ENEL 419: Probability and Random Variables

Midterm #2

Instructor: Dr. Abu Sesay November 21, 2017

Room: ENA 103

Time: 09:30 – 10:30AM

Last Name (printed):	First Name:	ID #:

Signature:	

Instructions:

- All the University of Calgary regulations apply to this exam.
- Answer all three questions in the booklet provided.
- You are allowed to use a non-programmable calculator for this exam. No other electronic device, including music devices or anything with wireless capability, will be allowed during the exam.
- The test is closed-book and closed-notes. Formulas are provided on the last for your convenience.
- To reduce distraction to other students, you are not allowed to leave during the last ten minutes of the exam.
- Please print or write your answers legibly. What cannot be read cannot be marked.
- If you write anything you do not want marked, put a large "X" through it and write "rough work" beside it.

Marks Summary

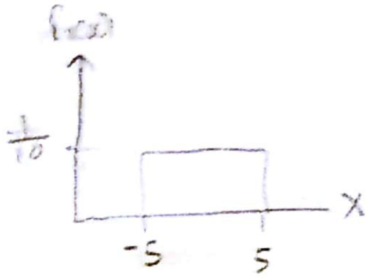
	Q1	Q2	Q3	Total
Marks	30	30	34	94
Out of	32	32	36	100%

Question 1:

Marks		
	(a)	The voltage developed across a 1 kΩ resistor is a continuous uniform random variable, denoted X, in the range $-5 \le X \le 5$ volts.
/5		(i) Write the formula for the PDF and sketch the graph of the PDF, $f_X(x)$, as a function of x? Label all axis.
/8		(ii) Find the mean and variance of the voltage.
/8		(iii) Find the probability $P[\lvert X-1 \rvert < 2]$.
	(b)	On a laboratory assignment, if the equipment is working, the PDF of the observed random output Y, is $f_Y(y) = \begin{cases} c(2-y), & 0<y<2 \\ 0, & \text{otherwise.} \end{cases}$
/3	(i)	What is the value of the coefficient c?
/8	(ii)	Determine the conditional probability $P[Y \le 0.75 \mid Y \ge 0.5]$.
/32		

a)

i) $f_X(x) = \frac{1}{5-(-5)} = \frac{1}{10}$ $\quad f_X(x) = \begin{cases} \frac{1}{10} & -5 \le X \le 5 \\ 0 & \text{otherwise} \end{cases}$



(5/5)

ii) $E[X] = \int_{-5}^{5} \left(\frac{1}{10}\right) x\,dx = \left(\frac{1}{10}\right)\left(\frac{x^2}{2}\Big|_{-5}^{5}\right) = \left(\frac{1}{20}\right)(25-(25)) = 0$

$E[X^2] = \frac{1}{10}\int_{-5}^{5} x^2\,dx = \left(\frac{1}{10}\right)\left(\frac{1}{3}\right)\left(x^3\Big|_{-5}^{5}\right) = \left(\frac{1}{30}\right)(125+125) = \boxed{\frac{250}{30}}$

$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{250}{30}$ (8/8)

iii) $P[|X-1|<2]$

$-2 < |x-1| < 2$

$-1 < x < 3$

$\int_{-1}^{3} \frac{1}{10}\,dx = \left(\frac{1}{10}\right)(3-(1)) = \cancel{\frac{6}{10}}$

$\frac{4}{10}$

(6/8)

b) i) $\int_0^2 c(2-y)\,dy = 1$

$c\left(2y-\frac{y^2}{2}\Big|_0^2\right) = 1$

$c(4-2) = 1$, $\boxed{c = \frac{1}{2}}$ $\left(\frac{3}{3}\right)$

ii) $P[Y \leq 0.75 \mid Y \geq 0.5]$

$$= \frac{\int_{0.5}^{0.75} \left(\frac{1}{2}\right)(2-y)\,dy}{\int_{0.5}^{2} \left(\frac{1}{2}\right)(2-y)\,dy} = \frac{\left(\frac{1}{2}\right)\left(2y-\frac{y^2}{2}\right)\Big|_{0.5}^{0.75}}{1-\left(\frac{1}{2}\right)\left(2y-\frac{y^2}{2}\right)\Big|_{0.5}^{2}} =$$

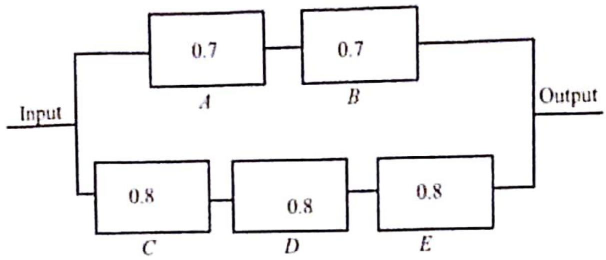
$$= \frac{0.1719}{1-\left(2y-\frac{y^2}{2}\right)\Big|_{0.5}^{2}}$$

$$\frac{0.1719}{0.5625} = 0.3056 \quad \left(\frac{8}{8}\right)$$

Q1 $-\left(\frac{30}{32}\right)$

Question 2:

Marks		
	(a)	Consider the circuit system in the figure below. Assume the components fail independently. The number in each block represents the reliability of that block.
/8	(i)	(a) What is the probability that the entire system works?
/8	(ii)	(b) Given that the system works, what is the probability that the component A is not working?
/8	(iii)	(c) It is known that the system does not work. What is the probability that the component A also does not work?
/8	(b)	Assume that a new light bulb will burn out after t hours, where t has a failure PDF given by $f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & 0 < t < \infty, \\ 0, & \text{elsewhere.} \end{cases}$ Assume $\lambda = 0.01$ and find the probability that the bulb will not burn out before t_0 hours.
/32		



a) $P[\text{sys works}] = R_s$

$$1 - R_s = (1-(0.7)(0.7))(1-(0.8)(0.8)(0.8))$$

$R_s = P[A\cap B \cup (C\cap D\cap E)]$

$$R_s = 1 - (1-0.7^2)(1-0.8^3)$$

$$\boxed{R_s = 0.75112}$$ (8/8)

b) $P[\bar{A}|F] = \dfrac{\bar{A}\cap((A\cap B)\cup(C\cap D\cap E))}{0.75112} = \dfrac{(\bar{A}\cap A\cap B)\cup(\bar{A}\cap C\cap D\cap E)}{0.75112}$

$$P[\bar{A}|F] = \frac{(0.3)(0.8)(0.8)(0.8)}{0.75112} = \boxed{0.2045}$$ (8/8)

c) $P[\bar{A}|\bar{F}]$ $\quad \bar{F} = (\bar{A}\cup\bar{B})\cap(\bar{C}\cup\bar{D}\cup\bar{E})$

$$= \frac{(\bar{A})\cap(\bar{A}\cup\bar{B})\cap(\bar{C}\cup\bar{D}\cup\bar{E})}{1-0.75112} = (\bar{A}\cup\bar{A}\cap\bar{B})\cap(\bar{C}\cup\bar{D}\cup\bar{E})$$

How did you calculate this??

What is this??

(6/8)

$= 0.58$

Q2 − (30/32)

b) $R_T(t) = e^{-\lambda t}$

$$\boxed{R_T(t) = e^{-(0.01)(t_0)}}$$

→ This formula was from the notes

(8/8)

~~$\bar{A} \cap (\bar{A} \cup \bar{B}) \cap (\bar{C} \cup \bar{D} \cup \bar{E}) = (\bar{A} \cap \bar{A}) \cup (\bar{A} \cap \bar{B}) \cap (\bar{C} \cup \bar{D} \cup \bar{E})$~~

~~$\bar{A}$ $\bar{A}\bar{B}\bar{C}\bar{D}\bar{E}$~~

~~0.24888~~

~~$(\bar{A}) \cap ($~~

Question 3:

Marks	
	In ENEL 419, the fraction of students who receive grades below 85%, each year, is a random variable denoted X and the fraction of students who receive 85% and above is also a random variable denoted Y. The fraction of students who receive grades below 85% is always greater than the fraction that receive grades of 85% and above. The two groups of students are described by the joint PDF function $f_{XY}(x,y) = cxy$.

Marks		Part	
1.5	/2	(a)	Find the value of c
10	/10	(a)	Find the marginal probability density functions.
6	/6	(b)	Find the expected values of X and Y.
8	/8	(c)	Find the standard deviations of X and Y.
5.5	/6	(d)	Find the correlation $r_{XY} = E[XY]$, between X and Y
3	/4	(e)	Determine the correlation coefficient.
34	/36		

$0 \le Y \le X \le 1$

a) $\int_0^1 \int_0^1 cxy \, dx dy = 1$

$1 = \int_0^1 \int_0^x cxy \, dx dy$
$= c\int_0^1 x \left[\frac{1}{2}y^2\right]_0^x dx$
$= \frac{c}{2}\int_0^1 x^3 dx$
$\to = \frac{c}{2}\left[\frac{1}{4}x^4\right]_0^1$
$= \frac{c}{8} \Rightarrow \boxed{c=8}$

-0.5

$(c)\left(\frac{x^2}{2}\right)\Big|_0^1 \left(\frac{y^2}{2}\right)\Big|_0^1 = (c)\left(\frac{1}{4}\right) = 1, \boxed{c=4}$

b) $f_X(x) = \int_0^x 4xy \, dy = \frac{4xy^2}{2}\Big|_0^x = \begin{cases} 2x^3 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$4x^3$

