Chapter 1: Calculus on Euclidean Space

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1.1.1

(a) $fg^2 = x^2y(y\sin z)^2 = x^2y^3\sin^2 z$

(b) $g\partial_x f + f\partial_y g = y\sin z(2xy) + x^2y(\sin z) = (2xy^2 + x^2y)\sin z$

(c) $\partial^2_{yz}(fg) = \partial^2_{yz}(x^2y^2\sin z) = 2x^2y\cos z$

(d) $\partial_y(\sin f) = \partial_y \sin(x^2y) = x^2\cos(x^2y)$

1.1.2

(a) 0

(b) $3^2(-1) - (1)0.5 = -9 - 0.5 = 9.5$

(c) $a^2 - (1-a) = a^2 + a - 1$

(d) $t^2t^2 - t^4t^3 = t^4 - t^7$

1.1.3

(a) $\partial_x(x\sin(xy) + y\cos(xz)) = \sin(xy) + xy\cos(xy) - yz\sin(xz)$

(b)
$$\partial_x f = \frac{\partial f}{\partial g}\frac{\partial g}{\partial h}\frac{\partial h}{\partial x} = (\cos g)(e^h)(2x) = 2xe^{x^2+y^2+z^2}\cos\left(e^{x^2+y^2+z^2}\right)$$

1.1.4

Since $h = x^2 - yz$, so $h(g_1, g_2, g_3) = g_1^2 - g_2 g_3$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial h}{\partial g_1}\frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial g_2}\frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial g_3}\frac{\partial g_3}{\partial x} \\ &= 2g_1\frac{\partial g_1}{\partial x} - g_3\frac{\partial g_2}{\partial x} - g_2\frac{\partial g_3}{\partial x}\end{aligned}$$

(a) $2(x+y)(1) - (x+z)(0) - y^2(1) = 2(x+y) - y^2$

(b) $2e^z(0) - e^x(e^{x+y}) - e^{x+y}(e^x) = -2e^x e^{x+y}$

(c) $2x(1) - x(-1) + x(1) = 0$

1.2.1

(a) $3v_p - 2w_p = 3(-2,1,-1) - 2(0,1,3) = (-6,1,-9) = -6U_1 + U_2 - 9U_3$

1.2.2

$W - xV = 2x^2U_2 - U_3 - x(x^2U_1 + xyU_2) = -x^3U_1 + (2x^2 - x^2y)U_2 - U_3$

At $p = (-1,0,2)$,

$(W - xV)(p) = -(-1)^3U_1(p) + (2(-1)^2 - (-1)^2 0)U_2(p) - U_3(p) = U_1(p) + 2U_2(p) - U_3(p) = (1,2,-1)$

1.2.3

(a) $V = \frac{1}{7}(2z^2U_1 - xyU_3) = \frac{2z^2}{7}U_1 - \frac{xy}{7}U_3$

(b) $V = p_1U_1 + (p_3 - p_1)U_2$

(c) $V = xU_1 + 2yU_2 + xy^2U_3$

(d) $V = (1+p_1, p_2p_3, p_2) - (p_1, p_2, p_3) = (1, p_2(p_3-1), p_2-p_3) = U_1 + p_2(p_3-1)U_2 + (p_2-p_3)U_3$

(e) $V = 0 - p = -p_1U_1 - p_2U_2 - p_3U_3$

1.2.4

If $fV + gW = f(y^2U_1 - x^2U_3) + g(x^2U_1 - zU_2)$, the coefficient for U_1 is $fy^2 + gx^2 = 0$.

1.2.5

(a) Suppose $aV_1 + bV_2 + cV_3 = 0$, then $(a + cx)U_1 + bU_2 + (-ax + c)U_3 = 0$. By independence of the natural basis, We can see $b = 0$. Moreover $a + cx = 0$ and $ax - c = 0$ for all x. Take $x = 0$, we get $c = 0$ and $a = 0$ follows. Therefore they are linearly independent at each point.

(b) We can write V in terms of U in matrix form.

