Chapter 1: Manifolds

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Exercise 1.1.1

(a) Let $g_0(x) = -x^{-1}$, and g_k be the k-th derivatives of g_0. Then $f(x) = \exp(-1/x) = \exp(g_0(x))$.

$$f^{(k)}(x) = \exp(g_0(x)) \sum_{i=1}^{k} g_i(x) = f(x) \sum_{i=1}^{k} g_i(x)$$

Note that $g_i(0) = 0$ and $f(0) = 0$. Therefore $f^{(k)}(0) = 0$.

$\blacksquare$

(b) Suppose f is analytic at 0, then we can expand f in a neighbordhood centering at 0, say $(-\epsilon, \epsilon)$. $fx = \sum_{k=0}^{\infty} (f^{(k)}0) x^k / k! = 0$ for all $x \in (-\epsilon, \epsilon)$. However $f(x) = \exp(-1/x) \neq 0$ for $x \in (0, \epsilon)$. Hence f is not analytic at 0.

$\blacksquare$

Exercise 1.2.1

Denote our manifold $\mathbb{R}^3$ as M, cylindrical coordinate $u_c = (r, \theta, z) : M \to \mathbb{R}^3$, spherical coordinate $u_s = (\rho, \theta, \phi) : M \to \mathbb{R}^3$. For them to be admissible, we need to check if $u_c \circ u_s^{-1}$ and $u_s \circ u_c^{-1}$ are C^∞. Note that

$$u_c \circ u_s^{-1} = \begin{cases} r &= \rho \sin \phi \\ \theta &= \theta \\ z &= \rho \cos \phi \end{cases}$$

$$|J| = \left| \frac{\partial(r, \theta, z)}{\partial(\rho, \theta, \phi)} \right| = |\rho|$$

So as long as $\rho > 0$, $u_c \circ u_s^{-1}$ is C^∞.

$$u_s \circ u_c^{-1} = \begin{cases} \rho & = \sqrt{r^2 + z^2} \\ \theta & = \theta \\ \phi & = \arctan(r/z) \end{cases}$$

$$|J| = \left| \frac{\partial(\rho, \theta, \phi)}{\partial(r, \theta, z)} \right| = \begin{vmatrix} \frac{r}{\sqrt{r^2+z^2}} & \frac{z}{\sqrt{r^2+z^2}} \\ \frac{z}{r^2+z^2} & -\frac{r}{r^2+z^2} \end{vmatrix} = \frac{1}{\sqrt{r^2 + z^2}}$$

$u_s \circ u_c^{-1}$ is C^∞ except $r = z = 0$. Therefore all points except for origin can be added to domains of systems.

Exercise 1.2.2

Standard structure on $\mathbb{R}$ consists of a single chart which is the identity map i. If u^3 is admissible to the standard structure. Then $i \circ u^{1/3}$ must be C^∞. But $(u^{1/3})' = \frac{1}{3u^{2/3}}$ is singular at 0. So $u^{1/3}$ is not admissible to standard structure.