Chapter 3: Euclidean Geometry

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1 Isometries of $\Re^3$

1

Consider

$$\begin{aligned}|C(p+a)-C(p)-C(a)|^2 &= C(p+a)\cdot C(p+a)+C(p)\cdot C(p)+C(a)\cdot C(a)\\ &\quad -2C(p+a)\cdot C(p)-2C(p+a)\cdot C(a)+2C(p)\cdot C(a)\\ &= (p+a)^2+p^2+a^2-2(p+a)p-2(p+a)a+2pa\\ &= p^2+2pa+a^2+p^2+a^2-2p^2-2pa-2pa-2a^2+2pa\\ &= 0\end{aligned}$$

Therefore $C(p+a)=C(p)+C(a)$. It follows that $CT_a(p)=C(p+a)=C(p)+C(a)=T_{C(a)}C(p)$ ■

2

From the result in problem 1.1 $FG=T_aAT_bB=T_aT_{A(b)}AB$ and $GF=T_bBT_aA=T_bT_{B(a)}BA$. The transnational parts are $T_{a+A(b)}$ and $T_{b+B(a)}$ respectively.

3

Suppose $Cp=Cq$, Then

$$\begin{aligned}&\Leftrightarrow \langle Cp-Cq, Cp-Cq\rangle = 0\\ &\Leftrightarrow CpCp-2CpCq-CqCq=0\\ &\Leftrightarrow p^2-2pq-q^2=0\\ &\Leftrightarrow p=q\end{aligned}$$

C is 1-1. Therefore there exists inverse C^{-1}. To show C^{-1} is orthogonal transformation. Suppose p,q such that $C^{-1}p=\tilde{p}$ and $C^{-1}q=\tilde{q}$

$$\langle C^{-1}p, C^{-1}q\rangle = \langle \tilde{p},\tilde{q}\rangle = \langle C\tilde{p}, C\tilde{q}\rangle = \langle p,q\rangle$$

So C^{-1} is orthogonal transformation. We can define the inverse of F. $F^{-1} = (T_a C)^{-1} = C^{-1} T_{-a}$. F^{-1} is isometry.

4

$$C = \frac{1}{3} \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

It's trivial to check orthogonality after factoring out 1/3.

$Cp = \frac{1}{3}(2, 19, -7)$ and $Cq = \frac{1}{3}(-5, -4, 7)$. Then $\langle Cp, Cq \rangle = \frac{1}{9}(-135) = -15 = \langle p, q \rangle$.

5

(a) $q = F(p) = T_a C(p) = (-3\sqrt{2} + 1, 1, 5\sqrt{2} - 1)^T$

(b) $q = F^{-1}(p) = (T_a C)^{-1}(p) = C^{-1} T_{-a}(p) = C^T T_{-a}(p) = (5\sqrt{2}, -5, 4\sqrt{2})^T$

(c) $q = (C T_a)(p) = (5\sqrt{2}, 1, 2\sqrt{2})^T$

6

(a) $C = \text{diag}(-1, -1, -1)$ and $a = (0, 0, 0)$.

(b) Not isometry. If $p \perp a$, then $d(F(p), 0) = d(0, 0) = 0 \neq d(p, 0)$.

(c) $C = I$, $a = (-1, -2, -3)$.

(d) $C = \text{diag}(1, 1, 0)$, $a = (0, 0, 1)$.

7

For $F_1, F_2 \in \text{Iso}(3)$, $F_1 F_2 = T_a C_1 T_b C_2 = T_a T_{C_1(b)} C_1 C_2 \in \text{Iso}(3)$. Associative is trivial since they are functions. Inverse exists for every F as proven in problem 3.

8

Only Identity is in both subgroups.

9

(a) For an orthgonal matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it satisfies

$$\begin{cases} ac + bd = 0 \\ a^2 + b^2 = 1 \\ c^2 + d^2 = 1 \end{cases}$$

We have a free parameter. Let $d = \pm \sin\theta$, then

$$\begin{cases} d = \pm \sin\theta \\ c = \cos\theta \\ b = \mp \cos\theta \\ a = \sin\theta \end{cases}$$

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sin\theta & \mp\cos\theta \\ \cos\theta & \pm\sin\theta \end{pmatrix}$

(b) $F = T_a C$. $CpCp = p^2 \Rightarrow c^2 p^2 = p^2 \Rightarrow c = 1$. So an isometry in $\Re$ is just a displacement by a constant a.

