ENEL 419: Probability and Random Variables

Midterm #2

Instructor: Dr. Abu Sesay

November 17, 2016

Room: EDC 179 **Time: 18:00 – 19:30**

Last Name (printed):	First Name:	ID #:

Signature:

Instructions:

- All the University of Calgary regulations apply to this exam
- Answer all three questions in the blue booklet provided.
- You are allowed to use a non-programmable calculator for this exam. No other electronic device, including music devices or anything with wireless capability, will be allowed during the exam.
- The exam is closed-book and closed-notes. Formulas are provided below for your convenience.
- To reduce distraction to other students, you are not allowed to leave during the last ten minutes of the exam.
- Please print or write your answers legibly. What cannot be read cannot be marked.
- If you write anything you do not want marked, put a large X through it and write "rough work" beside it.

Marks Summary

	Q1	Q2	Q3	Total
Marks				
Out of	31	35	34	100

$E\left[X^n\right]=\int_{-\infty}^{\infty} x^n f_X(x)\,dx$	$f_Y(y)=f_X\left(g^{-1}(y)\right)\left\lvert\dfrac{dg^{-1}(y)}{dy}\right\rvert$	$E\left[h(X)\right]=\int_{-\infty}^{\infty} h(x) f_X(x)\,dx$ $\sigma_{h(X)}^2=E\left[\left(h(X)-\mu_{h(X)}\right)^2\right]$
$\sigma_X^2=E\left[\left(X-\mu_X\right)^2\right]$ $=E\left[X^2\right]-\mu_X^2$	$\dfrac{1}{\sqrt{2\pi}\sigma_X}\int_{\gamma}^{\infty}\exp\left(-\dfrac{\left(x-\mu_X\right)^2}{2\sigma_X^2}\right)dx=Q\left(\dfrac{\gamma-\mu_X}{\sigma_X}\right)$	$ax^2+bx+c=0\Rightarrow x=\dfrac{\left(-b\pm\sqrt{b^2-4ac}\right)}{2a}$
$p_X(x)=\sum_y p_{XY}(x,y)$	$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$	$p_{X\mid Y}(x\mid y)=p_{XY}(x,y)/p_Y(y)$
$f_{X\mid Y}(x\mid y)=f_{XY}(x,y)/f_Y(y)$	$C_{XY}=E\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right]=r_{XY}-\mu_X\mu_Y$	$r_{XY}=E\left[XY\right]$
$\rho_{XY}=C_{XY}/\left(\sigma_X\sigma_Y\right)$; $\lvert\rho_{XY}\rvert\le 1$	$P\left[\left(X,Y\right)\in A\right]=\iint_A f_{XY}(x,y)\,dxdy$, A is a region in the XY-plane for which $P\left[\left(X,Y\right)\in A\right]$ is to be evaluated.	

Question 1:

Marks		
/7 /5	(a)	The voltage developed across a $1\,\text{k}\Omega$ resistor is a continuous uniform random variable, denoted X, in the range $-10 \le X \le 10$ volts. (i) Write the formula for the PDF and sketch the graph of the PDF, $f_X(x)$, as a function of x? (ii) Find the probability $P[\lvert X\rvert < 3]$.
/14 /4	(b)	On a laboratory assignment, if the equipment is working, the PDF of the observed random output Y, is $f_Y(y) = \begin{cases} c(1-y), & 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$ (i) Determine the conditional probability $P[Y \le 0.75 \mid Y \ge 0.5]$. (ii) Find the PDF of the new random variable $Z = 2Y-1$. Write the expression for the PDF and indicate the range of values of z.
/30		

Solutions:

(a)

(i) $$f_X(x) = \begin{cases} \dfrac{1}{20}, & -10 \le x \le 10, \\ 0, & \text{elsewhere.} \end{cases}$$

(ii) $$P[\lvert X\rvert < 3] = P[-3 < X < 3] = \int_{-3}^{3} f_X(x)\,dx = \frac{1}{20}\int_{-3}^{3} dx = \frac{3}{10}$$

(b) First, we need to find the value of c to make $f_Y(y)$ a valid PDF,

$$\int_{-\infty}^{\infty} f_Y(y)dy = c\int_0^1 (1-y)dy = c\left(y - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1} = \frac{c}{2} = 1 \Rightarrow c = 2 \Rightarrow f_Y(y) = 2(1-y)$$

$$P\left[Y \le \frac{3}{4}\middle| Y \ge \frac{1}{2}\right] = \frac{P\left[Y \le \frac{3}{4}, Y \ge \frac{1}{2}\right]}{P\left[Y \ge \frac{1}{2}\right]};$$

$$P\left[Y \le \frac{3}{4}, Y \ge \frac{1}{2}\right] = P\left[\frac{1}{2} \le Y \le \frac{3}{4}\right] = 2\int_{1/2}^{3/4}(1-y) = 2\left(y - \frac{y^2}{2}\right)\Bigg|_{y=1/2}^{y=3/4} = \frac{3}{16}$$

$$P\left[Y \ge \frac{1}{2}\right] = 2\int_{1/2}^{1}(1-y)\,dy = 2\left(y - \frac{y^2}{2}\right)\Bigg|_{y=1/2}^{y=1} = \frac{1}{4}$$

$$P\left[Y \le \frac{3}{4}\middle| Y \ge \frac{1}{2}\right] = \frac{3/16}{1/4} = \frac{12}{16} = \frac{3}{4}$$

