

Problem 1)

$$f_x(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{other wise} \end{cases}$$

$$\int 2x dx = x^2 \Big|_0^1 = 1$$

$$Y = 0.5X + 0.25$$

a) range of y $(0.5(0) + 0.25) \leq y \leq (0.5(1) + 0.25)$

$$0.25 \leq y \leq 0.75$$

b) CDF of $X = F_X(x) = P(X \leq x)$

$$F_X(x) = \int_{-\infty}^x f_X(u) du$$

for $x < 0$

$$F_X(x) = \int_{-\infty}^0 0 du = 0$$

for $0 \leq x \leq 1$

$$F_X(x) = \int_0^x 2u du = u^2 \Big|_0^x = x^2$$

$$F_X(x) = 1 \quad \text{for} \quad x > 1$$

$$\text{So } F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

c CDF of Y

$$F_Y(y) = P[Y \leq y = 0.5x + 0.25]$$

$$x = \frac{y - 0.25}{0.5} = 2y - 0.5$$

$$F_Y(y) = P[X \leq x = 2y - 0.5] = F_X(2y - 0.5)$$

So,

$$F_Y(y) = \begin{cases} 0 & y < 0.25 \\ 4y^2 - 2y + 0.25 & 0.25 \leq y \leq 0.75 \\ 1 & 0.75 < y \end{cases}$$

d] PDF of Y

$$\text{since } F_Y(y) = F_X(2y - 0.5)$$

$$f_Y(y) = \frac{dF_X(2y - 0.5)}{dy} = f_X(2y - 0.5) \cdot 2 = 2 \cdot 2(2y - 0.5) = 8y - 2$$

$$f_Y(y) = \begin{cases} 8y - 2 & 0.25 \leq y \leq 0.75 \\ 0 & \text{otherwise} \end{cases}$$

e] Y's PDF is valid if

$$\int_{-\infty}^{\infty} f_Y(y) dy = \underline{1}$$

$$= \int_{0.25}^{0.75} 8y - 2 dy = \left[4y^2 - 2y \right]_{0.25}^{0.75} = \frac{3}{4} - \left(-\frac{1}{4} \right) = 1$$

Furthermore,

$$f_Y(y) \geq 0 \text{ for any } y$$

$$7] E[X] = \int_0^1 x \cdot 2x \, dx = \frac{2}{3} x^3 \Big|_0^1 = \boxed{\frac{2}{3}} = \mu_x$$

$$E[Y] = 0.5 \left(\frac{2}{3} \right) + 0.25 = \int_{0.25}^{0.75} (8y^2 - 2y) \, dy$$

$$= \frac{4}{12} + \frac{3}{12} = \boxed{\frac{7}{12}} = \mu_y$$

$$8] E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_0^1 2x^3 \, dx = \frac{1}{2} x^4 \Big|_0^1$$

$$= \boxed{\frac{1}{2}}$$

$$E(Y^2) = \int_{0.25}^{0.75} (8y^3 - 2y^2) \, dy = 2y^4 - \frac{2}{3} y^3 \Big|_{0.25}^{0.75} = \frac{45}{128} - \left(-\frac{1}{384} \right)$$

$$= \frac{45}{128} + \frac{1}{384} = \boxed{\frac{17}{48}}$$

h] Standard deviation, σ

$$\sigma_x = \sqrt{E[X^2] - \mu_x^2} = \sqrt{\frac{1}{2} - \left(\frac{2}{3} \right)^2}$$

$$= \sqrt{\frac{1}{18}} = \frac{\sqrt{2}}{6} = \boxed{0.236}$$

$$\sigma_y = \sqrt{E[Y^2] - \mu_y^2} = \sqrt{\frac{17}{48} - \left(\frac{7}{12} \right)^2} = \frac{\sqrt{143}}{12} = \frac{\sqrt{2}}{12}$$

$$= \boxed{0.118}$$

Problem 2

$$P[\mu_x - 2\sigma_x < X < \mu_x + 2\sigma_x] =$$

$$P[X < \mu_x + 2\sigma_x] - P[X < \mu_x - 2\sigma_x] =$$

$$F(\mu_x + 2\sigma_x) - F(\mu_x - 2\sigma_x) =$$

$$\mu_x = \int_0^1 x \cdot 6x(1-x) dx = \int_0^1 6x^2 - 6x^3 dx = 2x^3 - \frac{3}{2}x^4 \Big|_0^1 = \frac{1}{2} - 0 \\ = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 \cdot 6x(1-x) dx = \int_0^1 6x^3 - 6x^4 dx = \frac{3}{2}x^4 - \frac{6}{5}x^5 \Big|_0^1 = \frac{3}{10} - 0 \\ = \frac{3}{10}$$

$$\sigma_x = \sqrt{\frac{3}{10} - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{10} = 0.224$$

$$\mu_x + 2\sigma_x = \frac{1}{2} + 2\left(\frac{\sqrt{5}}{10}\right) = \frac{5 + 2\sqrt{5}}{10} = 0.947$$

$$\mu_x - 2\sigma_x = \frac{1}{2} - 2\left(\frac{\sqrt{5}}{10}\right) = \frac{5 - 2\sqrt{5}}{10} = 0.0528$$

$$P[0.0528 < X < 0.947] = \int_{0.0528}^{0.947} 6x(1-x) dx = \int_{0.0528}^{0.947} 6x - 6x^2 dx \\ = 3x^2 - 2x^3 \Big|_{0.0528}^{0.947} = 0.44187 - 0.0080691$$

