Option Pricing with Infinite Activity exp-Lévy Models using Fourier-based Methods

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Abstract. This paper is dedicated to the analysis and application of Lévy processes, in particular the variance gamma process; an infinite activity finite variation process, which provides control over the skewness and kurtosis of the distribution. This is especially useful in financial modelling of risk-neutral asset returns, as you can accommodate for the discrepancy between the Black-Scholes model and option pricing data.

The treatment of Lévy processes is given from a foundational perspective, by providing analysis on Poisson-type processes from a measure-theoretic standpoint and studying the essence of the Fourier transform, with its implications in probability theory and option pricing. Subsequently, we are equipped with the mathematical tools required for the analysis in the sequel.

We then consider the financial and computational applications of Lévy processes, finding that the exp-Lévy model exhibits market incompleteness and that Fourier-based methods have their limitations, however provide results that converge very fast.

Keywords. Lévy processes; Fourier transform; exponential martingales; option pricing.

Contents

CHAPTER 1

Introduction

1. Historical Background

Ever since BACHELIER, a pioneer in mathematical finance, introduced a mathematical model for pricing options in 1900 [**3**, **18**], the modelling and analysis of stochastic processes, especially Browninan motion, has flourished in mathematical finance and many other fields.

In 1933, KOLMOGOROV released his revolutionary book, the *Grundbegriffe*, laying the axiomatic foundations of modern probability theory via the use of measure theory [**1**, **38**]. This was then furthered by mathematicians, such as LÉVY and KHINTCHINE, who made important connections between *infinitely divisible* distributions and Lévy processes. In the subsequent years, ITÔ formalised the intuitive ideas of LÉVY using the ideas of KOLMOGOROV; in the process he developed the foundations of stochastic calculus [**26**].

In 1973, BLACK and SCHOLES with the help of MERTON, made a great leap in options pricing when they introduced the Black-Scholes model [**8**]. They made use of geometric Brownian motion to describe the risk-neutral dynamics of asset prices, and in doing so, they found that the value of an option is given by the solving the *heat transfer equation* developed by FOURIER in 1822 [**24**].

Nearing the end of the second millennium, EBERLEIN et al. [**22**] and BAKSHI et al. [**4**] found that the Black-Scholes model was not sufficient in fitting the risk-neutral distribution of asset returns, which exhibited negative skew and excess kurtosis. It was during this time that MADAN et al. [**32**] proposed the use of the variance gamma model in accounting for skew and kurtosis in modelling asset prices, by doing so one could substantially correct the observed 'smile and skew' in the Black-Scholes implied volatility curve.

It was observed by MADAN et al. [**33**, **37**] that the variance gamma process had many desirable properties, namely *infinite activity* and no continuous martingale component, it was noted that this was 'a departure from existing option pricing literature, where the main mode of analysis is diffusion' [**32**]. In turn, this provided more evidence for the conjecture that market indices we devoid of a diffusion component and jumps in asset prices lead to non-normal returns [**13**].

Soon afterwards, Carr et al. [**11**] confirmed this conjecture in an incredible study on the structure of asset returns. They concluded that statistical and risk-neutral processes for asset prices are pure jump processes that possess infinite activity and finite variation. In order to test the conjecture, they generalised the variance gamma model to allow for infinite activity and a diffusion component, hence creating the *CGMY* model. In their paper, they used Fourier-based methods for option pricing introduced by Carr & Madan [**12**], namely the fast Fourier transform developed by Cooley & Tukey in 1965 [**17**].

2. Outline

There are three main aims of this paper: (1) to develop the foundations and theory underlying Lévy processes in a natural and intuitive manner; (2) to apply the exp-Lévy model in the financial modelling of options; and (3) to develop a Fourier-based method for option pricing and measure its performance.

I begin by introducing the Fourier transform in Chapter 2, a tool that is ubiquitous in mathematics, which will enable the analysis and modelling of Lévy processes in the sequel.

Then, in Chapter 3, I establish the foundation to the analysis of Lévy processes, by studying Poisson-type processes and developing a measure-theoretic approach to the analysis of jump processes, which will be required in the analysis of Lévy processes in Chapter 4.

In Chapter 4, I introduce Lévy processes and their fundamental connection to infinitely divisible distributions, which is used to develop two crucial theorems used in the analysis of Lévy processes: the *Lévy-Itô decomposition* and *Lévy-Khintchine representation*. We then explore the pathwise and distributional properties of Lévy processes, hence finding the requirements to transform them into martingales.

In Chapter 5, we will use the theory developed in prior chapters, in order to introduce the exp-Lévy model for asset pricing and establish the market incompleteness that Lévy processes exhibit. Then we will use the Fourier-based method, introduced in Chapter 2, to present a method [**12**] for pricing options using the exp-Lévy model. Then, we use the variance gamma process, introduced in Chapter 4, in the exp-Lévy model to assess the performance of Fourier-based methods and provide analysis on the effect on option prices by using this new model.

In Chapter 6, we conclude by noting on the development of the material throughout the paper, namely the advances in understanding Lévy processes from a theoretical, financial and computational perspective. The results of Chapter 5 are summarised and we consider further work regarding Lévy processes that this paper does not touch on.

CHAPTER 2

The Fourier Transform

"Fourier's analytical theory of heat ... is the ultimate source of much modern work in the theory of functions of a real variable and in the critical examination of the foundation of mathematics."

– Eric Temple Bell, *The Development of Mathematics*

Often we find that we do not have the distribution function of a random variable in closed form, however we are able to find the characteristic function explicitly [**40**, p. 4]. Thus, before we begin the study of Lévy processes in the sequel, we must start by introducing the concept of the Fourier transform and its probabilistic analogue: the characteristic function.

