ENEL 419: Probability and Random Variables

Midterm

Instructor: Dr. Abu Sesay
November 8, 2018

Room: ENA 201

Time: 12:30 – 1:45

Last Name (printed):	First Name:	ID #:
[redacted]	[redacted]	[redacted]

Signature: [redacted]

Instructions:

- All the University of Calgary regulations apply to this exam.
- Answer all three questions in the booklet provided.
- You are allowed to use a non-programmable calculator for this exam. No other electronic device, including music devices or anything with wireless capability, will be allowed during the exam.
- The test is closed-book and closed-notes. Formulas and a Q-function table are provided on the last two pages. You may detach them for your convenience.
- To reduce distraction to other students, you are not allowed to leave during the last ten minutes of the exam.
- Please print or write your answers legibly. What cannot be read cannot be marked.
- If you write anything you do not want marked, put a large "X" through it and write "rough work" beside it.

Marks Summary

	Q1	Q2	Q3	Total
Marks	34	33	30.5	97.5
Out of a max of	34	33	33	100%

Note:
If combinations or permutations are involved, you may use the corresponding functions on your calculator and write down your answer.

Question 1:

Marks	(a)	From a box containing 6 black balls and 4 green balls, 3 balls are drawn.
5 /5	(i)	Suppose the 3 balls are drawn in succession with each ball color recorded and the ball is not placed back into the box before the next draw. What is the probability that all 3 balls drawn are of the same color?
5 /5	(ii)	Suppose the 3 balls are drawn in succession with each ball color recorded and the ball is not placed back into the box before the next draw. What is the probability that each color is represented?

6 B 4 G 3 drawn

i) $n_s = \binom{10}{3} = 120$

$P[\text{same color}] = \dfrac{\underbrace{\binom{6}{3}\binom{4}{0}}_{\text{all white}} + \underbrace{\binom{6}{0}\binom{4}{3}}_{\text{all green}}}{\binom{10}{3}} = \dfrac{20+4}{120}$

$P(\text{same color}) = \boxed{0.2}$

ii) $P[\text{each color rep.}] = \dfrac{\underbrace{\binom{6}{1}\binom{4}{2}}_{\text{at least 1 blue}} + \underbrace{\binom{6}{2}\binom{4}{1}}_{\text{at least 1 green}}}{\binom{10}{3}} = \dfrac{36+60}{120}$ $\boxed{0.8}$

at least 1 of each.

Marks		
	(b)	A lot containing 7 electronic components is sampled by a quality control engineer. The lot contains 4 good components and 3 defective components. A sample of 3 is taken by the engineer.
4 /9	(i)	Find the PMF of the good components and write it in tabular form.
3 /3	(ii)	Find the expected value of the number of good components.
5 /5	(iii)	Find the variance of the number of good components.
2 /2	(iv)	Find the probability that the number of good components is either 1 or 2.
5 /5	~~(vi)~~ (V)	Find the probability that the ~~number of good components is between 1 and 2, that is,~~ $P[1 < X \le 3]$.
24 /34		

i) $P_X(x) = \dfrac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}$

$P_X(0) = \dfrac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} = 0.0286$ $P_X(1) = \dfrac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} = 0.343$

$P_X(2) = \dfrac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = 0.514$ $P_X(3) = \dfrac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = 0.114$

X	0	1	2	3
$P_X(x)$	0.0286	0.343	0.514	0.114

ii) $\mu_X = E[X] = 0 \times 0.0286 + 1 \times 0.343 + 2 \times 0.514 + 3 \times 0.114 = \boxed{1.71}$

iii) $\sigma_X^2 = E[X^2] - \mu_X^2 \rightarrow E[X^2] = 0^2 \times 0.0286 + 1^2 \times 0.343 + 2^2 \times 0.514 + 3^2 \times 0.114$

$E[X^2] = \boxed{3.425}$

$\sigma_X^2 = 3.425 - 1.71^2 = \boxed{0.5}$

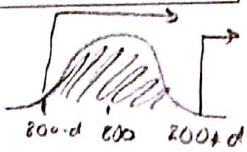
iv) $P[1 \le X \le 2] = \underset{P_X(1)}{0.343} + \underset{P_X(2)}{0.514} = \boxed{0.857}$

V) $P[1 < X \le 3] = P_X(2) + P_X(3) = 0.514 + 0.114 = \boxed{0.628}$

Question 2:

Marks		
	(a)	An electrical firm manufactures light bulbs that have a life, before burn-out, that is Gaussian distributed with a mean value equal to 800 hours and a variance of 1600 $(\text{hours})^2$.
12 /12	(i)	Find the value of a time interval d, such that the probability that a bulb burns out between $800-d$ and $800+d$ hours is equal to 0.5485.
6 /6	(ii)	Find the value of k such that $P[X>k]=0.77337$.

i) $P[800-d < x < 800+d] = 0.5485$



$= P[x > 800-d] - P[x > 800+d]$

$= Q\left(\frac{800-d-800}{40}\right) - Q\left(\frac{800+d-800}{40}\right)$

$\sigma_x^2 = 1600$

$\sigma_x = \sqrt{1600} = 40$

$= Q\left(\frac{-d}{40}\right) - Q\left(\frac{d}{40}\right) = 1 - Q\left(\frac{d}{40}\right) - Q\left(\frac{d}{40}\right)$

$0.5485 = 1 - 2Q\left(\frac{d}{40}\right) \quad Q\left(\frac{d}{40}\right) = \frac{1-0.5485}{2} = 0.22575$

$\frac{d}{40} = 0.74 \rightarrow \boxed{d = 29.6} \rightarrow$

$800 - 29.6 = 770.4$

$800 + 29.6 = 829.6$

ii) $P[X>k] = 0.77337 = Q\left[\frac{k-800}{40}\right] \rightarrow 1 - 0.77337 = 1 - Q\left(\frac{k-800}{40}\right)$

$0.22663 = Q\left(\frac{800-k}{40}\right) \quad 0.75 = \frac{800-k}{40} \quad 30 = 800-k$

$k = 800 - 30$

$\boxed{k = 770}$ ✓

Marks		
	(b)	Consider a random variable X, that is uniform over the range $a \le X \le b$.
6 /6	(i)	Show that the mean is equal to $\mu_X = \frac{b+a}{2}$.
9 /9	(ii)	Show that the variance is equal to $\sigma_X^2 = \frac{(b-a)^2}{12}$.
33 /33		

i) $f_x(X) = \int_a^b c\,dx = cx\Big|_a^b = c(b-a) = 1 \rightarrow c = \frac{1}{b-a}$

$\mu_x(x) = \int_a^b \frac{1}{b-a} x\,dx = \frac{1}{b-a}\frac{x^2}{2}\Big|_a^b = \frac{1}{b-a}\frac{(b^2-a^2)}{2} = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$

$\mu_x(x) = \boxed{\frac{b+a}{2}}$ ✓

ii) $E[x^2] = \int_a^b \frac{1}{b-a} x^2\,dx = \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b = \frac{1}{b-a}\frac{(b^3-a^3)}{3} = \frac{(b-a)(b^2+ba+a^2)}{(b-a)\,3}$

$E[x^2] = \frac{b^2+ba+a^2}{3}$

$\sigma_x^2 = E[x^2] - \mu_x^2 = \frac{b^2+ba+a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{4}{4}\cdot\frac{b^2+ba+a^2}{3} - \frac{b^2+2ab+a^2}{4}\cdot\frac{3}{3}$

$= \frac{4b^2+4ba+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)(b-a)}{12}$

$\boxed{\sigma_x^2 = \frac{(b-a)^2}{12}}$ ✓

Question 3:

Marks		
		The voltage at the input of an amplifier is a random variable X, with a PDF given by $$f_X(x)=\begin{cases}2x, & 0\le x\le 1,\\ 0, & \text{otherwise.}\end{cases}$$ The output of the amplifier is a new random variable $Y=4X+1$.
5 /5	(i)	What is the CDF of X?
2 /2	(ii)	What is the range of values of the output voltage Y?
1 /2	(iii)	What is the CDF of Y?
3 /4	(iv)	What is the PDF of Y?
5 /5	(v)	Find the mean values of X and Y.
5 /5	(vi)	Find the mean-square values of X and Y.
5 /5	(vii)	Find the standard deviations of X and Y.
4.5 /5	(viii)	What is the probability that the output is at least 1.5 volts.
30.5/33		

i) $F_X(x) = \int_0^x 2x\,dx = x^2\Big|_0^x = x^2 \rightarrow F_X(x) = \begin{cases} 0 & x<0 \\ x^2 & 0\le x\le 1 \\ 1 & x>1 \end{cases}$

ii) $Y = 4X+1$ at $x=0 \rightarrow y = 4(0)+1 \rightarrow y=1$

at $x=1$ $\quad y = 4(1)+1 = y=5$ $\qquad 1\le Y\le 5$

iii) $Y=4X+1 \rightarrow x = \frac{y-1}{4}$ $\qquad F_Y(y) = F_X\left(\frac{y-1}{4}\right) = \left(\frac{y-1}{4}\right)^2$

