DGD 10

Q1. COUNTING: BINARY STRINGS **i.** How many **binary strings of length 9** are there?

Procedure:
Build a binary string of length 9

Task T_i: choose ith entry

$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$

2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1

There are $2^9 = 512$ binary strings of length 9.

ii. How many **binary strings of length 9** are there that begin with 4 zeros?

Procedure:
Build a binary string of length 9 that begins with four 0's.

$0\ 0\ 0\ 0\ \_\ \_\ \_\ \_\ \_$

1 choice: 0 | 1 choice: 0 | 1 choice: 0 | 1 choice: 0 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1 | 2 choices: 0 or 1

Task T_i: choose ith entry

There are $1^4 \cdot 2^5 = 32$ such strings.

iii. How many **binary strings of length 9** are there that contain **exactly** 4 zeros?

Task T_1: Choose 4 of the 9 entries in the string to place the zeros

(in one of $C(9,4) = \binom{9}{4} = \frac{9!}{4!5!} = \frac{\cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot 6}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 21 \cdot 6 = 126$ ways)

Task T_2: fill the other 5 entries with ones (in 1 way)

$\therefore$ there are $\binom{9}{4} \cdot (1) = 126$ such binary strings.

iv. How many **binary strings of length 9** are there that contain **at most** 4 zeros?

We will break this up into 5 separate cases:

Case 0: String contains no zeros (hence 9 ones)

There are $\binom{9}{0} = 1$ binary strings of length 9 with no zeros

Case 1: String contains exactly 1 zero (hence 8 ones)

There are $\binom{9}{1} = 9$ binary strings of length 9 with exactly 1 zero

Case 2: String contains exactly 2 zeros (hence 7 ones)

There are $\binom{9}{2} = 36$ binary strings of length 9 with exactly 2 zeros

Case 3: String contains exactly 3 zeros (hence 6 ones)

There are $\binom{9}{3} = 84$ binary strings of length 9 with exactly 3 zeros

Case 4: String contains exactly 4 zeros (hence 5 ones)

There are $\binom{9}{5} = 126$ binary strings of length 9 with exactly 4 zeros

$\therefore$ there are $1 + 9 + 36 + 84 + 126 = 256$ binary strings with at most 4 zeros.

Q2. COUNTING: RELATIONS AND FUNCTIONS

i. How many relations are there on the set $A = \{1, 2, 3\}$?

Recall that a relation $\mathcal{R}$ on A is simply a subset of $A \times A$

In this example, $|A| = 3$ $\therefore$ $|A \times A| = 9$

Procedure: build a subset of a 9-element set $A \times A = \{(1,1), (1,2), \ldots, (3,3)\}$

Task T_i: either put the ith element of $A \times A$ in the relation or not (2 ways)

There are $2^{|A \times A|} = 2^9 = 512$ relations on A.

ii. How many relations on the set $A = \{1, 2, 3\}$ are **reflexive**?

Recall that a relation $\mathcal{R}$ on A is reflexive if, for all $a \in A$, $(a,a) \in \mathcal{R}$.

In this example, $|A| = 3$ $\therefore$ $|A \times A| = 9$

and to be reflexive, $\mathcal{R}$ needs to contain the three pairs $(1,1)$, $(2,2)$, and $(3,3)$

Thus, there are $9 - 3 = 6$ other pairs $(a,b) \in A \times A$ for which each has 2 choices with respect to belonging to $\mathcal{R}$: either $(a,b) \in \mathcal{R}$ or $(a,b) \notin \mathcal{R}$.

To be explicit (since this set A is reasonably small), here is $A \times A$:

$$A \times A = \{\underbrace{(1,1), (2,2), (3,3)}_{T_1}, \underbrace{(1,2)}_{T_2}, \underbrace{(2,1)}_{T_3}, \underbrace{(3,1)}_{T_4}, \underbrace{(1,3)}_{T_5}, \underbrace{(2,3)}_{T_6}, \underbrace{(3,2)}_{T_7}\}$$

T_1: put $(1,1)$, $(2,2)$, $(3,3)$ in $\mathcal{R}$ (1 way)

T_2: put $(1,2)$ in $\mathcal{R}$ or don't. (2 ways)

T_3: put $(2,1)$ in $\mathcal{R}$ or don't. (2 ways)

T_4 ⋮ ← same idea

T_5 ⋮

T_6 ⋮

T_7: put $(3,2)$ in $\mathcal{R}$ or don't. (2 ways)

We can build a reflexive relation $\mathcal{R}$ on A using the procedure $(T_1, T_2, \ldots, T_7)$

$$\boxed{\therefore \text{ there are } (1)(2)(2)(2)(2)(2)(2) = 2^6 = 64 \text{ reflexive relations on } A}$$

iii. How many functions are there from A to $B = \{u, v\}$?

