5. Continuous RVs and Probability Density Function:

(Reading Exercises: Montgomery and Runger Chapter 4.1-4.2 & Class notes)
(Yates and Goodman Chapter 4)

You will

- Understand the concept of a probability density function of a continuous random variable.
- Know the formal definition of a probability density function of a continuous random variable.
- Know how to find the probability that a continuous random variable takes on a value inside some interval.
- Know the formal definition of a cumulative distribution function of a continuous random variable.
- Know how to find the cumulative distribution function of a continuous random variable from its probability density function.
- Know how to extend the definitions of the mean, variance and standard deviation to functions of a continuous random variable.
- Be able to apply the methods learned in the section to new problems.

5.1 Probability Distribution Function

Definition: A random variable is continuous if its set of possible values belong to an entire interval of numbers.

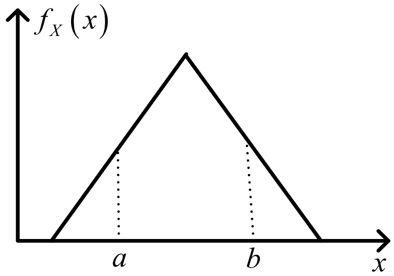
Definition: A continuous random variable is said to have a probability density function (PDF), denoted $f_X(x)$, if

- $f_X(x) \geq 0$ (non-negative) for all $x \in \mathbb{R}$ (all real numbers x)
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $P[a < x < b] = P[a < x \leq b] = P[a \leq x < b] = P[a \leq x \leq b] = \int_a^b f_X(x)dx$

Remarks:

- A continuous random variable is described by a probability density function
- With continuous random variables we talk about the probability of X being in some interval, for example, $a \leq x \leq b$

$$P[a \leq X \leq b] = \int_a^b f_X(x)\, dx$$



- The probability $P[a \le X \le b]$, is the area under the PDF in the interval$a \le X \le b$.
- A continuous random variable has a probability of zero, to assume exactly any one of its values, $P[X = b] = 0$.

<u>Examples:</u> Suppose the error in the reaction temperature, in $^\circ C$, for a controlled laboratory experiment is a continuous random variable x having the PDF

$$f_X(x) = \begin{cases} \frac{x^2}{3}, & 1 < x < 2 \\ 0, & otherwise. \end{cases}$$

Verify that $f_X(x)$ is a valid pdf and then find $P[0 < X \le 1]$.

<u>Solution:</u>
It is a valid PDF because $f_X(x) \ge 0$ and the area is equal to 1. The probability is

$$P[0 < X \le 1] = \int_{-\infty}^{1} \frac{x^2}{3} dx = \left. \frac{x^3}{9} \right|_0^1 = \frac{1}{9}$$

5.2 Cumulative Distribution Function

<u>Definition:</u> The cumulative distribution function (CDF), denoted $F_X(x)$, of a continuous random variable X, with pdf $f_X(x)$ is defined as the probability that X is less than or equal to a given value x, that is,

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(y) dy$$

- For every x, $F_X(x)$is the area under the PDF curve to the left of x.
- If the derivative exists, then the PDF is related to the CDF as follows:
- $f_X(x) = \frac{dF_X(x)}{dx}$

Properties of CDF:

- $F_X(x)$is a non-decreasing function from left to right,
- $F_X(x)$ is a continuous function,
- $F_X(\infty) = 1$
- $F_X(-\infty) = 0$
- $P[a < X \leq b] = F_x(b) - F_x(a)$
- $P[X > x] = 1 - P[X \leq x] = 1 - F_X(x)$

Example: A random variable has the probability density function

$$f_X(x) f_X(x) = \begin{cases} \frac{x^2}{3}, & 1 < x < 2 \\ 0, & otherwise. \end{cases}$$

Find the CDF $F_X(x)$, and evaluate the probability $P[0 < X \leq 1]$.

