1\. a) $f(x) = g(1-\|x\|_2^2)$, $x \in \mathbb{R}^d$, where:

$$g(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Hence, $f(x) = \begin{cases} e^{\frac{-1}{1-\|x\|_2^2}}, & 1-\|x\|_2^2 > 0 \\ 0, & 1-\|x\|_2^2 \leq 0 \end{cases}$

$$f(x) = \begin{cases} e^{\frac{-1}{1-\|x\|_2^2}}, & \|x\|_2^2 < 1 \\ 0, & \|x\|_2^2 \geq 1 \end{cases}$$

$$f(x) = \begin{cases} e^{\frac{-1}{1-\|x\|_2^2}}, & \|x\|_2 < 1 \\ 0, & \|x\|_2 \geq 1 \end{cases}$$

$$\operatorname{supp} f(x) \underset{\text{by definition}}{=} \overline{\{x \in X : f(x) \neq 0\}} \underset{\text{from the fctn form above}}{=} \overline{\{x \in \mathbb{R}^d \mid \|x\|_2 < 1\}} = \{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1\}$$

$= \overline{B_1(0)}$ ✓

$f'(x) = g'(1-\|x\|_2^2) \cdot (2\|x\|_2)$ (by chain rule)

$f''(x) = 2g'(1-\|x\|_2^2) \cdot (2\|x\|_2)$

$f^{(n)}(x) = 2^n \|x\|_2 \cdot g^{(n)}(1-\|x\|_2^2) \Rightarrow$ since $g(x) \in C^\infty$, $f(x) \in C^\infty$ ✓

$$\left.\begin{array}{l} f(x) \in C^{\infty} \\ supp(f) = \overline{B_1(0)} \end{array}\right\} \Rightarrow f \in C_0^{\infty}(\mathbb{R}^d) \quad V$$

b) $\forall \varepsilon > 0 \quad f_\varepsilon(x) = \varepsilon^{-d} \cdot f(\varepsilon^{-1} \cdot x) \quad , x \in \mathbb{R}^d.$

Prove $supp(f_\varepsilon) = \{x \in \mathbb{R}^d \mid \|x\|_2 \leq \varepsilon\} =: \overline{B_\varepsilon(0)}$

and $\displaystyle\int_{\mathbb{R}^d} f_\varepsilon(x)dx = \int_{\mathbb{R}^d} f(x)dx$

Proof : $f_\varepsilon(x) = \begin{cases} \varepsilon^{-d} \cdot e^{\frac{-1}{1-\|\varepsilon^{-1} \cdot x\|_2^2}} & , \|\varepsilon^{-1} \cdot x\|_2 < 1 \\ \varepsilon^{-d} \cdot 0, & \|\varepsilon^{-1} \cdot x\|_2 \geq 1 \end{cases}$

$$\|\varepsilon^{-1} \cdot x\|_2^2 = (\varepsilon^{-1} x_1)^2 + (\varepsilon^{-1} x_2)^2 + \ldots + (\varepsilon^{-1} x_d)^2 = \varepsilon^{-2} \|x\|_2^2$$

$$f_\varepsilon(x) = \begin{cases} \varepsilon^{-d} \cdot e^{\frac{-1}{1-\varepsilon^{-2}\|x\|_2^2}} & , \frac{1}{\varepsilon} \cdot \|x\|_2 < 1 \\ 0 & , \frac{1}{\varepsilon} \cdot \|x\|_2 \geq 1 \end{cases}$$

$$f_\varepsilon(x) = \begin{cases} \varepsilon^{-d} \cdot e^{\frac{-1}{1-\frac{1}{\varepsilon^2}\|x\|_2^2}} & , \|x\|_2 < \varepsilon \\ 0, & \|x\|_2 \geq \varepsilon \end{cases}$$

$$\operatorname{supp} f(x) = \overline{\{x \in X : f(x) \neq 0\}} = \overline{\{x \in \mathbb{R}^d \mid \|x\|_2 < \varepsilon\}} = \{x \in \mathbb{R}^d \mid \|x\|_2 \leq \varepsilon\}$$

by definition

from the fctn form above

$$= \overline{B_\varepsilon(0)} \quad \checkmark$$

$$\int_{\mathbb{R}^d} f_\varepsilon(x)\,dx = \int_{\overline{B_\varepsilon(0)}} \frac{1}{\varepsilon^d} \cdot e^{\frac{1}{\frac{1}{\varepsilon^2}\|x\|_2^2 - 1}}\,dx =$$

$\operatorname{supp} f_\varepsilon(x) = \overline{B_\varepsilon(0)}$

$$= \int_{\overline{B_\varepsilon(0)}} \left(\varepsilon^d e^{\frac{1}{1-\frac{1}{\varepsilon^2}\|x\|_2^2}}\right)^{-1} dx = \left[\begin{array}{ll} y = \frac{x}{\varepsilon} & \|x\|_2^2 = \varepsilon^2 \|y\|_2^2 \\ \overline{B_\varepsilon(0)} \to \overline{B_1(0)} & dx = \varepsilon^d dy \end{array}\right] =$$

$$= \int_{\overline{B_1(0)}} \cancel{\varepsilon^{d-d}}\, e^{\frac{-1}{1-\|y\|_2^2}}\,dy = \int_{\overline{B_1(0)}} e^{\frac{-1}{1-\|y\|_2^2}}\,dy = \int_{\overline{B_1(0)}} f(y)\,dy =$$

$$= \int_{\mathbb{R}^d} f(y)\,dy.$$

$\operatorname{supp} f(x) = \overline{B_1(0)}$