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

Then

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Then

$$\begin{aligned} xU_1 + yU_2 + zU_3 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \\ &= \frac{1}{1+x^2} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} -x^2 & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \\ &= -\frac{x^3 + xz}{1+x^2} V_1 + \frac{y}{1+x^2} V_2 + \frac{x^2 + z}{1+x^2} V_3 \end{aligned}$$

■

1.3.1

(a) $v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0} = \frac{d}{dt} f(2 + 2t, -t, -1 + 3t)|_{t=0} = \frac{d}{dt}(t^2(3t - 1))|_{t=0} = 0$

(b) $v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0} = \frac{d}{dt} f(2 + 2t, -t, -1 + 3t)|_{t=0} = 14(2 + 2t)^6|_{t=0} = 896$

(c) $v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0} = \frac{d}{dt} f(2 + 2t, -t, -1 + 3t)|_{t=0} = \frac{d}{dt} e^{2+2t} \cos t|_{t=0} = 2e^{2+2t} \cos t|_{t=0} = 2e^2$

1.3.2

(a) $v_p[f] = \partial_x f(p)v_1 + \partial_y f(p)v_2 + \partial_z f(p)v_3 = 0 + 2yz(p)(-1) + y^2(p)(3) = 0$

(b) $v_p[f] = \partial_x f(p)v_1 + \partial_y f(p)v_2 + \partial_z f(p)v_3 = 7x^6(p)2 = 896$

(c) $v_p[f] = e^x \cos y|_p v_1 - e^x \sin y|_p v_2 = 2e^2$

1.3.3

Note that $V = y^2 U_1 - xU_3 = y^2\partial_x - x\partial_z$

(a) $V[f] = y^2\partial_x(xy) - x\partial_z(xy) = y^3$

(b) $V[g] = y^2\partial_x z^3 - x\partial_z z^3 = 3xz^2$

(c) $V[fg] = y^2\partial_x(xyz^3) - x\partial_z(xyz^3) = y^2yz^3 - x(3xyz^2) = y^3z^3 - 3x^2yz^2$

(d) $fV[g] - gV[f] = xy(3xz^2) - z^3y^3 = 3x^2yz^2 - y^3z^3$

(e) $V[f^2 + g^2] = V[f^2] + V[g^2] = 2fV[f] + 2gV[g] = 2xy(y^3) + 2z^3(3xz^2) = 2xy^4 + 6xz^5$

(f) $V[V[f]] = V[y^3] = y^2\partial_x y^3 - x\partial_z y^3 = 0$

1.3.4

For any point p, $V_p = \sum_i v_i(p)U_i(p)$. Then $V_p[x_j] = \sum_i v_i(p)U_i(p)[x_j] = \sum_i v_i(p)\delta_{ij} = v_j(p)$.

$\blacksquare$

1.3.5

Note that $V = \sum_i v_i U_i$ and $W = \sum_i w_i U_i$. Since $V[f] = W[f]$ for every f, take $f = x_j$, we get $v_j = w_j$ for every j. Hence $V = W$

$\blacksquare$

1.4.1

Since $\alpha(t) = (1 + \cos t, \sin t, 2\sin(t/2))$. $\alpha'(t) = (\sin t, \cos t, \cos(t/2))$.

$t = 0, \alpha'(0) = (0, 1, 1)$

$t = \frac{\pi}{2}, \alpha'(\pi/2) = (1, 0, \sqrt{2}/2)$

$t = \pi, \alpha'(\pi) = (0, -1, 0)$

1.4.2

$\alpha(t) = \int \alpha'(t)dt = (t^3/3, t^2/2, e^t) + C.$

Since $\alpha(0) = (0, 0, 1) + C = (1, 0, 5)$, so $C = (1, 0, 4)$.

