SOLUTIONS Version A

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Math for Computing — Instructor: Elizabeth Maltais

MAT1348X – Midterm 1 – Monday, June 4, 2018

- ▷ Clearly write your name and student number on this test, and **sign it** below to confirm that you have read, understood and agreed to follow these **instructions**:
- ▷ This is a 75-minute **closed-book** test. No notes. No calculators.
- ▷ **Put away everything** except for a few pens or pencils, an eraser, and your student id. card.
- ▷ The exam consists of 9 pages. Page 9 contains the **Table of Logical Equivalences**.
- ▷ The maximum points possible = 50 points.
- ▷ Questions 1–4 are short-answer problems. Write the final answer in the appropriate box, space, or circle your answer. You do not need to show any other work.
- ▷ Questions 5–9 are long-answer problems. To receive full marks, your solution must be complete, correct, and contain all relevant details.
- ▷ Read all questions carefully and be sure to follow the instructions for the individual problems. You may ask for reasonable clarifications.
- ▷ For additional work space, you may use the backs of pages.
  Do not use any of your own scrap paper.
- ▷ You must use **proper mathematical notation and terminology**. Make sure that your notation is consistent with the notation used in class.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read and understood, and will comply with the above instructions.**

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Do not write in this table.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Total
Maximum points	1 pt	6 pts	6 pts	9 pts	5 pts	5 pts	6 pts	6 pts	6 pts	50 points
Marks obtained										

Short-answer Questions.

Write your final answer in the answer box, the space provided, or circle your answer.
No justification is needed.

Q1. [1 point] Fill in the following truth table.

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$	$P \oplus Q$	$\neg P$
T	**T**	T	T	T	T	F	F
T	**F**	F	T	F	F	T	F
F	**T**	F	T	T	F	T	T
F	**F**	F	F	T	T	F	T

Q2. [6 points] For each proposition below, circle the appropriate response to indicate whether the given compound proposition is a tautology, contradiction, or contingency, or whether you do not have sufficient information to determine its type.

The variables X, Y, and Z represent *mystery* compound propositions of the following types:

X is a **tautology** Y is a **contradiction**, and Z is a **contingency**.

i. $X \rightarrow Y$ $\equiv T \rightarrow F \equiv F$ tautology contradiction contingency unable to determine

ii. $(X \wedge Y) \rightarrow Z$ tautology contradiction contingency unable to determine

$\equiv T \wedge F \rightarrow Z$

$\equiv F \rightarrow Z \equiv T$

iii. $Z \rightarrow Z$ tautology contradiction contingency unable to determine

$\equiv \neg Z \vee Z$

$\equiv T$

iv. $X \vee Y$ tautology contradiction contingency unable to determine

$\equiv T \vee F \equiv T$

v. $X \wedge Z$ tautology contradiction contingency unable to determine

$\equiv T \wedge Z$

$\equiv Z$

vi. $Y \wedge Z$ tautology contradiction contingency unable to determine

$\equiv F \wedge Z$

$\equiv F$

Q3a. [1 point] Translate the statement X below into propositional logic using the following propositional variables and appropriate logical connectives:

H : "The tiger is hungry."

E : "The tiger will eat you."

A : "You say *abracadabra*."

X : "If the tiger is hungry, then (it will eat you unless you say *abracadabra*.) "

Translation of X into propositional logic: $H \rightarrow (E \vee A)$

Q3b. [1 point] Write a compound proposition that is logically equivalent to the **contrapositive** of X, using the variables defined in **Q3a.**

Contrapositive of X in propositional logic: $\neg(E \vee A) \rightarrow \neg H$

$(\equiv \neg E \wedge \neg A \rightarrow \neg H)$

Q3c. [2 points] In English, write the **contrapositive** of X below:

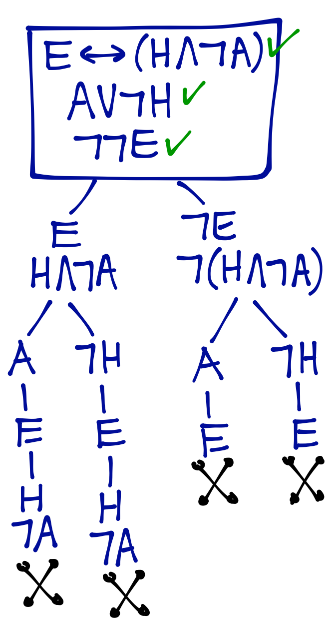
Contrapositive of X in English:

If the tiger will not eat you and you do not say abracadabra, then the tiger is not hungry.