2 The tangent map of an isometry

1

Translation is an isometry, so $T(v_p) = I(v)_{Tp} = v_{T(p)}$ which has the same Euclidean coordinates as v_p.

2

Given isometries $G = T_g C_g, F = T_f C_f$, $(GF)_*(v_p) = (T_g C_g T_f C_f)_*(v_p) = (T_g T_{C_g(f)} C_g C_f)_*(v_p) = C_g C_f(v)_{G\circ F(p)} = G_* F_*(v)$

3

$F = T_a C$, $p = (0, 1, 0)$, $q = (3, -1, 1)$

we have $[e] = A[u] = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}[u]$ and $[f] = B[u] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$

To transform from coordinates of e to f.

$$C = B^t A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & -1\end{pmatrix}\frac{1}{3}\begin{pmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ -2/3 & 1/3 & 2/3\\ \sqrt{2}/6 & 2\sqrt{2}/3 & -\sqrt{2}/6\end{pmatrix}$$

$F(p) = T_a C(p) = a + Cp = q$. So $a = q - Cp = (3, -1, 1) - (0, 1/3, 2\sqrt{2}/3) = (3, -4/3, 1 - 2\sqrt{2}/3)$

4

(a) A plane is defined by $\langle (x-p)_p, q_p\rangle = 0$. If an isometry $F = T_a C$, then

$$\begin{aligned}
&\langle (x-p)_p, q_p\rangle = 0\\
&\Leftrightarrow \langle F_*(x-p)_p, F_* q_p\rangle = 0\\
&\Leftrightarrow \langle C(x-p)_{F(p)}, Cq_{F(p)}\rangle = 0\\
&\Leftrightarrow \langle C(T_{C(a)}x - T_{C(a)}p)_{F(p)}, Cq_{F(p)}\rangle = 0\\
&\Leftrightarrow \langle (F(x) - F(p))_{F(p)}, Cq_{F(p)}\rangle = 0
\end{aligned}$$

Note that $(T_{C(a)}x - T_{C(a)}p = x - p$ since translation is canceled out.

■

(b) Let $e_1 = (0, 1, 0), e_2 = (1/\sqrt{2}, 0, -1/\sqrt{2})$, then $e_3 = e_1 \times e_2 = (-1/\sqrt{2}, 0, -1/\sqrt{2})$ form a frame. From e_1 to e_2, we simply need to perform a 90 degree rotation along e_3. The transformation is $C_e = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ wrt to the frame. Then it is $A^t C_e A$ in the canonical frame where A is the attitude matrix. We get

$$C_u = A^t C_e A = \begin{pmatrix}0 & 1 & 0\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ -1/\sqrt{2} & 0 & -1/\sqrt{2}\end{pmatrix}^t \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ -1/\sqrt{2} & 0 & -1/\sqrt{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}$$

Since $F(1/2, -1, 0) = TC(1/2, -1, 0) = (1, -2, 1)$, we get $T = (3/4 - \sqrt{2}/2, -2 + \sqrt{2}/4, 3/4 - \sqrt{2}/2)$

3 Orientation

1

$\mathrm{Sgn}(FG) = \mathrm{Sgn}(T_aC_1T_bC_2) = \mathrm{Sgn}(T_aT_{C_1(b)}C_1C_2) = \det(C_1C_2) = \det(C_1)\det(C_2) = \mathrm{Sgn}F \cdot \mathrm{Sgn}G$

Let $G = F^{-1}$, then $\mathrm{Sgn}F \cdot \mathrm{Sgn}F^{-1} = \mathrm{Sgn}I = 1$. Therefore $\mathrm{Sgn}F = \mathrm{Sgn}(F^{-1})$

2

Suppose H_1 is orientation reversing isometry, let $H_1 = H_0F$, then $F = H_1H_0^{-1}$. H_0^{-1} is an isometry so it has unique inverse. Then F is also unique and $\mathrm{Sgn}F = \mathrm{Sgn}H_1\mathrm{Sgn}H_0^{-1} = 1$ which is orientation preserving.