(i) $$Y = \frac{Z+1}{2} \Rightarrow f_Z(z) = \frac{1}{2} f_Y\left(\frac{z+1}{2}\right) = \begin{cases} \dfrac{1}{2}(1-z), & -1 \le z \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Question 2:

Marks		
 /8 /4 /6	(a)	A random variable X has a Gaussian distribution with a mean of zero and probability $P\left[\lvert X\rvert \le 10\right] = 0.38292$ (i) Find the standard deviation of X. (ii) Find the mean-square value, $E\left[X^2\right]$, of X, that is. (iii) Find the probability of X lying in the range $0 \le X \le 5$.
/6 /8	(b)	A professor of "Probability and Random Variables" pays 25 cents, for each blackboard error made in lectures, to the student who points out the error. In a career of n years with blackboard errors, the total amount (in dollars) paid can be approximated by a Gaussian random variable X_n with expected value $40n$ and variance $100n$. (i) At least, how many years would the professor have to work so that the probability that the amount he/she pays out, X_n, exceeds 1000 dollars is 3.3977×10^{-6}? (ii) For how many years would the professor have to teach in order to guarantee that the probability of paying out no more than 1000 dollars is at least 0.9906133?
/32		

Solutions:

(a) Given: $\mu_X = 0$ and $P\left[\lvert X\rvert \le 10\right]$

(i)

$$P\left[\lvert X\rvert \le 10\right] = P\left[-10 \le X \le 10\right] = P\left[X > -10\right] - P\left[X \ge 10\right]$$

$$= 1 - Q\left(\frac{10}{\sigma_X}\right) - Q\left(\frac{10}{\sigma_X}\right) = 1 - 2Q\left(\frac{10}{\sigma_X}\right) = 0.38292$$

$$1 - 2Q\left(\frac{10}{\sigma_X}\right) = 0.38292 \Rightarrow Q\left(\frac{10}{\sigma_X}\right) = \frac{1-0.38292}{2} = 0.30854$$

From Q-function table $\frac{10}{\sigma_X} = 0.5 \Rightarrow \sigma_X = 20$

(ii) $\sigma_X^2 = E\left[X^2\right] - \mu_X^2 \Rightarrow E\left[X^2\right] = \sigma_X^2 + \mu_X^2 = \sigma_X^2 = 20$

(iii) $P\left[0 \le X \le 5\right] = Q\left(\frac{0-0}{\sigma_X}\right) - Q\left(\frac{5-0}{\sigma_X}\right) = Q(0) - Q(0.25)$

$$= 0.5 - 0.40129 = 0.0987$$

(b) Given for random variable X_n: $\mu_{X_n} = 40n$ and $\sigma_{X_n}^2 = 100n$ where n is the number of years.

(i) $P\left[X_n \ge 1000\right] = Q\left(\frac{1000-40n}{10\sqrt{n}}\right) = 3.3977\times10^{-6}$

(ii) From the Q-function table, we have

$\frac{1000-40n}{10\sqrt{n}} = 4.5 \Rightarrow$ we need to solve for n

$$\frac{1000-40n}{10\sqrt{n}} = 4.5 \Rightarrow (25-n)^2 = \left(\frac{4.5}{4}\right)^2 n = (1.25)^2 n$$

$$n^2 - 51.2656n + 625 = 0 \Rightarrow n = \frac{51.2656 \pm \sqrt{(51.2656)^2 - 4\times 625}}{2} = \frac{51.2656 \pm 11.3209}{2}$$

$n = 20$ or 31 years. Pick the lower value (at least) $n = 20$ years.

(iii) $$P\left[X_n \le 1000\right] = 1 - P\left[X_n > 1000\right] = 1 - Q\left(\frac{1000-40n}{10\sqrt{n}}\right) \ge 0.9906133$$

