Chapter 8: Hypothesis Testing

January 28, 2024

Exercise 8.1

Let H_0 be the hypothesis that the coin is fair, aka $\theta_0 = 0.5$.

Likelihood ratio test

The likelihood method for independent Bernoulli trial is $L(\theta|x) = \theta^{560}(1-\theta)^{1000-560}$ where 560 is the number of head. We know that $\theta = \frac{560}{1000}$ is the empirical estimator of θ that maximizes the likelihood function. So the ratio test gives

$$\log \lambda(x) = \log \frac{L(0.5|x)}{L(0.56|x)} = 1000 \log 0.5 - \{560 \log 0.56 + 440 \log 0.44\} \Rightarrow \lambda(x) \approx 0.00073$$

0.00073 is too small so H_0 can be rejected. Therefore the coin is not fair.

Check the probability of such event

Assume coin is fair $\theta = 0.5$, then the CDF of the process is

$$P(X \geq x) = \sum_{i=x}^{1000} P(X = i) = \sum_{i=x}^{1000} \binom{1000}{i} 0.5^i 0.5^{1000-i}$$

Then we can check if the event $X \geq 560$ is a small event for this θ. Indeed it is $\approx 0.08\%$. So the coin is not fair.

Exercise 8.2

Let H_0 be the null hypothesis that the incident number of this year is generated from $Pois(\lambda)$ where $\lambda < 15$. To estimate whether the generating distribution has decreased in λ, we let $\pi(\lambda) = \mathcal{N}(\mu = \frac{10+15}{2} = 12.5, \sigma^2 = (15-10)^2) = \frac{1}{5\sqrt{2\pi}} \exp\left(-0.5 \frac{(12.5-\lambda)^2}{5^2}\right)$ (we choose midpoint between 15 and 10 is because 10 is the MLE for the latest year's data point)

$$\begin{aligned}
P(\lambda < 15|x=10) &= \sum_{\lambda=0}^{14} P(\lambda|x=10) \\
&= \frac{\sum_0^{14} P(x=10|\lambda)\pi(\lambda)}{\sum_0^{\infty} P(x=10|\lambda)\pi(\lambda)} \\
&= \frac{\sum_0^{14} P(x=10|\lambda)}{\sum_0^{30} P(x=10|\lambda)} \text{ (Let the prior } P(\lambda) = Uniform(0,30)) \\
&= \frac{\sum_{i=0}^{14} i^{10}e^{-i}}{\sum_{i=0}^{30} i^{10}e^{-i}} \approx 0.87
\end{aligned}$$

Type I Error is about 1- 0.87 = 0.13, not small. If we compute $P(x <= 10|\lambda = 15) \approx 0.11$, so $\lambda = 15$ is still capable of producing such result. It is inconclusive.

Exercise 8.3

H_0 region is $\theta \leq \theta_0$ and H_1's region is $\theta > \theta_0$. Then define $b = m\theta_0$ to be the expected success count if $\theta = \theta_0$.

A Bernoulli trial $f(y|\theta) = I_{Y=1}\theta + I_{Y=0}(1-\theta)$. Then the likelihood function

$$L(\theta|y) = \prod_1^m f(y_i|\theta) = \binom{m}{k}\theta^k(1-\theta)^{m-k}$$

where $k = \sum_i Y_i$

To maximize L, we can use the MLE which is the $\theta_{\text{max}} = \frac{k}{m}$. To reject H_0, we need the MLE to stay out H_0 region, so $\frac{k}{m} > \theta_0 \Rightarrow \sum_i Y_i = k > m\theta_0 = b$

Exercise 8.5

(a) The likelihood function

$$L(\theta, v|x) = \prod_{i=1}^n f(x_i|\theta, v) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}} \prod_i I_{[v,\infty)}(x_i) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}}, \text{ (given } v \leq x_{\text{min}}, 0 \text{ otherwise)}$$

Holding θ fixed, L is a monotonic polynomial function of v. So $v_0 = x_{(1)}$ the boundary of v maximizes L.

