

# Chapter 8: Hypothesis Testing

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## Exercise 8.1

Let  $H_0$  be the hypothesis that the coin is fair, aka  $\theta_0 = 0.5$ .

### Likelihood ratio test

The likelihood method for independent Bernoulli trial is  $L(\theta|x) = \theta^{560}(1 - \theta)^{1000-560}$  where 560 is the number of head. We know that  $\theta = \frac{560}{1000}$  is the empirical estimator of  $\theta$  that maximizes the likelihood function. So the ratio test gives

$$\log \lambda(x) = \log \frac{L(0.5|x)}{L(0.56|x)} = 1000 \log 0.5 - \{560 \log 0.56 + 440 \log 0.44\} \Rightarrow \lambda(x) \approx 0.00073$$

0.00073 is too small so  $H_0$  can be rejected. Therefore the coin is not fair.

### Check the probability of such event

Assume coin is fair  $\theta = 0.5$ , then the CDF of the process is

$$P(X \geq x) = \sum_{i=x}^{1000} P(X = i) = \sum_{i=x}^{1000} \binom{1000}{i} 0.5^i 0.5^{1000-i}$$

Then we can check if the event  $X \geq 560$  is a small event for this  $\theta$ . Indeed it is  $\approx 0.08\%$ . So the coin is not fair.

## Exercise 8.2

Let  $H_0$  be the null hypothesis that the incident number of this year is generated from  $Pois(\lambda)$  where  $\lambda < 15$ . To estimate whether the generating distribution has decreased in  $\lambda$ , we let  $\pi(\lambda) = \mathcal{N}(\mu = \frac{10+15}{2} = 12.5, \sigma^2 = (15 - 10)^2) = \frac{1}{5\sqrt{2\pi}} \exp\left(-0.5 \frac{(12.5-\lambda)^2}{5^2}\right)$  (we choose midpoint between 15 and 10 is because 10 is the MLE for the latest year's data point)

$$\begin{aligned}
P(\lambda < 15 | x = 10) &= \sum_{\lambda=0}^{14} P(\lambda | x = 10) \\
&= \frac{\sum_{\lambda=0}^{14} P(x = 10 | \lambda) \pi(\lambda)}{\sum_{\lambda=0}^{\infty} P(x = 10 | \lambda) \pi(\lambda)} \\
&= \frac{\sum_{\lambda=0}^{14} P(x = 10 | \lambda)}{\sum_{\lambda=0}^{30} P(x = 10 | \lambda)} \text{ (Let the prior } P(\lambda) = \text{Uniform}(0, 30)) \\
&= \frac{\sum_{i=0}^{14} i^{10} e^{-i}}{\sum_{i=0}^{30} i^{10} e^{-i}} \approx 0.87
\end{aligned}$$

Type I Error is about  $1 - 0.87 = 0.13$ , not small. If we compute  $P(x \leq 10 | \lambda = 15) \approx 0.11$ , so  $\lambda = 15$  is still capable of producing such result. It is inconclusive.

### Exercise 8.3

$H_0$  region is  $\theta \leq \theta_0$  and  $H_1$ 's region is  $\theta > \theta_0$ . Then define  $b = m\theta_0$  to be the expected success count if  $\theta = \theta_0$ .

A Bernoulli trial  $f(y|\theta) = I_{Y=1}\theta + I_{Y=0}(1 - \theta)$ . Then the likelihood function

$$L(\theta|y) = \prod_1^m f(y_i|\theta) = \binom{m}{k} \theta^k (1 - \theta)^{m-k}$$

where  $k = \sum_i Y_i$

To maximize  $L$ , we can use the MLE which is the  $\theta_{\max} = \frac{k}{m}$ . To reject  $H_0$ , we need the MLE to stay out  $H_0$  region, so  $\frac{k}{m} > \theta_0 \Rightarrow \sum_i Y_i = k > m\theta_0 = b$

### Exercise 8.5

(a) The likelihood function

$$L(\theta, v|x) = \prod_{i=1}^n f(x_i|\theta, v) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}} \prod_i I_{[v, \infty)}(x_i) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}}, \text{ (given } v \leq x_{\min}, 0 \text{ otherwise)}$$

Holding  $\theta$  fixed,  $L$  is a monotonic polynomial function of  $v$ . So  $v_0 = x_{(1)}$  the boundary of  $v$  maximizes  $L$ .

Let  $\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \log\left(x_{(1)}^n\right) - \log(\prod_i x_i) = 0$ , then we get

$$\theta_0 = \frac{n}{\log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)} = \frac{n}{T(x)}$$

where  $T \equiv \log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)$

(b)  $H_0 = \{(\theta = 1, v)\}$ , So the rejection region of  $H_0$  is

$$\lambda(x) = \frac{\sup_{\theta=1} L(\theta, v|x)}{\sup_{\theta} L(\theta, v|x)} = \frac{T^n}{n^n} \exp(n - T) \leq c$$

We take derivative of  $\lambda$ ,

$$\partial_T \lambda = \left(\frac{T}{n}\right)^{n-1} e^{n-T} \left(1 - \frac{T}{n}\right)$$

So the monotonicity of  $\lambda$  is determined by  $(1 - T/n)$ . When  $T = n$ ,  $\lambda$  reaches maximum of 1, when  $T < n$ ,  $\lambda$  increases monotonically and when  $T > n$ ,  $\lambda$  decreases monotonically. Therefore, if  $\lambda(x) < c$  for  $0 < c \leq 1$ , we will have two values  $c_1$  and  $c_2$  (on left/right side of  $n$  respectively) where  $T \leq c_1 \leq n$  or  $n \leq c_2 \leq T$ .