Technische Universität Hamburg
Mathematical Image Processing (Winter Term 2022/2023)
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Exercise Sheet 6

Deadline: 28.11.22, 12:00pm

application 2D wavelets: image compression

Exercise 1 (Haar wavelet)**.**
Consider the scaling function

$$\varphi(x) := \chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the scaled / shifted versions

$$\varphi_k^{(j)} = 2^{j/2} \cdot \varphi(2^j \cdot x - k) \qquad \text{for all } j, k \in \mathbb{Z}. \qquad x \in \mathbb{R}^d$$

a) Show that

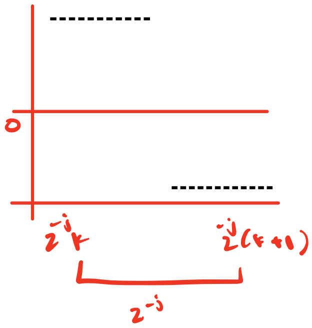
use def regarding closure

$$\text{supp}\left(\varphi_k^{(j)}\right) = \left[2^{-j} \cdot k, 2^{-j} \cdot (k+1)\right] \qquad \text{for all } j, k \in \mathbb{Z}.$$

Conclude that we have

$$\langle \varphi_k^{(j)}, \varphi_l^{(j)} \rangle_{L^2(\mathbb{R})} = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{if } k \ne l \end{cases}$$

for fixed $j \in \mathbb{Z}$.



j↑ dist get ↓

We define the scale space of order $j \in \mathbb{Z}$ as

$$V_j = \overline{\text{span}\,\{\varphi_k^{(j)} \mid k \in \mathbb{Z}\}}.$$

b) Prove the refinement equation

$$\varphi_k^{(j-1)} = 2^{-1/2} \cdot \left(\varphi_{2k}^{(j)} + \varphi_{2k+1}^{(j)}\right) \qquad \text{for all } j, k \in \mathbb{Z}.$$

and use this equation to verify the inclusion

$$V_{j-1} \subset V_j \qquad \text{for all } j \in \mathbb{Z}.$$

It can be shown that

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\} \qquad \text{and} \qquad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{1}$$

More precisely, the sequence $(V_j)_{j\in\mathbb{Z}}$ forms a **Multiresolution analysis** of $L^2(\mathbb{R})$. Given an $f \in L^2(\mathbb{R})$, let $\Pi_{V_j}(f)$ denote the best approximation to f from V_j, i.e.

$$\|f - \Pi_{V_j}(f)\|_{L^2(\mathbb{R})} = \min_{v \in V_j} \|f - v\|_{L^2(\mathbb{R})} \qquad \text{for all } j \in \mathbb{Z}.$$

The second part of (1) then implies that

$$\lim_{j \to \infty} \Pi_{V_j}(f) = f,$$

which means that we can reconstruct any function $f \in L^2(\mathbb{R})$ arbitrarily well. For our signal analysis, we want to decompose the space $L^2(\mathbb{R})$ into an orthogonal sum

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j\,,$$

↑ j ↑ resolution

where W_j represents the part that corresponds to the resolution level $j \in \mathbb{Z}$. Therefore, we introduce the *Haar wavelet* on $\mathbb{R}$

$$\psi(x) := \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x) = \begin{cases} 1, & \text{if } 0 \le x < 1/2 \\ -1, & \text{if } 1/2 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

and its shifted / scaled versions

$$\psi^{(j)} = 2^{j/2} \cdot \psi(2^j \cdot x - k) \qquad \text{for all } j, k \in \mathbb{Z}.$$

Analogously, we set

$$W_j = \overline{\text{span}\,\{\psi_k^{(j)} \mid k \in \mathbb{Z}\}} \qquad \text{for all } j \in \mathbb{Z}.$$

c) Prove the refinement equations

$$\psi(x) = \varphi(2x) - \varphi(2x-1) \quad \text{for all } x \in \mathbb{R}, \qquad \psi_k^{(j-1)} = 2^{-1/2} \cdot \left(\varphi_{2k}^{(j)} - \varphi_{2k+1}^{(j)}\right) \quad \text{for all } j, k \in \mathbb{Z}$$

and the orthogonality relation

$$\langle \varphi_k^{(j-1)}, \psi_l^{(j-1)} \rangle_{L^2(\mathbb{R})} = 0 \qquad \text{for all } j, k, l \in \mathbb{Z}.$$

