Chapter 1: Probability Theory Exercises

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Exercise 1.1

(a) $S = \{ssss | s \in \{H, T\}\}$, a string of length 4 with alphabet H and T

(b) $S = \mathbb{N} \cup \{0\}$ since damaged leaves are non-negative whole number

(c) $S = \mathbb{N} \cup \{0\}$ since we count in hours which is a non-negative whole number

(d) $S = \mathbb{R}^+$ since weight can be any positive real number

(e) $S = [0, 1]$ fraction is between 0 and 1

Exercise 1.4

(a) $P(A \cup B \cup (A \cap B)) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Note $A \cap B \subset A$

(b)
$$\begin{aligned} P((A \cup B) \cap (A \cap B)^c) &= P(A \cup B) + P((A \cap B)^c) - P(A \cup B \cup (A \cap B)^c) \\ &= P(A \cup B) + 1 - P(A \cap B) - 1 \\ &= P(A) + P(B) - 2P(A \cap B) \end{aligned}$$

(c) $P(A \cup B) = P(A) + P(B) - P(\cup B)$

(d) $P(A) + P(B) - 2P(A \cap B)$

Exercise 1.5

(a) Note that $A \subset C$. Therefore $A \cap C = A$. Hence $A \cap B \cap C = A \cap B$ = { a U.S. birth results in identica twins and both being females }.

(b) Let D by the event of a twin birth.

$$P(A \cap B \cap C) = P(A \cap B) = P(A|B, D)P(B|D)P(D) = \frac{1}{2}\frac{1}{3}\frac{1}{90} = \frac{1}{540}$$

Exercise 1.6

We have $p_0 = (1-u)(1-w)$, $p_1 = u(1-w) + w(1-u)$, $p_2 = uw$. Also $p_0 = p_1 = p_2 = p$. Therefore we have 3 variables and 3 equations.

$$\begin{aligned} uw - u - w + 1 &= p \\ -2uw + u + w &= p \\ uw &= p \end{aligned}$$

We get $p = \frac{1}{3}$, $uw = \frac{1}{3}$, $u + w = 1$. This only has imaginary solution. Hence there is no solution for w and u which satisfy the conditions.

Exercise 1.7

(a) The dart board has area of πr^2. The probability of scoring i points is

$$P(X = i) = \begin{cases} \frac{A - \pi r^2}{A}, & i = 0 \\ \frac{\pi r^2}{A}\left[\left(\frac{6-i}{5}\right)^2 - \left(\frac{5-i}{5}\right)^2\right], & i = 1, 2, 3, 4, 5 \end{cases}$$

(b)

$$P(\text{i points}|\text{board hit}) = \frac{P(\text{i points and hit board})}{P(\text{hit board})} = \frac{\frac{\pi r^2}{A}\left[\left(\frac{6-i}{5}\right)^2 - \left(\frac{5-i}{5}\right)^2\right]}{\pi r^2/A} = \left(\frac{6-i}{5}\right)^2 - \left(\frac{5-i}{5}\right)^2$$

Exercise 1.8

(a) See 1.7 (a)

(b) The probability of scoring $i < j$ points corresponds to area of rings with area $A_i > A_j$. therefore the probability is decreasing function.

(c) Summing the distribution in 1.7 (a) shows it is equal to 1. And the probability is greater or equal to zero since areas can't be negative.

Exercise 1.13

Note that $P(A\cup B) = P(A)+P(B)-P(A\cap B) \leq 1$. Therefore $P(A\cap B) \geq P(A)+P(B)-1 = \frac{1}{12} \neq 0$, can't be disjoint.

Exercise 1.14

Given $|S| = n$, we can order the elements such that $S = \{a_1, \dots, a_n\}$, There exists a bijection from the set of binary string of length n, $B = \{b^n | b \in \{0, 1\}\}$ to elements in power set of S where 0 at the ith position means the i-th element is not present in the subset and 1 means otherwise. For each bit in the binary string, there are two possible states 0 and 1. Therefore the total n-binary string count is 2^n. By property of bijection, power set of S also has the same number of elements.

Exercise 1.5

Suppose $n = k$ holds, a job consists of k separate tasks, the entire job can be done in $c_1 \times \cdots \times c_k$ ways. Then when $n = k+1$, denote the ways to do the job when $n = k$ as C_k. Then we have two jobs that can be done in ways C_k and c_{k+1}. Using the same argument when $k = 2$, for each way the job $n = k$ is done, there are c_{k+1} ways to do the new job, the total is $C_k \times c_{k+1} = c_1 \times \cdots \times c_{k+1}$.

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Exercise 1.16

(a) A person's name has 3 parts, so we have 3 character as initials, each with 26 choices. So 26^3 possibilities.

(b) If a person can have only 1 given name, then we have 26^2 for 2 character initials. The total possibilities together with (a) is $26^2 + 26^3$.

(c) By the same argument, for 3 given name, we have 26^4. In total $26^2 + 26^3 + 26^4$.

