2. Conditional Probability and Independence

(Reading Exercises: Montgomery and Runger Chapter 2 - Sections 2.5-2.8)

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial or prior information.** Here are some examples of situations we have in mind:

(1) In an experiment involving two successive rolls of a die, you are told that the sum of the two rolls is 9. How likely is it that the first roll was a 6?
(2) In a word guessing game, the first letter of the word is a "t". What is the likelihood that the second letter is an "h"?
(3) How likely is it that a person has a disease given that a medical test was negative?
(4) A spot shows up on a radar screen. How likely is it that it corresponds to an aircraft?

2.1 Conditional Probability

Example: An electrical engineering lab has 20 probes of which 3 are defective. A student selects 2 probes randomly (one after the other), what is the probability that both are defective?

Solution: Define events $A = \{\text{First probe is defective}\}$ and $B = \{\text{Second probe is defective}\}$.

The probability that both are defective is

$$P[A \cap B] = P[A]P[B \mid A];$$

When we first probe, the probability is $P[A] = \frac{3}{20}$. After picking the first, there are 19 probes remaining, of which 2 are bad, therefore, the probability that the second is defective given that the first is defective, is

$$P[B \mid A] = \frac{2}{19}$$

$$P[A \cap B] = \frac{3}{20} \times \frac{2}{19} = \frac{3}{190}.$$

Definition:

- Consider two events A and B. The probability of A occurring, given that B has occurred is the ***conditional probability*** defined as

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} \Rightarrow P[A \cap B] = P[A \mid B]P[B]$$

- Similarly, the probability of B occurring, given that A has occurred is the ***conditional probability*** defined as

$$P[B \mid A] = \frac{P[A \cap B]}{P[A]} \Rightarrow P[A \cap B] = P[B \mid A]P[A]$$

- The joint probability of A and B is defined as the intersection,

  $$P[A\cap B]=P[A\mid B]P[B]=P[B\mid A]P[A]$$

Multiplication Rule:

Consider a set A comprising subsets $A_1, A_2, \ldots, A_n$. According to the multiplication rule, the joint probability is given by the following product:

$$P[A_1A_2\ldots A_n]=P[A_1]P[A_2\mid A_1]P[A_3\mid A_1A_2]\cdots P[A_n\mid A_1A_2\ldots A_{n-1}]$$

Example: Consider events $D=\{\text{flight departs on time}\}$ and $A=\{\text{flight arrives on time}\}$. The probability of a flight departing on time is $P[D]=0.83$ and the probability of arriving on time is $P[A]=0.82$. The joint probability that a flight departs and arrives on time is $P[D\cap A]=0.78$. (a) What is the probability that the flight would arrive on time if it departed on time? (Answer: 0.94) (b) What is the probability that a flight departed on time if it arrived on time? (Answer: 0.95)

Example: A box contains 5000 IC chips, of which 1000 are manufactured by company X and the rest by company Y. Ten percent of the chips made by company X are defective and 5% of the chips made by company Y are defective. If a randomly select chip is found to be defective, what is the probability that it came from company X?

	X	Y	Total
Total # of chips	1000	4000	5000
Probability of events	$P[X]=0.2$	$P[Y]=0.8$	1
# of Defectives: $0.1\times X+0.05\times Y$	100	200	300 $P[D]=0.06$
Conditional probability of defect	$P[D\mid X]=0.1$	$P[D\mid Y]=0.05$	

$P[D\cap X]=0.02$	$P[D\cap Y]=0.04$
$P[X\mid D]=\dfrac{P[D\cap X]}{P[D]}=0.33$	$P[Y\mid D]=\dfrac{P[D\cap Y]}{P[D]}=0.67$

2.2 Independence

Definitions:

- Two events, A and B, are independent if one of the following is true:

$$P[A \mid B] = P[A] \text{ or } P[B \mid A] = P[B] \text{ or } P[A \cap B] = P[A]P[B]$$

- Multiple events $\{A_i\}_{i=1}^{n}$ are independent if

$$P[A_1 \cap A_2 \cdots \cap A_n] = P[A_1]P[A_2]\cdots P[A_n]$$

Example: Consider three events with corresponding probabilities given by the table:

Event A: College student	Event B: Smoker	Event C: Heart disease
$P[A] = 0.7$	$P[B] = 0.1$	$P[C] = 0.05$
$P[A \cap C] = 0.035$	$P[B \cap C] = 0.03$	

$$P[C \mid A] = \frac{P[C \cap A]}{P[A]} = \frac{0.035}{0.7} = 0.05 = P[C] \Rightarrow A \;\&\; C \text{ independent}$$

$$P[C \mid B] = \frac{P[C \cap B]}{P[B]} = \frac{0.03}{0.1} = 0.3 \neq P[C] \Rightarrow B \;\&\; \text{C are dependent}$$

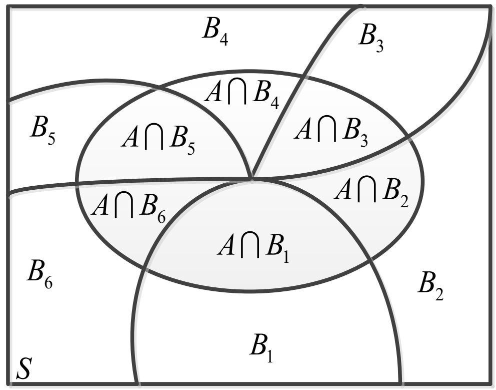
2.3 Theorem of Total Probability

Objective: To compute the probability of an event $A \in S$, in terms of conditional probabilities.

Divide and conquer

- Partition the sample space into disjoint sample spaces $B_1, B_2, \cdots, B_n$ as shown in the Venn diagram below. Then the event A may be expressed as the sum

$$A = \sum_{i=1}^{n} A \cap B_i$$



- Suppose we are given the probabilities $P[A \mid B_i]$, and $P[B_i]$ for each i. Then

$$P[A \cap B_i] = P[A \mid B_i]P[B_i],$$

- Compute $P[A]$ in terms of $P[A \mid B_i]$, and $P[B_i]$ for $i = 1, 2\cdots, n$, that is,

$$\boxed{P[A] = \sum_{k=1}^{n} A \cap B_k = \sum_{k=1}^{n} P[A \mid B_k]P[B_k]}$$

  - The above expression is known as the ***Law of total probabilities***

Example: Two balls are drawn in succession (without replacement) from an urn containing 2 black balls and 3 white balls. Find the probability the second ball drawn is white.

Solution: Define the following events:

$$B_1 = \{1^{st} \text{ ball is black}\}, \quad W_1 = \{1^{st} \text{ ball is white}\}, \quad W_2 = \{2^{nd} \text{ ball is white}\}$$

Define event $W = \{B_1 \cap W_2 \text{ or } W_1 \cap W_2\}$. Then

$$P[W] = P[W_2 \mid B_1]P[B_1] + P[W_2 \mid W_1]P[W_1] = \frac{3}{4}\times\frac{2}{5} + \frac{2}{4}\times\frac{3}{5} = 0.6$$

Example: In a manufacturing plant, three machines, B_1, B_2, and B_3 make 30%, 30% and 40%, respectively, of the products. It is known that some of these products are defective. The defective rates from the three machines, B_1, B_2, and B_3, are 10%, 4% and 7%, repectively. Using the theorem of total probability, find the probability a selected product is defective. (answer: $P[D] = 0.07$). Hint: Define event $D = \{\text{selected product is defective}\}$. From the given information we have the following:

$P[B_1]=0.3$ $P[B_2]=0.3$ $P[B_2]=0.4$

$P[D\mid B_1]=0.1$ $P[D\mid B_2]=0.04$ $P[D\mid B_3]=0.07$

2.4 Bayes' Rule

We are given disjoint events $B_i,\ i=1,2,\ldots,n$, that are partitions of of event A. Suppose we know the following probabilities:

- Probabilities $P[B_i],\ i=1,2,\ldots,n$, are called **a priori probabilities.** They are measures of initial beliefs
- Probabilities $P[B_i\mid A],\ i=1,2,\ldots,n$, are conditional probabilities called **a posteriori probabilities.** They are measures of belief given that A occurred.

Given conditional probabilities probabilities $P[A\mid B_i],\ i=1,2,\ldots,n$, we wish to compute the a posteriori probability $P[B_j\mid A],\ j=1,2,\ldots,n$. The expression for finding this probability is known as Bayes' Rule

Bayes' Rule:

$$P[B_j\mid A]=\frac{P[A\cap B_j]}{P[A]}=\frac{P[A\mid B_j]P[B_j]}{P[A]}=\frac{P[A\mid B_j]P[B_j]}{\sum_{k=1}^{n}P[A\mid B_k]P[B_k]},\quad 1\le j\le n$$

