

ENEL 419: Probability and Random Variables

Midterm #2

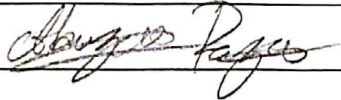
Instructor: Dr. Abu Sesay November 21,  
2017

Room: ENA 103

Time: 09:30 – 10:30AM

Last Name (printed):	First Name:	ID #:
Roza	Abuzar	30012156

Signature:



Instructions:

- All the University of Calgary regulations apply to this exam.
- Answer all three questions in the booklet provided.
- You are allowed to use a non-programmable calculator for this exam. No other electronic device, including music devices or anything with wireless capability, will be allowed during the exam.
- The test is closed-book and closed-notes. Formulas are provided on the last for your convenience.
- To reduce distraction to other students, you are not allowed to leave during the last ten minutes of the exam.
- Please print or write your answers legibly. What cannot be read cannot be marked.
- If you write anything you do not want marked, put a large "X" through it and write "rough work" beside it.

Marks Summary

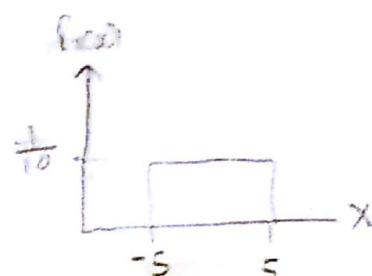
	Q1	Q2	Q3	Total
Marks	30	30	34	94
Out of	32	32	36	100%

Question 1:

Marks	
/5	(a) The voltage developed across a 1 k $\Omega$ resistor is a continuous uniform random variable, denoted $X$ , in the range $-5 \leq X \leq 5$ volts.
/8	(i) Write the formula for the PDF and sketch the graph of the PDF, $f_X(x)$ , as a function of $x$ ? Label all axis.
/8	(ii) Find the mean and variance of the voltage.
/8	(iii) Find the probability $P[ X-1  < 2]$ .
/3	(b) On a laboratory assignment, if the equipment is working, the PDF of the observed random output $Y$ , is
/8	$f_Y(y) = \begin{cases} c(2-y), & 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$
/32	(i) What is the value of the coefficient $c$ ?
	(ii) Determine the conditional probability $P[Y \leq 0.75   Y \geq 0.5]$ .

a)

$$i) f_X(x) = \frac{1}{5 - (-5)} = \frac{1}{10} \quad f_X(x) = \begin{cases} \frac{1}{10} & -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$



$$ii) E[X] = \int_{-5}^5 \left(\frac{1}{10}\right) x dx = \left(\frac{1}{10}\right) \left(\frac{x^2}{2}\right) \Big|_{-5}^5 = \left(\frac{1}{20}\right) (25 - (-25)) = 0$$

$$E[X^2] = \int_{-5}^5 \frac{1}{10} x^2 dx = \left(\frac{1}{10}\right) \left(\frac{1}{3}\right) (x^3) \Big|_{-5}^5 = \left(\frac{1}{30}\right) (125 - (-125)) = \frac{250}{30}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{250}{30} \quad \left(\frac{5}{3}\right)$$

iii)  $P[|X-1| < 2]$

$$-2 < x-1 < 2$$

$$-1 < x < 3$$

$$-1 \int_{-1}^3 \frac{1}{10} dx = \left(\frac{1}{10}\right) (3 - (-1)) = \frac{4}{10} = \frac{2}{5}$$

$$b) i) \int_0^1 c(1-y) dy = 1$$

$$c \left( 2y - \frac{y^2}{2} \right) \Big|_0^1 = 1$$

$$c(4-2) = 1, \quad \boxed{c = \frac{1}{2}}$$

(3/3)

$$ii) P[Y \leq 0.75 | Y \geq 0.5]$$

$$= \frac{\int_{0.5}^{0.75} \left( \frac{1}{2} \right) (2-y) dy}{\int_{0.5}^1 \left( \frac{1}{2} \right) (2-y) dy} = \frac{\left( \frac{1}{2} \right) \left( 2y - \frac{y^2}{2} \right) \Big|_{0.5}^{0.75}}{1 - \left( \frac{1}{2} \right) \left( 2y - \frac{y^2}{2} \right) \Big|_{0.5}^1} =$$

$$= \frac{0.1719}{0.5625}$$

$$1 - \left( 2y - \frac{y^2}{2} \right) \Big|_{0.5}^1$$

$$\frac{0.1719}{0.5625} = 0.3056 \checkmark \left( \frac{8}{8} \right)$$

$$Q_1 - \left( \frac{30}{32} \right)$$

Question 2:

<p>Marks</p> <p>/8 (i)</p> <p>/8 (ii)</p> <p>/8 (iii)</p>	<p>(a) Consider the circuit system in the figure below. Assume the components fail independently. The number in each block represents the reliability of that block.</p> <p>(a) What is the probability that the entire system works?</p> <p>(b) Given that the system works, what is the probability that the component A is not working?</p> <p>(c) It is known that the system does not work. What is the probability that the component A also does not work?</p> <div data-bbox="481 376 1085 631" data-label="Diagram"> </div>
<p>/8 (b)</p>	<p>Assume that a new light bulb will burn out after <math>t</math> hours, where <math>t</math> has a failure PDF given by</p> $f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & 0 < t < \infty, \\ 0, & \text{elsewhere.} \end{cases}$ <p>Assume <math>\lambda = 0.01</math> and find the probability that the bulb will not burn out before <math>t_0</math> hours.</p>
<p>/32</p>	

a)  $P[\text{sys works}] = R_s$

$$1 - R_s = (1 - (0.7 \times 0.7)) \times (1 - (0.8 \times 0.8 \times 0.8))$$

$$R_s = 1 - (1 - 0.7^2)(1 - 0.8^3)$$

$$R_s = 0.75112 \quad (5/8)$$

$$R_s = P[A \cap B \cup (C \cap D \cap E)]$$

$$b) P[\bar{A}|F] = \frac{\bar{A} \cap (A \cap B \cup (C \cap D \cap E))}{0.75112} = \frac{(\bar{A} \cap A \cap B) \cup (\bar{A} \cap C \cap D \cap E)}{0.75112}$$

$$P[\bar{A}|F] = \frac{0.7 \times 0.8 \times 0.8 \times 0.8}{0.75112} = 0.2045 \quad (6/8)$$

$$c) P[\bar{A}|F] \quad \bar{F} = (\bar{A} \cup \bar{B}) \cap (\bar{C} \cup \bar{D} \cup \bar{E})$$

$$= \frac{(\bar{A}) \cap (\bar{A} \cup \bar{B}) \cap (\bar{C} \cup \bar{D} \cup \bar{E})}{1 - 0.75112} = \frac{(\bar{A} \cup \bar{A} \cap \bar{B}) \cap (\bar{C} \cup \bar{D} \cup \bar{E})}{1 - 0.75112}$$

$$= 0.58$$

$$Q_2 = \left( \frac{30}{32} \right)$$



$$b) R_T(t) = e^{-\lambda t}$$

$$R_T(t) = e^{-(0.01)(t)}$$

Ⓟ This formula was from the notes

$$\left(\frac{8}{t}\right)$$

$$\overline{A} \cap (\overline{A} \cup \overline{B}) \cap (\overline{C} \cup \overline{D} \cup \overline{E}) = \overline{A} \cap \overline{A} \cup (\overline{A} \cap \overline{B}) \cap (\overline{C} \cup \overline{D} \cup \overline{E})$$

0.24555



### Question 3:

Marks			In ENEL 419, the fraction of students who receive grades below 85%, each year, is a random variable denoted $X$ and the fraction of students who receive 85% and above is also a random variable denoted $Y$ . The fraction of students who receive grades below 85% is always greater than the fraction that receive grades of 85% and above. The two groups of students are described by the joint PDF function $f_{XY}(x, y) = cxy$ .
15	/2	(a)	Find the value of $c$
10	/10	(a)	Find the marginal probability density functions.
6	/6	(b)	Find the expected values of $X$ and $Y$ .
8	/8	(c)	Find the standard deviations of $X$ and $Y$ .
5.5	/6	(d)	Find the correlation $r_{XY} = E[XY]$ , between $X$ and $Y$
3	/4	(e)	Determine the correlation coefficient.
34	/36		

a)  $\int_0^1 \int_0^x cxy \, dx \, dy = 1$

$1 = \int_0^1 \int_0^x cxy \, dx \, dy$   
 $= c \int_0^1 x \left[ \frac{1}{2} y^2 \right]_0^x \, dx$   
 $= \frac{c}{2} \int_0^1 x^3 \, dx$   
 $= \frac{c}{2} \left[ \frac{1}{4} x^4 \right]_0^1 = \frac{c}{8} \Rightarrow c = 8$

$(c) \left( \frac{x^2}{2} \right) \Big|_0^1 \left( \frac{y^2}{2} \right) \Big|_0^1 = (c) \left( \frac{1}{4} \right) = 1, \boxed{c=4}$

b)  $f_X(x) = \int_0^x 4xy \, dy = 4x \left( \frac{y^2}{2} \right) \Big|_0^x = 2x^3 \quad 0 < x < 1$   
 $0$  otherwise

