

Chapter 4: Multiple Random Variables

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Exercise 4.1

- (a) Since $f_{X,Y}(x, y)$ is constant. $X^2 + Y^2 < 1$ is circle of radius 1. Therefore $P(X^2 + Y^2 < 1) = \pi/4$
- (b) $2X - Y = 0$ divides the unit square into two region of equal area and f is constant. Therefore $P(2X - Y > 0) = 1/2$
- (c) $P(|X + Y| < 2) = P(-2 < X + Y < 2)$. The area covers the entire square. Therefore $P(|X + Y| < 2) = 1$.

Exercise 4.4

- (a) Since $\int_0^1 \int_0^2 f(x, y) dx dy = \int_0^1 \int_0^2 C(x + 2y) dx dy = 4C = 1$ So $C = 1/4$.
- (b) $f(x) = \int_0^1 f(x, y) dy = (1/4)(xy + y^2)|_0^1 = \frac{x+1}{4}$, $x \in (0, 2)$
- (c) For $(x, y) \in (0, 2) \times (0, 1)$:

$$F(x, y) = P(X < x, Y < y) = \int_{-\infty}^x \int_{-\infty}^y \frac{t + 2s}{4} ds dt = \int_0^x \int_0^y \frac{t + 2s}{4} ds dt = \frac{1}{8}(x^2 y + 2xy^2)$$

For $(x, y) \in (0, 2) \times [1, \infty)$:

$$F(x, y) = P(X < x, Y < y) = \int_0^x \int_0^1 \frac{t + 2s}{4} ds dt = \frac{1}{8}(x^2 + 2x)$$

For $(x, y) \in (-\infty, 2] \times (0, 1)$:

$$F(x, y) = P(X < x, Y < y) = \int_0^2 \int_0^y \frac{t + 2s}{4} ds dt = \frac{1}{2}(y + y^2)$$

(d) from (b), we have $f(x) = \frac{x+1}{4}$. And $z = \frac{9}{(x+1)^2}$ is monotonic for $x \in [0, 2]$ with $z \in [1, 9]$. So we can take $x = \frac{\sqrt{z}}{3} - 1$. Then

$$f(z) = f(x^{-1}(z)) \left| \frac{dx}{dz} \right| = \frac{3}{4}(z^{-1/2})\left(\frac{3}{2}z^{-3/2}\right) = \frac{9}{8}z^{-2}$$

Exercise 4.5

(a) The area for integration is $0 < x < 1$ and $0 < y < x^2$.

$$P(X > \sqrt{Y}) = \int_0^1 \int_0^{x^2} x + y dy dx = \int_0^1 x^3 + \frac{x^4}{2} dx = 0.35$$

(b) The area of integration is $0 < x < 1$ and $x^2 < y < x$.

$$P(X^2 < Y < X) = \int_0^1 \int_{x^2}^x 2x dy dx = \int_0^1 2x^2 - 2x^3 dx = \frac{1}{6}$$

Exercise 4.6

Let X, Y be the time A and B arrive in time interval $[0, 1]$. Since they are independent, $f(x, y) = f(x)f(y) = 1$ for $(x, y) \in [0, 1] \times [0, 1]$.

Let T be the length of time A waits for B. Then $T = \max(Y - X, 0)$ because $T = 0$ when $Y < X$.

$$P(T < t) = P(\max(Y - X, 0)) = P(Y - X < t, Y \geq X) + P(Y < X)$$

For term $P(Y - X < t, Y \geq X)$, The area of integration is the area between $y - x = t$ and $y \geq x$ bounded by unit square. We can find the complement area which is an isosceles right triangle with side of $1 - t$, which gives

$$P(Y - X < t, Y \geq X) = \frac{1}{2} - \frac{1}{2}(1 - t)^2$$

$P(Y < X)$ is the lower half triangle of the unit square which has area of $\frac{1}{2}$ Therefore

$$P(T < t) = P(Y - X < t, Y \geq X) + P(Y < X) = 1 - \frac{1}{2}(1 - t)^2$$

Exercise 4.7

We can formulate the problem as such: $X \in [0, 30], Y \in [40, 50]$, find $P(X + Y < 60)$. We want to find the intersection of $x + y = 60$ with $[0, 30] \times [40, 50]$. We get $(10, 50), (20, 40)$. Since the distributions are uniform, we can simply find area of the trapezoid and divide it by the total area.

$$P(X + Y < 60) = \frac{10(10 + 20)0.5}{10(30)} = 150/300 = 0.5$$

Exercise 4.9

For interval $[a, b] \times [c, d]$.

$$\begin{aligned} P(a \leq X \leq b)P(c \leq Y \leq d) &= [F_X(b) - F_X(a)][F_Y(d) - F_Y(c)] \\ &= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c) \\ &= F(b, d) - F(b, c) - F(a, d) + F(a, c) \end{aligned}$$

If we define the regions $A_1 = [a, b] \times [c, d]$, $A_2 = [a, b] \times (-\infty, c)$, $A_3 = (-\infty, a) \times (-\infty, c)$, $A_4 = (-\infty, a) \times [c, d]$. Then

$$\begin{aligned} F(b, d) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ F(b, c) &= P(A_3) + P(A_2) \\ F(a, d) &= P(A_3) + P(A_4) \\ F(a, c) &= P(A_3) \end{aligned}$$

Hence

$$\begin{aligned} P(a \leq X \leq b)P(c \leq Y \leq d) &= F(b, d) - F(b, c) - F(a, d) + F(a, c) \\ &= P(A_1) \\ &= P([a, b] \times [c, d]) \\ &= P(X \in [a, b], Y \in [c, d]) \end{aligned}$$

Exercise 4.10

(a) Summing up the columns and rows, we have $P(X = 1) = 1/4$, $P(X = 2) = 1/2$, $P(X = 3) = 1/4$. $P(Y = 2) = P(Y = 3) = P(Y = 4) = 1/3$.

(b) We can build up a table for independent U, V just by multiplying the marginal probability.

U/V	1	2	3
2	1/12	1/6	1/12
3	1/12	1/6	1/12
4	1/12	1/6	1/12

Exercise 4.11

U and V are dependent. Consider $P(V|U = n)$ and $P(V)$, knowing $U = n$ means there is only one toss of head in the first n trials which means $P(V \leq n|U = n) = 0 \neq P(V \leq n)$.

Exercise 4.12

Let X and Y be uniform(0,1). Then $f_{X,Y} = 1$ is a unit square on $[0, 1] \times [0, 1]$. By symmetry, we only need to consider the probability conditioned on $X > Y$ which is the lower half triangle of the unit square. Then the 3 segments are $1 - X$, $X - Y$ and Y . For the segments to be a triangle, denoted by event T , it must satisfy the sum of two sides is larger than the other side. Therefore

$$\begin{aligned}1 - X + X - Y &> Y \\1 - X + Y &> X - Y \\X - Y + Y &> 1 - X\end{aligned}$$

Simplifying the expression, we have event T given $X > Y$ as the area of a region bounded by

$$\begin{aligned}\frac{1}{2} &> Y \\Y &> X - \frac{1}{2} \\X &> \frac{1}{2} \\X &> Y\end{aligned}$$

Note that area of $X > Y$ is $\frac{1}{2}$ and the area of the region for T is $\frac{1}{8}$. So $P(T|X > Y) = \frac{1/8}{1/2} = \frac{1}{4}$ and .

Finally, we have $P(T) = P(T|X > Y)P(X > Y) + P(T|X \leq Y)P(X \leq Y) = \frac{1}{4}$