Exercise 1 (Separable Masks).
Let $A \in \mathbb{R}^{m\times n}$ be a filter mask. We say that A is **separable**, if there are vectors $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ such that

$$A = v \cdot w^T.$$

m x1

a) Prove that a filter mask A is separable if and only if A has rank one.

b) Decide whether the following filter masks are separable or not:

$$A_L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_B = \frac{1}{16}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

c) Suppose that $A = v \cdot w^T \in \mathbb{R}^{m\times n}$ is separable. Show that for any image u, we have

$$A \boxed{*} u = w^T \boxed{*} (v \boxed{*} u),$$

where $\boxed{*}$ denotes the correlation.

(a) show separability => rank 1 and rank 1 => separability

proof: =>) Assume A separable. let $u \in \mathbb{R}^n$

$A = vw^T$

$Au = vw^T u$

$= (u^T w) v$

=> that A maps any vector to just a scalar multiple of v

the dim(image(A)) = 1

thus A has rank 1 ✓

<=) Assume A has rank 1

then $Aw = kv$ for $k \in \mathbb{R}$, $w \in \mathbb{R}^n$, $v \in \mathbb{R}^m$

since every column of A is just a multiple of v,

$$A\begin{pmatrix} w_1 v & w_2 v & \cdots & w_n v \end{pmatrix} = v\begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}$$

$$= vw^T$$

thus A is separable.

Hence A is separable if and only if A has rank 1

$\blacksquare$

(b) Given $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

reducing gives $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \dim(\text{Image}(A)) = 2 \neq 1$

thus A_2 is not separable

$\blacksquare$

Given $A_B = \frac{1}{16}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

reducing gives $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \dim(\text{Image}(A)) = 1$

thus A_B has rank 1 and is separable $\blacksquare$

(c) show $A \boxed{*} u = w^T \boxed{*} (v \boxed{*} u)$

<u>proof:</u> $A \boxed{*} u = (v w^T) \boxed{*} u$

$= (w^T * v) \boxed{*} u$

$= (w^T * v) * u$

$= (w * v^T) * u$

$= w * (v^T * u)$

$= w^T \boxed{*} (v^T * u)$

$= w^T \boxed{*} (v \boxed{*} u)$

$\blacksquare$