Chapter 0: Set Theory and Topology

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Problem 0.1.2.1

Since $A \triangle B = A \cup B - A \cap B$. Then

$$\begin{aligned} A \triangle B &= A \cup B - A \cap B \\ &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= (A \cap A^c) \cup (B \cap B^c) \cup (A \cap B^c) \cup (B \cap A^c) \\ &= (A \cap B^c) \cup (B \cap A^c) \\ &= (A - B) \cup (B - A) \end{aligned}$$

$$\begin{aligned} A \cap C \triangle B \cap C &= (A \cap C - B \cap C) \cup (B \cap C - A \cap C) \\ &= [(A \cap C) \cap (B^c \cup C^c)] \cup [(B \cap C) \cup (A^c \cup C^c)] \\ &= [A \cap C \cap B^c \cup A \cap C \cap C^c] \cup [B \cap C \cap A \cup B \cap C \cap C^c] \\ &= [A \cap C \cap B^c \cup \emptyset] \cup [B \cap C \cap A \cup \emptyset] \\ &= A \cap B^c \cap C \cup B \cap A^c \cap C \\ &= (A - B) \cap C \cup (B - A) \cap C \\ &= [(A - B) \cup (B - A)] \cap C \\ &= (A \triangle B) \cap C \end{aligned}$$

Exercise 0.1.3.1

$A \times B \neq B \times A$ Since Cartesian product is a set of ordered pair.

Exercise 0.1.4.1

Since $f : A \to B$ and There exists g such that $f \circ g = i_B$. Since the domain of $f \circ g$ is B. Then for each $y \in B$, $f \circ g(y) = i_B(y) = y$ which means there exists $x \in A$ such that $g(y) = x$ and

$f(x) = y$. Therefore f is onto.

$\blacksquare$

If there exists y_1, y_2 such that $g(y_1) = g(y_2)$. Then

$$\begin{aligned} f \circ g(y_1) = f \circ g(y_2) &\Leftrightarrow i_B(y_1) = i_B(y_2) \\ &\Leftrightarrow y_1 = y_2 \end{aligned}$$

Therefore g is 1-1.

$\blacksquare$

Let $h = f|_{gB}$, Since $f \circ g = i_B$, for each $y \in B$, $f \circ g(y) = i_B(y) = y$ which means there exists an $x \in g(B)$ such that $f(x) = y$. Therefore $h = f|_{gB}$ is onto.

Note that $f \circ g$ can be written as $f|_{gB} \circ g = h \circ g = i_B$ since f can only take on values in $g(B)$. g is 1-1 means there is inverse g^{-1} that is also 1-1. Hence $h = h \circ g \circ g^{-1} = i_B \circ g^{-1}$. Both i_B and g^{-1} are 1-1, so h is also 1-1.

$\blacksquare$

Let $x \in g(B)$ and consider $g \circ h(x)$. There exists $y \in B$ such that $y = h(x)$. We know $h \circ g(y) = i_B(y) = y$. Suppose some $x_1 = g(y)$, $h \circ g(y) = h(x_1) = y = h(x) \Rightarrow x_1 = x$ since h is 1-1. So $g(y) = x$. Therefore $g \circ h(x) = g(y) = x \Leftrightarrow g \circ h = i_{gB} \Leftrightarrow g = i_{gB}h^{-1}$

$\blacksquare$

f need not be 1-1. Example: $A = \{1, 2\}, B = \{3\}$. $f(1) = f(2) = 3$, $g(3) = 2$ and $h = f|_{g(B)=\{2\}}$.

$\blacksquare$

Exercise 0.1.4.2

Suppose $f : A \to B$ is 1-1 and onto, then for each $y \in B$ there corresponds a unique $x \in A$ such that $f(x) = y$. Define $g : B \to A$ such that for each $y \in B$, $g(y) = x$ where $f(x) = y$. g is a function since each y corresponds to a unique x guaranteed by f. Therefore $g \circ f = i_A$ and $f \circ g = i_B$.

$\blacksquare$

Suppose There is a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$. For $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. Applying g on both side, we have $x_1 = x_2$. Therefore f is 1-1.

