1. Foundations of Probability

(Reading Exercises: Montgomery and Runger Chapter 1 -Sections 2.1, 2.3 and 2.4)

Learning outcomes:

You will be able to do the following:

- Distinguish between a population and a sample
- Distinguish between a random and a deterministic experiment.
- Distinguish between a simulation model, a deterministic model, and a probability model.
- Distinguish between probability theory and statistics
- Understand why probability is so critically important to the advancement of most kinds of electrical engineering research and design.
- Define an event.
- Derive new events by taking subsets, unions, intersections, and/or complements of already existing events.
- State the definitions of specific kinds of events, namely empty events, mutually exclusive (or disjoint) events, and exhaustive events.
- State the formal definition of probability.
- State the three ways of assigning a probability to an event.
- State the probability axioms and their corollaries,
- Apply the axioms to determine probabilities of various events.
- Get lots of practice calculating probabilities of various events

1.1 Basic Definitions: Sample spaces and events

Definitions:

- **Experiment:** "A scientific procedure undertaken to make a discovery or demonstrate a known fact.
- **Outcome**: the result of an experiment
- **Deterministic experiment:** Outcomes are predictable when the experiment is repeated under the same conditions.
- **Random experiment:** Outcomes are unpredictable when the experiment is repeated under the same conditions.
- **Sample space:** Denoted S, is a collection of all possible outcomes of an experiment.
- **Discrete sample space:** S contains discrete numbers of elements (outcomes)
    - **Finite sample space** contains a finite number of discrete elements.
    - **Infinite sample space** contains an infinite number of discrete elements.
- **Continuous sample space,** S contains intervals, for example, $S = \{[0,1],[2,3]\}$. An outcome can fall anywhere in an interval, for instance $[0,1]$

Models of Systems

Definition:

- **Model:** A description of a real physical system, process or phenomenon that we want to analyze. A model is used to help us explain observed behavior of a physical system, process or phenomenon using a set of simple and understandable rules. These rules can be used to predict the outcome of experiments involving a given physical situation.
- **Deterministic models (Mathematical models):** the solution of a set of mathematical equations specifies the exact outcome of the experiment.
- **Probabilistic Models:** In many practical situations, we are not able to accurately model some or all aspects of a physical system or process due to **inaccuracies, uncertainties, randomness in measurements,** etc.
- **Computer simulation model:** A computer program is used to simulate or mimic the dynamics of a system

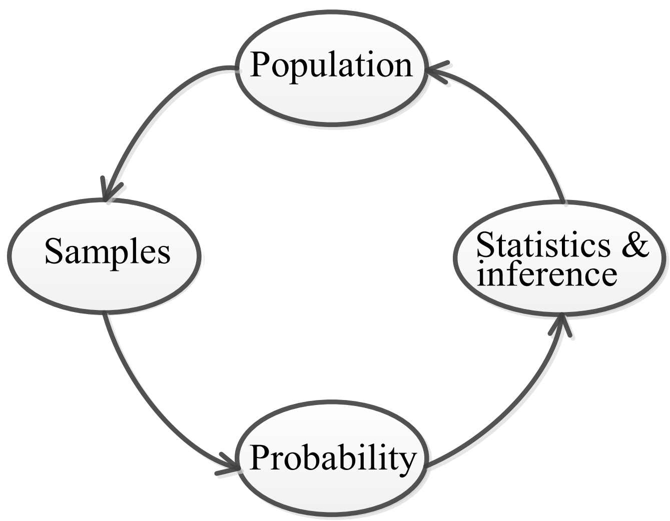
Populations and Samples

Definitions:

- **Population:** a collection of all objects or elements under study
- **Sample:** a subset of a population selected for studying or testing
- **Probability theory:** The study of the mathematical rules that govern random events
- **Statistics:** The application of probability theory to the collection, analysis, and description of random data for the purpose of making inferences, judgements, or conclusions about a population

<u>Example:</u> An electronic manufacturing company of transistors can tolerate, in the long run, 5% defective production. A quality control engineer in that company must determine whether the defective rate is within a tolerable range. The engineer accomplishes this by taking, for example, 1000 transistors from the assembly line and analyzing them.

