18. Counting: PIE & The Sum Rule
Proof by Induction

factorial: $\boxed{0! = 1}$ and for $n \geq 1$, $\boxed{n! = n(n-1)! = n(n-1)\cdots(2)(1)}$

 # r-permutations of an n-set: $\boxed{P(n,r) = n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}}$ $\boxed{P(n,n) = n!}$

 # r-combinations of an n-set: $\boxed{C(n,r) = \dbinom{n}{r} = \dfrac{P(n,r)}{P(r,r)} = \dfrac{n!}{(n-r)!r!}}$

THE PRINCIPLE OF INCLUSION–EXCLUSION (PIE)

Theorem 18.1. Let A and B be sets. Then

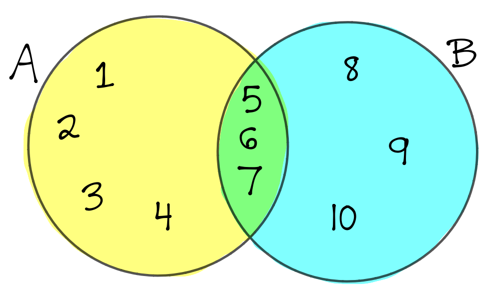
$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|\overline{A \cup B}| = |\mathcal{U}| - |A \cup B|$$
$$= |\mathcal{U}| - |A| - |B| + |A \cap B|$$

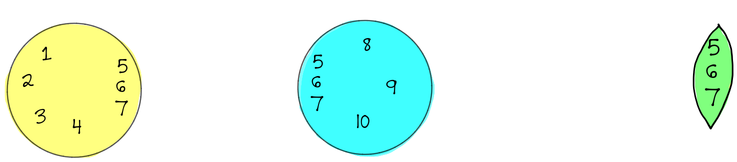
Example 18.2. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and let $B = \{5, 6, 7, 8, 9, 10\}$.

$|A \cup B| = 10$



By PIE,

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 7 + 6 - 3 = 10$$



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* These notes are solely for the personal use of students registered in MAT1348.

Example 18.3. How many 5-letter "words" (strings of length 5 consisting of lowercase letters of the English alphabet) start with the letter **a** or end with the letters **zz** ?

Procedure: create 5-letter words with given properties (a sequence of 5 tasks)

T_1	T_2	T_3	T_4	T_5

Let $A=\{$5-letter words starting with 'a'$\}$

a				
1 way	26 ways	26 ways	26 ways	26 ways

Let $B=\{$5-letter words ending with 'zz'$\}$

			z	z
26 ways	26 ways	26 ways	1 way	1 way

Then $A\cap B=\{$5-letter words starting with 'a' and ending with 'zz'$\}$

a			z	z
1 way	26 ways	26 ways	1 way	1 way

and $A\cup B=\{$5-letter words that start with 'a' or end with 'zz'$\}$

Thus, we want $|A\cup B|$. By PIE,

$$\boxed{\begin{aligned}|A\cup B| &= |A|+|B|-|A\cap B|\\ &= 26^4+26^3-26^2\end{aligned}}$$

Example 18.4. How many 5-letter "words" do not start with the letter **a** nor end with the letters **zz** ?

Observe that $\overline{A\cup B}=\{$5-letter words that do not start with 'a' and do not end with 'zz'$\}$

and, in this scenario, the universal set is $\mathcal{U}=\{$5-letter words$\}$

26 ways	26 ways	26 ways	26 ways	26 ways

Thus, we want $|\overline{A\cup B}|$

By PIE,
$$\begin{aligned}|\overline{A\cup B}| &= |\mathcal{U}| - |A\cup B|\\ &= 26^5-[26^4+26^3-26^2]\\ &= 11\,407\,500\end{aligned}$$

Example 18.5. A photographer is taking group photos of 6 inhabitants of The Island of K&K, namely A, B, C, D, E, and F.

In how many different ways can the photographer arrange 5 out of 6 of these inhabitants in a line so that

A and B are both in the photo or B, C, and E are all in the photo ?

