ENEL 419: Probability and Random Variables

Midterm Exam

Instructor: Dr. Abu Sesay

October 20, 2023

Start Time: 12:00 pm Duration: 70 minutes.

Last Name (printed):	First Name:	ID #:

Signature:

Instructions:

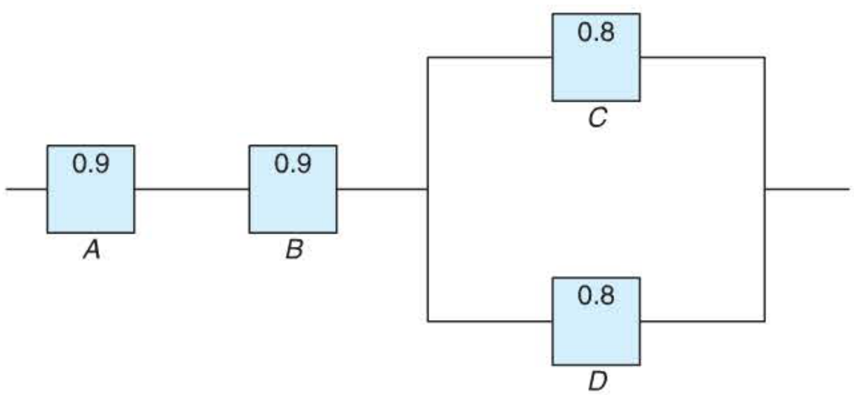
- All the University of Calgary regulations apply to this exam.
- Answer all three questions in the booklet provided.
- You are allowed to use any calculator for this exam. No other electronic device, including music devices or anything with wireless capability, will be allowed during the exam.
- The test is closed-book and closed-notes. Formulas are provided on the last page. Please detach for your convenience.
- To reduce distraction to other students, you are not allowed to leave during the last ten minutes of the exam.
- Please print or write your answers legibly. What cannot be read cannot be marked.
- If you write anything you do not want marked, put a large "X" through it and write "rough work" beside it.

Marks Summary

	Q1	Q2	Q3	Total
Marks				
Out of	35	30	35	100

Question 1:

Marks		
		An electrical system consists of four components, A, B, C and D as illustrated in the figure below. The probability of working of each component is also shown in the corresponding component block. For instance, the probability that block A works is $P[A] = 0.9$. The entire system is considered to be working only if a signal applied to the input terminal on the left side can propagate right through to the output terminal on the right side. Assuming that all four components work (or fail) independently, answer the following questions:
/5	(a)	Write a logic expression, in terms of union, intersection or complement operations, to show the conditions under which the entire system works.
/10	(b)	Find the probability that the entire system works.
/10	(c)	Find the probability that the component C does not work, given that the entire system works.
/10	(d)	Suppose component A is moved and appended to the right side of the system. What is the probability that the entire system works?
/35		



Question 1 Solutions:

(a) $A \cap B \cap (C \cup D)$ (5 marks)

(b) $P[entire\ system\ works] = P[A \cap B \cap (C \cup D)]$(3 marks)

Due to independence

$P[entire\ system\ works] = P[A] \times P[B] \times P[C \cup D]$, (3 marks)

$P[C \cup D] = P[C] + P[D] - P[C] \times P[D] = 0.8 + 0.8 - 0.8 \times 0.8 = 0.96$ (2 marks)

$P[system\ works] = 0.9 \times 0.9 \times 0.96 = 0.7776$(2 marks)

(c) Probability that the component C does not work, given that the entire system works:

$P[\bar{C}|entire\ system\ works] = \frac{P[entire\ system\ works\ \cap C\ does\ not\ work]}{P[entire\ system\ works]}$ (3 marks)

$P[\bar{C}|entire\ system\ works] = \frac{P[A \cap B \cap \bar{C} \cap D]}{P[entire\ system\ works]}$(2 marks)

Due to independence

$\frac{P[A \cap B \cap \bar{C} \cap D}{P[entire\ system\ works]} = \frac{P[A] \times P[B] \times P[\bar{C}] \times P[D]}{P[entire\ system\ works]}$ (5 marks)

$= \frac{0.9 \times 0.0 \times (1-0.8) \times 0.8}{0.7776} = 0.1667$(2 marks)

(d) If A is moved to the end we have

$P[entire\ system\ works] = P[B \cap (C \cup D) \cap A]$(3 marks)

By commutative rule

$P[entire\ system\ works] = P[B \cap (C \cup D) \cap A]$(2 marks)
$= P[A \cap B \cap (C \cup D)]$(2 marks)

The procedure is the same as in part (c); give the remainder of the 6 marks if the answer is correct. (total 10 marks)

$= 0.7776$

Question 2:

Marks		
	(a)	One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. Suppose one ball is drawn from the first bag and placed unseen in the second bag. Answer the following questions:
/10	(i)	Suppose a ball is now drawn from the second bag. What is the probability that it is black?
/6	(ii)	Suppose a ball is now drawn from the first bag. What is the probability that it is white?

