ENEL 419 Probability and Random Variables Fall 2023

Quiz #1	Friday, September 22, 2023	Duration: 30 minutes Start Time: 12:40 pm
ID #: 30155686	Last Name: Mansour	First Name: Hazem

Formulas:

$A \cup B = A + B - A \cap B$; $\bar{A} = S - A$; $P[A|B] = \frac{P[A \cap B]}{P[B]}$; $\overline{A \cup B} = \bar{A} \cap \bar{B}$; $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

ANSWER ALL THREE QUESTIONS.

20/20

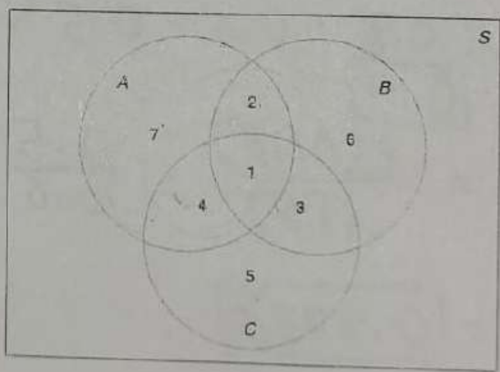
Question 1:

Consider the Venn diagram given below the table. Write down the corresponding numbered regions represented by the set operations listed in the table. Note that a set operation may represent more than one numbered region.

	Set Operations	Regions	Marks
(a)	$A \cup C$	7, 2, 1, 4, 5, 3	/1
(b)	$\bar{B} \cap A$	7, 4	/1
(c)	$A \cap B \cap C$	1	/1
(d)	$(A \cup B) \cap \bar{C}$	7, 2, 6	/1
		Total	/4

(7, 2, 1, 6, 3, 4)

4/4



Question 2:

	Marks
A final year ECE student is applying for graduate studies, with funding, at two top universities (A and B). His personal assessment of the situation is that the probability of getting an offer from university A is 0.8 and the probability of getting an offer from university B is 0.6. If he believes that the probability that he will get offers from both universities is 0.5. Answer the following questions:	
(a) What is the probability that he gets at least one offer from these universities?	/2
(b) Given that he gets an offer from B, what is the probability that he will receive an offer from A?	/2
(c) Given that he does not get an offer from B, what is the probability that he will receive an offer from A?	/2
(d) Given that he does not get an offer from B, what is the probability that he will not receive an offer from A?	/3
(e) What is the probability that he will receive no offer at all?	/2
Total	/11

$P[A]=0.8 \quad P[B]=0.6 \quad P[A\cap B]=0.5$

(a) $P[A\cup B] = P[A]+P[B]-P[A\cap B]$

$= 0.8 + 0.6 - 0.5$

$P[A\cup B] = \boxed{0.9}$

(b) $P[A|B] = \dfrac{P[A\cap B]}{P[B]} = \dfrac{0.5}{0.6}$

$P[A|B] = \boxed{0.8333}$

(c) $P[A|\bar{B}] = \dfrac{P[A\cap \bar{B}]}{P[\bar{B}]} = \dfrac{P[A\cap(S-B)]}{1-P[B]}$

$= \dfrac{P[A]-P[A\cap B]}{1-P[B]} = \dfrac{0.8-0.5}{1-0.6}$

11/11

$P[A|\bar{B}] = \boxed{0.75}$

(d) $P[\bar{A}|\bar{B}] = \dfrac{P[\bar{A}\cap\bar{B}]}{P[\bar{B}]} = \dfrac{P[\overline{A\cup B}]}{1-P[B]} = \dfrac{1-P[A\cup B]}{1-P[B]}$

$P[\bar{A}|\bar{B}] = \dfrac{1-0.9}{1-0.6} = \boxed{0.25}$

(e) $P[\overline{A\cup B}] = 1-P[A\cup B] = \boxed{0.1}$

Question 3:

