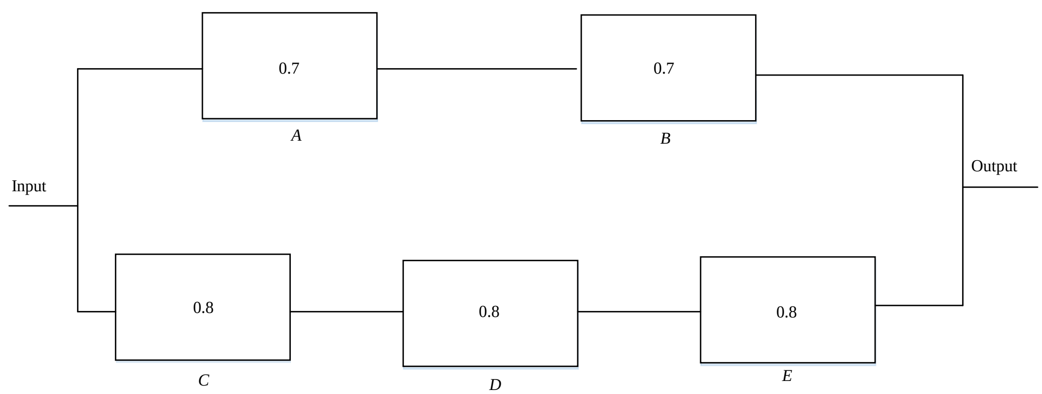
Question 1:

Marks		
/3	(a)	Prove that $1+P[A\cap B]-P[A]-P[B]=P[\overline{A}\cap\overline{B}]$
/3	(b)	Prove that $1-P[A\cap B]=P[\overline{A}\cup\overline{B}]$
   /5 /5 /5	(c)	In a high school graduating class of 100 students, 54 studied mathematics, 69 studied history and 35 studied both mathematics and history. If one of these students is selected at random, find the probability that. (i) The student took mathematics or history; (ii) The student did not take either of these subjects; (iii) The student took history but not mathematics.
   /5 /5	(d)	Consider the circuit system shown below. The number inside a block denotes the probability that the component works properly. Assume the components fail independently. (i) What is the probability that the entire system works? (ii) Given that the system works, what is the probability that component A is not working.
/31		



Solutions:

(a) Can be proven in one of two ways

$$1+P[A\cap B]-P[A]-P[B]=1-\left(P[A]+P[B]-P[A\cap B]\right)$$

$$=1-P[A\cup B]=P[\overline{A\cup B}]=P[\overline{A}\cap\overline{B}]$$

$$\boxed{\text{or } P[\overline{A}\cap\overline{B}]=P[\overline{A\cup B}]=1-P[A\cup B]=1+P[A\cap B]-P[A]-P[B]}$$

(b) Can be proven in one of two ways

$$1-P[A\cap B]=P[\overline{A\cap B}]=P[\overline{A}\cup\overline{B}]$$

$$\boxed{\text{or } P[\overline{A}\cup\overline{B}]=P[\overline{A\cap B}]=1-P[A\cap B]}$$

(c) Define events, M: student took math, H: student took history,

(i) $P[M\cup H]=P[M]+P[H]-P[M\cap H]=\frac{54}{100}+\frac{69}{100}-\frac{35}{100}=\frac{88}{100}=0.88$ (ok if not simplified)

(ii) $P[\overline{M\cup H}]=P[\overline{M}\cap\overline{H}]=1-\frac{22}{25}=\frac{3}{25}=0.12$ (ok if not simplified)

(iii)

$$P\left[H\cap\overline{M}\right]=P\left[H\cap(S-M)\right]=P[H\cap S]-P[H\cap M]$$
$$=P[H]-P[H\cap M]=\frac{69}{100}-\frac{35}{100}=\frac{34}{100}=0.34$$ (ok if not simplified)

(d)

(i) **Approach 1:** Use the complement.

Define events, S_1: top rail works, S_2: bottom rail works and
S_T: overall system works

$$P\left[\overline{S_1}\right]=1-P[A\cap B]=1-0.7\times 0.7=0.51$$
$$P\left[\overline{S_2}\right]=1-P[C\cap D\cap E]=1-0.8\times 0.8\times 0.8=0.488$$

For overall system not to work, both rails must not work (failures are independent)

$$P\left[\overline{S_T}\right]=P\left[\overline{S_1}\right]P\left[\overline{S_2}\right]=0.51\times 0.488=0.24888$$
$$P[S_T]=1-P\left[\overline{S_T}\right]=1-0.24888=0.75112$$

Approach 2: Use the union of the two rails and the independence of all components.