Use these properties and part b) to prove the decomposition

$$V_j = V_{j-1} \oplus W_{j-1} \qquad \text{for all } j \in \mathbb{Z}.$$

In total, we get

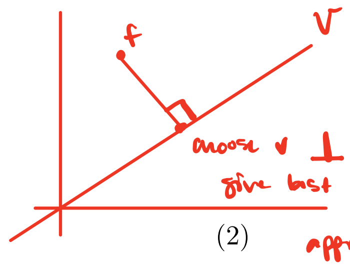
$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j,$$

such that $\{\psi_k^{(j)} \mid j, k \in \mathbb{Z}\}$ is a complete orthonormal system of $L^2(\mathbb{R})$.

try later

~~(⋆) **Exercise 2** (Orthogonal Projections).~~
Let H be a real Hilbert space and $V \subset H$ be a closed subspace. We want to prove that the projection operator

$$\Pi_V(f) := \underset{v \in V}{\text{argmin}}\, \|f - v\|_H \qquad \text{for all } f \in H$$



choose v ⊥ to give best approx.

is well-defined.

a) Show that there is an element $v^* \in V$ such that

$$\|f - v^*\|_H = \inf_{v \in V} \|f - v\|. \tag{2}$$

To this end, consider a sequence $(v_k)_{k \in \mathbb{N}}$ in V that satisfies

$$\lim_{k \to \infty} \|f - v_k\|_H = \inf_{v \in V} \|f - v\|_H$$

and use the parallelogram identity

$$\|v + w\|_H^2 + \|v - w\|_H^2 = 2 \cdot \|v\|_H^2 + 2 \cdot \|w\|_H^2 \qquad \text{for all } v, w \in H$$

to show that this sequence is a Cauchy sequence.

b) Prove that $v^* \in V$ satisfies property (2) if and only if

$$\langle f - v^*, v \rangle_H = 0 \qquad \text{for all } v \in V. \tag{3}$$

c) Let $v_1^*, v_2^* \in V$ satisfy (3). Show that this implies $v_1^* = v_2^*$.

Hence, there is a unique best approximation $\Pi_V(f)$ to each $f \in H$ from V, the so-called *orthogonal projection*.

Exercise 3 (Heat equation).
We consider the *heat equation*

$$\partial_t u(x,t) - \Delta u(x,t) = 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0, \infty). \tag{4}$$

a) We define the *heat kernel* via

$$\Phi(x,t) = f_t(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,,$$

eg $f_t(x) = G_{\sqrt{2t}}(x)$

where f_t is the scaled Gaussian function from exercise sheet 5 for every $t > 0$:

f_h

$$f_t(x) := (4\pi t)^{-d/2} \cdot e^{-\|x\|_2^2/4t} \quad \text{for } x \in \mathbb{R}^d$$

$\frac{\partial}{\partial t} f = \Delta_x f$

Show that Φ solves the heat equation ($\boxed{4}$).

Let $g \in L^\infty(\mathbb{R}^d)$ be a continuous function. If we add the initial condition

$$u(x,0) = g(x) \qquad \text{for all } x \in \mathbb{R}^d,$$

it can be shown that the (continuous) solution of the heat equation is then given by the convolution

$$u(x,t) = (f_t * g)\,(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,. \tag{5}$$

b) Show that the solution ($\boxed{5}$) is bounded on $\mathbb{R}^d \times [0,\infty)$ in this case. You can use that

$$\int_{\mathbb{R}^d} \Phi(x,t)\; dx = 1 \qquad \text{for all } t > 0.$$

mass preservation

c) For every time $t > 0$, we define the respective slice of the solution as

$$u_t : \mathbb{R}^d \to \mathbb{R},\; x \mapsto u(x,t).\,.$$

what does u_t look like?

Prove the following equation:

eg rearrange brackets

$$u_{t+h} = f_h * u_t \qquad \text{for all } t, h > 0.$$

Hint: Take a look at Sheet 5 Exercise 1c).

=> iterative application of Gaussian is application of smoother Gaussian

d) Let $B_r(0)$ denote the d-dimensional open ball around zero, i.e.

$$B_r(0) = \{x \in \mathbb{R}^d \mid \|x\| < r\},$$

for any radius $r \in \mathbb{R}_+$. We assume that there is a radius $R > 0$ such that $g(x) > 0$ for $x \in B_R(0)$ and $g(x) = 0$ for $x \in \mathbb{R}^d \setminus B_R(0)$. Prove that we have

$$u(x,t) > 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)$$

in this case. This phenomenon is called *infinite propagation speed*, since the positiveness of g on an arbitrarily small ball translates to the whole domain at all further times.

