7. Bivariate Random Variables

(Reading Exercises: Montgomery and Runger – Chapter 5 and Yates and Goodman – Chapter 5)

In some experiments, it may be desirable to simultaneously measure outcomes of two or more random variables$X = \{X_1, X_2, \cdots, X_n\} \in \mathbb{R}^n$. Often in electrical engineering, we want to investigate the joint behavior of multiple random variables. The main focus in this course, however, is on bivariate (two) random variables. In order to study the joint behavior of two random variables we must know how to handle joint probability distributions.

Learning outcomes:

You will

- Know the formal definition of a joint probability mass function of two discrete random variables.
- Know how to use a joint probability mass function to find the probability of a specific event.
- Know how to find a marginal probability mass function of a discrete random variable X from the joint probability mass function of X and Y.
- Know the conditions for independence of two random variables X and Y.
- Be able to find the expectation of a function of the discrete random variables X and Y using their joint probability mass function.
- Be able to find the means and variances of the discrete random variables X and Y using their joint probability mass function.
- Be able to compute covariance and correlation coefficient inorder to infer the kind of relationship between two random variables.
- Be able to apply the methods learned in the section to solve new problems.

7.1 Joint Probability Distribution of Discrete Random Variables

Joint PMF

Definition: Consider two discrete random variables X and Y, defined on the sample space of an experiment. Their joint PMF, denoted $p_{XY}(x, y)$ is defined, for all real numbers x and Y, as

$$p_{XY}(x, y) = P[X = x \text{ and } Y = y]$$

Properties of the joint PMF:

1. $p_{XY}(x, y) \geq 0 \ \forall (x, y)$
2. $\sum_x \sum_y p_{XY}(x, y) = 1$

- Any function that satisfies these two properties is a joint PMF.

Joint CDF of Discrete Random Variables

Definition: Consider two discrete random variables X and Y, defined on the sample space of an experiment. Their joint CDF, denoted $F_{XY}(x,y)$ is defined, for all real numbers x and y, as

$$F_{XY}(x,y) = P[X \le x, Y \le y] = \sum_{v=-\infty}^{y} \sum_{u=-\infty}^{x} p_{XY}(u,v)$$

Example: For the joint PMF in the table below, calculate$P[X + Y \le 2]$.

PMF	$X=0$	$X=1$
$Y=1$	0.09	0.06
$Y=2$	0.05	0.08
$Y=3$	0.08	0.24
$Y=4$	0.10	0.30

Solution:

$$P[X + Y \le 2] = P[X = 0, Y = 1] + P[X = 0, Y = 2] + P[X = 1, Y = 1]$$
$$= 0.09 + 0.05 + 0.06 = 0.2$$

Example: Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find the following:

(a) The joint probability mass density function $p_{XY}(x,y)$.
(b) $P[(X,Y) \in A]$, where A is the region $\{(x,y)|x + y \le 1\}$

Solution: Since only two ball pens are selected, the possible pairs of values (the region) are

(x,y): $(0,0), (0,1), (1,0), (1,1), (0,2), (2,0)$.

(a) The probability $p_{XY}(x,y)$ represents the probability that x blue and y red pens are selected. There are 3 distinguishable partitions – blue, red and green.
- The total number of equally likely ways of selecting any 2 pens from the 8 pens is $n = \binom{8}{2} = 28$.
- The number of equally likely ways of selecting x blue pens from 3 is $n_1 = \binom{3}{x}$
- The number of equally likely ways of selecting y red pens from 2 is $n_2 = \binom{2}{y}$.
- The number of equally likely ways of selecting green pens from 3 after selecting x blue and y red pens is $n_3 = \binom{3}{2 - x - y}$.

Therefore, the PMF is

$$p_{XY}(x,y) = \frac{n_1 n_2 n_3}{n} = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}, x = 0, 1, 2 \; and \; y = 0, 1, 2; 0 \le x + y \le 2.$$

$p_{XY}(x,y)$	$x = 0$	1	2
$y = 0$	$p_{XY}(0,0) = \frac{3}{28}$	$p_{XY}(1,0) = \frac{9}{28}$	$p_{XY}(2,0) = \frac{3}{28}$
1	$p_{XY}(0,1) = \frac{3}{14}$	$p_{XY}(1,1) = \frac{3}{14}$	$p_{XY}(2,1) = 0$
2	$p_{XY}(0,2) = \frac{1}{28}$	$p_{XY}(1,2) = 0$	$p_{XY}(2,2) = 0$

(b) $P[(X,Y) \in A] = P[X + Y \le 1] = p_{XY}(0,0) + p_{XY}(0,1) + p_{XY}(1,0) = \frac{9}{14}$

The tabulated PMF is a valid PMF since the sum of all the entries is equal to one.

