1 Dynamical Systems

1.1 Introduction

We start this book at the very beginning, by asking ourselves the question, *What is a dynamical system?*

Disregarding for a moment the dynamical aspects—forgetting about time—we are immediately led to ponder the more basic issue, *What is a mathematical model?* What does it tell us? What is its mathematical nature? Mind you, we are not asking a philosophical question: we will not engage in an erudite discourse about the relation between reality and its mathematical description. Neither are we going to elucidate the methodology involved in actually deriving, setting up, postulating mathematical models. What we are asking is the simple question, *When we accept a mathematical expression, a formula, as an adequate description of a phenomenon, what mathematical structure have we obtained?*

We view a mathematical model as an *exclusion law*. A mathematical model expresses the opinion that some things can happen, are possible, while others cannot, are declared impossible. Thus Kepler claims that planetary orbits that do not satisfy his three famous laws are impossible. In particular, he judges nonelliptical orbits as unphysical. The second law of thermodynamics limits the transformation of heat into mechanical work. Certain combinations of heat, work, and temperature histories are declared to be impossible. Economic production functions tell us that certain amounts of raw materials, capital, and labor are needed in order to manufacture a finished product: it prohibits the creation of finished products unless the required resources are available.

We formalize these ideas by stating that a mathematical model selects a certain subset from a universum of possibilities. This subset consists of the occurrences that the model allows, that it declares possible. We call the subset in question the *behavior* of the mathematical model.

True, we have been trained to think of mathematical models in terms of equations. *How do equations enter this picture?* Simply, an equation can be viewed as a law excluding the occurrence of certain outcomes, namely, those combinations of variables for which the equations are not satisfied. This way, equations define a behavior. We therefore speak of *behavioral equations* when mathematical equations are intended to model a phenomenon. It is important to emphasize already at this point that behavioral equations provide an effective, but at the same time highly nonunique, way of specifying a behavior. Different equations can define the same mathematical model. One should therefore not exaggerate the intrinsic significance of a specific set of behavioral equations.

In addition to behavioral equations and the behavior of a mathematical model, there is a third concept that enters our modeling language *ab initio: latent variables*. We think of the variables that we try to model as *manifest* variables: they are the attributes at which the modeler in principle focuses attention. However, in order to come up with a mathematical model for a phenomenon, one invariably has to consider other, *auxiliary*, variables. We call them *latent* variables. These may be introduced for no other reason than in order to express in a convenient way the laws governing a model. For example, when modeling the behavior of a complex system, it may be convenient to view it as an interconnection of component subsystems. Of course, the variables describing these subsystems are, in general, different from those describing the original system. When modeling the external terminal behavior of an electrical circuit, we usually need to introduce the currents and voltages in the internal branches as auxiliary variables. When expressing the first and second laws of thermodynamics, it has been proven convenient to introduce the internal energy and entropy as latent variables. When discussing the synthesis of feedback control laws, it is often imperative to consider models that display their internal state explicitly. We think of these internal variables as latent variables. Thus in first principles modeling, we distinguish two types of variables. The terminology first principles modeling refers to the fact that the physical laws that play a role in the system at hand are the elementary laws from physics, mechanics, electrical circuits, etc.

This triptych—*behavior/behavioral equations/manifest and latent variables*—is the essential structure of our modeling language. The fact that we take the behavior, and not the behavioral equations, as the central object specifying a mathematical model has the consequence that basic system properties (such as time-invariance, linearity, stability, controllability, observability) will also refer to the behavior. The subsequent problem then always arises how to deduce these properties from the behavioral equations.

1.2 Models

1.2.1 The universum and the behavior

Assume that we have a phenomenon that we want to model. To start with, we cast the situation in the language of mathematics by assuming that the *phenomenon* produces outcomes in a set $\mathbb{U}$, which we call the *universum*. Often $\mathbb{U}$ consists of a product space, for example a finite dimensional vector space. Now, a (deterministic) mathematical model for the phenomenon (viewed purely from the black-box point of view, that is, by looking at the phenomenon only from its terminals, by looking at the model as descriptive but not explanatory) claims that certain outcomes are possible, while others are not. Hence a model recognizes a certain subset $\mathfrak{B}$ of $\mathbb{U}$. This subset is called the *behavior* (of the model). Formally:

Definition 1.2.1 A *mathematical model* is a pair $(\mathbb{U}, \mathfrak{B})$ with $\mathbb{U}$ a set, called the *universum*—its elements are called *outcomes*—and $\mathfrak{B}$ a subset of $\mathbb{U}$, called the *behavior*. □

Example 1.2.2 During the ice age, shortly after Prometheus stole fire from the gods, man realized that H_2O could appear, depending on the temperature, as liquid water, steam, or ice. It took a while longer before this situation was captured in a mathematical model. The generally accepted model, with the temperature in degrees Celsius, is $\mathbb{U} = \{\text{ice, water, steam}\} \times [-273, \infty)$ and $\mathfrak{B} = ((\{\text{ice}\} \times [-273, 0]) \cup (\{\text{water}\} \times [0, 100]) \cup (\{\text{steam}\} \times [100, \infty))$. □

