Exercise sheet 7

Exercise 1 1D Perona Malik Equation

$$\partial_t u = (g(|u'|).u')' \quad = \partial_t u = \underbrace{(g(u')\,u')'}_{①} \quad \text{as } u'(x_0,t_0)>0$$

given: $t_0>0$

Edge conditions

(i) $u'(x_0,t_0)>0$ (ii) $u''(x_0,t_0)=0$ (iii) $u'''(x_0,t_0)<0$

(a) $u''''(x_0,t_0)=0$

flux function $f(s) = s.g(s) \ \forall\ s\in[0,\infty)$.

then prove: $\partial_t u' = f'(u')\,u'''$

Soln: $f(s) = s\,g(s)$

$\underbrace{f(u') = u' g(u')}_{②}$ from ① and ②

$$\partial_t u = (f(u'))'$$

$\Rightarrow \partial_t u = f'(u').u''$

$\Rightarrow (\partial_t u)' = (f'(u').u'')'$

$\Rightarrow \partial_t u' = f'(u').u''' + \underbrace{f''(u').(u'')^2}_{0}$ as $u''=0$ (given)

$\Rightarrow \partial_t u' = f'(u').u'''$

Proved!

(b) $u''''(x_0,t_0)>0$ $g=g_k$ for fixed $k>0$

$$g_k(s) = \frac{1}{1+\frac{s^2}{k^2}} \qquad s\in[0,\infty)$$

To show:

(1) $\partial_t u'>0$ if $u'>k$ and $\partial_t u'<0$ if $u'<k$

(2) $\partial_t u''=0$

(3) $\partial_t u'''<0$ if $k<u'<\sqrt{3}k$

we know, $\partial_t u = (g(u').u')' = \partial_t u = (g_k(u').u')'$

$$g_k(u') = \frac{1}{1+\frac{(u')^2}{k^2}} = \frac{k^2}{k^2+(u')^2}$$

$$\Rightarrow \partial_t u = \left(\frac{k^2}{k^2+(u')^2} \cdot u' \right)'$$

$$\left(\frac{u}{v}\right)' \cancel{= \frac{u'v - v'u}{v^2}}$$

$$\Rightarrow (\partial_t u)' = \frac{k^2 u'' (k^2+(u')^2) - (2u' . u'')(k^2 u')}{(k^2+(u')^2)^2}$$

$$= \frac{k^4 u'' + k^2 (u')^2 u'' - 2k^2 (u')^2 u''}{(k^2+(u')^2)^2}$$

$$= \frac{k^4 u'' - k^2 (u')^2 u''}{(k^2+(u')^2)^2} \cancel{= \frac{k^2 u'' (k^2 - (u')^2)}{(k^2+(u')^2)^2}}$$

(as $u'' = 0$?)

$$(\partial_t u)' = \partial_t u' = \frac{\left[k^4 u''' - \overbrace{k^2 (2u'(u'')^2}^{0} + u''' k^2 (u')^2)\right] (k^2+(u')^2)^2 - \left[2(k^2+(u')^2) . 2\underbrace{u' u''}_{0}\right] \left[\underbrace{k^4 u''}_{0} - k^2 \underbrace{(u')^2 u''}_{0}\right]}{(k^2+(u')^2)^4}$$

$$= \frac{(k^4 u''' + k^2 (u')^2 u''') \cancel{(k^2+(u')^2)^2}}{(k^2+(u')^2)^{\cancel{4}\, 2}}$$

$$= \frac{k^4 u''' + k^2 (u')^2 u'''}{(k^2+(u')^2)^2} = \frac{k^2 u''' \cancel{(k^2+(u')^2)}}{(k^2+(u')^2)^{\cancel{2}}}$$

$$= \frac{k^2 u'''}{k^2+(u')^2}$$

(i) If $u' > k$ then $\partial_t u' > 0$

(ii) If $u' < k$ then $\partial_t u' < 0$

(as $u'''' = 0$)

$$(2)\ (\partial_t u')' = \partial_t u'' = \frac{k^2 u'''' \cancel{(k^2+(u')^2)} - \overbrace{2u' u'' (k^2 u''')}^{0}}{(k^2+(u')^2)^{\cancel{2}}}$$

$$= \frac{k^2 u''''}{k^2+(u')^2}$$

since $u'''' = 0$

$\Rightarrow \partial_t u'' = 0$

(3)

$$(\partial_t u'')' = \partial_t u''' = \frac{\overbrace{(k^2 u''''')(k^2+(u')^2)}^{0 \ (\text{as } u'''''=0)} - (2u'u'')(k^2 u'''')}{(k^2+(u')^2)^2}$$

$$= \frac{-2k^2 u'.u''.u''''}{(k^2+(u')^2)^2}$$

if $k < u' < \sqrt{3}k$

$\Rightarrow \partial_t u''' < 0$

<u>Exercise 3</u> $u_0 \in L^2(\mathbb{R}^2) \rightarrow$ noisy image

$$\phi_\lambda(u) = \frac{1}{2}\|u_0 - u\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2}\left(\|\partial_{x_1}u\|^2_{L^2(\mathbb{R})} + \|\partial_{x_2}u\|^2_{L^2(\mathbb{R})}\right)$$

$\forall u \in H^1(\mathbb{R}^2)$

Regularization parameter $\lambda > 0$

$$u^* = \min_{u \in H^1(\mathbb{R}^2)} \phi(u)$$ s.t. u^* is denoised version of u_0

(a) Plancherel's ~~Theorem~~ Theorem to show.

$$\phi_\lambda(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\mathcal{F}(u_0)(\xi) - \mathcal{F}(u)(\xi)|^2 + \frac{\lambda}{2}|\xi|^2 \cdot |\mathcal{F}(u)(\xi)|^2 d\xi$$

$\mathcal{F} \rightarrow$ fourier transform on $L^2(\mathbb{R}^2)$