- **Population:** The totality of all possible transistors coming out of the assembly line.
- **Sample**: The 1000 transistors taken from the assembly line.
- The engineer uses **probability theory** to determine whether the defective transistors, in the long run, meet the 5% rate or less.
- The engineer can also determine the chances of obtaining more than 5% defective transistors.
- The engineer uses **Statistics** to make **inference** regarding the acceptance or rejection of the production process at a certain confidence level.



1.2 Basic Set Theory:

Set theory provides a basic mathematical tool for studying probability. Set theory may be considered as the **algebra of probability theory**. The math involved in probability theory relies on elements of sets. The table below summarizes some correspondence of set algebra and probability theory.

Set Algebra	Probability Theory
Set	Event
Universal set	Sample space
Element	Outcome

Basic Set Definitions and notations

Definition:

- **Set:** An unordered collection of events (or objects or elements). Sets can contain items of mixed types or other sets.
- **Subsets:** If every element of a set A is also an element of set B, we say that A is a subset of B, and we write, $A \subset B$ or $B \supset A$.
- **Equal sets:** $A = B$. That is all the elements of A are also the elements of B and vice versa.

Definitions:

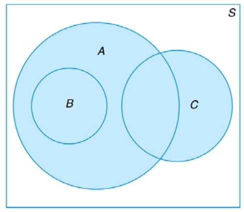
- **Universal Set**: The same as a sample space. All other sets are subsets of the universal set.
- **Event:** A subset of the universal set or sample space S and is usually denoted with capital letters, such as $A, B, C,$ …
- **Empty or null Set:** $S = \varnothing$
- $x \in S$
- $x \notin S$

Examples:

- $A = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$
- $B = \{1,\ 2,\ 3,\ 4\}$ and $C = \{4,\ 1,\ 3,\ 2\}$
- $x = 5$ and $x = 7$
- The event A that the chip fails before the end of the third year (the warranty period) is the subset $A = \{t \mid 0 \le t < 3\}$.

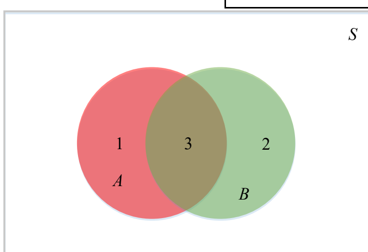
1.3 Basic Set Operations and Venn Diagrams

- Venn diagrams
  Venn diagrams are used to graphically represent (or visualize) sets and set operations.

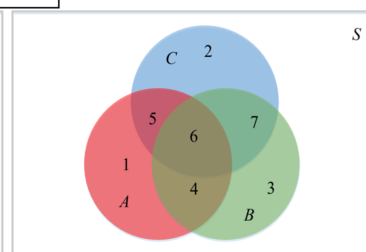


- **Union operation:** The **union** of two disjoint sets A and B is a set that contains every element that is either in A or in B or in both. We denote the union operation of the sets A and B as

$$\boxed{C = A \bigcup B}$$



$A \bigcup B$ = all shaded regions



$A \bigcup B \bigcup C$ = all shaded regions

<u>**Example:**</u>

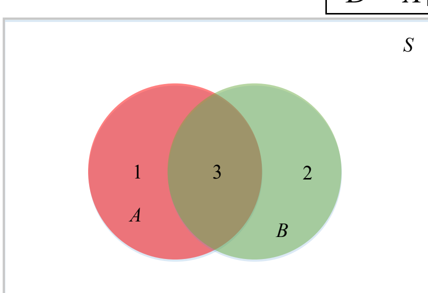
$A = \{1,\ 2,\ 6\}$ and $B = \{1,\ 3,\ 4,\ 5,\ 6\}$

