Chapter 5: Properties of Random Samples

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Exercise 5.1

The samples are drawn from Bernoulli trial with success rate 0.01. The probability of n samples not containing color-blind is 0.99^n. We want to find N such that for $n \geq N$, $0.99^n \leq 1 - 0.95$. $N \approx 299$.

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Exercise 5.2

(a) Let T be the number of years until the first year's rainfall is exceeded. Then

$$\begin{aligned}
P(T = k) &= P(X_2 \leq X_1, \ldots, X_{k-1} \leq X_1, X_k > X_1) \\
&= \int_x P(X_2 \leq x, \ldots, X_{k-1} \leq x, X_k > x | X_1 = x) f(x) dx \\
&= \int_x P(X_k > x) f(x) \prod_{i=2}^{k-1} P(X_i \leq x) dx \\
&= \int_x (1 - F(x)) f(x) F(x)^{k-1} dx \\
&= \int_x F(x)^{k-1} f(x) dx - \int_x F(x)^k f(x) dx \\
&= \frac{1}{k} F(x)^k |_{-\infty}^{\infty} - \frac{1}{k+1} F(x)^{k+1} |_{-\infty}^{\infty} \\
&= \frac{1}{k} - \frac{1}{k+1} \\
&= \frac{1}{k(k+1)}
\end{aligned}$$

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(b)

$$\mathbf{E}T = \sum_k kP(T = k) = \sum_k \frac{1}{k+1} = \infty$$

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Exercise 5.3

Since $\{X_i\}$ are i.i.d $\sim F_X(x)$ and Y_i is hierachical wrt to X_i. $Y_i \sim \text{Bernoulli}(P(X_i > \mu))|X_i$. So Y_i are i.i.d. Therefore the sum of $Y_i \sim \text{Binomial}(n, P(X_i > \mu)) = \text{Binomial}(n, 1 - F_X(\mu))$.

$$P(Y_i = k) = \binom{n}{k}(1 - F_X(\mu))^k F_X(\mu)^{n-k}$$

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Exercise 5.4

(a) $X_i|P \sim \text{Bernoulli}(P)$ are i.i.d and $P \sim \text{Uniform}(0, 1)$. Let $T = \sum_{i=1}^k X_i$.

$$\begin{aligned}
P(X_1 = x_1, \ldots, X_k = x_k) &= \int_0^1 P(X_1 = x_1, \ldots, X_k = x_k|P = p)f(p)dp \\
&= \int_0^1 \prod_{i=1}^k P(X_i = x_i|P = p)f(p)dp, \text{ Since } X_i|P \text{ are i.i.d} \\
&= \int_0^1 \prod_{i=1}^k p^{x_i}(1-p)^{1-x_i} f(p)dp \\
&= \int_0^1 p^{\sum_i x_i}(1-p)^{1-\sum_i x_i} f(p)dp \\
&= \int_0^1 p^t(1-p)^{1-t} f(p)dp
\end{aligned}$$

where $t = \sum_i x_i$.

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(b) From (a),

$$P(X_1 = x_1, \ldots, X_n = x_n) = \int_0^1 p^t(1-p)^{1-t} f(p)dp$$

where $t = \sum_i^n x_i$.

On the other hand,

$$\prod_i^n P(X_i = x_i) = \prod_i^n \int_0^1 P(X_i = x_i | P = p) f(p) dp = \prod_i^n \int_0^1 p^{x_i}(1-p)^{1-x_i} dp$$

Therefore $P(X_1 = x_1, \ldots, X_n = x_n) \neq \prod_i^n P(X_i = x_i)$.

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Exercise 5.5

let $Y = \sum_i X_i$ then $\bar{X} = Y/n$. Suppose we have $f_Y(y)$, then

$$f_{\bar{X}}(\bar{x}) = f_Y(y) = f_Y(n\bar{x}) \left| \frac{dy}{d\bar{x}} \right| = n f_Y(n\bar{x})$$

Exercise 5.6

*Book has typos, it should be 5.2.9 instead of 5.2.3.