We begin by presenting the discrete Fourier transform in Section 2.1, as it will provide an intuitive example of how the Fourier transform works in practice and will be used for option valuation in Chapter 5. We establish the inverse Fourier transform in Section 2.2, commenting on the nature of the transform and the Parseval relation, which will become crucial in deriving the options pricing method in the sequel. Then we will introduce the characteristic function in Section 2.3, this will enable us to represent a random variable in an alternative way, thus providing one of the tools we need for the study of Poisson-type processes in Chapter 3 and Lévy processes in Chapter 4.

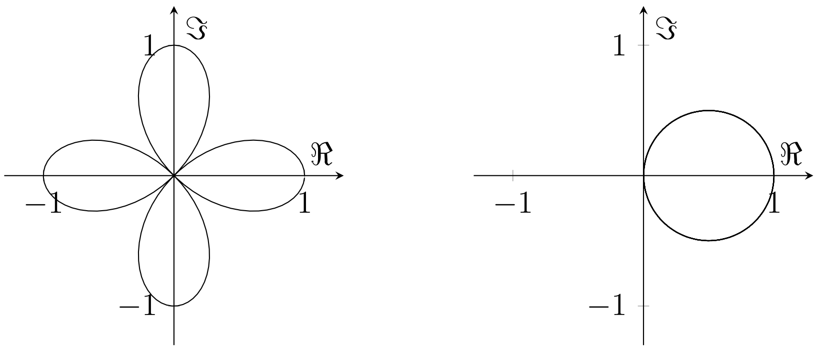
We begin by introducing the indispensable *Fourier transform* in the following definition.

Definition 2.1 (cf. [**41**, §7.2]). Let $f : \mathbb{R}^d \mapsto \mathbb{C}$ be a measurable function that satisfies

$$\int_{\mathbb{R}^d} |f(x)|dx < \infty,$$

or equivalently $f \in L^1(\mathbb{R}^d)$. Then, the **Fourier transform** $\mathcal{F}[f](z)$ of f is

$$\forall z \in \mathbb{R}^d : \mathcal{F}[f](z) = \int_{\mathbb{R}^d} e^{-iz.x} f(x)dx.$$



(A) $z = 1$ (B) $z = 2$

FIGURE 2.1. Integrand of $\mathcal{F}[f](z)$, with $f : [0, 2\pi] \mapsto [0, 1]$ and $f(x) = \cos(2x)$. $\Im$ and $\Re$ refer to the imaginary and real axis respectively.

1. Discrete Fourier Transform

The Fourier transform may seem abstract to the uninitiated, however, by using it in Example 2.1, one may be able to grasp how the transform works in practice.

EXAMPLE 2.1. Let $\mathcal{F}[f]$ be the Fourier transform of $f : [0, 2\pi] \mapsto [0, 1]$, with $f(x) = \cos(2x)$. Figure 2.1 shows the integrand of $\mathcal{F}[f](z)$ for $z \in \{1, 2\}$, which we then integrate over $[0, 2\pi]$. We can think of this (in the discrete sense) as taking samples of the integrand at equidistant angles on the complex plane, then the sum of these points gives an approximation of $\mathcal{F}[f]$; this is analogous to taking the centre of mass of these points. By dividing $[0, 2\pi]$ such that $\Theta = \{\theta = n\Delta : \forall n \in \{0, \ldots, N-1\}\}$ with $\Delta = \frac{2\pi}{N}$, we have

$$\mathcal{F}[f](z) = \int_{\mathbb{R}} e^{-izx} f(x)dx \approx \sum_{\theta \in \Theta} e^{-iz\theta} f(\theta) = \sum_{n=0}^{N-1} e^{-izn\Delta} f(n\Delta). \tag{2.1}$$

As $N \to \infty$, the approximation converges to $\mathcal{F}[f](z)$. Hence, by Figure 2.1, it is clear that $|\mathcal{F}[f](1)| = 0$ and $|\mathcal{F}[f](2)| > 0$. In fact, for all $z \in [0, 2\pi] \setminus \{-2, 2\}$ (as cosine is an even function), the real part $\Re$ of the Fourier transform $|\Re\{\mathcal{F}[f](z)\}| \ll \Re\{\mathcal{F}[f](2)\}$, as shown by Figure 2.2. This is what we expect as $\Re[e^{-izx} f(x)] = \cos zx \cdot \cos 2x$, thus $\Re[e^{-i2x} f(x)] = \cos^2(2x) \geq 0$ and $\int_0^{2\pi} cos^2(2x)dx = \pi$.

The discretisation in (2.1) is the **discrete Fourier transform** (DFT), which has two types of error associated with it: ε_{trunc} and ε_{discr}; the truncation and discretisation error respectively. In Example 2.1, there is no

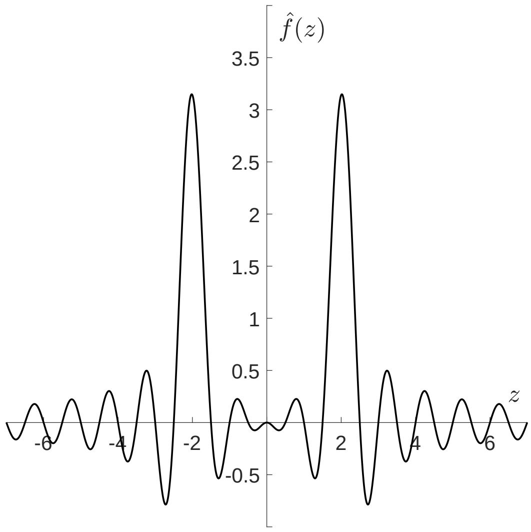


FIGURE 2.2. The Fourier transform $\mathcal{F}[f](z)$ of $f : [0, 2\pi] \mapsto [0, 1]$, with $f(x) = \cos(2x)$. Note the peaks at 2 and -2, indicating that f oscillates at 2 cycles/2π.

