Geometry Note

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1 Definitions

Space of linear function $L(V, W)$ vector space of linear functions from V to W.

Dual Space $V^* = L(V, R)$. For each basis $\{e_i\}$ of V, there exists unique $\{e^i\}$ of V^* such that $e^i(e_j) = \delta^i_j$

Tensor Space $T^r_s = \underbrace{V \otimes V}_{\text{r times}} \otimes \underbrace{V^* \otimes V^*}_{\text{s times}}$ is space of multilinear functions on

$$\underbrace{V^* \times \ldots \times V^*}_{\text{r times}} \times \underbrace{V \times \ldots \times V}_{\text{s times}}$$

Tensor Product between A of (r, s) and B of (t, u), is

$$\begin{aligned} A \otimes B(\tau^1, \ldots, \tau^{r+t}, v_1, \ldots, v_{s+u}) \\ = A(\tau^1, \ldots, \tau^r, v_1, \ldots, v_s) \\ B(\tau^{r+1}, \ldots, \tau^{r+t}, v_{s+1}, \ldots, v_{s+u}) \end{aligned}$$

Vector Field X on coordinate neighborhood U of a manifold M, with coordinate x^i. For each point p, $X = X^i \partial_i$. $X[f] = X^i \partial_i f$

Change of Coordinates If Y has coordinate neighborhood V of y^i, then $Y^i = X^j \frac{\partial y^i}{\partial x^j}$

Map Differential(Pushforward) F_* is induced map $F_* : TM \to TN$ of C^∞ map $F : M \to N$. $F_*(v_p) = (F_* v)_{F(p)}$. With coordinate, $F_* = [\partial_j (y^i \circ F)]$, the Jacobian of F. Note that $y^i \circ F = F^i(x^1, \ldots, x^m)$

Tensor Bundle $T^r_s M$ of type (r, s) is the union of all tensor spaces $M^r_s(p)$ at each point $p \in M$.

Tangent Bundle $TM = T^1_0 M$,

Scalar Bundle $T^0_0 M = M \times \Re$,

Cotangent Bundle/ Differentials / Phase space $T^0_1 M$

Tensor Field T of type (r, s), $T(p) \in T^r_s M(p)$ for each p. $(1, 0)$ is vector field, $(0, 0)$ gives real-valued function. $(0, 1)$ gives differential.

Tensor Coordinate of T^r_s wrt coordinate x^i are d^{r+s} real-valued functions

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = T(dx^{i_1}, \ldots dx^{i_r}, \partial_{j_1}, \ldots, \partial_{j_s})$$

Tensor Product

Exterior Product

Differential forms p-form is C^∞ skew-symmetric covariant tensor field of degree p (type $(0, p)$). Local basis has $\binom{d}{p}$ p-forms $dx^{i_1} \cdots dx^{i_p}$ where $(i_1, \ldots, i_p)$ is increasing.

Case Study 1: Surface of a sphere

The surface of sphere of radius 1 is a manifold

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

We can define a chart (U, ψ) for S^2 where $U \subseteq M$ with spherical coordinate. Let

$$U = \{(\theta, \phi) \in [0, 2\pi] \times [0, \pi]\}$$

and

$$\psi(x, y, z) : \begin{cases} \theta = \arccos(z) \\ \phi = \text{sng}(y) \arccos \frac{x}{\sqrt{x^2+y^2}} \end{cases} , \psi^{-1}(\theta, \phi) : \begin{cases} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{cases}$$

Then $\psi(U) \subseteq \mathbb{R}^2$ is a homeomorphism from U to $\psi(U)$. ψ is called a **Locale coordinate map**. And the component functions (θ, ϕ) defined by $\psi(p) = (\theta(p), \phi(p))$ for $p \in S^2$ are called **local coordinates** on U.

One can think of this as giving a temporary identification between U and $\psi(U)$. When we work in this chart, we can think of U as an open subsets of the manifold and as an open subset of $\mathbb{R}^2$. Thus, we can represent a point $p \in U \subseteq S^2$ by its coordinate $(\theta, \phi) = \psi(p)$ and think of it as being the point p. We say (θ, ϕ) is the local coordinate for p or $p = (\theta, \phi)$ in local coordinates. *(See Lee's Smooth Manifold Local Coordinate Representations section)*

Given the same chart, the coordinate vectors $\partial_\theta, \partial_\phi$ form a basis for T_pS^2. If $v \in T_pS^2$, then

$$v = v^1 \frac{\partial}{\partial\theta}\bigg|_p + v^2 \frac{\partial}{\partial\phi}\bigg|_p = v^1\partial_\theta + v^2\partial_\phi = v^i\partial_i$$