$\{1,\ 2,\ 3\} \bigcup \{6,\ 7,\ 8\} = \{1,\ 2,\ 3,\ 6,\ 7,\ 8\}$

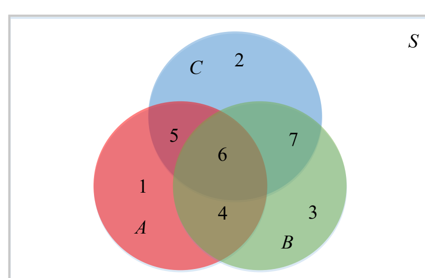
- Analogy with logic circuit operations

- **Intersection operation:** The intersection of two sets A and B contains only elements that are in A and in B. We denote the intersection of the sets A and B as

$$\boxed{D = A \bigcap B}$$



$A \bigcap B$ = region 3



$A \bigcap B \bigcap C$ = Region 6

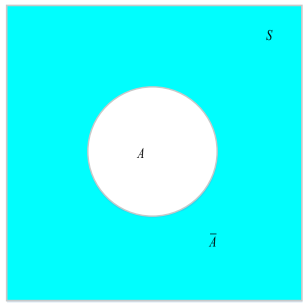
Example:

$\{1,\ 2,\ 3,\ 7,\ 8\} \cap \{6,\ 7,\ 8\} \ = \ \{7,\ 8\}$

Let $A = \{1,2,5,6\}$ and $B = \{2,4,6\}$. Then, $A \cap B = \{2,6\}$.

- Analogy with logic operations

- **Complement operation:** The complement of the set A, denoted A^c or $\overline{A}$, is the set containing all elements in S that are not in A
  Denoted A^c or $\overline{A}$



$\overline{A} = A^c = A' =$ complement (shaded area)

Example:

$S \ = \ \{1,\ 2,\ \ldots,\ 10\}$

$\{1,\ 2,\ 3,\ 4,\ 5\}^c = \{6,\ 7,\ 8,\ 9,\ 10\}$

$\{2,\ 4,\ 6,\ 8,\ 10\}^c = \{1,\ 3,\ 5,\ 7,\ 9\}$

- Analogy with logic operations

1.4 Basic Set Identities or Laws

They are used to simply complex set operations.

- **Commutative property:**

1. $A \bigcup B = B \bigcup A$	2. $A \bigcap B = B \bigcap A$

Example: Use Venn diagrams

- **Associative property:**

1. $(A \bigcup B) \bigcup C = A \bigcup (B \bigcup C)$	2. $(A \bigcap B) \bigcap C = A \bigcap (B \bigcap C)$

Example: Use Venn diagrams

- **Distributive property:**

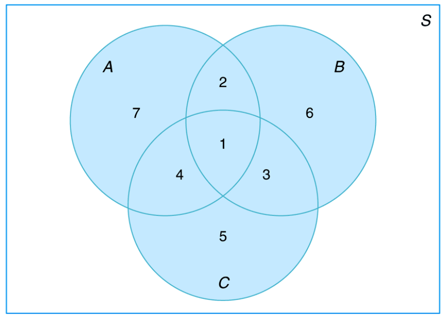
$(A \bigcup B) \bigcap C = (A \bigcap C) \bigcup (B \bigcap C)$

Example: Use Venn diagrams

- **DeMorgan's Laws:**

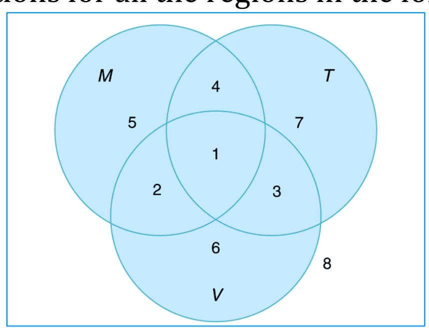
1. $\overline{A \bigcup B} = \overline{A} \bigcap \overline{B}$	2. $\overline{A \bigcap B} = \overline{A} \bigcup \overline{B}$

Example: Write the set operations for each region represented by the numbers 1, 2, 3, 4, 5, 6, 7 and 8 in the Venn diagrams below. The solutions are summarized in the table below.