truncation error as the function f has a finite domain, thus there is only discretisation error. Then, in order to keep ε_{discr} small, one must use a large value for N. The computational complexity of the DFT algorithm is $\mathcal{O}(N^2)$, which is reduced to $\mathcal{O}(N \log N)$ when using the fast Fourier Transform (FFT) by Cooley & Tukey [**17**]; this is a tremendous increase in efficiency and makes FFT viable in option pricing.

In Chapter 5, we will utilise the discrete Fourier transform in the option pricing method, however this is by no means the limits of the DFT, which is used widely in many fields, namely spectral analysis [**34**] and polynomial multiplication [**10**].

2. Inverse Fourier Transform

Now, we introduce the inverse Fourier transform in Definition 2.2; it will become vital for option pricing in Chapter 5.

Definition 2.2. Let $f : \mathbb{R} \mapsto \mathbb{C}$ with $f \in L^1(\mathbb{R})$, the **Inverse Fourier transform** $\mathcal{F}^{-1}[f](x)$ is given by

$$f(x) = \mathcal{F}^{-1}\mathcal{F}[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iz.x}\mathcal{F}[f](z)dz, \tag{2.2}$$

by the Fourier inversion theorem [**41**, Theorem 7.5].

The representation (2.2) explicitly shows how one can take the Fourier transform $\mathcal{F}[f](z)$ of a function f, thus mapping it to the *frequency domain*, and then take the inverse transform $\mathcal{F}^{-1}\mathcal{F}[f](x)$ resulting in f again. In fact, this holds in the opposite direction, i.e. take the inverse transform and then the forwards transform, thus illustrating the cyclic nature of the Fourier transform and the fact that the inverse transform is *equivalent* to forwards transform (with a sign change and $\frac{1}{2\pi}$ normalisation factor).

In Chapter 5, we will find that the risk-neutral density ϱ is not analytically tractable, however the characteristic function ϕ is is known in closed form. I have not yet introduced the characteristic function (see Section 2.3), hence (for the sake of simplicity) assume it is the Fourier transform for now. I introduce the *Parseval relation* in Theorem 2.1, which makes an interesting connection between product of functions f and g and their Fourier transform counterpart, it will be used in Chapter 5 in deriving the options pricing method.

Theorem 2.1 (cf. [**41**, §5.4]). Let $\mathcal{F}[f](z)$ and $\mathcal{F}[g](z)$ be the Fourier transform of $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ respectively, then the **Parseval relation** takes the form

$$\int_{\mathbb{R}} f(x)g(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}[f](z) \cdot \mathcal{F}[g](z)dz.$$

3. Characteristic Function

Now we extend Definition 2.1 such that we can apply it to a real-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; it is called the *characteristic function* of X.

Definition 2.3 (cf. [**31**]). The **characteristic function** $\Phi_X : \mathbb{R}^d \to \mathbb{C}$ of a $\mathbb{R}^d$-valued random variable X in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined by

$$\forall z \in \mathbb{R}^d : \Phi_X(z) \equiv E(e^{iz.X}) = \int_{\mathbb{R}^d} e^{iz.x}d\mu_X(x),$$

where μ_X is the probability measure on $\mathbb{R}^d$ given by $\mu_X(A) = \mathbb{P}(X \in A)$ for all $A \in \Omega$.

Note the sign change in the exponential term of the integrand in Definition 2.3. This is equivalent to an inverse Fourier transform $2\pi \cdot \mathcal{F}^{-1}[f](x)$, but now defined for a random variable X in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The result is due to the fact that the inverse of a Fourier transform is a Fourier transform with a sign change, by the Fourier inversion theorem (see Definition 2.2).

The probability density function $f_X(x)$ of X is equivalent to the Radon-Nikodym derivative $\frac{d\mu_X}{d\lambda}$, with the Lebesgue measure λ (see [**7**, Theorem A.52]), and can be used to compute $\Phi_X(z)$ in Definition 2.3. Example 2.2 shows how we can use the analytical properties of the characteristic function to study random variables, in order to do this we must first prove Proposition 2.1 and Proposition 2.2.

PROPOSITION 2.1. For independent random variables X and Y, the characteristic function of a linear combination $aX + bY$ is given by

$$\forall a, b, z \in \mathbb{R}^d : \Phi_{aX+bY}(z) = \Phi_X(az)\Phi_Y(bz).$$

For the general case where we have a linear combination of n independent random variables $S_n = a_1X_1 + \cdots + a_nX_n$, the characteristic function is $\Phi_{S_n}(z) = \Phi_{X_1}(a_1z)\cdots\Phi_{X_n}(a_nz)$. (see [**15**, p. 30]).

