Yousif AL-Khoury
UCID: 30080225

ENEL 419
Assignment 3

Problem 1)

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int 2x\,dx = x^2 \Big|_0^1 = 1$$

$$Y = 0.5X + 0.25$$

a) range of y $\quad (0.5(0)+0.25) \le y \le (0.5(1)+0.25)$

$$\boxed{0.25 \le y \le 0.75}$$

b) CDF of $X = F_X(x) = P(X \le x)$

$$F_X(x) = \int_{-\infty}^{x} f_x(u)\,du$$

for $x < 0$

$$F_X(x) = \int_{-\infty}^{0} 0\,du = 0$$

For $0 \le x \le 1$

$$F_X(x) = \int_0^x 2u\,du = u^2 \Big|_0^x = x^2$$

$F_x(x) = 1$ for $x > 1$

So $$F_x(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

c) CDF of Y

$F_y(y) = P[Y \le y = 0.5x + 0.25]$

$x = \frac{y - 0.25}{0.5} = 2y - 0.5$

$F_y(y) = P[X \le x = 2y - 0.5] = F_x(2y - 0.5)$

So,

$$F_y(y) = \begin{cases} 0 & y < 0.25 \\ 4y^2 - 2y + 0.25 & 0.25 \le y \le 0.75 \\ 1 & 0.75 < y \end{cases}$$

d] PDF of Y

since $F_y(y) = F_x(2y - 0.5)$

$$f_y(y) = \frac{dF_x(2y-0.5)}{dy} = f_x(2y-0.5)\cdot 2$$

$$= 2\cdot 2(2y-0.5) = 8y-2$$

$$\boxed{f_y(y) = \begin{cases} 8y-2 & 0.25 \leq y \leq 0.75 \\ 0 & \text{otherwise} \end{cases}}$$

e] Y's PDF is valid if

$$\int_{-\infty}^{\infty} f_y(y)\, dy = 1$$

$$= \int_{0.25}^{0.75} 8y-2\, dy = \left[4y^2 - 2y\right]\Big|_{0.25}^{0.75} = \frac{3}{4} - \left(-\frac{1}{4}\right) = 1$$

Furthermore,

$f_y(y) \geq 0$ for any y

f] $E[X] = \int_0^1 x \cdot 2x \, dx = \frac{2}{3}x^3 \Big|_0^1 = \boxed{\frac{2}{3}} = \mu_x$

$E[Y] = 0.5\left(\frac{2}{3}\right) + 0.25 = \int_{0.25}^{0.75} 8y^2 - 2y \, dy$

$= \frac{4}{12} + \frac{3}{12} = \boxed{\frac{7}{12}} = \mu_y$

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g] $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_0^1 2x^3 \, dx = \frac{1}{2} x^4 \Big|_0^1$

$\boxed{= \frac{1}{2}}$

$E(Y^2) = \int_{0.25}^{0.75} (8y^3 - 2y^2) \, dy = 2y^4 - \frac{2}{3}y^3 \Big|_{0.25}^{0.75} = \frac{45}{128} - \left(-\frac{1}{384}\right)$

$= \frac{45}{128} + \frac{1}{384} = \boxed{\frac{17}{48}}$

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h] Standard deviation, σ

$\sigma_x = \sqrt{E[X^2] - \mu_x^2} = \sqrt{\frac{1}{2} - \left(\frac{2}{3}\right)^2}$

$= \sqrt{\frac{1}{18}} = \frac{\sqrt{2}}{6} = \boxed{0.236}$

$\sigma_y = \sqrt{E[Y^2] - \mu_y^2} = \sqrt{\frac{17}{48} - \left(\frac{7}{12}\right)^2} = \frac{\sqrt{143}}{12} = \frac{\sqrt{2}}{12}$

$\boxed{= 0.118}$

Problem 2

$$P[\mu_x - 2\sigma_x < X < \mu_x + 2\sigma_x] =$$

$$P[X < \mu_x + 2\sigma_x] - P[X < \mu_x - 2\sigma_x] =$$

$$F(\mu_x + 2\sigma_x) - F(\mu_x - 2\sigma_x) =$$

$$\mu_x = \int_0^1 x \cdot 6x(1-x)\,dx = \int_0^1 6x^2 - 6x^3\,dx = 2x^3 - \frac{3}{2}x^4 \Big|_0^1 = \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 6x(1-x)\,dx = \int_0^1 6x^3 - 6x^4 = \frac{3}{2}x^4 - \frac{6}{5}x^5 \Big|_0^1 = \frac{3}{10} - 0$$

$$= \frac{3}{10}$$

$$\sigma_x = \sqrt{\frac{3}{10} - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{10} = 0.224$$

$$\mu_x + 2\sigma_x = \frac{1}{2} + 2\left(\frac{\sqrt{5}}{6}\right) = \frac{5 + 2\sqrt{5}}{10} = 0.947$$

$$\mu_x - 2\sigma_x = \frac{1}{2} - 2\left(\frac{\sqrt{5}}{10}\right) = \frac{5 - 2\sqrt{5}}{10} = 0.0528$$

$$P[0.0528 < X < 0.947] = \int_{0.0528}^{0.947} 6x(1-x)\,dx = \int_{0.0528}^{0.947} 6x - 6x^2\,dx$$