$$\boxed{= 0.4338}$$

Problem 3

$$f_X(x) = \begin{cases} \frac{1}{5} e^{-\frac{x}{5}} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$a) E[X] = \int_0^{\infty} x \left(\frac{1}{5} e^{-\frac{x}{5}} \right) dx = \frac{1}{5} \left[\int_0^{\infty} x e^{-\frac{x}{5}} dx \right]$$

$$\int_0^{\infty} x e^{-\frac{x}{5}} dx = \left[x(-5e^{-\frac{x}{5}}) - \int -5e^{-\frac{x}{5}} dx \right]_0^{\infty}$$

$$= \left[-5x e^{-\frac{x}{5}} + 5(-5e^{-\frac{x}{5}}) \right]_0^{\infty}$$

$$= \left[-5e^{-\frac{x}{5}}(x+5) \right]_0^{\infty}$$

$$E[X] = \frac{1}{5} \left[-5e^{-\frac{x}{5}}(x+5) \right]_0^{\infty} =$$

$$= \frac{1}{5} \lim_{t \rightarrow \infty} \left[-5e^{-\frac{t}{5}}(t+5) - (-25) \right]$$

$$= \frac{1}{5} \lim_{t \rightarrow \infty} \left[-5e^{-\frac{t}{5}} t \left(1 + \frac{5}{t} \right) + 25 \right]$$

$$\lim_{t \rightarrow \infty} -5e^{-\frac{t}{5}} t = -5 \lim_{t \rightarrow \infty} \frac{t}{e^{\frac{t}{5}}} = -5 \lim_{t \rightarrow \infty} \frac{1}{\frac{1}{5} e^{\frac{t}{5}}}$$

$$= -5 \lim_{t \rightarrow \infty} \left(\frac{1}{5} e^{-\frac{t}{5}} \right) = 0$$

$$\lim_{t \rightarrow \infty} \left(1 + \frac{5}{t} \right) = 1 + 0 = 1$$

$$E[X] = \frac{1}{5} [0 \times 1 + 25] = 5$$

$$\boxed{E[X] = 5}$$

$$b) E[X^2] = \int_0^{\infty} x^2 \left(\frac{1}{5} e^{-\frac{x}{5}} \right) dx = \frac{1}{5} \int_0^{\infty} x^2 e^{-\frac{x}{5}} dx$$

$$\begin{aligned} \int_0^{\infty} x^2 e^{-\frac{x}{5}} dx &= \left[x^2 \left(-5 e^{-\frac{x}{5}} \right) - \int \left(-5 e^{-\frac{x}{5}} \right) 2x dx \right]_0^{\infty} \\ &= \left[-5x^2 e^{-\frac{x}{5}} + 10 \int x e^{-\frac{x}{5}} dx \right]_0^{\infty} \end{aligned}$$

$$E[X^2] = \left[-5x^2 e^{-\frac{x}{5}} + 10 \left(-5 e^{-\frac{x}{5}} (x+5) \right) \right]_0^{\infty} \cdot \frac{1}{5}$$

$$E[X^2] = \left[-5 e^{-\frac{x}{5}} x^2 \left(1 + \frac{10}{x} + \frac{50}{x^2} \right) \right]_0^{\infty} \cdot \frac{1}{5}$$

$$-5 \lim_{t \rightarrow \infty} e^{-\frac{x}{5}} x^2 = -5 \lim_{t \rightarrow \infty} \frac{x^2}{e^{\frac{x}{5}}} = -5 \lim_{t \rightarrow \infty} \frac{2x}{\frac{1}{5} e^{\frac{x}{5}}} = -5 \lim_{t \rightarrow \infty} \frac{2}{\frac{1}{25} e^{\frac{x}{5}}} = 0$$

$$E[X^2] = [0 - (-250)] \cdot \frac{1}{5} = 50$$

$$\text{Variance, } \sigma^2 = E[X^2] - (E[X])^2 = 50 - 5^2 = 25$$

$$\boxed{\sigma^2 = 25}$$

$$\text{standard deviation, } \sigma = \sqrt{\sigma^2} = \sqrt{25} = 5$$

$$\sigma = 5$$

$$c) \quad Y = X + 5 \quad X = Y - 5$$

range of Y

$$Y > 0 + 5 \Rightarrow Y > 5$$

$$f_Y(y) = \frac{dF_X(y-5)}{dy} = f_X(y-5) \cdot 1$$

$$f_X(y-5) = \frac{1}{5} e^{-\left(\frac{y-5}{5}\right)} = \frac{1}{5} e^{-\frac{y}{5} + 1} = \frac{e}{5} e^{-\frac{y}{5}}$$

$$f_Y(y) = \begin{cases} \frac{e}{5} e^{-\frac{y}{5}}, & y > 5 \\ 0, & \text{elsewhere} \end{cases}$$

$$E[Y^2] = \frac{e}{5} \int_5^{\infty} y^2 e^{-\frac{y}{5}} dy$$

$$= \left[-5y^2 e^{-\frac{y}{5}} + 10(-5e^{-\frac{y}{5}}(x+5)) \right]_5^{\infty} \cdot \frac{e}{5}$$

$$= [0 - (-229.425)] \cdot \frac{e}{5} = \underline{\underline{125}}$$

$$E[(X+5)^2] = 125$$