Find the solution to $\frac{d^2x}{dt^2} + 2\sqrt{3}\frac{dx}{dt} + 4x = 0$

try solution $x = e^{bt}$

$$b^2 e^{bt} + 2\sqrt{3}\, b\, e^{bt} + 4 e^{bt} = 0$$

$$(b^2 + 2\sqrt{3}\, b + 4)e^{bt} = 0$$

$$(b + \sqrt{3} + j)(b + \sqrt{3} - j)e^{bt} = 0 \quad \Rightarrow b = -\sqrt{3} \pm j$$

general solution:

$$x(t) = c_1 e^{-(\sqrt{3}+j)t} + c_2 e^{-(\sqrt{3}-j)t}$$

to find particular solution, use initial conditions:

$$x(0) = c_1 + c_2 = 1 \quad \Rightarrow c_1 = 1 - c_2.$$

$$x'(0) = -(\sqrt{3}+j)c_1 - (\sqrt{3}-j)c_2 = 0$$

$$+(\sqrt{3}+j)(1-c_2) + (\sqrt{3}-j)c_2 = 0$$

$$+\sqrt{3} - \cancel{\sqrt{3}c_2} + j - jc_2 + \cancel{\sqrt{3}c_2} - jc_2 = 0$$

$$+2jc_2 = \sqrt{3} + j$$

multiply both sides by j

$$-2c_2 = -1 + j\sqrt{3} \quad \Rightarrow c_2 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$\Rightarrow c_1 = 1 - c_2$$

$$= 1 - \frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} + j\frac{\sqrt{3}}{2}$$

particular solution is:

$$x(t) = \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)e^{-(\sqrt{3}+j)t} + \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)e^{-(\sqrt{3}-j)t}$$

write in terms of sinusoids:

$$x(t) = \frac{1}{2}e^{-\sqrt{3}t}\left((1+j\sqrt{3})e^{-jt} + (1-j\sqrt{3})e^{jt}\right)$$

$$= \frac{1}{2}e^{-\sqrt{3}t}\left(2e^{j\frac{\pi}{3}}e^{-jt} + 2e^{-j\frac{\pi}{3}}e^{jt}\right)$$

$$= \frac{2}{2}e^{-\sqrt{3}t}\, 2\left(\frac{e^{j(\frac{\pi}{3}-t)} + e^{-j(\frac{\pi}{3}-t)}}{2}\right)$$

$$= 2e^{-\sqrt{3}t}\cos\left(t - \frac{\pi}{3}\right)$$

What is the inverse Laplace trasform of $\frac{4s+6}{s^2+4s+3}$?

$$\mathcal{L}^{-1}\left[\frac{4s+6}{s^2+4s+3}\right]$$

$$\cancel{A}\ \frac{4s+6}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

find A, B:

multiply both sides of equation by $(s+1)(s+3)$:

$$4s+6 = A(s+3) + B(s+1)$$

let $s=-1$: $\quad -4+6 = A(-1+3) \quad A=1$

let $s=-3$: $\quad -12+6 = B(-3+1) \quad B=3$

$$\mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{3}{s+3}\right] = e^{-t} + 3e^{-3t}$$

Calculate inverse laplace transform of $\frac{s^2+4s+4}{(s+1)(s^2+2s+2)}$.

$$\frac{s^2+4s+4}{(s+1)(s+1+j)(s+1-j)} = \frac{A}{s+1} + \frac{B}{s+1+j} + \frac{C}{s+1-j}$$

$$s^2+4s+4 = A(s^2+2s+2) + B(s+1)(s+1-j) + C(s+1)(s+1+j)$$

Let $s=-1$: $\quad 1-4+4 = A(1-2+2) \Rightarrow A=1$

Let $s=-1+j$: $\quad (-1+j)(-1+j)+4(-1+j)+4 = C(-1+j+1)(-1+j+1+j)$

$$\cancel{1}-j-j-\cancel{1}+(\cancel{-4}+4j)+\cancel{4} = C(j)(2j)$$

$$j2 = C(-2) \qquad C=-j$$

Let $s=-1-j$: $\quad (-1-j)(-1-j)+4(-1-j)+4 = B(-1-j+1)(-1-j+1-j)$

$$\cancel{1}+j+j-\cancel{1}+(\cancel{-4}-j4)+\cancel{4} = B(-j)(-2j)$$

$$-j2 = B(-2) \qquad B=j$$

So we have:

$$\mathcal{L}\left[\frac{1}{s+1} + \frac{j}{s+1+j} + \frac{-j}{s+1-j}\right] = e^{-t} + je^{-(1+j)t} - je^{-(1-j)t}$$

$$= e^{-t} + 2e^{-t}\left(\frac{e^{-jt}-e^{jt}}{2j}\right)$$

$$= e^{-t} - 2e^{-t}\sin(-t)$$