($\star$) **Exercise 4** (Test function II).
In this exercise, we want to show that the function

$$g : \mathbb{R} \to \mathbb{R},\ t \mapsto \begin{cases} e^{-t^{-1}} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

is indeed in $C^\infty(\mathbb{R})$.

a) As a composition of functions from $C^\infty((0,\infty))$, the restricted function

$$\tilde{g} : (0,\infty) \to \mathbb{R},\ t \mapsto e^{-t^{-1}}$$

is also in $C^\infty((0,\infty))$. Show that we can write the n-th derivative of $\tilde{g}$ as

$$\tilde{g}^{(n)}(t) = P_n(-t^{-1}) \cdot e^{-t^{-1}} \qquad \text{for } t \in (0,\infty),$$

where P_n is a polynomial with $\deg(P_n) = 2n$ for each $n \in \mathbb{N} \cup \{0\}$.

P_n is of the form

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ for $a_0, \dots, a_n \in \mathbb{R}$

calculating derivatives of $\tilde{g}(t)$

$n=1$ $\tilde{g}'(t) = t^{-2} e^{-t^{-1}}$

$= (-t^{-1})^2 e^{-t^{-1}} = P_1(-t^{-1}) e^{-t^{-1}}$

$\deg(P_1) = 2(1) = 2$ ✓

$n=2$ $\tilde{g}''(t) = t^{-2}(t^{-2} e^{-t^{-1}}) + (-2t^{-3} e^{-t^{-1}})$

$= ((-t^{-1})^4 + 2(-t^{-1})^3) e^{-t^{-1}}$

$= P_2(-t^{-1}) e^{-t^{-1}}$

$\deg(P_2) = 2(2) = 4$ ✓

$n=3$ $\tilde{g}'''(t) = (t^{-4} - 2t^{-3})(t^{-2} e^{-t^{-1}}) + (-4t^{-5} - 6t^{-4}) e^{-t^{-1}}$

$$= \left((-t^{-1})^6 + 6(-t^{-1})^5 - 6(-t^{-1})^4 \right) e^{-t^{-1}}$$

$$= P_3(-t^{-1}) e^{-t^{-1}}$$

$$\deg(P_3) = 2(3) = 6 \checkmark$$

$n = 4$ $$\tilde{g}^{(4)}(t) = (t^{-6} - 6t^{-5} - 6t^{-4})(t^{-2} e^{-t^{-1}}) +$$

$$(-6t^{-5} + 30t^{-4} + 24t^{-3}) e^{-t^{-1}}$$

$$= \left((-t^{-1})^8 + \cdots + 24(-t^{-1})^3 \right) e^{-t^{-1}}$$

$$= P_4(-t^{-1})$$

$$\deg(P_4) = 2(4) = 8 \checkmark$$

$\Rightarrow \tilde{g}^{(n)}(t)$ can always be written in the form

$$\tilde{g}^{(n)}(t) = P_n(-t^{-1}) e^{-t^{-1}}$$

since the derivative of $e^{-t^{-1}}$

$$e^{-t^{-1}} = t^{-2} e^{-t^{-1}} = (-t^{-1})^2 e^{-t^{-1}}$$

$$\deg(P_{n+1}) = \deg(P_n) + 2$$

$$= 2n + 2 = 2(n+1)$$

$\Rightarrow$ the degree of P_n increases by 2 each time n increases

$\therefore \deg(P_n) = 2n$ for $n \in \mathbb{N} \cup \{0\}$

$\blacksquare$