Technische Universität Hamburg
Mathematical Image Processing (Winter Term 2022/2023)
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Exercise Sheet 9

Deadline: 19.12.22, 12:00pm

Exercise 1 (Cyclic Toeplitz Matrices)**.**
A *cyclic Toeplitz matrix* is a matrix that has the form

$$C = \begin{pmatrix} c_1 & c_N & \dots & c_3 & c_2 \\ c_2 & c_1 & \ddots & \vdots & c_3 \\ \vdots & c_2 & \ddots & c_N & \vdots \\ c_{N-1} & \vdots & \ddots & c_1 & c_N \\ c_N & c_{N-1} & \dots & c_2 & c_1 \end{pmatrix} \in \mathbb{C}^{N \times N},$$

where $c = (c_1, ..., c_N)^T \in \mathbb{C}^N$ is called the generating vector of C. In this exercise, we want to take a look at an efficient way to solve the linear system

$$C \cdot x = b \tag{1}$$

for given $b \in \mathbb{C}^N$ via deconvolution with Fourier methods.

a) For given $x \in \mathbb{C}^N$, verify that

$$C \cdot x = x \otimes c,$$

where $\otimes$ denotes the circular convolution from Exercise Sheet 4.

Additionally, we assume that

$$F_N(c)_j \neq 0 \qquad \text{for all } j \in \{1, ..., N\},$$

where F_N is the Discrete Fourier Transform (DFT) from Exercise Sheet 5.

b) Define the vector $d \in \mathbb{C}^N$ via

$$d_j := \frac{F_N(b)_j}{\sqrt{N} \cdot F_N(c)_j} \qquad \text{for all } j \in \{1, ..., N\}.$$

Use the discrete convolution theorem from Exercise Sheet 5, Exercise 4 to prove that the solution of the linear system (1) is given by $x = F_N^{-1}(d)$.

Exercise 2 (Laplace sharpening)**.**
In this exercise, we want to implement the Laplace sharpening, which is given by the formula

$$\tilde{u} = u - \tau \cdot \Delta u, \tag{2}$$

where u is the initial grayscale image and $\tau > 0$ is the sharpening parameter. Make sure to convert to type 'double' before computations. Due to the sharpening (2), it might happen that some of the color values of $\tilde{u}$ are outside of the initial color range $[0, 255]$, such that a simple conversion to 'uint8' would erase some effects of the sharpening. Therefore, we set

$$k_{\max} = \max\{\max(\tilde{u}), 255\} \qquad k_{\min} = \min\{\min(\tilde{u}), 0\}$$

and tranform the color values $k \in [\min(\tilde{u}), \max(\tilde{u})]$ via the map

$$\varphi(k) := \frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot 255. \tag{3}$$

The resulting image $\hat{u} = \varphi \circ \tilde{u}$ can then be converted to 'uint8'. Write a function

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function [B] = laplaceSharpening(A, tau, N)
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which performs the aforementioned Laplace sharpening of a grayscale image A with sharpening parameter τ. The function should be able to repeat the sharpening (2) for a given number $N \in \mathbb{N}$ of iterations and apply the color tranformation (3) in the end. Note that we will ignore the smoothing of the image prior to applying the Laplace Filter (see lecture) in this exercise.

Exercise 3 (Ill-posed problem on $\ell^2(\mathbb{N})$)**.**
Consider the linear operator

$$A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}),\ (x_n)_{n\in\mathbb{N}} \mapsto \left(n^{-1} \cdot x_n\right)_{n\in\mathbb{N}}.$$

a) Show that A is well-defined, i.e. $Ax \in \ell^2(\mathbb{N})$ for $x \in \ell^2(\mathbb{N})$. Moreover, show that A is injective.

b) Prove that there is a constant $C > 0$ such that

$$\|Ax\|_2 \leq C \cdot \|x\|_2 \qquad \text{for all } x \in \ell^2(\mathbb{N}).$$

This means that A is a bounded linear operator and therefore continuous.

c) Due to the last part of a), we can find an inverse linear operator

$$A^{-1} : \operatorname{im}(A) \to \ell^2(\mathbb{N}).$$

Give an explicit formula for $A^{-1}y$, where $y \in \operatorname{im}(A)$. For every $C > 0$, show that there is $y \in \operatorname{im}(A)$ with $\|y\|_2 = 1$ and $\|A^{-1}y\|_2 > C$. Hence, the inverse operator A^{-1} is not bounded / continuous.

($\star$) Exercise 4 (Ill-posed problems and compact operators)**.**
Let X, Y be infinite-dimensional normed spaces. Moreover, let $A : X \to Y$ be compact and invertible. Prove that the inverse mapping A^{-1} is not continuous.