

Chapter 8: Hypothesis Testing

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Exercise 8.1

Let H_0 be the hypothesis that the coin is fair, aka $\theta_0 = 0.5$.

Likelihood ratio test

The likelihood method for independent Bernoulli trial is $L(\theta|x) = \theta^{560}(1 - \theta)^{1000-560}$ where 560 is the number of head. We know that $\theta = \frac{560}{1000}$ is the empirical estimator of θ that maximizes the likelihood function. So the ratio test gives

$$\log \lambda(x) = \log \frac{L(0.5|x)}{L(0.56|x)} = 1000 \log 0.5 - \{560 \log 0.56 + 440 \log 0.44\} \Rightarrow \lambda(x) \approx 0.00073$$

0.00073 is too small so H_0 can be rejected. Therefore the coin is not fair.

Check the probability of such event

Assume coin is fair $\theta = 0.5$, then the CDF of the process is

$$P(X \geq x) = \sum_{i=x}^{1000} P(X = i) = \sum_{i=x}^{1000} \binom{1000}{i} 0.5^i 0.5^{1000-i}$$

Then we can check if the event $X \geq 560$ is a small event for this θ . Indeed it is $\approx 0.08\%$. So the coin is not fair.

Exercise 8.2

Let H_0 be the null hypothesis that the incident number of this year is generated from $Pois(\lambda)$ where $\lambda < 15$. To estimate whether the generating distribution has decreased in λ , we let $\pi(\lambda) = \mathcal{N}(\mu = \frac{10+15}{2} = 12.5, \sigma^2 = (15 - 10)^2) = \frac{1}{5\sqrt{2\pi}} \exp\left(-0.5 \frac{(12.5-\lambda)^2}{5^2}\right)$ (we choose midpoint between 15 and 10 is because 10 is the MLE for the latest year's data point)

$$\begin{aligned}
P(\lambda < 15 | x = 10) &= \sum_{\lambda=0}^{14} P(\lambda | x = 10) \\
&= \frac{\sum_{\lambda=0}^{14} P(x = 10 | \lambda) \pi(\lambda)}{\sum_{\lambda=0}^{\infty} P(x = 10 | \lambda) \pi(\lambda)} \\
&= \frac{\sum_{\lambda=0}^{14} P(x = 10 | \lambda)}{\sum_{\lambda=0}^{30} P(x = 10 | \lambda)} \quad (\text{Let the prior } P(\lambda) = \text{Uniform}(0, 30)) \\
&= \frac{\sum_{i=0}^{14} i^{10} e^{-i}}{\sum_{i=0}^{30} i^{10} e^{-i}} \approx 0.87
\end{aligned}$$

Type I Error is about $1 - 0.87 = 0.13$, not small. If we compute $P(x \leq 10 | \lambda = 15) \approx 0.11$, so $\lambda = 15$ is still capable of producing such result. It is inconclusive.

Exercise 8.3

H_0 region is $\theta \leq \theta_0$ and H_1 's region is $\theta > \theta_0$. Then define $b = m\theta_0$ to be the expected success count if $\theta = \theta_0$.

A Bernoulli trial $f(y|\theta) = I_{Y=1}\theta + I_{Y=0}(1 - \theta)$. Then the likelihood function

$$L(\theta|y) = \prod_1^m f(y_i|\theta) = \binom{m}{k} \theta^k (1 - \theta)^{m-k}$$

where $k = \sum_i Y_i$

To maximize L , we can use the MLE which is the $\theta_{\max} = \frac{k}{m}$. To reject H_0 , we need the MLE to stay out H_0 region, so $\frac{k}{m} > \theta_0 \Rightarrow \sum_i Y_i = k > m\theta_0 = b$

Exercise 8.5

(a) The likelihood function

$$L(\theta, v|x) = \prod_{i=1}^n f(x_i|\theta, v) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}} \prod_i I_{[v, \infty)}(x_i) = \frac{\theta^n v^{n\theta}}{(\prod_i x_i)^{\theta+1}}, \text{ (given } v \leq x_{\min}, 0 \text{ otherwise)}$$

Holding θ fixed, L is a monotonic polynomial function of v . So $v_0 = x_{(1)}$ the boundary of v maximizes L .

