1. The CDF of a random variable W

$$F_W(w) = \begin{cases} 0 & w < -5, \\ \frac{w+5}{8} & -5 \le w < -3, \\ \frac{1}{4} & -3 \le w < 3, \\ \frac{1}{4} + \frac{3(w-3)}{8} & 3 \le w < 5, \\ 1 & w \ge 5. \end{cases}$$

(a) What is $P[W \le 4]$?

(b) What is $P[-2 < W \le 2]$?

(c) What is $P[W > 0]$?

(d) What is the value of a such that $P[W \le a] = 1/2$?

(a)

$$P[W \le 4] = F_W(4) = 1/4 + 3/8 = 5/8.$$

(b)

$$P[-2 < W \le 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0.$$

(c)

$$P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 3/4.$$

(d) By inspection of $F_W(w)$, we observe that $P[W \le a] = F_W(a) = 1/2$ for a in the range $3 \le a \le 5$. In this range,

$$F_W(a) = 1/4 + 3(a-3)/8 = 1/2.$$

This implies $a = 11/3$.

2. Find the PDF $f_U(u)$ of the random variable U

We find the PDF by taking the derivative of $F_U(u)$ on each piece that $F_U(u)$ is defined. The CDF and corresponding PDF of U are

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \le u < -3, \\ 1/4 & -3 \le u < 3, \\ 1/4 + 3(u-3)/8 & 3 \le u < 5, \\ 1 & u \ge 5, \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5, \\ 1/8 & -5 \le u < -3, \\ 0 & -3 \le u < 3, \\ 3/8 & 3 \le u < 5, \\ 0 & u \ge 5. \end{cases}$$

3. Random variable X has PDF

$$f_X(x) = \begin{cases} 1/4 & -1 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Define the random variable $Y = h(X) = X^2$.

(a) Find $E[X]$ and $\mathrm{Var}[X]$

(b) Find $E[h(X)]$

(c) Find $E[Y]$ and $\mathrm{Var}[Y]$

We recognize that X is a uniform random variable from [-1,3].

(a) $\mathrm{E}[X] = 1$ and $\mathrm{Var}[X] = \frac{(3+1)^2}{12} = 4/3$.

(b) The new random variable Y is defined as $Y = h(X) = X^2$. Therefore

$$\mathrm{E}[h(X)] = \mathrm{E}\left[X^2\right] = \mathrm{Var}[X] + \mathrm{E}[X]^2 = 4/3 + 1 = 7/3.$$

(c) Finally

$$\mathrm{E}[Y] = \mathrm{E}[h(X)] = \mathrm{E}\left[X^2\right] = 7/3,$$

$$\mathrm{Var}[Y] = \mathrm{E}\left[X^4\right] - \mathrm{E}\left[X^2\right]^2 = \int_{-1}^{3} \frac{x^4}{4}\,dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9}.$$

4. The CDF of a random variable U is

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \le u < -3, \\ 1/4 & -3 \le u < 3, \\ 1/4 + 3(u-3)/8 & 3 \le u < 5, \\ 1 & u \ge 5. \end{cases}$$

(a) Find $E[U]$ and $\mathrm{Var}[U]$

(b) Find $E\left[2^U\right]$

To find the moments, we first find the PDF of U by taking the derivative of the CDF. The corresponding PDF is

$$f_U(u) = \begin{cases} 0 & u < -5, \\ 1/8 & -5 \le u < -3, \\ 0 & -3 \le u < 3, \\ 3/8 & 3 \le u < 5, \\ 0 & u \ge 5. \end{cases}$$

(a) The expected value of U is

$$\begin{aligned} \mathrm{E}[U] = \int_{-\infty}^{\infty} u f_U(u)\, du &= \int_{-5}^{-3} \frac{u}{8}\, du + \int_3^5 \frac{3u}{8}\, du \\ &= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_3^5 = 2. \end{aligned}$$