PROOF. By Definition 2.3, we have

$$\begin{aligned}\Phi_{aX+bY}(z) &= E(e^{iz.(aX+bY)})\\ &= E(e^{iaz.X}e^{ibz.Y})\\ &= E(e^{iaz.X})E(e^{ibz.Y})\\ &= \Phi_X(az)\Phi_Y(bz).\end{aligned}$$

□

PROPOSITION 2.2. The characteristic function of a random variable $X \sim \text{Pois}(\lambda)$, where $\text{Pois}(\lambda)$ denotes a Poisson distribution with intensity λ, is given by

$$\forall z \in \mathbb{R} : E(e^{izX}) = \exp[\lambda(e^{iz} - 1)].$$

PROOF. By Definition 2.3, we have

$$\begin{aligned}E(e^{izX}) &= \int_{\mathbb{R}} e^{izx} d\mu_X(x)\\ &= \int_{\mathbb{R}} e^{izx}\frac{\lambda^x e^{-\lambda}}{x!}dx\\ &= e^{-\lambda}\int_{\mathbb{R}} \frac{(\lambda e^{iz})^x}{x!}dx\end{aligned}$$

$$= e^{-\lambda}e^{\lambda e^{iz}} = \exp[\lambda(e^{iz} - 1)].$$

□

Example 2.2. Suppose we have a linear combination of independent random variables $X + Y$, with $X \sim \mathrm{Pois}(\lambda_X)$ and $Y \sim \mathrm{Pois}(\lambda_Y)$. From Proposition 2.2, we know that the characteristic function of a Poisson random variable with intensity λ is $\exp[\lambda(e^{iz} - 1)]$, and by using Proposition 2.1 we know the characteristic function of $X + Y$, which is given by

$$\forall z \in \mathbb{R} : \Phi_{X+Y}(z) = \Phi_X(z)\Phi_Y(z) = e^{\lambda_X(e^{iz}-1)}e^{\lambda_Y(e^{iz}-1)} = e^{(\lambda_X+\lambda_Y)(e^{iz}-1)}.$$

$\Phi_{X+Y}(z)$ is equivalent to the characteristic function of a Poisson random variable with intensity $\lambda_X + \lambda_Y$, therefore we conclude that $X + Y \sim \mathrm{Pois}(\lambda_X + \lambda_Y)$ because two random variables with the same characteristic function are identically distributed (see [**15**, p. 30]).

CHAPTER 3

Poisson-type Processes

Poisson-type processes naturally provide the framework for describing the jumps within a stochastic process, making it indispensible in the analysis of Lévy processes in Chapter 4.

In Section 3.1 we introduce the Poisson process, an increasing process that is foundational to the rest of the chapter. Then we provide a measure-theoretic interpretation of Poisson processes, in Section 3.2 on Poisson random measures, which we will utilise in Section 3.3 and 3.4 to construct the compensated Poisson process and jump process respectively.

1. Poisson Process

We begin by introducing Poisson processes and the concept of a random measure, as it will provide the tools for us to build simple Lévy processes; allowing us to study the properties and structure of Lévy processes in the sequel.

DEFINITION 3.1 (cf. [**27**, Ch. VI, §1a]). A function $f : [0,T] \to \mathbb{R}^d$ is said to be **càdlàg** if it is right-continuous with left limits for all $t \in [0,T]$. One can also say $f \in \mathbb{D}([0,T],\mathbb{R}^d)$, the collection of all càdlàg functions with the mapping $[0,T] \mapsto \mathbb{R}^d$ known as the *Skorokhod space.*

PROPERTY 3.1. A càdlàg function has a countable number of discontinuities, and $\forall\varepsilon > 0$ there is a finite number of discontinuities greater than ε. $\Delta f(t) = f(t) - f(t-)$ is referred to as the 'jump' of f at t, hence $\Delta f(t) > 0$ if and only if there is a discontinuity point at t (see [**15**, p. 38]).

EXAMPLE 3.1. Figure 3.1 illustrates a simple example of a càdlàg function that has 5 points of discontinuity, and for any $\varepsilon > 0$ there is a finite number of discontinuities in $f(t)$, i.e. $\forall t \in [0,T]$: $|\{t : \Delta f(t) > \varepsilon\}| < \infty$.

DEFINITION 3.2. Given a sequence of i.i.d. exponential random variables $(\tau_i)_{i\geq 1}$ that have parameter λ, a **Poisson process** $(N_t)_{t\geq 0}$ with intensity λ is given by

$$N_t = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n},$$

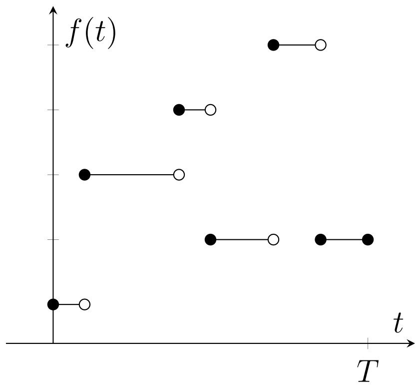


FIGURE 3.1. Example of a càdlàg function $f : [0, T] \to \mathbb{R}$.

and $T_n = \sum_{i=1}^{n} \tau_i$.

A Poisson process is counting process; a stochastic process that is integer, non-negative and monotonically increasing. The process counts the number of random times $T_n \in [0, t]$, with each increment $T_n - T_{n-1}$, for all $n > 1$, being independent and exponentially distributed with parameter λ.

PROPERTY 3.2. The Poisson process (N_t) has independent and homogeneous increments: for a sequence $(t_i)_{i\geq 1}$, $N_{t_i} - N_{t_{i-1}}$ are independent random variables with the same distribution as $N_{t_i - t_{i-1}}$. See [**15**, Propostion 2.12] for a proof.

Property 3.2 is a consequence of **memorylessness** in the exponential random variables that build the Poisson process (see [**39**, §11.2.1] and [**39**, p. 465]).

