DGD 5

Q1. PROOF STRATEGIES — PROOF BY CONTRADICTION

Prove the following theorem using a **proof by contradiction**:

Theorem 5.1. Let x and y integers. If $\underbrace{x + y = 2018}_{P}$, then $\underbrace{x < 1010 \text{ or } y < 1010}_{Q}$.

$P \longrightarrow Q$

To prove $P \to Q$ by contradiction, we assume $\neg(P \to Q)$ is true
(which is $\equiv P \wedge \neg Q$)

$\neg Q$ says: "It is not the case that $(\underbrace{x<1010}_{Q_1}$ or $\underbrace{y<1010}_{Q_2})$"
ie $\neg(Q_1 \vee Q_2)$
(which is $\equiv \neg Q_1 \wedge \neg Q_2$)

In other words: $\neg Q$ says "$x \geq 1010$ and $y \geq 1010$" (De Morgan's Law)

Proof.
Let x and y be integers.
Assume P is true and assume $\neg Q$ is true.
That is, assume $x+y=2018$ and assume "$x \geq 1010$ and $y \geq 1010$."

$$
\begin{aligned}
\text{Then } 2018 &= x+y && \text{(since } P \text{ is true)}\\
&\geq 1010+1010 && \text{(since } \neg Q \text{ is true)}\\
&= 2020
\end{aligned}
$$

so $2018 \geq 2020$ ↯

$\therefore$ The assumption $\neg(P \to Q)$ is T leads to contradiction.
$\therefore P \to Q$ must be true. ∎

Q2. PROOF STRATEGIES — PROOF OF EQUIVALENCE

Prove the following theorem:

Theorem 5.2. Let n be an integer. Then $\underbrace{5 \mid n^2}_{P}$ if and only if $\underbrace{5 \mid n}_{Q}$.

$P \longleftrightarrow Q$

We must prove $P \to Q$ and $Q \to P$.

Indirect proof of $P \to Q$.

Let n be an integer.

Assume $\neg Q$ is true. ie Assume $5 \nmid n$. (goal: prove $\neg P$ is true ie prove $5 \nmid n^2$)

Then there are 4 possibilities for a nonzero remainder when n is divided by 5: remainder 1, 2, 3, or 4.

Case 1 (remainder 1). Assume $n = 5k+1$ for some integer k. (goal: prove $\neg P$ ie prove $5 \nmid n^2$)

$$\begin{aligned} \text{Then } n^2 &= (5k+1)^2 \\ &= 25k^2 + 10k + 1 \\ &= 5[5k^2 + 2k] + 1 \\ &= 5j + 1 \quad \text{where } j = 5k^2 + 2k \text{ so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 1 (ie $\neg P$ is true).

Case 2 (remainder 2). Assume $n = 5k+2$ for some integer k. (goal: prove $\neg P$ ie prove $5 \nmid n^2$)

$$\begin{aligned} \text{Then } n^2 &= (5k+2)^2 \\ &= 25k^2 + 20k + 4 \\ &= 5[5k^2 + 4k] + 4 \\ &= 5j + 4 \quad \text{where } j = 5k^2 + 4k \text{ so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 2 (ie $\neg P$ is true).

<u>Case 3</u> (remainder 3). Assume $n=5k+3$ for some integer k. (goal: prove $\neg P$ ie prove $5 \nmid n^2$)

$$\begin{aligned} \text{Then } n^2 &= (5k+3)^2 \\ &= 25k^2+30k+9 \\ &= 5[5k^2+6k+1]+4 \\ &= 5j+4 \quad \text{where } j=5k^2+6k+1 \text{ so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 3 (ie $\neg P$ is true).

<u>Case 4</u> (remainder 4). Assume $n=5k+4$ for some integer k. (goal: prove $\neg P$ ie prove $5 \nmid n^2$)

$$\begin{aligned} \text{Then } n^2 &= (5k+4)^2 \\ &= 25k^2+40k+16 \\ &= 5[5k^2+8k+3]+1 \\ &= 5j+1 \quad \text{where } j=5k^2+8k+3 \text{ so } j \in \mathbb{Z} \end{aligned}$$

Thus 5 does not divide n^2 in case 4 (ie $\neg P$ is true).

Since $\neg Q \equiv (\text{Case }1) \vee (\text{Case }2) \vee (\text{Case }3) \vee (\text{Case }4)$ and we proved

Case $1 \to \neg P$
Case $2 \to \neg P$
Case $3 \to \neg P$
Case $4 \to \neg P$

it follows that we proved $\neg Q \to \neg P$.

$\therefore P \to Q$ is true. $\square$

<u>Direct proof of $Q \to P$.</u>

Let n be an integer. Assume Q is true. ie Assume $5 \mid n$. (goal: prove P is T ie prove $5 \mid n^2$)

Then $n=5k$ for some integer k.

Thus $n^2=(5k)^2=25k^2=5(5k^2)=5j$ where $j=5k^2$ so $j \in \mathbb{Z}$.