For $y \in B$, there exists an $x \in A$ such that $g(y) = x$ since g is a function. Applying f to both side, we have $f(g(y)) = f(x) \Leftrightarrow i_B(y) = y = f(x)$. So we have found an x for every y such that $y = f(x)$. Therefore f is onto.

$\blacksquare$

Exercise 0.1.5.1

Suppose f is onto, $B_1, B_2 \in P(B)$ and $f^{-1}(B_1) = f^{-1}(B_2)$. If $y \in B_1$, then there exists $x \in A$ such that $f(x) = y$ since f is onto. By definition of complete inverse image map, $x \in f^{-1}(B_1) = f^{-1}(B_2)$ implies $y = f(x) \in B_2$. The same argument applies to B_2. Then we have $B_1 = B_2$. Therefore f^{-1} is 1-1.

Suppose f^{-1} is 1-1. For $\{y\} \in P(B)$, there exists a unique $\{x\} \in P(A)$ such that $f^{-1}(\{y\}) = \{x\}$. This implies for every $y \in B$ there exists x such that $f(x) = y$.

$\blacksquare$

Exercise 0.1.5.2

(a)

$$\begin{aligned} x \in f^{-1}(D_1 \cap D_2) &\Leftrightarrow \exists y \in D_1 \cap D_2, f(x) = y \\ &\Leftrightarrow x \in (f^{-1}D_1) \cap (f^{-1}D_2) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}(D_1 \cup D_2) &\Leftrightarrow \exists y \in D_1 \cup D_2, f(x) = y \\ &\Leftrightarrow x \in (f^{-1}D_1) \text{ if } y \in D_1, x \in (f^{-1}D_2) \text{ if } y \in D_2 \\ &\Leftrightarrow x \in (f^{-1}D_1) \cup (f^{-1}D_2) \end{aligned}$$

(c)

$$\begin{aligned} y \in f(C_1 \cap C_2) &\Rightarrow \exists x \in C_1 \cap C_2, f(x) = y \\ &\Rightarrow y \in (fC_1) \cap (fC_2) \end{aligned}$$

(d)

$$\begin{aligned} y \in f(C_1 \cup C_2) &\Leftrightarrow \exists x \in C_1 \cup C_2, f(x) = y \\ &\Leftrightarrow y \in fC_1 \text{ if } x \in C_1, y \in fC_2 \text{ if } x \in C_2 \\ &\Leftrightarrow y \in (fC_1) \cup (fC_2) \end{aligned}$$

Exercise 0.1.5.3

Let $A = \{1, 2\}$, $B = \{3\}$, $f(1) = f(2) = 3$. If $C_1 = \{1\}, C_2 = \{2\}$. Then $fC_1 \cap fC_2 = \{3\} \neq f(C_1 \cap C_2) = f(\emptyset)$

Exercise 0.1.5.4

For $B, C \in P(A)$, $\Phi C = \Phi B \Rightarrow \phi_C = \phi_B$. If $\phi_C(x) = 1$, then $\phi_B(x) = 1$ which means $x \in C$ implies $x \in B$ and vice versa. By the same argument on $\phi_C(x) = 0$, we have $B = C$. So Φ is 1-1.

For a characteristic function $\phi_D \in 2^A$. By definition $D \subset A \Rightarrow D \in P(A)$. So Φ is onto.

Exercise 0.1.5.5

If A is finite ($A = \{a_1, \ldots, a_n\}$), there exists a bijection between $P(A)$ and $\{(b_1, \ldots, b_n) | b_i \in \{0, 1\}\}$ where b_i is 0 if a_i is absent in the subset, 1 if a_i is present. We have two choice for each i and there are n of them. So $|P(A)| = 2^n$. From exercise 0.1.5.4, $|2^A| = |P(A)| = 2^n$.

Exercise 0.1.5.6

$F : A \to 2^A$. For each $a \in A$, $(Fa)(a)$ is either 1 or 0. We can define $f \in 2^A$ such that $fa = (1 - (Fa))a$ for all a. Then $fa \neq (Fa)a$ for all $a \in A$.

