17. Counting: Product Rule, Permutations & Combinations

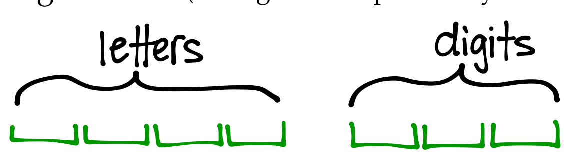
The Product Rule for k Tasks

Suppose a procedure can be broken down into a sequence of k tasks $T_1, T_2, \ldots, T_k$, such that

- there are n_1 ways to carry out task T_1
- and, for $i = 2, \ldots, k$, there are n_i ways to carry out task T_i after each of the tasks $T_1, T_2, \ldots, T_{i-1}$ has been done in a certain way.

Then there are $(n_1)(n_2)\cdots(n_k)$ ways to carry out the procedure.

Example 17.1. How many (new) Ontario licence plates are there consisting of four uppercase letters followed by 3 digits? (let's ignore the possibility that some words/sequences might be forbidden)



T_1: task of choosing 1st letter (26 ways)
T_2: task of choosing 2nd letter (26 ways)
T_3: task of choosing 3rd letter (26 ways)
T_4: task of choosing 4th letter (26 ways)
T_5: task of choosing 1st digit (10 ways)
T_6: task of choosing 2nd digit (10 ways)
T_7: task of choosing 3rd digit (10 ways)

$\therefore$ there are $(26)(26)(26)(26)(10)(10)(10) = 26^4\,10^3$ ways to create such a licence plate.

* These notes are solely for the personal use of students registered in MAT1348.

Example 17.2. How many (new) Ontario licence plates are there such that the first letter is an 'A' and the second digit is a '5' ?

tasks: select each letter/digit

$$\overbrace{\underset{\uparrow\ 1\text{ way}}{\underline{A}}\ \underset{\uparrow\ 26\text{ ways}}{\underline{\ \ }}\ \underset{\uparrow\ 26\text{ ways}}{\underline{\ \ }}\ \underset{\uparrow\ 26\text{ ways}}{\underline{\ \ }}}^{\text{letters}}\quad \overbrace{\underset{\uparrow\ 10\text{ ways}}{\underline{\ \ }}\ \underset{\uparrow\ 1\text{ way}}{\underline{5}}\ \underset{\uparrow\ 10\text{ ways}}{\underline{\ \ }}}^{\text{digits}}$$

$\therefore$ there are $(1)(26)(26)(26)(10)(1)(10) = 26^3 \cdot 10^2$ ways to create such a licence plate.

Example 17.3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y, z\}$.

How many distinct functions are there from A to B?

Procedure: build a function $f : A \to B$ by choosing images $\underbrace{f(1)}_{\uparrow\ 3\text{ways}}, \underbrace{f(2)}_{\uparrow\ 3\text{ways}}, \underbrace{f(3)}_{\uparrow\ 3\text{ways}}, \underbrace{f(4)}_{\uparrow\ 3\text{ways}}, \underbrace{f(5)}_{\uparrow\ 3\text{ways}}$

For $1 \leq i \leq 5$, let T_i be the task

T_i: choose the image $f(i) \in B$ in one of 3 ways

$\therefore$ there are $3^5 = 243$ functions from A to B.

Example 17.4. Let A and B be finite sets with $|A| = m$ and $|B| = n$.

How many functions $f : A \to B$ are there? Let $A = \{a_1, \dots, a_m\}$

Procedure: to create a function $f : A \to B$

$\hookrightarrow$ choose $f(a_1), \dots, f(a_m) \in B$

<u>ie</u> choose exactly one image in B for each element in the domain A.

task T_1: choose $f(a_1) \in B$ (in one of $|B| = n$ ways)

task T_2: choose $f(a_2) \in B$ (in one of $|B| = n$ ways)

$\vdots$

task T_m: choose $f(a_m) \in B$ (in one of $|B| = n$ ways)

$$\begin{aligned}\therefore \#\text{ ways to perform } (T_1, T_2, \dots, T_m) &= |B| \cdot |B| \cdots |B| \\ &= (n)(n) \cdots (n) \\ &= n^m \\ &= |B|^{|A|}\end{aligned}$$

$\boxed{\therefore \text{ there are } |B|^{|A|}\ (= n^m) \text{ functions } f : A \to B}$

Example 17.5. Let A and B be finite sets with $|A| = m$ and $|B| = n$.

