ENEL 419 — **Probability and Random Variables** — **Fall 2020**

**Quiz #3**

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| ID #: | Last Name: | First Name: |
| Quiz #3 | Start time<br>Thursday, October 29, 2020 | Due date and time<br>Friday 30, 2020, 5:0pm |

## Question 1:

A random variable $X$, has a PDF given by

$$f_X(x) = \begin{cases} cx^2, & -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of $c$ and use it to

(b) Find the CDF

(c) Evaluate the probability $P\left[\left|X - \frac{1}{2}\right| < 1\right]$.

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## Question 2:

Suppose men's shirt sizes are approximately Gaussian distributed with mean 16.2 inches and variance 0.81 square inches.

(a) Find the probability that the neck size of a randomly selected man lies in the range 13.5" and 18.9". Express your answer to four decimal places.

(b) Suppose we change the mean to 16.0 inches. Find the value of the variance such that the probability of the neck size of a randomly selected man lying in the range 13.5" and 18.5" remains the same as in part (a).

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a) $f_x = \begin{cases} cx^2 & -2 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1 \Rightarrow \int_{-2}^{2} cx^2\,dx = 1$$

$$\frac{1}{3}x^3 \cdot c \Big|_{-2}^{2} = \frac{8}{3}c - \left(-\frac{8}{3}c\right) = \frac{16}{8} \cdot c = 1$$

$$\boxed{C = \frac{3}{16}}$$

b) CDF of X $= F_X(x) = P(X \leq x)$

$$F_x(x) = \int_{-\infty}^{x} f_u(u)\,du$$

for $x < -2$

$$F_x(x) = \int_{-\infty}^{x} f_u(u)\,du = \int_{-\infty}^{x} 0\,du = 0$$

for $-2 < x < 2$

$$F_x(x) = \int_{-2}^{x} f_u(u)\,du = \int_{-2}^{x} cu^2\,du = \int_{-2}^{x} \frac{3}{16}u^2\,du = \frac{1}{16}\left[u^3\Big|_{-2}^{x}\right] = \frac{1}{16}x^3 - \left(\frac{1}{16} \cdot -8\right)$$

$$= \frac{1}{16}x^3 + \frac{1}{2} = \boxed{\frac{x^3+8}{16}}$$

for $x > 2$

$$F_x(x) = \int_{2}^{x} 0\,du = 0$$

CDF for X is:

$$\boxed{F_x(x) = \begin{cases} 0 & x < -2 \\ \dfrac{x^3+8}{16} & -2 < x < 2 \\ 0 & 2 < x \end{cases}}$$

c) $P\left[|X-\frac{1}{2}|<1\right] = P\left[-1 < X-\frac{1}{2} < 1\right]$

$Y = X - \frac{1}{2} \qquad X = Y + \frac{1}{2}$ range of $Y \Rightarrow \; -\frac{5}{2} < y < \frac{3}{2}$

$$f_Y(y) = \frac{d F_X(y+\frac{1}{2})}{dy} = f_X(y+\tfrac{1}{2}) \cdot 1 = \frac{3}{16}(y+\tfrac{1}{2})^2 = \frac{3}{16}\left[y^2+y+\tfrac{1}{4}\right]$$
for $-\frac{5}{2} < y < \frac{3}{2}$

$$P\left[|X-\tfrac{1}{2}|<1\right] = \int_{-1}^{1} f_Y(y)\,dy = \frac{3}{16}\cdot\int_{-1}^{1}\left(y^2+y+\tfrac{1}{4}\right)dy = \frac{3}{16}\left[\tfrac{1}{3}y^3+\tfrac{1}{2}y^2+\tfrac{1}{4}y\right]\Big|_{-1}^{1}$$

$$= \frac{3}{16}\left[\left(\tfrac{1}{3}+\tfrac{1}{2}+\tfrac{1}{4}\right)-\left(-\tfrac{1}{3}+\tfrac{1}{2}-\tfrac{1}{4}\right)\right] = \frac{3}{16}\left[\frac{2}{3}+\frac{2}{4}\right] = \frac{3}{16}\cdot\frac{7}{6} = \frac{7}{32}$$

$$\boxed{P\left[|X-\tfrac{1}{2}|<1\right] = \frac{7}{32}}$$

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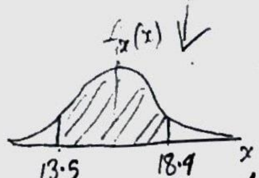
Q2 $\quad \mu = 16.2 \qquad \sigma^2 = 0.81 \qquad \sigma = 0.9$

assume Random Variable $X$ represents neck size

a) $P[13.5 < X < 18.9] = P[X < 18.9] - P[X < 13.5]$

$$= P\left[Z < \frac{18.9-16.2}{0.9}\right] - P\left[Z < \frac{13.5-16.2}{0.9}\right]$$

$Z = \frac{X-\mu}{\sigma}$




$$= P[Z<3] - P[Z<-3]$$

also $= \int_{13.5}^{18.9} f_X(x)\,dx = \int_{13.5}^{18.9} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$$P[Z<3] - P[Z<-3] = (1-Q(3)) - (1-Q(-3))$$
$$= (1-Q(3)) - (1-(1-Q(3)))$$
$$= 1 - Q(3) - Q(3) = 1 - 2Q(3)$$
$$= 1 - 2(0.0013499)$$ From Q-function table

$$\boxed{P[13.5 < X < 18.9] = 0.9973002}$$

$\boxed{b}$ $\mu = 16$ , Find $\sigma^2$,

$P[13.5 < X < 18.5] = P[13.5 < X < 18.9]_{\text{from part (a)}} = 0.9973002$

$P\left[Z < \frac{18.5-\mu}{\sigma}\right] - P\left[Z < \frac{13.5-\mu}{\sigma}\right] = 0.9973002$

$\frac{18.5-16}{\sigma} = 3$ $\qquad$ $\frac{13.5-16}{\sigma} = -3$

$\boxed{\sigma = \frac{5}{6}}$

$\boxed{\sigma^2 = \frac{25}{36}}$