4.b) $\tilde{g}(t) = \begin{cases} e^{-\frac{1}{t}}, & t>0 \\ 0, & t\leq 0 \end{cases}$, $g:(0,\infty)\to\mathbb{R}$

$$g^{(n)}(t) = P_n\left(-\frac{1}{t}\right)e^{-\frac{1}{t}} \quad ; \deg P_n = 2n$$

Conclude $g(t)\in C^\infty(\mathbb{R})$, $g:\mathbb{R}\to\mathbb{R}$

Proof:

$$P_n\left(-\frac{1}{t}\right) = -\frac{1}{P_n(t)}$$

Taking one-sided limit:

$$\lim_{t\downarrow 0} g(t) = \lim_{t\to 0}\frac{-e^{-1/t}}{P_n(t)} = \lim_{t\to 0}\frac{-e^{-1/t}}{t^{2n}} =$$

$$= \left[\text{since } \frac{1}{t^m} = t\cdot\left(\frac{1}{t}\right)^{m+1} \leq (m+1)!\,t\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{x}\right)^n = (m+1)!\,te^{\frac{1}{t}}\right] = \lim_{t\to 0}\left(-\cancel{e^{-1/t}}\underbrace{(2n+1)!}_{<\infty}\cdot t\cdot \cancel{e^{1/t}}\right) = 0 \Rightarrow$$

$\Rightarrow g^{(n)}(t)$ is continious at 0 $\forall n\in\mathbb{N} \Rightarrow g(t)\in C^\infty(\mathbb{R})$

$\blacksquare$