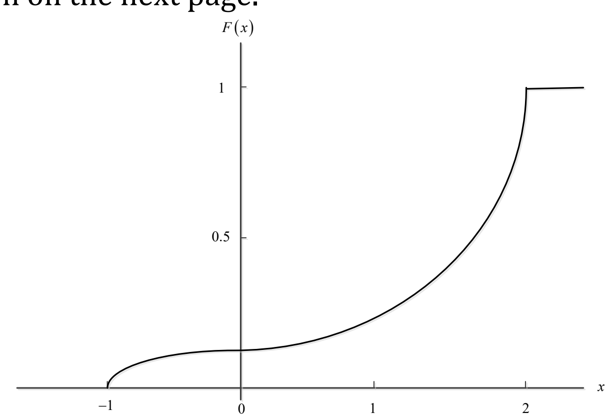
Solution: For the interval $-1 < X < 2$, we have

$$F_X(x) = \int_{-\infty}^{x} f(y)\, dy = \int_{-1}^{x} \frac{y^2}{3}\, dy = \left.\frac{y^3}{9}\right|_{-1}^{x} = \frac{x^3 + 1}{9}$$

Therefore,

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{x^3+1}{9}, & -1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

A rough sketch is shown on the next page.



$$P[0 < X \leq 1] = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Example: The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b. The DOE has determined that the pdf of the winning (low) bid is,

$$f_X(x) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \le x \le 2b, \\ 0, & otherwise. \end{cases}$$

Find the CDF $F_X(x)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate b.

Solution: For the interval $\frac{2}{5}b \le x < 2b$,

$$F_X(x) = \int_{\frac{2b}{5}}^{x} \frac{5}{8b}\,dy = \frac{5y}{8b}\bigg|_{\frac{2b}{5}}^{x} = \frac{5x}{8b} - \frac{1}{4}.$$

$$F_X(x) = \begin{cases} 0, & x < \frac{2}{5}b \\ \frac{5x}{8b} - \frac{1}{4}, & \frac{2}{5}b \le x < 2b \\ 1, & x \ge 2b \end{cases}$$

For the interval $x \ge 2b$, $F_X(x) = 1$

$$P[X \le b] = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}$$

5.3 PDF of the Transformation of Random Variables

A random variable X has PDF $f_X(x)$. Consider a new random variable, which is a linear function of X, $Y = g(X) \Rightarrow y = g(x)$. The PDF of the transformation Y is determined as follows:

$$f_Y(y) = f_X\big(g^{-1}(y)\big)\left|\frac{dg^{-1}(y)}{dy}\right|$$

- $x = g^{-1}(y)$ is the inverse function,
- $\left|\frac{dg^{-1}(y)}{dy}\right|$ is called the Jacobian of the transformation

Example: Given the PDF $f_X(x)$ of X, find the PDF of $Y = aX + b$.

Solution:

$$y = g(x) = ax + b \Rightarrow g^{-1}(y) = x = \frac{y-b}{a} \Rightarrow \frac{g^{-1}(y)}{dy} = \frac{dx}{dy} = \frac{1}{a}$$

Therefore, the PDF of Y is given by,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

The CDF of a Function of a RV:

A random variable X has CDF $F_X(x)$. Consider a new random variable $Y = g(X) \Rightarrow y = g(x)$. The CDF of Y is determined as follows:

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = P[X \leq g^{-1}(y)] = F_X\left(g^{-1}(y)\right)$$

<u>**Examples:**</u>

1. Adding a scale factor and an offset (DC value). Let $Y = aX + b$ where a and b are constants. Find $F_Y(y)$

5.4 Expected Value of a Continuous Random Variable:

<u>**Definition:**</u> Consider a continuous random variable X, with PDF $f_X(x)$. The expected (mean or average) value of X, denoted $E[X]$ or μ_X is defined as

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

<u>**Examples:**</u> Let random variable X represent the life expectancy (in hours) of a certain electronic component and has probability density function

$$f_X(x) = \begin{cases} \dfrac{20{,}000}{x^3}, & x > 100 \\ 0, & \text{elsewhere} \end{cases}$$

What is the expected life of this component? Otherwise, if I purchase many of these components, on average, how long will they last?