$\therefore 5 \mid n^2$ (by def of divides) so P is true.

$\therefore Q \to P$ is true $\square$

Overall, we proved $P \to Q$ and $Q \to P$ $\therefore$ we proved $P \leftrightarrow Q$ is true

ie we proved Theorem 5.1. $\blacksquare$

Q3. PROOF STRATEGIES

Prove the following theorem:

$$\textbf{Theorem 5.3. } \underbrace{\text{The equation } x^3 + 3x + 5 = 0 \text{ has no rational roots.}}_{P}$$

proof by contradiction

Assume $\neg P$ is true (goal: show this leads to contradiction).

ie Assume it is not the case that the eqⁿ $x^3+3x+5=0$ has no rational roots.

Then there must exist at least one root of this equation that is rational.

Assume r is a rational root of the equation.

Then, there exist integers n and d such that $r=\frac{n}{d}$ and $d\neq 0$ (def of rational)

We may assume without loss of generality, that n and d have no common factors other than ± 1, ie the fraction $\frac{n}{d}$ is in lowest terms.

* In particular, n and d are not both even.

Since $r=\frac{n}{d}$ is a root, it follows that

$$\left(\frac{n}{d}\right)^3+3\left(\frac{n}{d}\right)+5=0$$

consequently, $\frac{n^3}{d^3}+\frac{3n}{d}+5=0$ $\therefore$ $\boxed{n^3+3nd^2+5d^3=0}$ (multiply both sides of equation by d^3)

We will consider all possible cases for the parity of n and d:

Case 1 (n even d odd)

Assume n is even and assume d is odd.

Then $n=2k$ for some $k\in\mathbb{Z}$ and $d=2j+1$ for some $j\in\mathbb{Z}$ (def of even/odd)

$$\begin{aligned}
\text{Thus } 0 &= n^3+3nd^2+5d^3 \\
&= (2k)^3+3(2k)(2j+1)^2+5(2j+1)^3 \\
&= 8k^3+6k(4j^2+4j+1)+5(8j^3+12j^2+6j+1) \\
&= 8k^3+6k(4j^2+4j+1)+40j^3+60j^2+30j+5 \\
&= 2\underbrace{[4k^3+3k(4j^2+4j+1)+20j^3+30j^2+15j+2]}+1 \\
&= 2m+1 \text{ where } m = \text{this integer}
\end{aligned}$$

$\therefore$ $0=2m+1$ for some $m\in\mathbb{Z}$

$\therefore$ 0 is odd $\lightning$ ($0=2(0)$ so 0 is definitely even)

Case 2 (n odd d odd)

Assume n is odd and assume d is odd.

Then $n=2k+1$ for some $k\in\mathbb{Z}$ and $d=2j+1$ for some $j\in\mathbb{Z}$ (def of odd)

Thus $0=n^3+3nd^2+5d^3$

$$=(2k+1)^3+3(2k+1)(2j+1)^2+5(2j+1)^3$$

$$=8k^3+12k^2+6k+1+(6k+3)(4j^2+4j+1)+5(8j^3+12j^2+6j+1)$$

$$=8k^3+12k^2+6k+1+6k(4j^2+4j+1)+12j^2+12j+3+40j^3+60j^2+30j+5$$

$$=8k^3+12k^2+6k+24j^2k+24jk+6k+12j^2+12j+40j^3+60j^2+30j+9$$

$$=2\underbrace{[4k^3+6k^2+3k+12j^2k+12jk+3k+6j^2+6j+20j^3+30j^2+15j+4]}_{\uparrow\ m \text{ is integer}}+1$$

$= 2m+1$ where $m =$ m is integer

$\therefore\ 0=2m+1$ for some $m\in\mathbb{Z}$

$\therefore\ 0$ is odd ↯ ($0=2(0)$ so 0 is definitely even)

Case 3 (n odd d even)

Assume n is odd and assume d is even.

Then $n=2k+1$ for some $k\in\mathbb{Z}$ and $d=2j$ for some $j\in\mathbb{Z}$ (def of odd/even)

Thus $0=n^3+3nd^2+5d^3$

$$=(2k+1)^3+3(2k+1)(2j)^2+5(2j)^3$$

$$=8k^3+12k^2+6k+1+(6k+3)(4j^2)+5(8j^3)$$

$$=2\underbrace{[4k^3+6k^2+3k+(6k+3)(2j^2)+5(4j^3)]}_{\uparrow\ m \text{ is integer}}+1$$

$= 2m+1$ where $m =$ m is integer

$\therefore\ 0=2m+1$ for some $m\in\mathbb{Z}$

$\therefore\ 0$ is odd ↯ ($0=2(0)$ so 0 is definitely even)

In all possible cases for the parity of n and d (given that $\frac{n}{d}$ is in lowest terms so they are not both even) we arrive at a contradiction.

$\therefore$ it was wrong to assume $\neg P$ is true

$\therefore$ P itself must actually be true, that is, the eq^{n} has no rational roots. $\blacksquare$