Chapter 1: Calculus on Euclidean Space

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1.1.1

(a) $fg^2 = x^2y(y\sin z)^2 = x^2y^3\sin^2 z$

(b) $g\partial_x f + f\partial_y g = y\sin z(2xy) + x^2y(\sin z) = (2xy^2 + x^2y)\sin z$

(c) $\partial^2_{yz}(fg) = \partial^2_{yz}(x^2y^2\sin z) = 2x^2y\cos z$

(d) $\partial_y(\sin f) = \partial_y \sin(x^2y) = x^2\cos(x^2y)$

1.1.2

(a) 0

(b) $3^2(-1) - (1)0.5 = -9 - 0.5 = 9.5$

(c) $a^2 - (1-a) = a^2 + a - 1$

(d) $t^2t^2 - t^4t^3 = t^4 - t^7$

1.1.3

(a) $\partial_x(x\sin(xy) + y\cos(xz)) = \sin(xy) + xy\cos(xy) - yz\sin(xz)$

(b)

$$\partial_x f = \frac{\partial f}{\partial g}\frac{\partial g}{\partial h}\frac{\partial h}{\partial x} = (\cos g)(e^h)(2x) = 2xe^{x^2+y^2+z^2}\cos\left(e^{x^2+y^2+z^2}\right)$$

1.1.4

Since $h = x^2 - yz$, so $h(g_1, g_2, g_3) = g_1^2 - g_2g_3$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial h}{\partial g_1}\frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial g_2}\frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial g_3}\frac{\partial g_3}{\partial x} \\ &= 2g_1\frac{\partial g_1}{\partial x} - g_3\frac{\partial g_2}{\partial x} - g_2\frac{\partial g_3}{\partial x}\end{aligned}$$

(a) $2(x+y)(1) - (x+z)(0) - y^2(1) = 2(x+y) - y^2$

(b) $2e^z(0) - e^x(e^{x+y}) - e^{x+y}(e^x) = -2e^xe^{x+y}$

(c) $2x(1) - x(-1) + x(1) = 0$

1.2.1

(a) $3v_p - 2w_p = 3(-2, 1, -1) - 2(0, 1, 3) = (-6, 1, -9) = -6U_1 + U_2 - 9U_3$

1.2.2

$W - xV = 2x^2U_2 - U_3 - x(x^2U_1 + xyU_2) = -x^3U_1 + (2x^2 - x^2y)U_2 - U_3$

At $p = (-1, 0, 2)$,

$(W-xV)(p) = -(-1)^3U_1(p)+(2(-1)^2-(-1)^2 0)U_2(p)-U_3(p) = U_1(p)+2U_2(p)-U_3(p) = (1, 2, -1)$

1.2.3

(a) $V = \frac{1}{7}(2z^2U_1 - xyU_3) = \frac{2z^2}{7}U_1 - \frac{xy}{7}U_3$

(b) $V = p_1U_1 + (p_3 - p_1)U_2$

(c) $V = xU_1 + 2yU_2 + xy^2U_3$

(d) $V = (1+p_1, p_2p_3, p_2)-(p_1, p_2, p_3) = (1, p_2(p_3-1), p_2-p_3) = U_1+p_2(p_3-1)U_2+(p_2-p_3)U_3$

(e) $V = 0 - p = -p_1U_1 - p_2U_2 - p_3U_3$

1.2.4

If $fV + gW = f(y^2U_1 - x^2U_3) + g(x^2U_1 - zU_2)$, the coefficient for U_1 is $fy^2 + gx^2 = 0$.

1.2.5

(a) Suppose $aV_1 + bV_2 + cV_3 = 0$, then $(a + cx)U_1 + bU_2 + (-ax + c)U_3 = 0$. By independence of the natural basis, We can see $b = 0$. Moreover $a + cx = 0$ and $ax - c = 0$ for all x. Take $x = 0$, we get $c = 0$ and $a = 0$ follows. Therefore they are linearly independent at each point.

