Exercise 3 (Heat equation).
We consider the *heat equation*

$$\partial_t u(x,t) - \Delta u(x,t) = 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,.$$

a) We define the *heat kernel* via

$$\Phi(x,t) = f_t(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,,$$

$\Leftrightarrow f_t(x) = G_{\sqrt{2t}}(x)$

where f_t is the scaled Gaussian function from exercise sheet 5 for every $t > 0$:

$$f_t(x) := (4\pi t)^{-d/2} \cdot e^{-\|x\|_2^2/4t} \quad \text{for } x \in \mathbb{R}^d$$

$\frac{\partial}{\partial t} f = \Delta_x f$

Show that Φ solves the heat equation (4).

$$\frac{\partial}{\partial t}\Phi = \frac{\partial}{\partial t}\left((4\pi t)^{-d/2} \cdot e^{-\|x\|_2^2/4t} \right)$$

$$\frac{\partial}{\partial t}\left((4\pi t)^{-d/2} \right) = -\frac{d}{2}(4\pi t)^{-\frac{d}{2}-1} (4\pi)$$

$$= -2\pi d (4\pi t)^{\frac{-d-2}{2}}$$

$$\frac{\partial}{\partial t}\left(e^{-\|x\|_2^2 t^{-1}/4} \right) = e^{-\|x\|_2^2/4t} \cdot \frac{\|x\|_2^2}{4} t^{-2}$$

$$= \frac{\|x\|_2^2}{4t^2} e^{-\|x\|_2^2/4t}$$

$$\frac{\partial}{\partial t}\Phi = (4\pi t)^{-d/2} \cdot \frac{\|x\|_2^2}{4t^2} e^{-\|x\|_2^2/4t} - 2\pi d (4\pi t)^{\frac{-d-2}{2}} e^{-\|x\|_2^2/4t}$$

$$\frac{\partial}{\partial t}\Phi = \left[(4\pi t)^{-d/2} \frac{\|x\|_2^2}{4t^2} - 2\pi d (4\pi t)^{\frac{-d-2}{2}} \right] e^{-\|x\|_2^2/4t}$$

$$\frac{\partial}{\partial x}\Phi = (4\pi t)^{-d/2} e^{-\|x\|_2^2/4t} \cdot \left(-\frac{1}{4t} \cdot \sum_{i=1}^{d} 2x_i \frac{\partial}{\partial x_i} \right)$$

$$\|x\|_2^2 = \sum_{i=1}^{d} x_i^2 \qquad \frac{\partial}{\partial x} = \sum_{i=1}^{d} 2x_i \cdot \frac{\partial}{\partial x_i} \qquad \Rightarrow 2x$$

$$= \frac{-(4\pi t)^{-d/2}}{4t} \cdot e^{-\|x\|_2^2/4t} \cdot 2x$$

$$\frac{\partial}{\partial x}\Phi = -\frac{(4\pi t)^{-d/2}}{4t} \underbrace{2x_i e^{-\|x\|_2^2/4t}}$$

$$\frac{\partial^2}{\partial x^2}\phi = \frac{-(4\pi t)^{-d/2}}{4t}\left[2\left(e^{-\|x\|_2^2/4t} \cdot \frac{-1}{4t} \cdot 2x + 2e^{-\|x\|_2^2/4t}\right]\right.$$

$$= -\frac{(4\pi t)^{-d/2}}{2t} e^{-\|x\|_2^2/4t}\left[-\frac{x}{2t} + d\right]$$

plugging into $\frac{\partial}{\partial t}\phi - \Delta_x \phi = 0$ gives:

$$\left[(4\pi t)^{-d/2}\frac{\|x\|_2^2}{4t^2} - 2\pi d(4\pi t)^{\frac{-d-2}{2}}\right] e^{-\|x\|_2^2/4t} + \left[-\frac{x}{2t} + d\right]\frac{(4\pi t)^{-d/2}}{2t} e^{-\|x\|_2^2/4t} = 0$$

$$(4\pi t)^{-d/2}\frac{\|x\|_2^2}{4t^2} - 2\pi d(4\pi t)^{\frac{-d-2}{2}} - \frac{(4\pi t)^{-d/2}}{4t^2}x^2 + \frac{d(4\pi t)^{-d/2}}{2t} = 0$$

$$0 = 0$$

thus, $\phi(x,t)$ satisfies the heat equation ∎

Let $g \in L^\infty(\mathbb{R}^d)$ be a continuous function. If we add the initial condition

$$u(x,0) = g(x) \qquad \text{for all } x \in \mathbb{R}^d,$$

it can be shown that the (continuous) solution of the heat equation is then given by the convolution

$$u(x,t) = (f_t * g)(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,. \tag{5}$$

b) Show that the solution (5) is bounded on $\mathbb{R}^d \times [0,\infty)$ in this case. You can use that

$$f \text{ bounded } \Leftarrow \left[\ \int_{\mathbb{R}^d} \Phi(x,t)\,dx = 1 \qquad \text{for all } t > 0. \right.$$

mass preservation

& continuous

$\varphi(x,t) =: f_t$

$g(x) = u(x,0)$

eg is $|u(x,t)| < M$ for $M \in \mathbb{R}$

$$(f_t * g)(x) = \int_{\mathbb{R}^d} f_t(x-y)\, g(y)\,dy$$

$$|(f_t * g)(x)| \leq \left| \int_{\mathbb{R}^d} f_t(x-y)\, g(y)\,dy \right|$$

$$\leq \int_{\mathbb{R}^d} |f_t(x-y)\, g(y)|\,dy$$

$$\leq \int_{\mathbb{R}^d} |g(y)|\,dy$$

$\leq M$ since $g \in L^\infty$ and f_t is bounded

$\therefore$ $u(x,t)$ is bounded ∎

c) For every time $t > 0$, we define the respective slice of the solution as

$$u_t : \mathbb{R}^d \to \mathbb{R}, \ x \mapsto u(x,t).$$

Prove the following equation:

what does u_t look like?

eg rearrange brackets

$$u_{t+h} = f_h * u_t \qquad \text{for all } t, h > 0.$$

Hint: Take a look at Sheet 5 Exercise 1c).

=> iterative application of Gaussian is

$u(x,t) = (f_t * g)(x)$ $\qquad$ $u(x,0) = g(x)$

f_h another scaled Gaussian

$u_{t+h} = u(x, t+h)$

show $(f_{t+h} * g)(x) = f_h * (f_t * g)$

$u(x, t+h) := (f_{t+h} * g)(x)$

$f_h * u_t = f_h * (f_t * g)$

$= (f_h * f_t) * g$

From Exercise sheet 5, problem 1 part c, we know a family of multivariate scaled Gaussians satisfies

$f_a * f_b = f_{a+b}$ for $a, b \in \mathbb{R}_+$

Thus for $t, h > 0$

$f_t * f_h = f_{t+h}$

Then,

$$f_h * v_t = f_{t+h} * g$$

$$= v_{t+h}$$

$\blacksquare$