	Marks
An urn contains 10 balls numbered zero through 9. We conduct an experiment where A ball is pulled out of the urn at random.	
(a) List the elements of the sample space.	/1
(b) List the elements corresponding to an event A that the number drawn is divisible 2.	/1
(c) List the elements corresponding to an event B that the number drawn is divisible by 3.	/1
(d) List the elements corresponding to the event $A \cup B$.	/1
(e) List the elements corresponding to the event $A \cap B$.	/1
Total	/5

(a) $S = \{0,1,2,3,4,5,6,7,8,9\}$

(b) $A = \{0,2,4,6,8\}$

(c) $B = \{0,3,6,9\}$

$5/5$

(d) $A \cup B = \{0,2,3,4,6,8,9\}$

(e) $A \cap B = \{0,6\}$

ENEL 419 $\boxed{\frac{18.5}{20}}$ **Probability and Random Variables** **Fall 2023**

Quiz #2	**Friday, October 13, 2023**	**Duration: 30 minutes** **Start Time: 12:40 pm**
ID #: 30155686	**Last Name:** Mansour	**First Name:** Hazem

Formulas:

Permutation: $n_s = \frac{n!}{(n-k)!}$; $k \le n$ $0! = 1$.	Combination $n_s = \binom{n}{k} = \frac{n!}{(n-k)!k!}$; $k \le n$
Distinguishable permutations (k distinct partitions) $n_s = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots n_{K-1}}{n_K} = \frac{n!}{n_1!\,n_2!\cdots n_K!}$	Multiplication rule: $n_s = n_1 \times n_2 \times \cdots \times n_K$ Independent sequential experiments: $n_s = \left[\binom{K_1}{k_1} \times \ldots \times \binom{K_n}{k_n}\right] + \left[\binom{M_1}{n_1} \times \ldots \times \binom{M_n}{n_n}\right]$
$F_X(x) = P[X \le x] = \sum_{\forall x} p_X(x)$ $\forall$ *is logic symbol meaning "for all"*	$p_X(x) = P[X = x], \forall x$
$P[A \mid B] = \frac{P[A \cap B]}{P[B]}$	$P[B_i \mid A] = \frac{P[A \mid B_i]P[B_i]}{P[A]}P[A]$, $P[A] = \sum_{i}^{n} P[A \mid B_i]P[B_i],\ i = 1,2,\ldots,n$

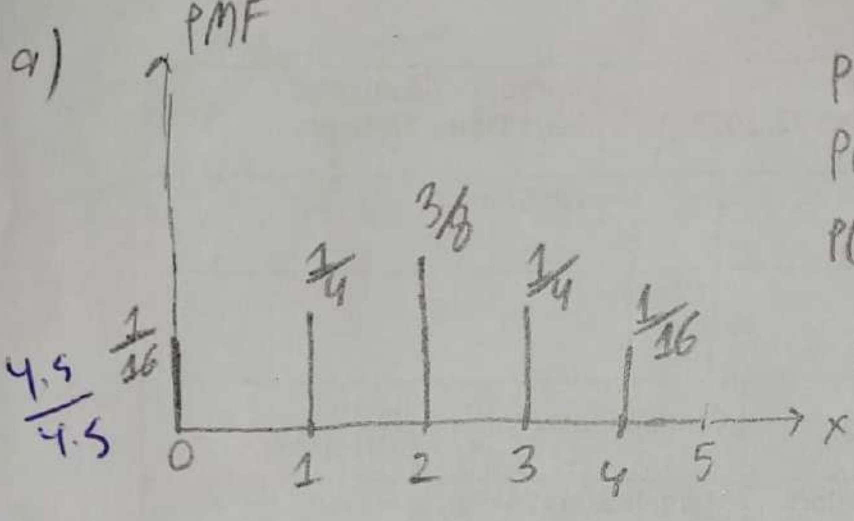
ANSWER ALL THREE QUESTIONS in available the blank space.

