Exercise 2 (Discrete Fourier Transform).
Let $f = (f_1, ..., f_N) \in \mathbb{C}^N$ be a finite signal. In this exercise, we want to derive the discrete version of the Fourier Transform.

a) For $j \in \mathbb{Z}$, we set $\omega_N^{(j)} = 2\pi \cdot \frac{j}{N}$. Consider the space $\mathbb{C}^N$ with the standard inner product

$$\langle x, y\rangle = \sum_{k=1}^{N} x_k \cdot \overline{y_k} \qquad \text{for all } x, y \in \mathbb{C}^N.$$

Show that the vectors

$$b_j := \left(N^{-1/2} \cdot e^{i\cdot\omega_N^{(j-1)}\cdot(k-1)}\right)_{1\leq k\leq N} \in \mathbb{C}^N \qquad \text{for } j \in \{1, ..., N\}$$

form an orthonormal basis of $\left(\mathbb{C}^N, \langle\cdot,\cdot\rangle\right)$. Find coefficients $\hat{f} = \left(\hat{f}_1, ..., \hat{f}_N\right)$ such that

$$f = \sum_{j=1}^{N} \hat{f}_j \cdot b_j.$$

The mapping $F_N : \mathbb{C}^N \to \mathbb{C}^N,\ f \mapsto \hat{f}$ is called *Discrete Fourier Transform (DFT).*

orthonormality $\Rightarrow \langle b_i, b_j\rangle = \delta_{ij}$ where δ is dirac-delta

let $i = j$, then

$$\langle b_j, b_j\rangle = \sum_{k=1}^{N} \left(N^{-1/2} e^{i\omega_N^{(j-1)}(k-1)}\right)\left(N^{-1/2} e^{-i\omega_N^{(j-1)}(k-1)}\right)$$

$$= \frac{1}{N} \cdot \sum_{k=1}^{N} e^{0}$$

$$= \frac{1}{N} \cdot N = 1$$ thus b_j form an orthonormal basis

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