EX 2 $B = \{b_i : \Omega \to \mathbb{R} \mid i = 1, \dots, n\}$ $\Omega \subset \mathbb{R}$ 1

$S(x) = \sum_{i=1}^{n} c_i b_i(x) \ \forall \ x \in \Omega$

$c = (c_1, \dots, c_n) \in \mathbb{R}^n \setminus \{0\}$ has atmost $n-1$ zeroes in Ω

$S_B = \text{span}\{b_1, \dots, b_n\} \to$ Haar spae

(a) $B = \{b_1, \dots, b_n\}$

$X = \{x_1, \dots, x_n\}$ (pairwise distinct)

$$V_{B,X} = \begin{pmatrix} b_1(x_1) & \dots & b_n(x_1) \\ \vdots & & \\ b_1(x_n) & \dots & b_n(x_n) \end{pmatrix} \in \mathbb{R}^{n \times n} \text{ is invertible ?}$$

Proof: To show $V_{B,X} \to$ the vandermonde Matrix is invertible we can show the columns of this matrix are linearly independent

Let us assume that

$l_1 v_1 + l_2 v_2 + l_3 v_3 + \dots + l_n v_n = \vec{0} = (0, \dots, 0)$

where $v_j = (b_j(x_1), \dots, b_j(x_n))$ is the j^{th} column written as a vector and $l_1, \dots, l_n \in \mathbb{R}$

for the k^{th} coordinate it would be

$l_1 b_1(x_k) + l_2 b_2(x_k) + \dots + l_n b_n(x_k) = 0 \qquad k = 1, \dots, n$

$= \sum_{i=1}^{n} l_i b_i(x_k) = 0$

Thus x_k's ($k = 1, \dots, n$) have to be pairwise distinct zeroes for this linear combination. However as per the haar system condition, it can have atmost $n-1$ zeroes, thus it can be true only if all the coefficient $l_1, \dots, l_n$ are 0

$\Rightarrow v_1, \dots, v_n$ are linearly independent

$\Rightarrow V_{B,X}$ is invertible

(b) Let y be n-dimensional column vector

$y^T A^T A y = (Ay)^T (Ay) = \|Ay\|^2 \geq 0$

$\Rightarrow A^T A$ is positive semi-definite

here A is an $m \times n$ matrix

$A = (b_j(x_i)) \in \mathbb{R}^{m \times n}$

(2)

A has a submatrix which is the Vandermonde matrix from part (a) which has rank n $\Rightarrow$ A also has rank n $\Rightarrow$ it has n linearly independent column vectors

and rank (A) = rank $(A^T A)$

$\Rightarrow$ $A^T A$ is invertible

$\Rightarrow$ It is positive definite.