Chapter 2: Transformation and Expectations Exercises

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Exercise 2.1

(1) $y = g(x) = x^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. We will find $F_Y(y)$ instead.

$$F_X(x) = \int_{t=0}^{x} 42t^5(1-t)dt = 7t^6 - 6t^7|_0^x = 7x^6 - 6x^7$$

Since g is an increasing function,

$$F_Y(y) = F_X(g^{-1}(y)) = 7y^2 - 6y^{\frac{7}{3}}$$

Then pdf for y, $f_y(y) = \frac{d}{dy}F_Y = 14y - 14y^{\frac{4}{3}}$ where $y \in (0,1)$

(2) g is increasing, $f_X = 7e^{-7x}$ is continuous on $[0,\infty)$ and $g^{-1} = (y-3)/4$ has continuous derivative on $[3,\infty)$. Therefore, by theorem 2.1.5,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{7}{4}\exp(-\frac{7}{4}(y-3)), \quad y \in [3,\infty)$$

(3) We will find F_Y instead.

$$F_X(x) = \int_0^x 30t^2(1-t)^2dt = 10x^3 + 6x^5 - 15x^4$$

Since $g(x) = x^2$ is increasing on $(0,1)$. Therefore

$$F_Y(y) = F_X(g^{-1}(y)) = 10y^{3/2} + 6y^{5/2} - 15y^2$$

Pdf $f_Y(y) = \frac{d}{dy} = 15y^{1/2} + 15y^{3/2} - 30y$ for $y \in (0,1)$

Exercise 2.2

(b) $y = g(x) = -\log x$ is monotonic, $f_X(x)$ is continuous on $(0,1)$ and $x = g^{-1}(y) = e^{-y}$ has continuous derivative on $(0,\infty)$. Therefore we can use the theorem 2.1.5,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{(n+m+1)!}{n!m!}e^{-(n+1)y}(1-e^{-y})^m, \quad y \in (0,\infty)$$

(c) $y = g(x) = e^x$ is monotonic, $f_X(x)$ is continuous on $(0,\infty)$ and $x = g^{-1}(y) = \log x$ has continuous derivative of $1/x$ on $(0,\infty)$. Therefore we can use theorem 2.1.5

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{\log y}{y\sigma^2}\exp\left(-\frac{(\log y)^2}{2\sigma^2}\right)$$

Exercise 2.3

$f_Y(y) = P(Y = y) = P(\frac{X}{X+1} = y) = P(X = \frac{y}{1-y}) = \frac{1}{3}\left(\frac{2}{3}\right)^{y/(1-y)}$

Exercise 2.4

(a) $\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty}\frac{1}{2}\lambda e^{-\lambda x}dx + \int_{-\infty}^{0}\frac{1}{2}\lambda e^{\lambda x}dx = -\frac{1}{2}e^{-\lambda x}|_0^{\infty} + \frac{1}{2}e^{\lambda x}|_{-\infty}^{0} = 1$

(b)

$$\text{For } x < 0, \quad F_X(t) = \int_{-\infty}^{t}\frac{1}{2}\lambda e^{\lambda x}dx = \frac{1}{2}e^{\lambda t}$$

$$\text{For } x \geq 0, \quad F_X(t) = F_X(0) + \int_0^t \frac{1}{2}\lambda e^{-\lambda x}dx = 1 - \frac{1}{2}e^{-\lambda t})$$

(c)

$$P(|X| < t) = P(-|t| < X < |t|) = F_X(|t|) - F_X(-|t|) = 1 - \frac{1}{2}e^{-\lambda|t|}) - \frac{1}{2}e^{-\lambda|t|} = 1 - e^{-\lambda|t|})$$

Exercise 2.6

(a) $f_Y(y) = 1/2\exp(-|y|^{1/3})|\frac{y^{-2/3}}{3}| = \frac{1}{6}y^{-2/3}\exp(-|y|^{1/3})$