3

$v = 3U_1 + U_2 - U_3$ and $w = -3U_1 - 3U_2 + U_3$.

$$v \times w = \begin{vmatrix} U_1 & U_2 & U_3 \\ 3 & 1 & -1 \\ -3 & -3 & 1 \end{vmatrix} = -2U_1 - 6U_3$$

$$C_*(v \times w) = \frac{1}{3}\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix} = \frac{1}{3}(10e_1 + 8e_2 - 14e_3)$$

On the right hand side

$$C_*(v) = -e_1 + 3e_2 + e_3$$

$$C_*(w) = \frac{1}{3}(-e_1 - 11e_2 - 7e_3)$$

$$\mathrm{Sgn}(C)C_*(v) \times C_*(w) = (-1)\frac{1}{3}\begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & 3 & 1 \\ -1 & -11 & -7 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & 3 & 1 \\ 1 & 11 & 7 \end{vmatrix} = \frac{1}{3}(10e_1 + 8e_2 - 14e_3)$$

4

Since $\det C = +1$ is the product of all eigenvalues of C, so it has at least 1 eigenvalue of value 1, let e_3 be the corresponding eigenvector. Then $C(e_3) = e_3$. So C is a rotation around e_3 by θ. Now pick e_1 and e_2 in the plane A perpendicular to e_3 such that $e_1 \perp e_2$. Then e_1, e_2 form a basis for A.

By right hand rule, $C(e_1)$ rotates e_1 counterclock wise by θ and $C(e_2)$ rotates e_2 the same amount. Coordinate vector $(1,0)$ gets rotated to $(\cos\theta, \sin\theta)$. We can then work out $C(e_1) = \cos\theta e_1 + \sin\theta e_2$ and $C(e_2) = -\sin_\theta e_1 + \cos\theta e_2$.

5

Let a be a point such that $\|a\| = 1$.

$$\begin{aligned}
C(p) \cdot C(q) &= [a \times p + (p \cdot a)a] \cdot [a \times q + (q \cdot a)a] \\
&= (a \times p) \cdot (a \times q) + (q \cdot a)(a \times p) \cdot a + (p \cdot a)a \cdot a \times q + (p \cdot a)a \cdot (q \cdot a)a \\
&= (a \times p) \cdot (a \times q) + (p \cdot a)(q \cdot a)\|a\| \\
&= a \cdot (q \times (a \times p)) + (p \cdot a)(q \cdot a) \\
&= a \cdot ((q \cdot p)a - (q \cdot a)p) + (p \cdot a)(q \cdot a) \\
&= (q \cdot p)\|a\| - (q \cdot a)(a \cdot p) + (p \cdot a)(q \cdot a) \\
&= q \cdot p
\end{aligned}$$

Therefore C is an orthogonal transformation.

■

6

(a) $O^+(3)$ is not empty (obviously). By definition in Ex 3.3.4, a rotation A is orthogonal such that $\det A = 1$. Then for A, B, the product AB is orthogonal since $O(3)$ is a group and $\det(AB) = \det A \det B = 1$ is also a rotation. For each A, there exist an orthogonal inverse, and $\det A^{-1} = \frac{1}{\det A} = 1$ which is also a rotation.

(b) By Ex 3.3.1, $\text{Sgn}(FG) = \text{Sgn}F\text{Sgn}G = 1$ for orientation preserving isometry F and G. So it is closed under multiplication. And $\text{Sgn}F = \text{Sgn}F^{-1}$ means the inverse is also orientation preserving.

4 Euclidean Geometry

1

2

$\bar{\alpha} = C(\alpha) = -\cos tU_1 + \frac{1}{\sqrt{2}}(\sin t - 2t)U_2 + \frac{1}{\sqrt{2}}(\sin t + 2t)U_3$

$\alpha'' = -\cos tU_1 - \sin tU_2$

$Y' = U_1 - 2tU_2 + 2tU_3$

$\bar{Y} = C_*(Y) = -tU_1 - \sqrt{2}t^2U_2 + \sqrt{2}U_3$

Then

$C_*(Y') = -U_1 - 2\sqrt{2}tU_2$ and $(\bar{Y})' = -U_1 - 2\sqrt{2}tU_2$

$C_*(\alpha'') = \cos tU_1 - \frac{1}{\sqrt{2}}\sin tU_2 - \frac{1}{\sqrt{2}}\sin tU_3$ and $(\bar{\alpha})'' = \cos tU_1 - \frac{1}{\sqrt{2}}\sin tU_2 - \frac{1}{\sqrt{2}}\sin tU_3$

$Y' \cdot \alpha'' = -\cos t + 2t\sin t$ and $\bar{Y}' \cdot \bar{\alpha}'' = -\cos t + 2t\sin t$

3

Let the triangle vertex be A,B and C. The sides of triangle one are of length $(AB, BC, CA) = (4, 3, 5)$. The sides for triangle two are of length $(AB, BC, CA) = (5, 3, 4)$. Therefore $(3, 1)$ maps to $(2, 0)$, $(7, 1)$ maps to $(-2/5, 16/5)$ and $(7, 4)$ maps to $(2, 5)$.

