Exercise 3 (Geteaux derivative).
Let X, Y be normed spaces. We call a function $\Phi : X \to Y$ *Gateaux differentiable* if the limit

$$d\Phi(u, v) := \lim_{t \to 0} \frac{\Phi(u + t \cdot v) - \Phi(u)}{t}$$

exists for all $u, v \in X$.

a) Let $\Phi : X \to \mathbb{R}$ be convex and Gateaux differentiable. Prove that for any $u, v \in X$, we have the estimate

$$d\Phi(u, v - u) \leq \Phi(v) - \Phi(u).$$

φ is convex if:

$$\varphi(\lambda u + (1-\lambda)v) \leq \lambda\varphi(u) + (1-\lambda)\varphi(v)$$

for $u, v \in X$, $0 \leq \lambda \leq 1$

proof:

$$d\varphi(u, v-u) = \lim_{t \to 0} \frac{\varphi(u + t(v-u)) - \varphi(u)}{t}$$

$$= \lim_{t \to 0} \frac{1}{t}(\varphi(u + tv - tu) - \varphi(u))$$

$$= \lim_{t \to 0} \frac{1}{t}(\varphi(tv + (1-t)u) - \varphi(u))$$

by convexity $$\leq \lim_{t \to 0} \frac{1}{t}(t\varphi(v) + (1-t)\varphi(u) - \varphi(u))$$

$$= \lim_{t \to 0} \varphi(v) + \cancel{\frac{1}{t}\varphi(u)} - \varphi(u) - \cancel{\frac{1}{t}\varphi(u)}$$

$$= \varphi(v) - \varphi(u)$$

thus $d\varphi(u, v-u) \leq \varphi(v) - \varphi(u)$ ■

b) In the setting of part a), prove that $u^* \in X$ minimizes the functional Φ over the space X if and only if $d\Phi(u^*, v) = 0$ for all $v \in X$.

If φ is differentiable, the Gateaux derivative is the standard directional derivative

$$d\varphi(u,v) = \nabla\varphi(u)^T \cdot v \qquad \forall\ u,v \in X$$

proof:

$\Rightarrow$) Assume $u^* \in X$ is a minimizer of φ

then by definition of a minimizer,

$$d\varphi(u^*, v) = 0 \quad \forall v \in X$$

$\Leftarrow$) Assume $d\varphi(u^*, v) = 0$

from part (a)

$$d\varphi(u, v-u) = \nabla\varphi(u)^T(v-u) \leq \varphi(v) - \varphi(u)$$

thus,

$$\varphi(v) \geq \varphi(u^*) + \cancelto{0}{\nabla\varphi(u^*)^T v}$$

hence $\varphi(u^*) \leq \varphi(v) \quad \forall v \in X$

$\Rightarrow$ u^* is the unique minimizer of φ

Hence, u^* minimizes φ if and only if

$$d\varphi(u^*, v) = 0 \qquad \forall\, v \in X.$$

$\blacksquare$