$f_Y(y) = \int_y^1 4xy \, dx = (4y) \left( \frac{x^2}{2} \right) \Big|_y^1 = 2y(1-y^2) \quad 0 \leq y \leq 1$   
 $0$  otherwise

c)  $E[X] = \int_0^1 2x^4 \, dx = \frac{2}{5} x^5 \Big|_0^1 = \frac{2}{5}$   
 $E[X^2] = \int_0^1 2x^6 \, dx = \frac{2}{7} x^7 \Big|_0^1 = \frac{2}{7}$   
 $\sigma_X = \sqrt{\frac{1}{3} - \left( \frac{2}{5} \right)^2} = 0.416$   
 $\boxed{E[X] = \frac{2}{5}}$   
 $\boxed{E[X^2] = \frac{2}{7}}$   
 $\boxed{\sigma_X^2 = \frac{2}{75}}$   
 $\boxed{\sigma_X = \sqrt{\frac{2}{75}}}$

$E[Y] = \int_0^1 (2)(y^2 - y^4) \, dy = (2) \left( \frac{y^3}{3} - \frac{y^5}{5} \right) \Big|_0^1 = (2) \left( \frac{1}{3} - \frac{1}{5} \right) = 0.26$   
 $E[Y^2] = \int_0^1 (2)(y^3 - y^5) \, dy = (2) \left( \frac{y^4}{4} - \frac{y^6}{6} \right) \Big|_0^1 = (2) \left( \frac{1}{4} - \frac{1}{6} \right) = 0.167$   
 $\sigma_Y = \sqrt{0.167 - (0.26)^2} = 0.315$   
 $\boxed{E[Y] = \frac{8}{15}}$   
 $\boxed{E[Y^2] = \frac{11}{15}}$   
 $\boxed{\sigma_Y^2 = \frac{11}{225}}$   
 $\boxed{\sigma_Y = \sqrt{\frac{11}{225}}}$

$$d) E[xy] = \int_0^1 \int_0^1 xy \cdot f_{xy}(x,y) = 4 \int_0^1 x^2 dx \int_0^1 y^2 dy$$

$$E[xy] = (4) \left( \frac{x^3}{3} \Big|_0^1 \right) \left( \frac{y^3}{3} \Big|_0^1 \right) = (4) \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) = 0.44$$

$$r_{xy} = \int_0^1 \int_0^1 8x^2 y^2 dx dy$$

$$r_{xy} = \frac{4}{9}$$

$$e) = C_{xy} = r_{xy} - \mu_x \mu_y$$

$$C_{xy} = 0.44 - \left( \overset{0.26}{0.416} \right) \left( \overset{0.26}{0.315} \right) = 0.313$$

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} = \frac{0.313}{(0.416)(0.315)} = 2.39$$

$$C_{xy} = r_{xy} - \mu_x \mu_y$$

$$= \frac{4}{9} - \left( \frac{4}{5} \right) \left( \frac{8}{15} \right)$$

$$= \frac{4}{225}$$

$$\rho_{xy} = \frac{\frac{4}{225}}{\left( \frac{\sqrt{2}}{\sqrt{99}} \right) \left( \frac{\sqrt{11}}{\sqrt{15}} \right)}$$

$$\rho_{xy} = \frac{4}{\sqrt{66}}$$

# Formula Sheet

$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$	$f_Y(y) = f_X(g^{-1}(y)) \left  \frac{dg^{-1}(y)}{dy} \right $	$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$ $\sigma_{h(X)}^2 = E\left[\left(h(X) - \mu_{h(X)}\right)^2\right]$
$\sigma_X^2 = E\left[(X - \mu_X)^2\right]$ $= E[X^2] - \mu_X^2$	$\frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \mu_X)^2}{2\sigma_X^2}\right) dx = Q\left(\frac{\gamma - \mu_X}{\sigma_X}\right)$	$p_X(x) = \sum_y p_{XY}(x, y)$
$f_Y(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$	$p_{X Y}(x y) = p_{XY}(x, y) / p_Y(y)$	$f_{X Y}(x y) = \frac{f_{XY}(x, y)}{f_Y(y)}$
$r_{XY} = E[XY]$	$C_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = r_{XY} - \mu_X \mu_Y$	$\rho_{XY} = C_{XY} / (\sigma_X \sigma_Y)$
$P[(X, Y) \in A] = \iint_A f_{XY}(x, y) dx dy$ , $A$ is a region in the $XY$ -plane for which $P[(X, Y) \in A]$ is to be evaluated.		

Do not provide any answers beyond this table. Any answers on this page and beyond will not be marked