7.2 Continuous Random Variables

You will
- Know the formal definition of a joint probability density function of two continuous random variables.
- Be able to use a joint probability density function to find the probability of a specific event.
- Know how to find a marginal probability density function of a continuous random variable from the joint probability density function of two random variables.
- Know how to find the means and variances of the continuous random variables using their joint probability density function.
- Know the formal definition of a conditional probability density function of a continuous random variable Y given a continuous random variable X.
- Know how to calculate the conditional mean and conditional variance of a continuous random variable Y given a continuous random variable X.
- Be able to compute covariance and correlation coefficient inorder to infer the kind of relationship between two random variables
- Be able to apply the methods learned in the section to solve new problems.

Joint PDF:

Definition: Let X and Y be two continuous random variables defined on the sample space of an experiment. The joint pdf, denoted $f_{XY}(x,y)$, is a function with the following properties:

$$(1)\, f_{XY}(x,y) \geq 0, \forall x$$

$$(2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

$$(3)\, P[(x,y) \in A] = \iint_A f_{XY}(x,y)dxdy$$

- Any function that satisfies these two properties is a joint PDF.

Joint CDF

Definition: The joint CDF for two continuous random variables, X and Y, is defined as

$$F_{XY}(x,y) = P[X \leq x, Y \leq y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)dudv$$

- The joint PDF is the second partial derivative with respect to the two random variables,

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

Properties of the Joint CDF:

$F_{XY}(-\infty,y) = 0$ $F_{XY}(\infty,\infty) = 1$	$F_{XY}(x,-\infty) = 0$ $F_{XY}(x,\infty) = F_X(x)$	$F_{XY}(-\infty,-\infty) = 0$ $F_{XY}(\infty,y) = F_Y(y)$

7.3 Marginal PDFs:

Definition: For discrete random variables the marginal or individual probability mass functions of X and Y are defined, respectively, as

$$p_X(x) = \sum_x p_{XY}(x,y); \qquad p_Y(y) = \sum_x p_{XY}(x,y)$$

Example: Find the marginal PDFs of the random variables X and Y for the joint PMF given by the table below.

$p_{XY}(x,y)$	$x=0$	1	2
$y=0$	$p_{XY}(0,0)=\frac{3}{28}$	$p_{XY}(1,0)=\frac{9}{28}$	$p_{XY}(2,0)=\frac{3}{28}$
1	$p_{XY}(0,1)=\frac{3}{14}$	$p_{XY}(1,1)=\frac{3}{14}$	$p_{XY}(2,1)=0$
2	$p_{XY}(0,2)=\frac{1}{28}$	$p_{XY}(1,2)=0$	$p_{XY}(2,2)=0$

Solution:

x	0	1	2
$p_X(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$p_Y(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

Definition: For continuous random variables, the marginal or individual probability density functions of X and Y are defined, respectively, as

$$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

Example*:* Find the marginal pdfs for $f_{XY}(x,y)=\frac{2}{3}(2x+3y),\ 0\le x\le 1\ and\ 0\le y\le 1$. For

Answers*:* $f_X(x)=\frac{1}{5}(4x+3)\ and\ f_Y(y)=\frac{2}{5}(1+3y)$.