Q3d. [2 points] Use any method to determine whether the following argument is **valid** or not. No explanation is required.

$$E \leftrightarrow (H \wedge \neg A)$$
$$A \vee \neg H$$
$$\therefore \neg E$$

Circle: **VALID** **INVALID**



Q4. Here is a truth table for *mystery* compound propositions P, Q, and R, each consisting of the propositional variables w, x, y.

w	x	y	P	Q	R	$(P \wedge R) \to Q$
T	**T**	**T**	**F**	**T**	**F**	T
T	**T**	**F**	**T**	**F**	**T**	F
T	**F**	**T**	**T**	**F**	**T**	F
T	**F**	**F**	**F**	**T**	**T**	T
F	**T**	**T**	**F**	**T**	**T**	T
F	**T**	**F**	**T**	**T**	**F**	T
F	**F**	**T**	**F**	**F**	**F**	T
F	**F**	**F**	**F**	**F**	**T**	T

Q4a. [2 points] Give a **DNF** for P:

$(w \wedge x \wedge \neg y) \vee (w \wedge \neg x \wedge y) \vee (\neg w \wedge x \wedge \neg y)$

Q4b. [2 points] Is the set $\{Q, R\}$ **consistent**? *Circle:* **YES** **NO**

If you circled **YES**, give *all* truth assignments that certify your answer:

Q and R are both T for the following truth assignments:

① $w=T, x=F, y=F$

② $w=F, x=T, y=T$

Q4b. [2 points] Is the set $\{P, Q, R\}$ **consistent**? *Circle:* **YES** **NO**

If you circled **YES**, give *all* truth assignments that certify your answer:

Q4d. [3 points] Is the argument $(P \wedge R) \to Q$ a **valid** argument?

Circle: **YES** **NO**

If you circled **NO**, give *all* counterexamples:

All premises are true but the conclusion is F

when ① $w=T, x=T, y=F$

② $w=T, x=F, y=T$

Long-Answer Questions. Detailed solutions are required.

Q5. [5 points] On the Island of Knights & Knaves, you meet an inhabitant A who is either a knight or a knave. A says: "I am a knave or I know where the gold is hidden."

What (if anything) can you conclude? Be as specific as possible about A's type and whether A knows where the gold is hidden. Clearly define any variables that enter into your solution.

Solution using the Truth Table method:
(alternative solution method given in Version B solutions)

Define the Variables:
a: "A is a Knight."
g: "A knows where the gold is hidden."

Translate:
A says: "$\neg a \vee g$"

Table:

a	g	$\neg a \vee g$	
T	T	T	✓ (possible)
T	F	F	↯ a Knight cannot lie.
F	T	T	↯ a Knave cannot tell the truth
F	F	T	↯ a Knave cannot tell the truth

Conclusion:
A is a Knight
and
A knows where the gold is hidden

Q6. [5 points] Using the laws in the **Table of Logical Equivalences** (on Page 10), show that

$$w \to \neg(x \vee y) \equiv (\neg w \vee \neg x) \wedge \neg(w \wedge y)$$

Justify each step by giving the name or number of the corresponding equivalence on Page 9. *Do not skip steps. Do not combine several equivalences into a single step. Do not omit necessary parentheses.*

$$w \to \neg(x \vee y) \equiv \neg w \vee \neg(x \vee y) \quad \text{(implication law \# 1)}$$

$$\equiv \neg w \vee (\neg x \wedge \neg y) \quad \text{(De Morgan's law \# 20)}$$

$$\equiv (\neg w \vee \neg x) \wedge (\neg w \vee \neg y) \quad \text{(distributive law \# 17)}$$

