Graph Theory Note

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1 Definitions

Graph G is a finite set V(vertex set) with irreflexive, **symmetric** relation R on V. E the edge set is the set of symmetric pairs in R. $|V|$ is **order** of G and $|E|$ is size of G. A (p, q) graph is a graph with order p and size q.

Subgraph H of G is when $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

$G-e$ is subgraph of G where $V(G) = V(G-e)$ and $E(G) - \{e\} = E(G-e)$.

$G - v$ is subgraph of G where $V(G) - \{v\} = V(G - v)$ and $E(G) - \{(v, u) \in E(G) | \forall u \in V(G)\} = E(G - v)$

Degree of vertice v denoted by $\deg_G v$ is the number of edges incident with v. v is odd or even is when its degree is odd or even.

Adjacent vertices v and w means $(v, w) \in E(G)$. Adjacent edges (v, w_1) and (v, w_2) are when $w_1 \neq w_2$.

Digraph(Directed Graph) G has a relation R that is not necessarily symmetric. $(u, v) \in E$ is called a directed edge or an arc.

Network is a graph/digraph with a function $f : E \to \Re$. When $f : E \to \{\pm 1\}$ it is called a signed graph.

Multigraph is a network when f is a multi map, e.g. $f = \{(v_1, v_2, 1), (v_1, v_2, 2)\}$

Loop-graph is when R is no longer irreflexive.

Isomorphism from G_1 to G_2 is a bijection $\phi : V(G_1) \to V(G_2)$ s.t $(v_1, v_2) \in E(G_1) \iff (\phi(v_1), \phi(v_2)) \in E(G_2)$.

Graph traversal

1. A u_1-u_n **walk** is a sequence $\{u_1, \ldots, u_n\}$ where (u_i, u_{i+1}) is an edge.
2. A u_1-u_n **trail** is a walk with no repeating edges.
3. A u_1-u_n **path** is a walk with no repeating vertices.
4. u-u trail that contains at least 3 edges is a **circuit**
5. A **cycle** is a circuit with no repeating vertices.

Connected graph G is when u-v path exists for any $u \neq v \in V(G)$. Otherwise a graph is disconnected.

Component H of a graph G is the largest connected subgraph that contains itself.

Cut-vertex is a vertex v in connected graph G such that $G - v$ is disconnected.

Bridge is an edge e in connected graph G such that $G - e$ is disconnected.

2 Examples Modelings

Friendship can be represented as a graph.

City can be represented as a digraph where road intersection are vertices and arcs as one-way or two- way streets.

Employer/Employee hierarchy can be represented as diagraph with people as vertices and arc connecting subordinate with their supervisor.

3 Results

1. For (p, q) graph, $\sum_v \deg v = 2q$.

2. Every graph has even number of odd vertices.

3. Let G be connected graph, e is a bridge iff e not in any cycle of G.

4 Graph Algorithms