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)dy &= \int_0^{\infty} \frac{1}{6}y^{-2/3}\exp(-y^{1/3})dy + \int_{-\infty}^{0} \frac{1}{6}y^{-2/3}\exp(y^{1/3})dy \\ &= \frac{1}{6}(-3)\exp(-y^{1/3})|_0^{\infty} + \frac{1}{6}(3)\exp(y^{1/3})|_{-\infty}^{0} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

Exercise 2.15

Note that $X \wedge Y \leq X \vee Y$. They can be either X or Y and they are not equal as long as $X \neq Y$. Therefore $X + Y = (X \vee Y) + (X \wedge Y) \to (X \vee Y) = X + Y - (X \wedge Y)$. Apply expectation to both side. We get $\mathbf{E}(X \vee Y) = \mathbf{E}X + \mathbf{E}Y + \mathbf{E}(X \wedge Y)$

Exercise 2.23

(a) Note that $y \in [0, 1)$,

$$F_Y(y) = P(Y < y) = P(X^2 < y) = P(-\sqrt{y} < X < \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} (1 + x)/2dx = \sqrt{y}$$

Then

$$f_Y(y) = \frac{d}{dy}F_Y = \frac{1}{2}\frac{1}{\sqrt{y}}$$

(b)

$$EY = \int_0^1 \frac{y}{2}\frac{1}{\sqrt{y}} = \frac{1}{3}$$

$EY^2 = \int_0^1 \frac{y^2}{2}\frac{1}{\sqrt{y}} = \frac{1}{5}$

$$VarY = EY^2 - (EY)^2 = 1/5 - (1/3)^2 = 4/45$$

Exercise 2.30

(a) $M_X(t) = Ee^{tX} = \int_0^c e^{tx}/c dx = e^{tx}/(tc)|_0^c = \frac{e^{tc}-1}{tc}$

(b) $M_X(t) = Ee^{tX} = \int_0^c e^{tx}\frac{2x}{c^2}dx = \frac{2(tce^{tc}-e^tc+1)}{(ct)^2}$

(c)

$$\begin{aligned}
M_X(t) = Ee^{tX} &= \int \frac{1}{2\beta}\exp(tx - |x-\alpha|/\beta)dx \\
&= \frac{1}{2\beta}\int_{-\infty}^{\alpha}\exp(tx + \frac{x}{\beta} - \frac{\alpha}{\beta}) + \frac{1}{2\beta}\int_{\alpha}^{\infty}\exp(tx - \frac{x}{\beta} + \frac{\alpha}{\beta}) \\
&= \frac{\exp(-\alpha/\beta)}{2\beta}\frac{1}{t+1/\beta}\exp((t+1/\beta)x)|_{-\infty}^{\alpha} + \frac{\exp(\alpha/\beta)}{2\beta}\frac{1}{t-1/\beta}\exp((t-1/\beta)x)|_{\alpha}^{\infty} \quad (1) \\
&= \frac{e^{\alpha t}}{1-(\beta t)^2}
\end{aligned}$$

Note that in order for the two integral to be finite t must satisfies $-1/\beta \le t \le 1/\beta$.

(d)

$$M_X(t) = Ee^{tX} = \sum_{x=0}^{\infty} e^{tx}\binom{r+x-1}{x}p^r(1-p)^x = \sum_{x=0}^{\infty}\binom{r+x-1}{x}p^r(e^t(1-p))^x$$

Note that the pmf sums to 1.

$$\text{i.e. } \sum_{x=0}^{\infty}\binom{r+x-1}{x}p^r(1-p)^x = 1$$

We can use this as an identity by adjusting p. Replace $p \leftarrow 1-e^t(1-p)$. Then we have

$$\sum_{x=0}^{\infty}\binom{r+x-1}{x}(1-e^t(1-p))^r(e^t(1-p))^x = 1$$

Which gives $\sum_{x=0}^{\infty}\binom{r+x-1}{x}(e^t(1-p))^x = \frac{1}{(1-e^t(1-p))^r}$. Then

$$M_X(t) = \sum_{x=0}^{\infty}\binom{r+x-1}{x}p^r(e^t(1-p))^x = \frac{p^r}{(1-e^t(1-p))^r}$$

Where $0 < 1-e^t(1-p) \le 1 \Rightarrow t < -\log(1-p)$