Exercise 3

$A: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $(x_n)_{n\in\mathbb{N}} \mapsto (n^{-1}x_n)_{n\in\mathbb{N}}$

(a) A is well defined i.e $Ax \subset \ell^2(\mathbb{N})$ for $x \in \ell^2(\mathbb{N})$

$$x = (x_1, x_2, \dots, x_n)$$

$$Ax = \left(\frac{x_1}{1}, \frac{x_2}{2}, \dots, \frac{x_n}{n}\right)$$

TS: A is well defined:

$$\|Ax\|_2 = \left(\sum |Ax|^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} \left|\frac{x}{n}\right|^2\right)^{1/2} \leq \left(\sum |x|^2\right)^{1/2} \in \ell^2(\mathbb{N})$$

$\Rightarrow$ A is wel defined.

A is injective

consider any sequence $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ s.t.

$$A x_n = A y_n$$

i.e $\quad n^{-1}x_n = n^{-1}y_n$

$$\Rightarrow x_n = y_n$$

$\Rightarrow$ A is injective

(b) T.S there is a constant $c > 0$ s.t.

$\|Ax\|_2 \leq c\|x\|_2 \quad \forall \; x \in l^2(\mathbb{N})$

for $x = 0$ $\quad Ax = 0$

for $x \neq 0$ $\quad \|Ax\| \leq c\|x\|$

$\Rightarrow \dfrac{\|Ax\|}{\|x\|} \leq c$

$\Rightarrow \left\{ \dfrac{\|Ax\|}{\|x\|} : x \in l^2(\mathbb{N}) - \{0\} \right\}$ is bounded above by some $c \in \mathbb{R}$

$\therefore$ supremum exists.

The supremum is $\|A\| = \sup \left\{ \dfrac{\|Ax\|}{\|x\|} : x \in l^2(\mathbb{N}) - \{0\} \right\}$

$\dfrac{\|Ax\|}{\|x\|} \leq \|A\| \quad x \neq 0)$

$\Rightarrow \|Ax\| \leq \|A\|\|x\|$ $\quad \Rightarrow \|A\|$ is the smallest possible value of c.

To show: A is cont.

Let $(x_n)_{n\in\mathbb{N}} \to x$ in $l^2(\mathbb{N})$

$\Rightarrow x_n - x \to 0 \Rightarrow \|x_n - x\| \to 0$

$\because x_n, x \in l^2(\mathbb{N}) \Rightarrow (x_n - x) \in l^2(\mathbb{N})$

Now since A is bounded

$\|A(x_n - x)\| \leq C\|x_n - x\| \to 0$ as $n\to\infty$

$\Rightarrow \|Ax_n - Ax\| \to 0$

$\Rightarrow Ax_n - Ax \to 0$

$\Rightarrow Ax_n \to Ax$ in $l^2(\mathbb{N})$

so, A is cont.

Exercise 4

~~Let T be bounded~~. Let $A \to$ compact operator

Let $A^{-1} \to$ bounded operator

~~Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. the $A^{-1}x_n$ is also a bounded sequence $\therefore A(A^{-1}x_n)$ is a convergent subsequence.~~

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence

$A \to$ compact $\Rightarrow$ there is a subsequence $\{(x_n)_r\}$ such that $A x_{n(r)}$ converges.

~~Since $(x_n)_{(r)}$ is bounded and~~

And since A^{-1} is bounded $\Rightarrow$ continuous the sequence $A^{-1}(A x_{n(r)})$ converges

$\Rightarrow A^{-1}A$ is compact

$\Rightarrow I$ is compact

however it is not possible since $A: X \to Y$ where X and Y are infinite dimensional

$\therefore$ our assumption is wrong.