(a) Probability that randomly chosen pixel is in $C_i(s)$ equals to $P_1(i) = \frac{1}{|\Omega|} \sum_{j=0}^{\lfloor s \rfloor} H_u(j)$

$$P_2(i) = \frac{1}{|\Omega|} \sum_{j=\lfloor s \rfloor+1}^{255} H_u(j)$$

Proof:

$P_1(i) =$ (by definition of probability) $= \frac{|C_1(s)|}{|\Omega|} = \frac{1}{|\Omega|} \cdot |\{(i,j) \in \Omega \mid u(i,j) \leq s\}| =$

$= \frac{1}{|\Omega|} \cdot \sum_{k \in F} \left(H_u(k) \cdot \mathbb{1}_{k \leq s} \right) =$ (since $k \in \mathbb{N}_0$, $s \in \mathbb{R}$) $= \frac{1}{|\Omega|} \cdot \sum_{k \in [0,255]} \left(H_u(k) \cdot \mathbb{1}_{k \leq \lfloor s \rfloor} \right) =$

$= \frac{1}{|\Omega|} \cdot \sum_{k=0}^{\lfloor s \rfloor} H_u(k)$.

$P_2(i) =$ (by definition of probability) $= \frac{|C_2(s)|}{|\Omega|} = \frac{1}{|\Omega|} \cdot |\{(i,j) \in \Omega \mid u(i,j) \in (s, 255]\}| =$

$= \frac{1}{|\Omega|} \cdot \sum_{k \in F} \left(H_u(k) \cdot \mathbb{1}_{k > s} \right) =$ (since $k \in \mathbb{N}_0$, $s \in \mathbb{R}$) $= \frac{1}{|\Omega|} \cdot \sum_{k \in [0,255]} \left(H_u(k) \cdot \mathbb{1}_{k \geq \lfloor s \rfloor + 1} \right) =$

$= \frac{1}{|\Omega|} \cdot \sum_{k=\lfloor s \rfloor+1}^{255} H_u(k)$. $\blacksquare$

(b) Prove that $\forall s \in [0, 255]$: $P_1(s)\mu_1(s) + P_2(s)\mu_2(s) = \mu$,

$$\mu = \frac{1}{|\Omega|} \sum_{i=0}^{255} i \cdot H_u(i)$$

Proof:

I) if $C_1(s) = \emptyset$: then $C_2(s) = \Omega$ and $\mu_1(s)P_1(s) + \mu_2(s)P_2(s) =$

$$= 0 + \frac{1}{|\Omega|} \cancel{\sum_{j=\lfloor s \rfloor+1}^{255} H_u(j)} \cdot \cancel{\frac{1}{|C_2(s)|}} \cdot \sum_{j=\lfloor s \rfloor+1}^{255} j \cdot H_u(j) = \left(\text{in part a) was proven } |C_2(s)| = \sum_{i=\lfloor s \rfloor+1}^{255} H_u(i)\right) =$$

$$= \frac{1}{|\Omega|} \sum_{j=\lfloor s \rfloor+1}^{255} j \cdot H_u(j) = \left(\text{since } C_1(s) = \emptyset, \text{ then } H_u(j) = 0 \ \forall j \leq \lfloor s \rfloor\right) =$$

$$= \frac{1}{|\Omega|} \left(\sum_{j=\lfloor s \rfloor+1}^{255} j \cdot H_u(j) + \underbrace{\sum_{j=0}^{\lfloor s \rfloor} j \cdot H_u(j)}_{=0} \right) = \frac{1}{|\Omega|} \cdot \sum_{j=0}^{255} j \cdot H_u(j) = \mu.$$

II) <u>if $C_2(s) = \emptyset$</u>: then $C_1(s) = \Omega$ and $\mu_1(s)P_1(s) + \mu_2(s)P_2(s) =$

$$= 0 + \frac{1}{|\Omega|} \cancel{\sum_{j=0}^{\lfloor s \rfloor} H_u(j)} \cdot \cancel{\frac{1}{|C_1(s)|}} \cdot \sum_{j=0}^{\lfloor s \rfloor} j \cdot H_u(j) = \left(\text{in part a) was proven } |C_1(s)| = \sum_{i=0}^{\lfloor s \rfloor} H_u(i)\right) =$$

$$= \frac{1}{|\Omega|} \sum_{j=0}^{\lfloor s \rfloor} j \cdot H_u(j) = \left(\text{since } C_2(s) = \emptyset, \text{ then } H_u(j) = 0 \ \forall j > s\right) =$$

$$= \frac{1}{|\Omega|} \left(\sum_{j=0}^{\lfloor s \rfloor} j \cdot H_u(j) + \underbrace{\sum_{j=\lfloor s \rfloor+1}^{255} j \cdot H_u(j)}_{=0} \right) = \frac{1}{|\Omega|} \cdot \sum_{j=0}^{255} j \cdot H_u(j) = \mu.$$