(b) We can write V in terms of U in matrix form.

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

Then

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Then

$$\begin{aligned} xU_1 + yU_2 + zU_3 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \\ &= \frac{1}{1+x^2} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} -x^2 & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \\ &= -\frac{x^3 + xz}{1+x^2} V_1 + \frac{y}{1+x^2} V_2 + \frac{x^2 + z}{1+x^2} V_3 \end{aligned}$$

■

1.3.1

(a) $v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0} = \frac{d}{dt} f(2 + 2t, -t, -1 + 3t)|_{t=0} = \frac{d}{dt}(t^2(3t - 1))|_{t=0} = 0$

(b) $v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0} = \frac{d}{dt} f(2 + 2t, -t, -1 + 3t)|_{t=0} = 14(2 + 2t)^6|_{t=0} = 896$

(c) $v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0} = \frac{d}{dt} f(2 + 2t, -t, -1 + 3t)|_{t=0} = \frac{d}{dt} e^{2+2t} \cos t|_{t=0} = 2e^{2+2t} \cos t|_{t=0} = 2e^2$

1.3.2

(a) $v_p[f] = \partial_x f(p)v_1 + \partial_y f(p)v_2 + \partial_z f(p)v_3 = 0 + 2yz(p)(-1) + y^2(p)(3) = 0$

(b) $v_p[f] = \partial_x f(p)v_1 + \partial_y f(p)v_2 + \partial_z f(p)v_3 = 7x^6(p)2 = 896$

(c) $v_p[f] = e^x \cos y|_p v_1 - e^x \sin y|_p v_2 = 2e^2$

1.3.3

Note that $V = y^2 U_1 - xU_3 = y^2\partial_x - x\partial_z$

(a) $V[f] = y^2\partial_x(xy) - x\partial_z(xy) = y^3$

(b) $V[g] = y^2\partial_x z^3 - x\partial_z z^3 = 3xz^2$

(c) $V[fg] = y^2\partial_x(xyz^3) - x\partial_z(xyz^3) = y^2yz^3 - x(3xyz^2) = y^3z^3 - 3x^2yz^2$

(d) $fV[g] - gV[f] = xy(3xz^2) - z^3y^3 = 3x^2yz^2 - y^3z^3$

(e) $V[f^2 + g^2] = V[f^2] + V[g^2] = 2fV[f] + 2gV[g] = 2xy(y^3) + 2z^3(3xz^2) = 2xy^4 + 6xz^5$

(f) $V[V[f]] = V[y^3] = y^2\partial_x y^3 - x\partial_z y^3 = 0$

1.3.4

For any point p, $V_p = \sum_i v_i(p)U_i(p)$. Then $V_p[x_j] = \sum_i v_i(p)U_i(p)[x_j] = \sum_i v_i(p)\delta_{ij} = v_j(p)$.

■

1.3.5

Note that $V = \sum_i v_i U_i$ and $W = \sum_i w_i U_i$. Since $V[f] = W[f]$ for every f, take $f = x_j$, we get $v_j = w_j$ for every j. Hence $V = W$

■

1.4.1

Since $\alpha(t) = (1 + \cos t, \sin t, 2\sin(t/2))$. $\alpha'(t) = (\sin t, \cos t, \cos(t/2))$.

$t = 0, \alpha'(0) = (0, 1, 1)$

$t = \frac{\pi}{2}, \alpha'(\pi/2) = (1, 0, \sqrt{2}/2)$

$t = \pi, \alpha'(\pi) = (0, -1, 0)$

1.4.2

$\alpha(t) = \int \alpha'(t)dt = (t^3/3, t^2/2, e^t) + C$.

Since $\alpha(0) = (0,0,1) + C = (1,0,5)$, so $C = (1,0,4)$.