PROPOSITION 3.1. The characteristic function of N_t is

$$\forall z \in \mathbb{R} : E(e^{izN_t}) = \exp[\lambda t(e^{iz} - 1)].$$

PROOF. By Definition 2.3, we have

$$\begin{aligned} E(e^{izN_t}) &= \int_{\mathbb{N}} e^{izx} d\mu_{N_t}(x) \\ &= \sum_{x=0}^{\infty} e^{izx} e^{-\lambda t} \frac{(\lambda t)^x}{x!} \\ &= e^{-\lambda t} \sum_{x=0}^{\infty} \frac{(\lambda t e^{iz})^x}{x!} \\ &= e^{-\lambda t} e^{\lambda t e^{iz}} = \exp[\lambda t(e^{iz} - 1)]. \end{aligned}$$

□

EXAMPLE 3.2. Let there be an infinite ordered queue of cars waiting to pass a toll gate and suppose the time it takes each car to pass is an exponentially distributed independent random variable with parameter λ. Then the number of cars that pass the toll gate within the time period $[0, t]$ is given by a Poisson process N_t with intensity λ, and the waiting time for each car is given by T_n from Definition 3.2. Naturally we would also like to know the number of cars that pass in the time period $[s, t]$, $\forall s, t \geq 0$ and $s \neq t$, which is $N_t - N_s = N_{t-s}$ by Property 3.2.

2. Poisson Random Measure

Now we introduce counting measures in Definition 3.3, this is used in Example 3.3 and will help us build an understanding that allows us to define random measures in the sequel.

DEFINITION 3.3 (cf. [**7**, Example A.19]). A **counting measure** ν on the measurable space $(X, \mathcal{B}(X))$ is defined by

$$\forall A \in \mathcal{B}(X) : \nu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise.} \end{cases}$$

EXAMPLE 3.3. Let X be the set of natural numbers $\mathbb{N}$, then

$$\nu(A) = \text{ the number of points in } A$$

by Definition 3.3. Put differently, the measure ν gives every natural number in the set A unitary mass; as shown by Figure 3.2. Then the Lebesgue integral in the measure space $(X, \mathcal{B}, \nu)$ is

$$\int_A d\nu = \int \mathbf{1}_A d\nu = \nu(A).$$

Suppose we want to integrate the L^1-measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with respect to the counting measure ν, then we define a ν-measurable function $f_n : \mathbb{N} \to \mathbb{R}$ where

$$\forall x \in \mathbb{N} : f_n(x) = \begin{cases} f_n(x) & \text{if } 0 \leq x \leq n \\ 0 & \text{otherwise,} \end{cases}$$

with $\sum_{n=0}^{\infty} f(n) < \infty$. Then by the monotone convergence theorem [**42**, §5.1, Theorem 2], we have

$$\int f d\nu = \lim_{n\to\infty} \int f_n d\nu = \sum_{n=0}^{\infty} f(n),$$

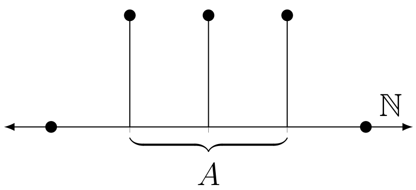


FIGURE 3.2. Counting measure ν assigning the value 1 to each element in A and 0 otherwise.

as f_n is non-decreasing and bounded below by 0. Hence we can treat the pointwise summation of a function f as an integral with respect to the counting measure ν.

In Example 3.3, we have f as a deterministic function, now given a finite natural number k, let us suppose $f : [0,T] \to \{0,1\}$ is given by

$$\forall t \in [0,T] : f(t) = \begin{cases} 1 & \text{for } t \in \{X_i : X_i \overset{\text{i.i.d.}}{\sim} \mathcal{U}_{[0,T]}, 1 \le i \le k\} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{U}_S$ denotes the uniform distribution on the set S. Then f is not deterministic and we have an alternative way of creating a Poisson process $(N_t)_{0 \le t \le T}$ with intensity $\frac{k}{T}$, given by

$$N_t = \int_0^t f d\mu,$$

where μ is the Lebesgue measure, as we are now dealing with uncountable sets (see [**42**, §2.6]). We can formalise this notion by using the concept of random measure in Definition 3.4 to construct the Poisson random measure in Definition 3.5.

DEFINITION 3.4 (cf. [**28**, p. 1]). For a measurable space $(X, \mathcal{B})$ and probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have a **random measure** $M : \Omega \times \mathcal{B} \to \mathbb{R}$. For a fixed $\omega \in \Omega$ we have a measure $M(\omega, \cdot)$ on the space $(X, \mathcal{B})$, and for a fixed $A \in \mathcal{B}$, we have a real-valued random variable $M(\cdot, A)$ that is $\mathcal{F}$-measurable in ω. For all bounded measurable sets A and (almost all) ω, the measure $M(\omega, A)$ must be finite, i.e. $\forall A \in \mathcal{B}, \omega \in \Omega$: $M(\omega, A) < \infty \quad \mathbb{P}$-a.e. For convenience, we may use the notation $M(A)$ in the sequel, which is equivalent to $M(\cdot, A)$.

PROPERTY 3.3 (cf. [**28**, Lemma 1.15]). For every non-negative $\mathcal{B}$-measurable function f on X, Definition 3.4 ensures that $M(\cdot, f) = \int f dM$ is a random variable.

We can immediately retrieve the definition of a Poisson random measure by imposing some restrictions on Definition 3.4. This will then be used in constructing the compensated Poisson random measure and studying Lévy processes in the sequel.

