Chapter 4: Multiple Random Variables

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Exercise 4.1

(a) Since $f_{X,Y}(x,y)$ is constant. $X^2+Y^2<1$ is circle of radius 1. Therefore $P(X^2+Y^2<1)=\pi/4$

(b) $2X-Y=0$ divides the unit square into two region of equal area and f is constant. Therefore $P(2X-Y>0)=1/2$

(c) $P(|X+Y|<2)=P(-2<X+Y<2)$. The area covers the entire square. Therefore $P(|X+Y|<2)=1$.

Exercise 4.4

(a) Since $\int_0^1\int_0^2 f(x,y)dxdy=\int_0^1\int_0^2 C(x+2y)dxdy=4C=1$ So $C=1/4$.

(b) $f(x)=\int_0^1 f(x,y)dy=(1/4)(xy+y^2)|_0^1=\frac{x+1}{4},\ \ x\in(0,2)$

(c) For $(x,y)\in(0,2)\times(0,1)$:

$$F(x,y)=P(X<x,Y<y)=\int_{-\infty}^{x}\int_{\infty}^{y}\frac{t+2s}{4}dsdt=\int_0^x\int_0^y\frac{t+2s}{4}dsdt=\frac{1}{8}(x^2y+2xy^2)$$

For $(x,y)\in(0,2)\times[1,\infty)$:

$$F(x,y)=P(X<x,Y<y)=\int_0^x\int_0^1\frac{t+2s}{4}dsdt=\frac{1}{8}(x^2+2x)$$

For $(x,y)\in(-\infty,2]\times(0,1)$:

$$F(x,y)=P(X<x,Y<y)=\int_0^2\int_0^y\frac{t+2s}{4}dsdt=\frac{1}{2}(y+y^2)$$

(d) from (b), we have $f(x)=\frac{x+1}{4}$. And $z=\frac{9}{(x+1)^2}$ is monotonic for $x\in[0,2]$ with $z\in[1,9]$. So we can take $x=\frac{\sqrt{z}}{3}-1$. Then

$$f(z)=f(x^{-1}(z))\left|\frac{dx}{dz}\right|=\frac{3}{4}(z^{-1/2})(\frac{3}{2}z^{-3/2})=\frac{9}{8}z^{-2}$$

Exercise 4.5

(a) The area for integration is $0<x<1$ and $0<y<x^2$.

$$P(X>\sqrt{Y})=\int_0^1\int_0^{x^2}x+ydydx=\int_0^1 x^3+\frac{x^4}{2}dx=0.35$$

(b) The area of integration is $0 < x < 1$ and $x^2 < y < x$.

$$P(X^2 < Y < X) = \int_0^1 \int_{x^2}^{x} 2x dy dx = \int_0^1 2x^2 - 2x^3 dx = \frac{1}{6}$$

Exercise 4.6

Let X, Y be the time A and B arrive in time interval $[0, 1]$. Since they are independent, $f(x, y) = f(x)f(y) = 1$ for $(x, y) \in [0, 1] \times [0, 1]$.

Let T be the length of time A waits for B. Then $T = \max(Y - X, 0)$ because $T = 0$ when $Y < X$.

$$P(T < t) = P(\max(Y - X, 0)) = P(Y - X < t, Y \geq X) + P(Y < X)$$

For term $P(Y - X < t, Y \geq X)$, The area of integration is the area between $y - x = t$ and $y \geq x$ bounded by unit square. We can find the complement area which is an isosceles right triangle with side of $1 - t$, which gives

$$P(Y - X < t, Y \geq X) = \frac{1}{2} - \frac{1}{2}(1 - t)^2$$

$P(Y < X)$ is the lower half triangle of the unit square which has area of $\frac{1}{2}$ Therefore

$$P(T < t) = P(Y - X < t, Y \geq X) + P(Y < X) = 1 - \frac{1}{2}(1 - t)^2$$

Exercise 4.7

We can formulate the problem as such: $X \in [0, 30], Y \in [40, 50]$, find $P(X + Y < 60)$. We want to find the intersection of $x + y = 60$ with $[0, 30] \times [40, 50]$. We get $(10, 50), (20, 40)$. Since the distributions are uniform, we can simply find area of the trapezoid and divide it by the total area.

