Geometry Note

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1 Definitions

Space of linear function $L(V, W)$ vector space of linear functions from V to W.

Dual Space $V^* = L(V, R)$. For each basis $\{e_i\}$ of V, there exists unique $\{e^i\}$ of V^* such that $e^i(e_j) = \delta^i_j$

Tensor Space $T^r_s = \underbrace{V \otimes V}_{\text{r times}} \otimes \underbrace{V^* \otimes V^*}_{\text{s times}}$ is space of multilinear functions on

$$\underbrace{V^* \times \ldots \times V^*}_{\text{r times}} \times \underbrace{V \times \ldots \times V}_{\text{s times}}$$

Tensor Product between A of (r, s) and B of (t, u), is

$$\begin{aligned} A \otimes B(\tau^1, \ldots, \tau^{r+t}, v_1, \ldots, v_{s+u}) \\ = A(\tau^1, \ldots, \tau^r, v_1, \ldots, v_s) \\ B(\tau^{r+1}, \ldots, \tau^{r+t}, v_{s+1}, \ldots, v_{s+u}) \end{aligned}$$

Vector Field X on coordinate neighborhood U of a manifold M, with coordinate x^i. For each point p, $X = X^i \partial_i$. $X[f] = X^i \partial_i f$

Change of Coordinates If Y has coordinate neighborhood V of y^i, then $Y^i = X^j \frac{\partial y^i}{\partial x^j}$

Map Differential(Pushforward) F_* is induced map $F_* : TM \to TN$ of C^∞ map $F : M \to N$. $F_*(v_p) = (F_* v)_{F(p)}$. With coordinate, $F_* = [\partial_j(y^i \circ F)]$, the Jacobian of F. Note that $y^i \circ F = F^i(x^1, \ldots, x^m)$

Tensor Bundle $T^r_s M$ of type (r, s) is the union of all tensor spaces $M^r_s(p)$ at each point $p \in M$.

Tangent Bundle $TM = T^1_0 M$,

Scalar Bundle $T^0_0 M = M \times \Re$,

Cotangent Bundle/ Differentials / Phase space $T^0_1 M$

Tensor Field T of type (r, s), $T(p) \in T^r_s M(p)$ for each p. $(1, 0)$ is vector field, $(0, 0)$ gives real-valued function. $(0, 1)$ gives differential.

Tensor Coordinate of T^r_s wrt coordinate x^i are d^{r+s} real-valued functions

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = T(dx^{i_1}, \ldots dx^{i_r}, \partial_{j_1}, \ldots, \partial_{j_s})$$

Tensor Product

Exterior Product

Differential forms p-form is C^∞ skew-symmetric covariant tensor field of degree p (type $(0, p)$). Local basis has $\binom{d}{p}$ p-forms $dx^{i_1} \cdots dx^{i_p}$ where $(i_1, \ldots, i_p)$ is increasing.