$\alpha(t) = (t^3/3 + 1, t^2/2, e^t + 4)$

1.4.3

Since $\alpha(t) = (1 + \cos t, \sin t, 2\sin(t/2))$ and $h(s) = \cos^{-1} s$. Therefore

$$\beta(s) = \alpha(h(s)) = \left(1 + s, \sin\cos^{-1} s, 2\sin\left(\frac{\cos^{-1} s}{2}\right)\right)$$

Note that $s \in (0, 1)$ meaning $\cos^{-1} s = h$ can be positive or negative. So $\sin\cos^{-1} s = \pm\sqrt{1 - s^2}$. Similarly, $2\sin\left(\frac{\cos^{-1} s}{2}\right) = \pm 2\sqrt{\frac{1-s}{2}}$ by half angle formula of $\sin$. By restricting $h \geq 0$, we get

$$\beta(s) = \left(1 + s, \sqrt{1 - s^2}, 2\sqrt{\frac{1 - s}{2}}\right)$$

1.4.4

$\beta = \alpha(h(s)) = (s, s^{-1}, \sqrt{2}\log s)$. Then $\beta'(s) = (1, -s^{-2}, \sqrt{2}s^{-1})$.

By lemma 4.5 $\beta'(s) = (dh/ds)\alpha'(h(s)) = s^{-1}(e^t, -e^{-t}, \sqrt{2})|_{t=h(s)} = (1, -s^{-2}, \sqrt{2}s^{-1})$

1.4.5

$l_1 : (1, -3, -1) + t(6 - 1, 2 + 3, 1 + 1) = (5t + 1, 5t - 3, 2t - 1).$

$l_2 : (-1, 1, 0) + s(-5 + 1, -1 - 1, -1) = (-4s - 1, -2s + 1, -s).$

Suppose they meet, then we have

$$\begin{align} 5t + 1 &= -4s - 1 \tag{1} \\ 5t - 3 &= -2s + 1 \tag{2} \\ 2t - 1 &= -s \tag{3} \\ & \tag{4} \end{align}$$

Solving the first two equation, we have $t = 2, s = -3$. Putting them into the 3rd equation, we get $3 = 3$ which is consistent. They do meet.

1.4.6

For any curve $\alpha(t)$ with initial velocity of v_p. Then $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$ by Lemma 4.6. Evaluating at 0, we get

$$\begin{aligned}
\alpha'(0)[f] = v_p[f] &= \frac{d(f(\alpha))}{dt}(0) \\
&= \sum_i \frac{\partial f}{\partial x_i}(\alpha(0))\frac{d\alpha_i}{dt}(0) \\
&= \sum_i \frac{\partial f}{\partial x_i}(\alpha(0))\alpha_i'(0) \\
&= \sum_i \frac{\partial f}{\partial x_i}(p)[v_p]_i \\
&= \sum_i \frac{\partial f}{\partial x_i}(p)\frac{d([p+tv_p]_i)}{dt} \\
&= \sum_i \frac{\partial f}{\partial x_i}(p+tv_p)\frac{d([p+tv_p]_i)}{dt}\bigg|_{t=0} \\
&= \frac{df}{dt}(p+tv_p)\bigg|_{t=0}
\end{aligned}$$

1.4.7

(a) $\frac{d}{dt}(t, 1+t^2, t)\Big|_{t=0} = (1, 2(0), 1) = (1, 0, 1)$ at point $(0, 1, 0)$.

$\frac{d}{dt}(\sin t, \cos t, t)\Big|_{t=0} = (1, 0, 1)$ at point $(0, 1, 0)$.

$\frac{d}{dt}(\sinh t, \cosh t, t)\Big|_{t=0} = (\cosh(0), \sinh(0), 1) = (1, 0, 1)$ at point $(0, 1, 0)$.

(b) $f = x^2 - y^2 + z^2$

$f(t) = f(t, 1+t^2, t) = t^2 - (1+t^2)^2 + t^2$. Then $\frac{df}{dt}\Big|_0 = 4t - 4t(1+t^2)\Big|_0 = 0$

$f(t) = f(\sin t, \cos t, t) = \sin^2 t - \cos^2 t + t^2$. Then $\frac{df}{dt}\Big|_0 = 2\sin^t \cos t + 2\cos t \sin t + 2t\Big|_0 = 0$.

$f(t) = f(\sinh t, \cosh t, t) = \sinh^2 t - \cosh^2 t + t^2$. Then $\frac{df}{dt}\Big|_0 = 2\sinh t \cosh t - 2\cosh t \sinh t + 2t\Big|_0 = 2t\Big|_0 = 0$

1.4.8

(a)$x = \frac{1}{2}\cos t$, $y = \sin t$.