Procedure: build a function $f : A \to B$ by choosing images for each element of A

$$\underbrace{f(1)}_{\substack{\text{2 choices:}\\ u \text{ or } v}}, \underbrace{f(2)}_{\substack{\text{2 choices:}\\ u \text{ or } v}}, \underbrace{f(3)}_{\substack{\text{2 choices:}\\ u \text{ or } v}}$$

There are $2^3 = 8$ functions from A to B.

Q3. COUNTING A 9-member fellowship is to be formed by selecting its members from among a group of 15 people consisting of 1 wizard, 5 hobbits, 5 elves, 2 men, and 2 dwarves.

i. How many different 9-member fellowships could be formed from these people?

- choose 9 out of 15 people (in one of $\binom{15}{9} = \frac{15!}{9!6!} = 5005$ ways)

$\therefore$ there are 5005 different 9-member fellowships that could be formed

ii. Suppose one member will be distinguished from the rest as a "ring-bearer". How many different 9-member fellowships with one ring-bearer can be formed from these people?

T_1: choose the ring-bearer (in one of $\binom{15}{1} = 15$ ways)

T_2: choose 8 other members from among the remaining 14 people (in one of $\binom{14}{8} = 3003$ ways)

$\therefore$ there are $15 \cdot 3003 = 45045$ different 9-member fellowships that could be formed containing one ring-bearer.

iii. How many fellowships are there that include 1 wizard, 1 hobbit as the ring-bearer and 3 other hobbits, 1 elf, 2 men, and 1 dwarf?

T_1: choose 1 wizard (in $\binom{1}{1} = 1$ way)

T_2: choose one ring-bearer from among the 5 hobbits (in one of $\binom{5}{1} = 5$ ways)

T_3: choose 3 other hobbits from among the 4 remaining hobbits (in one of $\binom{4}{3} = 4$ ways)

T_4: choose an elf (in one of $\binom{5}{1} = 5$ ways)

T_5: choose 2 men (in $\binom{2}{2} = 1$ way)

T_6: choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways).

$\therefore$ there are $1 \cdot 5 \cdot 4 \cdot 5 \cdot 1 \cdot 2 = 200$ such fellowships that could be formed.

iv. After the fellowship has been selected in some way, its nine members will line up for a group photo before they depart on a perilous quest. How many ways can these 9 people be arranged for the photo?

There are $P(9,9) = 9! = 362\,880$ different arrangements (permutations) of the 9 members

v. How many fellowships are there that include 1 hobbit as the ring-bearer and an equal number of dwarves and elves?

We will break this up into cases where #dwarves = #elves.

<u>Case 1</u> 0 elves & 0 dwarves means we need

9 members from the 1 wizard, 5 hobbits, and 2 men (not possible)

<u>Case 2</u>: 1 dwarf + 1 elf means we need

7 members from the 1 wizard, 5 hobbits, and 2 men

T_0: choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1: choose 1 dwarf (in one of $\binom{2}{1} = 2$ ways)

T_2: choose 1 elf (in one of $\binom{5}{1} = 5$ ways)

T_3: choose 6 other members from among the 7 others (1 wizard, 4 hobbits, 2 men)

(in one of $\binom{7}{6} = 7$ ways)

<u>Case 3</u>: 2 dwarves + 2 eves means we need

5 members from the 1 wizard, 5 hobbits, and 2 men

T_0: choose one hobbit ring-bearer (in one of $\binom{5}{1} = 5$ ways)

T_1: choose 2 dwarves (in $\binom{2}{2} = 1$ way)

T_2: choose 2 elves (in one of $\binom{5}{2} = 10$ ways)

T_3: choose 4 other members from among the 7 others (1 wizard, 4 hobbits, 2 men)

(in one of $\binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ways)

$\therefore$ there are $5 \cdot 2 \cdot 5 \cdot 7 + 5 \cdot 1 \cdot 10 \cdot 35 = 350 + 1750 = 2100$ ways to form such a fellowship.