Exercise 1.17

Assume one piece can only have 2 different numbers and they are not order, we just need to choose 2 numbers from n. Therefore the number of combination is $\binom{n}{2} = \frac{n(n-1)}{2}$

Now consider the case when a piece have both side the same number, there are n possibilities. Therefore the total is $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$

Exercise 1.19

Taking rth partial derivatives of a n variable f, is just choosing r variables without order with replacement from n variables, which is $\binom{n-1+r}{r}$

Exercise 1.20

For the 12 calls, each call has a uniform distribution over the 7 days (with probability of 1/7). Then the count of calls grouped by day is multinomial distribution.

$$P(c_1, c_2, \dots, c_7) = \frac{12!}{c_1!c_2!\cdots c_7!}\left(\frac{1}{7}\right)^{\sum c_i} = \frac{12!}{c_1!c_2!\cdots c_7!}\left(\frac{1}{7}\right)^{12}$$

Where c_i is the number of calls in day i.

If we distribute the 12 calls such that every day gets 1 call. We have 5 calls left. So it comes down to summing the coefficient in the distribution for all possible distribution of the 5 calls.

We have

Distrubtion	Cases	Coefficient
5	7	$\frac{12!}{6!} = 665280$
4,1	$7 \times 6 = 42$	$\frac{12!}{5!2!} = 1995840$
3, 1,1	$\binom{7}{2} \times (7-2) = 105$	$\frac{12!}{4!2!2!} = 4989600$
3, 2	$7 \times 6 = 42$	$\frac{12!}{4!3!} = 3326400$
2, 2, 1	$\binom{7}{2} \times (7-2) = 105$	$\frac{12!}{3!3!2!} = 6652800$
2, 1, 1, 1	$\binom{7}{3} \times (7-3) = 140$	$\frac{12!}{3!2!2!2!} = 9979200$
1, 1, 1, 1, 1	$\binom{7}{5} = 21$	$\frac{12!}{2!2!2!2!2!} = 14968800$

If we add up the cases $\times$ coefficient and divide by 7^{12}, we get 0.22845.

Exercise 1.21

For no matching shoes, we can only choose 1 shoe from a pair, therefore we need to choose $2r$ pairs so $2r$ must be less than n. First we choose $2r$ shoes from n pairs: $\binom{n}{2r}$. For each pair, we can choose the left shoe or right shoe and we have $2r$ chosen pairs: 2^{2r}. So the total way to choose non matching shoes is $\binom{n}{2r}2^{2r}$. Total way of choosing is $\binom{2n}{2r}$. So the probability is the $\binom{n}{2r}2^{2r}/\binom{2n}{2r}$.

Exercise 1.26

Let T be the number of toss until a 6 appears. $P(T > 5) = 1 - P(T <= 5) = 1 - \sum_{t=1}^{5} \frac{5^{t-1}}{6^t} \approx 0.40$

Exercise 1.52

Integrating $g(x)$, we have

$$G(x) = \int_{-\infty}^{x} g(t) = \begin{cases} \frac{F(x)-F(x_0)}{1-F(x_0)}, x \geq x_0 \\ 0, x < x_0 \end{cases}$$

$\lim\limits_{x \to -\infty} G(x) = 0$ and $\lim\limits_{x \to \infty} G(x) = \lim\limits_{x \to \infty} \frac{F(x)-F(x_0)}{1-F(x_0)} = \frac{1-F(x_0)}{1-F(x_0)} = 1$

Since $F(x_0) < 1$ and $F(x)$ is right continuous, so $G(x)$ is also right continuous.

Exercise 1.53

$\lim\limits_{y \to -\infty} F_Y(y) = \lim\limits_{y \to 1}(1 - \frac{1}{y^2}) = 1 - 1 = 0$,

$\lim\limits_{y \to \infty} F_Y(y) = \lim\limits_{y \to 1}(1 - \frac{1}{y^2}) = 1 - 0 = 1$

$(1 - 1/x^2) - (1 - 1/y^2) = 1/y^2 - 1/x^2 > 0$ for $x > y$ Therefore F_Y is non-decreasing.

$1 - 1/y^2$ is smooth on $[1, \infty]$ hence right continuous. Therefore F_Y is a cdf.

$f_Y(y) = \frac{dF_y}{dy} = \frac{2}{y^3}$

When $Z = 10(Y - 1)$,

$$F_Z(z) = P(Z \leq z) = P(10(Y-1) \leq z) = P(Y \leq \frac{z}{10} + 1) = 1 - \frac{1}{(0.1z+1)^2}$$

,where $0 \leq z < \infty$. 0 otherwise.

Exercise 1.54

$c \int_0^{\pi/2} \sin x = c(-0 + 1) = 1$ which gives $c = 1$

$c \int_{-\infty}^{\infty} \exp -|x| = 2c \int_0^{\infty} \exp -x = 2c = 1$ which gives $c = \frac{1}{2}$

Exercise 1.55

$F_T(t) = \int_0^t 1/1.5 \exp(-s/1.5) ds = 1 - \exp(-t/1.5)$

Note that $V \in [5, \infty)$

$F_V(5) = P(V \leq 5) = P(V = 5) = P(T < 3) = 1 - \exp(-2)$

When $v \in [5, 6)$, $t \in [2.5, 3)$, Therefore $P(5 < V < 6) = 0$. By Cdf property, $P(V < 6) = P(V = 5) = 1 - \exp(-2)$

When $v \in [6, \infty)$, $t \in [3, \infty)$, therefore $F_V(v) = P(V < v) = P(2T \leq v) = P(T \leq v/2) = 1 - \exp(-v/3)$

Note that the cdf is continuous at $V = 6$.