<u>**Solution:**</u>

$$\mu_X = E[X] = \int_{100}^{\infty} x \frac{20{,}000}{x^3} dx = -\frac{20{,}000}{x}\Bigg|_{100}^{\infty} = 200 \text{ hours}$$

Expected Value of a Function of a Continuous RV:

<u>**Definition:**</u> Consider a continuous random variable X, with PDF $f_X(x)$. The expected value of the function $h(X)$ of the random X, is defined as

$$\mu_{h(X)} = E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$$

<u>**Examples:**</u> Let a continuous random variable X, have probability density function

$$f_X(x) = \begin{cases} \dfrac{x^2}{3}, & -1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of $h(X) = 4X + 3$

Solution:

$$E[4X+3] = \int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} (4x^3+3x^2)dx = \frac{1}{3}(x^4+x^3)|_{-1}^{2} = 8$$

5.5 Variance and Standard Deviation of Continuous Random Variables:

Definition: Consider a continuous random variable X with PDF $f_X(x)$ and mean μ_X. The variance of X, denoted $Var(X)$ or σ_X^2 is defined as

$$\sigma_X^2 = Var(X) = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx$$

Definition: The standard deviation of a random variable X is the square root of the variance,

$$\sigma_X = E[(X-\mu_X)^2] = \sqrt{\sigma_X^2}$$

- The variance is also known as the second central moment of X
- The variance (or standard deviation) of a random variable characterizes the variability or spread of the distribution.
- It gives a description of the shape of the distribution.

Shortcut formula for variance:

The shortcut formula is obtained by simplifying the expression

$$\sigma_X^2 = E[(X-\mu_X)^2] = E[X^2] - 2\mu_X E[X] + \mu_X^2 = E[X^2] - \mu_X^2$$

- $E[X^2]$ is the second moment or mean-square value of X

Example: The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f_X(x) = \begin{cases} 2(x-1), & 1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the mean and variance of X

Solution:

$$\mu_X = 2\int_1^2 x(x-1)\,dx = \tfrac{5}{3};\; E[X^2] = 2\int_1^2 x^2(x-1)\,dx = \tfrac{17}{6}$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}$$

Definition: Let X be a random variable with probability density function $f_X(x)$. Let $h(X)$ be a function of X with expected value $\mu_{h(X)}$. The variance of the function of a continuous random variable $h(X)$ is defined as

$$\sigma_{h(X)}^2 = E\left[\left(h(X) - \mu_{h(X)}\right)^2\right] = \int_{-\infty}^{\infty} \left(h(X) - \mu_{h(X)}\right)^2 f_X(x)dx$$

Example: Let a continuous random variable X have probability density function

$$f_X(x) = \begin{cases} \frac{x^2}{3}, & -1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the variance of $h(X) = 4X + 3$.

Solution: From the previous example $\mu_{h(x)} = 8$

$$\sigma_{h(X)}^2 = E[(4X + 3 - 8)^2] = \int_{-1}^{2} (4x - 5)^2 \frac{x^2}{3} dx = \int_{-1}^{2} \frac{16x^4 - 40x^3 + 25x^2}{3} dx$$

$$= \frac{16 \times 11}{5} - 10 \times 5 + 25 = \frac{176 - 125}{15} = \frac{51}{5}$$

Properties of expectations:

Consider two random variables X and Y with expected values μ_X and μ_Y. Let a, b and c be constants. Then for

- $Z = a \Rightarrow E[Z] = \mu_X = a$ and $\sigma_Z^2 = 0$
- $Z = aX + b \Rightarrow \mu_Z = E[aX + b] = a\mu_X + b$ and $\sigma_Z^2 = a^2\sigma_X^2$

5.6 Important Continuous Random Variables in Electrical Engineering:

Uniform Random Variable:

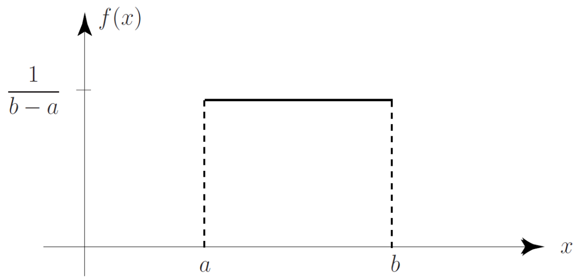
Definition: A continuous uniform random variable X, is one that is equally likely (with equal probability) to assume values in a finite interval $a \le X \le b$. The PDF is equal to a constant c, in the interval $a \le X \le b$,

$$f_X(x) = \begin{cases} c & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