Let $\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \log\left(x_{(1)}^n\right) - \log(\prod_i x_i) = 0$, then we get

$$\theta_0 = \frac{n}{\log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)} = \frac{n}{T(x)}$$

where $T \equiv \log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)$

(b) $H_0 = \{(\theta = 1, v)\}$, So the rejection region of H_0 is

$$\lambda(x) = \frac{\sup_{\theta=1} L(\theta, v|x)}{\sup_\theta L(\theta, v|x)} = \frac{T^n}{n^n}\exp(n-T) \leq c$$

We take derivative of λ,

$$\partial_T \lambda = \left(\frac{T}{n}\right)^{n-1} e^{n-T}\left(1-\frac{T}{n}\right)$$

So the monotonicity of λ is determined by $(1-T/n)$. When $T=n$, λ reaches maximum of 1, when $T<n$, λ increases monotonically and when $T>n$, λ decreases monotonically. Therefore, if $\lambda(x)<c$ for $0<c\leq 1$, we will have two values c_1 and c_2 (on left/right side of n respectively) where $T\leq c_1\leq n$ or $n\leq c_2\leq T$.

Exercise 8.6

(a) Let

$$L(\theta,\mu|x,y) = f(x_1,\ldots,x_n,y_1,\ldots,y_m|\theta,\mu) = \prod_i^n f(x_i|\theta)\prod_i^m f(y_i|\mu) = \theta^n\mu^m \exp\left(-\theta\sum_i^n x_i - \mu\sum_i^m y_i\right)$$

be the likelihood function of the joint distribution. Then

$$\ln(L(\theta,\mu)) = n\ln(\theta) + m\ln(\mu) - \theta\sum_i^n x_i - \mu\sum_i^m y_i$$

. For H_0 where $\theta=\mu$, we solve $\frac{d\ln(L(\theta,\mu|\theta=\mu))}{d\theta}=0$ and get

$$\hat{\theta}_0 = \frac{n+m}{\sum_i^n x_i + \sum_i^m y_i}$$

as the MLE under the constraint.

For H_1, we solve $\frac{\partial \ln L}{\partial\theta}=0$ and $\frac{\partial \ln L}{\partial\mu}=0$ and get

$$\hat{\theta}_1 = \frac{n}{\sum_i^n x_i}, \quad \hat{\mu_1} = \frac{n}{\sum_i^m y_i}$$

.

Therefore

$$\lambda((x,y)) = \frac{\sup_{\theta=\mu} L(\theta,\mu|x,y)}{\sup_{\theta,\mu} L(\theta,\mu|x,y)} = \frac{L(\hat{\theta_0},\hat{\theta_0}|x,y)}{L(\hat{\theta_1},\hat{\mu_1})} = \frac{(n+m)^{n+m}}{n^n m^m}\frac{\left(\sum_i^n x_i\right)^n\left(\sum_i^m y_i\right)^m}{\left(\sum_i^n x_i+\sum_i^m y_i\right)^{n+m}}$$

(b) To show that $T=\frac{\sum X}{\sum X+\sum Y}$ can also give the same LRT, we just need to express the LRT in terms of T. Let $C=\frac{(n+m)^{n+m}}{n^n m^m}$, then

$$\lambda((x,y)) = C\frac{\left(\sum_i^n x_i\right)^n\left(\sum_i^m y_i\right)^m}{\left(\sum_i^n x_i+\sum_i^m y_i\right)^{n+m}} = C\left(\frac{\sum_i^n x_i}{\sum_i^n x_i+\sum_i^m y_i}\right)^n\left(\frac{\sum_i^m y_i}{\sum_i^n x_i+\sum_i^m y_i}\right)^m = CT^n(1-T)^m$$

(c) Let $U=\sum_1^n X_i$, then we calculate the MGF, $M_U(t)=E\left[e^{\sum_i t}\right]=\prod E\left[e^{X_i t}\right]=\prod M_{X_i}(t)=\frac{1}{(1-\theta t)^n}$ since H_0 is true. It matches the gammar distribution's MGF, therefore $U=\sum_i X_i\sim$ Gamma(n,θ). Similarly $V=\sum_1^m Y_i\sim$ Gamma(m,θ).

