*Technische Universität Hamburg*
*Mathematical Image Processing (Winter Term 2022/2023)*
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# Exercise Sheet 6

Deadline: 28.11.22, 12:00pm

**Exercise 1** (Haar wavelet)**.**
Consider the scaling function

$$\varphi(x) := \chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the scaled / shifted versions

$$\varphi_k^{(j)} = 2^{j/2} \cdot \varphi(2^j \cdot x - k) \qquad \text{for all } j, k \in \mathbb{Z}.$$

a) Show that

$$\operatorname{supp}\left(\varphi_k^{(j)}\right) = \left[2^{-j} \cdot k, 2^{-j} \cdot (k+1)\right] \qquad \text{for all } j, k \in \mathbb{Z}.$$

Conclude that we have

$$\langle \varphi_k^{(j)}, \varphi_l^{(j)} \rangle_{L^2(\mathbb{R})} = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases}$$

for fixed $j \in \mathbb{Z}$.

We define the scale space of order $j \in \mathbb{Z}$ as

$$V_j = \overline{\operatorname{span}\,\{\varphi_k^{(j)} \mid k \in \mathbb{Z}\}}.$$

b) Prove the refinement equation

$$\varphi_k^{(j-1)} = 2^{-1/2} \cdot \left(\varphi_{2k}^{(j)} + \varphi_{2k+1}^{(j)}\right) \qquad \text{for all } j, k \in \mathbb{Z}.$$

and use this equation to verify the inclusion

$$V_{j-1} \subset V_j \qquad \text{for all } j \in \mathbb{Z}.$$

It can be shown that

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\} \qquad \text{and} \qquad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{1}$$

More precisely, the sequence $(V_j)_{j \in \mathbb{Z}}$ forms a **Multiresolution analysis** of $L^2(\mathbb{R})$. Given an $f \in L^2(\mathbb{R})$, let $\Pi_{V_j}(f)$ denote the best approximation to $f$ from $V_j$, i.e.

$$\|f - \Pi_{V_j}(f)\|_{L^2(\mathbb{R})} = \min_{v \in V_j} \|f - v\|_{L^2(\mathbb{R})} \qquad \text{for all } j \in \mathbb{Z}.$$

The second part of (1) then implies that

$$\lim_{j \to \infty} \Pi_{V_j}(f) = f,$$

which means that we can reconstruct any function $f \in L^2(\mathbb{R})$ arbitrarily well. For our signal analysis, we want to decompose the space $L^2(\mathbb{R})$ into an orthogonal sum

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j\ ,$$

where $W_j$ represents the part that corresponds to the resolution level $j \in \mathbb{Z}$. Therefore, we introduce the *Haar wavelet* on $\mathbb{R}$

$$\psi(x) := \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1/2 \\ -1, & \text{if } 1/2 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

and its shifted / scaled versions

$$\psi^{(j)} = 2^{j/2} \cdot \psi(2^j \cdot x - k) \qquad \text{for all } j, k \in \mathbb{Z}.$$

Analogously, we set

$$W_j = \overline{\text{span}\,\{\psi_k^{(j)} \mid k \in \mathbb{Z}\}} \qquad \text{for all } j \in \mathbb{Z}.$$

c) Prove the refinement equations

$$\psi(x) = \varphi(2x) - \varphi(2x-1) \quad \text{for all } x \in \mathbb{R}, \qquad \psi_k^{(j-1)} = 2^{-1/2} \cdot \left(\varphi_{2k}^{(j)} - \varphi_{2k+1}^{(j)}\right) \quad \text{for all } j, k \in \mathbb{Z}$$

and the orthogonality relation

$$\langle \varphi_k^{(j-1)}, \psi_l^{(j-1)} \rangle_{L^2(\mathbb{R})} = 0 \qquad \text{for all } j, k, l \in \mathbb{Z}.$$

Use these properties and part b) to prove the decomposition

$$V_j = V_{j-1} \oplus W_{j-1} \qquad \text{for all } j \in \mathbb{Z}.$$

In total, we get

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j,$$

such that $\{\psi_k^{(j)} \mid j, k \in \mathbb{Z}\}$ is a complete orthonormal system of $L^2(\mathbb{R})$.

