Chapter 2: Transformation and Expectations Exercises

September 19, 2021

Exercise 2.1

(1) $y = g(x) = x^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. We will find $F_Y(y)$ instead.

$$F_X(x) = \int_{t=0}^{x} 42t^5(1-t)dt = 7t^6 - 6t^7|_0^x = 7x^6 - 6x^7$$

Since g is an increasing function,

$$F_Y(y) = F_X(g^{-1}(y)) = 7y^2 - 6y^{\frac{7}{3}}$$

Then pdf for y, $f_y(y) = \frac{d}{dy}F_Y = 14y - 14y^{\frac{4}{3}}$ where $y \in (0,1)$

(2) g is increasing, $f_X = 7e^{-7x}$ is continuous on $[0,\infty)$ and $g^{-1} = (y-3)/4$ has continuous derivative on $[3,\infty)$. Therefore, by theorem 2.1.5,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{7}{4}\exp(-\frac{7}{4}(y-3)), \quad y \in [3,\infty)$$

(3) We will find F_Y instead.

$$F_X(x) = \int_0^x 30t^2(1-t)^2dt = 10x^3 + 6x^5 - 15x^4$$

Since $g(x) = x^2$ is increasing on $(0,1)$. Therefore

$$F_Y(y) = F_X(g^{-1}(y)) = 10y^{3/2} + 6y^{5/2} - 15y^2$$

Pdf $f_Y(y) = \frac{d}{dy} = 15y^{1/2} + 15y^{3/2} - 30y$ for $y \in (0,1)$

Exercise 2.2

(b) $y = g(x) = -\log x$ is monotonic, $f_X(x)$ is continuous on $(0,1)$ and $x = g^{-1}(y) = e^{-y}$ has continuous derivative on $(0,\infty)$. Therefore we can use the theorem 2.1.5,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{(n+m+1)!}{n!m!}e^{-(n+1)y}(1-e^{-y})^m, \quad y \in (0,\infty)$$

(c) $y = g(x) = e^x$ is monotonic, $f_X(x)$ is continuous on $(0,\infty)$ and $x = g^{-1}(y) = \log x$ has continuous derivative of $1/x$ on $(0,\infty)$. Therefore we can use theorem 2.1.5

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{\log y}{y\sigma^2}\exp\left(-\frac{(\log y)^2}{2\sigma^2}\right)$$

Exercise 2.3

$f_Y(y) = P(Y=y) = P(\frac{X}{X+1} = y) = P(X = \frac{y}{1-y}) = \frac{1}{3}\left(\frac{2}{3}\right)^{y/(1-y)}$

Exercise 2.23

(a) Note that $y \in [0,1)$,

$$F_Y(y) = P(Y<y) = P(X^2<y) = P(-\sqrt{y}<X<\sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}}(1+x)/2dx = \sqrt{y}$$

Then

$$f_Y(y) = \frac{d}{dy}F_Y = \frac{1}{2}\frac{1}{\sqrt{y}}$$

(b)

$$EY = \int_0^1 \frac{y}{2}\frac{1}{\sqrt{y}} = \frac{1}{3}$$

$EY^2 = \int_0^1 \frac{y^2}{2}\frac{1}{\sqrt{y}} = \frac{1}{5}$

$$VarY = EY^2 - (EY)^2 = 1/5 - (1/3)^2 = 4/45$$