DEFINITION 3.5 (cf. [**29**, Definition 2.3]). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The random measure $M : \Omega \times \mathcal{B} \to \mathbb{N}_0 \cup \{\infty\}$ is called a **Poisson random measure** on X with intensity measure μ, if it satisfies the following:

(1) $\forall A \in \mathcal{B}$: $M(A)$ is a Poisson random variable with parameter $\mu(A)$. Note if $\mu(A) = 0 \Leftrightarrow M(A) = 0$ $\mathbb{P}$-a.s. and $\mu(A) = \infty \Leftrightarrow M(A) = \infty$ $\mathbb{P}$-a.s.
(2) for $1 \leq i \leq n$, the random variables $M(A_i)$ are independent if the sets $(A_i) \in \mathcal{B}$ are disjoint.
(3) $M(\cdot)$ is a measure on $(X, \mathcal{B})$ $\mathbb{P}$-a.s., i.e. $\mathbb{P}(\{\omega \in \Omega : M(\omega, \cdot) \in \mathcal{B}\}) = 1$.

3. Compensated Poisson Process

Now we must provide some motivation for introducing the compensated Poisson process. In order to do this we must first introduce the idea of martingales.

DEFINITION 3.6 (cf. [**2**, §2.1.2]). For $t \in [0, T]$, a càdlàg and $\mathcal{F}_t$-adapted process (X_t) is a **martingale** if $\forall t \in [0, T] : E(|X_t|) < \infty$ and

$$\forall s \leq t : E(X_t|\mathcal{F}_s) = X_s.$$

Otherwise if $\forall s \leq t$:

- $E(X_t|\mathcal{F}_s) \geq X_s$ then (X_t) is a **submartingale**.
- $E(X_t|\mathcal{F}_s) \leq X_s$ then (X_t) is a **supermartingale**.

PROPOSITION 3.2. An $\mathcal{F}_t$-adapted Poisson process (N_t) with intensity λ is a submartingale.

PROOF. We have $\forall s \leq t : E(N_t|\mathcal{F}_s) = E(N_t - N_s + N_s|\mathcal{F}_s)$. By Property 3.2, $N_t - N_s = N_{t-s}$ has a Poisson distribution with intensity $\lambda(t - s)$ and is independent of N_s, then $E(N_t - N_s + N_s|\mathcal{F}_s) = E(N_{t-s} + N_s|\mathcal{F}_s) = \lambda(t - s) + N_s \geq N_s$ □

In finance we desire processes that satisfy the *first fundamental theorem of asset pricing*, which states that a market on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is arbitrage-free if there exists at least one equivalent martingale measure Q to $\mathbb{P}$ (see [**7, 21**]).

DEFINITION 3.7. Let there be a measure Q on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, then Q is an **equivalent martingale measure** (EMM) if

- Q is equivalent to $\mathbb{P}$, i.e. $Q \sim \mathbb{P}$, and
- the *discounted* process $\widetilde{S} = (e^{-rt} S_t)_{t \geq 0}$ is a martingale with respect to Q.

This provides motivation for us to introduce the compensated Poisson process in Definition 3.8, as the Poisson process is not a martingale by Proposition 3.2.

DEFINITION 3.8 (cf. [**15**, §2.5.4]). A **compensated Poisson process** is defined by

$$\widetilde{N}_t = N_t - \lambda t,$$

where N_t is a Poisson process with intensity λ from Definition 3.2.

The compensated Poisson process is a martingale, this is shown by Proposition 3.3.

PROPOSITION 3.3. An $\mathcal{F}_t$-adapted compensated Poisson process $(\widetilde{N}_t)$ with intensity λ is a martingale.

PROOF. By Proposition 3.2, we have $\forall s \leq t : E(N_t|\mathcal{F}_s) = \lambda(t-s) + N_s$, therefore $E(\widetilde{N}_t|\mathcal{F}_s) = E(N_t - \lambda t|\mathcal{F}_s) = E(N_t|\mathcal{F}_s) - \lambda t = N_s - \lambda s = \widetilde{N}_s$. $\square$

PROPOSITION 3.4. The characteristic function of $\widetilde{N}_t$ is

$$\forall z \in \mathbb{R} : E(e^{iz\widetilde{N}_t}) = \exp[\lambda t(e^{iz} - 1 - iz)].$$

PROOF. By Definition 2.3, we have

$$\begin{aligned}
E(e^{iz\widetilde{N}_t}) &= E(e^{iz(N_t - \lambda t)}) \\
&= E(e^{izN_t} e^{iz\lambda t}) \\
&= e^{iz\lambda t} E(e^{izN_t}) \\
&= e^{iz\lambda t} e^{\lambda t(e^{iz}-1)} && \text{(by Proposition 3.1)} \\
&= \exp[\lambda t(e^{iz} - 1 - iz)].
\end{aligned}$$

$\square$

We now construct a compensated Poisson random measure in Definition 3.9; proving that it is indeed a random measure by showing that it satisfies Definition 3.5 of a Poisson random measure for the case of a compensated Poisson process.

Definition 3.9 (cf. [**15**, §2.6.2]). The **compensated Poisson random measure** $\widetilde{M}$ is defined by

$$\widetilde{M}(A) = M(A) - \mu(A),$$

with M as the Poisson random measure and μ its intensity measure from Definition 3.5. For disjoint sets $(A_i)_{1\leq i\leq n} \in \mathcal{B}$, we have $\widetilde{M}(A_1), \ldots, \widetilde{M}(A_n)$ as independent random variables, as $M(A_i)$ and $\mu(A_i)$ are independent for all i. Furthermore, we have $E(\widetilde{M}(A_i)) = E(M(A_i) - \mu(A_i)) = 0$, as $M(A_i)$ is a Poisson random variable with parameter $\mu(A_i)$.