Example 1.2.3 Economists believe that there exists a relation between the amount P produced of a particular economic resource, the capital K invested in the necessary infrastructure, and the labor L expended towards its production. A typical model looks like $\mathbb{U} = \mathbb{R}_+^3$ and $\mathfrak{B} = \{(P, K, L) \in \mathbb{R}_+^3 \mid P = F(K, L)\}$, where $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the *production function*. Typically, $F : (K, L) \mapsto \alpha K^\beta L^\gamma$, with $\alpha, \beta, \gamma \in \mathbb{R}_+$, $0 \le \beta \le 1$, $0 \le \gamma \le 1$, constant parameters depending on the production process, for example the type of technology used. Before we modeled the situation, we were ready to believe that every triple $(P, K, L) \in \mathbb{R}_+^3$ could occur. After introduction of the production function, we limit these possibilities to the triples satisfying $F = \alpha K^\beta L^\gamma$. The subset of $\mathbb{R}_+^3$ obtained this way is the behavior in the example under consideration. □

1.2.2 Behavioral equations

In applications, models are often described by equations (see Example 1.2.3). Thus the behavior consists of those elements in the universum for which "balance" equations are satisfied.

Definition 1.2.4 Let $\mathbb{U}$ be a universum, $\mathbb{E}$ a set, and $f_1, f_2 : \mathbb{U} \to \mathbb{E}$. The mathematical model $(\mathbb{U}, \mathfrak{B})$ with $\mathfrak{B} = \{u \in \mathbb{U} \mid f_1(u) = f_2(u)\}$ is said to be described by *behavioral equations* and is denoted by $(\mathbb{U}, \mathbb{E}, f_1, f_2)$. The set $\mathbb{E}$ is called the *equating space*. We also call $(\mathbb{U}, \mathbb{E}, f_1, f_2)$ a *behavioral equation representation* of $(\mathbb{U}, \mathfrak{B})$. □

Often, an appropriate way of looking at $f_1(u) = f_2(u)$ is as *equilibrium conditions:* the behavior $\mathfrak{B}$ consists of those outcomes for which two (sets of) quantities are in balance.

Example 1.2.5 Consider an electrical resistor. We may view this as imposing a relation between the voltage V across the resistor and the current I through it. Ohm recognized more than a century ago that (for metal wires) the voltage is proportional to the current: $V = RI$, with the proportionality factor R called the resistance. This yields a mathematical model with universum $\mathbb{U} = \mathbb{R}^2$ and behavior $\mathfrak{B}$, induced by the behavioral equation $V = RI$. Here $\mathbb{E} = \mathbb{R}$, $f_1 : (V, I) \mapsto V$, and $f_2(V, I) : I \mapsto RI$. Thus $\mathfrak{B} = \{(I, V) \in \mathbb{R}^2 \mid V = RI\}$.

Of course, nowadays we know many devices imposing much more complicated relations between V and I, which we nevertheless choose to call (non-Ohmic) resistors. An example is an (ideal) diode, given by the (I, V) characteristic $\mathfrak{B} = \{(I, V) \in \mathbb{R}^2 \mid (V \geq 0 \text{ and } I = 0) \text{ or } (V = 0 \text{ and } I \leq 0\}$. Other resistors may exhibit even more complex behavior, due to hysteresis, for example. □

Example 1.2.6 Three hundred years ago, Sir Isaac Newton discovered (better: deduced from Kepler's laws since, as he put it, *Hypotheses non fingo*) that masses attract each other according to the inverse square law. Let us formalize what this says about the relation between the force F and the position vector q of the mass m. We assume that the other mass M is located at the origin of $\mathbb{R}^3$. The universum $\mathbb{U}$ consists of all conceivable force/position vectors, yielding $\mathbb{U} = \mathbb{R}^3 \times \mathbb{R}^3$. After Newton told us the behavioral equations $F = -k\frac{mMq}{\|q\|^3}$, we knew more: $\mathfrak{B} = \{(F, q) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid F = -k\frac{mMq}{\|q\|^3}\}$, with k the gravitational constant, $k = 6.67 \times 10^{-8}$ cm^3/g.sec^2. Note that $\mathfrak{B}$ has three degrees of freedom–down three from the six degrees of freedom in $\mathbb{U}$. □

In many applications models are described by *behavioral inequalities*. It is easy to accommodate this situation in our setup. Simply take in the above definition $\mathbb{E}$ to be an ordered space and consider the behavioral inequality $f_1(u) \leq f_2(u)$. Many models in operations research (e.g., in linear programming) and in economics are of this nature. In this book we will not pursue models described by inequalities.

