv \pmod 7$ is transitive

Since it's reflexive, symmetric and transitive, $\equiv \pmod 7$ is an equivalence relation on A. ■