$$P(X + Y < 60) = \frac{10(10 + 20)0.5}{10(30)} = 150/300 = 0.5$$

Exercise 4.9

For interval $[a, b] \times [c, d]$.

$$\begin{aligned}
P(a \leq X \leq b)P(c \leq Y \leq d) &= [F_X(b) - F_X(a)][F_Y(d) - F_Y(c)] \\
&= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c) \\
&= F(b, d) - F(b, c) - F(a, d) + F(a, c)
\end{aligned}$$

If we define the regions $A_1 = [a, b] \times [c, d]$, $A_2 = [a, b] \times (-\infty, c)$, $A_3 = (-\infty, a) \times (-\infty, c)$, $A_4 = (-\infty, a) \times [c, d]$. Then

$$\begin{aligned}
F(b, d) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) \\
F(b, c) &= P(A_3) + P(A_2) \\
F(a, d) &= P(A_3) + P(A_4) \\
F(a, c) &= P(A_3)
\end{aligned}$$

Hence

$$\begin{aligned}
P(a \leq X \leq b)P(c \leq Y \leq d) &= F(b, d) - F(b, c) - F(a, d) + F(a, c) \\
&= P(A_1) \\
&= P([a, b] \times [c, d]) \\
&= P(X \in [a, b], Y \in [c, d])
\end{aligned}$$

Exercise 4.10

(a) Summing up the columns and rows, we have $P(X = 1) = 1/4$, $P(X = 2) = 1/2$, $P(X = 3) = 1/4$. $P(Y = 2) = P(Y = 3) = P(Y = 4) = 1/3$.

(b) We can build up a table for independent U, V just be multiplying the marginal probability.

U/V	1	2	3
2	1/12	1/6	1/12
3	1/12	1/6	1/12
4	1/12	1/6	1/12

Exercise 4.11

U and V are dependent. Consider $P(V|U = n)$ and $P(V)$, knowing $U = n$ means there is only one toss of head in the first n trials which means $P(V \le n|U = n) = 0 \neq P(V \le n)$.

Exercise 4.12

Let X and Y be uniform(0,1). Then $f_{X,Y} = 1$ is a unit square on $[0, 1] \times [0, 1]$. By symmetry, we only need to consider the probability conditioned on $X > Y$ which is the lower half triangle of the unit square. Then the 3 segments are $1 - X, X - Y$ and Y. For the segments to be a triangle, denoted by event T, it must satisfy the sum of two sides is larger than the other side. Therefore

$$\begin{aligned} 1 - X + X - Y &> Y \\ 1 - X + Y &> X - Y \\ X - Y + Y &> 1 - X \end{aligned}$$

Simplifying the expression, we have event T given $X > Y$ as the area of a region bounded by

$$\begin{aligned} \frac{1}{2} &> Y \\ Y &> X - \frac{1}{2} \\ X &> \frac{1}{2} \\ X &> Y \end{aligned}$$

Note that area of $X > Y$ is $\frac{1}{2}$ and the area of the region for T is $\frac{1}{8}$. So $P(T|X > Y) = \frac{1/8}{1/2} = \frac{1}{4}$ and .

Finally, we have $P(T) = P(T|X > Y)P(X > Y) + P(T|X \le Y)P(X \le Y) = \frac{1}{4}$

Exercise 4.27

Suppose $X \sim n(\mu, \sigma^2)$, $Y \sim n(\gamma, \sigma^2)$ and $X \perp Y$. Let $U = X + Y$ and $V = X - Y$. We have $X = \frac{U+V}{2}$ and $Y = \frac{U-V}{2}$. then

$$
\begin{aligned}
g(u,v) &= f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \\
&= \frac{1}{2} f(x(u,v)) f(y(u,v)) \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{(\frac{u+v}{2} - \mu)^2}{2\sigma^2} \right) \exp\left(-\frac{(\frac{u-v}{2} - \gamma)^2}{2\sigma^2} \right) \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{4\sigma^2} (u - (\mu+\gamma))^2 \right) \exp\left(-\frac{1}{4\sigma^2} (v - (\mu-\gamma))^2 \right)
\end{aligned}
$$

