Exercise 1 (Morphological Operations).
In the lecture, we introduced the closing operation $\bullet$ and the opening operation $\circ$. We want to prove some properties of these two operations in this exercise.

a) Let $A, B \subset \mathbb{R}^n$. Show that $A \circ B \subset A$ and $A \subset A \bullet B$.

b) Prove the following statement: If $A_1 \subset A_2 \subset \mathbb{R}^n$, we have

$$A_1 \oplus B \subset A_2 \oplus B \qquad \text{and} \qquad A_1 \ominus B \subset A_2 \ominus B.$$

Use this property and part a) to verify the equations

$$(A \bullet B) \bullet B = A \bullet B \qquad \text{and} \qquad (A \circ B) \circ B = A \circ B.$$

(a) (i) by definition: $A \circ B := (A \ominus B) \oplus B$

let $x \in A \ominus B$.

Then $x + b \in A$ for $b \in B$, thus $x \in A$

Now let $x \in A \circ B = (A \ominus B) \oplus B$

Then $x = y + b$ for $y \in A \ominus B$ and $b \in B$

Because $x \in A$ from $A \supset A \ominus B$,

$x \in A$ from $x \in A \circ B$

thus $A \circ B \subset A$ ■

(ii) by definition: $A \bullet B := (A \oplus B) \ominus B$

let $x \in A$, then $x \in A \oplus B$ by definition

$(A \oplus B) \ominus B := \{x : x + b \in A \oplus B \text{ for } b \in B\}$

thus $x \in A \oplus B \Rightarrow x \in (A \oplus B) \ominus B$

hence, $x \in A \Rightarrow x \in (A \oplus B) \ominus B$

and $A \subset A \bullet B$ ■

(b) $A_1 \oplus B := \{ a_1 + b : a_1 \in A_1, b \in B \}$

$\subset \{ a_1 + b : a_1 \in A_2, b \in B \}$ since $A_1 \subset A_2$

$= A_2 \oplus B$

thus $A_1 \oplus B \subset A_2 \oplus B$

$\blacksquare$

$A_1 \ominus B := \{ x : x + B \subset A_1 \} \subset \{ x : x + B \subset A_2 \} = A_2 \ominus B$ since $A_1 \subset A_2$

thus $A_1 \ominus B \subset A_2 \ominus B$

$\blacksquare$

(i) show $(A \circ B) \circ B = A \circ B$:

from part (a), $A \circ B \subset A$

thus $(A \circ B) \ominus B \subset A \ominus B$ by monotonicity above

also from part (a), $A \subset A \bullet B$

thus, $A \ominus B \subset (A \ominus B) \bullet B = (A \circ B) \ominus B$

so $A \ominus B = (A \circ B) \ominus B$

dilating both sides:

$((A \circ B) \ominus B) \oplus B = (A \ominus B) \oplus B$

$\Rightarrow (A \circ B) \circ B = A \circ B$

$\blacksquare$

(ii) show $(A \bullet B) \bullet B = A \bullet B$

$(A \bullet B) \bullet B = ((A \bullet B)^c \circ B^c)^c$

$= ((A^c \circ B^c) \circ B^c)^c$

$= (A^c \circ B^c)^c$

$= A \bullet B$

thus $(A \bullet B) \bullet B = A \bullet B$ $\blacksquare$

Exercise 3 (Discrete Convolution of Finite Sequences).
In this exercise, we want to take a look at the convolution of finite sequences. For infinite sequences from the space

$$\ell^1(\mathbb{Z}) = \left\{ x = (x_n)_{n\in\mathbb{Z}} \,\middle|\, \|x\|_1 = \sum_{n\in\mathbb{Z}} |x_n| < \infty \right\},$$

the convolution $* : \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ is defined by the formula

$$(x * y)_n := \sum_{k\in\mathbb{Z}} x_k \cdot y_{n-k} \qquad \text{for } n \in \mathbb{Z}$$

for all $x, y \in \ell^1(\mathbb{Z})$. However, we will mostly deal with finite data in practical cases, so that we want to take a look at suitable convolution methods for finite sequences.

a) Given a finite sequence $x = (x_1, ...x_l)$ with $L \in \mathbb{N}$, we can define the *zero-padded* version $x^{(0)} \in \ell^1(\mathbb{Z})$ of x as

$$x_n^{(0)} = \begin{cases} x_n & \text{if } n \in \{1, ..., L\} \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{1, ..., L\}. \end{cases}$$

For finite sequences $x = (x_1, ...x_L)$ and $y = (y_1, ...y_M)$, we can perform a convolution by computing the convolution $x^{(0)} * y^{(0)}$. Show that we have

$$(x^{(0)} * y^{(0)})_n = 0 \qquad \text{for all } n \in \mathbb{Z} \setminus \{2, ..., L + M\}.$$

let $y_n^{(0)} = \begin{cases} y_n & n \in \{1, ..., M\} \\ 0 & n \in \mathbb{Z} \setminus \{1, ..., M\} \end{cases}$

$$(x^{(0)} * y^{(0)})_n = \sum_{k \in \mathbb{Z}} x_k^{(0)} \cdot y_{n-k}^{(0)}$$

$$= \sum_{k \in \{2,...,L+M\}} x_k^{(0)} \cdot y_{n-k}^{(0)} + \sum_{\substack{k \in \mathbb{Z} \setminus \\ \{2,...,L+M\}}} x_k^{(0)} \cdot y_{n-k}^{(0)}$$

but $x_k^{(0)} = 0$ for $k \in \{2, ..., L+M\}$

thus, $(x^{(0)} * y^{(0)})_n = 0$ for $n \in \mathbb{Z} \setminus \{2, ..., L+M\}$

