Exercise 1 (Cyclic Toeplitz Matrices).
A *cyclic Toeplitz matrix* is a matrix that has the form

$$C = \begin{pmatrix} c_1 & c_N & \dots & c_3 & c_2 \\ c_2 & c_1 & \ddots & \vdots & c_3 \\ \vdots & c_2 & \ddots & c_N & \vdots \\ c_{N-1} & \vdots & \ddots & c_1 & c_N \\ c_N & c_{N-1} & \dots & c_2 & c_1 \end{pmatrix} \in \mathbb{C}^{N \times N},$$

where $c = (c_1, ..., c_N)^T \in \mathbb{C}^N$ is called the generating vector of C. In this exercise, we want to take a look at an efficient way to solve the linear system

$$C \cdot x = b \tag{1}$$

for given $b \in \mathbb{C}^N$ via deconvolution with Fourier methods.

a) For given $x \in \mathbb{C}^N$, verify that

$$C \cdot x = x \otimes c,$$

where $\otimes$ denotes the circular convolution from Exercise Sheet 4.

the jth element of $Cx = b$ is given by:

$$(Cx)_j = b_j = \sum_{k=1}^{N} C_{j,k} \cdot x_k$$

the circular convolution, nth element is defined:

$$(x \otimes y)_n = \sum_{k=1}^{N} x_k \cdot y^{(p)}_{n-k+1} \qquad x, y \in \mathbb{C}^N$$

with $y^{(p)}$ the periodic continuation of y

$$y^{(p)}_n := y_{(n-1)\% N + 1}$$

gives $y^{(p)}_{n-k+1} = y_{(n-k+1-1)\% N + 1}$

$$= y_{(n-k)\% N + 1}$$

need to check that $C_{j,k} = c^{(p)}_{j-k+1}$

$$= c_{(j-k)\% N + 1}$$

looking at the first column of G

$k=1$: $G_{1,1} = c_{(1-1)\%N+1} = c_1$ ✓

$G_{2,1} = c_{(2-1)\%N+1} = c_2$ ✓

$G_{3,1} = c_{(3-1)\%N+1} = c_3$ ✓

the second column,

$k=2$: $G_{1,2} = c_{(1-2)\%N+1} = c_N$ ✓

$G_{2,2} = c_{(2-2)\%N+1} = c_1$ ✓

$G_{3,2} = c_{(3-2)\%N+1} = c_2$ ✓

this can be checked for the remaining values of G, hence

$$b_j = (G \cdot x)_j = \sum_{k=1}^{N} G_{j,k} \cdot x_k$$

$$= \sum_{k=1}^{N} x_k \cdot c_{(j-k)\%N+1}$$

$$= \sum_{k=1}^{N} x_k \cdot c^{(p)}_{j-k+1}$$

$$= (x \circledast c)_j$$

$$\therefore \quad Gx = x \circledast c$$

$\blacksquare$

Additionally, we assume that

$$F_N(c)_j \neq 0 \qquad \text{for all } j \in \{1, ..., N\},$$

where F_N is the Discrete Fourier Transform (DFT) from Exercise Sheet 5.

b) Define the vector $d \in \mathbb{C}^N$ via

$$d_j := \frac{F_N(b)_j}{\sqrt{N} \cdot F_N(c)_j} \qquad \text{for all } j \in \{1, ..., N\}.$$

Use the discrete convolution theorem from Exercise Sheet 5, Exercise 4 to prove that the solution of the linear system (1) is given by $x = F_N^{-1}(d)$.

the discrete convolution property is given by:

$$F_N(f \circledast g)_j = N^{1/2} \cdot F_N(f)_j \cdot F_N(g)_j \quad \text{for } f, g \in \mathbb{C}^N$$

$$j \in \{1, \dots, N\}$$

From part (a), $b = x \circledast c$

$$F_N(b)_j = F_N(x \circledast c)_j = N^{1/2} F_N(x)_j \cdot F_N(c)_j$$

gives: $$d_j := \frac{F_N(b)_j}{\sqrt{N} \cdot F_N(c)_j} = \frac{\sqrt{N}\, F_N(x)_j \cdot F_N(c)_j}{\sqrt{N}\, F_N(c)_j}$$

$$= F_N(x)_j$$

$$\Rightarrow x_j = F_N^{-1}(d)_j \quad , \quad j = \{1, \dots, N\}$$

thus $x = F_N^{-1}(d)$

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