Technische Universität Hamburg
Mathematical Image Processing (Winter Term 2022/2023)
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Exercise Sheet 13

Deadline: 02.02.23, 12:00pm

Remark: This is an additional exercise sheet, which means that the following 3 exercises do not add to the total amount of exercises relevant for the exam bonus. But you can earn additional homework points in case you still need homework points to receive the bonus. Note that the deadline for the submission of your solution is 02.02.23, 12:00pm.

Exercise 1 (RGB2Grayscale conversion).
Given an RGB image $u : \Omega \to \{0, ..., 255\}^3$, we want to compute a respective grayscale image. Therefore, we consider the following methods:

(i) **Mean value:**

$$u_{gray}(x, y) = 1/3 \cdot (u_{red}(x, y) + u_{green}(x, y) + u_{blue}(x, y)) \qquad \text{for all } (x, y) \in \Omega$$

(ii)) **Digicam:**

$$u_{gray}(x, y) = u_{green}(x, y) \qquad \text{for all } (x, y) \in \Omega.$$

(iii) **Luma:**

$$u_{gray}(x, y) = 0.2126 \cdot u_{red}(x, y) + 0.7152 \cdot u_{green}(x, y) + 0.0722 \cdot u_{blue}(x, y) \qquad \text{for all } (x, y) \in \Omega$$

Write a function

```
function [B] = RGB2Grayscale(A, method)
```

that converts a RGB image A (class 'uint8') to a grayscale image B (class 'uint8') with one of the above methods (formulas), which are denoted by the strings 'mean', 'digicam' and 'luma'.

Exercise 2 (Isodata Algorithm).
In this exercise, we want to implement the *Isodata method* for B/W conversion. So let

$$u : \{1, ..., M\} \times \{1, ..., N\} \to \{0, ..., 255\}$$

be a grayscale image and $H_u : \{0, ..., 255\} \to \mathbb{N}_0$ denote its histogram. For a given threshold $s \in [0, 255]$, we divide $\Omega = \{1, ..., M\} \times \{1, ..., N\}$ into the two classes

$$C_1(s) = \{(i, j) \in \Omega \mid u(i, j) \in [0, s]\} \quad \text{and} \quad C_2(s) = \{(i, j) \in \Omega \mid u(i, j) \in (s, 255]\}.$$

Note that $C_1(s)$ contains all pixels that will be colored black if we perform a B/W-conversion with the threshold s, whereas C_2 contains all white pixels. In this setup, we consider the functions

$$\mu_1(s) = \begin{cases} \frac{1}{|C_1(s)|} \sum\limits_{i=0}^{\lfloor s \rfloor} i \cdot H_u(i), & \text{if } C_1(s) \neq \emptyset \\ 0, & \text{if } C_1(s) = \emptyset \end{cases} \quad \text{and} \quad \mu_2(s) = \begin{cases} \frac{1}{|C_2(s)|} \sum\limits_{i=\lfloor s \rfloor + 1}^{255} i \cdot H_u(i), & \text{if } C_2(s) \neq \emptyset \\ 255, & \text{if } C_2(s) = \emptyset. \end{cases}$$

Now the Isodata method aims to find a threshold $s^* \in [0, 255]$ that satisfies $s^* = f(s^*)$, where the function f is given by

$$f(s) := \frac{\mu_1(s) + \mu_2(s)}{2} \qquad \text{for all } s \in [0, 255].$$

Notice that we have already proven the existence of such an element as well as the convergence of the simple fixed point iteration

$$s_{n+1} = f(s_n) \qquad \text{for all } n \in \mathbb{N} \tag{1}$$

and for an arbitrary initial value $s_0 \in [0, 255]$ in Exercise Sheet 3, Exercise 4. Write a function

```
function [B, s] = IsodataMethod(A, s0)
```

that converts a grayscale image A (class 'uint8') to a B/W image B (class 'logical') via the Isodata method. Use the fixed point iteration (1) with initial value $s0$ to find a fixed point s after a finite number of iterations. The function should also return the threshold in the end. For the computation of the histogram of A, you may use the built-in function from Octave/Matlab.

Exercise 3 (Convolution Theorem)**.**
Let $f, g \in L^1(\mathbb{R}^d)$. Prove the following statements:

a) We have the estimate

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1},$$

such that we can conclude $f * g \in L^1(\mathbb{R}^d)$.

b) For every $\omega \in \mathbb{R}^d$, we have the equality

$$\mathcal{F}(f * g)(\omega) = (2\pi)^{d/2} \cdot \mathcal{F}(f)(\omega) \cdot \mathcal{F}(g)(\omega),$$

where $\mathcal{F}$ denotes the Fourier Transform on $L^1(\mathbb{R}^d)$.