# Chapter 8: Hypothesis Testing

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## Exercise 8.1

Let $H_0$ be the hypothesis that the coin is fair, aka $\theta_0 = 0.5$.

### Likelihood ratio test

The likelihood method for independent Bernoulli trial is $L(\theta|x) = \theta^{560}(1-\theta)^{1000-560}$ where $560$ is the number of head. We know that $\theta = \frac{560}{1000}$ is the empirical estimator of $\theta$ that maximizes the likelihood function. So the ratio test gives

$$\log \lambda(x) = \log \frac{L(0.5|x)}{L(0.56|x)} = 1000 \log 0.5 - \{560 \log 0.56 + 440 \log 0.44\} \Rightarrow \lambda(x) \approx 0.00073$$

$0.00073$ is too small so $H_0$ can be rejected. Therefore the coin is not fair.

### Check the probability of such event

Assume coin is fair $\theta = 0.5$, then the CDF of the process is

$$P(X \geq x) = \sum_{i=x}^{1000} P(X = i) = \sum_{i=x}^{1000} \binom{1000}{i} 0.5^i 0.5^{1000-i}$$

Then we can check if the event $X \geq 560$ is a small event for this $\theta$. Indeed it is $\approx 0.08\%$. So the coin is not fair.

## Exercise 8.2

Let $H_0$ be the null hypothesis that the incident number of this year is generated from $Pois(\lambda)$ where $\lambda < 15$. To estimate whether the generating distribution has decreased in $\lambda$, we let $\pi(\lambda) = \mathcal{N}(\mu = \frac{10+15}{2} = 12.5, \sigma^2 = (15-10)^2) = \frac{1}{5\sqrt{2\pi}} \exp\left(-0.5\frac{(12.5-\lambda)^2}{5^2}\right)$ (we choose midpoint between 15 and 10 is because 10 is the MLE for the latest year's data point)

$$
\begin{aligned}
P(\lambda < 15|x = 10) &= \sum_{\lambda=0}^{14} P(\lambda|x = 10) \\
&= \frac{\sum_0^{14} P(x = 10|\lambda)\pi(\lambda)}{\sum_0^{\infty} P(x = 10|\lambda)\pi(\lambda)} \\
&= \frac{\sum_0^{14} P(x = 10|\lambda)}{\sum_0^{30} P(x = 10|\lambda)} \text{ (Let the prior } P(\lambda) = Uniform(0, 30)) \\
&= \frac{\sum_{i=0}^{14} i^{10} e^{-i}}{\sum_{i=0}^{30} i^{10} e^{-i}} \approx 0.87
\end{aligned}
$$

Type I Error is about 1- 0.87 = 0.13, not small. If we compute $P(x <= 10|\lambda = 15) \approx 0.11$, so $\lambda = 15$ is still capable of producing such result. It is inconclusive.

# Exercise 8.3

$H_0$ region is $\theta \leq \theta_0$ and $H_1$'s region is $\theta > \theta_0$. Then define $b = m\theta_0$ to be the expected success count if $\theta = \theta_0$.

A Bernoulli trial $f(y|\theta) = I_{Y=1}\theta + I_{Y=0}(1 - \theta)$. Then the likelihood function

$$
L(\theta|y) = \prod_1^m f(y_i|\theta) = \binom{m}{k} \theta^k (1 - \theta)^{m-k}
$$

where $k = \sum_i Y_i$

To maximize $L$, we can use the MLE which is the $\theta_{\text{max}} = \frac{k}{m}$. To reject $H_0$, we need the MLE to stay out $H_0$ region, so $\frac{k}{m} > \theta_0 \Rightarrow \sum_i Y_i = k > m\theta_0 = b$

# Exercise 8.5

**(a)** The likelihood function

$$
L(\theta, v|x) = \prod_{i=1}^n f(x_i|\theta, v) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}} \prod_i I_{[v,\infty)}(x_i) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}}, \text{(given } v \leq x_{\text{min}}, 0 \text{ otherwise)}
$$

Holding $\theta$ fixed, $L$ is a monotonic polynomial function of $v$. So $v_0 = x_{(1)}$ the boundary of $v$ maximizes $L$.