Now we can show f is not in the range of F. Suppose there exists $b \in A$ such that $Fb = f$. But then $(Fb)b = 1 - fb = fb \Leftrightarrow 1 = 0$ which is a contradiction.

Exercise 0.2.1.1

Let $X = \{a, b\}$, we have $T = \{\{a\}, \emptyset, X\}$ and $T = \{\{b\}, \emptyset, X\}$ with concrete and discrete topologies. Therefore 4 distinct topologies.

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Let $X = \{a, b, c\}$,

For 2 elements topolgy, we have the concrete topology $\{X, \emptyset\}$. Total of 1.

For 3 elements topology, we have $T = \{\{a\}, \emptyset, X\}$ (3 of this kind). $T = \{\{a, b\}, \emptyset, X\}$ (3 of this kind). Total of 6.

For 4 elements topology, we have $T = \{\{a, b\}, \{a\}, \emptyset, X\}$ ($3 \times 2 = 6$ of this kind). $T = \{\{a, b\}, \{c\}, \emptyset, X\}$ (3 of this kind). Total of 9.

For 5 elements topology, $T = \{\{a, b\}, \{a, c\}, \{a\}, \emptyset, X\}$ (3 of this kind). Total of 3. $T = \{\{a, b\}, \{a\}, \{b\}, \emptyset, X\}$ (3 of this kind). Total of 6.

For 6 elements topology, $T = \{\{a, b\}, \{a, c\}, \{a\}, \{b\}, \emptyset, X\}$ ($3 \times 2 = 6$ of this kind). Total of 6

For 8 elements topology, there's only 1 which is $P(X)$.

Therefore X has 1 + 6 + 9 + 6 + 6 + 1 = 29 distinct topologies.

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Exercise 0.2.1.3

(a) If $x \in A^- \cup B^-$, then x is in the all closed set that contain A or all closed sets that contain B which implies x is in all close sets that contain $A \cup B$ since any close sets that contains $A \cup B$ contain A and B. Therefore $x \in (A \cup B)^-$.

Let $x \in (A \cup B)^-$. Suppose $x \notin A^-$ and $x \notin B^-$, then there exists closed set D_A and D_B containing A and B respectively such that $x \notin D_A$ and $x \notin D_B$. But $A \cup B \subset D_A \cup D_B$ and finite union of closed set is also closed. So $D_A \cup D_B$ is a closed set that covers $A \cup B$. So x must be in $D_A \cup D_B$. We have reached contradiction that $x \in D_A$ or $x \in D_B$.

Therefore $(A \cup B)^- = A^- \cup B^-$.

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(b) By definition of closure.

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(c) A^- is closed since arbitrary intersection of closed set is closed. A^- is the smallest closed set that contains itself. So $A^- = (A^-)^-$

■

(d) $X = X - \emptyset$ is open, therefore $\emptyset$ is closed. By the same argument in (c). $\emptyset^- = \emptyset$.

■

Exercise 0.2.1.2

We state the dual proposition for interior operator ^{0}.

(a) $(A \cap B)^0 = A^0 \cap B^0$

Proof: If $x \in A^0 \cap B^0$, then $x \in O_A \cap O_B$ for some open set $O_A \subset A$ and $O_B \subset B$. Finite intersect of open sets is open and $O_A \cap O_B \subset A \cap B$. Therefore $x \in (A \cap B)^0$.

If $x \in (A \cap B)^0$, then $x \in O$ for some open set $O \subset A \cap B$. Note that $O \subset A$ and $O \subset B$. Therefore $x \in A^0$ and $x \in B^0$. So $x \in A^0 \cap B^0$.

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(b) $A^0 \subset A$

Proof: Follow by definition of interior.

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(c) $(A^0)^0 = A^0$

Proof: The interior of A^0 is the largest open set that is contained in A^0 which is itself.

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(d) $\emptyset^0 = \emptyset$

Proof: $\emptyset$ is open set and by argument in (c). It follows.