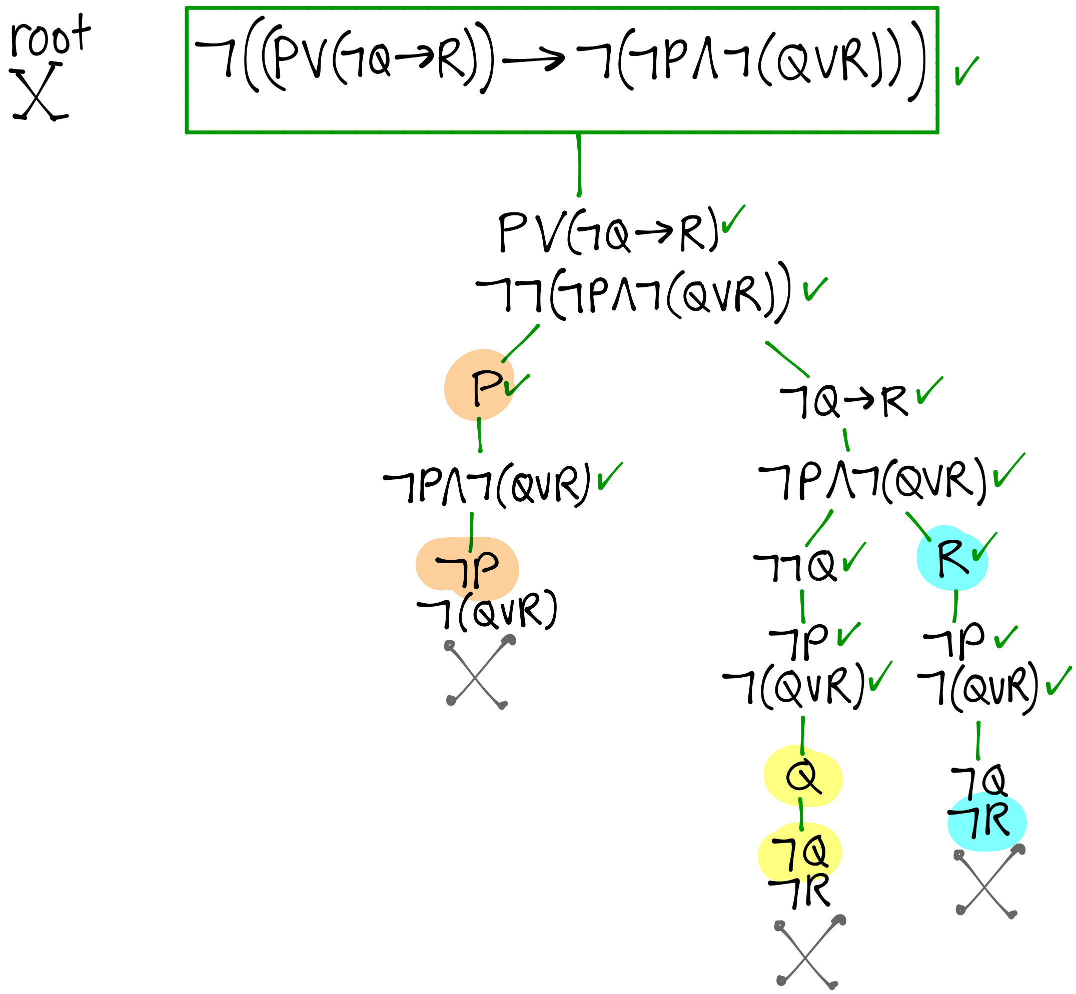
$$\equiv (\neg w \vee \neg x) \wedge \neg(w \wedge y) \quad \text{(De Morgan's Law \# 19)}$$

Q7. [6 points] Using an appropriate truth tree, determine whether or not the compound proposition X, defined as follows, is a **contradiction**.

$$X: \quad \neg\Big(\big(P \vee (\neg Q \to R)\big) \to \neg\big(\neg P \wedge \neg(Q \vee R)\big) \Big)$$

Grow a **complete** truth tree. Clearly label each path as either "active" or "dead". Apply the branching rules to the propositions as written (i.e. do not rewrite a proposition using logical equivalences before branching).

Complete Truth Tree:



Is X a **contradiction**? *Circle:* **YES** **NO**

Briefly explain making reference to your tree and its root.

All paths are dead in the tree with root X

∴ the root X can never be true.

Q8. [6 points] Let d and k be integers.

Recall that $d \mid k$ is notation for "d divides k", which means that k is an integer multiple of d.

Give a **direct proof** of the following theorem:

Theorem 1. Let n be an integer. If $\underbrace{3 \mid n}_{P}$, then $\underbrace{18 \mid (2n^2 + 6n)}_{Q}$.