7.4 Conditional Probability Distribution:

Conditional CDFs

Recall the condition probability expression for two events $A\ and\ B$,

$$P[B|A]=\frac{P[A\cap B]}{P[A]}$$

This definition also applies to the conditional CDF of X and Y. Joint CDF is given below.

$$F_{X|Y}[x|Y] = P[X \le x|Y] = \frac{P[X \le x, Y]}{P[Y]}$$

Similarly, the CDF of Y conditioned on X is given by,

$$F_{Y|X}[y|X] = P[Y \le y|X] = \frac{P[Y \le y, X]}{P[X]}$$

Conditional PDF

Similarly, like the joint conditional CDF, the joint conditional PDF is defined as

$$p_{X|Y}(x|Y) = \frac{p_{XY}(x,Y)}{P[Y]};\ p_{Y|X}(y|X) = \frac{p_{XY}(X,y)}{P[X]},\quad (Discrete)$$

$$f_{X|Y}(x|Y) = \frac{f_{XY}(x,Y)}{P[Y]};\ f_{Y|X}(y|X) = \frac{f_{XY}(X,y)}{P[X]},\quad (continuous)$$

Example: Use the table below to find the conditional PMF of $X\ and\ Y$ given that $Y = 1$ and use it to find $P[X = 0|Y = 1]$.

$p_{XY}(x,y)$	$x = 0$	1	2
$y = 0$	$p_{XY}(0,0) = \frac{3}{28}$	$p_{XY}(1,0) = \frac{9}{28}$	$p_{XY}(2,0) = \frac{3}{28}$
1	$p_{XY}(0,1) = \frac{3}{14}$	$p_{XY}(1,1) = \frac{3}{14}$	$p_{XY}(2,1) = 0$
2	$p_{XY}(0,2) = \frac{1}{28}$	$p_{XY}(1,2) = 0$	$p_{XY}(2,2) = 0$

Solution:

$$p_{X|1}(x|1) = \frac{p_{XY}(x,1)}{p_Y[1]}$$

$$p_Y(1) = \sum_{x=0}^{2} p_{XY}(x,1) = \frac{3}{14} + \frac{3}{14} = \frac{3}{7} \Longrightarrow p_{X|1}(x|1) = \frac{p_{XY}(x,1)}{p_Y[1]} = \frac{7}{3} p_{XY}(x,1)$$

$$p_{X|1}(0|1) = \frac{p_{XY}(0,1)}{p_Y[1]} = \frac{7}{3} \times \frac{3}{14} = \frac{1}{2} \Longrightarrow P[X = 0|Y = 1] = \frac{1}{2}$$

$$p_{X|1}(x|1) = \frac{p_{XY}(1,1)}{p_Y[1]} = \frac{7}{3} \times \frac{3}{14} = \frac{1}{2} \Longrightarrow P[X = 1|Y = 1] = \frac{1}{2}$$

$$p_{X|1}(2|1) = \frac{p_{XY}(2,1)}{p_Y[1]} = \frac{7}{3} \times 0 = 0 \Longrightarrow P[X = 2|Y = 1] = 0$$

Example: The joint pdf is

$$f_{XY}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & elsewhere \end{cases}$$

Solution:

$$f_X(x) = 10\int_x^1 xy^2 dy = \frac{10}{3}x(1 - x^3), 0 < x < 1$$

$$f_Y(y) = 10\int_0^y xy^2 dy = 5y^4, 0 < y < 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{3y^2}{1 - x^3}, \qquad 0 < x < 1, 0 < y < 1$$

$$P\left[Y > \frac{1}{2}|X = \frac{1}{4}\right] = \int_{\frac{1}{2}}^1 f_{Y|X}\left(y|x = \frac{1}{4}\right) dy = \int_{\frac{1}{2}}^1 \frac{3y^2}{1 - \left(\frac{1}{4}\right)^3} dy = \frac{8}{9}$$

Example: Consider the following joint density function:

$$f_{XY}(x,y) = \begin{cases} xy^2, & 0 < x < 1, 0 < y < 1 \\ 0, & elsewhere \end{cases}$$

Find the marginal density functions $f_X(x)$ *and* $f_Y(y)$, and the conditional density function $f_{X|Y}(x|y)$.