Question 1:
The CDF of a discrete random variable X is given below.

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{1}{16}, & \text{for } 0 \le x < 1, \\ \frac{5}{16}, & \text{for } 1 \le x < 2, \\ \frac{11}{16}, & \text{for } 2 \le x < 3, \\ \frac{15}{16}, & \text{for } 3 \le x < 4, \\ 1 & \text{for } x \ge 4. \end{cases}$$

(a) Find and sketch the PMF of the random variable X.
(b) Evaluate the probability $P[X \ge 1 \mid X < 4]$

a)



$P(0) = \frac{1}{16}$

$P(1) = \frac{5}{16} - \frac{1}{16} = \frac{1}{4}$

$P(2) = \frac{11}{16} - \frac{5}{16} = \frac{3}{8}$

$P(3) = \frac{15}{16} - \frac{11}{16} = \frac{1}{4}$

$P(4) = 1 - \frac{15}{16} = \frac{1}{16}$

b) $P[X \geq 1 \mid X < 4] = \dfrac{P[X \geq 1] \cap P[X < 4]}{P[X < 4]}$

← from CDF

$= \dfrac{P[1 \leq X < 4]}{P[X < 4]} = \dfrac{\frac{15}{16} - \frac{5}{16}}{1} = \dfrac{\frac{5}{8}}{1} = \frac{5}{8}$

$\frac{1}{2.5}$

$\hookrightarrow \dfrac{P(1) + P(2) + P(3)}{P(0) + \cdots + P(3)}$

Question 2:

When shipping diesel engines abroad, it is common to pack 12 engines in one container that is then loaded on a rail car and sent to a port. Suppose that a company has received complaints from its customers that many of the engines arrive in nonworking condition. To help solve this problem, the company decides to make a spot check of containers after loading. The company will test 3 engines from the container at random. If any of the 3 are nonworking, the container will not be shipped until each engine in it is checked. Suppose that a given container has 2 nonworking engines.

Find the probability that a container will not be shipped.

$$n_S = \binom{12}{3} = \frac{12!}{(12-3)!\,(3)!} = \frac{12\times 11\times 10\times 9!}{9!\,(3\times 2\times 1)}$$

$$n_S = 220$$

$$n_A = \binom{2}{1}\binom{10}{2} = \frac{2!}{1!\,1!} \times \frac{10!}{(10-2)!\,2!}$$

$$= \frac{2\times 1}{1} \times \frac{10\times 9\times 8!}{8!\,2!}$$

$$= 2\times \frac{10\times 9}{2\times 1}$$

$$n_A = 90$$

$$n_B = \binom{2}{2}\binom{10}{1}$$

$$= 10$$

$$\frac{6}{6}$$

$$P[A] = \frac{n_A + n_B}{n_S} = \frac{90+10}{220} = 0.4545$$

Question 3:

To keep up with demand, a cell phone battery manufacturing company utilizes three machines, B_1, B_2, and B_3. Machines, B_1, B_2, and B_3 make 30%, 45%, *and* 25%, respectively, of the batteries. It is known from experience that 2%, 3%, *and* 2% of the batteries made by machines B_1, B_2, and B_3, respectively, are defective. Now, suppose that a manufactured battery is randomly selected. Answer the following questions:

(a) List the events and their respective probabilities.
(b) What is the probability that a selected battery is defective?
(c) If a battery was selected randomly and found to be defective, what is the probability that it was made by machine B_3?

a) $B_1 \rightarrow$ Machine 1 , $B_2 \rightarrow$ Machine 2 , $B_3 \rightarrow$ Machine 3

$D \rightarrow$ defective battery , $D|B_i \rightarrow$ defective from Machine i, $i=1,2,3$

