20. Binomial Coefficients & The Binomial Theorem

Recall: $C(n,r) = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$C(n,r)$: # of r-combinations (ie # of r-element subsets) of an n-element set

$\binom{n}{r}$: called a "binomial coefficient" read "n choose r"

Simple Observations on Binomial Coefficients

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{(1)(n)!} = 1 \qquad \binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n\cdot(n-1)!}{(1)(n-1)!} = n$$

For any $r \in \{0, 1, \ldots, n\}$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \binom{n}{n-r}$$

Thus,

$$\binom{n}{n} = \binom{n}{n-n} = \binom{n}{0} = 1 \qquad \binom{n}{n-1} = \binom{n}{n-(n-1)} = \binom{n}{1} = n$$

PASCAL'S IDENTITY & PASCAL'S TRIANGLE

Theorem 20.1. (PASCAL'S IDENTITY) Let n and k be positive integers with $n \geq k$. Then

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof of Pascal's Identity.

$$LS = \binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!}$$

$$= \frac{n!}{k!(n-k)(n-k-1)!} + \frac{n!}{(k+1)\cdot k!\cdot(n-k-1)!}$$

* These notes are solely for the personal use of students registered in MAT1348.

$$= \frac{n!\,(k+1)}{(k+1)k!\,(n-k)(n-k-1)!} + \frac{n!\,(n-k)}{(k+1)\cdot k!\,(n-k)(n-k-1)!}$$

$$= \frac{n!\left[(k+1)+(n-k)\right]}{(k+1)k!\,(n-k)(n-k-1)!} = \frac{n!\,(n+1)}{(k+1)!\,(n-k)!}$$

$$= \frac{(n+1)!}{(k+1)!\,(n+1-(k+1))!} = \binom{n+1}{k+1} = RS$$

Pascal's Triangle	**(in terms of binomial coefficients)**
$n=0$	$\binom{0}{0}$
$n=1$	$\binom{1}{0}$ $\binom{1}{1}$
$n=2$	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
$n=3$	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
$n=4$	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$\vdots$

Pascal's Triangle	**(with evaluated coefficients)**
$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
$n=7$	1 7 21 35 35 21 7 1

$\vdots$

Row Sums of Pascal's Triangle

row #		row sum
$n = 0$	1	$= 1$
$n = 1$	$1 + 1$	$= 2$
$n = 2$	$1 + 2 + 1$	$= 4$
$n = 3$	$1 + 3 + 3 + 1$	$= 8$

Theorem 20.2. For all integers $n \geq 0$,

$$\sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

Proof of Theorem 20.2. (by induction)

⋆ For each $n \in \mathbb{N}$, let $P(n)$: " $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$ "

⋆⋆ B.I. $n_0 = 0$

$P(0)$ says " $\binom{0}{0} = 2^0$ "

$LS = \binom{0}{0} = 1$ $\quad RS = 2^0 = 1$ $\quad \therefore P(0)$ is true.

⋆⋆⋆ I.S. Let k be an integer such that $k \geq n_0 = 0$.

We must prove $P(k) \longrightarrow P(k+1)$.

⋆⋆⋆⋆ I.H. Assume $P(k)$ is true:

ie assume $\boxed{\binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k} = 2^k}$ ← Induction Hypothesis

(goal: prove $P(k+1)$ follows from $P(k)$)

$P(k+1)$ says $\binom{k+1}{0} + \binom{k+1}{1} + \dots + \binom{k+1}{k+1} = 2^{k+1}$

RS of $P(k+1) = 2^{k+1}$

$$= 2^k + 2^k \quad \text{(since } 2^{k+1} = 2\cdot 2^k = 2^k + 2^k\text{)}$$

$$= \underbrace{\binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k}} + \underbrace{\binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k}} \quad \text{(using I.H. twice!)}$$

$$= \binom{k}{0} + \underbrace{\binom{k}{0} + \binom{k}{1}} + \underbrace{\binom{k}{1} + \binom{k}{2}} + \dots + \underbrace{\binom{k}{k-1} + \binom{k}{k}} + \binom{k}{k} \quad \text{(rearranging so that like terms are side-by-side)}$$

$$= \binom{k}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \dots + \binom{k+1}{k} + \binom{k}{k} \quad \text{(using Pascal's Identity } k \text{ times!)}$$

$$= \underbrace{\binom{k+1}{0}} + \binom{k+1}{1} + \binom{k+1}{2} + \dots + \binom{k+1}{k} + \underbrace{\binom{k+1}{k+1}}$$

since $\binom{k}{0} = 1 = \binom{k+1}{0}$ and $\binom{k}{k} = 1 = \binom{k+1}{k+1}$

$= \text{LS of } P(k+1)$ $\qquad \therefore P(k) \to P(k+1)$ is true.

⋆⋆⋆⋆⋆ Conclusion Since $P(0)$ is true and since we proved $P(k) \to P(k+1)$, it follows from Mathematical Induction that $P(n)$ is true for all integers $n \geq 0$.

Another Proof of Theorem 20.2. Let S be an n-element set.

