Graph Theory Note

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1 Definitions

Graph G is a finite set V(vertex set) with irreflexive, **symmetric** relation R on V. E the edge set is the set of symmetric pairs in R. $|V|$ is **order** of G and $|E|$ is size of G. A (p, q) graph is a graph with order p and size q.

Subgraph H of G is when $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

$G - e$ is subgraph of G where $V(G) = V(G - e)$ and $E(G) - \{e\} = E(G - e)$.

$G - v$ is subgraph of G where $V(G) - \{v\} = V(G - v)$ and $E(G) - \{(v, u) \in E(G) | \forall u \in V(G)\} = E(G - v)$

Degree of vertice v denoted by $\deg_G v$ is the number of edges incident with v. v is odd or even is when its degree is odd or even.

Adjacent vertices v and w means $(v, w) \in E(G)$. Adjacent edges (v, w_1) and (v, w_2) are when $w_1 \neq w_2$.

Digraph(Directed Graph) G has a relation R that is not necessarily symmetric. $(u, v) \in E$ is called a directed edge or an arc.

Network is a graph/digraph with a function $f : E \to \Re$. When $f : E \to \{\pm 1\}$ it is called a signed graph.

Multigraph is a network when f is a multi map, e.g. $f = \{(v_1, v_2, 1), (v_1, v_2, 2)\}$

Loop-graph is when R is no longer irreflexive.

Isomorphism from G_1 to G_2 is a bijection $\phi : V(G_1) \to V(G_2)$ s.t $(v_1, v_2) \in E(G_1) \iff (\phi(v_1), \phi(v_2)) \in E(G_2)$.

Graph traversal

1. A u_1-u_n **walk** is a sequence $\{u_1, \ldots, u_n\}$ where (u_i, u_{i+1}) is an edge.

2. A u_1-u_n **trail** is a walk with no repeating edges.
3. A u_1-u_n **path** is a walk with no repeating vertices.
4. u-u trail that contains at least 3 edges is a **circuit**
5. A **cycle** is a circuit with no repeating vertices.

Eulerian circuit is a circuit that contains all vertices and edges of a multigraph

Connected graph G is when u-v path exists for any $u \neq v \in V(G)$. Otherwise a graph is disconnected.

Traversable Graph is a graph that has a Eulerian trial (containing all edges and vertices).

Component H of a graph G is the largest connected subgraph that contains itself.

Cut-vertex is a vertex v in connected graph G such that $G - v$ is disconnected.

Bridge is an edge e in connected graph G such that $G - e$ is disconnected.

Hamiltonian Graph is a graph with a cycle that contains every vertex.

2 Examples Modelings

Friendship can be represented as a graph.

City can be represented as a digraph where road intersection are vertices and arcs as one-way or two- way streets.

Employer/Employee hierarchy can be represented as diagraph with people as vertices and arc connecting subordinate with their supervisor.

3 Results

1. For (p, q) graph, $\sum_v \text{deg}v = 2q$.

2. Every graph has even number of odd vertices.

3. Let G be connected graph, e is a bridge iff e not in any cycle of G.

4. A multigraph is eulerian iff it is connected and every vertex is even.

5. A multigraph is traversable iff it is connected and has exactly two odd vertices. Any eulerian trail starts on one and ends on the other.

6. Eulerian or traversable graph can be drawn without lifting pencil. (Connected with odd vertices count either 0 or 2)

7. G is hamiltonian if order $p \geq 3$ and $\deg v \geq p/2$ for every vertex of G.

4 Graph Algorithms