Chapter 2: Transformation and Expectations Exercises

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Exercise 2.1

(1) $y = g(x) = x^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. Note that $\frac{d}{dy}g^{-1} = \frac{1}{3}y^{-\frac{2}{3}}$ is not continuous on $(0,1)$ when it is near 0. We can't use the pmf change of variable theorem. We need to find $F_Y(y)$ first.

$$F_X(x) = \int_{t=0}^{x} 42t^5(1-t)dt = 7t^6 - 6t^7|_0^x = 7x^6 - 6x^7$$

Since g is an increasing function,

$$F_Y(y) = F_X(g^{-1}(y)) = 7y^2 - 6y^{\frac{7}{3}}$$

Then pdf for y, $f_y(y) = \frac{d}{dy}F_Y = 14y - 14y^{\frac{4}{3}}$ where $y \in (0,1)$

(2) g is increasing, $f_X = 7e^{-7x}$ is continuous on $[0,\infty)$ and $g^{-1} = (y-3)/4$ has continuous derivative on $[3,\infty)$. Therefore, by theorem 2.1.5,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{7}{4}\exp(-\frac{7}{4}(y-3)), \quad y \in [3,\infty)$$

(3) g doesn't have a continuous inverse. So we will find F_Y instead.

$$F_X(x) = \int_0^x 30t^2(1-t)^2dt = 10x^3 + 6x^5 - 15x^4$$

Since $g(x) = x^2$ is increasing on $(0,1)$. Therefore

$$F_Y(y) = F_X(g^{-1}(y)) = 10y^{3/2} + 6y^{5/2} - 15y^2$$

Pdf $f_Y(y) = \frac{d}{dy} = 15y^{1/2} + 15y^{3/2} - 30y$ for $y \in (0,1)$