Chapter 4: Multiple Random Variables

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Exercise 4.1

(a) Since $f_{X,Y}(x, y)$ is constant. $X^2 + Y^2 < 1$ is circle of radius 1. Therefore $P(X^2 + Y^2 < 1) = \pi/4$

(b) $2X - Y = 0$ divides the unit square into two region of equal area and f is constant. Therefore $P(2X - Y > 0) = 1/2$

(c) $P(|X + Y| < 2) = P(-2 < X + Y < 2)$. The area covers the entire square. Therefore $P(|X + Y| < 2) = 1$.

Exercise 4.4

(a) Since $\int_0^1 \int_0^2 f(x, y)dxdy = \int_0^1 \int_0^2 C(x + 2y)dxdy = 4C = 1$ So $C = 1/4$.

(b) $f(x) = \int_0^1 f(x, y)dy = (1/4)(xy + y^2)|_0^1 = \frac{x+1}{4}, \;\; x \in (0, 2)$

(c) For $(x, y) \in (0, 2) \times (0, 1)$:

$$F(x, y) = P(X < x, Y < y) = \int_{-\infty}^{x} \int_{\infty}^{y} \frac{t + 2s}{4} dsdt = \int_0^x \int_0^y \frac{t + 2s}{4} dsdt = \frac{1}{8}(x^2 y + 2xy^2)$$

For $(x, y) \in (0, 2) \times [1, \infty)$:

$$F(x, y) = P(X < x, Y < y) = \int_0^x \int_0^1 \frac{t + 2s}{4} dsdt = \frac{1}{8}(x^2 + 2x)$$

For $(x, y) \in (-\infty, 2] \times (0, 1)$:

$$F(x, y) = P(X < x, Y < y) = \int_0^2 \int_0^y \frac{t + 2s}{4} dsdt = \frac{1}{2}(y + y^2)$$

(d) from (b), we have $f(x) = \frac{x+1}{4}$. And $z = \frac{9}{(x+1)^2}$ is monotonic for $x \in [0, 2]$ with $z \in [1, 9]$. So we can take $x = \frac{\sqrt{z}}{3} - 1$. Then

$$f(z) = f(x^{-1}(z)) \left| \frac{dx}{dz} \right| = \frac{3}{4}(z^{-1/2})(\frac{3}{2}z^{-3/2}) = \frac{9}{8}z^{-2}$$

Exercise 4.5

(a) The area for integration is $0 < x < 1$ and $0 < y < x^2$.

$$P(X > \sqrt{Y}) = \int_0^1 \int_0^{x^2} x + y dy dx = \int_0^1 x^3 + \frac{x^4}{2} dx = 0.35$$

(b) The area of integration is $0 < x < 1$ and $x^2 < y < x$.

$$P(X^2 < Y < X) = \int_0^1 \int_{x^2}^{x} 2x dy dx = \int_0^1 2x^2 - 2x^3 dx = \frac{1}{6}$$

Exercise 4.6

Let X, Y be the time A and B arrive in time interval $[0, 1]$. Since they are independent, $f(x, y) = f(x)f(y) = 1$ for $(x, y) \in [0, 1] \times [0, 1]$.

Let T be the length of time A waits for B. Then $T = \max(Y - X, 0)$ because $T = 0$ when $Y < X$.

$$P(T < t) = P(\max(Y - X, 0)) = P(Y - X < t, Y \geq X) + P(Y < X)$$

For term $P(Y - X < t, Y \geq X)$, The area of integration is the area between $y - x = t$ and $y \geq x$ bounded by unit square. We can find the complement area which is an isosceles right triangle with side of $1 - t$, which gives

$$P(Y - X < t, Y \geq X) = \frac{1}{2} - \frac{1}{2}(1 - t)^2$$

$P(Y < X)$ is the lower half triangle of the unit square which has area of $\frac{1}{2}$ Therefore

$$P(T < t) = P(Y - X < t, Y \geq X) + P(Y < X) = 1 - \frac{1}{2}(1 - t)^2$$

Exercise 4.7

We can formulate the problem as such: $X \in [0, 30], Y \in [40, 50]$, find $P(X + Y < 60)$. We want to find the intersection of $x + y = 60$ with $[0, 30] \times [40, 50]$. We get $(10, 50), (20, 40)$. Since the distributions are uniform, we can simply find area of the trapezoid and divide it by the total area.

$$P(X + Y < 60) = \frac{10(10 + 20)0.5}{10(30)} = 150/300 = 0.5$$