$$P[S_T]=P[A\cap B]+P[C\cap D\cap E]-P\left[(A\cap B)\cap(C\cap D\cap E)\right]$$
$$=P[A]P[B]+P[C]P[D]P[E]-P[A]P[B]P[C]P[D]P[E]$$
$$=0.7^2+0.8^3-0.7^2\times 0.8^3=0.49+0.512-0.25088=0.75112$$

(ii) For system to work when A does not work C, D and E must work

System works is event $F=C\cap D\cap E$

$$P\left[\overline{A}\,|\,F\right]=\frac{P\left[\overline{A}\cap C\cap D\cap E\right]}{P[C\cap D\cap E]}=\frac{0.3\times 0.8\times 0.8\times 0.8}{0.8\times 0.8\times 0.8}=0.3$$ (ok if not simplified)

Note: Accept if student provides this solution

$$P\left[\overline{A}\,|\,S_T\right]=\frac{P\left[\overline{A}\cap C\cap D\cap E\right]}{P[S_T]}=\frac{0.3\times 0.8\times 0.8\times 0.8}{0.75112}\simeq 0.2045$$

(ok if not simplified)

Question 2:

Marks		
	(a)	A lab technician receives an order of 7 micro-chips of which 2 are known to be defective (but not exactly which ones). The technician randomly selects 3 micro-chips for examination. Let random variable X, represent the number of defective chips selected.
/8		(i) Find a formula for the probability distribution (PMF) of X,
/5		(ii) Express the PMF in table form,
/3		(iii) Sketch the probability histogram,
/5		(iv) Find and sketch the cumulative distribution function (CDF),
/2		(v) Evaluate the probability $P[0.5 < X < 2.5]$,
/3		(vi) Evaluate the probability $P[\{0.5 < X < 2.5\} \cap \{X \le 3\}]$
/4		(vii) Evaluate the probability $P[\{0.5 < X < 2.5\} \mid \{X \le 3\}]$
/5	(b)	Determine the value of c such that the function $f(x)$ is a valid probability distribution function (PMF) of a discrete random variable X: $f(x) = c\binom{2}{x}\binom{3}{3-x}$, 0, 1, 2.
/35		

Solutions:

(a)

(i) We can select x defective chips from 2 and $3-x$ from 5 non-defective chips in n_A ways, where

$$n_A = \binom{2}{x}\binom{5}{3-x}, \; x = 0,1,2$$

A random selection of 3 from 7 chips, yields the number of elements in the sample space for this experiment

$$n_s = \binom{7}{3}$$

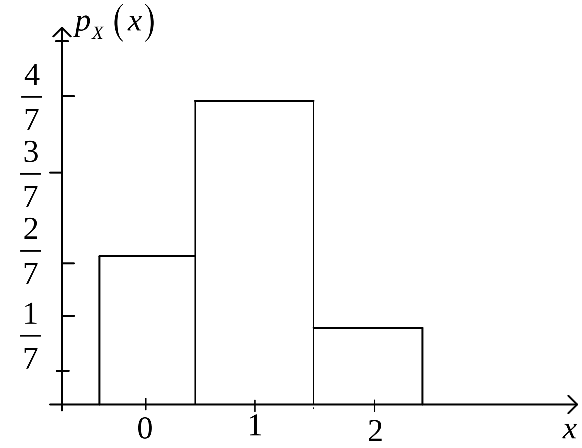
The formula for the PMF is

$$p_X[x] = P[X = x] = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}}, \quad x = 0,1,2$$

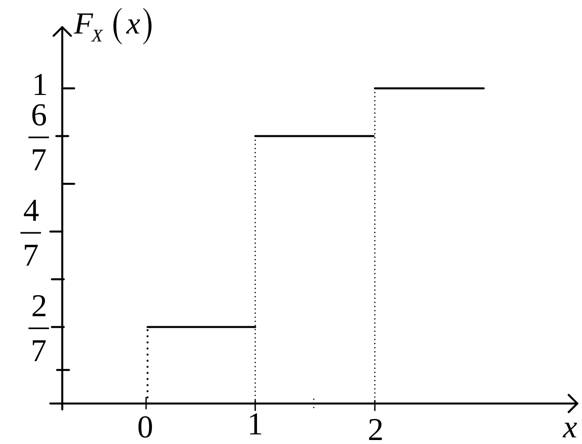
(ii) In tabular form, we have

x	0	1	2
$p_X(x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

(iii) Histogram form



(iv) CDF



(v) $P[0.5 < X < 2.5] = F_X(2.5) - F_X(0.5) = 1 - \frac{2}{7} = \frac{5}{7}$

(vi) $P\left[\{0.5 < X < 2.5\} \cap \{X \le 3\}\right] = \frac{5}{7}$

(vii) $P\left[\{0.5 < X < 2.5\} \mid \{X \le 3\}\right] = \frac{P\left[\{0.5 < X < 2.5\} \cap \{X \le 3\}\right]}{P[X \le 3]} = \frac{5/7}{1} = \frac{5}{7}$