**($\star$) Exercise 2** (Orthogonal Projections)**.**
Let $H$ be a real Hilbert space and $V \subset H$ be a closed subspace. We want to prove that the projection operator

$$\Pi_V(f) := \operatorname*{argmin}_{v \in V} \|f - v\|_H \qquad \text{for all } f \in H$$

is well-defined.

a) Show that there is an element $v^* \in V$ such that

$$\|f - v^*\|_H = \inf_{v \in V} \|f - v\|. \tag{2}$$

To this end, consider a sequence $(v_k)_{k \in \mathbb{N}}$ in $V$ that satisfies

$$\lim_{k \to \infty} \|f - v_k\|_H = \inf_{v \in V} \|f - v\|_H$$

and use the parallelogram identity

$$\|v + w\|_H^2 + \|v - w\|_H^2 = 2 \cdot \|v\|_H^2 + 2 \cdot \|w\|_H^2 \qquad \text{for all } v, w \in H$$

to show that this sequence is a Cauchy sequence.

b) Prove that $v^* \in V$ satisfies property (2) if and only if

$$\langle f - v^*, v \rangle_H = 0 \qquad \text{for all } v \in V. \tag{3}$$

c) Let $v_1^*, v_2^* \in V$ satisfy (3). Show that this implies $v_1^* = v_2^*$.

Hence, there is a unique best approximation $\Pi_V(f)$ to each $f \in H$ from $V$, the so-called *orthogonal projection*.

**Exercise 3** (Heat equation)**.**
We consider the *heat equation*

$$\partial_t u(x,t) - \Delta u(x,t) = 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0, \infty). \tag{4}$$

a) We define the *heat kernel* via

$$\Phi(x,t) = f_t(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,,$$

where $f_t$ is the scaled Gaussian function from exercise sheet 5 for every $t > 0$:

$$f_t(x) := (4\pi t)^{-d/2} \cdot e^{-\|x\|_2^2/4t} \quad \text{for } x \in \mathbb{R}^d$$

Show that $\Phi$ solves the heat equation (4).

Let $g \in L^\infty(\mathbb{R}^d)$ be a continuous function. If we add the initial condition

$$u(x,0) = g(x) \qquad \text{for all } x \in \mathbb{R}^d,$$

it can be shown that the (continuous) solution of the heat equation is then given by the convolution

$$u(x,t) = (f_t * g)\,(x) \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\,. \tag{5}$$

b) Show that the solution (5) is bounded on $\mathbb{R}^d \times [0,\infty)$ in this case. You can use that

$$\int\limits_{\mathbb{R}^d} \Phi(x,t)\;dx = 1 \qquad \text{for all } t > 0.$$

c) For every time $t > 0$, we define the respective slice of the solution as

$$u_t : \mathbb{R}^d \to \mathbb{R},\;\; x \mapsto u(x,t).$$

Prove the following equation:

$$u_{t+h} = f_h * u_t \qquad \text{for all } t,h > 0.$$

**Hint:** Take a look at Sheet 5 Exercise 1c).

d) Let $B_r(0)$ denote the $d$-dimensional open ball around zero, i.e.

$$B_r(0) = \{x \in \mathbb{R}^d \mid \|x\| < r\},$$

for any radius $r \in \mathbb{R}_+$. We assume that there is a radius $R > 0$ such that $g(x) > 0$ for $x \in B_R(0)$ and $g(x) = 0$ for $x \in \mathbb{R}^d \setminus B_R(0)$. Prove that we have

$$u(x,t) > 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,\infty)$$

in this case. This phenomenon is called *infinite propagation speed*, since the positiveness of $g$ on an arbitrarily small ball translates to the whole domain at all further times.

**Exercise 4** (Gaussian Filter)**.**
In this exercise, we want to implement a Gaussian filter for image smoothing. Consider the non-normalized Gaussian functions

$$\tilde{f}_t(x) := e^{-\|x\|_2^2/4t} \quad \text{for } x \in \mathbb{R}^d.$$

Given a parameter $t > 0$, we set the filter width to $W_t = \left\lceil 3 \cdot \sqrt{2t} \right\rceil \in \mathbb{N}$. Hence, the filter mask is given via the matrix

$$\tilde{F}_t = \left(\tilde{f}_t(i,j)\right)_{-W_t \le i,j \le W_t} \in \mathbb{R}^{2W_t+1 \times 2W_t+1}.$$

and the normalized filter mask is given via

$$F_t = M^{-1} \cdot \tilde{F}_t \qquad \text{where} \qquad M = \sum_{i=-W_t}^{W_t} \sum_{j=-W_t}^{W_t} \tilde{f}_t(i,j).$$

Write a function

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function [B, F] = GaussianFilter(A, t)
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that performs a convolution of the grayscale image $A$ with the normalized filter mask $F_t$ for a given parameter $t > 0$. Here, the image should be padded outside the image by reflecting the values inside the image. For the convolution, you can use the built-in function `imfilter` in Matlab / Octave. Moreover, the function should return the filter mask $F = F_t$.