<u>**Example:**</u> In the previous example,

$P[B_1]=0.3$ $P[B_2]=0.3$ $P[B_2]=0.4$

$P[D\mid B_1]=0.1$ $P[D\mid B_2]=0.04$ $P[D\mid B_3]=0.07$

what is the probability that a defective product is from machine B_1?

$$P[B_1\mid \mathrm{D}]=\frac{P[\mathrm{D}\mid B_1]\times P[B_1]}{P[D]}=\frac{0.1\times 0.3}{0.07}=0.428$$

<u>**Example:**</u>

Coin A is Fair, that is, 50% heads and 50% tails, that is,

$P[\text{heads}\mid A]=0.5$ and $P[\text{tails}\mid A]=0.5$

Coin B is rigged with 75% heads and 25% tails, that is,
$P\left[\text{heads} \mid B\right] = 0.75$ and $P\left[\text{tails} \mid B\right] = 0.25$

Coin A or coin B is selected, with equal probability, it is tossed, and heads appear. What is the probability that coin B had been selected given that heads appeared? In other words, find $P\left[B \mid \text{heads}\right]$.

Solution: Apply generalized Bayes' Rule:

$$P\left[B \mid \text{heads}\right] = \frac{P\left[\text{heads} \mid B\right] \times P\left[B\right]}{P\left[\text{heads} \mid A\right] \times P\left[A\right] + P\left[\text{heads} \mid B\right] \times P\left[B\right]}$$

Coin A or coin B is selected with equal probability $P\left[A\right] = P\left[B\right] = 0.5$. Therefore

$$P\left[B \mid \text{heads}\right] = \frac{0.75 \times 0.5}{0.5 \times 0.5 + 0.75 \times 0.5} = \frac{3}{5} \neq P\left[B\right]$$

Example: Two machines: A produces 60% of the products of which 2% is defective and B produces 40% of the products of which 4% is defective. A product is examined and found to be defective. (a) Find the probability $P\left[D\right]$, that it is defective. (b) Find the probability $P\left[A \mid D\right]$ that it came from A given that it is defective. (e) Show that the probability that it came from B given that it is defective is $P\left[B \mid D\right] = 0.429$.

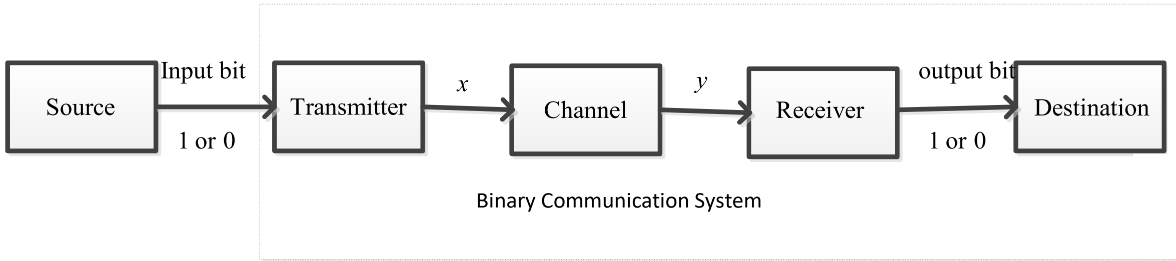
Answers: Define event D, the product is defective.

$P\left[A\right] = 0.6, \quad P\left[B\right] = 0.4. \; P\left[D \mid A\right] = 0.02, \quad P\left[D \mid B\right] = 0.04.$

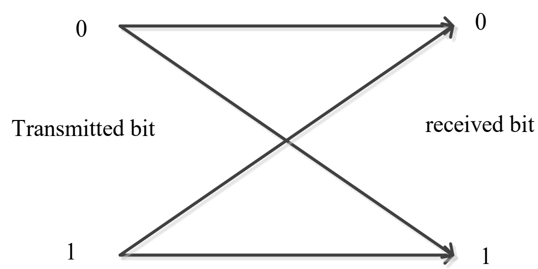
$P\left[D\right] = P\left[A\right]P\left[D \mid A\right] + P\left[B\right]P\left[D \mid B\right] = 0.012 + 0.16 = 0.028$

$$P\left[A \mid D\right] = \frac{P\left[D \mid A\right]P\left[A\right]}{P\left[D\right]} = \frac{0.02 \times 0.6}{0.028} = 0.43, \quad P\left[B \mid D\right] = \frac{P\left[D \mid B\right]P\left[B\right]}{P\left[D\right]} = \frac{0.04 \times 0.4}{0.028} = 0.57$$

Example: The block diagram below is a simple digital, binary communication system. The system transmits a bit 0 or a bit 1. The channel corrupts the signal, which may cause errors at the receiver. The errors may cause bits to be switched.



