Exercise 1

$$\left.\begin{array}{l}\phi(0)=1 \qquad\qquad s_f(k)=f_k \ \forall k=1..N \\ \phi(i)=0\ ,\ i\in\mathbb{Z}/\{0\} \\ \phi^{(i)}(x):=\phi(x-i)\ \ \forall x\in\mathbb{R}\ \forall i\in\mathbb{Z}\end{array}\right\}\ \text{interpolation function}$$

a) $f=(f_i)_{i=1..N}\in\mathbb{R}^n$ - signal in 1D

Show that $S_f=\sum_{i=1}^{N} f_i\cdot\phi^{(i)}$ satisfies $\underbrace{S_f(k)=f_k\ \ \forall k=1..N}_{\text{interpolation condition}}$

Proof: $S_f(k)=\sum_{i=1}^{N} f_i\cdot\phi^{(i)}(k)=\sum_{i=1}^{N} f_i\cdot\phi(k-i)=[\phi(k-i)=0\ \forall i\neq k;\ \phi(k-i)=1$ for $i=k]=$
$=f_k\cdot 1=f_k.$

b) Show that if $s\in\operatorname{span}\{\phi^{(i)}\,|\,i\in\{1..N\}\}$ and $s(k)=f_k\ \forall k\in\{1..N\}$, then $s=s_f$

Proof: $s\in\operatorname{span}\{\phi^{(i)}\,|\,i\in\{1..N\}\}\Rightarrow s$ can be represented as $s=\sum_{i=1}^{N}a_i\phi^{(i)}$, where a_i are some coeffs. If $\forall k\in\{1..N\}$ $s(k)=f_k$, then

$$s(k)=\sum_{i=1}^{N}a_i\phi^{(i)}(k)=\sum_{i=1}^{N}a_i\phi(k-i)=a_k=f_k\quad\forall k\in\{1..N\}\Rightarrow$$

$$\Rightarrow s=\sum_{i=1}^{N}f_i\phi^{(i)}=s_f.$$

c) Verify that $\phi_0(x)=\chi_{[-0.5;0.5)}(x)=\begin{cases}1, & -0.5\leq x\leq 0.5\\ 0 & \text{otherwise}\end{cases}$ - nearest neighbor interpolation

and $\varphi_1(x)=\max(1-|x|,0)$ — bilinear interpolation

are interpolation functions.

Proof: $\left.\begin{array}{l}\phi(0)=1\\ \phi(i)=0\ ,\ i\in\mathbb{Z}/\{0\}\end{array}\right\}$ interpolation function

φ_0: $\phi_0(0)=1$
$\phi_0(i)=0\ \ \forall i\in\mathbb{Z}/\{0\}$ (since 0 is the only integer in $[-0.5,0.5]$)

φ_1 : $\varphi_1(0) = 1$

$\varphi_1(i) = 0 \quad \forall i \in \mathbb{Z}/\{0\}$ (since $1 - |i| < 0 \quad \forall i \in \mathbb{Z}/\{0\}$)

$$\sum_{i=1}^{N} \varphi_1(k-i) f_i = f_k$$

Another possible interpolation function: $\varphi_2(x) := \begin{cases} 0, & x = 1 \\ \left[\frac{1}{1-x}\right] & \text{otherwise} \end{cases}$

integer part of number

d) $u = (u_{i,j})_{\substack{1 \le i \le M \\ 1 \le j \le N}} \subset \mathbb{R}^{M \times N}$

$$S_u(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} u_{i,j} \varphi^{(i)}(x) \cdot \varphi^{(j)}(y), \quad (x,y) \in \mathbb{R}^2$$

Show that $S_u(k,l) = u_{k,l} \quad \forall (k,l) \in \{1 \dots M\} \times \{1 \dots N\}$

$$S_u(k,l) = \sum_{i=1}^{M} \sum_{j=1}^{N} u_{i,j} \varphi(k-i) \cdot \varphi(l-j) = \sum_{i=1}^{N} u_{i,l} \varphi(k-i) = u_{k,l}.$$