

12. Functions

Building new sets from old:

- power set of S Cartesian product of two (or more) sets
 $\mathcal{P}(S)$ $S \times T$ $S_1 \times S_2 \times \dots \times S_t$

Set Operations:

- union intersection complement difference symmetric difference
 $S \cup T$ $S \cap T$ \bar{S} $S - T$ $S \oplus T$

Set identities:

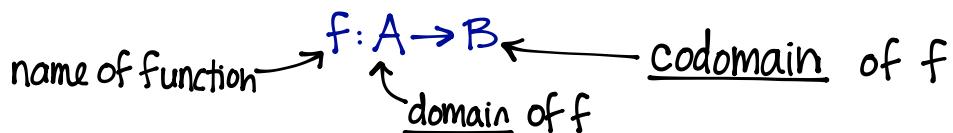
- verify using membership tables verify using a rigorous proof
 prove other identities using the laws from the Table of Important Set Identities

FUNCTIONS

Let A and B be sets.

A **function** f from A to B is an assignment of **exactly one** element of B to each element of A .

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element $a \in A$.

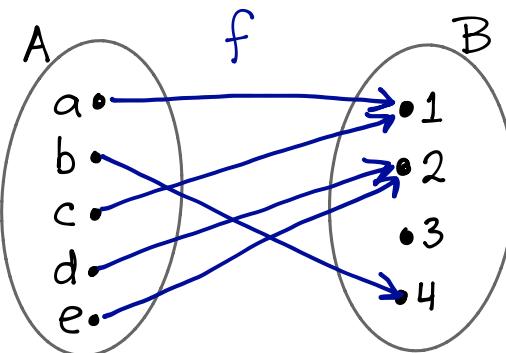


- for $a \in A$, $f(a)$ is the image of a (in particular, $f(a) \in B$)
- $f(A) = \{f(a) : a \in A\}$ is the set of images of all $a \in A$
 called the image of the domain ("range" of f)
- for any subset $S \subseteq A$, we may consider the set $f(S) = \{f(a) : a \in S\}$
 the image of the set $S \subseteq A$
- for each $b \in B$, define $f^{-1}(b) = \{a \in A : f(a) = b\}$
 the set $f^{-1}(b)$ is called the preimage of b

Example 12.1. Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$.

Define $f: A \rightarrow B$ as follows:

$$\begin{aligned}f(a) &= 1 \\f(b) &= 4 \\f(c) &= 1 \\f(d) &= 2 \\f(e) &= 2\end{aligned}$$



$$f(A) = \{1, 2, 4\}$$

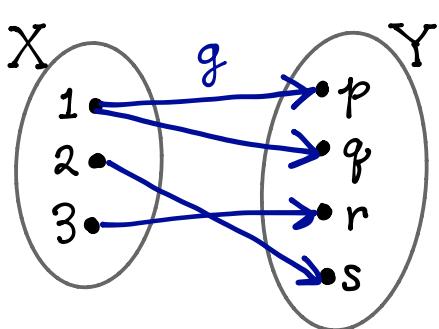
$$f(\{c, d, e\}) = \{1, 2\}$$

$$f^{-1}(2) = \{d, e\}$$

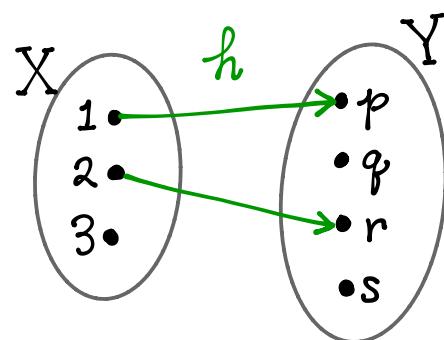
$$f^{-1}(3) = \emptyset$$

$$f^{-1}(4) = \{b\}$$

Example 12.2. Let $X = \{1, 2, 3\}$ and $Y = \{p, q, r, s\}$.



g is not a function from X to Y because $1 \in X$ is assigned to more than one element of Y



h is not a function from X to Y because $3 \in X$ is not assigned to any element of Y .

Examples of functions given by rules (instead of arrow diagrams).

$$\begin{aligned}\text{Ex. } f: \mathbb{R} &\rightarrow \mathbb{R} \\f(x) &= x^2\end{aligned}$$

$$\begin{aligned}\text{Ex. } f: \mathbb{Z} \times \mathbb{Z}^+ &\rightarrow \mathbb{Q} \\f(m, n) &= \frac{m}{n}\end{aligned}$$