How many functions $f : A \to B$ are **injective**?

Let $A = \{a_1, \dots, a_m\}$

Procedure: to create an injective function $f: A \to B$

$\hookrightarrow$ choose $f(a_1), \dots, f(a_m) \in B$ all distinct (so that f is 1-1)

task T_1: choose $f(a_1) \in B$ in one of $|B| = n$ ways

task T_2: choose $f(a_2) \in B$ after $f(a_1)$ has been chosen in one of $n-1$ ways (so that $f(a_2) \neq f(a_1)$).

task T_3: choose $f(a_3) \in B$ after $f(a_1), f(a_2)$ have been chosen in one of $n-2$ ways (so that $f(a_3) \neq f(a_1)$ and $f(a_3) \neq f(a_2)$).

$\vdots$

task T_i: choose $f(a_i) \in B$ after $f(a_1), \dots, f(a_{i-1})$ have been chosen in one of $n-(i-1)$ ways (so that $f(a_i) \neq f(a_1), \dots, f(a_i) \neq f(a_{i-1})$).

$\vdots$

task T_m: choose $f(a_m) \in B$ after $f(a_1), \dots, f(a_{m-1})$ have been chosen in one of $n-(m-1)$ ways (so that $f(a_m) \neq f(a_1), \dots, f(a_m) \neq f(a_{m-1})$).

$\therefore$ # ways to carry out $(T_1, T_2, \dots, T_m) = n(n-1)(n-2)\cdots(n-(i-1))\cdots(n-(m-1))$

$\boxed{\therefore \text{ there are } n(n-1)\cdots(n-m+1) \text{ injective functions from } A \text{ to } B}$

**Note If $|B| < |A|$ (ie if $n < m$), then the answer is 0 (zero).

Permutations

A **permutation** is an ordered arrangement of some (or all) of the elements of a finite set.

Let $n \geq r \geq 0$ be integers.

An r**-permutation of an** n**-element set** S is an ordered arrangement of r elements from S.

Example 17.6. Write all 3-permutations of the 3-element set $S = \{1, 2, 3\}$

123	231
132	312
213	321

In total, there are six 3-permutations of S.

Example 17.7. How many 3-permutations are there of the 6-element set $S = \{1, 2, 3, 4, 5, 6\}$. Write some of these down.

Procedure: build a 3-permutation of a 6-element set:

1st element | 2nd element | 3rd element

6 ways to select | 5 ways to select | 4 ways

In total, there are $6 \cdot 5 \cdot 4 = 120$ 3-permutations of a 6-element set.

Ex 123 124 125 126 132 134 135 136 142 143 145 146 … 651 … etc.

Let $n \geq r \geq 0$ be integers.

$P(n, r) =$ # of r-permutations of an n-element set

$P(n, n) =$ # of permutations of an n-element set

$P(n,r) = ?$

Procedure: build an r-permutation of an n-element set:

1st element | 2nd element | 3rd element | … | rth element

n ways to select | n-1 ways to select | n-2 ways | | n-(r-1) ways

By the Product Rule:

$$\boxed{P(n,r) = (n)(n-1)(n-2)\cdots(n-r+1)}$$

In particular, for r=n,

$$\boxed{\begin{aligned} P(n,n) &= n(n-1)(n-2)\cdots(n-n+1) \\ &= n(n-1)(n-2)\cdots(1) \\ &= n! \end{aligned}}$$

Factorials

$1 = 0!$

"n factorial", denoted $n!$, is defined for each $n \in \mathbb{N}$ as follows:

$$0! = 1$$

For $n \geq 1$, $n!$ is defined recursively:

$$\begin{aligned} n! &= n(n-1)! \\ &= n(n-1)(n-2)\cdots(2)(1) \end{aligned}$$

Examples

$$\begin{aligned} 1! &= 1(0!) \\ &= 1(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} 2! &= 2(1!) \\ &= 2(1) \\ &= 2 \end{aligned}$$

$$\begin{aligned} 3! &= 3(2!) \\ &= 3(2)(1!) \\ &= 3(2)(1) \\ &= 6 \end{aligned}$$

$$\begin{aligned} 4! &= 4(3!) \\ &= 4(3)(2!) \\ &= 4(3)(2)(1) \\ &= 24 \end{aligned}$$

Example 17.8. List all possible 3-element subsets of the set $S = \{1, 2, 3, 4\}$

$$\left.\begin{array}{l}\{1,2,3\}\\ \{1,2,4\}\\ \{1,3,4\}\\ \{2,3,4\}\end{array}\right\}$$ 3-element subsets of S

"3-combinations from a 4-element set"

Let $n \geq r \geq 0$ be integers.