Next is to find the distribution of $T=\frac{U}{U+V}$. Since U,V are independent, so

$$f(u,v) = f(u)f(v) = \text{Gamma}(n,\theta)\text{Gamma}(m,\theta) = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}}u^{n-1}v^{m-1}e^{-\frac{1}{\theta}(u+v)}$$

Let $S = U + V$, then $T = \frac{U}{U+V} = \frac{U}{S}$. We have $U = TS, V = S(1-T)$. So the Jacobian $|J| = |S|$. By change of variables, we have

$$g(t,s) = f(u(t,s))f(v(t,s))|J| = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1}s^{n+m-1}e^{-\frac{1}{\theta}s}$$

Next we maginalize s,

$$\begin{aligned}
g(t) &= \int_0^\infty g(t,s)ds = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1}\int_0^\infty s^{n+m-1}e^{-\frac{1}{\theta}s}ds \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1}\int_0^\infty \frac{1}{\Gamma(n+m)\theta^{n+m}} s^{n+m-1}e^{-\frac{1}{\theta}s}ds \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1} \\
&= \text{Beta}(n,m)
\end{aligned}$$

Exercise 8.7

(a) For the integration of the pdf to converge, $\lambda > 0$.

$$LRT(x) = \frac{\sup_{\theta\le 0,\lambda} L(\lambda,\theta|x)}{\sup_{\theta,\lambda} L(\lambda,\theta|x)} = \frac{\sup_{\theta\le 0,\lambda} \frac{1}{\lambda^n}\exp\left(-\frac{1}{\lambda}\sum_i(x_i-\theta)\right)I(\theta\le x_{(1)})}{\sup_{\theta,\lambda} \frac{1}{\lambda^n}\exp\left(-\frac{1}{\lambda}\sum_i(x_i-\theta)\right)I(\theta\le x_{(1)})}$$

Let $f(\theta,\lambda) = \frac{1}{\lambda^n}\exp\left(-\frac{1}{\lambda}\sum_i(x_i-\theta)\right)$ and $\log f = -n\log\lambda - \frac{1}{\lambda}\sum_i(x_i-\theta)$. Take the partial derivative wrt θ, $\frac{\partial \log f}{\partial\theta} = \frac{n\theta}{\lambda}$ which means f is a monotonic increasing function along θ (attains maximum at $\theta = 0$).

Let $g(\theta,\lambda) = f(\theta,\lambda)I(\theta \le x_{(1)})$, then $g(\theta,\lambda)$ attains maximum at $\theta = x_{(1)}$ holding λ fixed. Then we have $g(x_{(1)},\lambda) = \frac{1}{\lambda^n}\exp\left(-\frac{1}{\lambda}\sum_i(x_i - x_{(1)})\right))$

Next we take the deritvative wrt to λ, $\frac{\partial \log g(x_{(1)},\lambda)}{\partial\lambda} = -\frac{n}{\lambda} + \frac{\sum_i(x_i-x_{(1)})}{\lambda^2} = 0$ implies when $\lambda = \bar{x} - x_{(1)}$, $g(x_{(1)},\lambda)$ attains maximum.

Now we have

$$\sup_{\theta,\lambda}\frac{1}{\lambda^n}\exp\left(-\frac{1}{\lambda}\sum_i(x_i-\theta)\right)I(\theta\le x_{(1)}) = g(x_{(1)},\bar{x}-x_{(1)})$$

If we constrain $\theta \le 0$, then by the same computation, g will attain maximum when $\theta = \min(0,x_{(1)})$ and $\lambda = \bar{x} - \min(0,x_{(1)})$.