Let $S = \{$ lineups that include both A and B $\}$

T_1: choose A and B to be in the photo (in $\binom{2}{2} = 1$ way)

T_2: choose 3 more inhabitants from among C, D, E, F to be in the photo (in $\binom{4}{3} = 4$ ways)

T_3: choose an ordering for the lineup of 5 inhabitants chosen in T_1, T_2. (in one of $P(5,5) = 5! = 120$ ways)

$\therefore |S| = (1)(4)(120) = 480$

Let $T = \{$ lineups that include B, C, and E $\}$

T_1: choose B, C, and E to be in the photo (in $\binom{3}{3} = 1$ way)

T_2: choose 2 more inhabitants from among A, D, and F to be in the photo (in $\binom{3}{2} = 3$ ways)

T_3: choose an ordering for the lineup of 5 inhabitants chosen in T_1, T_2. (in one of $P(5,5) = 5! = 120$ ways)

$\therefore |T| = (1)(3)(120) = 360$

Thus, $S \cap T = \{$ lineups that include A, B, C, and E $\}$

T_1: choose A, B, C, and E to be in photo ($\binom{4}{4} = 1$ way)

T_2: choose 1 more from among D and F ($\binom{2}{1} = 2$ ways)

T_3: choose an ordering of those 5 ($P(5,5) = 120$ ways)

$\therefore |S \cap T| = (1)(2)(120) = 240$

$S \cup T = \{$ lineups that include A & B or include B, C, & E $\}$

By PIE,

$|S \cup T| = |S| + |T| - |S \cap T| = 480 + 360 - 240 = 600$

$\boxed{\therefore \text{ there are 600 possible such photo lineups}}$

Suppose a procedure can be broken down into k tasks $T_1, T_2, \ldots, T_k$ such that

- for $i = 1, 2, \ldots, k$, the entire procedure can be done by task T_i which can be carried out in one of n_i ways,
- and the tasks $T_1, T_2, \ldots, T_k$ are mutually exclusive

ie none of the tasks share a common way to accomplish the procedure

Then there are $n_1 + n_2 + \cdots + n_k$ ways to carry out the procedure.

Example 18.6. How many binary strings of length 10 start with '000' and contain at least five 1's ?

0 0 0 _ _ _ _ _ _ _

Given that 3 entries are 0's and of the 7 remaining entries, at least 5 must be 1's,

We must consider the options for (# 1's):

① exactly 5 ones (hence '000' + 2 more 0's)
② exactly 6 ones (hence '000' + 1 more 0)
③ exactly 7 ones (hence '000' + no other 0's)

these are mutually exclusive.

① • choose 5 entries among the 7 remaining after '000' to place 5 1's into, then fill the rest with 0's.

$$\binom{7}{5} = \frac{7!}{5!2!} = \frac{7(6)(5!)}{5!(2)(1)} = 21 \text{ ways}$$

② · choose 6 entries among the 7 remaining after '000' to place 6 1's into, then fill the last with a 0.

$$\binom{7}{6} = \frac{7!}{6!1!} = 7 \text{ ways}$$

③ fill all 7 entries after '000' with 7 1's ($\binom{7}{7} = 1$ way)

$\boxed{\text{By the Sum Rule there are } 21+7+1 = 29 \text{ such binary strings in total.}}$

Exercise 18.7. How many binary strings of length 10 start with '000' or contain at least five 1's ?

Example 18.8. A 9-member fellowship is to be formed by selecting its members from among a group of 15 people consisting of 1 wizard, 5 hobbits, 5 elves, 2 men, and 2 dwarves.

In how many ways can the fellowship be formed if it must contain at least 2 hobbits **or** at least 2 men?

PIE

Let $A = \{\text{fellowships with at least 2 hobbits}\}$

$$\therefore |A| = \overbrace{\binom{5}{2}\cdot\binom{10}{7}}^{\substack{\text{2 hobbits and 7 non-hobbits} \\ \text{from the wizard, 5 elves, 2 men, and 2 dwarves}}} + \underbrace{\binom{5}{3}\binom{10}{6}}_{\text{3 hobbits and 6 non-hobbits}} + \overbrace{\binom{5}{4}\cdot\binom{10}{5}}^{\text{4 hobbits and 5 non-hobbits}} + \underbrace{\binom{5}{5}\cdot\binom{10}{4}}_{\text{5 hobbits and 4 non-hobbits}} = 4770$$

Let $B = \{\text{fellowships with at least 2 men}\}$

$$\therefore |B| = \underbrace{\binom{2}{2}\cdot\binom{13}{7}}_{\text{2 men and 7 others from the wizard, 5 hobbits, 5 elves, and 2 dwarves}} = 1716$$