$\alpha(t) = (t^3/3 + 1, t^2/2, e^t + 4)$

1.4.3

Since $\alpha(t) = (1 + \cos t, \sin t, 2\sin(t/2))$ and $h(s) = \cos^{-1} s$. Therefore

$$\beta(s) = \alpha(h(s)) = \left(1 + s, \sin\cos^{-1} s, 2\sin\left(\frac{\cos^{-1} s}{2}\right)\right)$$

Note that $s \in (0,1)$ meaning $\cos^{-1} s = h$ can be positive or negative. So $\sin\cos^{-1} s = \pm\sqrt{1-s^2}$. Similarly, $2\sin\left(\frac{\cos^{-1} s}{2}\right) = \pm 2\sqrt{\frac{1-s}{2}}$ by half angle formula of $\sin$. By restricting $h \geq 0$, we get

$$\beta(s) = \left(1 + s, \sqrt{1-s^2}, 2\sqrt{\frac{1-s}{2}}\right)$$

1.4.4

$\beta = \alpha(h(s)) = (s, s^{-1}, \sqrt{2}\log s)$. Then $\beta'(s) = (1, -s^{-2}, \sqrt{2}s^{-1})$.

By lemma 4.5 $\beta'(s) = (dh/ds)\alpha'(h(s)) = s^{-1}(e^t, -e^{-t}, \sqrt{2})|_{t=h(s)} = (1, -s^{-2}, \sqrt{2}s^{-1})$

1.4.5

$l_1 : (1, -3, -1) + t(6 - 1, 2 + 3, 1 + 1) = (5t + 1, 5t - 3, 2t - 1)$.

$l_2 : (-1, 1, 0) + s(-5 + 1, -1 - 1, -1) = (-4s - 1, -2s + 1, -s)$.

Suppose they meet, then we have

$$5t + 1 = -4s - 1 \quad (1)$$
$$5t - 3 = -2s + 1 \quad (2)$$
$$2t - 1 = -s \quad (3)$$
$$(4)$$

Solving the first two equation, we have $t = 2, s = -3$. Putting them into the 3rd equation, we get $3 = 3$ which is consistent. They do meet.

1.4.6

For any curve $\alpha(t)$ with initial velocity of v_p. Then $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$ by Lemma 4.6. Evaluating at 0, we get

$$\begin{aligned}
\alpha'(0)[f] = v_p[f] &= \frac{d(f(\alpha))}{dt}(0) \\
&= \sum_i \frac{\partial f}{\partial x_i}(\alpha(0))\frac{d\alpha_i}{dt}(0) \\
&= \sum_i \frac{\partial f}{\partial x_i}(\alpha(0))\alpha_i'(0) \\
&= \sum_i \frac{\partial f}{\partial x_i}(p)[v_p]_i \\
&= \sum_i \frac{\partial f}{\partial x_i}(p)\frac{d([p+tv_p]_i)}{dt} \\
&= \sum_i \frac{\partial f}{\partial x_i}(p+tv_p)\frac{d([p+tv_p]_i)}{dt}\bigg|_{t=0} \\
&= \frac{df}{dt}(p+tv_p)\bigg|_{t=0}
\end{aligned}$$

1.4.7

(a) $\frac{d}{dt}(t, 1+t^2, t)\Big|_{t=0} = (1, 2(0), 1) = (1, 0, 1)$ at point $(0, 1, 0)$.

$\frac{d}{dt}(\sin t, \cos t, t)\Big|_{t=0} = (1, 0, 1)$ at point $(0, 1, 0)$.

$\frac{d}{dt}(\sinh t, \cosh t, t)\Big|_{t=0} = (\cosh(0), \sinh(0), 1) = (1, 0, 1)$ at point $(0, 1, 0)$.

(b) $f = x^2 - y^2 + z^2$

$f(t) = f(t, 1+t^2, t) = t^2 - (1+t^2)^2 + t^2$. Then $\frac{df}{dt}\Big|_0 = 4t - 4t(1+t^2)\Big|_0 = 0$

$f(t) = f(\sin t, \cos t, t) = \sin^2 t - \cos^2 t + t^2$. Then $\frac{df}{dt}\Big|_0 = 2\sin^t \cos t + 2\cos t \sin t + 2t\Big|_0 = 0$.

