Mechanics

January 21, 2024

1 Basic Elementary Principles

1.1 Single particle

Let $\boldsymbol{r}$ be radius vector of a particle from given origin and $\boldsymbol{v}$ its velocity vector:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$$

acceleration is given by

$$\boldsymbol{a} = \frac{d^2\boldsymbol{r}}{dt^2}$$

Linear momentum $\boldsymbol{p} \equiv m\boldsymbol{v}$. Vector sum of forces exerted on the particle is total force $\boldsymbol{F}$,

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \equiv \dot{\boldsymbol{p}} \qquad \text{(Newton's second law)}$$

Conversation Theorem for linear momentum: If $\boldsymbol{F} = 0$, then $\dot{\boldsymbol{p}} = 0$ hence conserved.

Angular momentum of the particle around point O is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. Torque $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$. We can relationship

$$\boldsymbol{N} = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \frac{d\boldsymbol{L}}{dt} \equiv \dot{\boldsymbol{L}}$$

Conservation Theorem for angular momentum: If $\boldsymbol{N}$ is zero then $\dot{\boldsymbol{L}} = 0$, hence conserved.

If force field does the same work for any possible path between point 1 and 2 ($W_{12} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{s}$)), $\boldsymbol{F}$ is conservative or $\oint \boldsymbol{F} \cdot d\boldsymbol{s} = 0$. As a result, there exists a potential scalar field V such that

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r})$$

Energy Conservation Theorem for a particle: If $\boldsymbol{F}$ acting on a particle is conservative, then $E = T + V$ is conserved.

1.2 System of particles

Let $\boldsymbol{F}_i^{(e)}$ external force acting on particle i-th, and $\boldsymbol{F}_{ji}$ is the force exerted by j-th particle on i-th particle in the system,

$$\sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)} = \dot{\boldsymbol{p}}_i$$

Summing over all particles, we get

$$\frac{d^2}{dt^2}\sum_i m_i \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i^{(e)} + \sum_{i\neq j} \boldsymbol{F}_{ji} \tag{1}$$

$$\Rightarrow M\frac{d^2\boldsymbol{R}}{dt^2} = \sum_i \boldsymbol{F}_i^{(e)} \equiv \boldsymbol{F}^{(e)} \tag{2}$$

where $R = \frac{m_i \boldsymbol{r}_i}{\sum m_i} = \frac{m_i \boldsymbol{r}_i}{M}$ is the center of mass. The total linear momentum $\boldsymbol{P} = \sum m_i \frac{d\boldsymbol{r}_i}{dt} = M\frac{d\boldsymbol{R}}{dt}$.

Conservation theorem for the linear Momentum of a system of particles: if the total external force is zero, the total linear momentum is conserved.

Conservation Theorem for total angular momentum of a system of particles: $\boldsymbol{L}$ is constant in time if the applied external torque is zero. $\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}^{(e)}$

The total angular momentum of a system of particles is $\boldsymbol{L} = \boldsymbol{R} \times M\boldsymbol{v} + \sum_i \boldsymbol{r}_i' \times \boldsymbol{p}_i'$ where $\boldsymbol{r}_i' = \boldsymbol{R} - \boldsymbol{r}_i$ position relative to the center of mass.