Technische Universität Hamburg
Mathematical Image Processing (Winter Term 2022/2023)
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Exercise Sheet 4

Deadline: 14.11.22, 12:00pm

Exercise 1 (Morphological Operations).
In the lecture, we introduced the closing operation $\bullet$ and the opening operation $\circ$. We want to prove some properties of these two operations in this exercise.

a) Let $A, B \subset \mathbb{R}^n$. Show that $A \circ B \subset A$ and $A \subset A \bullet B$.

b) Prove the following statement: If $A_1 \subset A_2 \subset \mathbb{R}^n$, we have

$$A_1 \oplus B \subset A_2 \oplus B \qquad \text{and} \qquad A_1 \ominus B \subset A_2 \ominus B.$$

Use this property and part a) to verify the equations

$$(A \bullet B) \bullet B = A \bullet B \qquad \text{and} \qquad (A \circ B) \circ B = A \circ B.$$

Exercise 2 (Square Detection).
Suppose you are given a B/W image A which contains black squares on a white background (see figure below). Write a function

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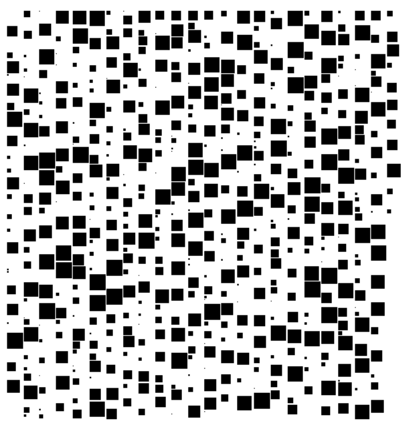
function [N] = countSquares(A),
```

that counts the number of black squares in A. The output should be an row vector N and the entry $N(s)$ should be the number of squares with size $s \times s$ for $s \in \mathbb{N}$. You can assume that all squares in the image have a side length which is less than or equal to 20 pixels. Moreover, you are allowed to use the following built-in Matlab / Octave functions: `imerode`, `imdilate`, `imopen`, `imclose`, `strel`



Exercise 3 (Discrete Convolution of Finite Sequences).
In this exercise, we want to take a look at the convolution of finite sequences. For infinite sequences from the space

$$\ell^1(\mathbb{Z}) = \left\{ x = (x_n)_{n\in\mathbb{Z}} \,\middle|\, \|x\|_1 = \sum_{n\in\mathbb{Z}} |x_n| < \infty \right\},$$

the convolution $* : \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ is defined by the formula

$$(x * y)_n := \sum_{k\in\mathbb{Z}} x_k \cdot y_{n-k} \qquad \text{for } n \in \mathbb{Z}$$

for all $x, y \in \ell^1(\mathbb{Z})$. However, we will mostly deal with finite data in practical cases, so that we want to take a look at suitable convolution methods for finite sequences.

a) Given a finite sequence $x = (x_1, ...x_l)$ with $L \in \mathbb{N}$, we can define the *zero-padded* version $x^{(0)} \in \ell^1(\mathbb{Z})$ of x as

$$x_n^{(0)} = \begin{cases} x_n & \text{if } n \in \{1, ..., L\} \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{1, ..., L\}. \end{cases}$$

For finite sequences $x = (x_1, ...x_L)$ and $y = (y_1, ...y_M)$, we can perform a convolution by computing the convolution $x^{(0)} * y^{(0)}$. Show that we have

$$(x^{(0)} * y^{(0)})_n = 0 \qquad \text{for all } n \in \mathbb{Z} \setminus \{2, ..., L + M\}.$$

b) In the previous part, we saw that the support of the convolution $x^{(0)} * y^{(0)}$ is contained in the set $\{2, ..., L + M\} \subset \mathbb{Z}$. Since we want the support of the convolution to be in $\{1, ..., L + M - 1\}$, we consider the **linear convolution** $\star$ which is defined as

$$(x \star y)_n := \sum_{k=1}^{L} x_k \cdot y^{(0)}_{n-k+1} \quad \text{for all } n \in \{1, ..., L + M - 1\}.$$

Prove the following equation:

$$(x \star y)_n = (x^{(0)} * y^{(0)})_{n+1} \qquad \text{for all } n \in \{1, ..., L + M - 1\}.$$

c) Now assume that we have two finite sequences $\tilde{x} = (\tilde{x}_1, ...\tilde{x}_N)$ and $y = (\tilde{y}_1, ...\tilde{y}_N)$ with the same length. The **circular convolution** $\tilde{x} \otimes \tilde{y}$ of $\tilde{x}$ and $\tilde{y}$ is then defined as

$$(\tilde{x} \otimes \tilde{y})_n = \sum_{k=1}^{N} \tilde{x}_k \cdot \tilde{y}^{(p)}_{n-k+1},$$

where $\tilde{y}^{(p)}$ is the periodic continuation of $\tilde{y}$, i.e.

$$\tilde{y}^{(p)}_n = \tilde{y}_{(n-1)\%N+1} \quad \text{with} \quad (n-1)\%N = (n-1) \bmod N \qquad \text{for all } n \in \mathbb{Z}.$$

For the sequences $x = (x_1, ...x_L)$ and $y = (y_1, ...y_M)$, where we might have $L \neq M$, we consider the zero-padded versions $\tilde{x}, \tilde{y}$ of length $L + M - 1$ given by

$$\tilde{x}_n = \begin{cases} x_n & \text{if } n \in \{1, ..., L\} \\ 0 & \text{if } n \in \{L + 1, ...L + M - 1\} \end{cases} \quad \text{and} \quad \tilde{y}_n = \begin{cases} y_n & \text{if } n \in \{1, ..., M\} \\ 0 & \text{if } n \in \{M + 1, ...L + M - 1\}. \end{cases}$$

Show that $x \star y = \tilde{x} \otimes \tilde{y}$ holds in this case, which means that we can compute the linear convolution of x and y by a circular convolution.