Exercise 4 (b) Theorem: Let $g : [a,b] \to [a,b]$ be a non decreasing function. Then there exists $s^* \in [a,b]$ with $s^* = g(s^*)$ i.e g has at least one fixed pt in $[a,b]$

Proof: given g is monotonically increasing

We know that $\mathbb{R}$ is complete and any subset $[a,b]$ of $\mathbb{R}$ is complete iff it is closed thus $[a,b]$ is complete, thus existence of x^* is guaranteed.

Consider a set $P = \{x \in [a,b] : x \leq g(x)\}$

Let $x^* = \sup(P)$

Let $x \in P$

Then $x \leq g(x)$ (by defn of P)

Also $x \leq x^*$

so, $g(x) \leq g(x^*)$ as g is monotonically increasing

so, $x \leq g(x^*)$

$\Rightarrow$ $g(x^*)$ is an upper bound

$\Rightarrow x^* \leq g(x^*) \longrightarrow (1)$

applying function g on both sides

$\Rightarrow g(x^*) \leq g(g(x^*))$

$\Rightarrow g(x^*) \in P$

$\Rightarrow g(x^*) \leq x^* \longrightarrow (2)$

from (1) and (2)

$\Rightarrow x^* = g(x^*)$

(c) Let $g : [a,b] \to [a,b]$ be a non decreasing function and consider the fixed pt iteration $s_{n+1} = g(s_n)$ $\forall n \in \mathbb{N}_0$ with starting pt $s_0 \in [a,b]$

Show that $(s_n)_{n \in \mathbb{N}_0}$ converges

Proof: sufficient and necessary conditions for existence and uniqueness of fixed pt are:

(i) If $g \in C[a,b]$ and $g(x) \in [a,b]$ $\forall x \in [a,b]$ then g has atleast one fixed pt in $[a,b]$

(ii) If $g'(x)$ exists on (a,b) and a +ve constant $k < 1$ exists with

$|g'(x)| \le k$ $\forall$ $x \in (a,b)$ then $\exists$ exactly one fixed pt in $[a,b]$

Since $g: [a,b] \to [a,b]$ the sequence $\{s_n\}_{n=0}^{\infty}$ is defined $\forall$ $n \ge 0$ and

$s_n \in [a,b]$ $\forall n$

Since $|g'(x)| \le k$ fore each n, we have

$$|s_n - s| = |g(s_{n-1}) - g(s)|$$

$$= |g'(c)|\,|s_{n-1} - s|$$ (by Mean value Theorem $c \in (a,b)$]

$$\le k\,|s_{n-1} - s|$$

using this inequality repeatedly.

$$|s_n - s| \le k\,|s_{n-1} - s| \le k^2\,|s_{n-2} - s| \cdots \le k^n\,|s_0 - s|$$

Since $0 < k < 1$ $\lim\limits_{n \to \infty} k^n = 0$ and

$$\lim_{n \to \infty} |s_n - s| \le \lim_{n \to \infty} k^n\,|s_0 - s| = 0$$

Hence $\{s_n\}_{n=0}^{\infty}$ converges to p.