Let $\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \log\left(x_{(1)}^n\right) - \log(\prod_i x_i) = 0$, then we get

$$
\theta_0 = \frac{n}{\log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)} = \frac{n}{T(x)}
$$

where $T \equiv \log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)$

**(b)** $H_0 = \{(\theta = 1, v)\}$, So the rejection region of $H_0$ is

$$
\lambda(x) = \frac{\sup_{\theta=1} L(\theta, v|x)}{\sup_\theta L(\theta, v|x)} = \frac{T^n}{n^n} \exp(n - T) \leq c
$$

We take derivative of $\lambda$,

$$\partial_T \lambda = \left(\frac{T}{n}\right)^{n-1} e^{n-T}\left(1 - \frac{T}{n}\right)$$

So the monotonicity of $\lambda$ is determined by $(1 - T/n)$. When $T = n$, $\lambda$ reaches maximum of 1, when $T < n$, $\lambda$ increases monotonically and when $T > n$, $\lambda$ decreases monotonically. Therefore, if $\lambda(x) < c$ for $0 < c \leq 1$, we will have two values $c_1$ and $c_2$ (on left/right side of $n$ respectively) where $T \leq c_1 \leq n$ or $n \leq c_2 \leq T$.

# Exercise 8.6

(a) Let

$$L(\theta, \mu|x, y) = f(x_1, \ldots, x_n, y_1, \ldots, y_m|\theta, \mu) = \prod_i^n f(x_i|\theta) \prod_i^m f(y_i|\mu) = \theta^n \mu^m \exp\left(-\theta \sum_i^n x_i - \mu \sum_i^m y_i\right)$$

be the likelihood function of the joint distribution. Then

$$\ln(L(\theta, \mu)) = n\ln(\theta) + m\ln(\mu) - \theta\sum_i^n x_i - \mu\sum_i^m y_i$$

. For $H_0$ where $\theta = \mu$, we solve $\frac{d\ln(L(\theta,\mu|\theta=\mu))}{d\theta} = 0$ and get

$$\hat{\theta}_0 = \frac{n+m}{\sum_i^n x_i + \sum_i^m y_i}$$

as the MLE under the constraint.

For $H_1$, we solve $\frac{\partial \ln L}{\partial \theta} = 0$ and $\frac{\partial \ln L}{\partial \mu} = 0$ and get

$$\hat{\theta}_1 = \frac{n}{\sum_i^n x_i}, \quad \hat{\mu_1} = \frac{n}{\sum_i^m y_i}$$

.

Therefore

$$\lambda((x,y)) = \frac{\sup_{\theta=\mu} L(\theta,\mu|x,y)}{\sup_{\theta,\mu} L(\theta,\mu|x,y)} = \frac{L(\hat{\theta_0}, \hat{\theta_0}|x,y)}{L(\hat{\theta_1}, \hat{\mu_1})} = \frac{(n+m)^{n+m}}{n^n m^m}\frac{\left(\sum_i^n x_i\right)^n \left(\sum_i^m y_i\right)^m}{\left(\sum_i^n x_i + \sum_i^m y_i\right)^{n+m}}$$

(b) To show that $T = \frac{\sum X}{\sum X + \sum Y}$ can also give the same LRT, we just need to express the LRT in terms of $T$. Let $C = \frac{(n+m)^{n+m}}{n^n m^m}$, then

$$\lambda((x,y)) = C\frac{\left(\sum_i^n x_i\right)^n \left(\sum_i^m y_i\right)^m}{\left(\sum_i^n x_i + \sum_i^m y_i\right)^{n+m}} = C\left(\frac{\sum_i^n x_i}{\sum_i^n x_i + \sum_i^m y_i}\right)^n \left(\frac{\sum_i^m y_i}{\sum_i^n x_i + \sum_i^m y_i}\right)^m = CT^n(1-T)^m$$

(c) Let $U = \sum_1^n X_i$, then we calculate the MGF, $M_U(t) = E\left[e^{\sum_i t}\right] = \prod E\left[e^{X_i t}\right] = \prod M_{X_i}(t) = \frac{1}{(1-\theta t)^n}$ since $H_0$ is true. It matches the gammar distribution's MGF, therefore $U = \sum_i X_i \sim \text{Gamma}(n,\theta)$. Similarly $V = \sum_1^m Y_i \sim \text{Gamma}(m,\theta)$.

