Assignment 3

Consider a random variable, X, has a PDF given by

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let us construct another random variable, $Y = 0.5X + 0.25$ (a linear function of random variable X).

(a) Find the range of Y.
(b) Find the CDF of X.
(c) Find the CDF of Y.
(d) Find the PDF of Y.
(e) Show that the function obtained in part (d) is a valid PDF.
(f) Find the mean values of X and Y.
(g) Find the mean-square values of X and Y.
(h) Find the standard deviations of X and Y

(a) $\min Y = 0.5(0) + 0.25 = 0.25$

$\max Y = 0.5(1) + 0.25 = 0.75$

range of $Y = [0.25, 0.75]$

(b) $CDF = \int_0^x f_x(x)\,dx$

$\int_0^x 2x\,dx = x^2$

so... $CDF = \begin{cases} 0, & x<0 \\ x^2, & 0 \le x \le 1 \\ 1, & 1 \le x \end{cases}$

(c) $F_Y(y) = P(Y \le y) = P(0.5x + 0.25 \le y)$

solve for $x \Rightarrow 0.5x < y - 0.25 \Rightarrow \boxed{x \le 2(y - 0.25)}$

using CDF of x:

$P(x \le 2(y-0.25)) \Rightarrow F_Y(y) = F_x(2(y-0.25))$

$\therefore F_Y(Y) = \begin{cases} 0 & , y < 0.25 \\ (2(y-0.25))^2 & , 0.25 \le y \le 0.75 \\ 1 & , 0.75 \le y \end{cases}$

(d) $f_Y(Y) = \dfrac{d F_y(Y)}{dy} = 4(y - 0.25) \Rightarrow f_Y(Y) = \begin{cases} 8(y-0.25) & , 0.25 \le y \le 0.75 \\ 0 & , \text{otherwise} \end{cases}$

(e) PDF of Y is valid?

- non-negative over range; $[0.25, 0.75] \rightarrow 8(y-0.25) \rightarrow 0 \Rightarrow 4$ (non-negative)
- integral over range $= = 1$; $\int_{0.25}^{0.75} f_y(y)\,dy = 1$ $\therefore$ it is valid

(f) $\mu_x = \Sigma x\, p_x(x)$ and $\mu_y = \Sigma y\, p_y(y)$

so...

$\mu_x = \int_0^1 x\, f_x(x) = \int_0^1 2x^2\,dx = \frac{2}{3}$ $\qquad \mu_y = 0.5\mu_x + 0.25 = \frac{7}{12}$

(g) $E[x^2] = \int_0^1 x^2 f_x(x) = \int_0^1 2x^3\,dx = 0.5 = \frac{1}{2}$

$E[y^2] = \int_{0.25}^{0.75} y^2 F_y(y) = \int_{0.25}^{0.75} 8y^2(y-0.25)\,dy = \frac{17}{48} = 0.354$

(h) $\sigma_x^2 = E[x^2] - \mu_x^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} \Rightarrow \sigma_x = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}} = 0.236$

$\sigma_y^2 = E[y^2] - \mu_y^2 = \frac{17}{48} - \left(\frac{7}{12}\right)^2 = \frac{1}{72} \Rightarrow \sigma_y = \sqrt{\frac{1}{72}} = \frac{1}{6\sqrt{2}} = 0.118$