(b)

$$f(x) = c\binom{2}{x}\binom{3}{3-x} = c\left[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}\right]$$

$$= c[1+6+3] = 10c = 1 \Rightarrow c = \frac{1}{10}$$

Question 3:

Marks		
		A producer of a certain type of electronic component chips to suppliers in lots of 20. Suppose 60% of the lot contain no defective component, 30% contain one defective component and 10% contain 2 defective components. A lot is picked, two components are randomly selected and tested, and neither is defective. Define the following events: A: 2 non-defective components are selected, O: a lot contains one defective component, N: a lot does not contain defective components, T: a lot contains two defective component.
 /4 /4 /4	(a)	Find the conditional probabilities: (i) $P[A\|O]$, (ii) $P[A\|T]$, (iii) $P[A\|N]$.
/8	(b)	What is the probability that the lot contains zero defective components given that the two selected components are non-defective (i.e., $P[N\|A]$)?
/4	(c)	What is the probability that the lot contains one defective component given that the two selected components are non-defective (i.e., $P[O\|A]$)?
/4	(d)	What is the probability that the lot contains two defective components given that the two selected components are non-defective (i.e., $P[T\|A]$) ?
 /2 /2 /2	(e)	Select the correct answer to the following questions: (i) We toss a coin n times and count the number of times a head appears. We can model the random variable that counts the number of heads by a Bernoulli distribution, a Binomial distribution a Geometric distribution or a Poisson distribution. (ii) We toss a coin several times until one head appears. We can model the resulting random variable by a Bernoulli distribution, a Binomial distribution a Geometric distribution or a Poisson distribution. (iii) In digital wireless communication, we transmit a single wave. Due to multiple paths in the channel, the receiver sees several replicas of the wave, which arrive with varying relative delays. The random variable that counts the number of arrivals can be modeled by a Bernoulli distribution, a Binomial distribution, a Geometric distribution or a Poisson distribution.
/34		

Solutions: First convert percentages to numbers:

$$60\% \text{ of } 20 = 12 \text{ contain } 0 \text{ defective component} \Rightarrow \mathrm{P}[N] = 0.6$$

$$30\% \text{ of } 20 = 6 \text{ contain } 1 \text{ defective component} \Rightarrow \mathrm{P}[O] = 0.3$$

$$10\% \text{ of } 20 = 12 \text{ contain } 2 \text{ defective component} \Rightarrow \mathrm{P}[T] = 0.1$$

(a) The experiment selects 2 out of 20 therefore, there are $n_s = \binom{20}{2}$ ways (sample space).

(i) Event A - select 2 chips out of 19 (since we already know that there is 1 defective). There are $n_A = \binom{19}{2}$ ways. The probability is

$$P[A|O] = \frac{n_A}{n_s} = \frac{\binom{19}{2}}{\binom{20}{2}} = \frac{9}{10}$$

(ii) Event A - select 2 chips out of 18 (since we already know that there are 2 defectives). There are $n_A = \binom{18}{2}$ ways. The probability is

$$P[A|T] = \frac{\binom{18}{2}}{\binom{20}{2}} = \frac{153}{190}$$

(iii) Event A - select 2 chips out of 20 (since we already know that there are 0 defectives). There are $n_A = \binom{20}{2}$ ways. The probability is

$$P[A|N] = \frac{\binom{20}{2}}{\binom{20}{2}} = 1$$

For parts (a), (b) and (c) use total probability and Bayes' rule:

$$P[A] = P[A|N] \times P[N] + P[A|O] \times P[O] + P[A|T] \times P[T]$$
$$= 0.6 \times 1 + \frac{0.3 \times 9}{10} + \frac{0.1 \times 153}{1190} = 0.9505$$

(b) $P[N|A] = \dfrac{P[A|N]P[N]}{P[A]} = \dfrac{1 \times 0.6}{0.9505} = 0.6312$

(c) $P[O|A] = \dfrac{P[A|O]P[O]}{P[A]} = \dfrac{0.3 \times 0.9}{0.9505} \simeq 0.2841$

(d) $P[T|A] = \dfrac{P[A|T]P[T]}{P[A]} = 1 - 0.6312 - 0.2841 = 0.0847$ (or calculate as above)

(e)

(i) Binomial

(ii) Geometric

(iii) Poisson