Note further that whereas behavioral equations specify the behavior uniquely, the converse is obviously not true. Clearly, if $f_1(u) = f_2(u)$ is a set of behavioral equations for a certain phenomenon and if $f : \mathbb{E} \to \mathbb{E}'$ is any bijection, then the set of behavioral equations $(f \circ f_1)(u) = (f \circ f_2)(u)$ form another set of behavioral equations yielding the same mathematical model. Since we have a tendency to think of mathematical models in terms of behavioral equations, most models being presented in this form, it is important to emphasize their ancillary role: *it is the behavior, the solution set of the behavioral equations, not the behavioral equations themselves, that is the essential result of a modeling procedure.*

1.2.3 *Latent variables*

Our view of a mathematical model as expressed in Definition 1.2.1 is as follows: identify the outcomes of the phenomenon that we want to model (specify the universum $\mathbb{U}$) and identify the behavior (specify $\mathfrak{B} \subseteq \mathbb{U}$). However, in most modeling exercises we need to introduce other variables in addition to the attributes in $\mathbb{U}$ that we try to model. We call these other, auxiliary, variables *latent variables*. In a bit, we will give a series of instances where latent variables appear. Let us start with two concrete examples.

Example 1.2.7 Consider a one-port resistive electrical circuit. This consists of a graph with nodes and branches. Each of the branches contains a resistor, except one, which is an external port. An example is shown in Figure 1.1. Assume that we want to model the port behavior, the relation between the voltage drop across and the current through the external port. Introduce as auxiliary variables the voltages $(V_1, \ldots, V_5)$ across and the currents $(I_1, \ldots, I_5)$ through the internal branches, numbered in the obvious way as indicated in Figure 1.1. The following relations must be satisfied:

- *Kirchhoff's current law:* the sum of the currents entering each node must be zero;
- *Kirchhoff's voltage law:* the sum of the voltage drops across the branches of any loop must be zero;
- The *constitutive laws of the resistors* in the branches.

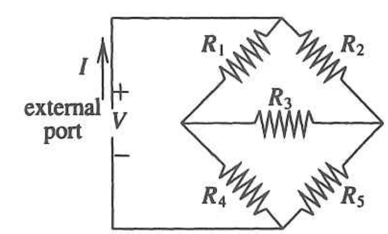

FIGURE 1.1. Electrical circuit with resistors only.

These yield:

Constitution laws	*Kirchhoff's current laws*	*Kirchhoff's voltage laws*
$R_1I_1 = V_1,$	$I = I_1 + I_2,$	$V_1 + V_4 = V,$
$R_2I_2 = V_2,$	$I_1 = I_3 + I_4,$	$V_2 + V_5 = V,$
$R_3I_3 = V_3,$	$I_5 = I_2 + I_3,$	$V_1 + V_4 = V_2 + V_5,$
$R_4I_4 = V_4,$	$I = I_4 + I_5,$	$V_1 + V_3 = V_2,$
$R_5I_5 = V_5,$		$V_3 + V_5 = V_4.$

Our basic purpose is to express the relation between the voltage across and current into the external port. In the above example, this is a relation of the form $V = RI$ (where R can be calculated from R_1, R_2, R_3, R_4, and R_5), obtained by eliminating $(V_1, \ldots, V_5, I_1, \ldots, I_5)$ from the above equations. However, the basic model, the one obtained from *first principles*, involves the variables $(V_1, \ldots, V_5, I_1, \ldots, I_5)$ in addition to the variables (V, I) whose behavior we are trying to describe. The node voltages and the currents through the internal branches (the variables $(V_1, \ldots, V_5, I_1, \ldots, I_5)$ in the above example) are thus latent variables. The port variables (V, I) are the manifest variables. □

Example 1.2.8 An economist is trying to figure out how much of a package of n economic goods will be produced. As a firm believer in equilibrium theory, our economist assumes that the production volumes consist of those points where, product for product, the supply equals the demand. This equilibrium set is a subset of $\mathbb{R}_+^n$. It is the behavior that we are looking for. In order to specify this set, we can proceed as follows. Introduce as latent variables the price, the supply, and the demand of each of the n products. Next determine, using economic theory or experimentation, the supply and demand functions $S_i : \mathbb{R}_+^n \to \mathbb{R}_+$ and $D_i : \mathbb{R}_+^n \to \mathbb{R}_+$. Thus $S_i(p_1, p_2, \ldots, p_n)$ and $D_i(p_1, p_2, \ldots, p_n)$ are equal to the amount of product i that is bought and produced when the going market prices are $p_1, p_2, \ldots, p_n$. This yields the behavioral equations

$$\begin{aligned} s_i &= S_i(p_1, p_2, \ldots, p_n), \\ d_i &= D_i(p_1, p_2, \ldots, p_n), \\ s_i &= d_i = P_i, \quad i = 1, 2, \ldots, n. \end{aligned}$$

These behavioral equations describe the relation between the prices p_i, the supplies s_i, the demands d_i, and the production volumes P_i. The P_is for which these equations are solvable yield the desired behavior. Clearly, this behavior is most conveniently specified in terms of the above equations, that is, in terms of the behavior of the variables p_i, s_i, d_i, and $P_i (i = 1, 2, \ldots, n)$ jointly. The manifest behavioral equations would consist of an equation involving $P_1, P_2, \ldots, P_n$ only. □

These examples illustrate the following definition.