Now we can write the LRT as

$$\begin{aligned}
LRT(x) &= \frac{\sup_{\theta\le 0,\lambda} L(\lambda,\theta|x)}{\sup_{\theta,\lambda} L(\lambda,\theta|x)} \\
&= \frac{g(\min(0,x_{(1)}),\bar{x}-\min(0,x_{(1)}))}{g(x_{(1)},\bar{x}-x_{(1)})} \\
&= \begin{cases} 1 & x_{(1)} < 0 \\ \left(1 - \frac{x_{(1)}}{\bar{x}}\right)^n & x_{(1)} \ge 0 \end{cases}
\end{aligned}$$

Exercise 8.12

(a) By Exercise 8.37 below, when σ^2 is known, we can write the power function of the test as

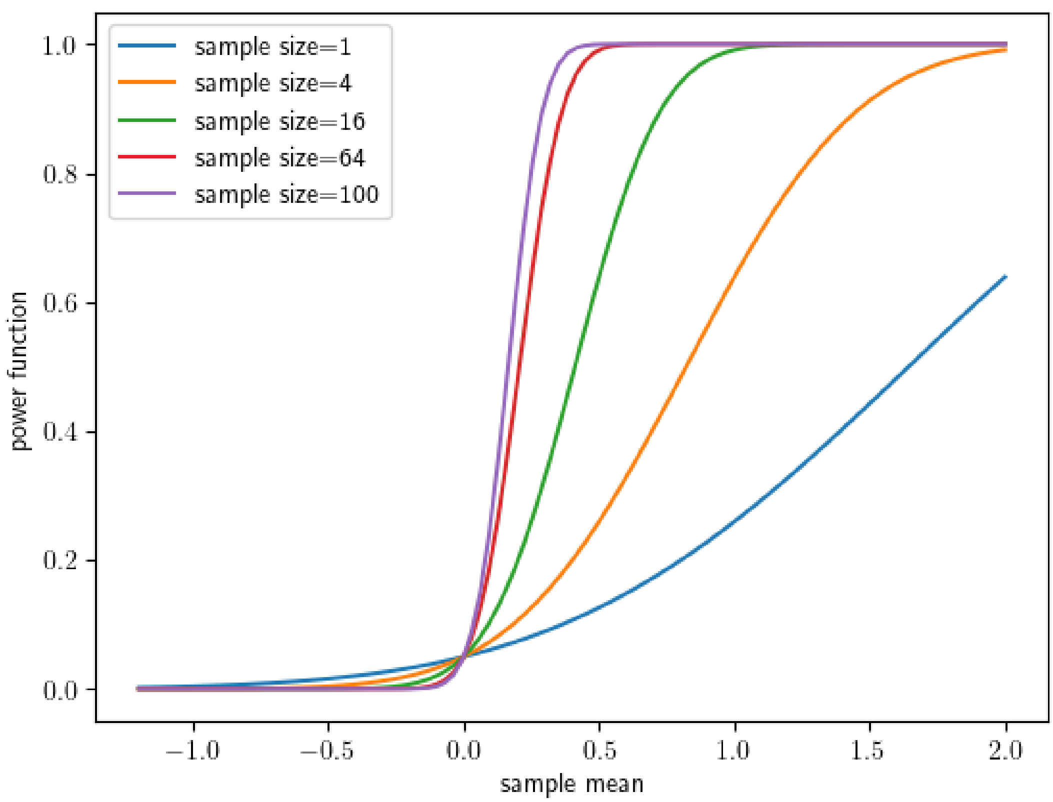
$$\beta(\mu) = P_\mu(X \in R) = P_\mu(\overline{X} > 0 + z_\alpha\frac{\sigma}{\sqrt{n}})$$

where $P(Z > z_a lpha) = z_\alpha$. By subtracting both side with μ, we have

$$\beta(\mu) = P_\mu(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > z_\alpha - \frac{\mu}{\frac{\sigma}{\sqrt{n}}}) = P_\mu(Z > z_\alpha - \frac{\mu}{\sigma/\sqrt{n}})$$

