Exercise 3 $\nabla u_\sigma = \nabla(G_\sigma * u)$ $\sigma > 0$

$G_\sigma \leftarrow$ 2D gaussian function

$$J_0(\nabla u_\sigma(x)) = \nabla u_\sigma(x) \cdot \nabla u_\sigma(x)^T \in \mathbb{R}^{2\times 2}$$

(a) $\nabla u_\sigma(x) \neq 0$

finding eigen values of structure tensor $J_0(\nabla u_\sigma(x))$

$$J_0(\nabla u_\sigma(x)) = \begin{pmatrix} \partial x_1 u_\sigma(x) \\ \partial x_2 u_\sigma(x) \end{pmatrix} \begin{pmatrix} \partial x_1 u_\sigma(x) & \partial x_2 u_\sigma(x) \end{pmatrix}$$

$$= \begin{pmatrix} \partial x_1^2 u_\sigma^2(x) & \partial x_1 \partial x_2 u_\sigma^2(x) \\ \partial x_1 \partial x_2 u_\sigma^2(x) & \partial x_2^2 u_\sigma^2(x) \end{pmatrix}$$

$$\det(J_0(\nabla u_\sigma(x)) - \lambda I) = 0$$

$$\Rightarrow \det \begin{pmatrix} \partial x_1^2 u_\sigma^2(x) - \lambda & \partial x_1 \partial x_2 u_\sigma^2(x) \\ \partial x_1 \partial x_2 u_\sigma^2(x) & \partial x_2^2 u_\sigma^2(x) - \lambda \end{pmatrix} = 0$$

$$\cancel{(\partial x_1^2 u_\sigma^2(x) - \lambda)^2}$$

$$(\partial x_1^2 u_\sigma^2(x) - \lambda)(\partial x_2^2 u_\sigma^2(x) - \lambda) - (\partial x_1 \partial x_2 u_\sigma^2(x))^2 = 0$$

$$\Rightarrow \cancel{\partial x_1^2 \partial x_2^2 u_\sigma^4(x)} - \lambda \partial x_1^2 u_\sigma^2(x) - \lambda \partial x_2^2 u_\sigma^2(x) + \lambda^2 - \cancel{\partial x_1^2 \partial x_2^2 u_\sigma^4(x)} = 0$$

$$\Rightarrow \lambda^2 - \lambda u_\sigma^2(x)(\partial x_1^2 + \partial x_2^2) = 0$$

$$\Rightarrow \lambda(\lambda - u_\sigma^2(x)(\partial x_1^2 + \partial x_2^2)) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 0 \qquad \lambda_2 = \partial x_1^2 u_\sigma^2(x) + \partial x_2^2 u_\sigma^2(x) = |\nabla u_\sigma(x)|^2}$$

finding eigen vectors of the structure tensor $J_0(\nabla u_\sigma(x))$

for $\lambda_1 = 0$

$$(J_0(\nabla u_\sigma(x)) - \lambda_1 I)(X) = 0 \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \partial x_1^2 u_\sigma^2(x) & \partial x_1 \partial x_2 u_\sigma^2(x) \\ \partial x_1 \partial x_2 u_\sigma^2(x) & \partial x_2^2 u_\sigma^2(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned}\partial x_1^2 U_\sigma^2(x)\, x_1 + \partial x_1 \partial x_2 U_\sigma^2(x)\, x_2 &= 0\\ \partial x_1 \partial x_2 U_\sigma^2(x)\, x_1 + \partial x_2^2 U_\sigma^2(x)\, x_2 &= 0\end{aligned}\right\}$$

$$\Rightarrow \partial x_1 x_1 + \partial x_2 x_2 = 0$$

$$\partial x_1 x_1 + \partial x_2 x_2 = 0$$

$$\Rightarrow -\partial x_1 x_1 = \partial x_2 x_2$$

If $x_1 = \dfrac{-\partial x_2 U_\sigma(x)}{|\nabla U_\sigma(x)|}$ then $x_2 = \dfrac{\partial x_1 U_\sigma(x)}{|\nabla U_\sigma(x)|}$

$\Rightarrow$ the corresponding eigen vector $= \dfrac{1}{|\nabla U_\sigma(x)|}\begin{pmatrix} -\partial x_2 U_\sigma(x) \\ \partial x_1 U_\sigma(x)\end{pmatrix}$