$$= 3x^2 - 2x^3 \Big|_{0.0528}^{0.947} = 0.99187 - 0.0080691$$

$$\boxed{= 0.9838}$$

Problem 3

$$f_X(x) = \begin{cases} \frac{1}{5} e^{-\frac{x}{5}} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

a) $E[X] = \int_0^\infty x\left(\frac{1}{5} e^{-\frac{x}{5}}\right) dx = \frac{1}{5}\left[\int_0^\infty x e^{-\frac{x}{5}} dx\right]$

$$\int_0^\infty x e^{-\frac{x}{5}} dx = \left[x(-5e^{-\frac{x}{5}}) - \int -5e^{-\frac{x}{5}} dx\right]_0^\infty$$

$$= \left[-5x e^{-\frac{x}{5}} + 5(-5e^{-\frac{x}{5}})\right]_0^\infty$$

$$= \left[-5e^{-\frac{x}{5}}(x+5)\right]_0^\infty$$

$$E[X] = \frac{1}{5}\left[-5e^{-\frac{x}{5}}(x+5)\right]_0^\infty =$$

$$= \frac{1}{5} \lim_{t \to \infty} \left[-5e^{-\frac{t}{5}}(t+5) - (-25)\right]$$

$$= \frac{1}{5} \lim_{t \to \infty} \left[-5e^{-\frac{t}{5}} t\left(1 + \frac{5}{t}\right) + 25\right]$$

$$\lim_{t \to \infty} -5e^{-\frac{t}{5}} t = -5 \lim_{t \to \infty} \frac{t}{e^{\frac{t}{5}}} = -5 \lim_{t \to \infty} \frac{1}{\frac{1}{5} e^{\frac{t}{5}}}$$

$$= -5 \lim_{t \to \infty} \left(5 e^{-\frac{t}{5}}\right) = 0$$

$$\lim_{t \to \infty} \left(1 + \frac{5}{t}\right) = 1 + 0 = 1$$

$$E[X] = \frac{1}{5}\left[0\times 1 + 25\right] = 5$$

$$\boxed{E[X] = 5}$$

b) $$E[X^2] = \int_0^\infty x^2\left(\frac{1}{5}e^{-\frac{x}{5}}\right)dx = \frac{1}{5}\int_0^\infty x^2 e^{-\frac{x}{5}}\,dx$$

$$\int_0^\infty x^2 e^{-\frac{x}{5}}\,dx = \left[x^2\left(-5e^{-\frac{x}{5}}\right) - \int\left(-5e^{-\frac{x}{5}}\right)2x\,dx\right]_0^\infty$$

$$= \left[-5x^2 e^{-\frac{x}{5}} + 10\int x e^{-\frac{x}{5}}\,dx\right]_0^\infty$$

$$E[X^2] = \left[-5x^2 e^{-\frac{x}{5}} + 10\left(-5e^{-\frac{x}{5}}(x+5)\right)\right]_0^\infty \cdot \frac{1}{5}$$

$$E[X^2] = \left[-5e^{-\frac{x}{5}}x^2\left(1 + \frac{10}{x} + \frac{50}{x^2}\right)\right]_0^\infty \cdot \frac{1}{5}$$

$$-5\lim_{t\to\infty} e^{-\frac{x}{5}}x^2 = -5\lim_{t\to\infty}\frac{x^2}{e^{\frac{x}{5}}} = -5\lim_{t\to\infty}\frac{2x}{\frac{1}{5}e^{\frac{x}{5}}} = -5\lim_{t\to\infty}\frac{2}{\frac{1}{25}e^{\frac{x}{5}}} = 0$$

$$E[X^2] = \left[0 - (-250)\right]\cdot\frac{1}{5} = 50$$

Variance, $\sigma^2 = E[X^2] - (E[X])^2 = 50 - 5^2 = 25$

$$\boxed{\sigma^2 = 25}$$

standard deviation, $\sigma = \sqrt{\sigma^2} = \sqrt{25} = 5$

$$\boxed{\sigma = 5}$$

$\boxed{c}$ $Y = X + 5$ $\qquad X = Y - 5$

range of Y

$Y > 0 + 5 \Rightarrow Y > 5$

$$f_Y(y) = \frac{d F_X(y-5)}{dy} = f_X(y-5) \cdot 1$$

$$f_X(y-5) = \frac{1}{5} e^{-\left(\frac{y-5}{5}\right)} = \frac{1}{5} e^{-\frac{y}{5}+1} = \frac{e}{5} e^{-\frac{y}{5}}$$

$$f_Y(y) = \begin{cases} \frac{e}{5} e^{-\frac{y}{5}}, & y > 5 \\ 0, & \text{elsewhere} \end{cases}$$

$$E[Y^2] = \frac{e}{5} \int_5^{\infty} y^2 e^{-\frac{y}{5}} \, dy$$

$$= \left[-5y^2 e^{-\frac{y}{5}} + 10\left(-5e^{-\frac{y}{5}}(x+5)\right) \right]_5^{\infty} \cdot \frac{e}{5}$$

$$= \left[0 - (-229.925) \right] \cdot \frac{e}{5} = \underline{125}$$

$$\boxed{E[(X+5)^2] = 125}$$