Exercise 4 (Gaussian Filter)**.**
In this exercise, we want to implement a Gaussian filter for image smoothing. Consider the non-normalized Gaussian functions

$$\tilde{f}_t(x) := e^{-\|x\|_2^2/4t} \quad \text{for } x \in \mathbb{R}^d.$$

Given a parameter $t > 0$, we set the filter width to $W_t = \left\lceil 3 \cdot \sqrt{2t} \right\rceil \in \mathbb{N}$. Hence, the filter mask is given via the matrix

variance

$$\tilde{F}_t = \left(\tilde{f}_t(i,j)\right)_{-W_t \le i,j \le W_t} \in \mathbb{R}^{2W_t+1 \times 2W_t+1}.$$

and the normalized filter mask is given via

$$F_t = M^{-1} \cdot \tilde{F}_t \qquad \text{where} \qquad M = \sum_{i=-W_t}^{W_t} \sum_{j=-W_t}^{W_t} \tilde{f}_t(i,j).$$

Write a function

```
function [B, F] = GaussianFilter(A, t)
```

that performs a convolution of the grayscale image A with the normalized filter mask F_t for a given parameter $t > 0$. Here, the image should be padded outside the image by reflecting the values inside the image. For the convolution, you can use the built-in function `imfilter` in Matlab / Octave. Moreover, the function should return the filter mask $F = F_t$.

Exercise 3 (Heat equation).
We consider the *heat equation*

$$\partial_t u(x,t) - \Delta u(x,t) = 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty).$$

a) We define the *heat kernel* via

$$\Phi(x,t) = f_t(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty),$$

cf $f_t(x) = G_{\sqrt{2t}}(x)$

where f_t is the scaled Gaussian function from exercise sheet 5 for every $t > 0$:

$$f_t(x) := (4\pi t)^{-d/2} \cdot e^{-\|x\|_2^2/4t} \quad \text{for } x \in \mathbb{R}^d$$

$\frac{\partial}{\partial t} f = \Delta_x f$

Show that Φ solves the heat equation (4).

$$\frac{\partial}{\partial t}\Phi = \frac{\partial}{\partial t}\left((4\pi t)^{-d/2} \cdot e^{-\|x\|_2^2/4t} \right)$$

$$\frac{\partial}{\partial t}\left((4\pi t)^{-d/2} \right) = -\frac{d}{2}(4\pi t)^{-\frac{d}{2}-1}(4\pi)$$

$$= -2\pi d\,(4\pi t)^{\frac{-d-2}{2}}$$

$$\frac{\partial}{\partial t}\left(e^{-\|x\|_2^2\, t^{-1}/4} \right) = e^{-\|x\|_2^2/4t} \cdot \frac{\|x\|_2^2}{4}\, t^{-2}$$

$$= \frac{\|x\|_2^2}{4t^2}\, e^{-\|x\|_2^2/4t}$$

$$\frac{\partial}{\partial t}\Phi = (4\pi t)^{-d/2} \cdot \frac{\|x\|_2^2}{4t^2}\, e^{-\|x\|_2^2/4t} - 2\pi d\,(4\pi t)^{\frac{-d-2}{2}}\, e^{-\|x\|_2^2/4t}$$

$$\frac{\partial}{\partial t}\Phi = \left[(4\pi t)^{-d/2}\,\frac{\|x\|_2^2}{4t^2} - 2\pi d\,(4\pi t)^{\frac{-d-2}{2}} \right] e^{-\|x\|_2^2/4t}$$

$$\frac{\partial}{\partial x}\Phi = (4\pi t)^{-d/2}\, e^{-\|x\|_2^2/4t} \cdot \left(-\frac{1}{4t} \cdot \sum_{i=1}^{d} 2x_i \frac{\partial}{\partial x_i} \right)$$

$$\|x\|_2^2 = \sum_{i=1}^{d} x_i^2 \qquad \frac{\partial}{\partial x} = \sum_{i=1}^{d} 2x_i \cdot \frac{\partial}{\partial x_i} \quad \Rightarrow 2x$$