For $\lambda_2 = |\nabla U_\sigma(x)|^2$

$$(J_0(\nabla U_\sigma(x)) - (|\nabla U_\sigma(x)|^2 I))(X) = 0$$

$$\Rightarrow \begin{pmatrix} \partial x_1^2 U_\sigma^2(x) - |\nabla U_\sigma(x)|^2 & \partial x_1 \partial x_2 U_\sigma^2(x) \\ \partial x_1 \partial x_2 U_\sigma^2(x) & \partial x_2^2 U_\sigma^2(x) - |\nabla U_\sigma(x)|^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \left(\partial x_1^2 U_\sigma^2(x) - (\partial x_1^2 U_\sigma^2(x) + \partial x_2^2 U_\sigma^2(x))\right) x_1 + \partial x_1 \partial x_2 U_\sigma^2(x)\, x_2 = 0$$

$$\partial x_1 \partial x_2 U_\sigma^2(x)\, x_1 + \left(\partial x_2^2 U_\sigma^2(x) - (\partial x_1^2 U_\sigma^2(x) + \partial x_2^2 U_\sigma^2(x))\right) x_2 = 0$$

$$\Rightarrow -\partial x_2^2 U_\sigma^2(x)\, x_1 + \partial x_1 \partial x_2 U_\sigma^2(x)\, x_2 = 0$$

$$\partial x_1 \partial x_2 U_\sigma^2(x)\, x_1 - \partial x_1^2 U_\sigma^2(x)\, x_2 = 0$$

$$\Rightarrow -\partial x_2 x_1 + \partial x_1 x_2 = 0$$

$$\partial x_2 x_1 - \partial x_1 x_2 = 0$$

$$\Rightarrow \partial x_1 x_2 = \partial x_2 x_1$$

Let $x_1 = \dfrac{\partial x_1 U_\sigma(x)}{|\nabla U_\sigma(x)|}$ then $x_2 = \dfrac{\partial x_2 U_\sigma(x)}{|\nabla U_\sigma(x)|}$

The corresponding eigen vector is $\frac{1}{|\nabla U_\sigma(x)|} \cdot \nabla U_\sigma(x)$

since $\lambda_1, \lambda_2 \geq 0$

$\Rightarrow J_0(\nabla U_\sigma(x))$ is positive semi-definite

Also $J_0(\nabla U_\sigma(x)) = (J_0(\nabla U_\sigma(x)))^T$

$\therefore$ symmetric

(b) $J_\rho(\nabla U_\sigma(x))$

where $J_\rho(\nabla U_\sigma(x)) = (G_\rho * J_0(\nabla U_\sigma))(x) \in \mathbb{R}^{2\times 2}$

$$\Rightarrow J_\rho(\nabla U_\sigma(x)) = \begin{pmatrix} G_\rho * \partial x_1^2 U_\sigma^2(x) & G_\rho * \partial x_1 \partial x_2 U_\sigma^2(x) \\ G_\rho * \partial x_1 \partial x_2 U_\sigma^2(x) & G_\rho * \partial x_2^2 U_\sigma^2(x) \end{pmatrix}$$

finding eigen values

$$\det(J_\rho(\nabla U_\sigma(x)) - \lambda I) = 0$$

$$\det \begin{pmatrix} (G_\rho * \partial x_1^2 U_\sigma^2(x)) - \lambda & G_\rho * \partial x_1 \partial x_2 U_\sigma^2(x) \\ (G_\rho * \partial x_1 \partial x_2 U_\sigma^2(x)) & (G_\rho * \partial x_2^2 U_\sigma^2(x)) - \lambda \end{pmatrix} = 0$$

$$\cancel{((G_\rho * \partial x_1^2 U_\sigma^2(x)) - \lambda)((G_\rho * \partial x_2^2 U_\sigma^2(x)) - \lambda) - (G_\rho * \partial x_1 \partial x_2 U_\sigma^2(x))^2} = 0$$

$$\det \begin{pmatrix} J_{11} - \lambda & J_{12} \\ J_{21} & J_{22} - \lambda \end{pmatrix} = 0 \quad \Rightarrow \lambda^2 - \lambda(J_{11} + J_{22}) + (J_{11} J_{22} - J_{12} J_{21}) = 0$$

$$\Rightarrow \lambda = \frac{(J_{11} + J_{22}) \pm \sqrt{(J_{11} + J_{22})^2 - 4(J_{11} J_{22} - J_{12} J_{21})}}{2}$$

$$\Rightarrow \lambda = \frac{(J_{11} + J_{22}) \pm \sqrt{(J_{11} - J_{22})^2 + 4 J_{12} J_{21}}}{2}$$

$\lambda \geq 0 \Rightarrow$ positive semidefinite

Also, $J_e(\nabla U_G) = (J_e(\nabla U_G))^T$

$\therefore$ symmetric.