Thus, $A \cap B = \{\text{fellowships with at least 2 hobbits } \underline{\text{and}} \text{ at least 2 men}\}$

$$|A \cap B| = \underbrace{\binom{5}{2}\binom{2}{2}\binom{1+5+2}{5}}_{\substack{\text{2 hobbits, 2 men, and 5 others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} + \overbrace{\binom{5}{3}\binom{2}{2}\binom{1+5+2}{4}}^{\substack{\text{3 hobbits, 2 men, and 4 others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} + \underbrace{\binom{5}{4}\cdot\binom{2}{2}\binom{1+5+2}{3}}_{\substack{\text{4 hobbits, 2 men, and 3 others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}} + \overbrace{\binom{5}{5}\binom{2}{2}\binom{1+5+2}{2}}^{\substack{\text{5 hobbits, 2 men, and 2 others} \\ \text{from the wizard, 5 elves, and 2 dwarves}}}$$

$$\therefore |A \cap B| = 1568$$

$A \cup B = \{\text{fellowships with at least 2 hobbits } \underline{\text{or}} \text{ at least 2 men}\}$

By PIE,
$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 4770 + 1716 - 1568 \\ &= 4918 \end{aligned}$$

$\therefore$ there are 4918 different 9-member fellowships with at least 2 hobbits or at least 2 men.

More details on how we computed $|A|$, $|B|$ and $|A \cap B|$

Let $A = \{$fellowships with at least 2 hobbits$\}$

We will break A up into cases based on the number of hobbits

Case 1: exactly 2 hobbits

T_1: select 2 hobbits (in one of $\binom{5}{2} = 10$ ways)

T_2: select 7 other members from among the wizard, 5 elves, 2 men, 2 dwarves

(in one of $\binom{1+5+2+2}{7} = \binom{10}{7} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}^{4}}{\cancel{3}\cdot\cancel{2}\cdot 1} = 120$ ways)

Case 2: exactly 3 hobbits

T_1: select 3 hobbits (in one of $\binom{5}{3} = 10$ ways)

T_2: select 6 other members from among the wizard, 5 elves, 2 men, 2 dwarves

(in one of $\binom{1+5+2+2}{6} = \binom{10}{6} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 210$ ways)

Case 3: exactly 4 hobbits

T_1: select 4 hobbits (in one of $\binom{5}{4} = 5$ ways)

T_2: select 5 other members from among the wizard, 5 elves, 2 men, 2 dwarves

(in one of $\binom{1+5+2+2}{5} = \binom{10}{5} = \frac{\cancel{10}^{2}\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot 7\cdot 6}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 252$ ways)

Case 4: exactly 5 hobbits

T_1: select 5 hobbits (in one of $\binom{5}{5} = 1$ way)

T_2: select 4 other members from among the wizard, 5 elves, 2 men, 2 dwarves

(in one of $\binom{1+5+2+2}{4} = \binom{10}{4} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 210$ ways)

Let $B = \{$fellowships with at least 2 men$\}$

T_1: select both of the 2 men (in one of $\binom{2}{2} = 1$ way)

T_2: select 7 other members from among the wizard, 5 hobbits, 5 elves, 2 dwarves

(in one of $\binom{1+5+5+2}{7} = \binom{13}{7} = 1716$ ways)

Thus, $A \cap B = \{$fellowships with at least 2 hobbits <u>and</u> at least 2 men$\}$

We will break $A \cup B$ up into cases based on the number of hobbits

Case 1: exactly 2 hobbits

T_1: select 2 hobbits (in one of $\binom{5}{2} = 10$ ways)

T_2: select 2 men (in $\binom{2}{2} = 1$ way)

T_3: select 5 other members from among the wizard, 5 elves, 2 dwarves

(in one of $\binom{1+5+2}{5} = \binom{8}{5} = \frac{8\cdot 7\cdot\cancel{6}}{\cancel{3}\cdot\cancel{2}\cdot 1} = 56$ ways)

Case 2: exactly 3 hobbits

T_1: select 3 hobbits (in one of $\binom{5}{3} = 10$ ways)

T_2: select 2 men (in $\binom{2}{2} = 1$ way)

T_3: select 4 other members from among the wizard, 5 elves, 2 dwarves

(in one of $\binom{1+5+2}{4} = \binom{8}{4} = \frac{\cancel{8}^{2}\cdot 7\cdot\cancel{6}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 70$ ways)

Case 3: exactly 4 hobbits

T_1: select 4 hobbits (in one of $\binom{5}{4} = 5$ ways)

T_2: select 2 men (in $\binom{2}{2} = 1$ way)

T_3: select 3 other members from among the wizard, 5 elves, 2 dwarves

(in one of $\binom{1+5+2}{3} = \binom{8}{3} = \frac{8\cdot 7\cdot\cancel{6}}{\cancel{3}\cdot\cancel{2}\cdot 1} = 56$ ways)

Case 4: exactly 5 hobbits

T_1: select 5 hobbits (in one of $\binom{5}{5} = 1$ way)

T_2: select 2 men (in $\binom{2}{2} = 1$ way)

T_3: select 2 other members from among the wizard, 5 elves, 2 dwarves

(in one of $\binom{1+5+2}{2} = \binom{8}{2} = \frac{\cancel{8}^{4}\cdot 7}{\cancel{2}\cdot 1} = 28$ ways)

The (★★★★★)-Recipe for Mathematical Induction:

★ **A Proposition** $P(n)$ **that depends on** n

For $n \in \mathbb{N}$, define a proposition $P(n)$ (which says something about the number n).