$P[B_1] = 0.3$, $P[B_2] = 0.45$, $P[B_3] = 0.25$

$P[D|B_1] = 0.02$, $P[D|B_2] = 0.03$, $P[D|B_3] = 0.02$

$$P[D] = P[D|B_1]P[B_1] + P[D|B_2]P[B_2] + P[D|B_3]P[B_3]$$
$$= (0.02)(0.3) + (0.03)(0.45) + (0.02)(0.25)$$

$P[D] = 0.0245$

b) $\boxed{P[D] = 0.0245}$

1/1

c) $$P[B_3|D] = \frac{P[D|B_3]\,P[B_3]}{P[D]}$$
$$= \frac{0.02 \times 0.25}{0.0245}$$

$$\boxed{P[B_3|D] = 0.2041}$$

Quiz #2	Friday, November 3, 2023	Duration: 30 minutes Start Time: 12:40 pm
ID #: 30155686	Last Name: Mansour	First Name: Hazem

Formulas:

$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du;$ $P[a < x < b] = \int_a^b f_X(x)dx$	$\frac{1}{\sqrt{2\pi\sigma_X^2}}\int_\gamma^\infty \exp\left[-\frac{(x-\mu_X)}{2\sigma_X^2}\right] = Q\left(\frac{\gamma-\mu_X}{\sigma_X}\right)$

ANSWER BOTH QUESTIONS in available the blank space.

Question 1:

	Marks
The current, X in amperes, across a $1K\Omega$ is a random variable that has a PDF given by $f_X(x) = \begin{cases} cx^2, & -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$	
(a) Find the value of c.	/2
(b) Find the probability that the absolute value of the current does not exceed 1 amp, that is, $P[\lvert X\rvert < 1]$.	/3
(c) Find the CDF of the random variable. Indicate all the ranges.	/3
(d) Use the CDF to find the probability $P[\lvert X-1\rvert \le 1]$	/5
	/13

a) $\int_{-2}^{2} cx^2 dx = 1$

$x = 1\ 0\ -1$

$\frac{cx^3}{3}\Big|_{-2}^{2} = 1$

$\left[\frac{8c}{3}\right] - \left[-\frac{8c}{3}\right] = 1$

$\frac{8c+8c}{3} = 1$

$16c = 3$

$c = \frac{3}{16}$ ✓

$\frac{2}{2}$

(b) $P[X<1]$ OR $P[X<-1]$

$$P[|X|<1] = \int_{-2}^{1} \frac{3}{16}x^2\,dx + \int_{-2}^{-1} \frac{3}{16}x^2\,dx - P[X<1]\,P[X<-1]$$

$$= \left[\frac{1}{16}x^3\right]\Big|_{-2}^{1} + \left[\frac{1}{16}x^3\right]\Big|_{-2}^{-1}$$

$$= \left[\frac{1}{16} - \left(-\frac{8}{16}\right)\right] + \left[-\frac{1}{16} - \left(-\frac{8}{16}\right)\right]$$

$$= \frac{9}{16} + \frac{7}{16} - \left(\frac{9}{16}\right)\left(\frac{7}{16}\right) = 0.7539$$

X wrong final answer

$\frac{1.5}{3}$

c) $$F_X(x) = \int_{-2}^{x} \frac{3}{16}x^2\,dx = \left[\frac{1}{16}x^3\right]_{-2}^{x}$$

$$= \left[\frac{x^3}{16}\right] - \left[-\frac{8}{16}\right] = \frac{x^3+8}{16}$$

$$F_X(x) = \begin{cases} 0 & x \le -2 \\ \frac{x^3+8}{16} & -2 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

$\frac{2.75}{3}$

$0 \le x \le 2$

d) $P[|X-1| \le 1] = P[X \le 2]$ OR $P[X \le 0]$

$\frac{2.5}{5}$

$P[X \le 2] = F_X(2) - F_X(-2) = 1 - 0 = 1$

$P[X \le 0] = F_X(0) - F_X(-2) = 0.5$

$P[|X-1| \le 1] = P[X \le 2] + P[X \le 0] - P[X \le 2]\,P[X \le 0]$

$= 1 + 0.5 - (1)(0.5)$

$= 1$ X wrong final answer

$\frac{8.75}{13}$

Question 2:

	Marks
Suppose shirt sizes are approximately Gaussian distributed with mean 16.2" and variance 0.81 *square inches*.	
Find the probability that the neck size of a randomly selected individual lies in the range 13.5" *and* 18.9". Express your answer to four decimal places.	/7

$m_X = 16.2 \quad \sigma_X^2 = 0.81 \quad \sigma_X = 0.9$

$P[13.5 \le X \le 18.9] = P[X \le 18.9] - P[X \le 13.5]$

$= (1 - P[X > 18.9]) - (1 - P[X > 13.5])$

$= P[X > 13.5] - P[X > 18.9]$

$= Q\left(\frac{x - m_X}{\sigma_X}\right) - Q\left(\frac{x - m_X}{\sigma_X}\right)$

$= Q\left(\frac{13.5 - 16.2}{0.9}\right) - Q\left(\frac{18.9 - 16.2}{0.9}\right)$ ✓

$= Q(-3) - Q(3)$

$= (1 - Q(3)) - Q(3)$ ✓

$= (1 - 0.0013499) - 0.0013499$ ✓ $\frac{7}{7}$

$P[13.5 \le X \le 18.9] = 0.9973$

✓

$\frac{15.75}{20}$

ENEL 419 **Probability and Random Variables** **Fall 2020**

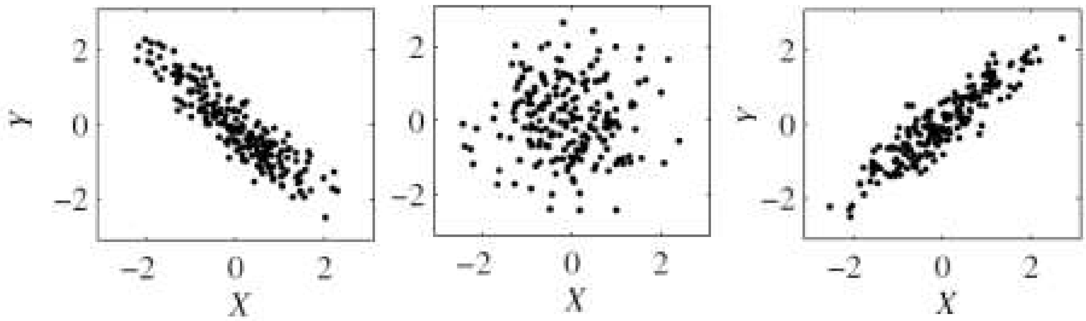
Quiz #4

Question 1:

Random variables X and Y are such that X has expected value $\mu_X = 0$ and standard deviation $\sigma_X = 3$, while Y has mean of $\mu_Y = 1$ and standard deviation of $\sigma_Y = 4$. In addition, X and Y have covariance $C_{XY} = -3$. Find the expected value and variance of $W = 2X + 2Y$.

Question 2:

Consider the three scatter plots of random variables (X, Y) shown in Figures (a), (b) and (c). Suppose the correlation coefficients for the three pairs of X and Y are $\rho_{XY} = 0$, $\rho_{XY} = 0.9$ and $\rho_{XY} = -0.9$, but we do not know which one belongs to what pair. State which correlation coefficient corresponds to which scatter plot.



(a) (b) (c)

Question 3:

Two random variables X and Y have joint PDF given by

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2}, & -1 \le x \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the region of possible pairs (x, y).