Let $\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \log\left(x_{(1)}^n\right) - \log(\prod_i x_i) = 0$, then we get

$$\theta_0 = \frac{n}{\log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)} = \frac{n}{T(x)}$$

where $T \equiv \log\left(\frac{\prod_i x_i}{x_{(1)}^n}\right)$

(b) $H_0 = \{(\theta = 1, v)\}$, So the rejection region of H_0 is

$$\lambda(x) = \frac{\sup_{\theta=1} L(\theta, v|x)}{\sup_{\theta} L(\theta, v|x)} = \frac{T^n}{n^n} \exp(n - T) \leq c$$

We take derivative of λ ,

$$\partial_T \lambda = \left(\frac{T}{n}\right)^{n-1} e^{n-T} \left(1 - \frac{T}{n}\right)$$

So the monotonicity of λ is determined by $(1 - T/n)$. When $T = n$, λ reaches maximum of 1, when $T < n$, λ increases monotonically and when $T > n$, λ decreases monotonically. Therefore, if $\lambda(x) < c$ for $0 < c \leq 1$, we will have two values c_1 and c_2 (on left/right side of n respectively) where $T \leq c_1 \leq n$ or $n \leq c_2 \leq T$.

Exercise 8.6

(a) Let

$$L(\theta, \mu | x, y) = f(x_1, \dots, x_n, y_1, \dots, y_m | \theta, \mu) = \prod_i^n f(x_i | \theta) \prod_i^m f(y_i | \mu) = \theta^n \mu^m \exp\left(-\theta \sum_i^n x_i - \mu \sum_i^m y_i\right)$$

be the likelihood function of the joint distribution. Then

$$\ln(L(\theta, \mu)) = n \ln(\theta) + m \ln(\mu) - \theta \sum_i^n x_i - \mu \sum_i^m y_i$$

. For H_0 where $\theta = \mu$, we solve $\frac{d \ln(L(\theta, \mu | \theta = \mu))}{d\theta} = 0$ and get

$$\hat{\theta}_0 = \frac{n + m}{\sum_i^n x_i + \sum_i^m y_i}$$

as the MLE under the constraint.

For H_1 , we solve $\frac{\partial \ln L}{\partial \theta} = 0$ and $\frac{\partial \ln L}{\partial \mu} = 0$ and get

$$\hat{\theta}_1 = \frac{n}{\sum_i^n x_i}, \quad \hat{\mu}_1 = \frac{m}{\sum_i^m y_i}$$

Therefore

$$\lambda((x, y)) = \frac{\sup_{\theta=\mu} L(\theta, \mu | x, y)}{\sup_{\theta, \mu} L(\theta, \mu | x, y)} = \frac{L(\hat{\theta}_0, \hat{\theta}_0 | x, y)}{L(\hat{\theta}_1, \hat{\mu}_1)} = \frac{(n + m)^{n+m}}{n^n m^m} \frac{(\sum_i^n x_i)^n (\sum_i^m y_i)^m}{(\sum_i^n x_i + \sum_i^m y_i)^{n+m}}$$

(b) To show that $T = \frac{\sum X}{\sum X + \sum Y}$ can also give the same LRT, we just need to express the LRT in terms of T . Let $C = \frac{(n+m)^{n+m}}{n^n m^m}$, then

$$\lambda((x, y)) = C \frac{(\sum_i^n x_i)^n (\sum_i^m y_i)^m}{(\sum_i^n x_i + \sum_i^m y_i)^{n+m}} = C \left(\frac{\sum_i^n x_i}{\sum_i^n x_i + \sum_i^m y_i} \right)^n \left(\frac{\sum_i^m y_i}{\sum_i^n x_i + \sum_i^m y_i} \right)^m = CT^n (1 - T)^m$$

(c) Let $U = \sum_1^n X_i$, then we calculate the MGF, $M_U(t) = E[e^{\sum_i X_i t}] = \prod E[e^{X_i t}] = \prod M_{X_i}(t) = \frac{1}{(1 - \theta t)^n}$ since H_0 is true. It matches the gamma distribution's MGF, therefore $U = \sum_i X_i \sim \text{Gamma}(n, \theta)$. Similarly $V = \sum_1^m Y_i \sim \text{Gamma}(m, \theta)$.