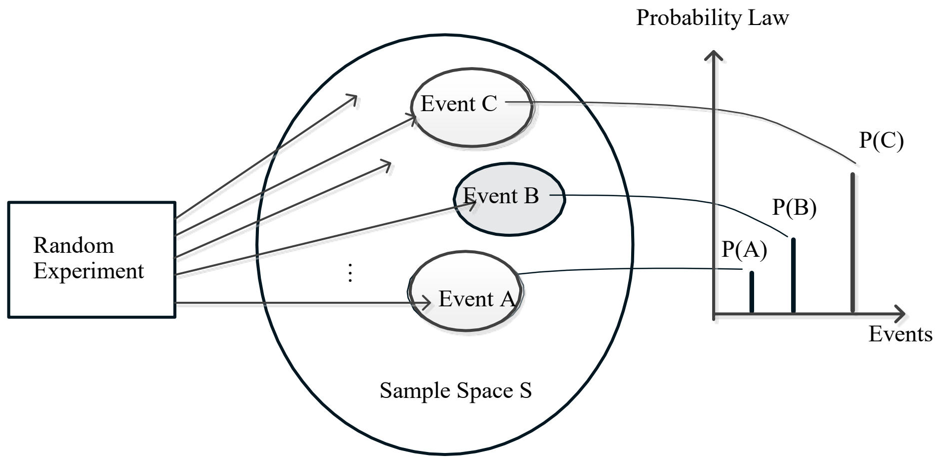
Region 1: $A\cap B\cap C$	Region 2: $A\cap B\cap\overline{C}$	Region 3: $\overline{A}\cap B\cap C$	Region 4: $A\cap C\cap\overline{B}$
Region 5: $\overline{A\cup B}\cap C$	Region 6: $\overline{A\cup C}\cap B$	Region 7: $A\cap\overline{B\cup C}$	

Exercise: Write the set operations for all the regions in the following Venn diagram



1.5 Probability Model:

In this section, we study the concept of **probability**, the **rules of probability**, and **probability models**. A probability model is illustrated in the diagram below.



Probability Model

A probability model comprises the following:

- A random experiment - this produces outcomes or sample points
- A sample space, S - This contains the set of all possible outcomes of the random experiment.
- Probabilities assigned to each of the sample points, making sure they sum up to 1.
- Events denoted $A_i,\ i=1,2,\ldots,n$ - A group or collection of outcomes or sample points that form subsets of the sample space.
- Probability rule - a function, which assigns a nonnegative number to each event. This number is called the probability of an event and is written as $P[\mathrm{A}_i]$, $i=1,2,\ldots,n$. The probability encodes our knowledge or belief about the collective "likelihood" of the event to occur. For example, if we are certain an event will occur, we say the probability is 100% - $P[\text{event}]=1$.

Example:

1.6 Probability Axioms and Corollaries

Axioms:

Definition:
Axioms are facts and do not need proofs.

- Axioms are properties that the probability of an event satisfies.
- Axioms do not provide a means of specifying the probabilities.

Probability and Axioms of Probability (Andrey N. Kolmogorov 1903-1987)

Definition:

Probability is a (real-valued) function, denoted $P[\cdot]$, that assigns to each event, A, in the sample space S, a number $P[A]$, called the probability of the event A.

A probability must satisfy the following three axioms:

- **Axiom 1:** For every event A, $P(A) \geq 0$.
- **Axiom 2:** For the sure or certain event S, $P[S] = 1$.
- **Axiom 3:** For any number of disjoint (mutually exclusive) events $A_1, A_2, A_3, \ldots$, $A_i \cap A_j = \emptyset$, for $i \neq j$,

$$P[A_1 \cup A_2 \cup \cdots] = \sum_{k=1}^{\infty} P[A_k]$$

Corollaries of Probability Axioms

Definition:

Corollaries are properties that can be derived from axioms.