An r**-combination from an** n**-element set** S is an r-element subset of an n-element set S.

$C(n, r) =$ # of r-combinations from an n-element set

= # of subsets T of S such that $|T| = r$ and $|S| = n$

Theorem 17.9. Let $n, r \in \mathbb{N}$ with $n \geq r \geq 0$.

Then

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof First, note that $\dfrac{n!}{r!\,(n-r)!} = \dfrac{P(n,r)}{P(r,r)}$ (verify this!)

We will now prove that $P(n,r) = C(n,r) \cdot P(r,r)$ [which is equivalent to proving $C(n,r) = \dfrac{P(n,r)}{P(r,r)}$]

Let's count $P(n,r)$ by the following 2-task procedure for building r-permutations of an n-element set.

task T_1: choose an r-element subset T of S (in one of $C(n,r)$ possible ways)

task T_2: arrange the r elements of T (in one of $P(r,r)$ possible ways)

By the Product Rule,

the # of ways to carry out $(T_1, T_2) = C(n,r) \cdot P(r,r)$

ie $P(n,r) = C(n,r) \cdot P(r,r)$

$\therefore C(n,r) = \dfrac{P(n,r)}{P(r,r)} = \dfrac{n!}{r!\,(n-r)!}$

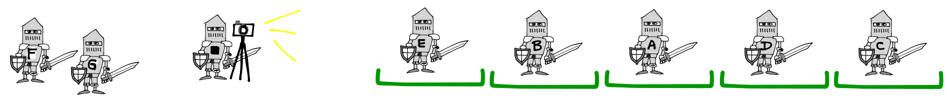
Example 17.10. A photographer is taking group photos of 7 inhabitants of The Island of K&K, namely A, B, C, D, E, F, and G.

i. In how many ways can the photographer select 5 out of 7 of these inhabitants to be in the photo?

The photographer wants to select a 5-element subset of $\{A, B, C, D, E, F, G\}$
(ie to select 5 inhabitants to be "in the photo".)

There are $C(7,5) = \frac{7!}{5!2!} = \frac{7\cdot 6\cdot \cancel{5!}}{\cancel{5!}\,2\cdot 1} = \frac{7\cdot \cancel{6}^{3}}{\cancel{2}} = 21$ ways to do this.

ii. How many distinct ways can the photographer arrange 5 out of 7 of these inhabitants in a line?



The photographer wants to select a 5-permutation of $\{A, B, C, D, E, F, G\}$
(ie to select 5 inhabitants to be "in the photo" and arrange them in some order).

There are $P(7,5) = \underline{7}\cdot\underline{6}\cdot\underline{5}\cdot\underline{4}\cdot\underline{3} = 2520$ ways to do this

iii. In how many distinct ways can the photographer line up 5 out of 7 of these inhabitants so that A and B are in the photo and A stands next to B?

T_1: choose 2 side-by-side positions in the lineup for A and B to stand (4 ways)

T_2: choose an ordering of A and B ($P(2,2) = 2! = 2$ ways, namely AB and BA)

T_3: choose a 3-permutation of the 5 remaining inhabitants to fill the remaining 3 places in the lineup ($P(5,3) = 5(4)(3) = 60$ ways)

$\therefore$ there are $(4)(P(2,2))(P(5,3)) = (4)(2)(60) = 480$ possible photo lineups in which A stands next to B.

Study Guide

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product rule for k tasks: there are $n_1 n_2 \cdots n_k$ ways to carry out the sequence of tasks $(T_1, T_2, \ldots, T_k)$ given that there are n_i ways to carry out task T_i after each of the previous tasks has been carried out in some way.

permutations	factorials	combinations
$P(n,k) = n(n-1)\cdots(n-(k-1))$	$n! = n(n-1)\ldots(2)(1)$	$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Exercises Sup.Ex. §8 # 2, 3, 4abcd, 12 Sup.Ex. §10 # 6a, 8a, 9abd, 10bc, 11
Rosen §6.1 # 1a, 7, 11, 16, 19, 25, 33, 34, 35, 37ab, 47