If we write $F(p) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} p = q$, plugging in point A and C from both triangle for p and q. We get 4 equations:

$$\begin{aligned} a + 3c + d &= 2 \\ b - 3d + c &= 0 \\ a + 7c + 4d &= 2 \\ b - 7d + 4c &= 5 \end{aligned}$$

where $c = \cos\theta$ and $d = \sin\theta$. Solving it, we get $a = 1, b = -3, c = \cos\theta = 0.6, d = \sin\theta = 0.8$. Therefore $F(p)$ in homogeneous coordinate is

$$\begin{pmatrix} 0.6 & 0.8 & 1 \\ -0.8 & 0.6 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$

4

We will show that F preserves length of any line segment. Given a line segment C parameterized by arc length from 0 to s_1, $\frac{dC}{ds}|_{C(s_1)}$ is the tangent of the curve at $C(s_1)$, and $F_*\frac{dC}{ds}|_{F(C(s_1))}$ is the tangent of the curve mapped by F at $F(C(s_1))$. Since F_* preserves dot product, so the length of the segments L_C and $L_{F\circ C}$ are equal.

$$L_C = \int_0^{s_1} \langle \frac{dC}{ds}, \frac{dC}{ds} \rangle^{1/2} ds = \int_0^{s_1} \langle F_*\frac{dC}{ds}, F_*\frac{dC}{ds} \rangle^{1/2} ds = L_{F\circ C}$$

5

By definition of covariant derivative,

$$\overline{\nabla_V W} = F_* \frac{d}{dt} W(p+tv)|_{t=0} = \frac{d}{dt} F_* W(p+tv)|_{t=0} = \frac{d}{dt} \overline{W}(F(p)+tF_*v)|_{t=0} = \nabla_{\overline{V}} \overline{W}$$

5 Congruence of Curves

1

Let $F = T$ be a translation of p, then F is isometry. Then α and β are congruent iff $F(\alpha) = p + \alpha = \beta$.

3

$$\begin{aligned}
\beta &= (t + \sqrt{3}\sin t, 2\cos t, \sqrt{3}t - \sin t) \\
\beta' &= (1 + \sqrt{3}\cos t, -2\sin t, \sqrt{3} - \cos t) \\
\beta'' &= (-\sqrt{3}\sin t, -2\cos t, \sin t) \\
\beta''' &= (-\sqrt{3}\cos t, 2\sin t, \cos t) \\
\beta' \times \beta'' &= (-2 + 2\sqrt{3}\cos t, -4\sin t, -2\cos t - 2\sqrt{3}) \\
\kappa &= \frac{|\beta' \times \beta''|}{|\beta'|^3} = \frac{\sqrt{20}}{\sqrt{8}^3} \neq 0 \\
\tau &= \frac{|(\beta' \times \beta'') \cdot \beta'''|}{|\beta' \times \beta''|^2} = -\frac{8}{20} \neq 0
\end{aligned}$$

Therefore β is a helix. To find an isometry F that maps helix $\alpha = (a\cos t, a\sin t, bt)$ to β, we consider when $t = 0$ the transformation between the two frames.

$$F \begin{cases} T_\beta(0) = \frac{1}{2\sqrt{2}}(1+\sqrt{3}, 0, \sqrt{3}-1) \\ B_\beta(0) = \frac{1}{2\sqrt{2}}(-1+\sqrt{3}, 0, -1-\sqrt{3}) \\ N_\beta(0) = (0, -1, 0) \end{cases} = \begin{cases} T_\alpha(0) = \frac{1}{\sqrt{a^2+b^2}}(0, a, b) \\ B_\alpha(0) = \frac{1}{\sqrt{a^2+b^2}}(0, -b, a) \\ N_\alpha(0) = (-1, 0, 0) \end{cases}$$

Then

$$F = C = \begin{pmatrix} 0 & \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ 0 & -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{-1+\sqrt{3}}{2\sqrt{2}} \\ \frac{-1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{-1-\sqrt{3}}{2\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}^T$$