3. Sequential Experiments and Principles of Counting (Combinatorics)

(Reading Exercises: Montgomery and Runger Chapter 2 - Sections 2.2 & Class notes) or (Yates and Goodman Chapter 2)

Applications include,

- The study of all possible arrangements of discrete objects,
- Algorithm complexity analysis,
- Resource allocation & scheduling, for example internet resources and frequency resources in communications (for example wireless communications),
- Security analysis (for example, assignment of IP and security codes),

> **Definition:** A sequential experiment is one that consists of a sequence of sub-experiments or tasks $A_1, A_2, \cdots, A_K$, where each subsequent task is dependent on the previous. Furthermore, there are n_i ways of completing sub-task A_i, for each $i = 1,2, \cdots K$.

Problem: We need to determine the probability of an event in a sequential experiment.

- The classical approach for computing the probability of an event A requires counting of all possible outcomes,

$$P[A] = \frac{\text{number of occurences of event } A}{\text{total number of occurrences of all events}} = \frac{n_A}{n_S}$$

For sequential experiments, we need the counts n_A andn_s, which can be extremely difficult.

Learning Objectives:

You will:

- Know and be able to apply the multiplication principle.
- Know how to count objects when the objects are sampled with replacement (repetition).
- Know how to count objects when the objects are sampled without replacement (no repetition).
- Know and be able to use the permutation formula to count the number of ordered arrangements of n objects taken n at a time.
- Know and be able to use the permutation formula to count the number of ordered arrangements of n objects taken $k < n$ at a time.
- Know and be able to use the combination formula to count the number of unordered subsets of k objects taken from n objects.
- Know and be able to use the combination formula to count the number of distinguishable permutations of $n$objects, in which k are of the objects are of one type and $n - k$ are of another type.

- Understand and be able to count the number of distinguishable permutations of n objects, when the objects are of more than two types.
- Know how to apply the methods learned in this section to new counting problems.

Counting methods: These are systematic methods used to count the total number of ways to complete an entire experiment, which then facilitates the calculation of the probability for the various ways of completing a sequential experiment.

The following is a list of counting methods most frequently used:

- Tree diagram
- Multiplication rule
- Permutations
- Combinations
- Distinguishable Permutations

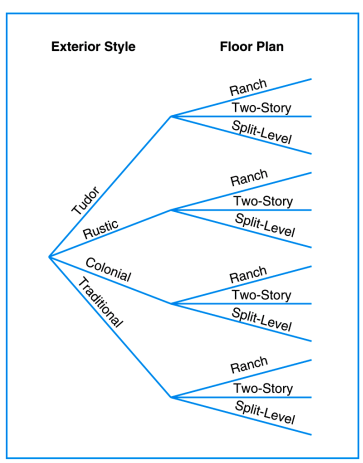
3.1 Method 1: Tree diagram

Example: A property developer offers prospective home buyers a choice of 4 exterior styles: Tudor, Rustic, Colonial and Traditional. The developer also offers 3 floor plans, Ranch, Two-storey, and Split level.

Solution: Choosing a property involves two sub-tasks:

(1) Sub-task A_1- selecting exterior styles in $n_2 = 4$ ways.
(2) Sub-task A_2- selecting floor plans in $n_1 = 3$ ways.

A tree diagram is shown below.



Counting the number of branches on the right-hand side of the tree, gives 12 different ways the buyer may choose a property. Each different way corresponds to an event.

3.2 Method 2: Multiplication rule

Multiplication Rule: Suppose an experiment (or procedure) consists of a sequence of sub-tasks$A_1, A_2, \cdots, A_K$; each completed one after the other. Furthermore, the sub-tasks can be completed in $n_1, n_2, \ldots, n_k$ ways, respectively. According to the multiplication rule, the total number of ways to complete the entire experiment is given by the product,

$$n_S = n_1 \times n_2 \times \cdots \times n_K$$

Conditions:
- Duplication is permissible
- Order is important

Example: Consider the previous example involving the choice of a property. The total of ways to select a property is

$$n_S = n_1 \times n_2 = 12$$

Example: The computer parts example - Different combinations. We want to build a computer from parts. We can buy the motherboard from two companies (A, B), RAM from four companies (A, C, D, E), hard drive from three companies (B, D, F) and graphics card from two companies (G, H). For each part, we are equally likely to buy from any one of the companies that manufactures that part. What is the probability of the event X, that we build a computer that has at least one part from company D?