From Q-function table, we have

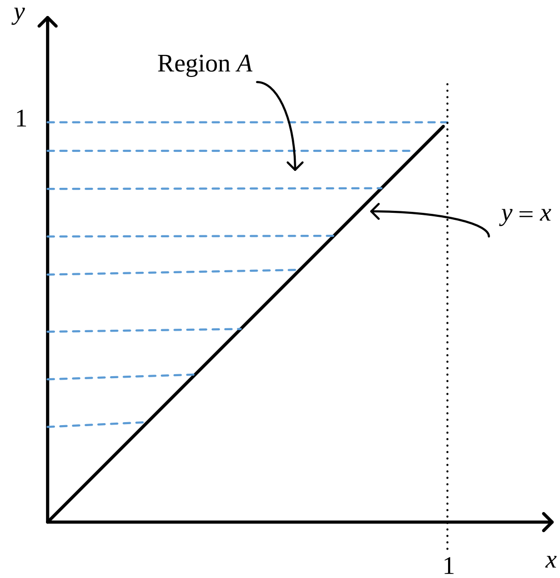
$Q\left(\frac{1000-40n}{10\sqrt{n}}\right) \le 1 - 0.9906133 = 0.0093867 \Rightarrow \frac{1000-40n}{10\sqrt{n}} \ge 2.35$. We need to solve for n.

$$\frac{1000-40n}{10\sqrt{n}} \ge 2.35 \Rightarrow (25-n)^2 \ge \left(\frac{2.35}{4}\right)^2 n = 0.3452n$$

$$n^2 - 50.3452n + 625 = 0 \Rightarrow n = \frac{50.3452 \pm \sqrt{(50.3452)^2 - 4\times 625}}{2} = \frac{50.3452 \pm 5.8855}{2}$$

$n \simeq 22$ or 28 years. We pick $n \simeq 22$ years. You can check if 28 would also work.

Question 3:

Marks		
		A commercial bank operates a drive-through and a walk-in facility. Let random variables X and Y, respectively, represent the proportion of time the drive through and walk-in facilities are busy on a randomly selected day. It is known that the joint PDF of X and Y is $f_{XY}(x,y)=2$. The region for which this PDF is valid is Region A (above the line $Y=X$) as shown in graph of y versus x below.  Answer the following questions:
/6	(a)	Find the marginal probability density functions.
/4	(b)	Find the expected values of X and Y.
/8	(c)	Find the standard deviations of X and Y.
/4 /2	(d)	(i) Find the correlation $r_{XY}=E[XY]$, between X and Y. (ii) Based on the result above (in part (d) (i)) for r_{XY}, would you be able to conclude that X and Y are orthogonal? Why or why not?
/4 /2	(e)	(i) Find the correlation coefficient for X and Y. (ii) Based on the result (in (i)) above, would you be able to confidently conclude that the relationship between X and Y is strong? Why or why not?
/2 /6	(f)	Suppose we create a new random variable $Z=2X-Y$. (i) What is the mean μ_Z of Z? (ii) What is the variance σ_Z^2, of Z?
/38		

Solutions:

(a) We need to determine the ranges of the values of the random variables from the given region A. We see that in region A, $Y \geq X$, therefore, $0 \leq x \leq y \leq 1$,

$$f_{XY}(x,y) = \begin{cases} 2, & 0 \leq x \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_X(x) = \int_x^1 2dy = 2y\Big|_{y=x}^{y=1} = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = \int_0^y 2dx = 2x\Big|_{x=0}^{x=y} = \begin{cases} 2y, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(b)

$$\mu_X = \int_0^1 2x(1-x)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}$$

$$\mu_Y = \int_0^1 2y^2 dy = 2\frac{y^3}{3}\Bigg|_0^1 = \frac{2}{3}$$

(c)

$$E\left[X^2\right] = 2\int_0^1 x^2(1-x)\,dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{6}$$

$$E\left[Y^2\right] = 2\int_0^1 y^2 y dy = 2\left(\frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{2}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \Rightarrow \sigma_X = \frac{1}{3\sqrt{2}} = 0.2357$$

$$\sigma_Y^2 = E\left[Y^2\right] - \mu_Y^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \Rightarrow \sigma_X = \frac{1}{3\sqrt{2}} = 0.2357$$

(d)

(i) $r_{XY} = E[XY] = \int_0^1\int_0^y 2xydxdy = \int_0^1 y^3 dy = \frac{1}{4}$

(ii) No, because $r_{XY} \neq 0$.

(e)

(i) $C_{XY} = r_{XY} - \mu_X\mu_Y = \frac{1}{4} - \frac{1}{3}\times\frac{2}{3} = \frac{1}{36} \Rightarrow \rho_{XY} = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{1/36}{\left(1/3\sqrt{2}\right)\times\left(1/3\sqrt{2}\right)} = 0.5$

(ii) No, because the correlation coefficient is only 0.5.

(f)

(i) $\mu_Z = 2\mu_X - \mu_Y = \frac{2}{3} - \frac{2}{3} = 0$.

(ii) $\sigma_Z^2 = E\left[Z^2\right] - \mu_Z^2 = 4E\left[X^2\right] - 4E[XY] + E\left[Y^2\right]$.

From parts (b) and (c), we have

$$\sigma_Z^2 = 4\times\frac{1}{6} - 4\times\frac{1}{4} + \frac{1}{2} = \frac{1}{6}$$