A uniform random variable X, being equally likely to assume any value in a finite interval $a \le X \le b$, implies that the PDF is a constant. Also, since the area under the PDF is equal to 1, we can show that the pdf is given by,

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

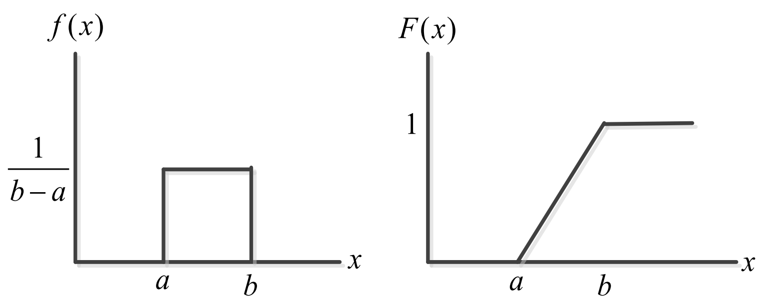
A sketch of the PDF is shown in the figure below.



$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

<u>CDF of Uniform Random Variables</u>:
The CDF is obtained by integrating the PDF, in three regions.

$$F_X(x) = \begin{cases} 0 & x < a \\ \displaystyle\int_a^x \frac{1}{b-a} dy = \frac{x-a}{b-a} & a \le x \le b \\ 1 & b > x \end{cases}$$



Uniform Probability Distribution

<u>Exercise:</u> Show that a random variable, uniformly distributed in the interval $b \le x \le a$, has mean and variance given by $E[X] = (a+b)/2$ and $\sigma_X^2 = (a-b)^2/12$.

<u>Example:</u> Resistors are manufactured with a tolerance of $\pm 10\%$. Within this tolerance region, they are approximately uniformly distributed. What is the probability that a nominal 1000Ω resistor has a value between 900Ω and 1010Ω?

Solution: For a nominal 1000Ω resistor, the tolerance region is $900 \leq X \leq 1100\Omega$, therefore, the PDF is

$$f_X(x) = \begin{cases} \frac{1}{200}, & 900 \leq X \leq 1100 \\ 0, & \text{otherwise} \end{cases}$$

The probability that the resistance lies in the range $900 \leq X \leq 1010\Omega$ is,

$$P[900 \leq X \leq 1010\Omega] = \int_{900}^{1010} \frac{1}{200} dx = 0.55.$$

Exponential Random Variables:

The exponential distribution is often used to model the time interval between occurrences of events or the lifetime of my devices or systems in practice.

The PDF and CDF of Exponential Random Variables:

PDF: The PDF of an exponential random variable, with parameter λ, is given by

$$f_X(x) = \begin{cases} 0, & 0 < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

CDF: The CDF of an exponential random variable, obtained by integrating the PDF, is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x) dx = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

Example: The interval of service (the duration from beginning to completion of service) for a customer in a line at the bank has an exponential distribution with parameter $\lambda = 0.033s^{-1}$. You are next to be served. (a) What is the probability that $15s$ or less will elapse until you are finished being served? (b) What is the probability that one minute or greater will elapse until you are finished being served?

Solution: Let the waiting time be Δt. Then the CDF of the waiting time is

$$F_{\Delta t}(\Delta t) = (1 - e^{-0.033\Delta t})u(\Delta t)$$

$$P[\Delta t \leq 15s] = F(15) - F(0) = 0.388$$

$$P[\Delta t \geq 60s] = 1 - P[\Delta t \leq 60s]$$

$$= 1 - F(60) + F(0) = 0.138$$

<u>Example:</u> The amount of time one spends in a bank is exponentially distributed with parameter $\lambda = 10$ minutes. (a) What is the probability that a customer will spend more than 15 minutes in the bank? (b) What is the probability that a customer will spend more than 15 minutes in the bank given that he/she is still in the bank after 10 minutes?