(b) $x = t$, $y = (1 - 3t)/4$.

(c) $x = t$, $y = e^t$.

1.4.9

$\alpha(t) = (2\cos t, 2\sin t, t)$, $\alpha'(t) = (-2\sin t, 2\cos t, 1)$.

Line at 0 is $u \to (2, 0, 0) + u(0, 2, 1)$

Line at $\pi/4$ is $v \to (\sqrt{2}, \sqrt{2}, \pi/4) + v(-\sqrt{2}, \sqrt{2}, 1)$

1.5.1

$p = (0, -2, 1)$, $v_p = (1, 2, -3)$.

(a) $(y^2 dx)(v_p) = y^2(p)dx(v_p) = (-2)^2(1) = 4$

(b) $(zdy - ydz)(v_p) = z(p)dy(v_p) - y(p)dz(v_p) = (1)(2) - (-2)(-3) = -4$

(c) $[(z^2 - 1)dx - dy + x^2 dz](v_p) = (z^2 - 1)(p)dx(v_p) - dy(v_p) + x^2(p)dz(v_p) = -2$

1.5.2

For any point p,

$$\begin{aligned}
\phi(V_p) &= (\sum_i f_i dx_i)(V_p) \\
&= \sum_i f_i(p) dx_i(V_p) \\
&= \sum_i f_i(p) dx_i(\sum_j v_j U_j) \\
&= \sum_i f_i(p) \sum_j v_j dx_i(U_j) \\
&= \sum_i f_i(p) \sum_j v_j \delta_{ij} \\
&= \sum_i f_i(p) v_i
\end{aligned}$$

Hence $\phi(V) = \sum_i f_i v_i$.

1.5.3

$\phi = x^2dx - y^2dz$

(a) For $V = xU_1 + yU_2 + zU_3$, $\phi(V) = x^3 - y^2z$.

(b) For $W = xy(U_1-U_3)+yz(U_1-U_2) = (xy+yz)U_1 - yzU_2 - xyU_3$, $\phi(W) = x^2y(x+z)+xy^3$

(c) For $T = (1/x)V + (1/y)W$,

$$\phi(T) = \phi((1/x)V + (1/y)W) = (1/x)\phi(V) + (1/y)\phi(W) = \frac{x^3 - y^2z}{x} + x^2(x+z) + xy^2$$

1.5.4

(a) Since $df^2 = 2fdf$. Suppose $df^n = nf^{n-1}df$,

$$df^{n+1} = d(ff^n) = f^ndf + fd(f^n) = f^ndf + f(nf^{n-1}df) = (n+1)f^ndf$$

. Therefore by induction, $df^n = nf^{n-1}df$. As a result,

$$df^4 = 4f^3df$$

(b) $df = d(\sqrt{f}\sqrt{f}) = 2\sqrt{f}d(\sqrt{f})$ Then

$$d(\sqrt{f}) = \frac{1}{2\sqrt{f}}df$$

(c) $d(\log(1+f^2)) = \frac{1}{1+f^2}d(1+f^2) = \frac{2f}{1+f^2}df$

1.5.5

(a)

$$df = \sum_i \partial_i f dx_i = \frac{xdx + ydy + zdz}{\sqrt{x^2+y^2+z^2}}$$

(b)

$$df = \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}dx + \frac{1}{x}dy) = \frac{1}{x^2+y^2}(-ydx + xdy)$$

1.5.6

$p = (0, -2, 1)$, $v_p = (1, 2, -3)$.

(a) $df = y^2dx + (2xy - z^2)dy - 2yzdz$. Then $df(v_p) = (-2)^2(1) + (-1)(2) - 2(-2)(-3) = 10$

(b) $df = \exp(yz)(dx + xzdy + xydz)$. Then $df(v_p) = \exp(-2)$.

