Chapter 3: Euclidean Geometry

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1 Isometries of $\Re^3$

1

Consider

$$\begin{aligned}|C(p+a)-C(p)-C(a)|^2 &= C(p+a)\cdot C(p+a)+C(p)\cdot C(p)+C(a)\cdot C(a)\\ &\quad -2C(p+a)\cdot C(p)-2C(p+a)\cdot C(a)+2C(p)\cdot C(a)\\ &= (p+a)^2+p^2+a^2-2(p+a)p-2(p+a)a+2pa\\ &= p^2+2pa+a^2+p^2+a^2-2p^2-2pa-2pa-2a^2+2pa\\ &= 0\end{aligned}$$

Therefore $C(p+a)=C(p)+C(a)$. It follows that $CT_a(p)=C(p+a)=C(p)+C(a)=T_{C(a)}C(p)$ ■

2

From the result in problem 1.1 $FG=T_aAT_bB=T_aT_{A(b)}AB$ and $GF=T_bBT_aA=T_bT_{B(a)}BA$. The transnational parts are $T_{a+A(b)}$ and $T_{b+B(a)}$ respectively.

3

Suppose $Cp=Cq$, Then

$$\begin{aligned}&\Leftrightarrow \langle Cp-Cq, Cp-Cq\rangle = 0\\ &\Leftrightarrow CpCp-2CpCq-CqCq=0\\ &\Leftrightarrow p^2-2pq-q^2=0\\ &\Leftrightarrow p=q\end{aligned}$$

C is 1-1. Therefore there exists inverse C^{-1}. To show C^{-1} is orthogonal transformation. Suppose p,q such that $C^{-1}p=\tilde{p}$ and $C^{-1}q=\tilde{q}$

$$\langle C^{-1}p, C^{-1}q\rangle = \langle \tilde{p},\tilde{q}\rangle = \langle C\tilde{p}, C\tilde{q}\rangle = \langle p,q\rangle$$

So C^{-1} is orthogonal transformation. We can define the inverse of F. $F^{-1} = (T_a C)^{-1} = C^{-1} T_{-a}$. F^{-1} is isometry.

4

$$C = \frac{1}{3} \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

It's trivial to check orthogonality after factoring out 1/3.

$Cp = \frac{1}{3}(2, 19, -7)$ and $Cq = \frac{1}{3}(-5, -4, 7)$. Then $\langle Cp, Cq \rangle = \frac{1}{9}(-135) = -15 = \langle p, q \rangle$.

5

(a) $q = F(p) = T_a C(p) = (-3\sqrt{2} + 1, 1, 5\sqrt{2} - 1)^T$

(b) $q = F^{-1}(p) = (T_a C)^{-1}(p) = C^{-1} T_{-a}(p) = C^T T_{-a}(p) = (5\sqrt{2}, -5, 4\sqrt{2})^T$

(c) $q = (C T_a)(p) = (5\sqrt{2}, 1, 2\sqrt{2})^T$

6

(a) $C = \text{diag}(-1, -1, -1)$ and $a = (0, 0, 0)$.

(b) Not isometry. If $p \perp a$, then $d(F(p), 0) = d(0, 0) = 0 \neq d(p, 0)$.

(c) $C = I$, $a = (-1, -2, -3)$.

(d) $C = \text{diag}(1, 1, 0)$, $a = (0, 0, 1)$.

7

For $F_1, F_2 \in \text{Iso}(3)$, $F_1 F_2 = T_a C_1 T_b C_2 = T_a T_{C_1(b)} C_1 C_2 \in \text{Iso}(3)$. Associative is trivial since they are functions. Inverse exists for every F as proven in problem 3.

8

Only Identity is in both subgroups.

9

(a) For an orthgonal matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it satisfies

$$\begin{cases} ac + bd = 0 \\ a^2 + b^2 = 1 \\ c^2 + d^2 = 1 \end{cases}$$

We have a free parameter. Let $d = \pm \sin\theta$, then

$$\begin{cases} d = \pm \sin\theta \\ c = \cos\theta \\ b = \mp \cos\theta \\ a = \sin\theta \end{cases}$$

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sin\theta & \mp\cos\theta \\ \cos\theta & \pm\sin\theta \end{pmatrix}$

(b) $F = T_a C$. $CpCp = p^2 \Rightarrow c^2 p^2 = p^2 \Rightarrow c = 1$. So an isometry in $\Re$ is just a displacement by a constant a.