The following is a list of corollaries of probability:

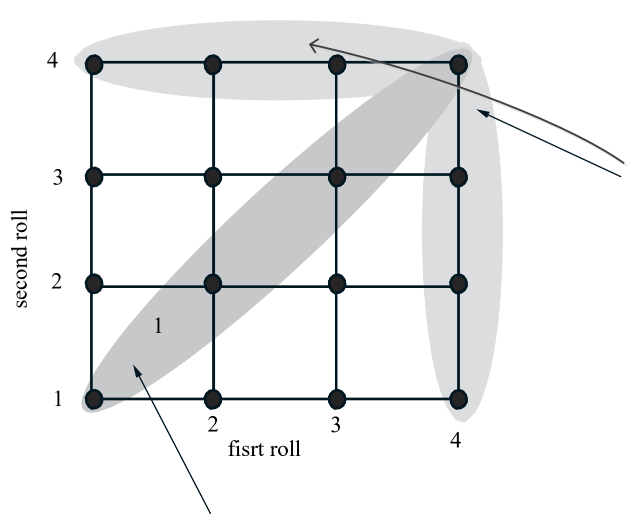
1. $P[A^c] = P[S] - P[A] = 1 - P[A]$ or $P[A] = 1 - P[A^c]$
2. For every event $0 \leq P[A] \leq 1$
3. $P[\varnothing] = 0$
4. If events A and B are such that $A \subseteq B$, then $P[A] \leq P[B]$ and $P[B - A] = P[B] - P[A]$.
5. For non-disjoint events, that is, $A \cap B \neq \varnothing \Rightarrow P[A \cap B] \neq 0$, we have $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

1.7 Methods of Assigning Probabilities

- **Subjective approach:**
  This is a judgement approach. Here, probability is defined as the degree of belief that we hold in the occurrence of an event (non-repeatable experiments). Examples are horse race and stock price, etc.
- **Relative frequency approach**:
  The relative frequency approach involves taking the following three steps in order to assign the probability of an event A:
  (1) Perform an experiment, an indefinite number of times, $n \to \infty$.
  (2) Count the number of times the event A of interest occurs and denote this number as n_A.
  (3) Then, the probability of event A occurring equals:
  $$P[A] = \lim_{n\to\infty} \frac{n_A}{n} = \lim_{n\to\infty} P_n[A]$$
- **Classical Approach**
  The classical approach assumes that all distinct (elementary) outcomes are equally likely, that is, they have equal probability of occurring. Let the sample space, S, be the set of all possible distinct outcomes to an experiment. The probability of some event, A, is defined as
  $$P[A] = \frac{\text{Number of times or ways event } A \text{ can occur}}{\text{Number of all possible outcomes in } S}$$

Example: Two 4-sided die rolls

Experiments involving rolling of 2, 4-sided dice



$$\text{Prob}\{\text{event that he fisrt roll is equal to the second}\} = 4/16$$
$$\text{Prob}\{\text{Event that at least one roll is a 4}\} = 7/16$$

Example: An urn contains 10 identical balls numbered $0,1,2,\ldots,9$. A random experiment involves selecting a ball from the urn. Find the probability of the following events:

1. $A = \{\text{number of balls selected is odd}\}$
2. $B = \{\text{number of balls selected is a multiple of } 3\}$
3. $C = \{\text{number of balls selected is less than } 5\}$
4. $D = A \bigcap B$
5. $E = A \bigcup B$

Example: A fair coin tossed three times. Find the probability of exactly two heads in three tosses.

$$P\left[\{\text{exactly 2 heads in 3 tosses}\}\right] = P\left[\{HHT, HTH, THH\}\right] = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

Exercise: (a) One head in first toss, (b) At least two heads,

<u>Some Major Pioneers of Probability and Statistics:</u>

Thomas Bayes 1701– 1761, English Statistician, Philosopher & Minister) - Wikipedia.com 	*Daniel Bernoulli (1700 – 1782, Mathematician & Physicist) - Wikipedia.com* 
Siméon Denis Poisson 1781 – 1840, French Mathematician, Engineer & Physicist) -Wikipedia.com 	*Johann Carl Friedrich Gauss (1777–1855, German Mathematician) - Wikipedia.com* 

Andrey NikolaevichKolmogorov (1903 – 1987, Russian Mathematician) - Wikipedia.com 	*William Sealy Gosset (1876 – 1937, English Statistician) – Wikipedia.com*