Joint, Marginal and Conditional Probability Distributions and Independence:

Problem 1: Consider two random variables X and Y. Determine the values of constants α and β so that the following functions are true joint probability distributions (PMFs):

(a) $f(x,y) = \alpha xy, \quad x = 1,2,3; \ \ y = 1,2,3$

(b) $f(x,y) = \beta|x-y|, \quad x = -2,0,2; \ \ y = -2,3$

Solution:

$$(a) \ \sum_{x=0}^{3}\sum_{y=0}^{3} p(x,y) = \sum_{x=0}^{3}\sum_{y=0}^{3} xy = 36\alpha = 1 \Rightarrow \alpha = \frac{1}{36}$$

$$(b) \ \sum_{x=0}^{3}\sum_{y=0}^{3} p(x,y) = \beta\sum_{x}\sum_{y}|x-y| = 15\beta = 1 \Rightarrow \beta = \frac{1}{15}$$

Problem 2: The joint probability distributions (PMFs) of two random variables X and Y is given by

$$p(x,y) = \frac{x+y}{30}, \quad x = 1,2,3; \ \ y = 0,1,2$$

Find

(a) $P[X \le 2, Y = 1]$	(b) $P[X > 2, Y \le 1]$
(c) $P[X > Y]$	(d) $P[X + Y = 4]$

Solution: First, we express the joint PMF of X and Y in table form

$p(x,y)$	$x=0$	$x=1$	$x=2$	$x=3$
$y=0$	0	1/30	2/30	3/30
$y=1$	1/30	2/30	3/30	4/30
$y=2$	2/30	3/30	4/30	5/30

$$(a) \ P[X \le 2, Y = 1] = p(0,1) + p(1,1) + p(2,1) = 1/30 + 2/30 + 3/30 = 1/5.$$
$$(b) \ P[X > 2, Y \le 1] = p(3,0) + p(3,1) = 3/30 + 4/30 = 7/30.$$
$$(c) \quad P[X > Y] = p(1,0) + p(2,0) + p(2,1) + p(3,0) + p(3,1) + p(3,2)$$
$$= 3/30 + 2/30 + 3/30 + 3/30 + 4/30 + 5/30 = 3/5.$$
$$(d) \quad P[X + Y = 4] = p(2,2) + p(3,1) = 4/30 + 4/30 = 4/15$$

Problem 3: The lifetimes, measured in years, of two components in an electronic system are represented by two random variables X and Y, with their joint PDF given by

$$f(x,y)=\begin{cases} e^{-(x+y)} & x>0;\ \ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional probability $P\left[0<X<1 \middle| Y=2\right]$

Answer:

$$P\left[0<X<1 \middle| Y=2\right]=0.6321.$$

Problem 4: An experiment is performed to determine the reaction time, in seconds to a certain stimulus and to also determine the temperature (in degrees Fahrenheit). Denote the reaction time by random variables X and reaction temperature by random variable Y. Their joint PDF is given by

$$f(x,y)=\begin{cases} 4xy, & 0<x<1,\ \ 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Find

(a) $P\left[0 \le X \le 0.5 \text{ and } 0.25 \le Y \le 0.5\right]$

(b) $P\left[X<Y\right]$

(c) Determine whether X and Y are independent or dependent.

Solution:

(a) $P\left[0 \le X < 0.5,\ 0.25 \le Y \le 0.5\right]=\dfrac{3}{64}.$

(b) $P\left[X<Y\right]=\dfrac{1}{2}.$

(c) $f_X\left(x\right)=2x,\ 0<x<1.$ $f_Y\left(y\right)=2y,\ 0<y<1.$

From above, $f\left(x,y\right)=f_X\left(x\right)f_Y\left(y\right)$. Therefore, X and Y are they are independent.

Problem 5: The diameter of an electric cable is a random variable X and the diameter of the mold that makes the electric cable is a random variable Y. Both X and Y are scaled so that they range between 0 and 1. Their joint PDF is given by

$$f(x,y)=\begin{cases} \dfrac{1}{y}, & 0<x<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $P\left[X+Y>0.5\right]$

Solution:

$$P[X+Y>0.5]=1-P[X+Y<0.5]=1-\int_0^{0.25}\int_x^{0.5-x}\frac{1}{y}dydx$$
$$=1+0.25\ln(0.25)=0.6534$$

Problem 6: The number of times a certain electronic device malfunctions is a random variable X and number of times a technician is called on an emergency basis is a random variable Y. Their joint PMF is given by by the table

$p(x,y)$	$x=1$	$x=2$	$x=3$
$y=1$	0.05	0.05	0.10
$y=2$	0.05	0.10	0.35
$y=3$	0.00	0.20	0.10

(a) What is the marginal PMF of X?

(b) What is the marginal PMF of Y?

(c) Find $P[Y=3 \mid X=2]$

(d) Determine whether X and Y are independent or dependent.