Definition 1.2.9 A *mathematical model with latent variables* is defined as a triple $(\mathbb{U}, \mathbb{U}_\ell, \mathfrak{B}_f)$ with $\mathbb{U}$ the *universum* of manifest variables, $\mathbb{U}_\ell$ the universum of *latent variables*, and $\mathfrak{B}_f \subseteq \mathbb{U} \times \mathbb{U}_\ell$ the *full behavior*. It defines the *manifest mathematical model* $(\mathbb{U}, \mathfrak{B})$ with $\mathfrak{B} := \{u \in \mathbb{U} \mid \exists \ell \in \mathbb{U}_\ell \text{ such that } (u, \ell) \in \mathfrak{B}_f\}$; $\mathfrak{B}$ is called the *manifest behavior* (or the *external behavior*) or simply the *behavior*. We call $(\mathbb{U}, \mathbb{U}_\ell, \mathfrak{B}_f)$ *a latent variable representation* of $(\mathbb{U}, \mathfrak{B})$. □

Note that in our framework we view the attributes in $\mathbb{U}$ as those variables that the model aims at describing. We think of these variables as *manifest*, as *external*. We think of the latent variables as auxiliary variables, as *internal*. In pondering about the difference between manifest variables and latent variables it is helpful *in the first instance* to think of the signal variables being directly *measurable;* they are *explicit*, while the latent variables are not: they are *implicit*, unobservable, or—better—only indirectly observable through the manifest variables. Examples: in pedagogy, scores of tests can be viewed as manifest, and native or emotional intelligence can be viewed as a latent variable aimed at explaining these scores. In thermodynamics, pressure, temperature, and volume can be viewed as manifest variables, while the internal energy and entropy can be viewed as latent variables. In economics, sales can be viewed as manifest, while consumer demand could be considered as a latent variable. We emphasize, however, that which variables are observed and measured through sensors, and which are not, is something that is really part of the instrumentation and the technological setup of a system. Particularly, in control applications one should not be nonchalant about declaring certain variables measurable and observed. Therefore, we will not further encourage the point of view that identifies *manifest* with *observable*, and *latent* with *unobservable*.

Situations in which basic models use latent variables either for mathematical reasons or in order to express the basic laws occur very frequently. Let us mention a few: *internal voltages* and *currents* in electrical circuits in order to express the external port behavior; *momentum* in Hamiltonian mechanics in order to describe the evolution of the position; *internal energy* and *entropy* in thermodynamics in order to formulate laws restricting the evolution of the temperature and the exchange of heat and mechanical work; *prices* in economics in order to explain the production and exchange of economic goods; *state variables* in system theory in order to express the memory of a dynamical system; the *wave function* in quantum mechanics underlying observables; and finally, the *basic probability space* Ω in probability theory: the big latent variable space in the sky, our example of a latent variable space *par excellence*.

Latent variables invariably appear whenever we model a system by the method of *tearing* and *zooming*. The system is viewed as an interconnection of subsystems, and the modeling process is carried out by *zooming* in on the individual subsystems. The overall model is then obtained by combining the models of the subsystems with the interconnection constraints. This ultimate model invariably contains latent variables: the auxiliary variables introduced in order to express the interconnections play this role.

Of course, equations can also be used to express the full behavior $\mathfrak{B}_f$ of a latent variable model (see Examples 1.2.7 and 1.2.8). We then speak of *full behavioral equations*.

1.3 Dynamical Systems

We now apply the ideas of Section 1.2 in order to set up a language for dynamical systems. The adjective *dynamical* refers to phenomena with a *delayed reaction*, phenomena with an *aftereffect*, with *transients*, *oscillations*, and, perhaps, an approach to *equilibrium*. In short, phenomena in which the *time evolution* is one of the crucial feature. We view a dynamical system in the logical context of Definition 1.2.1 simply as a mathematical model, but a mathematical model in which the objects of interest are functions of time: the universum is a function space. We take the point of view that a dynamical system constrains the time signals that the system can conceivably produce. The collection of all the signals compatible with these laws defines what we call the *behavior* of the dynamical system. This yields the following definition.

1.3.1 The basic concept

Definition 1.3.1 A *dynamical system* Σ is defined as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}),$$

with $\mathbb{T}$ a subset of $\mathbb{R}$, called the *time axis*, $\mathbb{W}$ a set called the *signal space*, and $\mathfrak{B}$ a subset of $\mathbb{W}^{\mathbb{T}}$ called the *behavior* ($\mathbb{W}^{\mathbb{T}}$ is standard mathematical notation for the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$). □

The above definition will be used as a *leitmotiv* throughout this book. The set $\mathbb{T}$ specifies the set of time instances relevant to our problem. Usually $\mathbb{T}$ equals $\mathbb{R}$ or $\mathbb{R}_+$ (in *continuous-time systems*), $\mathbb{Z}$ or $\mathbb{Z}_+$ (in *discrete-time systems*), or, more generally, an interval in $\mathbb{R}$ or $\mathbb{Z}$.