Next is to find the distribution of $T = \frac{U}{U+V}$. Since $U, V$ are independent, so

$$f(u,v) = f(u)f(v) = \text{Gamma}(n,\theta)\text{Gamma}(m,\theta) = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} u^{n-1} v^{m-1} e^{-\frac{1}{\theta}(u+v)}$$

Let $S = U + V$, then $T = \frac{U}{U+V} = \frac{U}{S}$. We have $U = TS, V = S(1-T)$. So the Jacobian $|J| = |S|$. By change of variables, we have

$$g(t,s) = f(u(t,s))f(v(t,s))|J| = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1} s^{n+m-1} e^{-\frac{1}{\theta}s}$$

Next we maginalize $s$,

$$\begin{aligned}
g(t) = \int_0^\infty g(t,s)ds &= \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1} \int_0^\infty s^{n+m-1} e^{-\frac{1}{\theta}s} ds \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1} \int_0^\infty \frac{1}{\Gamma(n+m)\theta^{n+m}} s^{n+m-1} e^{-\frac{1}{\theta}s} ds \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1} \\
&= \text{Beta}(n,m)
\end{aligned}$$

# Exercise 8.7

(a) For the integration of the pdf to converge, $\lambda > 0$.

$$LRT(x) = \frac{\sup_{\theta \le 0, \lambda} L(\lambda, \theta|x)}{\sup_{\theta, \lambda} L(\lambda, \theta|x)} = \frac{\sup_{\theta \le 0, \lambda} \frac{1}{\lambda^n} \exp\left(-\frac{1}{\lambda}\sum_i (x_i - \theta)\right) I(\theta \le x_{(1)})}{\sup_{\theta, \lambda} \frac{1}{\lambda^n} \exp\left(-\frac{1}{\lambda}\sum_i (x_i - \theta)\right) I(\theta \le x_{(1)})}$$

Let $f(\theta, \lambda) = \frac{1}{\lambda^n} \exp\left(-\frac{1}{\lambda}\sum_i (x_i - \theta)\right)$ and $\log f = -n \log \lambda - \frac{1}{\lambda}\sum_i (x_i - \theta)$. Take the partial derivative wrt $\theta$, $\frac{\partial \log f}{\partial \theta} = \frac{n\theta}{\lambda}$ which means $f$ is a monotonic increasing function along $\theta$ (attains maximum at $\theta = 0$).

Let $g(\theta, \lambda) = f(\theta, \lambda) I(\theta \le x_{(1)})$, then $g(\theta, \lambda)$ attains maximum at $\theta = x_{(1)}$ holding $\lambda$ fixed. Then we have $g(x_{(1)}, \lambda) = \frac{1}{\lambda^n} \exp\left(-\frac{1}{\lambda}\sum_i (x_i - x_{(1)})\right))$

Next we take the deritvative wrt to $\lambda$, $\frac{\partial \log g(x_{(1)}, \lambda)}{\partial \lambda} = -\frac{n}{\lambda} + \frac{\sum_i (x_i - x_{(1)})}{\lambda^2} = 0$ implies when $\lambda = \bar{x} - x_{(1)}$, $g(x_{(1)}, \lambda)$ attains maximum.

Now we have

$$\sup_{\theta, \lambda} \frac{1}{\lambda^n} \exp\left(-\frac{1}{\lambda}\sum_i (x_i - \theta)\right) I(\theta \le x_{(1)}) = g(x_{(1)}, \bar{x} - x_{(1)})$$

If we constrain $\theta \le 0$, then by the same computation, $g$ will attain maximum when $\theta = \min(0, x_{(1)})$ and $\lambda = \bar{x} - \min(0, x_{(1)})$.

Now we can write the LRT as

$$\begin{aligned}
LRT(x) &= \frac{\sup_{\theta \le 0, \lambda} L(\lambda, \theta|x)}{\sup_{\theta, \lambda} L(\lambda, \theta|x)} \\
&= \frac{g(\min(0, x_{(1)}), \bar{x} - \min(0, x_{(1)}))}{g(x_{(1)}, \bar{x} - x_{(1)})} \\
&= \begin{cases} 1 & x_{(1)} < 0 \\ \left(1 - \frac{x_{(1)}}{\bar{x}}\right)^n & x_{(1)} \ge 0 \end{cases}
\end{aligned}$$

# Exercise 8.8

(a) For $\mathcal{N}(\theta, a\theta)$, the likelihood function

$$L = \left(\frac{1}{a\theta\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum\left(\frac{x_i - \theta}{a\theta}\right)^2\right) = \left(\frac{1}{a\theta\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2a^2}\sum\left(\frac{x_i}{\theta} - 1\right)^2\right)$$

Then

$$\log L = -n(\log a + \log\theta + \log\left(\sqrt{2\pi}\right)) - \frac{1}{2a^2}\sum\left(\frac{x_i}{\theta} - 1\right)^2$$

Now we need to find $\sup_{a=1,\theta} L$ and $\sup_{a,\theta} L$. We will take the derivative wrt to $\theta$ first.