Solution:

(a)

	$x=1$	$x=2$	$x=3$
$f_X(x)$	0.10	0.35	0.55

(b)

	$y=1$	$y=2$	$y=3$
$f_Y(y)$	0.20	0.50	0.30

(c)

$$P[Y=3 \mid X=2]=0.5714$$

Problem 7: Random variables X and Y, denote the life in hours of two electronic components of a missile system. For the success of the system the two components must work in harmony. Their joint PDF is given by

$$f(x,y)=\begin{cases} ye^{-y(1+x)}, & x>0,\ y>0, \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$(a)\ f_X(x)=\dfrac{1}{(1+x)^2},\ \ x>0.\ \ f_Y(y)=e^{-y},\ \ y>0.$

$(b)\ \ P\left[X \geq 2,\ Y \geq 2\right] = \frac{1}{3}e^{-6}$

Problem 8: Random variables X and Y, have joint PDF is given by

$$f(x, y) = \begin{cases} \frac{3x - y}{9}, & 1 < x < 3,\ 1 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What are the marginal density functions of X and Y?

(b) Verify that X and Y whether are independent or not.

(c) Find $P\left[X > 2\right]$

Solution:

$(a)\ \ f_X\left(x\right) = \frac{x}{3} - \frac{1}{6},\ \ 1 < x < 3,\ \ f_Y\left(y\right) = \frac{4}{3} - \frac{2}{9}y,\ \ 1 < y < 2$

(b) No, since $f_X\left(x\right)f_Y\left(y\right) \neq f\left(x, y\right)$.

$(c)\ \ P\left[X > 2\right] = \int_2^3\left(\frac{x}{3} - \frac{1}{6}\right)dx = \left(\frac{x^2}{6} - \frac{x}{6}\right)\Bigg|_2^3 = \frac{2}{3}.$

Mathematical Expectations of Pairs of Random Variables - Covariance, Correlation Coefficient:

Problem 9: Tire experts A and B provide tire-quality ratings on a 3-point scale. Random variables X denotes rating provided by A and Y denotes rating provided by B. The two random variables have joint PMF is given by the table below.

$p(x,y)$	$y=1$	$y=2$	$y=3$
$x=1$	0.10	0.05	0.02
$x=2$	0.10	0.35	0.05
$x=3$	0.03	0.10	0.20

Find the mean values μ_X and μ_Y of X and Y, respectively.

	$x=1$	$x=2$	$x=3$
$f_X\left(x\right)$	0.17	0.50	0.33

	$y=1$	$y=2$	$y=3$
$f_Y\left(y\right)$	0.23	0.50	0.27

$$\mu_X = \sum_{x=1}^{3x} xf_X\left(x\right) = 2.16,\ \ \mu_Y = \sum_{x=1}^{3x} yf_Y\left(y\right) = 2.04$$

Problem 10: The number of times a certain electronic device malfunctions is a random variable X and number of times a technician is called on an emergency basis is a random variable Y. Their joint PMF is given by the table

$p(x,y)$	$x=1$	$x=2$	$x=3$
$y=1$	0.05	0.05	0.10
$y=3$	0.05	0.10	0.35
$y=5$	0.00	0.20	0.10

Find the covariance of X and Y.

Solutions:

	$x=1$	$x=2$	$x=3$
$f_X(x)$	0.10	0.35	0.55

	$y=1$	$y=3$	$y=5$
$f_Y(y)$	0.20	0.50	0.30

$$\mu_X = \sum_{x=1}^{3} x f_X(x) = 2.45., \quad \mu_Y = \sum_{x=1} y f_Y(y) = 3.20.$$

$$E[XY] = 7.85$$

$$\sigma_{XY}^2 = 0.01$$

Problem 11: Random variables X and Y, have joint PDF is given by

$$f(x,y) = \begin{cases} \dfrac{16y}{x^3}, & x > 2,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the correlation coefficient of X and Y.

Solution:

$$f_X(x) = \frac{8}{x^3}, \quad x > 2 \Rightarrow \mu_X = 4, \quad f_Y(y) = 2y, \quad 0 < y < 1 \Rightarrow \mu_Y = \frac{2}{3}.$$

$$E[XY] = \frac{8}{3} \Rightarrow Cov[XY] = 0.$$

Problem 12:

Random variables X and Y have joint PMF given by

$$p_{XY}(x,y)=\begin{cases} cxy & x=1,\ 2,\ 4;\ y=1,\ 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c,
(b) The marginal PMFs of X and Y,
(c) The expected values of X and Y,
(d) The standard deviations of X and Y.

$(a)\ \ c = 1/28$

$$(b)\ \ p_x(x)=\sum_{y=1,3} p(x,y)=\begin{cases} 4/28 & x=1 \\ 8/28 & x=2 \\ \underline{16/28 \quad x=4} \end{cases};\quad p_Y(y)=\sum_{x=1,2,4} p(x,y)=\begin{cases} 7/28 & y=1 \\ 21/27 & y=3 \end{cases}$$

$$(c)\ \ \mu_X = 3;\quad \mu_Y = \frac{5}{2}.$$

$$(d)\ \ E\left[X^2\right]=\frac{73}{7};\quad E\left[Y^2\right]=7;$$

$$\sigma_X^2 = \frac{10}{7};\quad \sigma_Y^2 = \frac{3}{4}$$