The set $\mathbb{W}$ specifies the way in which the outcomes of the signals produced by the dynamical system are formalized as elements of a set. These outcomes are the variables whose evolution in time we are describing. In what are called *lumped systems*, systems with a few well-defined simple components each with a finite number of degrees of freedom, $\mathbb{W}$ is usually a finite-dimensional vector space. Typical examples are electrical circuits and mass–spring–damper mechanical systems. In this book we consider almost exclusively lumped systems. They are of paramount importance in engineering, physics, and economics. In *distributed systems*, $\mathbb{W}$ is often an infinite-dimensional vector space. For example, the deformation of flexible bodies or the evolution of heat in media are typically described by partial differential equations that lead to an infinite-dimensional function space $\mathbb{W}$. In areas such as digital communication and computer science, signal spaces $\mathbb{W}$ that are finite sets play an important role. When $\mathbb{W}$ is a finite set, the term *discrete-event systems* is often used.

In Definition 1.3.1 the behavior $\mathfrak{B}$ is simply a family of time trajectories taking their values in the signal space. Thus elements of $\mathfrak{B}$ constitute precisely the trajectories compatible with the laws that govern the system: $\mathfrak{B}$ consists of all time signals which—according to the model—can conceivably occur, are compatible with the laws governing Σ, while those outside $\mathfrak{B}$ cannot occur, are prohibited. The behavior is hence the essential feature of a dynamical system.

Example 1.3.2 According to Kepler, the motion of planets in the solar system obeys three laws:

(K.1) planets move in elliptical orbits with the sun at one of the foci;

(K.2) the radius vector from the sun to the planet sweeps out equal areas in equal times;

(K.3) the square of the period of revolution is proportional to the third power of the major axis of the ellipse.

If a definition is to show proper respect and do justice to history, Kepler's laws should provide the very first example of a dynamical system. They do. Take $\mathbb{T} = \mathbb{R}$ (disregarding biblical considerations and modern cosmology: we assume that the planets have always been there, rotating, and will always rotate), $\mathbb{W} = \mathbb{R}^3$ (the position space of the planets), and $\mathfrak{B} = \{w : \mathbb{R} \to \mathbb{R}^3 \mid \text{Kepler's laws are satisfied}\}$. Thus the behavior $\mathfrak{B}$ in this example consists of the *planetary motions* that, according to Kepler, are possible, all trajectories mapping the time-axis $\mathbb{R}$ into $\mathbb{R}^3$ that satisfy his three famous laws. Since for a given trajectory $w : \mathbb{R} \to \mathbb{R}^3$ one can unambiguously decide whether or not it satisfies Kepler's laws, $\mathfrak{B}$ is indeed well-defined. Kepler's laws form a beautiful example of a dynamical system in the sense of our definition, since it is one of the few instances in which $\mathfrak{B}$ can be described explicitly, and not indirectly through differential equations. It took no lesser man than Newton to think up appropriate behavioral differential equations for this dynamical system. □

Example 1.3.3 Let us consider the motion of a particle in a *potential field* subject to an external force. The purpose of the model is to relate the position q of the particle in $\mathbb{R}^3$ to the external force F. Thus $\mathbb{W}$, the signal space, equals $\mathbb{R}^3 \times \mathbb{R}^3$: three components for the position q, three for the force F. Let $V\colon \mathbb{R}^3 \to \mathbb{R}$ denote the potential field. Then the trajectories (q, F), which, according to the laws of mechanics, are possible, are those that satisfy the differential equation

$$m\frac{d^2q}{dt^2} + V'(q) = F,$$

where m denotes the mass of the particle and V' the gradient of V. Formalizing this model as a dynamical system yields $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3$, and $\mathfrak{B} = \{(q, F) \mid \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \mid m\frac{d^2q}{dt^2} + V'(q) = F\}$. □

1.3.2 *Latent variables in dynamical systems*

The definition of a *latent variable model* is easily generalized to dynamical systems.

Definition 1.3.4 A *dynamical system with latent variables* is defined as $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_f)$ with $\mathbb{T} \subseteq \mathbb{R}$ the *time-axis*, $\mathbb{W}$ the *(manifest) signal space*, $\mathbb{L}$ the *latent variable space*, and $\mathfrak{B}_f \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ the *full behavior*. It defines a *latent variable representation* of the *manifest dynamical system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with *(manifest) behavior* $\mathfrak{B} := \{w : \mathbb{T} \to \mathbb{W} \mid \exists\, \ell : \mathbb{T} \to \mathbb{L}$ such that $(w, \ell) \in \mathfrak{B}_f\}$. □

Sometimes we will refer to the full behavior as the *internal* behavior and to the manifest behavior as the *external* behavior. Note that in a dynamical system with latent variables each trajectory in the full behavior $\mathfrak{B}_f$ consists of a pair (w, ℓ) with $w : \mathbb{T} \to \mathbb{W}$ and $\ell : \mathbb{T} \to \mathbb{L}$. The *manifest* signal w is the one that we are really interested in. The *latent variable* signal ℓ in a sense "supports" w. If $(w, \ell) \in \mathfrak{B}_f$, then w is a possible manifest variable trajectory since ℓ can occur simultaneously with w.