$$\begin{aligned}\frac{\partial \log L}{\partial\theta} &= 0\\ \Rightarrow a^2 &= \frac{n-1}{n}\frac{s^2}{\theta^2} - \frac{\overline{x}}{\theta}\end{aligned}$$

where $s^2$ is the sample variance and $\overline{x}$ is the sample mean.

$$\frac{\partial \log L}{\partial a} = 0 \Rightarrow a^2 = \frac{1}{n}\sum\left(\frac{x_i}{\theta} - 1\right)^2 = \frac{n-1}{n}\frac{s^2}{\theta^2} - \frac{2\overline{x}}{\theta} + 1$$

When $a = 1$, we solve

$$\begin{aligned}\frac{n-1}{n}\frac{s^2}{\theta^2} - \frac{\overline{x}}{\theta} &= a^2 = 1\\ \Rightarrow \theta^2 + \overline{x}\theta - \frac{n-1}{n}s^2 &= 0\\ \Rightarrow \hat{\theta}_R &= \frac{\sqrt{\overline{x}^2 + 4\frac{(n-1)}{n}s^2} - \overline{x}}{2}\end{aligned}$$

When $a$ is unrestricted, we combined the two derivative above and get $\hat{\theta} = \overline{x}$ and $\hat{a} = \sqrt{\frac{n-1}{n}\frac{s}{\overline{x}}}$

So

$$LRT(x) = \frac{\sup_{a=1,\theta} L}{\sup_{a,\theta} L} = \frac{L(a = 1, \theta = \hat{\theta}_R|x)}{L(a = \hat{a}, \theta = \hat{\theta}|x)}$$

# Exercise 8.12

(a) By Exercise 8.37 below, when $\sigma^2$ is known, we can write the power function of the test as

$$\beta(\mu) = P_\mu(X \in R) = P_\mu(\overline{X} > 0 + z_\alpha\frac{\sigma}{\sqrt{n}})$$

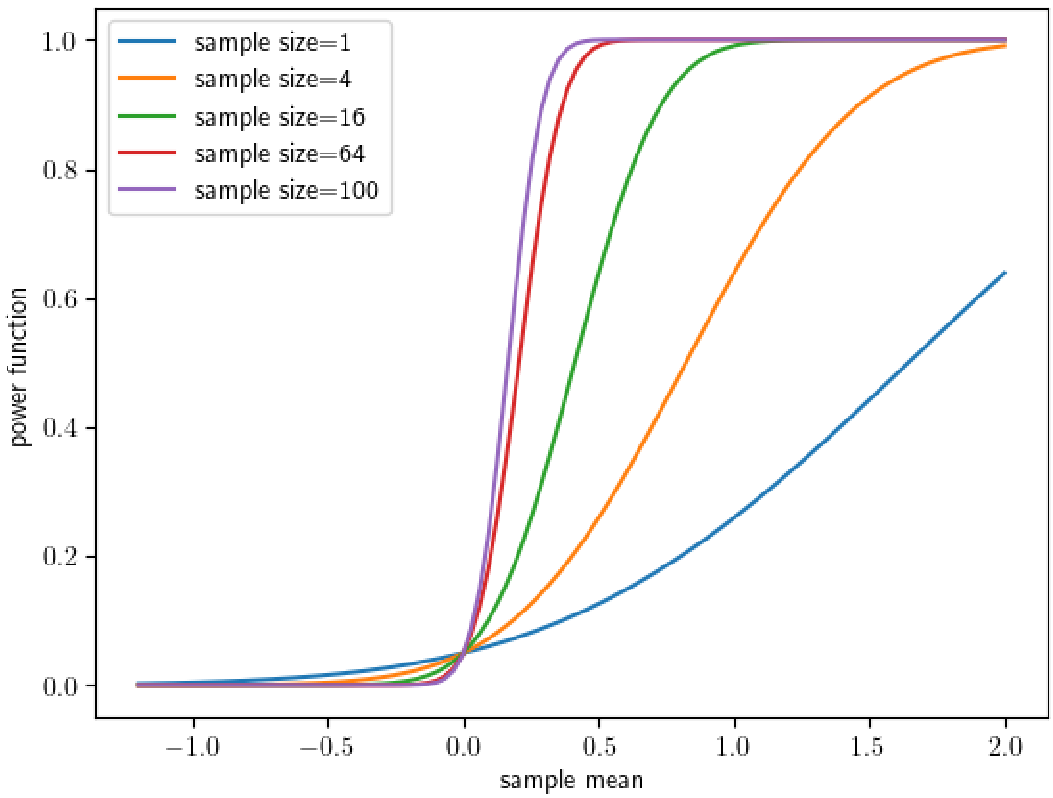
where $P(Z > z_alpha) = z_\alpha$. By subtracting both side with $\mu$, we have

$$\beta(\mu) = P_\mu(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > z_\alpha - \frac{\mu}{\frac{\sigma}{\sqrt{n}}}) = P_\mu(Z > z_\alpha - \frac{\mu}{\sigma/\sqrt{n}})$$