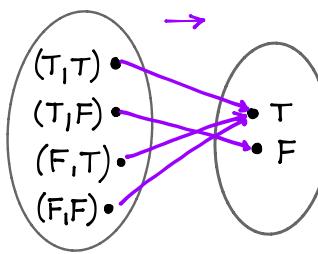
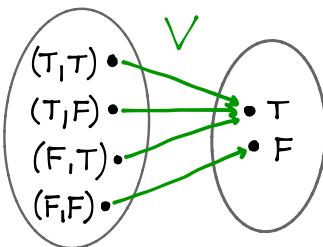
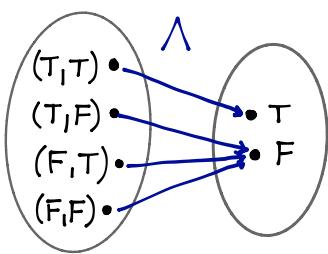
$$\begin{aligned}\text{Ex. } D: \mathbb{Z} \times \mathbb{Z} &\rightarrow \mathbb{Z} \\D(x, y) &= x - y\end{aligned}$$

$$\begin{aligned}\text{Ex. } S: \mathbb{N} \times \mathbb{N} &\rightarrow \mathbb{N} \\S(a, b) &= a + b\end{aligned}$$

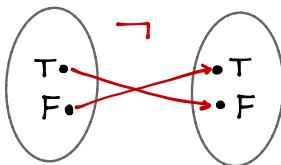
$$\begin{aligned}\text{Ex. } P: \mathbb{N} \times \mathbb{N} &\rightarrow \mathbb{N} \\P(n, m) &= nm\end{aligned}$$

$$\begin{aligned}\text{Ex. } f: \mathbb{R} &\rightarrow \mathbb{Z} \times \mathbb{Q} \\f(x) &= (8, 1.2) \\&\text{(a constant function)}\end{aligned}$$

More examples: domain $\{\text{T}, \text{F}\} \times \{\text{T}, \text{F}\}$ codomain $\{\text{T}, \text{F}\}$



Ex. domain $\{\text{T}, \text{F}\}$
codomain $\{\text{T}, \text{F}\}$



Example 12.3. Suppose X is a compound proposition consisting of n propositional variables.

We can think of X as a function with domain $\{\text{T}, \text{F}\}^n$ and codomain $\{\text{T}, \text{F}\}$

Thus, each element of X 's "domain" is a truth assignment to X 's n variables

The "image" of a truth assignment is X 's truth value for that truth assignment

Ex. Let X be the proposition $\neg(p \rightarrow (q \wedge r))$

Think of X as a function of 3 variables: $X(p, q, r)$ or $X : \{\text{T}, \text{F}\}^3 \rightarrow \{\text{T}, \text{F}\}$

Ex $X(\text{T}, \text{T}, \text{F}) = \text{T}$ because X is T when $p=\text{T}, q=\text{T}, r=\text{F}$

INJECTIVE (ONE-TO-ONE) FUNCTIONS

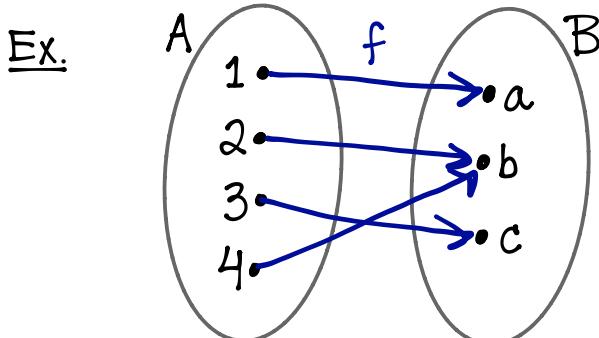
A function $f : A \rightarrow B$ is called **injective** or **one-to-one**, if for all $x, y \in A$, the implication

$$(f(x) = f(y)) \rightarrow (x = y) \quad \text{is True.}$$

Equivalently (contrapositive form)

for all $x, y \in A$, $(x \neq y) \rightarrow (f(x) \neq f(y))$ is True.

Thus, each distinct element of A is assigned its own distinct unique element of B .



f is not injective because $f(2) = f(4)$

Example 12.4. Is $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ injective?

Let $a, b \in \mathbb{R}^+$ (the domain of f)

Assume $f(a) = f(b)$. Then $a^2 = b^2$

$$\Rightarrow a^2 - b^2 = 0$$

$$\Rightarrow (a-b)(a+b) = 0$$

$$\begin{matrix} \downarrow \\ a=b \end{matrix} \text{ or } \begin{matrix} \downarrow \\ a=-b \end{matrix}$$

Since, $a, b \in \mathbb{R}^+$, the only possible solution is $a=b$

Note. Since, for this function, the domain is \mathbb{R}^+ , we know that neither a nor b are negative... so $a = -b$ is not actually possible.

Thus, we proved $(f(a) = f(b)) \rightarrow (a = b)$ is True for all $a, b \in \mathbb{R}^+$.

$\therefore f$ is injective (I-I).

Example 12.5. Is $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ injective?

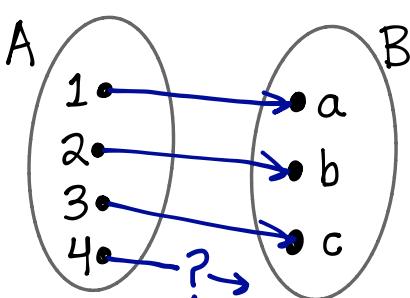
No! Counterexample: 1 and -1 are two distinct elements of the domain \mathbb{R} , yet $g(1) = g(-1) = 1$.