Solution: The number of choices for the individual events are

$$n_1 = 2; \;\; n_2 = 4; \;\; n_3 = 3; \;\; n_4 = 2$$

The total number of possible choices: $N = n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 2 = 48$.

For "at least one part from D ", consider the complement $\overline{D}$, "no part from D"; see table below.

Motherboard	RAM	HD	GPU
A, B	A, C, E; no D	B, F; no D	G, H
$n_1 = 2$	$n_2 = 3$	$n_3 = 2$	$n_4 = 2$

$$N_{\overline{D}} = n_1 \times n_2 \times n_3 \times n_4 = 2 \times 3 \times 2 \times 2 = 24 \Rightarrow P[X] = 1 - P[\overline{D}] = 1 - \frac{N_{\overline{D}}}{N} = \frac{24}{48} = 0.5$$

Exercise: Use the tree diagram approach to confirm the above result.

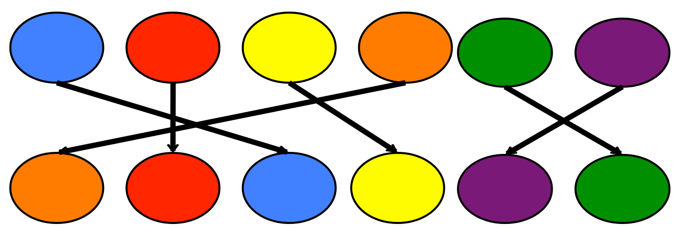
Example: Ways of assigning offices

<u>Example:</u> Ways of answering multiple choice questions**3.3**

Method 3: Permutations

<u>Definition:</u>
A permutation is the number of ordered arrangements of k objects selected from n distinct objects (k$\leq n$).

<u>Conditions:</u>
- Permutation arrangement
- Order matters
- Duplications, repetitions, or replacements are not permissible



<u>Example:</u> The set $\{3,1,2\}$, is a permutation of the set $S=\{1,2,3\}$.
Suppose we have n distinct objects to arrange in a particular order. The following are the permutations for the various situations:

- $k=n$: Ordering (arranging or choosing) n elements – no replacement.

$$n_S = n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$$

- $k \neq n$: Ordering (or choosing) k elements – no replacement.

$$n_S = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!} = {}_nP_k$$

- $k=n$: Ordering (arranging or choosing) n elements – with replacement.

$$n_S = n \times n \times \ldots \times n = n^k$$

Permutation can be thought of as a process of placing k objects (or people) into offices, positions, or slots with one object (or person) in each office, position, or slot. Repetition is not allowed because only one person can occupy a room. In that case, the first position can be filled in n ways, the second in $(n-1)$ ways and so on. The number of subsequent ways will reduce by one.

<u>Example:</u> How many possible Alberta license plates could be stamped if each license plate is required to have exactly 3 letters and 4 numbers?

Solutions: (a) Repetitions allowed: 175,760,000license plates, (b) No repetitions - 78,624,000

3.4 Method 4: Combinations

Definition:
A Combination is a permutation in which objects are arranged in any order. The number of unordered subsets of n objects taken k $(k \leq n)$ at a time, is written as

$$n_S = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} = {}_nC_k$$

- The notation $\binom{n}{k}$, is read "from n chose k."
- The k represents the number of objects one would like to select (without replacement and without regard to order) from the n objects one has.

Conditions:

- Combination arrangements
- Order is not important
- Duplication is not permissible

Example: A box contains 75 good IC chips and 25 defective chips. If 12 chips are selected at random, find the probability that at least one chip is defective.