$\blacksquare$

Exercise 0.2.4.1

(a) The metric $d_p(x,y) = \left(\sum_i |u^i x - u^i y|^p\right)^{1/p}$ for fixed x and y has the form of $d(p) = f(p)^{g(p)}$ where $f(p) = \sum_i (c_i)^p$, $c_i = |u^i x - u^i y| \geq 0$ and $g(p) = \frac{1}{p}$.

$$\begin{aligned}
d(p) &= f(p)^{g(p)} \\
\ln(d(p)) &= g(p) \ln f(p) \\
\frac{d'}{d} &= g' \ln f + \frac{gf'}{f} \\
d' &= d\left[g' \ln f + \frac{gf'}{f}\right] \\
d' &= d\left[-\frac{1}{p^2} \ln\left(\sum_i (c_i)^p\right) + \frac{\sum_i (c_i)^p \ln c_i}{p \sum_i (c_i)^p}\right] \\
&= \frac{d}{p}\left[\frac{\sum_i (c_i)^p \ln c_i}{\sum_i (c_i)^p} - \frac{1}{p} \ln\left(\sum_i (c_i)^p\right)\right] \\
&\leq \frac{d}{p}\left[\ln c_{\max} - \frac{1}{p} \ln\left(\sum_i (c_i)^p\right)\right] \\
&= \frac{d}{p^2}\left[\ln c_{\max}^p - \ln\left(\sum_i (c_i)^p\right)\right] \\
&= -\frac{d}{p^2}\left[\ln\left(\sum_i (c_i)^p\right) - \ln c_{\max}^p\right] \leq 0
\end{aligned}$$

Therefore $d_p(x,y)$ is non-increasing.

$\blacksquare$

(b) $d_1(x,y) = \sum_i c_i$ and $d_\infty(x,y) = c_{\max}$. Therefore

$$d_1(x,y) = \sum_i c_i \leq \sum_i c_{\max} = nc_{\max} = nd_\infty(x,y)$$

$\blacksquare$

Since we have

$$\begin{aligned}
d_p(x,y) &= (\sum_i c_i^p)^{1/p} \\
&\le (nc_{\max}^p)^{1/p} \\
\ln d_p(x,y) &= \frac{1}{p}\ln(nc_{\max}^p) = \frac{\ln n}{p} + \ln c_{\max} \\
\lim_{p\to\infty} \ln d_p(x,y) &= \lim_{p\to\infty} \frac{\ln n}{p} + \ln c_{\max} \\
&= \ln c_{\max}
\end{aligned}$$

Therefore $d_\infty(x,y) = \lim_{p\to\infty} d_p(x,y) = c_{\max}$

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(c) First we show d_t and d_∞ are strongly equivalent. By (b), we have shown one direction. Next we shown there exists k such that $kd_\infty \le d_t$.

Consider $\frac{d_t^t}{d_\infty^t}$.

$$\frac{d_t^t}{d_\infty^t} = \frac{\sum c_i^t}{\sum c_{\max}^t} = \sum^n \left(\frac{c_i}{c_{\max}}\right)^t \ge 1$$

Notice that every term is greater or equal to 0 and there would be a term that is equal to 1 ($c_i = c_{\max}$). So the sum is greater or equal to 1 which means $k = 1$. (We can get this conclusion from (a) as well.)

$$d_\infty \le d_s \le nd_\infty$$

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Now for any s, t, d_s, d_t are strongly equivalent. By definition, $d_s^s = \sum c_i^s$ and $d_t^t = \sum c_i^t$

Let $t = s + \delta$ and $\delta > 0$. So $t > s$.

$$\begin{aligned}
d_t^t = d_t^{s+\delta} &= \sum c_i^{s+\delta} \\
d_t^s &= \frac{\sum c_i^{s+\delta}}{d_t^\delta} \\
&\ge \frac{c_{\max}^\delta \sum c_i^s}{d_t^\delta} \\
&= \frac{d_\infty^\delta \sum c_i^s}{d_t^\delta} \\
&\ge \frac{d_\infty^\delta \sum c_i^s}{(nd_\infty)^\delta}, \text{ By (b)} \\
&= \frac{d_s^s}{n^\delta}
\end{aligned}$$

Therefore, $n^{\delta/s} d_t \ge d_s$

Furthermore, by (a), the metric is non-increase. We have

$$d_t \leq d_s \leq n^{t/s-1} d_t$$

So they are strongly equivalent.