$\blacksquare$

b) In the previous part, we saw that the support of the convolution $x^{(0)} * y^{(0)}$ is contained in the set $\{2, ..., L+M\} \subset \mathbb{Z}$. Since we want the support of the convolution to be in $\{1, ..., L+M-1\}$, we consider the **linear convolution** $\star$ which is defined as

$$(x \star y)_n := \sum_{k=1}^{L} x_k \cdot y^{(0)}_{n-k+1} \quad \text{for all } n \in \{1, ..., L+M-1\}.$$

Prove the following equation:

$$(x \star y)_n = (x^{(0)} * y^{(0)})_{n+1} \qquad \text{for all } n \in \{1, ..., L+M-1\}.$$

$$(x^{(0)} * y^{(0)})_{n+1} = \sum_{k \in \mathbb{Z}} x^{(0)}_k \cdot y^{(0)}_{n+1-k}$$

c) Now assume that we have two finite sequences $\tilde{x} = (\tilde{x}_1, ...\tilde{x}_N)$ and $y = (\tilde{y}_1, ...\tilde{y}_N)$ with the same length. The **circular convolution** $\tilde{x} \otimes \tilde{y}$ of $\tilde{x}$ and $\tilde{y}$ is then defined as

$$(\tilde{x} \otimes \tilde{y})_n = \sum_{k=1}^{N} \tilde{x}_k \cdot \tilde{y}^{(p)}_{n-k+1},$$

where $\tilde{y}^{(p)}$ is the periodic continuation of $\tilde{y}$, i.e.

$$\tilde{y}^{(p)}_n = \tilde{y}_{(n-1)\%N+1} \quad \text{with} \quad (n-1)\%N = (n-1) \bmod N \qquad \text{for all } n \in \mathbb{Z}.$$

For the sequences $x = (x_1, ...x_L)$ and $y = (y_1, ...y_M)$, where we might have $L \neq M$, we consider the zero-padded versions $\tilde{x}, \tilde{y}$ of length $L + M - 1$ given by

$$\tilde{x}_n = \begin{cases} x_n & \text{if } n \in \{1, ..., L\} \\ 0 & \text{if } n \in \{L+1, ...L+M-1\} \end{cases} \quad \text{and} \quad \tilde{y}_n = \begin{cases} y_n & \text{if } n \in \{1, ..., M\} \\ 0 & \text{if } n \in \{M+1, ...L+M-1\}. \end{cases}$$

Show that $x \star y = \tilde{x} \otimes \tilde{y}$ holds in this case, which means that we can compute the linear convolution of x and y by a circular convolution.

consider two general sequences, x_n and y_n.

let $x_n = \{x_1, x_2, x_3, x_4\}$ with $L = 4$

$y_n = \{y_1, y_2, y_3\}$ with $M = 3$

Computing $x * y$:

	x_1	x_2	x_3	x_4
y_1	x_1y_1	x_2y_1	x_3y_1	x_4y_1
y_2	x_1y_2	x_2y_2	x_3y_2	x_4y_2
y_3	x_1y_3	x_2y_3	x_3y_3	x_4y_3

$x * y = \{ x_1y_1,\ x_1y_2 + x_2y_1,\ x_1y_3 + x_2y_2 + x_3y_1,$

$x_2y_3 + x_3y_2 + x_4y_1,\ x_3y_3 + x_4y_2,\ x_4y_3 \}$

with length of output $L + M - 1 = 6$

Now, zero padding x_n and y_n as above gives:

$\tilde{x}_n = \{x_1, x_2, x_3, x_4, 0, 0\}$

$\tilde{y}_n = \{y_1, y_2, y_3, 0, 0, 0\}$

both with length $L - M - 1$

Computing $\tilde{x} \circledast \tilde{y}$:

$$\begin{bmatrix} x_1 & 0 & 0 & x_4 & x_3 & x_2 \\ x_2 & x_1 & 0 & 0 & x_4 & x_3 \\ x_3 & x_2 & x_1 & 0 & 0 & x_4 \\ x_4 & x_3 & x_2 & x_1 & 0 & 0 \\ 0 & x_4 & x_3 & x_2 & x_1 & 0 \\ 0 & 0 & x_4 & x_3 & x_2 & x_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} x_1 y_1 \\ x_2 y_1 + x_1 y_2 \\ x_3 y_1 + x_2 y_2 + x_1 y_3 \\ x_4 y_1 + x_3 y_2 + x_2 y_3 \\ x_4 y_2 + x_3 y_3 \\ x_4 y_3 \end{bmatrix}$$

Thus $x * y = \tilde{x} \circledast \tilde{y}$ for our example

but can be generalized for x_n and y_n of any

length L and M.

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