<u>Solution:</u>

$$P[X > 15] = e^{-15\lambda} = e^{-15/10} = 0.22$$
$$P[X > 15|X > 10] = P[X > 5] = e^{-5/10} = 0.604$$

Mean of and Mean-squared Value of Exponential Random Variables:

The mean value of an exponential random variable is given by

$$\mu_X = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{\lambda};\ E[X^2] = \int_0^{\infty} x^2 e^{-\lambda x} dx$$

Integrating by parts with $u = \lambda x^2$ and $dv = e^{-\lambda x} dx \Rightarrow du = 2\lambda x dx$ and $v = -\frac{1}{\lambda} e^{-\lambda x}$, yields

$$E[X^2] = \int_0^{\infty} x^2\, e^{-\lambda x} dx = \lim_{r \to \infty}\left(\left[-x^2 e^{-\lambda x}\right]\Big|_0^r + 2\int_0^r x e^{-\lambda x} dx\right)$$
$$= \lim_{r \to \infty}\left(\left[-x^2 e^{-\lambda x} - \frac{2}{\lambda} x e^{-\lambda x} - \frac{2}{\lambda^2} e^{-\lambda x}\right]\Big|_0^r\right) = \frac{2}{\lambda^2}$$

<u>Variance:</u> Using the short-cut formula, the variance is

$$\sigma_X^2 = E(X^2) - \left(E(X)\right)^2 = \frac{1}{\lambda^2}$$

Exponential random variables are sometimes used to model the time to failure of equipment or devices.

<u>Gaussian (or Normal) Random Variable:</u>

The Gaussian random variable is the single most widely used in practice.

- Many physical phenomena in electrical engineering are modeled very well by the Gaussian distribution.
- If the model of a continuous RV is unknown, we can conveniently use the Gaussian model.

Definition: A random variable X, with mean μ_X and variance σ_X^2, is said to be Gaussian if its PDF is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right), \quad -\infty \le x \le \infty$$

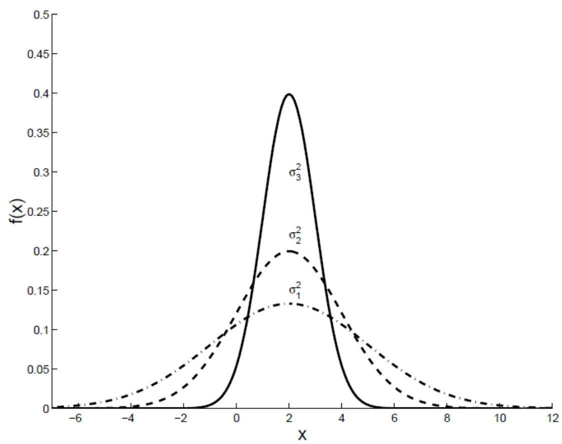
The Gaussian random variable is characterized by only two parameters:

- The mean μ_X and variance σ_X^2 or the standard deviation σ_X

Properties of the Gaussian PDF:

- The curve is bell shaped
- The curve is symmetric around the mean μ_X
- When the variance σ_X^2, decreases the shape of the curve is peakier.
- When the variance σ_X^2, increases the shape of the curve is more spread out

Example plots of a Gaussian PDF with, mean $\mu_X = 2$, and different variances are shown in the figure below.



Gaussian PDF: $\sigma_1 > \sigma_2 > \sigma_3$

Standard Normal (Standard Gaussian) Distribution:

For a Gaussian random variable Y with mean μ_Y and variance σ_y^2, the PDF is given by,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_y^2}\right), \quad -\infty \le y \le \infty$$