$\blacksquare$

Exercise 0.2.5.1

Suppose $(X, T_X), (Y, T_Y)$ are Hausdorff and $(x, y) \in X \times Y$. There exists neighorhoods $U_x, V_x \in T_X$ such that $U_x \cap V_x = \emptyset$ and $U_y, V_y \in T_Y$ such that $U_y \cap V_y = \emptyset$. By definition, $U_x \times U_y$ and $V_x \times V_y$ are neighorhood for (x, y). Since $U_x \times U_y \cap V_x \times V_y = \emptyset$ by definition of product. So $X \times Y$ is Hausdorff.

$\blacksquare$

Exercise 0.2.6.1

Let $f : X \to X$ be any function map from $(X, 2^X)$ to (X, T). For any $Y \in T$, $f^{-1}(Y) \in 2^X$ by definition of 2^X. f^{-1} maps open sets to open sets hence continueous.

$\blacksquare$

Let f be a continuous function from $(X, \{\emptyset, X\})$ to $(X, 2^X)$ and f is not constant. There exists $x_1, x_2 \in X$ and $y_1 = fx_1$, $y_2 = fx_2$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$. Since f is continous and point set are open in 2^X, $f^{-1}(\{y_1\}) = X = f^{-1}(\{y_2\})$. This means all $x \in X$ maps to y_1 or y_2. If $y_1 \neq y_2$ then f is not a function. Therefore $y_1 = y_2$ contradicts our assume. So f is constant.

$\blacksquare$

Exercise 0.2.6.2

$\tan$ and $\arctan$ are differentiable hence continous. They are also 1-1 and onto. Hence $\tan$ is homeomorphism.

Exercise 0.2.7.1

Let $f : X \to Y$ be homeomorphism and suppose $A \subset X$ is connected. If $f(A)$ is not connected, then there exists disjoint open sets $D_1 \cap f(A), D_2 \cap f(A)$ in relative topology of $f(A)$ such that their union is $f(A)$. Then

$$\begin{aligned} f^{-1}(D_1 \cap f(A) \cup D_2 \cap f(A)) &= f^{-1}(D_1 \cap f(A)) \cup f^{-1}(D_2 \cap f(A)) \\ &= f^{-1}(D_1) \cap f^{-1}(A) \cup f^{-1}(D_2) \cap f^{-1}(A) \\ &= f^{-1}(A) \cap \left[f^{-1}(D_1) \cup f^{-1}(D_2)\right] \\ &= A \end{aligned}$$

. Since f is homeomorphism, $f^{-1}(D_1 \cap f(A)) = f^{-1}(D_1) \cap A$ and $f^{-1}(D_2) \cap A$ are two disjoint open sets in relative topology of A and their union is A. This contradicts with the assumption that A is connected.

$\blacksquare$

Exercise 0.2.7.2

(a) Let $B = \{x | x \in A, x \text{ polygonal connected to } a \in A\}$. Obviously $B \subset A$. To prove B is open, we show that $B^0 = B$. Suppose $y \in B$ and $y \notin B^0$. Then y isn't in any open set contained in B.There exists an open ball $U \subset A$ containing y such that $U - B \neq \emptyset$. But $z \in U - B \subset A$ is polygonal connected to y and hence to a. So the points in $U - B$ are in B. By contradiction, $y \in B$ implies $y \in B^0$. Hence $B^0 = B$ means B is open.

$\blacksquare$

(b) Suppose A is not polygonally connected. Then for any $x \in A$ define U_x to be the set of points in A that is polygonal connected to x and V_x to be the set of points in A that is not polygonal connected to x. By our assumption, $V_x \neq \emptyset$ and $U_x \neq \emptyset$. Furthermore $V_x \cup U_x = A$ and $V_x \cap U_x = \emptyset$.