(b) Show that X and Y are orthogonal, that is, $E[XY] = 0$, but they are not independent, that is $C_{XY} \neq 0$

Q.1) $\mu_x = 0$, $\sigma_x = 3$, $\mu_y = 1$, $\sigma_y = 4$, $C_{xy} = -3$

$W = 2X + 2Y$

Expected value: $\mu_W = 2\mu_x + 2\mu_y$

$= 2(0) + 2(1)$

$\mu_W = 2$

$Var(2X+2Y) = 2^2 \sigma_x^2 + 2^2 \sigma_y^2 + 2(2)(2) C_{xy}$

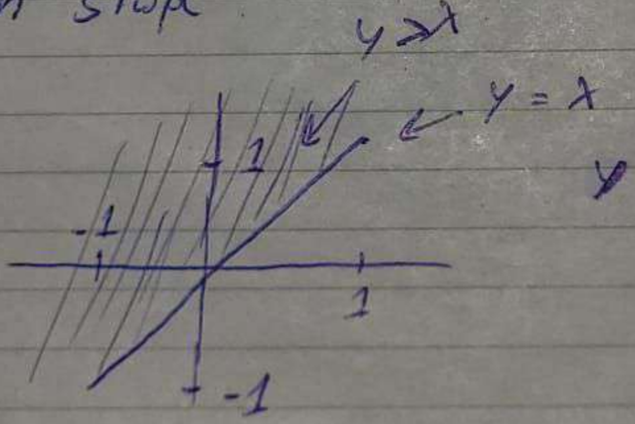
$= 4(9) + 4(16) + 8(-3)$

$= 76$

Q.2) $\rho_{xy} = -0.9$, $\rho_{xy} = 0$, $\rho_{xy} = 0.9$

based on slope

Q.3) a)



b)

$$E[XY] = \int_{-1}^{1}\int_{x}^{1} \frac{1}{2}xy\,dy\,dx$$

$$= \cancel{\frac{1}{4}}\ \frac{1}{2}\int_{-1}^{1}\int_{x}^{1} xy\ dy\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{xy^2}{2}\Big|_{x}^{1} \quad = \left(\frac{x}{2}\right) - \left(\frac{x^3}{2}\right)$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{x - x^3}{2}\,dx \quad = \frac{1}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[\cancel{\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)}\right] \quad = 0$$

$$0$$

$$C_{XY} = E[XY] - \mu_X \mu_Y$$

$$f_X(x) = \int_{x}^{1} \frac{1}{2}dy \quad = \frac{1}{2}y\Big|_{x}^{1} \quad = \frac{1}{2} - \frac{1}{2}x$$

$$\mu_X = \int_{-1}^{1} \frac{1}{2}x - \frac{1}{2}x^2 \quad = \left[\frac{x^2}{4} - \frac{x^3}{6}\right]_{-1}^{1} \quad = \left(\frac{1}{12}\right) - \left(\frac{5}{12}\right)$$

$$= -\frac{1}{3}$$

$$f_Y(y) = \int_{-1}^{y} \frac{1}{2}dx = \frac{1}{2}x\Big|_{-1}^{y} = \frac{1}{2}y + \frac{1}{2}$$

$$\mu_Y = \int_{-1}^{1} \frac{1}{2}y^2 + \frac{1}{2}y\,dy = \left[\frac{y^3}{6} + \frac{y^2}{4}\right]_{-1}^{1}$$

$$= \left(\frac{5}{12}\right) - \left(\frac{1}{12}\right) = \frac{1}{3}$$

$$C_{XY} = 0 - \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$$

ENEL 419 $\frac{15.5}{20}$ **Probability and Random Variables** **Fall 2023**

Quiz #5	Friday, December 1, 2023	Duration: 30 minutes Start Time: 12:40 pm
ID #: 30155686	Last Name: Mansour	First Name: Hazem