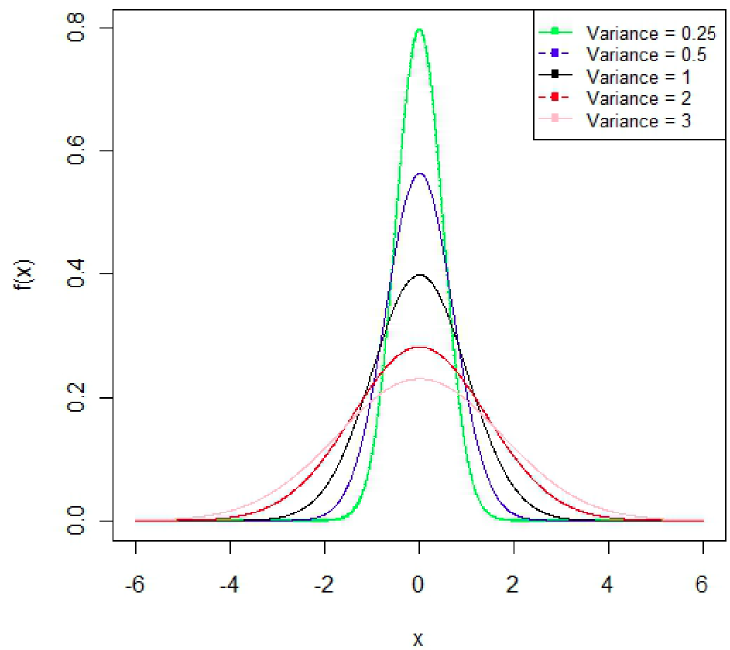
The above in the general Gaussian distribution. The normalized random variable X, given below, has a $\mu_X = 0$ and a variance $\sigma_X^2 = 1$.

$$X = \frac{Y - \mu_Y}{\sigma_Y}$$

Definition: A Gaussian random variable with zero mean and unit variance is referred to as standard Gaussian random variable and has PDF given by.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad -\infty \leq x \leq \infty$$

The plots below illustrate examples of the standard Gaussian random variable.



Example plots of Standard Gaussian random variables

CDF of Standard and Regular Gaussian Random Variables:

The CDF for a standard Gaussian random variable X, is given by,

$$F_X(x) = P[X \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx = \Phi(x) \quad \text{(standard Gaussian)}$$

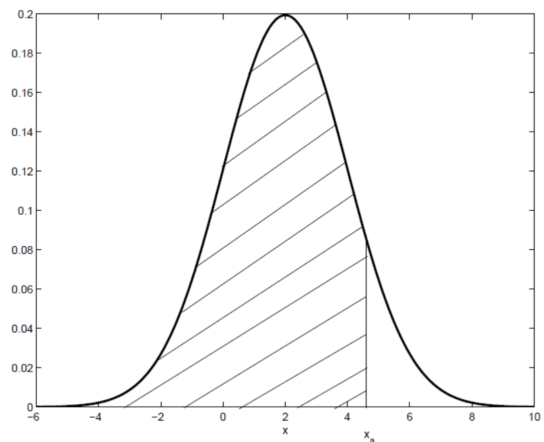
The CDF of the regular Gaussian random variable can be expressed in terms of $\Phi(\cdot)$ by making the following change of variable: $u = \frac{y-\mu_Y}{\sigma_Y} \Rightarrow dy = \sigma_Y du$, which yields,

$$F_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\int_{-\infty}^{y} \exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_y^2}\right)dy$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{y-\mu_Y}{\sigma_Y}} \exp\left(-\frac{u^2}{2}\right)du = \Phi\left(\frac{y-\mu_Y}{\sigma_Y}\right)$$
(regular Gaussian)

The Q-Function:
The Q-function or Marcum Q-function denoted $Q(\cdot)$, is the complement of the standard CDF, that is,

$$Q(x) = 1 - P[X \leq x] = P[X > x] = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} \exp\left(-\frac{u^2}{2}\right)du$$

The Q-function $Q(x_a)$, represents the area under the right tail of the standard normal function in the region $X > x_a$ as illustrated in the plot below.



Right tail:
$Tail\ region\ P[X > x_a]$; Shaded region: $P[X \leq x_a] = 1 - P[X > x_a]$

Remarks:
- There is no closed form solution for the Q-Function integral
- The Q-Function is tabulated and is used extensively in Electrical Engineering.
- The CDF of the regular Gaussian random variable can be easily expressed in terms of the Q-function as follows:

$$F_X(x) = P[X \leq x] = 1 - P[X > x] = \frac{1}{\sqrt{2\pi}\sigma_x}\int_x^\infty \exp\left(-\frac{(y-\mu_X)^2}{2\sigma_x^2}\right)dy$$
$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{x-\mu_X}{\sigma_x}}^\infty \exp\left(-\frac{y^2}{2}\right)dy = 1 - Q\left(\frac{x-\mu_Y}{\sigma_x}\right)$$

Properties of the Q-function:

- $Q(-\gamma) = 1 - Q(\gamma), \gamma > 0$
- $Q(\infty) = 0$
- $Q(-\infty) = 1$
- $Q(0) = 0.5$
- $Q(x)$ is monotonically decreasing with x

Applications of the Gaussian Distribution:

Example: A certain type of battery lasts, on average, 3 years with a standard deviation of 0.5 years. Assuming battery lives are Gaussian distributed. Find the probability that a given battery will last less than 2.3 years.