Next is to find the distribution of $T = \frac{U}{U+V}$. Since U, V are independent, so

$$f(u, v) = f(u)f(v) = \text{Gamma}(n, \theta) \text{Gamma}(m, \theta) = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} u^{n-1} v^{m-1} e^{-\frac{1}{\theta}(u+v)}$$

Let $S = U + V$, then $T = \frac{U}{U+V} = \frac{U}{S}$. We have $U = TS, V = S(1 - T)$. So the Jacobian $|J| = |S|$. By change of variables, we have

$$g(t, s) = f(u(t, s))f(v(t, s))|J| = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1} s^{n+m-1} e^{-\frac{1}{\theta}s}$$

Next we maginalize s ,

$$\begin{aligned} g(t) &= \int_0^\infty g(t, s) ds = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1} \int_0^\infty s^{n+m-1} e^{-\frac{1}{\theta}s} ds \\ &= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1} \int_0^\infty \frac{1}{\Gamma(n+m)\theta^{n+m}} s^{n+m-1} e^{-\frac{1}{\theta}s} ds \\ &= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1} \\ &= \text{Beta}(n, m) \end{aligned}$$

Exercise 8.37

(a) Given $Z \sim \mathcal{N}(0, 1)$ and $P(Z > z_\alpha) = \alpha$, Consider

$$\begin{aligned} &\sup_{\theta \in \Theta_0} P_\theta \left(\bar{X} > \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right) \\ &\sup_{\theta \in \Theta_0} P_\theta \left(\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha \right) \\ &\sup_{\theta \leq \theta_0} P_\theta \left(z > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha \right) \end{aligned}$$

the above probability is an increasing function of θ when $\theta \leq \theta_0$, therefore

$$\sup_{\theta \leq \theta_0} P_\theta \left(z > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha \right) = P_{\theta_0}(z > z_\alpha) = \alpha$$

So $\bar{X} > \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$ is indeed a test of size α that rejects H_0 . ■

To derive the test from LRT is, we have

$$\begin{aligned} \lambda(x) &= \frac{\sup_{\theta \leq \theta_0} L(x, |\theta, \sigma^2)}{\sup_{\theta \leq \theta_0} L(x, |\theta, \sigma^2)} = \frac{\exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \min(\bar{x}, \theta_0))^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \theta_0)^2\right\}} \\ &= \begin{cases} 1, & \theta_0 = \min(\bar{x}, \theta_0) \\ \exp\left\{-\frac{n}{2\sigma^2} (\bar{x} - \theta_0)^2\right\}, & \bar{x} = \min(\bar{x}, \theta_0) \end{cases} \end{aligned}$$

The rejection region for H_0 is $\{x \in R | \lambda(x) < c\}$ for $c \in [0, 1]$. We can write

$$\begin{aligned}
& \sup_{\theta \leq \theta_0} P(x \in R) \\
& \Rightarrow \sup_{\theta \leq \theta_0} P(\lambda(x) < c) \\
& \Rightarrow \sup_{\theta \leq \theta_0} P\left(\exp\left\{-\frac{n}{2\sigma^2}(\bar{x} - \theta_0)^2\right\} < c\right) \\
& \Rightarrow \sup_{\theta \leq \theta_0} P\left(\bar{x} > \theta_0 + \frac{\sigma}{\sqrt{n}}\sqrt{2\ln(1/c)}\right) \\
& \Rightarrow \sup_{\theta \leq \theta_0} P\left(\bar{x} > \theta_0 + \frac{\sigma}{\sqrt{n}}z_\alpha\right), \text{ (Simply choose } z_\alpha \equiv \sqrt{2\ln(1/c)}) \\
& \Rightarrow \sup_{\theta \leq \theta_0} P\left(\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha\right) \\
& \Rightarrow \sup_{\theta \leq \theta_0} P\left(z > \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + z_\alpha\right), \quad (Z \sim \mathcal{N}(0, 1)) \\
& \Rightarrow P(z > z_\alpha) = \alpha
\end{aligned}$$