Mechanics

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1 Basic Elementary Principles

Result 1 *(Newton's law)*

1. In an inertial reference frame, an object remains at rest or constant velocity unless acted upon by external force.
2. $$\dot{\boldsymbol{p}} = \frac{d(m\boldsymbol{v})}{dt} = \boldsymbol{F}^{(e)}$$
3. Two particles exert forces on each other $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$

1.1 Single particle

Result 2 (Conversation Theorem for linear momentum)

$$\textit{If } \boldsymbol{F} = 0, \ \textit{then } \dot{\boldsymbol{p}} = 0$$

Angular momentum of the particle around point O is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. Torque $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$.

$$\boldsymbol{N} = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \frac{d\boldsymbol{L}}{dt} \equiv \dot{\boldsymbol{L}}$$

Result 3 (Conservation Theorem for angular momentum)

$$\text{If } \boldsymbol{N} = 0, \text{then } \dot{\boldsymbol{L}} = 0$$

If force field $\boldsymbol{F}$ is conservative ($\oint \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ or $\int_a^b \boldsymbol{F} \cdot d\boldsymbol{s} = T_b - T_a$ is the same for any path between a and b). There exists a potential scalar field V such that

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r})$$

Result 4 (Energy Conservation Theorem for a particle)

$$E = T + V \text{ is conserved where } T \text{ and } V \text{ are kinetic energy and potential energy respectively}$$

1.2 System of particles

Center of mass:

$$\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum m_i} = \frac{\sum_i m_i \boldsymbol{r}_i}{M}$$

Let $\boldsymbol{F}_i^{(e)}$ external force acting on particle i-th, and $\boldsymbol{F}_{ji}$ is the force exerted by j-th particle on i-th particle in the system,

$$m_i \frac{d^2(\boldsymbol{r_i})}{dt^2} = \dot{\boldsymbol{p}}_{\boldsymbol{i}} = \sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)}$$

Summing over all particles, we get

$$\frac{d^2}{dt^2}\sum_i m_i \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i^{(e)} + \sum_{i\neq j} \boldsymbol{F}_{ji} \tag{1}$$

$$\Rightarrow M\frac{d^2\boldsymbol{R}}{dt^2} = \sum_i \boldsymbol{F}_i^{(e)} \equiv \boldsymbol{F}^{(e)} \tag{2}$$

where $R = \frac{m_i \boldsymbol{r}_i}{\sum m_i} = \frac{m_i \boldsymbol{r}_i}{M}$ is the center of mass. The total linear momentum $\boldsymbol{P} = \sum m_i \frac{d\boldsymbol{r}_i}{dt} = M\frac{d\boldsymbol{R}}{dt}$.

Conservation theorem for the linear Momentum of a system of particles: if the total external force is zero, the total linear momentum is conserved.

Conservation Theorem for total angular momentum of a system of particles: $\boldsymbol{L}$ is constant in time if the applied external torque is zero. $\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}^{(e)}$

The total angular momentum of a system of particles is $\boldsymbol{L} = \boldsymbol{R} \times M\boldsymbol{v} + \sum_i \boldsymbol{r}_i' \times \boldsymbol{p}_i'$ where $\boldsymbol{r}_i' = \boldsymbol{R} - \boldsymbol{r}_i$ position relative to the center of mass.