Note that A is open by definition of relative topology. Then we have U_x and V_x being open as well from the result of (a). Therefore A is not connected. By contrapositive, We have proven A connected $\Rightarrow$ A is polygonal connected.

$\blacksquare$

Exercise 0.2.8.1

Suppose $f : X \to Y$ is homeomorphism and $A \subset X$ is compact. For any open covering C_A of A. There exists a finite subcover C'_A of A. Define $D_A = \{f(C) | C \in C_A\}$. Then D_A is an open covering for $f(A)$ since f maps open set to open set. It follows that $D'_A = \{f(C') | C' \in C'_A\}$ is a finite subcovering for $f(A)$. Hence $f(A)$ is compact. Compactness is preserved.

$\blacksquare$

Exercise 0.2.9.1

(a) Let (X, T_X) be locally compact, and $(Y, Y \cap T_X)$ be a subspace where Y is closed. For any $x \in Y$, there exists compact neighborhood $U \subset X$ of x. We want to show $U \cap Y$ is the compact neighborhood for x in Y.

Suppose C is an open covering for $U \cap Y$, then $X - Y$ is open and covers the portion $U - Y$. Therefore $C \cup \{X - Y\}$ is an open covering for U. Given that U is compact, there is a finite subcovering $D \subset C \cup \{X - Y\}$. And D works as a finite subcovering for $U \cap Y$ since $X - Y$ doesn't need to be in D (it doesn't cover the portion $Y \cap U$). So $Y \cap U$ is compact. Y is locally compact.

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(b) In a discrete space X, for any $x \in X$ and any open covering of $\{x\}$, there exists an open set A in the covering such that $\{x\} \subset A$. Therefore A is the finite open subcovering. So $\{x\}$ is compact neighborhood of x. So X is locally compact.

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(c) For any $x \in \mathbb{R}^n$, there exists a closed ball centered at x. Closed ball are compact in $\mathbb{R}^n$. Therefore $\mathbb{R}^n$ is locally compact.

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Exercise 0.2.10.1

(a) Let A be countable subset of metric space X such that $A^- = X$. For $x \in X$ and a neighborhood U_x containing x. We want to show $A \cap U_x \neq \emptyset$.

Suppose $A \cap U_x = \emptyset$, then $U_x \subset X - A = A^- - A \subset \partial A$. This implies $x \in \partial A$. Since U_x is neighborhood of x, and by definition of boundary points, $U_x \cap A \neq \emptyset$. By contradiction, there exists some a from A such that $a \in U_x$.

Next we can construct an open ball $B_r(a) \subset U_x$ where $r \in \mathbb{R}$ and $r > 0$. For such r, there exists rational number r_0 such that $0 < r_0 < r$. Then we have constructed an open ball with rational radii inside any neighborhood of $x \in X$ which forms a basis of neighorhoods for X. Hence X is separable.

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(b) Let A be the set of rational in $\mathbb{R}$. Then $A^- = \mathbb{R}$. By (a), $\mathbb{R}$ is separable. And by Exercise 0.2.10.2 (below), product of separable spaces is separable, it follows that $\mathbb{R}^n$ is separable.

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Exercise 0.2.10.2

Let X, Y be separable and A_X, A_Y be the basis respectively. We want to show that $A_X \times A_Y$ forms a basis for $X \times Y$.

First $A_X \times A_Y$ is countable. Second, For any point $(x, y) \in X \times Y$ and any open neighbordhood U containing (x, y). U can be written as $U_X \times U_Y$ by definition of product space. Then U_X, U_Y are neighorhood that contains x, y respectively. Since A_X, A_Y are basis, there exist $V_X \in A_X, V_Y \in A_Y$ such that $x \in V_X \subset U_X$ and $y \in V_Y \subset U_Y$. Then $V_X \times V_Y \in A_X \times A_Y$ and $V_X \times V_Y \subset U$. Therefore $A_X \times A_Y$ forms a basis for $X \times Y$. $X \times Y$ is separable.

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