Solution: Let A denote the event that at least one chip is defective. Consider the complement $\bar{A}$, the event that no chip is defective. Then all 12 chips will come from the 75 good chips. The number of combinations of 12 from 75 good chips is

$$n_{\bar{A}} = \binom{75}{12}$$

The overall number of combinations of 12 good chips from a total of 100 (good and defective) chips is given by.

$$n_s = \binom{100}{12}$$

The probability of the event that there is no defective chip is

$$P[\bar{A}] = \frac{n_{\bar{A}}}{N} = \frac{75!}{12!\,63!} \times \frac{12!\,88!}{100!} = \frac{75!\,88!}{63!\,100!}$$

The probability of the event that there is at least one good chip is the complement,

$$P[A] = 1 - P[\bar{A}]$$

Example: Coin tossing 3 out of 10 tosses

Example: Lotto 649

3.5 Method 5: Distinguishable Permutations

Definitions: Consider a box containing n objects of which k_1 are of one kind, k_2 are of a second kind, and so on, $k_K$are of a K^{th} kind(let $n = k_1 + k_2 + \cdots + k_K$).

- Each k_i group is defined as a partition (so there are K partitions)
- In each partition, the objects are indistinguishable
- Objects in different partitions are distinguishable.

- To determine the overall number of distinguishable permutations of the n objects, we need to first compute the number of combinations$n_1, n_2, \cdots, n_K$, for the individual partitions, respectively, and then apply the multiplication rule gives,

$$n_S = n_1 \times n_2 \times \cdots \times n_K.$$

- Since the elements in each group are indistinguishable, order does not matter so we use combinations for each partition.

Example: A box contains a total of n fruits; $k_1$of which are oranges and k_2 are apples $(n = k_1 + k_2)$. How many distinguishable permutations are there?

1. Number of combinations in Partition A_1: k_1 oranges

$$n_1 = \binom{n}{k_1} = \frac{n!}{k_1!\,(n-k_1)!}$$

2. Number of combinations in PartitionA_2: k_2 apples

$$n_2 = \binom{n-k_1}{k_2} = \frac{(n-k_1)!}{k_2!\,(n-k_1-k_2)!}$$

3. Total number of distinct partitions is

$$n_S = n_1 \times n_2 = \frac{n!}{k_1!\,(n-k_1)!} \times \frac{(n-k_1)!}{k_2!\,(n-k_1-k_2)!} = \frac{n!}{k_1!\,k_2!}$$

Generalization: Consider a box containing n objects comprising Kdistinguishable partitions$A_1, A_2, \cdots, A_K$.

- Partition A_1 with k_1 identical elements
- Partition A_2 with k_2 identical elements

- And so, on
- PartitionA_K, with k_K identical elements

The total number of distinct permutations or number of ways to arrange a set of n elements involving K sets of sizes$k_1, k_2, \cdots, k_K$, respectively, is

$$n_S = \underbrace{\binom{n}{k_1}}_{\text{ways to place objecs of type 1}} \times \overbrace{\binom{n-k_1}{k_2}}^{\text{ways to place objecs of type 2}} \times \cdots \times \underbrace{\binom{n-k_1-k_2-\cdots-k_{K-1}}{k_K}}_{\text{ways to place objecs of type k}}$$

$$= \frac{n!}{k_1!\,k_2!\cdots k_k!};$$

where

$$(n-k_1-k_2\cdots-k_{K-1}-k_K)! = 0! = 1$$

Therefore, the probability of picking one permutation or choice is

$$P[\text{one choice}] = \frac{1}{n_S} = \frac{k_1!\,k_2!\cdots k_K!}{n!}$$

Exercise: A lab technician wants to assign 7 students to 2 double-seat benches and 1 triple-seat bench during an electronic circuit lab session. In how many was can these students be assigned?

Solution: There are three cells – two cells with $n_1 = n_2 = 2$ and one cell with $n_3 = 3$. The number of objects to be placed in the cells is $n = 7$. The total number of possible partitions is

$$n_s = \frac{n!}{n_1!n_2!n_3!} = \frac{7!}{2!2!3!} = 210$$

Exercise: In a bag, we have $k_1 = 1$ red ball, $k_2 = 2$ green balls, $k_3 = 3$ blue balls. What is the number of distinguishable permutations? What is the probability of a permutation?

Exercise: A batch of 50 chips contain 10 defective chips. Suppose 12 are selected at random and tested. What is the probability of selecting exactly 5 defective chips?

Example: How many strings can be formed by permuting the letters in SUCCESS?