4. Jump Processes

Suppose we have a function f that takes two arguments: time and a real-valued random variable. Given that f satisfies $E[M(|f|)] < \infty$, intuitively we have a function that gives the amplitudes of jumps. By extending on the notion of integration that we introduced in Example 3.3, integrating f with respect to the Poisson random measure yields a 'jump process' in Example 3.4; and integrating with respect to the compensated Poisson process yields a martingale.

Example 3.4 (cf. [**15**, §2.6.3]). For an $\mathcal{F}_t$-adapted Poisson random measure M on $E = [0,T] \times \mathbb{R}^d \setminus \{0\}$, let there be a measurable function $f : E \mapsto \mathbb{R}$. Let $\mu(|f|) = E[M(|f|)] < \infty$, then f is decomposable into its positive and negative part such that $f = f_+ - f_-$. Subsequently $\mu(f_+)$ and $\mu(f_-)$ are almost surely finite, thus satisfying Definition 3.5. Hence, Property 3.3 applies, which allows us to define $M(f)$ as a random variable with expectation

$$E[M(f)] = \mu(f) = \int_{[0,T]} \int_{\mathbb{R}^d} f(s,y)\mu(ds \times dy).$$

To define an $\mathcal{F}_t$-adapted stochastic process X_t, we must integrate f with respect to M up to time $t \in [0,T]$. Hence, by restricting the integral to $[0,t] \times \mathbb{R}^d \setminus \{0\}$, we have

$$X_t = M([0,t], f) = \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} f(s,y)M(ds\, dy), \tag{3.3}$$

which is referred to as a **jump process**. We can also integrate with respect to the compensated Poisson measure $\widetilde{M} = M - \mu$, and by [**15**, Proposition 2.16] we have the martingale

$$\widetilde{X}_t = \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} f(s,y)\widetilde{M}(ds\, dy) = X_t - \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} f(s,y)\mu(ds\, dy).$$

By constructing a jump process from a Poisson random measure in Example 3.4, we have indirectly introduced compound Poisson processes and jump measures, given by Definition 3.10 and Definition 3.11.

DEFINITION 3.10 (see [**15**, Definition 3.3]). A **compound Poisson process** $(X_t)_{0\leq t\leq T}$ with intensity $\lambda > 0$ and jump size distribution f is a stochastic process defined by

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where $(Y_i)_{i\geq 1}$ are i.i.d. random variables with law f and (N_t) is a Poisson process with intensity λ.

DEFINITION 3.11 (cf. [**15**, §3.3]). For a càdlàg process $(X_t)_{t\geq 0}$ one can describe the jumps of X by a **jump measure** J_X, a random measure on $\mathbb{R}^d \times [0,\infty)$ defined by

$$\forall A \in \mathcal{B}(\mathbb{R}^d) : J_X(A \times [t_1, t_2]) = |\{t \in [t_1, t_2] : \Delta X_t \neq 0,\, \Delta X_t \in A\}|.$$

Simply put, $J_X(A\times[t_1, t_2])$ counts the number of jumps of X in the interval $[t_1, t_2]$ for the jump sizes that are in A.

Notice that the process (X_t), defined by (3.3), is equivalent to an $\mathcal{F}_t$-adapted compound Poisson process. The jump measure of a compound Poisson process provides the information regarding the time and size of each jump, unlike the Poisson random measure in Example 3.4, which does not provide information regarding the size of jumps, but does contain information on the continuous component of X_t. Thus, we can represent a compound Poisson process as an integral with respect to the jump measure:

$$X_t = \int_0^t \int_{\mathbb{R}^d} x J_X(ds\, dx). \tag{3.4}$$

Now, by deriving the characteristic function of a compound Poisson process, we will obtain an essential representation of jump measures and an important component that will be used in the Lévy-Khintchine representation Theorem 4.2.

PROPOSITION 3.5. The characteristic function of a compound Poisson process X_t with jump size distribution f is

$$E(e^{iz.X_t}) = \exp\left[\int_{[0,t]} \int_{\mathbb{R}^d} (e^{izx} - 1)\lambda f(dx) ds\right].$$

PROOF. By the i.i.d. property of Y_i and the tower property, we have

$$E(e^{iz.X_t}) = E\left\{E\left[e^{iz.(Y_1+\cdots+Y_n)}\right]|N_t = n\right\} = E\left\{[\Phi_Y(z)]^n\,|N_t = n\right\}.$$

Hence, we have

$$\begin{aligned}
E(e^{iz.X_t}) &= E\left\{E\left[\Phi_Y(z)\right]^{N_t}\right\} \\
&= \int_{\mathbb{N}} E\left[\Phi_Y(z)\right]^x d\mu_{N_t}(x) && \text{(Definition 2.3)} \\
&= \exp\left[\lambda t(\Phi_Y(z)-1)\right] && \text{(Proposition 3.1)} \\
&= \exp\left[\lambda t \int_{\mathbb{R}^d}(e^{izx}-1)f(dx)\right] \\
&= \exp\left[\int_{[0,t]}\int_{\mathbb{R}^d}(e^{izx}-1)\lambda f(dx)ds\right].
\end{aligned}$$

□

By comparing the characteristic function of a compound Poisson process with the jump process construction in Example 3.4, we deduce that X_t has jump measure J_X that is a Poisson random measure with intensity measure $\mu(dx \times dt) = \lambda f(dx)dt$. In fact, we have also found the Lévy measure of a compound Poisson process, $\nu(dx) = \lambda f(dx)$, which can be interpreted as the expected number of jumps per unit time such that the jump size belongs to dx; this is found in Definition 4.3 in the sequel.

