Mechanics

May 20, 2025

1 Basic Elementary Principles

Result 1 *(Newton's law)*

1. In an inertial reference frame, an object remains at rest or constant velocity unless acted upon by external force.
2. $$\dot{\boldsymbol{p}} = \frac{d(m\boldsymbol{v})}{dt} = \boldsymbol{F}^{(e)}$$
3. Action and reaction: $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$ (Additional condition for strong version: $\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji} = 0$)

1.1 Single particle

Result 2 (Conservation of Linear Momentum)

$$\textit{If } \boldsymbol{F} = 0, \ \textit{then } \dot{\boldsymbol{p}} = 0$$

Angular momentum of the particle around point O is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. Torque $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$.

$$\boldsymbol{N} = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \frac{d\boldsymbol{L}}{dt} \equiv \dot{\boldsymbol{L}}$$

Result 3 (Conservation of Angular Momentum)

$$\text{If } \boldsymbol{N} = 0, \text{ then } \dot{\boldsymbol{L}} = 0$$

If force field $\boldsymbol{F}$ is conservative ($\oint \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ or $\int_a^b \boldsymbol{F} \cdot d\boldsymbol{s} = T_b - T_a$ is the same for any path between a and b). There exists a potential scalar field V such that

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r})$$

Result 4 (Conservation of Energy)

$$E = T + V \text{ is conserved where } T \text{ and } V \text{ are kinetic energy and potential energy respectively}$$

1.2 System of particles

Center of mass:

$$\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum m_i} = \frac{\sum_i m_i \boldsymbol{r}_i}{M}$$

Let $\boldsymbol{F}_i^{(e)}$ external force acting on particle i-th, and $\boldsymbol{F}_{ji}$ is the force exerted by j-th particle on i-th particle in the system,

$$m_i \frac{d^2(\boldsymbol{r_i})}{dt^2} = \dot{\boldsymbol{p}}_{\boldsymbol{i}} = \sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)}$$

Summing over all particles, we get

$$\dot{\boldsymbol{p}} = M\frac{d^2\boldsymbol{R}}{dt^2} = \sum_i \boldsymbol{F}_i^{(e)} \equiv \boldsymbol{F}^{(e)}$$

Result 5 (Conservation of Linear Momentum for a system of particles)

if the total external force $\boldsymbol{F}^{(e)} = 0$, the total linear momentum $\boldsymbol{p}$ is conserved.

The change of total angular momentum:

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{L} &= \sum_i \boldsymbol{r_i} \times \dot{\boldsymbol{p}}_{\boldsymbol{i}} = \sum_i \boldsymbol{r_i} \times (\boldsymbol{F}_i^{(e)} + \sum_j \boldsymbol{F}_{ji}) \\
&= \boldsymbol{N}^{(e)} + \sum_{i,j} \boldsymbol{r_i} \times \boldsymbol{F}_{ji} \\
&= \boldsymbol{N}^{(e)} + \sum_{i<j} \boldsymbol{r_i} \times \boldsymbol{F}_{ji} + \sum_{j<i} \boldsymbol{r_i} \times \boldsymbol{F}_{ji} \\
&= \boldsymbol{N}^{(e)} + \sum_{i<j} \boldsymbol{r_i} \times \boldsymbol{F}_{ji} + \sum_{i<j} \boldsymbol{r_j} \times \boldsymbol{F}_{ij} \\
&= \boldsymbol{N}^{(e)} + \sum_{i<j} \boldsymbol{r_i} \times \boldsymbol{F}_{ji} - \sum_{i<j} \boldsymbol{r_j} \times \boldsymbol{F}_{ji} \\
&= \boldsymbol{N}^{(e)} + \sum_{i<j} (\boldsymbol{r_i} - \boldsymbol{r_j}) \times \boldsymbol{F}_{ji} \\
&= \boldsymbol{N}^{(e)} + \sum_{i<j} \boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji}
\end{aligned}$$

When the internal force abides strong version of Newton's 3rd law, internal force pair has the same direction as the two particles $\boldsymbol{r}_{ij}$. Then $\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji} = 0$. Forces satisfy strong Newton's 3rd law are **central**.