Solution:

$$f_X(x) = x\int_0^1 (1 + 3y^2)dy = x(y + y^3)|_{y=0}^{y=1} = 2x$$

$$f_Y(y) = \int_0^1 x(1 + 3y^2)dx = \frac{x^2}{2}(1 + 3y^2)|_{x=0}^{x=1} = \frac{1 + 3y^2}{2}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$P\left[\frac{1}{4} < x < \frac{1}{2}\middle| Y = \frac{1}{3}\right] = \int_{1/4}^{1/2} 2x dx = \frac{3}{16}$$

Conditional Expectations:

Definition: If $X\ and\ Y$ are two random variables, the conditional expectation of $X\ given\ Y = y$, is defined as

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx; \quad (continuous)$$

$$E[X|Y = y] = \sum_{x} x\, p_{X|Y}(x|y); \quad (discrete)$$

Example:

A soft-drink machine has a random supply Y (with measurements in gallons) at the beginning of given day. Let X denote the amount (in dollars) of soft drink sold during the day. The machine is not resupplied during the day. It has been observed that $X\ and\ Y$ have joint PDF,

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2}, & 0 \le x \le y \le 2 \\ 0, & elsewhere \end{cases}$$

Find the conditional expectation of the amount of sales X given that $Y = 1$.

Solution:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \begin{cases} \frac{1}{y}, & 0 \le x \le y \le 2 \\ 0, & elsewhere \end{cases}$$

For $Y = 1$, we have

$$f_{X|Y}(x|y) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$$

$$E[X|Y = 1] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y = 1) dx = \int_{0}^{1} x dx = \frac{1}{2}$$

7.5 Independence of Random Variables and Functions:

Definition: Two random variables, $X \text{ and } Y$, are said to be statistically independent if one of the following holds:

$$F_{XY}(x,y) = F_X(x)F_Y(y) \Longrightarrow f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$F_{X|Y}(x|y) = F_X(x) \Longrightarrow f_{X|Y}(x,|y) = f_X(x)$$

$$F_{Y|X}(y|x) = F_Y(y) \Longrightarrow f_{Y|X}(y,|x) = f_Y(y)$$

Examples: In the previous example,

$$f_{X|Y}(x,|y) = f_X(x) = 2$$

Therefore, random variables $X \text{ and } Y$, are independent.

7.6 Expected Values of Functions of Bivariate Random Variables:

Definition: Consider two random variables $X \text{ and } Y$ with joint probability distribution function $p_{XY}(x,y)$ (PMF) or $f_{XY}(x,y)$ (PDF). If $g(X,Y)$ is a real-valued function of (X,Y), then the expected value of $g(X,Y)$ is,

$$E[\,g(X,Y)] = \sum_x \sum_y g(X,Y)p_{XY}(x,y)\,, \qquad (discrete)$$
$$E[\,g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(X,Y)f_{XY}(x,y)dxdy, \qquad (continuous$$

Definition: Consider two independent random variables $X \text{ and } Y$ with means $\mu_X \text{ and } \mu_Y$, respectively. Then the expected value of their product is

$$r_{XY} = E[XY] = \mu_X\mu_Y$$

Definition: If two random variables, $X \text{ and } Y$, are independent then, for any two functions $g(X) \text{ and } h(Y)$,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Covariance:

Definition: Consider two random variables $X \text{ and } Y$ with means $\mu_X \text{ and } \mu_Y$. The covariance of $X \text{ and } Y$, denoted $Cov(X,Y) \text{ or simply } C_{XY}$, is defined as

$$C_{XY}(x,y) = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (X-\mu_X)(Y-\mu_Y)p_{XY}(x,y), \qquad (discrete)$$

$$C_{XY}(x,y) = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (X-\mu_X)(Y-\mu_Y)f_{XY}(x,y)dxdy$$

$$C_{XY}(x,y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y = r_{XY} - \mu_X\mu_Y$$

- Covariance provides information regarding the nature of the relationship between the two random variables $X\ and\ Y$.
- If there is no relationship or the two random variables are uncorrelated, then $C_{XY} = 0$.

Correlation Coefficient:

<u>Definition:</u> Consider two random variables $X\ and\ Y$ with C_{XY} and individual standard deviations $\sigma_X\ and\ \sigma_Y$. The correlation coefficient of $X\ and\ Y$, denoted ρ_{XY} is defined as

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X\sigma_Y}, -1 \le \rho_{XY} \le 1$$

- has no unit and is used widely in Engineering.
  - $\rho_{XY} = +1$: $X\ and\ Y$ have a linear relationship (correlated) with a positive slope.
  - $\rho_{XY} = -1$: $X\ and\ Y$ have a linear relationship (correlated) with a negative slope.
  - $\rho_{XY} = 0$: $X\ and\ Y$ have no linear relationship (uncorrelated)
  - Statistically independent random variables are uncorrelated but the converse is not necessarily true.