The dual space to T_pS^2 is $T_p^*S^2$, if $w \in T_p^*S^2$,

$$w = w_1 d\theta + w_2 d\phi = w_i dx^i \text{(in generic coordinates)}$$

and $w(v) = w_i v^i$

S^2 is Riemannian with symmetric metric tensor defined as

$$\begin{aligned}
g &= g_{ij} dx^i \otimes dx^j \\
&= g_{11} d\theta \otimes d\phi + g_{12} d\theta \otimes d\phi + g_{21} d\phi \otimes d\theta + g_{22} d\phi \otimes d\phi \\
&= g_{11}(d\theta)^2 + \frac{1}{2}(g_{12} + g_{21}) d\theta \otimes d\phi + \frac{1}{2}(g_{21} + g_{12}) d\phi \otimes d\theta + g_{22}(d\phi)^2 \qquad , (g_{12} = g_{22}) \\
&= g_{11}(d\theta)^2 + \frac{g_{12}}{2}(d\theta \otimes d\phi + d\phi \otimes d\theta) + \frac{g_{21}}{2}(d\phi \otimes d\theta + d\theta \otimes d\phi) + g_{22}(d\phi)^2 \\
&= g_{11}(d\theta)^2 + g_{12} d\theta d\phi + g_{21} d\phi d\theta + g_{22}(d\phi)^2 \\
&= g_{ij} dx^i dx^j
\end{aligned}$$

We will now compute g. Since (θ, ϕ) are local coordinate of S^2, we can introduce a smooth immersion map $\iota = \psi^{-1} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ into $\mathbb{R}^3$. Since $\mathbb{R}^3$ has Euclidean metric $\bar{g} = (dx)^2 + (dy)^2 + (dz)^2$, then g is the pullback of $\bar{g}$,

$$\begin{aligned}
g &= \iota^* \bar{g} \\
&= (d(\sin\theta\cos\phi))^2 + (d(\sin\theta\sin\phi))^2 + (d(\cos\theta))^2 \\
&= (\cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi)^2 + (\cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi)^2 + (\sin\theta d\theta))^2 \\
&= (d\theta)^2 + \sin^2\theta (d\phi)^2
\end{aligned}$$

Case Study 2: Relativistic length contraction

Given a stationary frame, it has Minkowski flat metric of $(d\tau)^2 = (dt)^2 - (dx)^2$. A measure of length between location A and B along x-axis in a stationary frame are the distance between two simultaneous events $E_A = (t_0, x_a)^T$ and $E_B = (t_0, x_b)^T$. We want to calculate the distance with respect to a moving frame with constant velocity.

Let S be the stationary frame with axis (t, x) and S' with axis $(\bar{t}, \bar{x})$ be the moving frame in the x-direction with speed v. The line element $d\tau$ is invariant in different frames, therefore $(dt)^2 - (dx)^2 = (d\tau)^2 = (d\bar{t})^2 - (d\bar{x})^2$. So we have $(t)^2 - (x)^2 = (\bar{t})^2 - (\bar{x})^2$. The solution is given by

$$\begin{aligned} t &= \bar{t}\cosh\theta + \bar{x}\sinh\theta \\ x &= \bar{t}\sinh\theta + \bar{x}\cosh\theta \end{aligned}$$

The trajectory of the origin of S' along x axis is $x(t) = vt$ in frame S but $\bar{x}(\bar{t}) = 0$ in S' after the above transformation. So the above transformation is mapping $(\bar{t}, 0)$ to (t, vt). Substituting those in the solution, then we have $v = \tanh\theta \equiv \beta$. From that, we have $\cosh\theta = \frac{1}{\sqrt{1-v^2}} \equiv \gamma$, $\sinh\theta = \frac{v}{\sqrt{1-v^2}} = \gamma\beta$ and . We arrive at Lorentz transform from $S \to S'$:

$$\Lambda = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

Now consider the world line of A and B in S,

$$W_A(t) = \begin{pmatrix} t \\ x_a \end{pmatrix}, W_B(t) = \begin{pmatrix} t \\ x_b \end{pmatrix}$$

. After Lorentz transform,

$$\widehat{W}_A(t) = \begin{pmatrix} \gamma t - \gamma\beta x_a \\ -\gamma\beta t + \gamma x_a \end{pmatrix}, \widehat{W}_B(t) = \begin{pmatrix} \gamma t - \gamma\beta x_b \\ -\gamma\beta t + \gamma x_b \end{pmatrix}$$

As we can see when E_A and E_B that are simultaneous in S is not simultaneous in S' because their time component is not the same. To measure the distance in S', we will choose two events along the world line of A and B with the same time component in S' that is $\widehat{W}_A^{(0)}(t_a) = \widehat{W}_B^{(0)}(t_b)$. Therefore we have $\gamma t_a - \gamma\beta x_a = \gamma t_b - \gamma\beta x_b$. We can choose $t_a = \beta x_a$ and $t_b = \beta x_b$ which is their $\bar{x}$-intercept in S'.

Then the distance measured in S' is

$$\begin{aligned} \widehat{L} &= \widehat{W}_B^{(1)}(\beta x_b) - \widehat{W}_A^{(1)}(\beta x_a) \\ &= -\gamma\beta^2 x_b + \gamma x_b + \gamma\beta^2 x_a - \gamma x_a = \gamma(1-\beta^2)(x_b - x_a) \\ &= \frac{x_b - x_a}{\gamma} \\ &= \frac{L}{\gamma} \end{aligned}$$