We just need to get the marginal distribution for each u and v. Let $s = \frac{1}{2\sigma}(v - (\mu - \gamma))$, $ds = \frac{1}{2\sigma} dv$

$$
\begin{aligned}
g(u) = \int g(u,v) dv &= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{4\sigma^2} (u - (\mu+\gamma))^2 \right) \int \exp\left(-\frac{1}{4\sigma^2} (v - (\mu-\gamma))^2 \right) dv \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{4\sigma^2} (u - (\mu+\gamma))^2 \right) 2\sigma \int e^{-s^2} ds \\
&= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{4\sigma^2} (u - (\mu+\gamma))^2 \right) 2\sigma \sqrt{\pi} \\
&= \frac{1}{\sqrt{\pi (\sqrt{2}\sigma)^2}} \exp\left(-\frac{1}{2(\sqrt{2}\sigma)^2} (u - (\mu+\gamma))^2 \right)
\end{aligned}
$$

We get $u \sim \mathcal{N}(\mu + \gamma, 2\sigma^2)$, similarly, we get $v \sim \mathcal{N}(\mu - \gamma, 2\sigma^2)$. They are independent because $g(u,v) = g(u)g(v)$.

Exercise 4.31

We have hierarchical model

$$
\begin{aligned}
Y|X &\sim \text{Binomial}(n, X) \\
X &\sim \text{Uniform}(0, 1)
\end{aligned}
$$

(a)

$$
\begin{aligned}
\mathbf{E}(Y) &= \mathbf{E}(\mathbf{E}(Y|X)) = \mathbf{E}(nX) = n\mathbf{E}X = \frac{n}{2} \\
\text{Var}Y &= \text{Var}(\mathbf{E}(Y|X)) + \mathbf{E}(\text{Var}(Y|X)) \\
&= \text{Var}(nX) + \mathbf{E}(nX(1-X)) \\
&= n^2 \frac{(1-0)^2}{12} + n \int_0^1 x(1-x) dx \\
&= \frac{n^2}{12} + \frac{n}{6}
\end{aligned}
$$

(b) The joint density is

$$
\begin{aligned}
f(x,y) &= f(y|x) f(x) \\
&= \binom{n}{y} x^y (1-x)^{n-y}
\end{aligned}
$$

(c)

$$\begin{aligned}
f(y) &= \int_0^1 f(x,y)dx \\
&= \binom{n}{y} \int_0^1 x^y (1-x)^{n-y} dx \\
&= \binom{n}{y} \text{Beta}(y+1, n-y+1) \\
&= \binom{n}{y} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}
\end{aligned}$$

Exercise 4.32

(a) Note that $f(y,\Lambda) = f(y|\Lambda)f(\Lambda) = \frac{\Lambda^y e^{-\Lambda}}{y!} \frac{\beta^\alpha}{\Gamma(\alpha)} \Lambda^{\alpha-1} e^{-\beta\Lambda}$

$$\begin{aligned}
f(y) &= \int_0^\infty f(y,\Lambda) d\Lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha) y!} \int_0^\infty \Lambda^{y+\alpha-1} e^{-(\beta+1)\Lambda} d\Lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha) y!} \int_0^\infty \left(\frac{x}{\beta+1}\right)^{y+\alpha-1} e^{-x} \frac{dx}{\beta+1} \quad , \text{(Let } x = (\beta+1)\Lambda) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha) y! (\beta+1)^{y+\alpha}} \int_0^\infty x^{y+\alpha-1} e^{-x} dx \\
&= \frac{\beta^\alpha \Gamma(y+\alpha)}{y! (\beta+1)^{y+\alpha} \Gamma(\alpha)}
\end{aligned}$$

If α is positive integer, then $\Gamma(y+\alpha) = (y+\alpha)!$. So

$$f(y) = \frac{\beta^\alpha (y+\alpha)!}{(\beta+1)^{y+\alpha} y! \alpha!} = \binom{y+\alpha-1}{y} \frac{\beta^\alpha}{(\beta+1)^{y+\alpha}} = \binom{y+\alpha-1}{y} \left(1 - \frac{1}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^y$$

$$\mathbf{E}(Y) = \mathbf{E}(\mathbf{E}(Y|\Lambda)) = \mathbf{E}(\Lambda) = \alpha\beta$$

$$\text{Var}(Y) = \mathbf{E}(\text{Var}(Y|\Lambda)) + \text{Var}(\mathbf{E}(Y|\Lambda)) = \mathbf{E}(\Lambda) + \text{Var}(\Lambda) = \alpha\beta + \alpha\beta^2$$