(a) Let $Z = X + Y$, $V = X$, then

$$f_{V,Z}(v,z) = f_{X,Y}(v, z-v) \left| \frac{\partial(X,Y)}{\partial(V,Z)} \right| = f_{X,Y}(v, z-v) \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = f_{X,Y}(v, z-v)$$

Therefore

$$f_Z(z) = \int_v f_{X,Y}(v, z-v) dv = \int_v f_X(v) f_Y(z-v) dv$$

(b) Let $Z = XY$, $V = X$, then

$$f_{V,Z}(v,z) = f_{X,Y}(v, z/v) \left| \frac{\partial(X,Y)}{\partial(V,Z)} \right| = f_{X,Y}(v, z/v) \begin{vmatrix} 1 & 0 \\ -z/v^2 & 1/v \end{vmatrix} = f_{X,Y}(v, z-v) \left| \frac{1}{v} \right|$$

Therefore

$$f_Z(z) = \int_v f_{X,Y}(v, z/v) dv = \int_v f_X(v) f_Y(z/v) \left| \frac{1}{v} \right| dv$$

(c) Let $Z = X/Y$, $V = X$, then

$$\left| \frac{\partial(X,Y)}{\partial(V,Z)} \right| = \begin{vmatrix} 1 & 0 \\ 1/z & -v/z^2 \end{vmatrix} = \left| \frac{v}{z^2} \right|$$

Therefore

$$f_Z(z) = \int_v f_{V,Z}(v,z) dv = \int_v f_{X,Y}(v, v/z) \left| \frac{\partial(X,Y)}{\partial(V,Z)} \right| dv = \int_v f_X(v) f_Y(v/z) \left| \frac{v}{z^2} \right| dv$$

Exercise 5.7

(a) Combining the terms on the right side and order the term by power of w, we get

$$\begin{aligned}\left(\frac{A}{\tau^2}-\frac{C}{\sigma^2}\right)w^3&=0\\ \left(-\frac{2Az}{\tau^2}+\frac{B}{\tau^2}-\frac{D}{\sigma^2}\right)w^2&=0\\ \left(A+\frac{Az^2}{\tau^2}-\frac{2Bz}{\tau^2}-C\right)w&=0\\ B+\frac{Bz^2}{\tau^2}-D&=1\end{aligned}$$

We get linear equation of

$$\begin{pmatrix}\sigma^2 & 0 & -\tau^2 & 0\\ -2z\sigma^2 & \sigma^2 & 0 & -\tau^2\\ \tau^2+z^2 & -2z & -\tau^2 & 0\\ 0 & \tau^2+z^2 & 0 & -\tau^2\end{pmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\tau^2\end{pmatrix}$$

The determinant is $(-\sigma^2+\tau^2+z^2)^2+4z^2\sigma^2\neq 0$. So A,B,C,D exists.

(b) Skipping the trivial calculation.

Exercise 5.8

(a)

$$\begin{aligned}(n-1)S^2&=\sum_i(X_i-\bar{X})^2\\ &=\sum_i\left(X_i-\frac{1}{n}\sum_j X_j\right)^2\\ &=\sum_i\left(X_i-\frac{2}{n}X_i\sum_j X_j+\frac{1}{n^2}\left(\sum_j X_j\right)^2\right)\\ &=\sum_i X_i-\frac{2}{n}\sum_i X_i\sum_j X_j+\frac{1}{n^2}\sum_i\left(\sum_j X_j\right)^2\\ &=\sum_i X_i-\frac{2}{n}\sum_i\sum_j X_iX_j+\frac{1}{n}\sum_i\sum_j X_iX_j\end{aligned}$$

Multiply both side by $2n$, we get

$$\begin{aligned}
2n(n-1)S^2 &= 2n\sum_i X_i - 2\sum_i\sum_j X_iX_j \\
&= n\sum_i X_i - 2\sum_i\sum_j X_iX_j + n\sum_i X_i \\
&= n\sum_i X_i - 2\sum_i\sum_j X_iX_j + n\sum_j X_j \\
&= \sum_j\sum_i X_i - 2\sum_i\sum_j X_iX_j + \sum_i\sum_j X_j, \text{ (Note that } n = \sum_i 1 = \sum_j 1\text{)} \\
&= \sum_i\sum_j (X_i - X_j)^2
\end{aligned}$$