Chapter 7: Point Estimation

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Exercise 7.1

If we take the product of $f(x|\theta)$ as the likelihood function, all 3 values of θ attain maximum likelihood of 0.

Method 1: With perturbation on likelihood function

We will perturb the likelihood function by ϵ. Now even the probability 0 entries get a probability of ϵ. Then

$$\begin{aligned}
L(\theta = 1|x+\epsilon) &= \frac{\epsilon(1/3+\epsilon)^2(1/6+\epsilon)^2}{(1+5\epsilon)^5} \\
L(\theta = 2|x+\epsilon) &= \frac{\epsilon(1/4+\epsilon)^4}{(1+5\epsilon)^5} \\
L(\theta = 3|x+\epsilon) &= \frac{\epsilon^2(1/4+\epsilon)^2(1/2+\epsilon)}{(1+5\epsilon)^5}
\end{aligned}$$

We can compare the functions by taking the ratio and letting ϵ go to 0.

$$\frac{L(\theta = 2|x)}{L(\theta = 1|x)} = \lim_{\epsilon\to 0} \frac{L(\theta = 2|x+\epsilon)}{L(\theta = 1|x+\epsilon)} = \lim_{\epsilon\to 0} \frac{(1/4+\epsilon)^4}{(1/3+\epsilon)^2(1/6+\epsilon)^2} = 1.26$$

Similarly,

$$\frac{L(\theta = 2|x)}{L(\theta = 3|x)} = \lim_{\epsilon\to 0} \frac{(1/4+\epsilon)^4}{\epsilon(1/4+\epsilon)^2(1/2+\epsilon)} = \infty$$

Therefore $\theta = 2$ is the MLE

Method 2: With +1 smoothing

we can assume N trials are performed for each θ and apply laplace smoothing (+1 smoothing).

Then the likelihood functions become

$$\begin{aligned}
L(\theta = 1|x) &= \lim_{N\to\infty} \frac{(N/3+1)^2(N/6+1)^2}{(N+5)^5} \\
L(\theta = 2|x) &= \lim_{N\to\infty} \frac{(N/4+1)^4}{(N+5)^5} \\
L(\theta = 3|x) &= \lim_{N\to\infty} \frac{(N/4+1/N)^2(N/2+1)}{(N+5)^5}
\end{aligned}$$

This is essentially the same as method 1.