Binary Communication System



These events and their probabilities are summarized in the table below.

Events	Probabilities
A_0: Transmitted bit is 0	$P[A_0]=1-p$
A_1: Transmitted bit is 1	$P[A_1]=p$
$B_0\|A_0$: Received bit 0 when is 0 is transmitted	$P[B_0 \mid A_1]=\varepsilon \Rightarrow P[B_1 \mid A_1]=1-\varepsilon$
$B_1\|A_1$: Received bit 1 when 1 is transmitted	$P[B_1 \mid A_0]=\varepsilon \Rightarrow P[B_0 \mid A_0]=1-\varepsilon$
$B_1\|A_0$: Received bit 1 when is 0 is transmitted	
$B_0\|A_1$: Received bit 0 when is 1 is transmitted	

<u>Example:</u> Game show (Monty Hall Problem) – there are 3 doors labeled Door1, Door2 and Door3. Behind one door is a car, behind each other door is a goat. The objective is to win the car. You select a door which remains closed and the host opens one of the 2 remaining doors to show a goat. Would you remain with your first choice or switch to the other closed door?

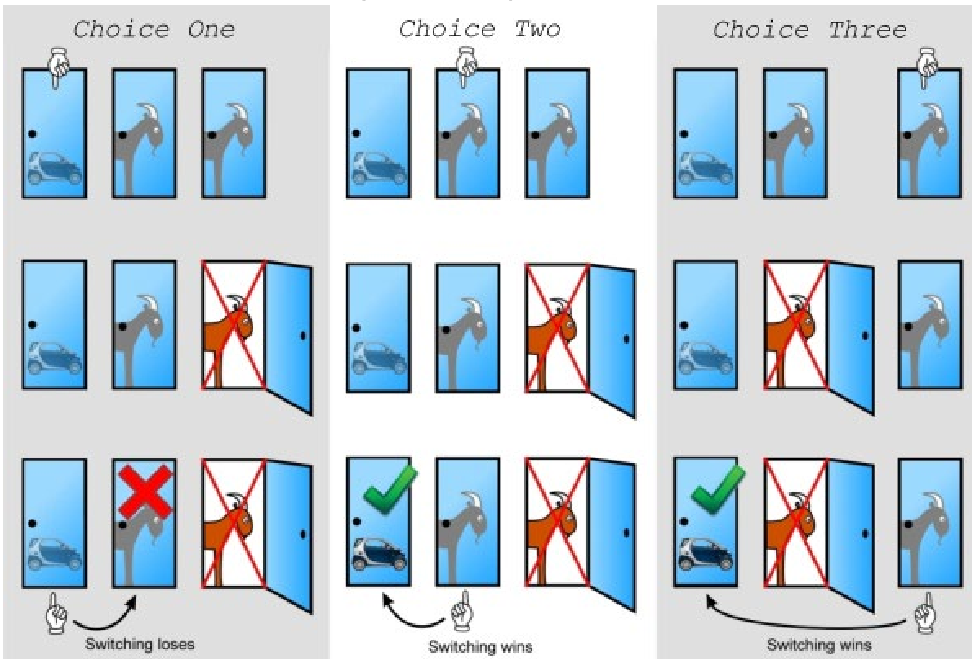
Fact:

Partitions	Events	Door1	Door2	Door3	Probability
Partition 1	B_1	C	G	G	1/3
Partition 2	B_2	G	C	G	1/3
Partition3	B_3	G	G	C	1/3

Assume you always choose Door1. The probability of the car is $P[C]=1/3$. Host opens door on goat (Door 2 or 3). The events are summarized in the table below.

Events	Door1	Door2	Door3	Probability
$B_1 \mid \text{Host}$	C	G	G	$1/3$
$B_2 \mid \text{Host}$	G	C	G	$1/3$
$B_3 \mid \text{Host}$	G	G	C	$1/3$

Reference: http://www.bing.com/images/search?q=monty+hall+problem



The probability of winning the car given that you do not switch is

$$P\left[C \mid \text{no switch and host}\right] = P\left[B_1 \mid \text{Host}\right] = 1/3$$

If you switch, there are two possibilities

$$P\left[C \mid \text{switch and host}\right] = P\left[B_2 \text{ or } B_3 \mid \text{Host}\right] = P\left[B_2\right] + P\left[B_3\right] = 2/3$$