	(b)	A manufacturer is studying the effects of cooking temperature, cooking time, and type of cooking oil for making potato chips. Three different temperatures, 4 different cooking times, and 3 different oils are to be used.
/8	(ii)	What is the total number of combinations to be studied?
/4	(iii)	How many combinations will be used for each type of oil?
/2	(iv)	Explain why permutations are or are not necessary in this problem.
/30		

Question 2 Solutions:

(a) Define the following events:

(i)

$B_1 = \{black\ ball\ drawn\ from\ bag\ 1\}$ (0.5)

$W_1 = \{black\ ball\ drawn\ from\ bag\ 1\}$ (0.5)

$B_2 = \{black\ ball\ drawn\ from\ bag\ 2\}$ (1)

$$We\ are\ interested\ in\ the\ union\ of\ the\ mutually\ exclusive\ events$$

$P[B_1 \cap B_2\ or\ W_1 \cap B_2] = P[B_1 \cap B_2] + P[W_1 \cap B_2]$ (0.5 each – total is 1 $(1\ mark\ for\ the\ left\ hand\ side)$

$$= P[B_1] \cap P[B_2|B_1] + P[W_1] \cap P[B_2|W_1] \text{(1 each – total is 2)}$$

$$= \frac{3}{7} \times \frac{6}{9} + \frac{4}{7} \times \frac{5}{9} = \frac{38}{63} \text{(2)}$$

(ii) If we first drew a black ball from bag 1, we have left 4 whites and 2 blacks. If we draw a black next, we have,

$$We\ are\ interested\ in\ the\ union\ of\ the\ mutually\ exclusive\ events$$

$P[B_1 \cap W_1\ or\ W_1 \cap W_1] = P[B_1 \cap W_1] + P[W_1 \cap W_1]$(1 each – total is 2)

$$= P[B_1]P[W_1|B_1] + P[W_1]P[W_1|W_1] \text{(1 each – total is 2)}$$

$$= \frac{3}{7} \times \frac{4}{6} + \frac{4}{7} \times \frac{3}{6} = \frac{4}{7} \text{ (2)}$$

(b) $n_1 = 3, n_2 = 4, n_3 = 3.$ (1 each – total is 3)

$n = n_1 \times n_2 \times n_3 = 3 \times 4 \times 3 = 36.$ (3)

(c) For one type of oil, we have $n_1 = 3, n_2 = 4, n_3 = 1.$(1)

$n = n_1 \times n_2 \times 1 = 3 \times 4 \times 1 = 12.$ (2)

(d) Permutations are not necessary (1) because order does not matter (1). (Total 2)

Question 3:

Marks		An investment firm offers its customers municipal bonds that mature after varying number s of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond , is $$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \le t < 3, \\ \frac{1}{2}, & 3 \le t < 5, \\ \frac{3}{4}, & 5 \le t < 7, \\ 1, & t \ge 7, \end{cases}$$
/2	(a)	Find $P[T = 5]$
/2	(b)	Find $P[T > 3]$
/4	(c)	Find $P[1.2 < T < 6]$
/6	(d)	Find $P[T \le 5 \mid T \ge 2]$
/8	(e)	Find the PMF
/13	(f)	Find the standard deviation of T.
/35		

Question 3 Solutions:

(a)

$$P[T=5]=F_X(5)-F_X(4) \text{ (1 mark)}$$
$$=\frac{3}{4}-\frac{1}{2} \text{ (0.5 marks)}$$
$$=\frac{1}{4}$$

(b)

$$P[T>3]=1-P[T\leq 3] \text{ (0.5 marks)}$$
$$=1-F_X(3)\text{(1 mark)}$$
$$=1-\frac{1}{2}=\frac{1}{2}$$

(c) $P[1.2<T<6]=F_X(6)-F_X(1.2)$(2 marks)

$$=\frac{3}{4}-\frac{1}{4}\text{(1.5 marks)}$$
$$=\frac{1}{2}$$

(d) $P[T\leq 5|T\geq 2]=\frac{P[2\leq T\leq 5]}{P[T\geq 2]}$(2 marks for numerator) and (2 marks for denominator)

$$=\frac{\frac{3}{4}-\frac{1}{4}}{1-\frac{1}{4}}$$ (0.5 marks for numerator) and (0.5 marks for denominator)

$$=\frac{2}{3}$$

(e) $p_T(t=1)=F_X(1)=\frac{1}{4}$; (2 marks)

$p_T(t=3)=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$; (2 marks)

$p_T(t=5)=\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$; (2 marks)

$p_T(t=7)=1-\frac{3}{4}=\frac{1}{4}$; (2 marks)

(f)

$\mu_T=\frac{1}{4}(1+3+5+7)=4$; (4 marks)

$E[T^2]=\frac{1}{4}(1+9+25+49)=21$; (4 marks)

$\sigma_T^2=E[T^2]-\mu_T^2=21-16=5$; (4 marks)

$\sigma_T=\sqrt{5}$(1 marks)