Result 6 (Conservation of Angular Momentum for a system of particles)

$\boldsymbol{L}$ is constant in time if applied external torque $\boldsymbol{N}^{(e)} = 0$ and internal forces are central

Let $\boldsymbol{r}_i = \boldsymbol{R} + \boldsymbol{r}_i'$ where $\boldsymbol{r}_i'$ is the position of ith particle wrt to $\boldsymbol{R}$ instead of O. Similarly, $\boldsymbol{v}_i = \boldsymbol{v} + \boldsymbol{v}_i'$ where $\boldsymbol{v} = \dot{\boldsymbol{R}}$. We can find $\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{M} = \frac{\sum_i m_i (\boldsymbol{R} + \boldsymbol{r}_i')}{M} = \boldsymbol{R} + \sum_i \boldsymbol{r}_i'$ which means

$$\sum_i \boldsymbol{r}_i' = 0, \qquad \sum_i \dot{\boldsymbol{r}}_i' = \sum_i \boldsymbol{v}_i' = 0$$

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Result 7 The total angular momentum about O is the angular momentum of center of mass + angular momentum of each particle around the center of mass.

$$\begin{aligned} \boldsymbol{L} &= \sum_i \boldsymbol{r_i} \times \boldsymbol{p_i} \\ &= \boldsymbol{R} \times M\boldsymbol{v} + \sum_i \boldsymbol{r}'_i \times \boldsymbol{p}'_i + \sum_i \boldsymbol{R} \times m_i \boldsymbol{v}'_i + \sum_i m_i \boldsymbol{r}'_i \times \boldsymbol{v} \\ &= \boldsymbol{R} \times M\boldsymbol{v} + \sum_i \boldsymbol{r}'_i \times \boldsymbol{p}'_i \end{aligned}$$

In general $\boldsymbol{L}$ depends on choice of O through $\boldsymbol{R}$ except when the center of mass is at rest $\boldsymbol{R} = 0$.

Result 8 The total kinetic energy of the system is

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\boldsymbol{v} + \boldsymbol{v}'_i)(\boldsymbol{v} + \boldsymbol{v}'_i) = \frac{1}{2} M v^2 + \sum_i \frac{1}{2} m_i (v'_i)^2$$

The ith particle has trajectory $\boldsymbol{r}_i(t)$ from time a to b. Then the work is the change in kinetic energy.

$$W_i = \int_a^b \boldsymbol{F}_i \cdot d\boldsymbol{r_i} = \int_a^b \frac{d(m_i v_i)}{dt} \cdot \frac{d\boldsymbol{r_i}}{dt} dt = \int_a^b m_i \dot{\boldsymbol{v}}_i \cdot \boldsymbol{v}_i dt = \int_a^b d(\frac{1}{2} m_i v_i^2) = T_i(b) - T_i(a)$$

Also we can write

$$W_i = \int_a^b \boldsymbol{F}_i \cdot d\boldsymbol{r_i} = \int_a^b \boldsymbol{F}_i^{(e)} \cdot d\boldsymbol{r_i} + \sum_j \int_a^b \boldsymbol{F}_{ji} \cdot d\boldsymbol{r_i}$$

If the external force $\boldsymbol{F}_i^{(e)}$ is conservative, then $\boldsymbol{F}_i^{(e)} = -\boldsymbol{\nabla} V_i$

$$\int_a^b \boldsymbol{F}_i^{(e)} \cdot d\boldsymbol{r_i} = -\int_a^b \boldsymbol{\nabla} V_i \cdot d\boldsymbol{r_i} = -\int_a^b \frac{\partial V_i}{\partial x_i} dx_i + \frac{\partial V_i}{\partial y_i} dy_i + \frac{\partial V_i}{\partial z_i} dz_i = -\int_a^b dV_i = -V_i|_a^b$$