Now we can plot the value of the power function for each $\mu \in [-1, 2]$.

```
import numpy as np
import scipy.stats as st
import matplotlib.pyplot as plt

sigma = 1
samples = [1, 4, 16, 64, 100]
test_size = 0.05
z_test_size = st.norm.ppf(1 - test_size)

```

```
def get_power_fun(n):
return lambda mean: 1 - st.norm.cdf(z_test_size - mean / (sigma / np.sqrt(n)))

data = []
mean_range = np.linspace(-1.2, 2, num=100)

ax = plt.subplot()
for n in samples:
values = get_power_fun(n)(mean_range)
ax.plot(mean_range, values, label=f"sample size={n}")

ax.set_title="Power function by sample sizes"
ax.set_ylabel("power function")
ax.set_xlabel("sample mean")
ax.legend()
```




# Exercise 8.37

(a) Given $Z \sim \mathcal{N}(0, 1)$ and $P(Z > z_\alpha) = \alpha$, Consider

$$\sup_{\theta \in \Theta_0} P_\theta \left( \overline{X} > \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right)$$

$$\sup_{\theta \in \Theta_0} P_\theta \left( \frac{\overline{X} - \theta}{\sigma/\sqrt{n}} > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha \right)$$

$$\sup_{\theta \leq \theta_0} P_\theta \left( z > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha \right)$$

the above probability is an increasing function of $\theta$ when $\theta \le \theta_0$, therefore

$$\sup_{\theta \le \theta_0} P_\theta \left( z > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha \right) = P_{\theta_0}(z > z_\alpha) = \alpha$$

So $\overline{X} > \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$ is indeed a test of size $\alpha$ that rejects $H_0$.

■

To derive the test from LRT is, we have

$$\begin{aligned}
\lambda(x) &= \frac{\sup_{\theta \le \theta_0} L(x, |\theta, \sigma^2)}{\sup_{\theta \le \theta_0} L(x, |\theta, \sigma^2)} = \frac{\exp\{-\frac{1}{2\sigma^2}\sum(x_i - \min(\bar{x}, \theta_0))^2\}}{\exp\{-\frac{1}{2\sigma^2}\sum(x_i - \theta_0)^2\}} \\
&= \begin{cases} 1, & \theta_0 = \min(\overline{x}, \theta_0) \\ \exp\{-\frac{n}{2\sigma^2}(\overline{x} - \theta_0)^2\}, & \overline{x} = \min(\overline{x}, \theta_0) \end{cases}
\end{aligned}$$

The rejection region for $H_0$ is $\{x \in R | \lambda(x) < c\}$ for $c \in [0, 1]$. We can write

$$\begin{aligned}
&\sup_{\theta \le \theta_0} P(x \in R) \\
&\Rightarrow \sup_{\theta \le \theta_0} P(\lambda(x) < c) \\
&\Rightarrow \sup_{\theta \le \theta_0} P\left(\exp\Big\{-\frac{n}{2\sigma^2}(\overline{x} - \theta_0)^2\Big\} < c\right) \\
&\Rightarrow \sup_{\theta \le \theta_0} P\left(\overline{x} > \theta_0 + \frac{\sigma}{\sqrt{n}}\sqrt{2\ln(1/c)}\right) \\
&\Rightarrow \sup_{\theta \le \theta_0} P\left(\overline{x} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\right) \qquad \text{, (Simply choose } z_\alpha \equiv \sqrt{2\ln(1/c)}) \\
&\Rightarrow \sup_{\theta \le \theta_0} P\left(\frac{\overline{X} - \theta}{\sigma/\sqrt{n}} > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha\right) \\
&\Rightarrow \sup_{\theta \le \theta_0} P\left(z > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha\right), \qquad (Z \sim \mathcal{N}(0, 1)) \\
&\Rightarrow P\left(z > z_\alpha\right) = \alpha
\end{aligned}$$