Let us now look at two typical examples of how dynamical models are constructed from first principles. We will see that latent variables are unavoidably introduced in the process. Thus, whereas Definition 1.3.1 is a good concept as a basic notion of a dynamical system, typical models will involve additional variables to those whose behavior we wish to model.

Example 1.3.5 Our first example considers the port behavior of the electrical circuit shown in Figure 1.2. We assume that the elements R_C, R_L, L, and C all have positive values. The circuit interacts with its environment through the external port. The variables that describe this interaction are the current I into the circuit and the voltage V across its external terminals. These are the *manifest* variables. Hence $\mathbb{W} = \mathbb{R}^2$. As time- axis in this example we take $\mathbb{T} = \mathbb{R}$. In order to specify the port behavior, we introduce as auxiliary variables the currents through and the voltages across the internal branches of the circuit, as shown in Figure 1.2. These are the *latent variables*. Hence $\mathbb{L} = \mathbb{R}^8$.

The following equations specify the laws governing the dynamics of this circuit. They define the relations between the manifest variables (the port current and volt-

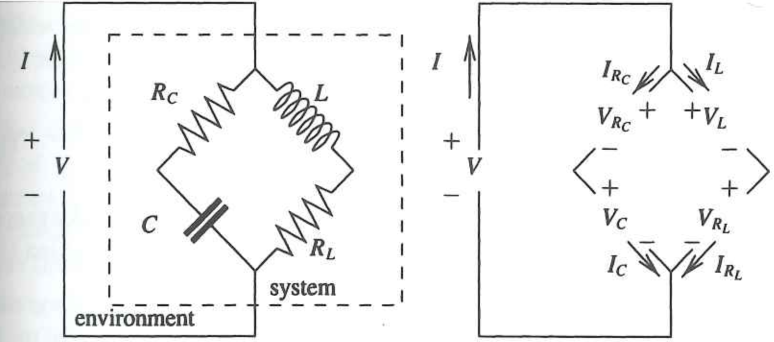

FIGURE 1.2. Electrical circuit.

age) and the latent variables (the branch voltages and currents). These equations constitute the *full behavioral equations*.

Constitutive equations:

$$V_{R_C} = R_C I_{R_C}, \qquad V_{R_L} = R_L I_{R_L}, \qquad C\frac{dV_C}{dt} = I_C, \qquad L\frac{dI_L}{dt} = V_L; \quad (1.1)$$

Kirchhoff's current laws:

$$I = I_{R_C} + I_L, \qquad I_{R_C} = I_C, \qquad I_L = I_{R_L}, \qquad I_C + I_{R_L} = I; \quad (1.2)$$

Kirchhoff's voltage laws:

$$V = V_{R_C} + V_C, \qquad V = V_L + V_{R_L}, \qquad V_{R_C} + V_C = V_L + V_{R_L}. \quad (1.3)$$

In what sense do these equations specify a manifest behavior? In principle this is clear from Definition 1.3.4. But is there a more explicit way of describing the manifest behavior other than through (1.1, 1.2, 1.3)? Let us attempt to eliminate the latent variables in order to come up with an explicit relation involving V and I only. In the example at hand we will do this elimination in an ad hoc fashion. In Chapter 6, we will learn how to do it in a systematic way.

Note first that the constitutive equations (1.1) allow us to eliminate V_{R_C}, V_{R_L}, I_C, and V_L from equations (1.2, 1.3). These may hence be replaced by

$$I = I_{R_C} + I_L, \qquad I_{R_C} = C\frac{dV_C}{dt}, \qquad I_L = I_{R_L}, \quad (1.4)$$

$$V = R_C I_{R_C} + V_C, \qquad V = L\frac{dI_L}{dt} + R_L I_{R_L}. \quad (1.5)$$

Note that we have also dropped the equations $I_C + I_{R_L} = I$ and $V_{R_C} + V_C = V_L + V_{R_L}$, since these are obviously redundant. Next, use $I_{R_L} = I_L$ and $I_{R_C} = \frac{V - V_C}{R_C}$ to eliminate I_{R_L} and I_{R_C} from (1.4) and (1.5) to obtain

$$R_L I_L + L\frac{dI_L}{dt} = V, \quad (1.6)$$

$$V_C + C_{R_C}\frac{dV_C}{dt} = V, \tag{1.7}$$

$$I = \frac{V - V_C}{R_C} + I_L. \tag{1.8}$$

We should still eliminate I_L and V_C from equations (1.6, 1.7, 1.8) in order to come up with an equation that contains only the variables V and I. Use equation (1.8) in (1.6) to obtain

$$V_C + \frac{L}{R_L}\frac{dV_C}{dt} = (1 + \frac{R_C}{R_L})V + \frac{L}{R_L}\frac{dV}{dt} - R_c I - \frac{LR_C}{R_L}\frac{dI}{dt}, \tag{1.9}$$

$$V_C + CR_C\frac{dV_C}{dt} = V. \tag{1.10}$$

Next, divide (1.9) by $\frac{L}{R_L}$ and (1.10) by CR_C, and subtract. This yields

$$(\frac{R_L}{L} - \frac{1}{CR_C})V_C = (\frac{R_C}{L} + \frac{R_L}{L} - \frac{1}{CR_C})V + \frac{dV}{dt} - \frac{R_C R_L}{L}I - R_C\frac{dI}{dt}. \tag{1.11}$$

Now it becomes necessary to consider two cases:

Case 1: $CR_C \neq \frac{L}{R_L}$. Solve (1.11) for V_C and substitute into (1.10). This yields, after some rearranging,

$$(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2})V = (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I \tag{1.12}$$

as the relation between V and I.

