Chapter 3: Euclidean Geometry

Ran Xie

March 12, 2022

1 Isometries of $\Re^3$

1

Consider

$$\begin{aligned}|C(p+a)-C(p)-C(a)|^2 &= C(p+a)\cdot C(p+a)+C(p)\cdot C(p)+C(a)\cdot C(a)\\ &\quad -2C(p+a)\cdot C(p)-2C(p+a)\cdot C(a)+2C(p)\cdot C(a)\\ &= (p+a)^2+p^2+a^2-2(p+a)p-2(p+a)a+2pa\\ &= p^2+2pa+a^2+p^2+a^2-2p^2-2pa-2pa-2a^2+2pa\\ &= 0\end{aligned}$$

Therefore $C(p+a)=C(p)+C(a)$. It follows that $CT_a(p)=C(p+a)=C(p)+C(a)=T_{C(a)}C(p)$ ■

2

From the result in problem 1.1 $FG=T_aAT_bB=T_aT_{A(b)}AB$ and $GF=T_bBT_aA=T_bT_{B(a)}BA$. The transnational parts are $T_{a+A(b)}$ and $T_{b+B(a)}$ respectively.

3

Suppose $Cp=Cq$, Then

$$\begin{aligned}&\Leftrightarrow \langle Cp-Cq, Cp-Cq\rangle=0\\ &\Leftrightarrow CpCp-2CpCq-CqCq=0\\ &\Leftrightarrow p^2-2pq-q^2=0\\ &\Leftrightarrow p=q\end{aligned}$$

C is 1-1. Therefore there exists inverse C^{-1}. To show C^{-1} is orthogonal transformation. Suppose p,q such that $C^{-1}p=\tilde{p}$ and $C^{-1}q=\tilde{q}$

$$\langle C^{-1}p, C^{-1}q\rangle=\langle\tilde{p},\tilde{q}\rangle=\langle C\tilde{p},C\tilde{q}\rangle=\langle p,q\rangle$$

So C^{-1} is orthogonal transformation. We can define the inverse of F. $F^{-1} = (T_a C)^{-1} = C^{-1} T_{-a}$. F^{-1} is isometry.

4

$$C = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

It's trivial to check orthogonality after factoring out 1/3.

$Cp = \frac{1}{3}(2, 19, -7)$ and $Cq = \frac{1}{3}(-5, -4, 7)$. Then $\langle Cp, Cq \rangle = \frac{1}{9}(-135) = -15 = \langle p, q \rangle$.

5

(a) $q = F(p) = T_a C(p) = (-3\sqrt{2} + 1, 1, 5\sqrt{2} - 1)^T$

(b) $q = F^{-1}(p) = (T_a C)^{-1}(p) = C^{-1} T_{-a}(p) = C^T T_{-a}(p) = (5\sqrt{2}, -5, 4\sqrt{2})^T$

(c) $q = (CT_a)(p) = (5\sqrt{2}, 1, 2\sqrt{2})^T$

6

(a) $C = \text{diag}(-1, -1, -1)$ and $a = (0, 0, 0)$.

(b) Not isometry. If $p \perp a$, then $d(F(p), 0) = d(0, 0) = 0 \neq d(p, 0)$.

(c) $C = I$, $a = (-1, -2, -3)$.

(d) $C = \text{diag}(1, 1, 0)$, $a = (0, 0, 1)$.

7

For $F_1, F_2 \in \text{Iso}(3)$, $F_1 F_2 = T_a C_1 T_b C_2 = T_a T_{C_1(b)} C_1 C_2 \in \text{Iso}(3)$. Associative is trivial since they are functions. Inverse exists for every F as proven in problem 3.

8

Only Identity is in both subgroups.

9

(a) For an orthgonal matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it satisfies

$$\begin{cases} ac + bd = 0 \\ a^2 + b^2 = 1 \\ c^2 + d^2 = 1 \end{cases}$$

We have a free parameter. Let $d = \pm \sin\theta$, then

$$\begin{cases} d = \pm \sin\theta \\ c = \cos\theta \\ b = \mp \cos\theta \\ a = \sin\theta \end{cases}$$

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sin\theta & \mp\cos\theta \\ \cos\theta & \pm\sin\theta \end{pmatrix}$

(b) $F = T_a C$. $CpCp = p^2 \Rightarrow c^2p^2 = p^2 \Rightarrow c = 1$. So an isometry in $\Re$ is just a displacement by a constant a.

2 The tangent map of an isometry

1

Translation is an isometry, so $T(v_p) = I(v)_{Tp} = v_{T(p)}$ which has the same Euclidean coordinates as v_p.

2

Given isometries $G = T_gC_g, F = T_fC_f$, $(GF)_*(v_p) = (T_gC_gT_fC_f)_*(v_p) = (T_gT_{C_g(f)}C_gC_f)_*(v_p) = C_gC_f(v)_{G\circ F(p)} = G_*F_*(v)$

3

$F = T_aC, p = (0,1,0), q = (3,-1,1)$

we have $[e] = A[u] = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}[u]$ and $[f] = B[u] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$

To transform from coordinates of e to f.

$$C = B^t A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -2/3 & 1/3 & 2/3 \\ \sqrt{2}/6 & 2\sqrt{2}/3 & -\sqrt{2}/6 \end{pmatrix}$$

$F(p) = T_a C(p) = a + Cp = q$. So $a = q - Cp = (3, -1, 1) - (0, 1/3, 2\sqrt{2}/3) = (3, -4/3, 1 - 2\sqrt{2}/3)$

4

(a) A plane is defined by $\langle (x-p)_p, q_p \rangle = 0$. If an isometry $F = T_a C$, then

$$\begin{aligned} &\langle (x-p)_p, q_p \rangle = 0 \\ &\Leftrightarrow \langle F_*(x-p)_p, F_* q_p \rangle = 0 \\ &\Leftrightarrow \langle C(x-p)_{F(p)}, Cq_{F(p)} \rangle = 0 \\ &\Leftrightarrow \langle C(T_{C(a)}x - T_{C(a)}p)_{F(p)}, Cq_{F(p)} \rangle = 0 \\ &\Leftrightarrow \langle (F(x) - F(p))_{F(p)}, Cq_{F(p)} \rangle = 0 \end{aligned}$$

Note that $(T_{C(a)}x - T_{C(a)}p = x - p$ since translation is canceled out.

$\blacksquare$

(b) Let $e_1 = (0,1,0), e_2 = (1/\sqrt{2}, 0, -1/\sqrt{2})$, then $e_3 = e_1 \times e_2 = (-1/\sqrt{2}, 0, -1/\sqrt{2})$ form a frame. From e_1 to e_2, we simply need to perform a 90 degree rotation along e_3. The transformation is $C_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ wrt to the frame. Then it is $A^t C_e A$ in the canonical frame where A is the attitude matrix. We get

$$C_u = A^t C_e A = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}^t \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

Since $F(1/2, -1, 0) = TC(1/2, -1, 0) = (1, -2, 1)$, we get $T = (3/4 - \sqrt{2}/2, -2 + \sqrt{2}/4, 3/4 - \sqrt{2}/2)$