Solution: Given $\mu_X = 3$ and $\sigma_X = 0.5$, find the probability that $X < 2.3$,

$$P[X \leq 2.3] = 1 - P[X > 2.3] = 1 - \frac{1}{\sqrt{2\pi}\times 0.5}\int_{2.3}^\infty \exp\left(-\frac{(x-3)^2}{2\times 0.5^2}\right)dx$$
$$= 1 - Q\left(\frac{2.3-3}{0.5}\right) = 1 - [1 - Q(1.4)] = 0.0808$$

Example: For the same battery life problem, what is the probability that a given battery will last between 1.5 to 2 years?

Solution: We need to find the following probability:

$$P[1.5 \leq X \leq 2] = P[X > 1.5] - P[X > 2] = Q\left(\frac{1.5-3}{0.5}\right) - Q\left(\frac{2-3}{0.5}\right)$$
$$= 1 - Q\left(\frac{1.5}{0.5}\right) - \left[1 - Q\left(\frac{1}{0.5}\right)\right] = Q(2) - Q(3)$$
$$= 0.0214$$

Example: What is the probability that a given battery will last between 3 to 4 years?

Solution:

$$P[3 \le X \le 4] = P[X > 3] - P[X > 4] = Q\left(\frac{3-3}{0.5}\right) - Q\left(\frac{4-3}{0.5}\right)$$
$$= 0.5 - Q(2) = 0.5 - 0.022275 = 0.47725$$

Example: A company manufactures electrical resistors with mean 3 Ohms and standard deviation 0.005 Ohms. A buyer sets specifications on the resistance to be 3.0 ± 0.01 Ohms. On the average, what percentage of resistors will be rejected?

Solution: We are required to find $P[3.01 < X < 2.99]$.

$$P[3.01 > X < 2.99] = P[x < 2.99] + P[x > 3.01] = 1 - P[x > 2.99] + P[x > 3.01]$$
$$= 1 - \left(1 - Q\left(\frac{3-2.99}{0.005}\right)\right) + Q\left(\frac{3.01-3}{0.005}\right) = 2Q(2) = 0.0455$$

The percentage of resistors rejected is 45.5%.

Example: A company manufactures electrical resistors with mean 3 Ohms. The standard deviation is 0.1Ω. Find a value of the tolerance d, such that 95% of all manufactured resistors fall in the range $3 \pm d\ \Omega$.

Solution: Assume Gaussian distribution. The region for 95% is $3 - d \le X \le 3 + d$. We need to find d such that the probability $P[3 - d \le X \le 3 + d]$ is equal to 95%`

$$P[3 - d \le X \le 3 + d] = P[X \ge 3 - d] - P[X \ge 3 + d]$$
$$= Q\left(\frac{3-d-3}{0.1}\right) - Q\left(\frac{3+d-3}{0.1}\right)$$
$$= 1 - Q\left(\frac{d}{0.1}\right) - Q\left(\frac{d}{0.1}\right) = 1 - 2Q\left(\frac{d}{0.1}\right) = 0.95 \Rightarrow Q\left(\frac{d}{0.1}\right) = 0.025$$

$$Q(10d) = 0.025 \Rightarrow d = 0.196\Omega$$

Example: Signal Detection - assume that in detection of a radar signal, with background noise, follow a Gaussian distribution with a mean of 0 V and a standard deviation of 0.45 V. The radar system assumes that an enemy airplane is detected if the received voltage exceeds 0.9 V. What is the probability of False Alarm?

Solution:

$$P_{FA} = P[\text{Received voltage} > 0.9V|\text{no enemy airplane exists}]$$

$$= P[\text{noise} > 0.9\text{V}] = Q\left[\frac{0.9}{0.45}\right] = Q(2) = 0.02275$$