$f(t) = f(\sinh t, \cosh t, t) = \sinh^2 t - \cosh^2 t + t^2$. Then $\frac{df}{dt}\Big|_0 = 2\sinh t\cosh t - 2\cosh t \sinh t + 2t\Big|_0 = 2t\Big|_0 = 0$

1.4.8

(a)$x = \frac{1}{2}\cos t$, $y = \sin t$.

(b) $x = t$, $y = (1 - 3t)/4$.

(c) $x = t$, $y = e^t$.

1.4.9

$\alpha(t) = (2\cos t, 2\sin t, t)$, $\alpha'(t) = (-2\sin t, 2\cos t, 1)$.

Line at 0 is $u \to (2, 0, 0) + u(0, 2, 1)$

Line at $\pi/4$ is $v \to (\sqrt{2}, \sqrt{2}, \pi/4) + v(-\sqrt{2}, \sqrt{2}, 1)$

1.5.1

$p = (0, -2, 1)$, $v_p = (1, 2, -3)$.

(a) $(y^2 dx)(v_p) = y^2(p)dx(v_p) = (-2)^2(1) = 4$

(b) $(zdy - ydz)(v_p) = z(p)dy(v_p) - y(p)dz(v_p) = (1)(2) - (-2)(-3) = -4$

(c) $[(z^2 - 1)dx - dy + x^2 dz](v_p) = (z^2 - 1)(p)dx(v_p) - dy(v_p) + x^2(p)dz(v_p) = -2$

1.5.2

For any point p,

$$\begin{aligned}
\phi(V_p) &= (\sum_i f_i dx_i)(V_p) \\
&= \sum_i f_i(p) dx_i(V_p) \\
&= \sum_i f_i(p) dx_i(\sum_j v_j U_j) \\
&= \sum_i f_i(p) \sum_j v_j dx_i(U_j) \\
&= \sum_i f_i(p) \sum_j v_j \delta_{ij} \\
&= \sum_i f_i(p) v_i
\end{aligned}$$

Hence $\phi(V) = \sum_i f_i v_i$.

1.5.3

$\phi = x^2dx - y^2dz$

(a) For $V = xU_1 + yU_2 + zU_3$, $\phi(V) = x^3 - y^2z$.

(b) For $W = xy(U_1 - U_3) + yz(U_1 - U_2) = (xy + yz)U_1 - yzU_2 - xyU_3$, $\phi(W) = x^2y(x+z) + xy^3$

(c) For $T = (1/x)V + (1/y)W$,

$$\phi(T) = \phi((1/x)V + (1/y)W) = (1/x)\phi(V) + (1/y)\phi(W) = \frac{x^3 - y^2z}{x} + x^2(x+z) + xy^2$$

1.5.4

(a) Since $df^2 = 2fdf$. Suppose $df^n = nf^{n-1}df$,

$$df^{n+1} = d(ff^n) = f^ndf + fd(f^n) = f^ndf + f(nf^{n-1}df) = (n+1)f^ndf$$

. Therefore by induction, $df^n = nf^{n-1}df$. As a result,

$$df^4 = 4f^3df$$

(b) $df = d(\sqrt{f}\sqrt{f}) = 2\sqrt{f}d(\sqrt{f})$ Then

$$d(\sqrt{f}) = \frac{1}{2\sqrt{f}}df$$

(c) $d(\log(1+f^2)) = \frac{1}{1+f^2}d(1+f^2) = \frac{2f}{1+f^2}df$

1.5.5

(a)

$$df = \sum_i \partial_i f dx_i = \frac{xdx + ydy + zdz}{\sqrt{x^2+y^2+z^2}}$$

(b)

$$df = \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}dx + \frac{1}{x}dy) = \frac{1}{x^2+y^2}(-ydx + xdy)$$

1.5.6

$p = (0, -2, 1)$, $v_p = (1, 2, -3)$.

