Exercise 3 (Convolution Theorem).
Let $f, g \in L^1(\mathbb{R}^d)$. Prove the following statements:

a) We have the estimate

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1},$$

such that we can conclude $f * g \in L^1(\mathbb{R}^d)$.

b) For every $\omega \in \mathbb{R}^d$, we have the equality

$$\mathcal{F}(f * g)(\omega) = (2\pi)^{d/2} \cdot \mathcal{F}(f)(\omega) \cdot \mathcal{F}(g)(\omega),$$

where $\mathcal{F}$ denotes the Fourier Transform on $L^1(\mathbb{R}^d)$.

(a) $\|f * g\|_{L^1} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \right| dx$

triangle inequality $\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\, dy\, dx$

by Fubini's $= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\, dx\, dy$

$= \|f\|_{L^1} \int_{\mathbb{R}^d} |g(y)|\, dy$

$= \|f\|_{L^1} \cdot \|g\|_{L^1}$

thus $\|f * g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}$

since $f, g \in L^1$, $\|f\|_{L^1} < \infty$ and $\|g\|_{L^1} < \infty$

$\Rightarrow \|f * g\|_{L^1} < \infty$

$\therefore f * g \in L^1(\mathbb{R}^d)$ $\blacksquare$

(b) show $\widehat{f * g} = (2\pi)^{d/2} \hat{f} \hat{g}$

$$\widehat{f*g} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (f*g)(x)\, e^{-i\langle z, x\rangle}\, dx$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle z, x\rangle} \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy\, dx$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle z, x\rangle} f(x-y)\, g(y)\, dy\, dx$$

Fubini's $\Rightarrow$ $$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle z, x\rangle} f(x-y)\, g(y)\, dx\, dy$$

$$= \underbrace{(2\pi)^{-d/2}} \int_{\mathbb{R}^d} e^{-i\langle z, y\rangle} g(y) \underbrace{\int_{\mathbb{R}^d} e^{i\langle z, x-y\rangle} f(x-y)\, dx}\, dy$$

$$= \int_{\mathbb{R}^d} e^{-i\langle z, y\rangle} g(y)\, \hat{f}(z)\, dy$$

multiplying both sides by $(2\pi)^{-d/2}$ gives

$$(2\pi)^{-d/2}\, \widehat{f*g} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle z, y\rangle} g(y)\, \hat{f}(z)\, dy$$

$$= \hat{f}(z)\, \hat{g}(z)$$

thus, $\widehat{f*g} = (2\pi)^{d/2} \hat{f} \hat{g}$

$\blacksquare$