$f_Y(y) = \int_y^1 4xy \, dx = (4y)\left(\frac{x^2}{2}\right)\Big|_y^1 = \begin{cases} 2y(1-y^2) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$4y(1-y^2)$

c) $E[X] = \int_0^1 2x^4 dx = \frac{2}{5}x^5\Big|_0^1 = \frac{2}{5}$, $\sigma_X = \sqrt{\frac{1}{3} - \left(\frac{2}{5}\right)^2} = 0.416$

$\boxed{E[X] = \frac{4}{5}}$ $\sigma_X^2 = \frac{2}{75}$ $\boxed{\sigma_X = \sqrt{\frac{2}{75}}}$

$E[X^2] = \int_0^1 2x^5 dx = \frac{2}{6}x^6\Big|_0^1 = \frac{2}{6}$ $E[X^2] = \frac{2}{3}$

$E[Y] = \int_0^1 (2)(y^2 - y^4) dy = (2)\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^1 = (2)\left(\frac{1}{3} - \frac{1}{5}\right) = 0.26$

$\boxed{E[Y] = \frac{8}{15}}$

$E[Y^2] = \int_0^1 (2)(y^3 - y^5) dy = (2)\left(\frac{y^4}{4} - \frac{y^6}{6}\right)\Big|_0^1 = (2)\left(\frac{1}{4} - \frac{1}{6}\right) = 0.167$

$\sigma_Y = \sqrt{0.167 - (0.26)^2} = \boxed{0.315}$

$E[Y^2] = \frac{1}{3}$ $\sigma_Y^2 = \frac{11}{225}$ $\boxed{\sigma_Y = \frac{\sqrt{11}}{15}}$

-0.5

d) $E[xy] = \int_0^1 \int_0^2 xy f_{xy}(x,y) = 4\int_0^1 x^2 dx \int_0^1 y^2 dy$

$E[xy] = (4)\left(\frac{x^3}{3}\Big|_0^1\right)\left(\frac{y^3}{3}\Big|_0^1\right) = (4)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = 0.44$

$r_{XY} = \int_0^1\int_0^2 8x^2y^2 \, dx\, dy$

$\boxed{r_{XY} = \frac{4}{9}}$

e) $= C_{xy} = r_{xy} - \mu_x\mu_y$

sub. error -0.5

$\frac{4}{5}$ 0.26

$C_{xy} = 0.44 - (0.416)(0.315) = 0.313$ -0.5

0.308 calc. error

$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} = \frac{0.313}{(0.416)(0.315)} = 2.39$

$C_{XY} = r_{XY} - \mu_X\mu_Y$

$= \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right)$

$= \frac{4}{225}$

$\rho_{XY} = \frac{\frac{4}{225}}{\left(\sqrt{\frac{2}{75}}\right)\left(\frac{\sqrt{11}}{15}\right)}$

$\boxed{\rho_{XY} = \frac{4}{\sqrt{66}}}$

Formula Sheet

$E[X^n]=\int_{-\infty}^{\infty} x^n f_X(x)\,dx$	$f_Y(y)=f_X\left(g^{-1}(y)\right)\left\lvert\frac{dg^{-1}(y)}{dy}\right\rvert$	$E[h(X)]=\int_{-\infty}^{\infty} h(x)f_X(x)\,dx$ $\sigma^2_{h(X)}=E\left[\left(h(X)-\mu_{h(X)}\right)^2\right]$
$\sigma_X^2=E\left[(X-\mu_X)^2\right]$ $=E\left[X^2\right]-\mu_X^2$	$\frac{1}{\sqrt{2\pi}\sigma_X}\int_{\gamma}^{\infty}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)dx=Q\left(\frac{\gamma-\mu_X}{\sigma_X}\right)$	$p_X(x)=\sum_y p_{XY}(x,y)$
$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$	$p_{X\mid Y}(x\mid y)=p_{XY}(x,y)/p_Y(y)$	$f_{X\mid Y}(x\mid y)=\frac{f_{XY}(x,y)}{f_Y(y)}$
$r_{XY}=E[XY]$	$C_{XY}=E\left[(X-\mu_X)(Y-\mu_Y)\right]=r_{XY}-\mu_X\mu_Y$	$\rho_{XY}=C_{XY}/(\sigma_X\sigma_Y)$
	$P[(X,Y)\in A]=\iint_A f_{XY}(x,y)\,dx\,dy$, A is a region in the XY-plane for which $P[(X,Y)\in A]$ is to be evaluated.	

Do not provide any answers beyond this table. Any answers on this page and beyond will not be marked