<u>Examples:</u> For the joint PMF in the table below, show that the two random variables are uncorrelated $C_{XY} = 0$ but they are dependent $p_{XY}(x,Y) \neq p_X(x)p_Y(y)$

	$X=-1$	$X=0$	$X=1$	Totals
$Y=-1$	1/8	1/8	1/8	3/8
$Y=0$	1/8	0	1/8	2/8
$Y=1$	1/8	1/8	1/8	3/8
Totals	3/8	2/8	3/8	1

Exercise: A travel agent keeps track of the number of customers who call and the number of trips booked on any one day. Let X denote the number of calls, Y the number of trips booked and $p_{XY}(x,y)$ the joint PMF. Records show the following:

$p_{XY}(x,y)$	$x=0$	$x=1$	$x=2$	$x=3$
$y=0$	0.04	0.08	0.12	0.10
$y=1$		0.06	0.20	0.16
$y=2$			0.12	0.10
$y=3$				0.02

Verify the following:

$$E[XY] = 2.20,$$

x	0	1	2	3
$p_X(x)$	0.04	0.14	0.44	0.28

y	0	1	2	3
$p_Y(y)$	0.34	0.42	0.22	0.02

$\mu_X = 1.86$	$\mu_Y = 0.92$	$C_{XY} = 0.4888$	$\sigma_X^2 = 0.9604$	$\sigma_Y^2 = 0.6336$	$\rho_{XY} = 0.6266$

Example: The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint PDF

$$f_{XY}(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1 \\ 0, & elsewhere \end{cases}$$

Find the covariance of X and Y, and the correlation coefficient.

Solution: First, compute marginal PDFs:

$$f_X(x) = \int_0^x 8xy dy = \begin{cases} 4x^3, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$$

$$f_Y(y) = \int_y^1 8xy dy = \begin{cases} 4y(1-y^2), & 0 \le y \le 1 \\ 0, & elsewhere \end{cases}$$

Next, compute the individual mean values

$$\mu_X = E[X] = \int_0^1 4x^4 dx = \frac{4}{5}, \qquad \mu_Y = E[Y] = \int_0^1 4y^2(1-y^2)dx = \frac{8}{15}$$

Using the joint PDF, we find

$$E[XY] = \int_0^1 \int_y^1 8x^2y^2 dxdy = \frac{4}{9}$$

The covariance is equal to

$$C_{XY} = E[XY] - \mu_X\mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}$$

To find the correlation coefficient, we need the individual variances. We can use the short-cut formula:

$$E[X^2] = \int_0^1 4x^5 dx = \frac{2}{3},\ \ E[Y^2] = \int_0^1 4y^3(1-y^2)dy = \frac{1}{3}$$

$$\sigma_X^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75},\ \ \sigma_Y^2 = \frac{1}{3} - \left(\frac{5}{15}\right)^2 = \frac{11}{225};$$

$$\rho_{XY} = \frac{4/225}{\sqrt{(2/225)(11/225)}}$$

7.7 Sums of Random Variables:

Mean and Variance of Sums of Two Random Variables

1. **Definition:** Consider two random variables $X\ and\ Y$. Then mean and variance of their sum are:
   - $\mu_Z = E[X \pm Y] = \mu_X \pm \mu_Y$
   - $\sigma_Z^2 = Var(X \pm Y) = \sigma_X^2 + \sigma_Y^2 \pm 2C_{XY}$
   - $\sigma_Z^2 = Var(X \pm Y) = \sigma_X^2 + \sigma_Y^2, if\ X\ and\ Y\ are\ independent$

Consider n independent random variables $X_1, X_2, \dots, X_n$ with individual PDFs $f_{X_i}(x_i)$, $i = 1, 2, \dots, n$. It is desired to determine the PDF and the statistics of the sum $X = \sum_{i=1}^{n} X_i$. This is best handled through the moment generating functions.

Learning Outcomes:

You will

- Know the definition of and how to calculate the moment-generating function of a linear combination of n independent random variables.
- Understand the steps involved in each of the proofs in this section.
- Be able to determine the PDF of a linear combination of n independent random variables from the moment-generating function.
- Understand and be able to apply the central limit theorem
- To be able to apply the methods learned to new problems.