$\therefore g$ is not injective (I-I).

Example 12.6. Let A and B be sets such that $|A| = 4$ and $|B| = 3$.

Does there exist an injective function $f : A \rightarrow B$?

No!



Informal explanation:

If $|A| > |B|$, then, at some point, B will "run out" of "new" distinct images for the elements of A .

Theorem 12.7. Let A and B be sets.

If there exists an injective function $f : A \rightarrow B$, then $|A| \leq |B|$.

P

Q

Theorem 12.7 in contrapositive form: Let A and B be sets.

If $|A| > |B|$, then there does not exist an injective function from A to B .

$\neg Q$

$\neg P$

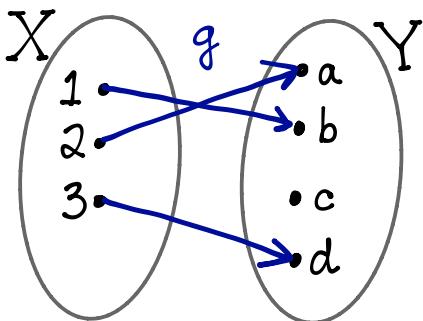
SURJECTIVE (ONTO) FUNCTIONS

A function $f : A \rightarrow B$ is called **surjective** or **onto** if, for every element $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$.

Equivalently, for all $b \in B$, $f^{-1}(B) \neq \emptyset$.

Thus, each element of the codomain B is the image of at least one element of A .

Ex.



g is not surjective because $c \in Y$ (the codomain) and yet $f^{-1}(c) = \{\} = \emptyset$.

Example 12.8. Is $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ defined by $f(m, n) = (2m, n)$ surjective?

Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ be an arbitrary element of the codomain of f .

Are we always guaranteed to be able to find some $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ (the domain of f) for which $f(m, n) = (a, b)$?

"
 $(2m, n)$

what if a
is odd?

No! f is not surjective (onto).

Counterexample:

No element of the domain $\mathbb{Z} \times \mathbb{Z}$ has $(1, 1)$ as its image because every element of the domain gets mapped to an element of the codomain whose 1st coordinate is even.

Example 12.9. Is $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f(x, y) = (2x, y)$ surjective?

Let $(a, b) \in \mathbb{R}^2$ (the codomain).

In order for $f(x, y) = (a, b)$ we need $(2x, y) = (a, b)$

$$\Rightarrow 2x = a \text{ and } y = b$$

$$\Rightarrow x = \frac{a}{2} (\in \mathbb{R}) \text{ and } y = b (\in \mathbb{R})$$

$$\Rightarrow \left(\frac{a}{2}, b\right) \in \mathbb{R} \times \mathbb{R} \text{ (the domain)}$$

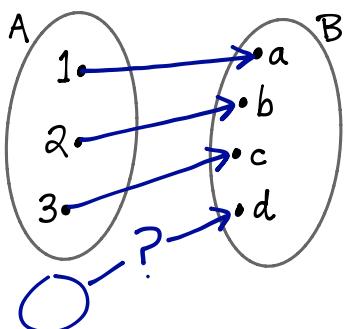
So for any $(a, b) \in \mathbb{R} \times \mathbb{R}$ (the codomain), we can find at least one element of the domain, namely $\left(\frac{a}{2}, b\right) \in \mathbb{R}^2$, such that $f\left(\frac{a}{2}, b\right) = (a, b)$

this is a "constructive" proof

$\therefore f$ is surjective (onto).

Example 12.10. Let A and B be sets such that $|A| = 3$ and $|B| = 4$. Does there exist a surjective function $f : A \rightarrow B$?

No!



Informal explanation:

If $|B| > |A|$, then A will "run out" of elements before all elements of B have been assigned as images.

Theorem 12.11. Let A and B be sets.

If there exists a surjective function $f : A \rightarrow B$, then $|A| \geq |B|$.

P

→

Q

Theorem 12.11 in contrapositive form: Let A and B be sets.

If $|B| > |A|$, then there does not exist a surjective function from A to B .

¬Q

→

¬P

STUDY GUIDE

Important terms and concepts:

- | | | | | |
|--|---------------------------------|---|--------------------------------|-----------------------------------|
| <input type="checkbox"/> function | <input type="checkbox"/> domain | <input type="checkbox"/> codomain | <input type="checkbox"/> image | <input type="checkbox"/> preimage |
| <input type="checkbox"/> injective (one-to-one) function | | <input type="checkbox"/> surjective (onto) function | | |

Exercises

Sup.Ex. §5 # 1abd, 3, 4, 5, 7, 9, 10

Rosen §2.3 # 1, 4ac, 7a, 8, 9, 10, 11, 12, 13, 15, 42a, 43a