Exercises on independence, basis, and dimension

Problem 9.1: (3.5 #2. *Introduction to Linear Algebra:* Strang) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution: Since $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$, the vectors $\mathbf{v}_4, \mathbf{v}_5$, and $\mathbf{v}_6$ are dependent on the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. To determine the relationship between the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ we apply row reduction to the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there are three pivots, the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are independent. Therefore the largest number of independent vectors among the given six vectors is **three.** This will be the rank of the 4 by 6 matrix of $\mathbf{v}$'s.

Problem 9.2: (3.5 #20.) Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

Solution: This plane is the nullspace of the matrix $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$ and also

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The special solutions to $A\mathbf{x} = \mathbf{0}$ are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

These form a basis for the nullspace of A and thus for the plane.

The intersection of this plane with the xy plane contains $\mathbf{v}_1$ and does not contain $\mathbf{v}_2$; the intersection must be a line. Since $\mathbf{v_1}$ lies on this line it also provides a basis for it.

Finally, we can use "inspection" or the cross product to find the vector

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix},$$

which is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. It is therefore perpendicular to the plane. Since the space of vectors perpendicular to a plane in $\mathbb{R}^3$ is one-dimensional, $\mathbf{v}_3$ serves as a basis for that space.