$$= -\frac{(4\pi t)^{-d/2}}{4t} \cdot e^{-\|x\|_2^2/4t} \cdot 2x$$

$$\frac{\partial}{\partial x}\Phi = -\frac{(4\pi t)^{-d/2}}{4t}\, 2x_i\, e^{-\|x\|_2^2/4t}$$

$$\frac{\partial^2}{\partial x^2}\phi = \frac{-(4\pi t)^{-d/2}}{4t}\left[2\left(e^{-\|x\|_2^2/4t} \cdot \frac{-1}{4t} \cdot 2x + 2e^{-\|x\|_2^2/4t}\right]\right.$$

$$= -\frac{(4\pi t)^{-d/2}}{2t} e^{-\|x\|_2^2/4t}\left[-\frac{x}{2t} + d\right]$$

plugging into $\frac{\partial}{\partial t}\phi - \Delta_x \phi = 0$ gives:

$$\left[(4\pi t)^{-d/2}\frac{\|x\|_2^2}{4t^2} - 2\pi d(4\pi t)^{\frac{-d-2}{2}}\right]e^{-\|x\|_2^2/4t} + \left[-\frac{x}{2t} + d\right]\frac{(4\pi t)^{-d/2}}{2t}e^{-\|x\|_2^2/4t} = 0$$

$$(4\pi t)^{-d/2}\frac{\|x\|_2^2}{4t^2} - 2\pi d(4\pi t)^{\frac{-d-2}{2}} - \frac{(4\pi t)^{-d/2}}{4t^2}x^2 + \frac{d(4\pi t)^{-d/2}}{2t} = 0$$

$$0 = 0$$

thus, $\phi(x,t)$ satisfies the heat equation $\blacksquare$

Let $g \in L^\infty(\mathbb{R}^d)$ be a continuous function. If we add the initial condition

$$u(x,0) = g(x) \qquad \text{for all } x \in \mathbb{R}^d,$$

it can be shown that the (continuous) solution of the heat equation is then given by the convolution

$$u(x,t) = (f_t * g)(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty). \tag{5}$$

b) Show that the solution (5) is bounded on $\mathbb{R}^d \times [0,\infty)$ in this case. You can use that

$$f \text{ bounded} \Leftarrow \left[\int_{\mathbb{R}^d} \Phi(x,t)\, dx = 1 \qquad \text{for all } t > 0. \right.$$

mass preservation

& continuous

$\varphi(x,t) =: f_t$

$g(x) = u(x,0)$

eq is $|u(x,t)| < M$ for $M \in \mathbb{R}$

$$(f_t * g)(x) = \int_{\mathbb{R}^d} f_t(x-y)\, g(y)\, dy$$

$$|(f_t * g)(x)| \leq \left| \int_{\mathbb{R}^d} f_t(x-y)\, g(y)\, dy \right|$$

$$\leq \int_{\mathbb{R}^d} |f_t(x-y)\, g(y)|\, dy$$

$$\leq \int_{\mathbb{R}^d} |g(y)|\, dy$$

$\leq M$ since $g \in L^\infty$ and f_t is bounded

$\therefore$ $u(x,t)$ is bounded $\blacksquare$

c) For every time $t > 0$, we define the respective slice of the solution as

$$u_t : \mathbb{R}^d \to \mathbb{R},\ x \mapsto u(x,t).$$

Prove the following equation:

$$u_{t+h} = f_h * u_t \qquad \text{for all } t, h > 0.$$

Hint: Take a look at Sheet 5 Exercise 1c).

what does u_t look like?
eg rearrange brackets
=> iterative application of Gaussian is

$$u(x,t) = (f_t * g)(x) \qquad u(x,0) = g(x)$$

f_h another scaled Gaussian

$$u_{t+h} = u(x, t+h)$$

show $(f_{t+h} * g)(x) = f_h * (f_t * g)$

$$u(x, t+h) := (f_{t+h} * g)(x)$$

$$f_h * u_t = f_h * (f_t * g)$$

$$= (f_h * f_t) * g$$

From Exercise sheet 5, problem 1 part c, we know a family of multivariate scaled Gaussians satisfies

$$f_a * f_b = f_{a+b} \quad \text{for } a, b \in \mathbb{R}_+$$

Thus for $t, h > 0$

$$f_t * f_h = f_{t+h}$$

Then,

$$
\begin{aligned}
f_h * v_t &= f_{t+h} * g \\
&= v_{t+h}
\end{aligned}
$$

∎