(a) $df = y^2dx + (2xy - z^2)dy - 2yzdz$. Then $df(v_p) = (-2)^2(1) + (-1)(2) - 2(-2)(-3) = 10$

(b) $df = \exp(yz)(dx + xzdy + xydz)$. Then $df(v_p) = \exp(-2)$.

(c) $df = (y\cos(xy)\cos(xz) - z\sin(xy)\sin(xy))dx + x\cos(xy)\cos(xz)dy - x\sin(xy)\sin(xz)dz$. Then $df(v_p) = y(p)dx(v_p) = (-2)1 = -2$.

1.5.7

$\phi(v_p) = \sum_i f_i(p)v_i$ for $\phi = \sum_i f_i dx_i$.

(a) Yes. $f_1(p) = 1$, $f_3(p) = -1$, $\phi = dx - dz$

(b) No. Because $dx_i(v_p)$ must involve v_i.

(c) Yes. $f_1(p) = p_3$, $f_2(p) = p_1$. So $\phi = zdx + xdy$.

(d) Yes. $df(v_p) = v_p(f) = v_p[x^2 + y^2]$. Therefore $f = x^2 + y^2$. $\phi = df = 2xdx + 2ydy$.

(e) Yes. $\phi = 0$.

(f) No. Because $dx_i(v_p)$ must involve v_i.

1.5.8

By definition of d. $df(v_p) = v_p[f]$. Then by theorem 3.3

$$d(fg)(v_p) = v_p(fg) = f(p)v_p(g) + g(p)v_p(f) = f(p)dg(v_p) + g(p)df(v_p)$$

1.5.9

Since $df = -2xydx + (1 - x^2 - 2yz)dy + (1 - y^2)dz$, we need to find points such that $xy = 0, 1 - x^2 - 2yz = 0, 1 - y^2 = 0$. From the last equation, we get $y = \pm 1$. Then the first equation gives $x = 0$. Putting the values into the 2nd equations, we have $z = \frac{1-x^2}{2y} = \pm\frac{1}{2}$. So the critical points are $(0, 1, 1/2)$ and $(0, -1, -1/2)$

1.5.10

Suppose p is a local maxima of f and p is not a critical point of f, then there exists a tangent vector v_p such that

$$df(v_p) = \frac{df}{dt}(p + tv_p)\bigg|_0 = \lim_{t\to 0} \frac{f(p+tv_p) - f(p)}{t} > 0$$

This is true in general as we can always take the opposite direction $-v_p$ if $df(v_p) < 0$. Then we have $\frac{f(p+t_0 v_p)-f(p)}{t_0} > \epsilon$ for some $\epsilon > 0$ and t_0 which contradicts with the assumption that p is local maxima of f. Same argument applies to local minima of f.

$\blacksquare$

1.5.11

(a) $(df)(v_p) = \sum_i \frac{\partial f}{\partial x_i} v_i$. Taylor expanding $f(u)$ around p, we get $f(u) = f(p) + \sum_i \partial_i f(p)(u_i - p_i) + O(u^2)$. Let $u = p + v$, then

$$f(p+v) = f(p) + \sum_i \partial_i f(p) v_i + O_2 = f(p) + df(v_p) + O_2$$

. Where O_2 is second order error. Therefore $f(p+v) - f(p) \approx df(v_p)$ in the first order.

$\blacksquare$

(b)

$$df = (2xy/z)dx + (x^2/z)dy - (x^2 y/z^2)dz$$

. $p = (1, 1.5, 1)$ and $v_p = (-0.1, 0.1, 0.2)$. We get $df(v_p) = 2(1.5)(-0.1) + 0.1 - (1.5)(0.2) = -0.3 + 0.1 - 0.3 = -0.5$.

Direct calculation gives $f(0.9, 1.6, 1.2) - f(1, 1.5, 1) = 1.08 - 1.5 = -0.42$