★★ **Basis of Induction (B.I.)**

For an initial **base value** $n_0 \in \mathbb{N}$, prove that $P(n_o)$ is true.

★★★ **Induction Step (I.S.)**

Let $k \geq n_0$ Prove that $P(k) \to P(k+1)$.

★★★★ **The Induction Hypothesis (I.H.)** Assume $P(k)$ is true.
(I.H. is the 1st step in a direct proof of the I.S.)
Goal: prove that $P(k+1)$ follows from $P(k)$.

★★★★★ **Conclusion** Since $P(n_0)$ is true, and since we proved that $P(k) \to P(k+1)$ for any $k \geq n_0$, it follows by **Mathematical Induction** that $P(n)$ is true for all $n \geq n_0$.

Example 18.9. Use **Mathematical Induction** to prove that the following formula holds for all integers $n \geq 1$:

$$\boxed{1 + 2 + \cdots + n \quad = \quad \frac{n(n+1)}{2}}$$

★ P(n) Define the proposition $\boxed{P(n):\ 1+2+\dots+n = \frac{n(n+1)}{2}}$

★★ B.I. base value: $\boxed{n_0 = 1}$ P(1) says "$\underbrace{1}_{LS} = \underbrace{\frac{1(1+1)}{2}}_{RS}$"

For P(1), $LS = 1$ and $RS = \frac{1(1+1)}{2} = 1$ $\therefore$ P(1) is true.

★★★ I.S. Let $k \geq n_0 = 1$. Now we prove $P(k) \longrightarrow P(k+1)$.

★★★★ I.H. Assume P(k) is true. (goal: prove P(k+1) follows from P(k))

P(k) says "$1+\dots+k = \frac{k(k+1)}{2}$"

Thus, our induction hypothesis is to assume

I.H. $\boxed{1+\dots+k = \frac{k(k+1)}{2}}$ for some integer $k \geq n_0 = 1$.

For the I.S. our goal is to show $P(k+1)$ follows from our I.H.

First, observe what $P(k+1)$ says: "$\underbrace{1+2+\dots+k+k+1}_{LS} = \underbrace{\frac{(k+1)(k+1+1)}{2}}_{RS}$"
(so we know what our goal is)

In $P(k+1)$, we have

$$LS = \underbrace{1+2+\dots+k}+(k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{(using I.H.)}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \quad \text{(common denom. so we can add fractions)}$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2} = RS \quad \therefore P(k+1) \text{ is true!} \quad \therefore \text{we proved } P(k) \to P(k+1)!$$

★★★★★ conclusion Since $P(1)$ is true and since we proved $P(k) \to P(k+1)$ is true for any $k \geq 1$, it follows from Mathematical Induction that $P(n)$ is true for all integers $n \geq 1$.

Study Guide

sum rule for k tasks:	there are $n_1 + n_2 + \cdots + n_k$ ways to carry out a procedure given that there are n_i ways to carry out the entire procedure with task T_i, and the tasks $T_1, \dots, T_k$ are mutually exclusive tasks that each carry out the entire procedure.
proof by induction:	1. Define $P(n)$ 2. **B.I.** Prove $P(n_0)$ 3. **I.S.** Let $k \geq n_o$. Prove $P(k) \to P(k+1)$. 4. **I.H.** Assume $P(k)$ is true (goal: prove $P(k+1)$ follows from I.H.) 5. **conclusion**

Exercises	Sup.Ex. §8 # 1, 5, 7, 9, 10, 11	Sup.Ex. §10 # 1, 2, 4, 6, 7, 8, 9, 10, 11
	Sup.Ex. §11 # 1, 9, 10	Rosen §6.1 # 9, 15, 29, 31, 49 (treat as inclusive or), 65
	Rosen §5.1 # 3, 5, 7, 9, 11, 13, 15	Rosen §6.3 # 19, 21, 27, 31, 33, 35, 37