(c) $df = (y\cos(xy)\cos(xz) - z\sin(xy)\sin(xy))dx + x\cos(xy)\cos(xz)dy - x\sin(xy)\sin(xz)dz$. Then $df(v_p) = y(p)dx(v_p) = (-2)1 = -2$.

1.5.7

$\phi(v_p) = \sum_i f_i(p)v_i$ for $\phi = \sum_i f_i dx_i$.

(a) Yes. $f_1(p) = 1$, $f_3(p) = -1$, $\phi = dx - dz$

(b) No. Because $dx_i(v_p)$ must involve v_i.

(c) Yes. $f_1(p) = p_3$, $f_2(p) = p_1$. So $\phi = zdx + xdy$.

(d) Yes. $df(v_p) = v_p(f) = v_p[x^2 + y^2]$. Therefore $f = x^2 + y^2$. $\phi = df = 2xdx + 2ydy$.

(e) Yes. $\phi = 0$.

(f) No. Because $dx_i(v_p)$ must involve v_i.

1.5.8

By definition of d. $df(v_p) = v_p[f]$. Then by theorem 3.3

$$d(fg)(v_p) = v_p(fg) = f(p)v_p(g) + g(p)v_p(f) = f(p)dg(v_p) + g(p)df(v_p)$$

1.5.9

Since $df = -2xydx + (1 - x^2 - 2yz)dy + (1 - y^2)dz$, we need to find points such that $xy = 0, 1 - x^2 - 2yz = 0, 1 - y^2 = 0$. From the last equation, we get $y = \pm 1$. Then the first equation gives $x = 0$. Putting the values into the 2nd equations, we have $z = \frac{1-x^2}{2y} = \pm\frac{1}{2}$. So the critical points are $(0, 1, 1/2)$ and $(0, -1, -1/2)$

1.5.10

Suppose p is a local maxima of f and p is not a critical point of f, then there exists a tangent vector v_p such that

$$df(v_p) = \frac{df}{dt}(p + tv_p)\bigg|_0 = \lim_{t \to 0} \frac{f(p + tv_p) - f(p)}{t} > 0$$

This is true in general as we can always take the opposite direction $-v_p$ if $df(v_p) < 0$. Then we have $\frac{f(p+t_0 v_p) - f(p)}{t_0} > \epsilon$ for some $\epsilon > 0$ and t_0 which contradicts with the assumption that p is local maxima of f. Same argument applies to local minima of f.

∎

1.5.11

(a) $(df)(v_p) = \sum_i \frac{\partial f}{\partial x_i} v_i$. Taylor expanding $f(u)$ around p, we get $f(u) = f(p) + \sum_i \partial_i f(p)(u_i - p_i) + O(u^2)$. Let $u = p + v$, then

$$f(p + v) = f(p) + \sum_i \partial_i f(p) v_i + O_2 = f(p) + df(v_p) + O_2$$

. Where O_2 is second order error. Therefore $f(p + v) - f(p) \approx df(v_p)$ in the first order.

∎

(b)

$$df = (2xy/z)dx + (x^2/z)dy - (x^2 y/z^2)dz$$

. $p = (1, 1.5, 1)$ and $v_p = (-0.1, 0.1, 0.2)$. We get $df(v_p) = 2(1.5)(-0.1) + 0.1 - (1.5)(0.2) = -0.3 + 0.1 - 0.3 = -0.5$.

Direct calculation gives $f(0.9, 1.6, 1.2) - f(1, 1.5, 1) = 1.08 - 1.5 = -0.42$

1.6.1

Given $\phi = yzdx + dz$, $\psi = \sin z dx + \cos z dy$, $\xi = dy + zdz$.