$F_Y(y) = \frac{(y-1)^2}{16}$ for $1\le y\le 5$

iv) $\left|\frac{dg^{-1}(y)}{dy}\right| = \frac{1}{4}$ $\qquad f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right| = 2\left(\frac{y-1}{4}\right)\cdot\frac{1}{4}$

$f_Y(y) = \frac{1}{2}\left(\frac{y-1}{4}\right) = \frac{1}{8}(y-1)$ for $1\le y\le 5$

v) $\mu_X = E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}(1-0) = \boxed{\frac{2}{3}} = 0.67$

$\mu_Y = E[Y] = \int_1^5 y f_Y(y)\,dy = \frac{1}{8}\int_1^5 (y^2-y)\,dy = \frac{1}{8}\left(\frac{y^3}{3}-\frac{y^2}{2}\right)\Big|_1^5$

$= \frac{1}{8}\left[\frac{5^3}{3} - \frac{5^2}{2} - \left(\frac{1^3}{3}-\frac{1^2}{2}\right)\right] = \frac{1}{8}\left[\frac{125}{3} - \frac{25}{2} - \frac{1}{3} + \frac{1}{2}\right]$

$= \frac{1}{8}\left[\frac{2}{2}\cdot\frac{124}{3} - \frac{24}{2}\cdot\frac{3}{3}\right] = \frac{1}{8}\left[\frac{248-72}{6}\right] = \frac{176}{8\cdot 6} = \frac{22}{6} = \boxed{\frac{11}{3}} \approx 3.67$

vi) $E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = \frac{2}{4}x^4\Big|_0^1 = \boxed{\frac{1}{2}} = 0.5$

$E[Y^2] = \int_1^5 y^2 f_Y(y) = \frac{1}{8}\int_1^5 (y^3-y^2)\,dy = \frac{1}{8}\left(\frac{y^4}{4}-\frac{y^3}{3}\right)\Big|_1^5$

$= \frac{1}{8}\left(\frac{5^4}{4} - \frac{5^3}{3} - \left(\frac{1^4}{4}-\frac{1^3}{3}\right)\right) = \frac{1}{8}\left(\frac{625}{4} - \frac{125}{3} - \frac{1}{4} + \frac{1}{3}\right)$

$= \frac{1}{8}\left(\frac{3}{3}\cdot\frac{624}{4} - \frac{124}{3}\cdot\frac{4}{4}\right) = \frac{1}{8}\left(\frac{1872-496}{12}\right) = \frac{172}{12} = \boxed{\frac{43}{3}} \approx 14.3$

vii)

$$\sigma_x = E[x^2] - \mu_x^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{9}{9}\cdot\frac{1}{2} - \frac{4}{9}\cdot\frac{2}{2} = \frac{9}{18} - \frac{8}{18}$$

$$\sigma_x^2 = \frac{1}{18} \rightarrow \sigma_x = \sqrt{\sigma_x^2} = \boxed{\frac{1}{\sqrt{18}}} \simeq 0.23$$

$$\sigma_y^2 = E[y^2] - \mu_y^2 = \frac{43}{3} - \left(\frac{11}{3}\right)^2 = \frac{9}{9}\cdot\frac{43}{3} - \frac{121}{9}\cdot\frac{3}{3} = \frac{387-363}{27}$$

$$\sigma_y^2 = \frac{24}{27} \rightarrow \sigma_y = \sqrt{\sigma_y^2} = \boxed{\sqrt{\frac{24}{27}}} \simeq 0.94$$

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viii) $P[Y \geq 1.5] = \int_{1.5}^{5} f_Y(y)\,dy = \frac{1}{8}\int_{1.5}^{5} (y-1)\,dy$

$$= \frac{1}{8}\left(\frac{y^2}{2} - y\right)\Big|_{1.5}^{5} = \frac{1}{8}\left(\frac{5^2}{2} - 5 - \left(\frac{1.5^2}{2} - 1.5\right)\right)$$

$$= \frac{1}{8}(12.5 - 5 - 2.25 + 1.5) = \boxed{\cancel{0.844}}$$

0.98