Solution: The word SUCCESS contains

$$3 \times S \rightarrow 3 \text{ indistinguishable letters} \rightarrow k_1 = 3$$
$$1 \times \text{U} \rightarrow 1 \text{ indistinguishable letter} \rightarrow k_2 = 1$$

$$2 \times C \rightarrow 2 \text{ indistinguishable letters} \rightarrow k_3 = 2$$
$$1 \times E \rightarrow 1 \text{ indistinguishable letter} \rightarrow k_4 = 1$$

The total number of strings (or distinct permutations) is

$$n_S = \underbrace{\binom{7}{3}}_{\text{ways to place 3 S}} \times \overbrace{\binom{4}{1}}^{\text{ways to place 1 U}} \times \underbrace{\binom{3}{2}}_{\text{ways to place 2 C}} \times \overbrace{\binom{1}{1}}^{\text{ways to place 1 E}} = \frac{7!}{3!\,1!\,2!\,1!} = 420$$

<u>Example:</u> The captain of a ship sends signals by arranging 4 orange flags and 3 blue flags on a vertical pole. How many different signals could the captain possibly send?

<u>Solution:</u> If the flags were numbered 1, 2, 3, 4, 5, 6, 7, then all the orange flags and blue flags would be distinguishable among themselves. In that case the captain could send $7! = 5{,}040$ possible signals. However, the flags are not numbered. The four orange flags are not distinguishable among themselves, and the three blue flags are not distinguishable among themselves. We need to count the number of distinguishable permutations when the two colors are the only features that make the flags distinguishable. The number of possible signals is

$$N = \frac{7!}{4!3!} = \frac{7 \times 6 \times 5}{3 \times 2} = 35$$

<u>Example:</u> A box contains 2 black balls and 3 white balls. Two balls are selected at random from the box.

(a) What is the probability that they are both black?

(b) If 3 balls are selected, what is the probability that two are black and the third is white?

<u>Solution:</u> - Use of partitions. Partition 1 contains 2 black balls ($n_1 = 2$) and Partition 2 contains 3 white balls ($n_2 = 3$). The total number of balls is $n = 5$.

(a) Overall experiment: pick 2 balls from a total $n = 5$,

$$n_s = \binom{n}{k} = \binom{5}{2} = \frac{5!}{2!\,3!} = 10$$

- Event A: pick 2 black balls from Partition 1 and 0 black balls from Partition 2
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$$n_A = n_1 \times n_2 = \binom{2}{2}\binom{3}{0} = 1$$

- The probability of selecting 2 black balls, both black, is
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$$P[\{\text{selecting 2 black balls}\}] = \frac{n_A}{n_S} = \frac{1}{10}.$$

(b)When 3 balls are selected at random from the total $n = 5$,

$$n_s = \binom{n}{k} = \binom{5}{2} = \frac{5!}{2!\,3!} = 10$$

When 2 black balls are selected from Partition 1 and 1 white ball is selected from Partition 2

$$n_A = \binom{2}{2}\binom{3}{1} = 3$$

The probability of selecting 2 black balls and 1 white ball is

$$P\left[\{\text{selecting } 2\}\right] = \frac{n_A}{n} = \frac{3}{10}.$$

Example: A box contains 2 black balls and 3 white balls. Two balls are selected (one at a time) at random from the box without replacement. What is the probability that they are both black? Solving the previous problem using conditional probability.

Solution: (Use of conditional probability).

Let$B_1 = \{1^{st}$ ball is black$\}$ and $B_2 = \{2^{nd}$ ball is black$\}$. Then $P[B_1] = \frac{2}{5}$ and $P[B_2|B_1] = \frac{1}{4}$ The event that they are both black is the intersection$B_2 \cap B_1$. We may now write

$$P[B_2 \cap B_1] = P[B_2|B_1] \times P[B_1] = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$$

Concluding Remarks on Permutation and Combinations:

- The main difference in the definition of a permutation and a combination is whether order is important.

    - **Permutation: order is important**
    - **Combination: order is not important**
    - **For distinguishable partitions, we need to compute thee number of combinations within each partition and apply the multiplication rule to obtain the overall permutation**