(a)

$\phi \wedge \psi = (yzdx + dz) \wedge (\sin z dx + \cos z dy) = yz \cos z dx \wedge dy - \sin z dx \wedge dz - \cos z dy \wedge dz$

$\psi \wedge \xi = (\sin z dx + \cos z dy) \wedge (dy + zdz) = \sin z dx \wedge dy + z \sin z dx \wedge dz + z \cos z dy \wedge dz$

$\xi \wedge \phi = (dy + zdz) \wedge (yzdx + dz) = -yzdx \wedge dy + dy \wedge dz - yz^2 dx \wedge dz$

(b)

$d\phi = d(yz) \wedge dx + d1 \wedge dz = (zdy + ydz) \wedge dx = -zdx \wedge dy - ydx \wedge dz$

$d\psi = d(\sin z) \wedge dx + d(\cos z) \wedge dy = \cos z dz \wedge dx - \sin z dz \wedge dy$

$d\xi = d1 \wedge dy + dz \wedge dz = 0$

1.6.2

Given $\phi = dx/y$, $\psi = zdy$. Then $\phi \wedge \psi = (z/y)dx \wedge dy$. Directly computing the differential gives

$$d(\phi \wedge \psi) = d(z/y) \wedge dx \wedge dy = (1/ydz - z/y^2dy) \wedge dx \wedge dy = 1/ydxdydz$$

Using theorem 6.4,

$$\begin{aligned} d(\phi \wedge \psi &= d\phi \wedge \psi - \phi \wedge d\psi \\ &= (-1/y^2dy \wedge dx) \wedge zdy - 1/ydx \wedge (dz \wedge dy) \\ &= 0 - 1/ydxdzdy \\ &= 1/ydxdydz \end{aligned}$$

1.6.3

For any function f, $df = \sum_i \partial_i f_i dx_i$.

$$\begin{aligned} d(df) &= \sum_i d(\partial_i f) \wedge dx_i \\ &= \sum_i (\sum_j \partial_i \partial_j f dx_j) \wedge dx_i \\ &= \sum_{i \neq j} \partial_i \partial_j f dx_j \wedge dx_i \\ &= -\sum_{i<j} \partial_i \partial_j f dx_i \wedge dx_j + \sum_{j<i} \partial_j \partial_i f dx_j \wedge dx_i \\ &= 0 \end{aligned}$$

$d(fdg) = df \wedge dg + fd(dg) = df \wedge dg$.

1.6.4

(a) $d(fdg + gdf) = df \wedge dg + dg \wedge df = 0$

(b) $d((f-g)(df+dg)) = d(f-g) \wedge (df+dg) = (df-dg) \wedge (df+dg) = df \wedge dg - dg \wedge df = 2df \wedge dg$

(c) $d(fdg \wedge gdf) = d(fdg) \wedge gdf - fdg \wedge d(gdf) = df \wedge dg \wedge gdf - fdg \wedge dg \wedge df = 0$

(d) $d(gfdf) + d(fdg) = d(gf) \wedge df + df \wedge dg = (fdg + gdf) \wedge df + df \wedge dg = (1-f)df \wedge dg$

1.6.5

$\phi_i = \sum_j f_{ij} dx_j$. If we do the wedge product,

$$\begin{aligned}
\bigwedge_i \phi_i &= \bigwedge_i \sum_j f_{ij} dx_j \\
&= \left[\sum_\sigma \prod_i f_{i,\sigma_i}\right] (dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_n}) \\
&= \left[\sum_\sigma \text{sgn}(\sigma) \prod_i f_{i,\sigma_i}\right] (dx_1 \wedge \cdots \wedge dx_n) \\
&= \det |f_{i,j}| dx_1 \wedge \cdots \wedge dx_n
\end{aligned}$$

Each term in the expanded expression is a product of picking a term in ϕ_i for each i. Since $dx_j \wedge dx_j$ vanishes, so we need to pick n terms with different j from ϕ_1 to ϕ_n and hence the expression above.

1.6.6

In cylindrical coordinate, $x = r\cos\theta, y = r\sin\theta, z = z$. The volume element in canonical coordinate is $dx \wedge dy \wedge dz$. Then

$$\begin{aligned}
dx \wedge dy \wedge dz &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \wedge dz \\
&= (r\cos^2 theta + r\sin^2\theta) dr \wedge d\theta \wedge dz \\
&= r dr \wedge d\theta \wedge dz
\end{aligned}$$

1.6.7

Given a one-form $\phi = \sum_i f_i dx_i$, then

$$d(d\phi) = d(d(\sum_i f_i dx_i)) = d(\sum_i df_i \wedge dx_i) = \sum_i d(1) \wedge df_i \wedge dx_i = 0$$

1.6.8

(a) $df = \sum_i \partial_i f dx_i$. dx_i 1-1 to U_i. Therefore df 1-1 to $\boldsymbol{\nabla} f$.

