(a) $f(x) = e^{-x^2/2}$ (Gaussian function)

f is a unique soln to D.E

$f'(x) = -x \cdot f(x) \quad \forall x \in \mathbb{R} \qquad f(0) = 1$

We know,

$\frac{df(x)}{dx} \xleftrightarrow{\text{fourier transform}} i\omega \hat{f}(\omega)$

$xf(x) \xleftrightarrow{\text{Fourier transform}} \frac{id(\hat{f}(\omega))}{d\omega}$

we are given $f(x) = e^{-x^2/2}$

$\frac{d(f(x))}{dx} = -xe^{-x^2/2} \longrightarrow (1)$

Taking fourier transform of above eq^n (1)

~~$i\omega$~~ $i\omega \hat{f}(\omega) = -\frac{id(\hat{f}(\omega))}{d\omega}$

$\Rightarrow \frac{1}{\hat{f}(\omega)} \cdot d(\hat{f}(\omega)) = -\omega d\omega \longrightarrow (2)$

integrating eq^n (2) we get

$\ln(\hat{f}(\omega)) = -\frac{\omega^2}{2}$

$f(\omega) = e^{-\omega^2/2}$

(b) $f_a(x) := (4\pi a)^{-d/2} e^{-\|x\|_2^2/4a} \quad x \in \mathbb{R}^d \qquad$ fixed $a > 0$

$\|x\|_2^2 = x_1^2 + \dots + x_d^2$

$e^{x+y} = e^x e^y$

$= \underbrace{\prod_{k=1}^{d} (4\pi a)^{-\frac{1}{2}} e^{-(x_k^2/4a)}}_{\cancel{g_1(x_k)}}$

$g_a(x_k)$

$\mathcal{F}(f_a) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f_a(x) e^{-i\langle \omega, x\rangle} dx$

$\langle \omega, x\rangle = \omega_1 x_1 + \dots + \omega_d x_d$

$e^{-i\langle \omega, x\rangle} = \prod_{k=1}^{d} e^{-i\omega_k x_k}$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \prod_{k=1}^{d} g_a(x_k) \cdot e^{-i\omega_k x_k} \, dx$$

$$= (2\pi)^{-d/2} \prod_{k=1}^{d} \int_{\mathbb{R}^d} g_a(x_k) e^{-i\omega_k x_k} \, dx_k$$

$$= (2\pi)^{-d/2} \prod_{k=1}^{d} \int_{\mathbb{R}} (4\pi a)^{-1/2} e^{-\left(\frac{x_k}{\sqrt{2a}}\right)^2/2} \, dx_k$$

$$= (2\pi)^{-d/2} \prod_{k=1}^{d} \int_{\mathbb{R}} (4\pi a)^{-1/2} e^{-(t^2/2)} \sqrt{2a} \, dt$$

$x_k/\sqrt{2a} = t_k$

$\frac{dx_k}{\sqrt{2a}} = dt_k$

$dx_k = \sqrt{2a} \, dt_k$

$$= \cancel{\sqrt{2a} \prod_{k=1}^{d} \int}$$

$$= (4\pi a)^{-d/2} \sqrt{2a} \prod_{k=1}^{d} \underbrace{\int_{\mathbb{R}} (2\pi)^{-1/2} e^{-(t^2/2)} \, dt_k}_{e^{-\frac{\omega_k^2}{2}}}$$

$$= (\sqrt{2a})(4\pi a)^{-d/2} e^{-\frac{\|\omega\|_2^2}{2}}$$

(c) T.S: $(f_a)_{a \in \mathbb{R}^+}$ satisfies

$$f_a * f_b = f_{a+b}$$

According to convolution thm.

$$\mathcal{F}(f * g) = (2\pi)^{d/2} \mathcal{F}(f) \cdot \mathcal{F}(g)$$

using this

$$\mathcal{F}(f_a * f_b) = (2\pi)^{d/2} \mathcal{F}(f_a) \mathcal{F}(f_b)$$

$$= (2\pi)^{d/2} (\sqrt{2a})(4\pi a)^{-d/2} e^{-\frac{\|\omega\|_2^2}{2}} \cdot \downarrow$$ (from part (b))

$$(\sqrt{2b})(4\pi b)^{-d/2} e^{-\frac{\|\omega\|_2^2}{2}}$$

$$= (2\pi)^{d/2} \left(4\pi (a+b)\right)^{-d/2} \sqrt{2}(a+b) \left(e^{-\|w\|_2^2}\right)^2$$

Also $f_{a+b} = \sqrt{2}(a+b) \left(4\pi (a+b)\right)^{-d/2} e^{-\frac{\|w\|_2^2}{2}}$