Chapter 7: Point Estimation

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Exercise 7.1

If we take the product of $f(x|\theta)$ as the likelihood function, all 3 values of θ attain maximum likelihood of 0.

Method 1: With perturbation on likelihood function

We will perturb the likelihood function by ϵ. Now even the probability 0 entries get a probability of ϵ. Then

$$\begin{aligned} L(\theta = 1|x+\epsilon) &= \frac{\epsilon(1/3+\epsilon)^2(1/6+\epsilon)^2}{(1+5\epsilon)^5} \\ L(\theta = 2|x+\epsilon) &= \frac{\epsilon(1/4+\epsilon)^4}{(1+5\epsilon)^5} \\ L(\theta = 3|x+\epsilon) &= \frac{\epsilon^2(1/4+\epsilon)^2(1/2+\epsilon)}{(1+5\epsilon)^5} \end{aligned}$$

We can compare the functions by taking the ratio and letting ϵ go to 0.

$$\frac{L(\theta=2|x)}{L(\theta=1|x)} = \lim_{\epsilon\to 0}\frac{L(\theta=2|x+\epsilon)}{L(\theta=1|x+\epsilon)} = \lim_{\epsilon\to 0}\frac{(1/4+\epsilon)^4}{(1/3+\epsilon)^2(1/6+\epsilon)^2} = 1.26$$

Similarly,

$$\frac{L(\theta=2|x)}{L(\theta=3|x)} = \lim_{\epsilon\to 0}\frac{(1/4+\epsilon)^4}{\epsilon(1/4+\epsilon)^2(1/2+\epsilon)} = \infty$$

Therefore $\theta = 2$ is the MLE

Method 2: With +1 smoothing

we can assume N trials are performed for each θ and apply laplace smoothing (+1 smoothing).

Then the likelihood functions become

$$\begin{aligned} L(\theta = 1|x) &= \lim_{N\to\infty}\frac{(N/3+1)^2(N/6+1)^2}{(N+5)^5} \\ L(\theta = 2|x) &= \lim_{N\to\infty}\frac{(N/4+1)^4}{(N+5)^5} \\ L(\theta = 3|x) &= \lim_{N\to\infty}\frac{(N/4+1/N)^2(N/2+1)}{(N+5)^5} \end{aligned}$$

This is essentially the same as method 1.

Exercise 7.2

Gamma(α, β) has pdf $f(x|\alpha,\beta) = \frac{x^{\alpha-1}e^{-\beta x}\beta^\alpha}{\Gamma(\alpha)}$, with α known, the likelihood function is

$$L(\alpha,\beta|x) = \prod_{i=1}^n f(x_i|\alpha,\beta) = \prod_{i=1}^n \frac{x_i^{\alpha-1}e^{-\beta x_i}\beta^\alpha}{\Gamma(\alpha)} = \frac{(\prod_i x_i)^{\alpha-1}}{\Gamma^n(\alpha)}\left[\beta^{n\alpha}e^{-\beta(\sum_i x_i)}\right]$$

Let $\frac{\partial L(\beta|x,\alpha)}{\partial\beta} = 0$, we get the MLE for β.

$$\hat{\beta} = \frac{\alpha}{\bar{x}}$$

If α is also unknown,

$$\frac{\partial L(\alpha,\beta|x)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\frac{A^{\alpha-1}\beta^{n\alpha}}{\Gamma^n(\alpha)} = 0$$

$$\Rightarrow \sum \ln x_i + n\ln\beta = \psi(\alpha) \Rightarrow \sum \ln x_i + n(\ln\alpha - ln\bar{x}) = \psi(\alpha)$$

where ψ is polygamma function

We can then solve for α numerically.

Exercise 7.3

$\log$ is monotonic increasing function, when L attains maximum values, $\log(L)$ also attains maximum.

Exercise 7.6

$f(x|\theta) = \theta x^{-2}$ with $0 < \theta \le x < \infty$. We need to extend the range of x in order to get rid of the θ dependency with indicator function (This will make x_i iid so we can compute the likelihood function by taking the product). So $f(x|\theta) = \mathbb{I}_{\theta\le x}\theta x^{-2}$ and $x \in \mathbb{R}$.

(a) For a sample $X_1, \ldots, X_n$, by factorization theorem, $f(x|\theta) = g(T(x)|\theta)h(x)$. So $h(x) = x^-2$ and $g(T(x)|\theta) = \mathbb{I}_{\theta\le x}\theta = \mathbb{I}_{\theta\le x_i,\forall i}\theta = \mathbb{I}_{\theta\le\min x_i}$. Therefore $T(x) = \min x_i$ is a sufficient statistics.

(b) The likelihood function $L(\theta|x) = \mathbb{I}_{\theta\le x_i}\theta^n(\prod_i x_i)^{-2}$. $L(\theta|x)$ is non zero only when $\theta \le \min x_i$. Then $L(\theta|x) = \theta^n(\prod_i x_i)^{-2}$ when $\theta \le \min x_i$ decreases monotonically as θ decreases. So it attains maximum when $\theta = \min x_i$

(c) Computing the first moment of the pdf gives

$$EX = \int_\theta^\infty x\theta x^{-2}dx = \theta\int_\theta^\infty x^{-1}dx = \theta\ln(x)|_\theta^\infty = \infty$$

Therefore moment doesn't exist for f. We cannot use moment to get an estimator.

Exercise 7.7

We have $L(\theta = 0|x) = \prod_i^n I_{x_i\in(0,1)}$ and $L(\theta = 1|x) = \prod_i^n \frac{1}{2\sqrt{x_i}}I_{x_i\in(0,1)}$. We can ignore the case when $x_i \notin (0,1)$ and take the log likelihood. We get $\log L(\theta = 0|x) = 0$ and $\log L(\theta = 1|x) = -n - \frac{1}{2}\log_2(s)$ where $s = \prod x_i$ and $s \in (0,1)$.

When $s^{1/n} < \frac{1}{4}$, $\theta = 1$ is the MLE otherwise 0 is the MLE.