III) <u>if $C_1(s) \neq \emptyset$ and $C_2(s) \neq \emptyset$:</u>

$$P_1(s)\mu_1(s) + P_2(s)\mu_2(s) = \frac{1}{|\Omega|} \cancel{\sum_{i=0}^{\lfloor s \rfloor} H_u(i)} \cdot \cancel{\frac{1}{|C_1(s)|}} \cdot \sum_{i=0}^{\lfloor s \rfloor} i \cdot H_u(i) +$$

$$+ \frac{1}{|\Omega|} \cancel{\sum_{i=\lfloor s \rfloor+1}^{255} H_u(i)} \cdot \cancel{\frac{1}{|C_2(s)|}} \cdot \sum_{i=\lfloor s \rfloor+1}^{255} i \cdot H_u(i) = \frac{1}{|\Omega|} \cdot \left[\sum_{i=0}^{\lfloor s \rfloor} i \cdot H_u(i) + \sum_{i=\lfloor s \rfloor+1}^{255} i \cdot H_u(i) \right] =$$

$$= \frac{1}{|\Omega|} \cdot \sum_{i=0}^{255} i \cdot H_u(i) = \mu.$$

Note: $C_1(s) = \emptyset \wedge C_2(s) = 0$ is impossible for any $\Omega \neq \emptyset$.

ⓒ $\forall s \in [0, 255]$ $\sigma^2(s) = \sum_{i=0}^{255} i^2 \cdot \tilde{H}_u(i) - (P_1(s) \cdot \mu_1(s)^2 + P_2(s) \cdot \mu_2(s)^2)$

Proof:

$$\sigma^2(s) = \sum_{i=0}^{\lfloor s \rfloor} (i - \mu_1(s))^2 \tilde{H}_u(i) + \sum_{i=\lfloor s \rfloor+1}^{255} (i - \mu_2(s))^2 \tilde{H}_u(i) = \sum_{i=0}^{\lfloor s \rfloor} i^2 \tilde{H}_u(i) +$$

$$+ \sum_{i=0}^{\lfloor s \rfloor} \mu_1(s)^2 \tilde{H}_u(i) - \sum_{i=0}^{\lfloor s \rfloor} 2i \mu_1(s) \tilde{H}_u(i) + \sum_{i=\lfloor s \rfloor+1}^{255} i^2 \tilde{H}_u(i) + \sum_{i=\lfloor s \rfloor+1}^{255} \mu_2(s)^2 \tilde{H}_u(i) -$$

$$- \sum_{i=\lfloor s \rfloor+1}^{255} 2i \mu_2(s) \tilde{H}_u(i) = \sum_{i=0}^{255} i^2 \tilde{H}_u(i) + \mu_1(s)^2 \underbrace{\sum_{i=0}^{\lfloor s \rfloor} \tilde{H}_u(i)}_{\frac{1}{|\Omega|} \cdot |C_1(s)|} + \mu_2(s)^2 \underbrace{\sum_{i=\lfloor s \rfloor+1}^{255} \tilde{H}_u(i)}_{\frac{1}{|\Omega|} \cdot |C_2(s)|} -$$

$$- 2\mu_1(s) \cdot \frac{1}{|\Omega|} \cdot \underbrace{\sum_{i=0}^{\lfloor s \rfloor} i \cdot H_u(i)}_{= |C_1(s)| \cdot \mu_1(s)} - 2\mu_2(s) \cdot \frac{1}{|\Omega|} \cdot \underbrace{\sum_{i=\lfloor s \rfloor+1}^{255} i \cdot H_u(i)}_{= |C_2(s)| \cdot \mu_2(s)} =$$

$$= \sum_{i=0}^{255} i^2 \tilde{H}_u(i) - \mu_1(s)^2 \cdot \underbrace{\frac{1}{|\Omega|} \cdot |C_1(s)|}_{= P_1(s)} - \mu_2(s)^2 \cdot \underbrace{\frac{1}{|\Omega|} \cdot |C_2(s)|}_{= P_2(s)} =$$

$$= \sum_{i=0}^{255} i^2 \tilde{H}_u(i) - \left(\mu_1(s)^2 \cdot P_1(s) + \mu_2(s)^2 \cdot P_2(s) \right).$$

Since $\sum_{i=0}^{255} i^2 \tilde{H}_u(i)$ doesn't depend on our only flexible variable s, minimizing $\sigma^2(s)$ is only possible by maximizing the subtrahend term $\mu_1(s)^2 P_1(s) + \mu_2(s)^2 \cdot P_2(s) = G(s)$.

– by Oleksandra Riabova –