If the action/reaction forces are conservative and central, $V_{ji}(|\boldsymbol{r_i} - \boldsymbol{r_j}|)$ only depends on distance between ith and jth particle, hence it is symmetric $V_{ji} = V_{ij}$. Also

$$\boldsymbol{F}_{ji} = -\boldsymbol{\nabla}_i V_{ij} = -\boldsymbol{F}_{ij} = \boldsymbol{\nabla}_j V_{ji} = \boldsymbol{\nabla}_j V_{ij}$$

We can find the gradients of V_{ij} with respect to $\boldsymbol{r_{ij}} = \boldsymbol{r_i} - \boldsymbol{r_j}$,

$$\boldsymbol{\nabla}_i V_{ij} = \frac{\partial V_{ij}}{\partial \boldsymbol{r_i}} = \frac{\partial V_{ij}}{\partial \boldsymbol{r_{ij}}} \frac{\partial \boldsymbol{r_{ij}}}{\partial \boldsymbol{r_i}} = \frac{\partial V_{ij}}{\partial \boldsymbol{r_{ij}}} = \boldsymbol{\nabla}_{ij} V_{ij} = -\boldsymbol{\nabla}_j V_{ij}$$

The work done by internal forces can be written in sums of pairs of potentials between ith and jth particle.

$$
\begin{aligned}
\sum_{i,j}\int_a^b \boldsymbol{F}_{ji}\cdot d\boldsymbol{r_i} &= -\sum_{i,j}\int_a^b \boldsymbol{\nabla}_j V_{ji}\cdot d\boldsymbol{r_i} \\
&= -\sum_{i<j}\int_a^b \boldsymbol{\nabla}_j V_{ji}\cdot d\boldsymbol{r_i} + \boldsymbol{\nabla}_i V_{ij}\cdot d\boldsymbol{r_j} \\
&= -\sum_{i<j}\int_a^b \boldsymbol{\nabla}_j V_{ij}\cdot d\boldsymbol{r_i} - \boldsymbol{\nabla}_j V_{ij}\cdot d\boldsymbol{r_j} \\
&= -\sum_{i<j}\int_a^b \boldsymbol{\nabla}_j V_{ij} d(\boldsymbol{r_i}-\boldsymbol{r_j}) \\
&= -\sum_{i<j}\int_a^b \boldsymbol{\nabla}_j V_{ij} d(\boldsymbol{r_{ij}}) \\
&= \sum_{i<j}\int_a^b \boldsymbol{\nabla}_{ij} V_{ij} d(\boldsymbol{r_{ij}}) \qquad \text{(note that } \boldsymbol{\nabla}_{ij}V_{ij} = -\boldsymbol{\nabla}_j V_{ij}) \\
&= \frac{1}{2}\sum_{i,j}\int_a^b \boldsymbol{\nabla}_{ij} V_{ij} d(\boldsymbol{r_{ij}}) \\
&= \frac{1}{2}\sum_{i,j} V_{ij}\big|_a^b
\end{aligned}
$$

Result 9 Total potential energy V when external and internal forces are conservative and internal forces also central is:

$$V = \sum_i V_i + \frac{1}{2}\sum_{i,j} V_{ij}$$

Such that total energy $T+V$ is conserved

On a rigid body, $|\boldsymbol{r_{ij}}|$ is constant between any two points. Then $0 = d(r_{ij}^2) = d(\boldsymbol{r}_{ij}\cdot\boldsymbol{r}_{ij}) = 2\boldsymbol{r}_{ij}\cdot d(\boldsymbol{r}_{ij})$. Therefore $d(\boldsymbol{r}_{ij})$ is perpendicular to $\boldsymbol{r}_{ij}$ which means the work done by internal force $\boldsymbol{F}_{ji}\cdot d(\boldsymbol{r}_{ij}) = 0$