Case 2: $CR_C = \frac{L}{R_L}$. Then (1.11) immediately yields

$$(\frac{R_C}{R_L} + CR_C\frac{d}{dt})V = (1 + CR_C\frac{d}{dt})R_C I \tag{1.13}$$

as the relation between V and I. We claim that equations (1.12, 1.13) specify the manifest behavior defined by the full behavioral equations (1.1, 1.2, 1.3). Indeed, our derivation shows that (1.1, 1.2, 1.3) imply (1.12, 1.13). But we should also show the converse. We do not enter into the details here, although in the case at hand it is easy to prove that (1.12, 1.13) imply (1.1, 1.2, 1.3). This issue will be discussed in full generality in Chapter 6.

This example illustrates a number of issues that are important in the sequel. In particular:

1. The full behavioral equations (1.1, 1.2, 1.3) are all linear differential equations. (Note: we consider algebraic relations as differential equations of order zero). The manifest behavior, it turns out, is also described by a linear differential equation, (1.12) or (1.13). A coincidence? Not really: in Chapter 6 we will learn that this is the case in general.

2. The differential equation describing the manifest behavior is (1.12) when $CR_C \neq \frac{L}{R_L}$. This is an equation of order two. When $CR_C = \frac{L}{R_L}$, however, it is given by (1.13), which is of order one. Thus the order of the differential equation describing the manifest behavior turns out to be a sensitive function of the values of the circuit elements.

3. We need to give an interpretation to the anomalous case $CR_C = \frac{L}{R_L}$, in the sense that for these values a discontinuity appears in the manifest behavioral equations. This interpretation, it turns out, is *observability*, which will be discussed in Chapter 5. □

Example 1.3.6 As a second example for the occurrence of latent variables, let us consider a Leontieff model for an economy in which several economic goods are transformed by means of a number of production processes. We are interested in describing the evolution in time of the total utility of the goods in the economy. Assume that there are N production processes in which n economic goods are transformed into goods of the same kind, and that in order to produce one unit of good j by means of the k th production process, we need at least a_{ij}^k units of good i. The real numbers a_{ij}^k, $k \in \underline{N} := \{1, 2, \ldots, N\}$, $i, j \in \underline{n} := \{1, 2, \ldots, n\}$, are called the *technology coefficients*. We assume that in each time unit one production cycle takes place.

Denote by

$q_i(t)$	the quantity of product i available at time t
$u_i^k(t)$	the quantity of product i assigned to the production process k at time t,
$y_i^k(t)$	the quantity of product i acquired from the production process k at time t.

Then the following hold:

$$\begin{aligned}\sum_{k=1}^{n} u_i^k(t) &\leq q_i(t) && \forall i \in \underline{n},\\ \sum_{j=1}^{n} a_{ij}^k y_j^k(t+1) &\leq u_i^k(t) && \forall k \in \underline{N}, i \in \underline{n},\\ q_i(t) &\leq \sum_{k=1}^{n} y^k(t) && \forall i \in \underline{n}.\end{aligned} \tag{1.14}$$

The underlying structure of the economy is shown in Figure 1.3. The differences between the right-hand and left-hand sides of the above inequalities are due to such things as inefficient production, imbalance of the available products, consumption, and other forms of waste. Now assume that the total utility of the goods in the

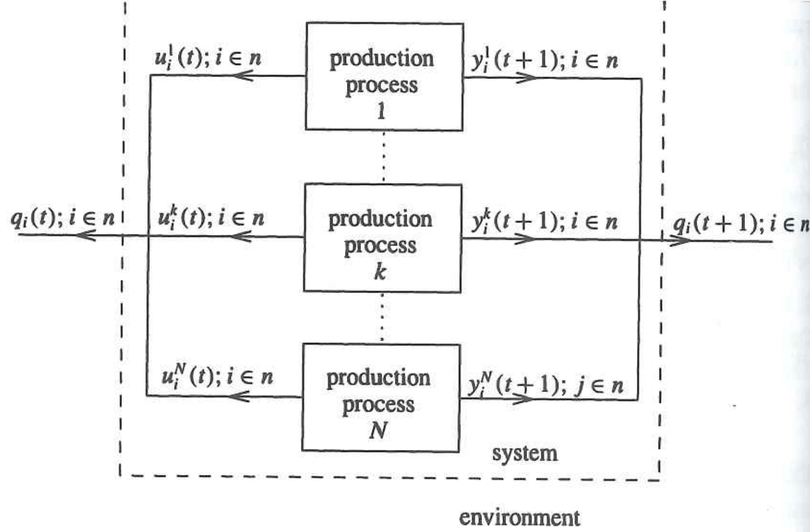

FIGURE 1.3. Leontieff economy.

