Chapter 0: Set Theory and Topology

Ran Xie

November 15, 2021

Problem 0.1.2.1

Since $A \triangle B = A \cup B - A \cap B$. Then

$$\begin{aligned}
A \triangle B &= A \cup B - A \cap B \\
&= (A \cup B) \cap (A \cap B)^c \\
&= (A \cup B) \cap (A^c \cup B^c) \\
&= (A \cap A^c) \cup (B \cap B^c) \cup (A \cap B^c) \cup (B \cap A^c) \\
&= (A \cap B^c) \cup (B \cap A^c) \\
&= (A - B) \cup (B - A)
\end{aligned}$$

$$\begin{aligned}
A \cap C \triangle B \cap C &= (A \cap C - B \cap C) \cup (B \cap C - A \cap C) \\
&= [(A \cap C) \cap (B^c \cup C^c)] \cup [(B \cap C) \cup (A^c \cup C^c)] \\
&= [A \cap C \cap B^c \cup A \cap C \cap C^c] \cup [B \cap C \cap A \cup B \cap C \cap C^c] \\
&= [A \cap C \cap B^c \cup \emptyset] \cup [B \cap C \cap A \cup \emptyset] \\
&= A \cap B^c \cap C \cup B \cap A^c \cap C \\
&= (A - B) \cap C \cup (B - A) \cap C \\
&= [(A - B) \cup (B - A)] \cap C \\
&= (A \triangle B) \cap C
\end{aligned}$$

Exercise 0.1.3.1

$A \times B \neq B \times A$ Since Cartesian product is a set of ordered pair.

Exercise 0.1.4.1

Since $f : A \to B$ and There exists g such that $f \circ g = i_B$. Since the domain of $f \circ g$ is B. Then for each $y \in B$, $f \circ g(y) = i_B(y) = y$ which means there exists $x \in A$ such that $g(y) = x$ and

$f(x) = y$. Therefore f is onto. ■

If there exists y_1, y_2 such that $g(y_1) = g(y_2)$. Then

$$\begin{aligned} f \circ g(y_1) = f \circ g(y_2) &\Leftrightarrow i_B(y_1) = i_B(y_2) \\ &\Leftrightarrow y_1 = y_2 \end{aligned}$$

Therefore g is 1-1. ■

Let $h = f|_{gB}$, Since $f \circ g = i_B$, for each $y \in B$, $f \circ g(y) = i_B(y) = y$ which means there exists an $x \in g(B)$ such that $f(x) = y$. Therefore $h = f|_{gB}$ is onto.

Note that $f \circ g$ can be written as $f|_{gB} \circ g = h \circ g = i_B$ since f can only take on values in $g(B)$. g is 1-1 means there is inverse g^{-1} that is also 1-1. Hence $h = h \circ g \circ g^{-1} = i_B \circ g^{-1}$. Both i_B and g^{-1} are 1-1, so h is also 1-1. ■

Let $x \in g(B)$ and consider $g \circ h(x)$. There exists $y \in B$ such that $y = h(x)$. We know $h \circ g(y) = i_B(y) = y$. Suppose some $x_1 = g(y)$, $h \circ g(y) = h(x_1) = y = h(x) \Rightarrow x_1 = x$ since h is 1-1. So $g(y) = x$. Therefore $g \circ h(x) = g(y) = x \Leftrightarrow g \circ h = i_{gB} \Leftrightarrow g = i_{gB} h^{-1}$ ■

f need not be 1-1. Example: $A = \{1, 2\}, B = \{3\}$. $f(1) = f(2) = 3$, $g(3) = 2$ and $h = f|_{g(B)=\{2\}}$. ■

Exercise 0.1.4.2

Suppose $f : A \to B$ is 1-1 and onto, then for each $y \in B$ there corresponds a unique $x \in A$ such that $f(x) = y$. Define $g : B \to A$ such that for each $y \in B$, $g(y) = x$ where $f(x) = y$. g is a function since each y corresponds to a unique x guaranteed by f. Therefore $g \circ f = i_A$ and $f \circ g = i_B$. ■

Suppose There is a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$. For $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. Applying g on both side, we have $x_1 = x_2$. Therefore f is 1-1.

For $y \in B$, there exists an $x \in A$ such that $g(y) = x$ since g is a function. Applying f to both side, we have $f(g(y)) = f(x) \Leftrightarrow i_B(y) = y = f(x)$. So we have found an x for every y such that $y = f(x)$. Therefore f is onto. ■