Exercise 2 To show error functionals are convex

$\phi : X \rightarrow \mathbb{R}$ on space X is convex if the inequality $\phi(\lambda . x + (1-\lambda) . y) \leq \lambda . \phi(x) + (1-\lambda) \phi(y)$

$x, y \in X$ and $\lambda \in [0,1]$

(a) $X \rightarrow$ linear space $\| . \|_X$

$\| \lambda x + (1-\lambda) y \| \leq \| \lambda x \| + \| (1-\lambda) y \|$ (using Δ inequality)

$= \lambda \| x \| + (1-\lambda) \| y \|$ (by absolutely scalable property of norm)

for $x, y \in X$

$\Rightarrow \| . \|_X$ is convex

(b) $L^2 - H^1$ denoising

$\phi : H^1(\mathbb{R}^2) \rightarrow \mathbb{R}$

$$u \rightarrow \frac{1}{2} \| u - u_0 \|_{L^2}^2 + \frac{\mu}{2} \| \nabla u \|_{L^2}^2$$

given image $u_0 \in L^2(\mathbb{R}^2)$

regularization parameter $\mu > 0$

$e(x) := x^2$ is monotone and convex on $[0, \infty)$

Sol^n (b): $\phi(\lambda u + (1-\lambda) v)$

$$= \frac{1}{2} \| (\lambda u + (1-\lambda)v) - u_0 \|_{L^2}^2$$

$$+ \frac{\mu}{2} \| \nabla (\lambda u + (1-\lambda) v \|_{L^2}^2$$

$$= \frac{1}{2} \| \lambda u + (1-\lambda)v - u_0 - \lambda u_0 + \lambda u_0 \|_{L^2}^2 + \frac{\mu}{2} \| \nabla(\lambda u) + \nabla((1-\lambda) v) \|_{L^2}^2$$

$$= \frac{1}{2} \| \lambda u + (1-\lambda)v - \lambda u_0 - (1-\lambda) u_0 \|_{L^2}^2 + \frac{\mu}{2} \| \lambda (\nabla u) + (1-\lambda)(\nabla v) \|_2^2$$

$$= \frac{1}{2} \| \lambda(u-u_0) + (1-\lambda)(v-u_0) \|_{L^2}^2 + \frac{\mu}{2} \| \lambda(\nabla u) + (1-\lambda)(\nabla v) \|_2^2 \quad \to (1)$$

Now, $f(x) = \|x\|_2$ is convex (from part (a))

$e(x) = x^2$ is monotone (non decreasing) and convex on $\underbrace{[0, \infty)}_{\text{range of } f(x)}$

$e \circ f = \|.\|_2^2$ is a composition of 2 convex functions which is convex too.

$\cancel{e(\lambda x + (1-\lambda) y)}$

$\cancel{= (\lambda x + (1-\lambda) y)^2}$

$\cancel{= \lambda^2 x^2 + (1-\lambda)^2 y^2}$

$\cancel{+ 2\lambda x (1-\lambda) y}$

$\cancel{\leq e(\lambda x) + e((1-\lambda) y)}$

Eq^n (1) is a summand of 2 convex functions which will again be a convex function.