$P \longrightarrow Q$

☐ For each variable that appears in your proof, you must state what that variable represents.

☐ For each step in your proof, it must be made clear to the reader whether it is an assumption, something you are about to prove, or something that follows from a previous step or a definition.

Direct Proof of Theorem 1.

Let n be an integer.

Assume P is true.

ie Assume 3 divides n.

Then $n = 3j$ for some integer j by def of divides.

$$\begin{aligned} \text{Thus } 2n^2 + 6n &= 2(3j)^2 + 6(3j) \\ &= 2(9j^2) + 18j \\ &= 18j^2 + 18j \\ &= 18(j^2 + j) \\ &= 18m \text{ where } m = j^2 + j \text{ so } m \text{ is an integer.} \end{aligned}$$

$\therefore$ $2n^2 + 6n = 18m$ for some integer m.

$\therefore$ 18 divides $2n^2 + 6n$ by def of divides (ie Q is true).

$\therefore$ We proved $P \rightarrow Q$ is true.

∎

Q9. [6 points] For this question, you will give an **indirect proof** of the following theorem:

Theorem 2. Let n be an integer. If $\underbrace{5n+7 \text{ is odd}}_{P}$, then $\underbrace{n \text{ is even}}_{Q}$.

$P \longrightarrow Q$

Start by writing Theorem 2 in its **contrapositive** form: $P \to Q \equiv \neg Q \to \neg P$

Contrapositive of Theorem 2 (in English):

Let n be an integer. If $\underbrace{n \text{ is odd}}_{\neg Q}$, then $\underbrace{5n+7 \text{ is even}}_{\neg P}$.

$\neg Q \longrightarrow \neg P$

Now give an **indirect proof** of Theorem 2.

- ☐ For each variable that appears in your proof, you must state what that variable represents.
- ☐ For each step in your proof, it must be made clear to the reader whether it is an assumption, something you are about to prove, or something that follows from a previous step or a definition.

Indirect Proof of Theorem 2.

Let n be an integer.

Assume $\neg Q$ is true.

ie Assume n is odd.

Then $n = 2k+1$ for some integer k (def of odd)

Thus
$$\begin{aligned} 5n+7 &= 5(2k+1)+7 \\ &= 10k+5+7 \\ &= 10k+12 \\ &= 2[5k+6] \\ &= 2j \text{ where } j = 5k+6 \text{ hence } j \text{ is an integer.} \end{aligned}$$

$\therefore$ $5n+7 = 2j$ for some integer j.

$\therefore$ $5n+7$ is even (by def of even). ie $\neg P$ is true.

$\therefore$ we proved $\neg Q \to \neg P$ which is $\equiv P \to Q$. $\therefore$ $P \to Q$ is true. ∎

Table of Logical Equivalences

	Equivalence	Name
1.	$P \to Q \equiv \neg P \lor Q$	Implication Law
2.	$P \leftrightarrow Q \equiv (P \land Q) \lor (\neg P \land \neg Q)$	Biconditional Laws
3.	$P \leftrightarrow Q \equiv (P \to Q) \land (Q \to P)$	
4.	$P \lor \neg P \equiv \mathbf{T}$	Negation Laws
5.	$P \land \neg P \equiv \mathbf{F}$	
6.	$P \lor \mathbf{F} \equiv P$	Identity Laws
7.	$P \land \mathbf{T} \equiv P$	
8.	$P \lor \mathbf{T} \equiv \mathbf{T}$	Domination Laws
9.	$P \land \mathbf{F} \equiv \mathbf{F}$	
10.	$P \lor P \equiv P$	Idempotent Laws
11.	$P \land P \equiv P$	
12.	$\neg\neg P \equiv P$	Double Negation Law
13.	$P \lor Q \equiv Q \lor P$	Commutative Laws
14.	$P \land Q \equiv Q \land P$	
15.	$(P \lor Q) \lor R \equiv P \lor (Q \lor R)$	Associative Laws
16.	$(P \land Q) \land R \equiv P \land (Q \land R)$	
17.	$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$	Distributive Laws
18.	$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$	
19.	$\neg(P \land Q) \equiv \neg P \lor \neg Q$	De Morgan's Laws
20.	$\neg(P \lor Q) \equiv \neg P \land \neg Q$	

(end of Midterm 1) — you may detach this page