economy is a function of the available amount of goods $q_1, q_2, \ldots, q_n$; i.e., $J : \mathbb{Z} \to \mathbb{R}_+$ is given by

$$J(t) = \eta(q_1(t), \ldots, q_n(t)),$$

with $\eta : \mathbb{R}_+^n \to \mathbb{R}_+$ a given function, the *utility*. For example, if we identify utility with resale value (in dollars, say), then $\eta(q_1, q_2, \ldots, q_n)$ is equal to $\sum_{k=1}^n p_i q_i$ with p_i the per unit selling price of good i.

How does this example fit in our modeling philosophy?

The first question to ask is, *What is the time-set?* It is reasonable to take $\mathbb{T} = \mathbb{Z}$. This does not mean that we believe that the products have always existed and that the factories in question are blessed with life eternal. What instead it says is that for the purposes of our analysis it is reasonable to assume that the production cycles have already taken place very many times before and that we expect very many more production cycles to come.

The second question is, What are we trying to model? *What is our signal space?* As die-hard utilitarians we decide that all we care about is the total utility J, whence $\mathbb{W} = \mathbb{R}_+$.

The third question is, *How is the behavior defined?* This is done by inequalities (1.14). Observe that these inequalities involve, in addition to the manifest variable J, as latent variables the us, qs, and ys. Hence $\mathbb{L} = \mathbb{R}_+^n \times \mathbb{R}_+^{n\times m} \times \mathbb{R}_+^{n\times p}$.

The full behavior is now defined as consisting of those trajectories satisfying the behavioral difference inequalities (1.14). These relations define the intrinsic dynamical system with $\mathbb{T} = \mathbb{Z}$, $\mathbb{W} = \mathbb{R}_+$, and the manifest behavior $\mathfrak{B} = \{J : \mathbb{Z} \to \mathbb{R}_+ \mid \exists\, q_i : \mathbb{Z} \to \mathbb{R}_+, u_i^k : \mathbb{Z} \to \mathbb{R}_+, y_i^k : \mathbb{Z} \to \mathbb{R}_+, i \in \underline{n}, k \in \underline{N}$, such that the inequalities (1.14) are satisfied for all $t \in \mathbb{Z}\}$. Note that in contrast to the previous example, where it was reasonably easy to obtain behavioral equations (1.12) or (1.13) explicitly in terms of the external attributes V and I, it appears impossible in the present example to eliminate the qs, us, and ys and obtain an explicit behavioral equation (or, more likely, inequality) describing $\mathfrak{B}$ entirely in terms of the J, the variables of interest in this example. □

1.4 Linearity and Time-Invariance

Until now we have discussed dynamical systems purely on a set-theoretic level. In order to obtain a workable theory it is necessary to impose more structure. Of particular importance in applications are linearity and time-invariance. These notions are now introduced.

Definition 1.4.1 A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be *linear* if $\mathbb{W}$ is a vector space (over a field $\mathbb{F}$: for the purposes of this book, think of it as $\mathbb{R}$ or $\mathbb{C}$), and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$ (which is a vector space in the obvious way by pointwise addition and multiplication by a scalar).

□

Thus linear systems obey the *superposition principle* in its ultimate and very simplest form: $\{w_1(\cdot), w_2(\cdot) \in \mathfrak{B}; \alpha, \beta \in \mathbb{F}\} \Rightarrow \{\alpha w_1(\cdot) + \beta w_2(\cdot) \in \mathfrak{B}\}$.

Time-invariance is a property of dynamical systems governed by laws that do not explicitly depend on time: if one trajectory is *legal* (that is, in the behavior), then the shifted trajectory is also *legal*.

Definition 1.4.2 A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$ is said to be *time-invariant* if $\sigma^t\mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{T}$ (σ^t denotes the *backward t-shift*: $(\sigma^t f)(t') := f(t'+t)$). If $\mathbb{T} = \mathbb{Z}$, then this condition is equivalent to $\sigma\mathfrak{B} = \mathfrak{B}$. If $\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{R}_+$, then time-invariance requires $\sigma^t\mathfrak{B} \subseteq \mathfrak{B}$ for all $t \in \mathbb{T}$. In this book we will almost exclusively deal with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and therefore we may as well think of time-invariance as $\sigma^t\mathfrak{B} = \mathfrak{B}$. The condition $\sigma^t\mathfrak{B} = \mathfrak{B}$ is called *shift-invariance* of $\mathfrak{B}$. □

Essentially all the examples that we have seen up to now are examples of time-invariant systems.

Example 1.4.3 As an example of a time-varying system, consider the motion of a point-mass with a time-varying mass $m(\cdot)$, for example, a burning rocket. The differential equation describing this motion is given by

$$\frac{d}{dt}(m(t)\frac{d}{dt}q) = F.$$