(b) $\{dx_i\}$ is 1-1 with $\{U_i\}$, for each $\phi = \sum_i f_i dx_i$ there exists $V = \sum_i f_i U_i$. Then

$$\begin{aligned} d\phi = \sum_i df_i \wedge dx_i &= \sum_i (\sum_j \partial_j f_i dx_j) \wedge dx_i \\ &= \sum_i \sum_j \partial_j f_i dx_j \wedge dx_i \\ &= \sum_{i \neq j} \partial_j f_i dx_j \wedge dx_i \\ &= -\sum_{i<j} \partial_j f_i dx_i \wedge dx_j + \sum_{j<i} \partial_j f_i dx_j \wedge dx_i \\ &= -\sum_{i<j} \partial_j f_i dx_i \wedge dx_j + \sum_{i<j} \partial_i f_j dx_i \wedge dx_j \quad \text{(swap i and j)} \\ &= \sum_{i<j} (\partial_i f_j - \partial_j f_i) dx_i \wedge dx_j \end{aligned}$$

This is equal to $\nabla \times \phi$ when the dimension is 3 because there exists 1-1 mapping between U_1, U_2, U_3 and $dx_1 \wedge dx_2, dx_2 \wedge dx_3, dx_1 \wedge dx_3$.

(c) For a vector field $V = f_1 U_1 + f_2 U_2 + f_3 U_3$, there exists $\eta = f_3 dx_1 dx_2 - f_2 dx_1 dx_2 + f_1 dx_2 dx_3$ by correspondence. Then

$$\begin{aligned} d\eta &= df_3 dx_1 dx_2 - df_2 dx_1 dx_2 + f_1 dx_2 dx_3 \\ &= \partial_3 f_3 dx_3 dx_1 dx_2 - \partial_2 f_2 dx_2 dx_1 dx_3 + \partial_1 f_1 dx_2 dx_3 \\ &= (\partial_3 f_3 + \partial_2 f_2 + \partial_1 f_1) dx_1 dx_2 dx_3 \\ &= \nabla \cdot V dx_1 dx_2 dx_3 \end{aligned}$$

1.6.9

$df = \partial_x f dx + \partial_y f dy$ and $dg = \partial_x g dx + \partial_y g dy$.

$$df \wedge dg = \partial_x f \partial_y g dx \wedge dy + \partial_y f \partial_x g dy \wedge dx = (\partial_x f \partial_y g - \partial_y f \partial_x g) dx \wedge dy = \begin{vmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{vmatrix} dx \wedge dy$$

1.7.1

$F(u,v) = (u^2 - v^2, 2uv)$.

(a) $F(p) = (0,0)$ then $(f_1(p), f_2(p)) = (p_1^2 - p_2^2, 2p_1p_2) = (0,0)$. Therefore $p_1 = 0, p_2 = 0$.

(b) $F(p) = (8,6)$ then $p_1^2 - p_2^2 = 8$ and $2p_1p_2 = 6$. Putting together two equations and factoring it, we have $(p_2^2 + 9)(p_2^2 - 1) = 0$. Therefore $p_1 = \pm 3, p_2 = \pm 1$.

(c) $F(p) = p$. Then $p_1^2 - p_2^2 = p_1$ and $2p_1p_2 = p_2$.

1.7.3

$$\begin{aligned}
F_*(v) &= \frac{d}{dt}\Big|_{t=0} F(p+tv) \\
&= \frac{d}{dt}\Big|_{t=0} ((p_1+tv_1)^2-(p_2+tv_2)^2, 2(p_1+tv_1)(p_2+tv_2)) \\
&= (2v_1(p_1+tv_1)-2v_2(p_2+tv_2), 2v_1(p_2+tv_2)+2(p_1+tv_1)v_2)\Big|_{t=0} \\
&= 2(v_1p_1-v_2p_2, v_1p_2+p_1v_2)
\end{aligned}$$

1.7.4

$$F_{*,p} = \begin{pmatrix} 2u & 2v \\ -2v & 2u \end{pmatrix}$$

$F_{*,p}$ only vanishes at the origin.

