Chapter 3: Common Families of Distribution

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Exercise 3.1

There are $N_1 - N_0 + 1$ numbers, therefore $P(x = n) = \frac{1}{N_1 - N_0 + 1}$.

$EX = \frac{N_1 + N_0}{2}$ which is just the midpoint.

Let $b = N_1, a = N_0$

$$\begin{aligned}
VarX &= EX^2 - (EX)^2 \\
&= \frac{1}{b-a+1}\sum_a^b x^2 - (EX)^2 \\
&= \frac{1}{b-a+1}\left[\sum_1^b x^2 - \sum_1^{a-1} x^2\right] - (EX)^2 \\
&= \frac{1}{b-a+1}\left[\frac{b(b+1)(2b+1)}{6} - \frac{(a-1)a(2a-1)}{6}\right] - \frac{(b+a)^2}{4} \\
&= \frac{2b(b+1)(2b+1) - 2(a-1)a(2a-1) - 3(b-a+1)(b+a)^2}{12(b-a+1)} \\
&= \frac{2b(b-a+1+a)(2b+1) + 2a(b-a+1-b)(2a-1) - 3(b-a+1)(b+a)^2}{12(b-a+1)} \\
&= \frac{2b(b-a+1)(2b+1) + 2a(b-a+1)(2a-1) - 3(b-a+1)(b+a)^2 - 4ab(b-a+1)}{12(b-a+1)} \\
&= \frac{2b(2b+1) + 2a(2a-1) - 3(b+a)^2 - 4ab}{12} \\
&= \frac{a^2 + b^2 - 2ab + 2b - 2a}{12} \\
&= \frac{(b-a)(b-a+2)}{12} \\
&= \frac{(N_1 - N_0)(N_1 - N_0 + 2)}{12}
\end{aligned}$$

Exercise 3.2

(a) Let X be the number of defective part in K samples and M be the total defective parts in 100 parts. Then

$$P(X = 0|M > 5) = \frac{\binom{100-M}{K}}{\binom{100}{K}}$$

is the probability of accepting a defective product given $M > 5$. To bound K, we can set $M = 6$ since defect parts becomes more prevalent which increases the chance for them to be sampled, setting $M = 6$ maximizes the false positive rate $P(X = 0|M)$.

Then

$$P(X = 0|M = 6) = \frac{\binom{94}{K}}{\binom{100}{K}} < 0.1$$

Solving for K numerically (polynomial of the 5th power), we get $K > 31$. We can choose $K = 32$.

(b) The false positive rate is now

$$P(X \leq 1|M = 6) = P(X = 0|M = 6) + P(X = 1|M = 6) = \frac{\binom{94}{K}}{\binom{100}{K}} + \frac{\binom{6}{1}\binom{94}{K-1}}{\binom{100}{K}} < 0.1$$

Solving for K numerically (same as above except there's an additional term $1 + \frac{6K}{95-K}$), We get $K > 50.24$ which means $K = 51$.