7.7.1 Moment Generating Functions

(Reading Exercises: Yates and Goodman – Section 9.2)

Single Random Variable:

Definition: Consider a random variable X, with PDF $f_X(x)$. The moment generating function of X denoted $\phi_X(jux)$, is defined as the expected value of the complex function e^{jux},

$$\phi_X(x) = E\left[e^{jux}\right] = \int_{-\infty}^{\infty} e^{jux} f_X(x) dx$$

According to the Fourier transform theory, the above integral is the inverse Fourier transform of the PDF $f_X(x)$.

Definition: The probability density function is the Fourier transform of the moment generating function,

$$f_X(x) = \int_{-\infty}^{\infty} e^{-jux} \phi_X(jux) dx$$

Remarks:

- The moment generating function and the pdf are Fourier transform pairs.
- It is often possible or easier to obtain the moment generating functions of random variables than it is to find the PDF directly.

Expected value of X:

Taking the first derivative of the moment generating function $\phi_X(ju)$ gives

$$\frac{d\phi_X(ju)}{du} = \int_{-\infty}^{\infty} f_X(x) \frac{d\left(e^{jux}\right)}{du} dx = j\int_{-\infty}^{\infty} x f_X(x) e^{jux} dx$$

Setting $u = 0$, we obtain the expression for the mean of X as

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = -j\frac{d\phi_X(ju)}{du}\bigg|_{u=0}$$

If we are interested in the m^{th} moment, $E[X^m]$, of a random variable X, we need to take the m^{th} derivative of the moment generating function and set $u=0$, that is,

Definition: Consider a random variable with moment generating function $\phi_X(ju)$. The m^{th} moment of X is obtained as the m^{th} derivative of the moment generating function evaluated at $u = 0$,

$$E[X^m] = \int_{-\infty}^{\infty} x^m f_X(x)dx = (-j)^m \frac{d^m\phi_X(ju)}{du^m}\bigg|_{u=0}$$

Two Random Variable:

Definition: The moment generating function of the sum of two independent random variables, $X\ and\ Y$, is defined as

$$\phi_{XY}(ju, jv) = E\left[e^{j(uX+vY)} = E\left[e^{juX}e^{juX}\right] = \phi_X(ju)\phi_Y(jv)\right]$$

The moment generating function of the sum of two independent random variables is equal to the product of their individual moment generating functions

Definition: The joint PDF, defined as the Fourier transform of $\phi_{XY}(ju, jv)$, is the convolution of the individual PDF's,

$$f_{XY}(x, y) = \int_{-\infty}^{\infty} e^{-j(ux+uy)}\phi_X(ju)\phi_Y(ju)dx = f_X(x) * f_Y(y)$$

- According to the Fourier transform theory, the Fourier transform of the product of two functions is equal to the convolution of their Fourier transforms.
- In general, given the PDFs of n independent random variables, $f_{X_i}(x_i), i = 1, 2, .., n$, the joint PDF is equal to the convolution of all the PDFs, that is,

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) * f_{X_2}(x_2) * \ldots * f_{X_n}(x_n)$$

7.8 The Central Limit Theorem:

Let $X_1, X_2, \dots, X_n$ be n independent random variables with arbitrary probability distribution functions. The central limit theorem tells us about the probability distribution of their sum as n approaches infinity.

Definition: Let $X_1, X_2, \dots, X_n$ be n independent random variables with arbitrary probability distribution functions and identical mean, $\mu_i = \mu$, and variance $\sigma_i^2 = \sigma^2$. Then, according to the central limit theorem, the distribution of their sum, $X = \sum_{i=1}^{n} X_i$ converges to the Gaussian distribution as n increases.

- The mean of the sum $\mu_X = \sum_{i=1}^{n} \mu = n\mu$
- The variance of the sum is

$$\sigma_X^2 = \sum_{i=1}^{n} \sigma^2$$

Definition: Consider the sum of the normalized random variables,

$$Y = \sum_{i=1}^{n} \frac{X_i - \mu_{X_i}}{\sqrt{n}\sigma_{X_i}}$$

Then, according to the central limit theorem, as the number of random variables increase, Y tends to the standard Gaussian random variable ($\mu_Y = 0 \ and \ \sigma_Y^2 = 1$).

Rule of Thumb: If $n > 30$, the Central Limit Theorem can be used.