Chapter 1: Probability Theory Exercises

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Exercise 1.1

(a) $S = \{ssss | s \in \{H, T\}\}$, a string of length 4 with alphabet H and T

(b) $S = \mathbb{N} \cup \{0\}$ since damaged leaves are non-negative whole number

(c) $S = \mathbb{N} \cup \{0\}$ since we count in hours which is a non-negative whole number

(d) $S = \mathbb{R}^+$ since weight can be any positive real number

(e) $S = [0, 1]$ fraction is between 0 and 1

Exercise 1.4

(a) $P(A \cup B \cup (A \cap B)) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$ Note $A \cap B \subset A$

(b)

$$\begin{aligned} P((A \cup B) \cap (A \cap B)^c) &= P(A \cup B) + P((A \cap B)^c) - P(A \cup B \cup (A \cap B)^c) \\ &= P(A \cup B) + 1 - P(A \cap B) - 1 \\ &= P(A) + P(B) - 2P(A \cap B) \end{aligned}$$

(c) $P(A \cup B) = P(A) + P(B) - P(\cup B)$

(d) $P(A) + P(B) - 2P(A \cap B)$

Exercise 1.6

We have $p_0 = (1-u)(1-w)$, $p_1 = u(1-w) + w(1-u)$, $p_2 = uw$. Also $p_0 = p_1 = p_2 = p$. Therefore we have 3 variables and 3 equations.

$$\begin{aligned} uw - u - w + 1 &= p \\ -2uw + u + w &= p \\ uw &= p \end{aligned}$$

We get $p = \frac{1}{3}$, $uw = \frac{1}{3}$, $u + w = 1$. This only has imaginary solution. Hence there is no solution for w and u which satisfy the conditions.

Exercise 1.13

Note that $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$. Therefore $P(A \cap B) \geq P(A) + P(B) - 1 = \frac{1}{12} \neq 0$, can't be disjoint.

Exercise 1.14

Given $|S| = n$, we can order the elements such that $S = \{a_1, \ldots, a_n\}$, There exists a bijection from the set of binary string of length n, $B = \{b^n | b \in \{0, 1\}\}$ to elements in power set of S where 0 at the ith position means the i-th element is not present in the subset and 1 means otherwise. For each bit in the binary string, there are two possible states 0 and 1. Therefore the total n-binary string count is 2^n. By property of bijection, power set of S also has the same number of elements.

Exercise 1.19

Taking rth partial derivatives of a n variable f, is just choosing r variables without order with replacement from n variables, which is $\binom{n-1+r}{r}$

Exercise 1.21

For no matching shoes, we can only choose 1 shoe from a pair, therefore we need to choose $2r$ pairs so $2r$ must be less than n. First we choose $2r$ shoes from n pairs: $\binom{n}{2r}$. For each pair, we can choose the left shoe or right shoe and we have $2r$ chosen pairs: 2^{2r}. So the total way to choose non matching shoes is $\binom{n}{2r}2^{2r}$. Total way of choosing is $\binom{2n}{2r}$. So the probability is the $\binom{n}{2r}2^{2r}/\binom{2n}{2r}$.

Exercise 1.26

Let T be the number of toss until a 6 appears. $P(T > 5) = 1 - P(T <= 5) = 1 - \sum_{t=1}^{5} \frac{5^{t-1}}{6^t} \approx 0.40$

Exercise 1.52

Integrating $g(x)$, we have

$$G(x) = \int_{-\infty}^{x} g(t) = \begin{cases} \frac{F(x)-F(x_0)}{1-F(x_0)}, x \geq x_0 \\ 0, x < x_0 \end{cases}$$

$\lim\limits_{x\to-\infty} G(x) = 0$ and $\lim\limits_{x\to\infty} G(x) = \lim\limits_{x\to\infty} \frac{F(x)-F(x_0)}{1-F(x_0)} = \frac{1-F(x_0)}{1-F(x_0)} = 1$

Since $F(x_0) < 1$ and $F(x)$ is right continuous, so $G(x)$ is also right continuous.

Exercise 1.53

$\lim\limits_{y\to-\infty} F_Y(y) = \lim\limits_{y\to 1}(1 - \frac{1}{y^2}) = 1 - 1 = 0$,

$\lim\limits_{y\to\infty} F_Y(y) = \lim\limits_{y\to 1}(1 - \frac{1}{y^2}) = 1 - 0 = 1$

$(1 - 1/x^2) - (1 - 1/y^2) = 1/y^2 - 1/x^2 > 0$ for $x > y$ Therefore F_Y is non-decreasing.

$1 - 1/y^2$ is smooth on $[1, \infty]$ hence right continuous. Therefore F_Y is a cdf.

$f_Y(y) = \frac{dF_y}{dy} = \frac{2}{y^3}$

When $Z = 10(Y - 1)$,

$$F_Z(z) = P(Z \leq z) = P(10(Y-1) \leq z) = P(Y \leq \frac{z}{10} + 1) = 1 - \frac{1}{(0.1z+1)^2}$$

,where $0 \leq z < \infty$. 0 otherwise.

Exercise 1.54

$c\int_0^{\pi/2} \sin x = c(-0 + 1) = 1$ which gives $c = 1$

$c\int_{-\infty}^{\infty} \exp -|x| = 2c\int_0^{\infty} \exp -x = 2c = 1$ which gives $c = \frac{1}{2}$

Exercise 1.55

$F_T(t) = \int_0^t 1/1.5 \exp(-s/1.5)ds = 1 - \exp(-t/1.5)$

Note that $V \in [5, \infty)$

$F_V(5) = P(V \leq 5) = P(V = 5) = P(T < 3) = 1 - \exp(-2)$

When $v \in [5, 6)$, $t \in [2.5, 3)$, Therefore $P(5 < V < 6) = 0$. By Cdf property, $P(V < 6) = P(V = 5) = 1 - \exp(-2)$

When $v \in [6, \infty)$, $t \in [3, \infty)$, therefore $F_V(v) = P(V < v) = P(2T \leq v) = P(T \leq v/2) = 1 - \exp(-v/3)$

Note that the cdf is continuous at $V = 6$.