1.7.5

$F_*(v_p) = \frac{d}{dt}\big|_{t=0} F(p+tv) = \frac{d}{dt}\big|_{t=0}(F(p)+tF(v)) = F(v)$

1.7.6

(a) Take $m = n = 1$, for any mapping that is a joint of two line segments with different slope, the mapping is not differential at the point of the joints hence it is not a diffeomorphism.

(b) Suppose $F : \Re^n \to \Re^n$ is 1-1 and onto, by inverse function theorem, we can take the open set as the entire $\Re^n$ since F is onto. Then F is diffeomorphism.

1.7.7

$$
\begin{aligned}
v_p[g(F)] &= \sum_i^n v_i U_i[g(F)] \\
&= \sum_i^n v_i \frac{\partial}{\partial x_i}[g(F)] \\
&= \sum_i^n v_i \sum_j^m \frac{\partial g}{\partial f_j} \frac{\partial f_j}{\partial x_i} \\
&= \sum_j^m \left[\sum_i^n v_i \frac{\partial f_j}{\partial x_i} \right] \frac{\partial g}{\partial f_j} \\
&= \sum_j^m \left[F_*(v_p) \right]_j \frac{\partial g}{\partial f_j} \\
&= F_*(v_p)[g]
\end{aligned}
$$

1.7.8

Given any curve $\alpha(t)$ such that $\alpha'(0) = v_p$.

$$
F_*(v_p) = \frac{d}{dt}\Big|_{t=0} F(\alpha(t)) = \sum_i^n \frac{\partial F}{\partial \alpha_i} \frac{d\alpha_i}{dt}\Bigg|_{t=0} = \sum_i^n [v_p]_i \frac{\partial F}{\partial \alpha_i}
$$

By proof of proposition 7.5, the expression above can give back the line definition of the push forward of F.

1.7.9

(a) $GF = (g_i \circ f_i)$. Since g_i, f_i are differentiable, so $g_i \circ f_i$ are differentiable. Hence GF is a mapping.

(b)

$$
[(GF)_*]_{ij} = \frac{\partial g_i \circ f_i}{\partial x_j} = \sum_k^m \frac{\partial g_i}{\partial f_k} \frac{\partial f_k}{\partial x_j} = [G_*]_{ik}[F_*]_{kj}
$$

(c) F is diffeomorpohism means F^{-1} exists and is differentiable. Then since F is differentiable itself, F^{-1} also has a differentiable inverse therefore F^{-1} is also a diffeomorpishm.

1.7.10

(a) Suppose $F(u_1, v_1) = F(u_2, v_2)$, then $v_1 \exp u_1 = v_2 \exp u_2$ and $2u_1 = 2u_2$. It is obvious that $v_1 = v_2$ as well. So F is 1-1.

For any point (y_1, y_2) in $\Re^2$, Let $u = y_2/2$, $v = y_1/\exp(-y_2/2)$, then $F(u, v) = (y_1, y_2)$. So F is onto.

$|F_*| = \begin{vmatrix} ve^u & e^u \\ 2 & 0 \end{vmatrix} = 2\exp(u) \neq 0$ So F is regular. These properties hold in $\Re^2$ so by inverse function theorem, F is diffeomorphism.

(b) $F(u, v) = (x, y)$. Then $ve^u = x$ and $2u = y$. Expressing u, v in terms of x, y, $u = y/2, v = x\exp(-y/2)$. We get the inverse

$$F^{-1}(x, y) = (0.5y, x\exp(-y/2))$$

$F^{-1} \circ F(u, v) = F^{-1}(ve^u, 2u) = ((0.5)2u, v\exp(u)\exp(-2u/2)) = (u, v)$

$F \circ F^{-1}(x, y) = F(0.5y, x\exp(-y/2)) = (x\exp(-y/2)\exp(0.5y), 0.5y(2)) = (x, y)$