Then

(1) $\binom{\#\text{ of subsets}}{\text{of } S} = |\mathcal{P}(S)| = 2^{|S|} = 2^n$

Also,

(2) $\binom{\#\text{ of subsets}}{\text{of } S} = \binom{\#\text{ of 0-element}}{\text{subsets of } S} + \binom{\#\text{ of 1-element}}{\text{subsets of } S} + \dots + \binom{\#\text{ of } n\text{-element}}{\text{subsets of } S}$

$$= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

Since (1) $= \binom{\#\text{ of subsets}}{\text{of } S} =$ (2), Theorem 21.2 is true.

Theorem 20.3. (THE BINOMIAL THEOREM) Let x and y be variables, and let $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^{i}$$

$$= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}x^0 y^n$$

Ex. $(x+y)^2 = (x+y)(x+y)$

$= xx + xy + yx + yy$

$= x^2 + 2xy + y^2$

← from each of these two factors, either x or y must contribute to one of the final terms in the expansion.

Example 20.4. Fully evaluate $\left(2 - \frac{1}{x}\right)^3$ first from scratch, then using the Binomial Theorem.

$$\left(2-\frac{1}{x}\right)^3 = \left(2-\frac{1}{x}\right)\left(2-\frac{1}{x}\right)\left(2-\frac{1}{x}\right)$$

$$= \left(4 - \frac{2}{x} - \frac{2}{x} + \frac{1}{x^2}\right)\left(2 - \frac{1}{x}\right)$$

$$= 8 - \frac{4}{x} - \frac{4}{x} + \frac{2}{x^2} - \frac{4}{x} + \frac{2}{x^2} + \frac{2}{x^2} - \frac{1}{x^3}$$

$$= 8 - \frac{12}{x} + \frac{6}{x^2} - \frac{1}{x^3}$$

$$\left(2-\frac{1}{x}\right)^3 = \sum_{i=0}^{3} \binom{3}{i} 2^{3-i} \cdot \left(-\frac{1}{x}\right)^i$$

$$= \binom{3}{0}2^3 \cdot \left(-\frac{1}{x}\right)^0 + \binom{3}{1} \cdot 2^2 \cdot \left(-\frac{1}{x}\right)^1 + \binom{3}{2} 2^1 \cdot \left(-\frac{1}{x}\right)^2 + \binom{3}{3} \cdot 2^0 \cdot \left(-\frac{1}{x}\right)^3$$

$$= (1)(8)(1) + (3)(4)\left(-\frac{1}{x}\right) + (3)(2)\left(\frac{1}{x^2}\right) + (1)(1)\left(-\frac{1}{x^3}\right)$$

$$= 8 + \left(-\frac{12}{x}\right) + \frac{6}{x^2} + \left(-\frac{1}{x^3}\right)$$

Example 20.5. Find the coefficients of $x^{12}y^{17}$ and $x^{13}y^{16}$ in the expansion of $(3x^2-5y)^{23}$

$$(3x^2-5y)^{23} = \sum_{i=0}^{23} \binom{23}{i} (3x^2)^{23-i} (-5y)^i$$

$$= \sum_{i=0}^{23} \binom{23}{i} \cdot 3^{23-i} \cdot (x^2)^{23-i} (-5)^i \cdot y^i$$

$$= \sum_{i=0}^{23} \underbrace{\binom{23}{i} \cdot 3^{23-i} \cdot (-5)^i}_{} \cdot x^{46-2i} \cdot y^i$$

for each $i \in \{0,1,\dots,23\}$, this is the coefficient of the term $x^{46-2i} \cdot y^i$

for the coefficient of the term $x^{12}y^{17}$, we need the index i such that

$x^{46-2i} \cdot y^i = x^{12} \cdot y^{17}$ thus $\begin{cases} 46-2i=12 \iff i=17 \\ i=17 \iff i=17 \end{cases}$ ← there is a solution that works for the exponents of both x and y, namely $i=17$

$\therefore$ the coefficient of $x^{12}y^{17}$ is $\binom{23}{17} 3^6 \cdot (-5)^{17}$ [plug in $i=17$ to $\binom{23}{i} 3^{23-i} \cdot (-5)^i$]

for the coefficient of the term $x^{13}y^{16}$, we need the index i such that

$x^{46-2i} \cdot y^i = x^{13} \cdot y^{16}$ thus $\begin{cases} 46-2i=13 \iff i=33/2 \\ i=16 \iff i=16 \end{cases}$ ← there is no solution that works for both x and y's exponents (even worse, the solution for x's exponent alone is not an integer...)

$\therefore$ $x^{13}y^{16}$ does not ever appear in the expansion of $(3x^2-5y)^{23}$

ie the coeffient of $x^{13}y^{16}$ is zero.

Study Guide

Important terms and concepts:

- ◇ binomial coefficient $\binom{n}{k}$
- ◇ Pascal's Identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$
- ◇ Row sums of Pascal's Triangle $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$
- ◇ The Binomial Theorem $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$
- ◇ coefficient of a specified term in the expansion of $(x+y)^n$

Supp. Exercise List (on Brightspace)	§10 # 3, 5
Rosen textbook	§6.4 # 3, 7, 9, 11, 19, 21