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i; \quad \sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n}$	$S_X^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2; \; S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$
$X = \sum_{i=1}^{n} X_i \Rightarrow \mu_X = \sum_{i=1}^{n} \mu_{X_i}; \; \sigma_X^2 = \sum_{i=1}^{n} \sigma_{X_i}^2$	$\bar{X} - \frac{\sigma_X}{\sqrt{n}} z_c \le \mu_X \le \bar{X} + \frac{\sigma_X}{\sqrt{n}} z_c; \; \bar{X} - \frac{S_X}{\sqrt{n}} t_c \le \mu_X \le \bar{X} + \frac{S_X}{\sqrt{n}} t_c$
$X \text{ is Gaussian} \Rightarrow P[X > \gamma] = Q\left[\frac{\gamma - \mu_X}{\sigma_X}\right]$	$Q[-\gamma] = 1 - Q[\gamma]$

Question 1:
The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, *and* 9.6 *liters*. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately Gaussian distribution. **Note: Use unbiased estimates. Show the steps how you arrive at your answer.**

$n = 7 \quad \alpha = 0.05 \quad v = 6$

$\bar{X} = \frac{1}{7}(9.8 + 10.2 + 10.4 + 9.8 + 10 + 10.2 + 9.6)$

$\bar{X} = 10$

$S_X^2 = \frac{1}{7-1}\sum_{i=1}^{7}(X_i - 10)^2$

$= \frac{1}{6}\left((-0.2)^2 + (0.2)^2 + (0.4)^2 + (-0.2)^2 + (0)^2 + (0.2)^2 + (-0.4)^2\right)$

$S_X^2 = 0.08$ $\quad t_c = 1.943$ $\quad$ 2.447 $\quad 10 - \frac{0.2828}{\sqrt{7}}(1.943) \le \mu_X \le 10 + \frac{0.2828}{\sqrt{7}}(1.943)$

$S_X = 0.2828$ $\quad$ X $\quad$ $9.7923 \le \mu_X \le 10.2077$

(-1) $\quad \frac{6}{7}$

Question 2:
A random sample of size 25 is taken from a Gaussian population having a mean of 80 and a standard deviation of 5. A second random sample of size 36 is taken from a different Gaussian

population having a mean of 75 and a standard deviation of 3. Let $\overline{X_1}$ denote the sample mean computed from the 25 measurements and $\overline{X_2}$ the sample mean computed from the 36 measurements.

(a) Find the mean and the standard deviation of the sample mean difference $\overline{X_1} - \overline{X_2}$. Assume the difference of the means to be measured to the nearest tenth (one decimal place). **Show the steps how you arrive at your answer.**

(b) Find the probability that the sample mean computed from the 25 measurements will exceed the sample mean computed from the 36 measurements by at least 3.55 but less than 5.9. Assume the difference of the means to be measured to the nearest tenth. **Show the steps how you arrive at your answer.**

$1: n_1 = 25, \mu_{x_1} = 80, 6_{x_1} = 5$ (-3.5)

$2: n_2 = 36, \mu_{x_2} = 75, 6_{x_2} = 3$

a) $\bar{X}_1 = \mu_{x_1} = 80 \quad \bar{X}_2 = \mu_{x_2} = 75$

$\bar{X_1} - \bar{X}_2 = 80 - 75$
$= 5$

b) $P[\mu_{x_2} + 3.55 < \mu_{x_1} < \mu_{x_2} + 5.9]$

$P[\mu_{x_1} > 78.55] = Q\left(\frac{78.55 - \mu_{x_1}}{6_{x_1}}\right)$
$= Q(-0.29)$
$= 1 - Q(0.29)$
$= 0.618$

$P[\mu_{x_1} < 80.9] = 1 - P[\mu_{x_1} > 80.9]$
$= 1 - Q\left(\frac{80.9 - \mu_{x_1}}{6_{x_1}}\right)$
$= 1 - Q(0.18)$
$= 1 - 0.42 = 0.58$

$P[\mu_{x_2} + 3.55 < \mu_{x_1} < \mu_{x_2} + 5.9]$
$= 0.618 + 0.58 - (0.618)(0.5$
$= 0.8396$