Now we can plot the value of the power function for each $\mu \in [-1, 2]$.

```
import numpy as np
import scipy.stats as st
import matplotlib.pyplot as plt

sigma = 1
samples = [1, 4, 16, 64, 100]
test_size = 0.05
z_test_size = st.norm.ppf(1 - test_size)

def get_power_fun(n):
return lambda mean: 1 - st.norm.cdf(z_test_size - mean / (sigma / np.sqrt(n)))

data = []
mean_range = np.linspace(-1.2, 2, num=100)

ax = plt.subplot()
for n in samples:
values = get_power_fun(n)(mean_range)
ax.plot(mean_range, values, label=f"sample size={n}")

ax.set_title="Power function by sample sizes"
ax.set_ylabel("power function")
ax.set_xlabel("sample mean")
ax.legend()
```



Exercise 8.37

(a) Given $Z \sim \mathcal{N}(0,1)$ and $P(Z > z_\alpha) = \alpha$, Consider

$$\begin{aligned}
&\sup_{\theta\in\Theta_0} P_\theta\left(\overline{X} > \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}\right)\\
&\sup_{\theta\in\Theta_0} P_\theta\left(\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} > \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_\alpha\right)\\
&\sup_{\theta\leq\theta_0} P_\theta\left(z > \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_\alpha\right)
\end{aligned}$$

the above probability is an increasing function of θ when $\theta \leq \theta_0$, therefore

$$\sup_{\theta\leq\theta_0} P_\theta\left(z > \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_\alpha\right) = P_{\theta_0}(z > z_\alpha) = \alpha$$

So $\overline{X} > \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$ is indeed a test of size α that rejects H_0.

■

To derive the test from LRT is, we have

$$\begin{aligned}
\lambda(x) &= \frac{\sup_{\theta\leq\theta_0} L(x,|\theta,\sigma^2)}{\sup_{\theta\leq\theta_0} L(x,|\theta,\sigma^2)} = \frac{\exp\{-\frac{1}{2\sigma^2}\sum(x_i - \min(\bar{x},\theta_0))^2\}}{\exp\{-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2\}}\\
&= \begin{cases} 1, & \theta_0 = \min(\overline{x},\theta_0) \\ \exp\{-\frac{n}{2\sigma^2}(\overline{x}-\theta_0)^2\}, & \overline{x} = \min(\overline{x},\theta_0) \end{cases}
\end{aligned}$$

The rejection region for H_0 is $\{x \in R | \lambda(x) < c\}$ for $c \in [0,1]$. We can write

$$\begin{aligned}
&\sup_{\theta\leq\theta_0} P(x\in R)\\
&\Rightarrow \sup_{\theta\leq\theta_0} P(\lambda(x) < c)\\
&\Rightarrow \sup_{\theta\leq\theta_0} P\left(\exp\left\{-\frac{n}{2\sigma^2}(\overline{x}-\theta_0)^2\right\} < c\right)\\
&\Rightarrow \sup_{\theta\leq\theta_0} P\left(\overline{x} > \theta_0 + \frac{\sigma}{\sqrt{n}}\sqrt{2\ln(1/c)}\right)\\
&\Rightarrow \sup_{\theta\leq\theta_0} P\left(\overline{x} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\right) \qquad, \text{(Simply choose } z_\alpha \equiv \sqrt{2\ln(1/c)})\\
&\Rightarrow \sup_{\theta\leq\theta_0} P\left(\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} > \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_\alpha\right)\\
&\Rightarrow \sup_{\theta\leq\theta_0} P\left(z > \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_\alpha\right), \qquad (Z\sim\mathcal{N}(0,1))\\
&\Rightarrow P\left(z > z_\alpha\right) = \alpha
\end{aligned}$$