Assignment 4

ENEL 419

1] $f_{XY}(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0<x<2, \quad 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

Independent if

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_X(x) = \int_0^1 f_{XY}(x,y)\,dy = \left[\frac{xy}{4} + \frac{xy^3}{4}\right]_0^1 = \left(\frac{x}{4} + \frac{x}{4}\right) - 0 = \frac{x}{2}$$

$$f_Y(y) = \int_0^2 f_{XY}(x,y)\,dx = \left[\frac{x^2}{8}(1+3y^2)\right]_0^2 = \frac{4}{8}(1+3y^2) - 0 = \frac{1+3y^2}{2}$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \frac{x}{2}\left(\frac{1+3y^2}{2}\right) = \frac{x(1+3y^2)}{4}$$

Random variables X and Y are independent

2]

$$f_{XY}(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $f_X(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\,dy = \left[\frac{3yx^2}{2} + \frac{1}{2}y^3\right]_0^1 = \frac{3x^2}{2} + \frac{1}{2}$

$f_Y(y) = \int_0^1 \frac{3}{2}(x^2+y^2)\,dx = \left[\frac{1}{2}x^3 + \frac{3y^2x}{2}\right]_0^1 = \frac{1}{2} + \frac{3y^2}{2}$

$f_{XY}(x,y) \ne f_X(x) f_X(y)$

$\boxed{\text{X \& Y NOT Independent}}$

b) $E(X+Y) = \mu_X + \mu_Y$

$\mu_X = \int_0^1 f_X(x)\,dx = \int_0^1 x\left(\frac{3x^2+1}{2}\right)dx = \left[\frac{3}{8}x^4 + \frac{x^2}{4}\right]_0^1 = \frac{5}{8}$

$\mu_Y = \int_0^1 f_Y(y)\,dy = \int_0^1 y\left(\frac{3y^2+1}{2}\right)dy = \left[\frac{3}{8}y^4 + \frac{y^2}{4}\right]_0^1 = \frac{5}{8}$

$$\boxed{E(X+Y) = \frac{5}{8} + \frac{5}{8} = \frac{10}{8} = 1.25}$$

c) $E(XY) = \int_0^1\int_0^1 \frac{3xy}{2}(x^2+y^2)\,dx\,dy$

$= \int_0^1 \left[\frac{3}{8}x^4y + \frac{3}{4}x^2y^3\right]_0^1 dy = \int_0^1 \frac{3}{8}y + \frac{3}{4}y^3\,dy$

$= \left[\frac{3}{16}y^2 + \frac{3}{16}y^4\right]_0^1 = \frac{3}{16} + \frac{3}{16} = \frac{6}{16} = \boxed{\frac{3}{8}}$

d) $Var(X) = \int_0^1 x^2 f_X(x)\,dx - E(x)^2 = \int_0^1 \frac{x^2(3x^2+1)}{2}dx - \left(\frac{5}{8}\right)^2$

$= \left[\frac{3}{10}x^5 + \frac{x^3}{6}\right]_0^1 = \frac{3}{10} + \frac{1}{6} - \frac{25}{64} = \boxed{\frac{73}{960} = 0.076}$

e) $Var(Y) = \int_0^1 y^2 f_Y(y)\,dy - E(y)^2 = \int_0^1 \frac{y^2(3y^2+1)}{2}dx - \left(\frac{5}{8}\right)^2$

$= \left[\frac{3}{10}y^5 + \frac{y^3}{6}\right]_0^1 = \frac{3}{10} + \frac{1}{6} - \frac{25}{64} = \frac{73}{960} = 0.076$

f) $Cov(X,Y) = C_{XY} = E[XY] - \mu_X\mu_Y = \frac{3}{8} - \left(\frac{5}{8}\cdot\frac{5}{8}\right)$

$\boxed{= -\frac{1}{64} = -0.015625}$

$\rho_{XY} = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{-\frac{1}{64}}{\sqrt{\frac{73}{960}}\cdot\sqrt{\frac{73}{960}}} = \frac{-960}{73\cdot 64} \boxed{= -0.20548}$

g) $Var(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2C_{XY}$

$= \left(\frac{73}{960}\right) + \left(\frac{73}{960}\right) + 2\left(-\frac{1}{64}\right)$

$\boxed{= 0.120833}$

3] Show $Cov(aX, bY) = ab\, Cov(X, Y)$

$$Cov(aX, bY) = E[abXY] - \mu_{aX}\mu_{bY}$$

$$E[abXY] = \iint abxy f_{XY}(x,y)\,dx\,dy = ab\underbrace{\iint xy f_{XY}(x,y)\,dx\,dy}_{E[XY]}$$
$$= ab\,E[XY]$$

$$\mu_{aX} = \int ax f_X(x)\,dx = a\int x f_X(x)\,dx = a\mu_X$$

$$\mu_{bY} = \int by f_Y(y)\,dy = b\int y f_Y(y)\,dy = b\mu_Y$$

$$Cov(aX, bY) = E[abXY] - \mu_{aX}\mu_{bY} = ab\,E[XY] - a\mu_X b\mu_Y$$
$$= ab(E[XY] - \mu_X\mu_Y) = \boxed{ab\,Cov(X, Y)}$$