(b) The second moment of U is

$$\begin{aligned} \mathrm{E}\left[U^2\right] = \int_{-\infty}^{\infty} u^2 f_U(u)\, du &= \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_3^5 \frac{3u^2}{8}\, du \\ &= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_3^5 = 49/3. \end{aligned}$$

The variance of U is $\mathrm{Var}[U] = \mathrm{E}[U^2] - (\mathrm{E}[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\, du = \int e^{(\ln 2)u}\, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2}.$$

The expected value of 2^U is then

$$\begin{aligned} \mathrm{E}\left[2^U\right] &= \int_{-\infty}^{\infty} 2^u f_U(u)\, du \\ &= \int_{-5}^{-3} \frac{2^u}{8}\, du + \int_3^5 \frac{3 \cdot 2^u}{8}\, du \\ &= \left.\frac{2^u}{8\ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8 \ln 2}\right|_3^5 = \frac{2307}{256 \ln 2} = 13.001. \end{aligned}$$

5. Radars detect flying objects by measuring the power reflected from them. The reflected power of an aircraft can be modeled as a random variable Y with PDF

$$f_Y(y) = \begin{cases} \dfrac{1}{P_0} e^{-y/P_0}, & \text{y} \ge 0 \\ 0, & \text{elsewhere} \end{cases} \qquad \text{where } P_0 = \text{constant.}$$

The aircraft is correctly identified by the radar if the reflected power of the aircraft is larger than its average value. What is the probability $P[C]$ that an aircraft is correctly identified?

$$F_Y(y) = P[Y \le y] = \begin{cases} 1 - e^{-\lambda y}, & 0 \le y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

The reflected power Y has an exponential PDF with $\lambda = 1/P_0$, therefore $E[Y] = P_0$. The probability that an aircraft is correctly identified is

$$P[Y > P_0] = 1 - P[Y \le P_0] = 1 - F_Y(P_0) = 1 - \left(1 - e^{-1}\right) = e^{-1}$$

6. An exponential random variable has parameter λ.
   (a) $P[X > 4]$, X is Gaussian with $\mu_X = 0,\ \sigma_X^2 = 4$.
   (b) $P[Y \le 2]$, Y is Gaussian with $\mu_Y = 2,\ \sigma_Y^2 = 25$
   (c) $P[Z \le \mu_Z + 1]$, Z is Gaussian with $\mu_Z = 2,\ \sigma_Z^2 = 4$
   (d) $P[W > 65]$, W is Gaussian with $\mu_W = 50,\ \sigma_W = 10$

Solutions:

(a) $P[X > 4] = Q\left(\dfrac{4-0}{2}\right) = Q(2) = 0.2272$

(b) $P[Y \le 2] = 1 - P[Y > 2] = 1 - Q\left(\dfrac{2-2}{5}\right) = 0.5$

(c) $P[Z \le \mu_Z + 1] = 1 - P[Z > \mu_Z + 1] = 1 - Q\left(\dfrac{\mu_Z + 1 - \mu_Z}{2}\right) = 1 - Q(0.5) = 1 - 0.30854 = 0.69$

(d) $P[W > 65] = Q\left(\dfrac{65-50}{10}\right) = 0.067$

7. In each of the following cases X is a Gaussian random variable. Find the value of μ_X.
   (a) $\sigma_X = 10$ and $P[X \le 10] = 0.933$ **(Answer: $\mu_X = -5$)**
   (b) $\sigma_X = 10$ and $P[X \le 0] = 0.067$ **(Answer: $\mu_X = 15$)**
   (c) σ_X is unknown and $P[X \le 10] = 0.977$. Express μ_X as a function of σ_X **(Answer: $\mu_X = 10 - 2\sigma_X$)**
   (d) $P[X > 5] = 0.5$ **(Answer: $\mu_X = 5$)**

8. The temperature T in a thermostatically controlled lecture hall is Gaussian random variable with expected value $\mu_T = 20^o\mathrm{C}$. In addition, $P[T < 18] = 0.1587$. Find the variance of T. **(Answer: $\sigma_X^2 = 4$)**