CHAPTER 4

Lévy Processes

Lévy processes, an important subclass of semimartingales, were first studied by French mathematician Lévy in the 1930s and 1940s, during the 'heroic age' of probability, in which mathematicians Khintchine and Itô developed the field further [**1**].

In Section 4.1, we will introduce Lévy processes and draw a connection to infinitely divisible distributions, which we will use in Section 4.2 to present the Lévy-Itô decompostion and Lévy-Khintchine representation theorems. We then use these theorems to study pathwise properties of Lévy processes (namely the finite variation type) in Section 4.3 and distributional properties in Section 4.4. Hence, we are able to find the conditions required for Lévy processes to be martingales. Lastly, in Section 4.5, a special class of Lévy processes, called *subordinators*, will be introduced. These are used in techniques, such as Brownian subordination, to produce infinite activity processes, namely the variance gamma process, which is used in the sequel (see Chapter 5).

1. Lévy Processes and Infinite Divisibility

Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We begin by defining stochastic continuity as it requisite to Definition 4.2 of Lévy processes; an important class of stochastic processes that will enable us to model market fluctuations and price options in the sequel.

DEFINITION 4.1 ([**36**, Definition 1.5]). X is **stochastically continuous** or **continuous in probability** if

$$\forall t \geq 0, \varepsilon > 0 : \quad \lim_{s\to t} \mathbb{P}(|X_s - X_t| \geq \varepsilon) = 0.$$

In other words, if a process has stochastic continuity, then as $h \to 0$ the probability of observing a jump in $[t, t+h]$ tends to 0. This is equivalent to saying that jumps only occur at random times. Hence, one can interpret stochastic continuity as a condition that excludes deterministic jumps, i.e. calendar effects.

DEFINITION 4.2 (cf. [**36**, Definition 1.6] and [**29**, Definition 1.1]). X is a **Lévy process** if it satisfies the following conditions:

(1) $X_0 = 0$ a.s., i.e. $\mathbb{P}(X_0 = 0) = 1$;
(2) X has *independent increments*, i.e. for $0 \leq s < t$: $X_t - X_s$ is independent of $\{X_u : 0 \leq u \leq s\}$;
(3) X has *stationary increments*, i.e. for $0 \leq s < t : X_t - X_s \sim X_{t-s}$;
(4) X is almost surely càdlàg, i.e. $\exists \Omega_0 \in \mathcal{F}$ s.t. $\mathbb{P}(\{\omega \in \Omega_0 : \forall t \geq 0, X_t(\omega) \in \mathbb{D}([0,\infty), \mathbb{R}^d)\}) = 1$; and
(5) X is continuous in probability.

One may come across a definition of Lévy processes that does not include condition (4); this is a Lévy process *in law* and can be used to construct an a.s. càdlàg process.

PROPERTY 4.1 (cf. [**2**, Proposition 1.3.1] and [**36**, Corollary 11.6]). For all $n \in \mathbb{Z}^+$, a Lévy process X_t can be expressed as the sum of n i.i.d. random variables with the probability distribution $\mu^{\frac{1}{n}}$; this property is known as **infinite divisibility**. Conversely, given any infinitely divisible distribution μ, there exists a Lévy process X_t such that X_1 has a distribution given by $\mu^{\frac{1}{n}}$.

Let there be Lévy processes X^a and X^b constructed from an infinitely divisible distribution μ, then $X^a \sim X^b$ by the *Uniqueness Theorem of Lévy* (see [**25**, p. 300]); thus one can say X is unique in distribution. In fact, the uniqueness theorem proves a stronger result: the canonical form of a Lévy process X, given by Theorem 4.1, is unique when constructed from an infinitely divisible distribution μ.

PROPOSITION 4.1. Let random variable $\xi : \mathbb{R}^d \mapsto \mathbb{C}$ have an infinitely divisible distribution μ with nth root of the characteristic function $\widehat{\mu}$. Then the **characteristic function** of ξ is given by

$$\forall z \in \mathbb{R}^d : E(e^{iz.\xi}) = [\widehat{\mu}(z)]^n,$$

$$\text{with} \quad \widehat{\mu}(z) = E(e^{iz.\mu^{\frac{1}{n}}}) = e^{\psi(z)}$$

and continuous function $\psi : \mathbb{R}^d \mapsto \mathbb{C}$, called the **characteristic exponent**.

PROOF. This is a direct consequence of Lemma 7.5 and Lemma 7.6 from [**36**] with Proposition 2.1. □

DEFINITION 4.3 (see [**15**, Defintion 3.4]). A **Lévy measure** ν on $\mathbb{R}^d$ for a Lévy process X is defined by

$$\forall A \in \mathcal{B}(\mathbb{R}^d) : \nu(A) = E[|\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}|].$$

In other words, $\nu(A)$ is the expected number of jumps per unit time such that the jump size belongs to A, and it immediately follows that

$$\forall t \geq 0 : E\{J_X(A \times [0,t])\} = t\nu(A).$$

2. The Lévy-Itô Decomposition and Lévy-Khintchine Representation Theorems

Now we introduce two fundamental theorems: the *Lévy-Itô decomposition* and the *Lévy-Khintchine representation*. These theorems will provide insight into the structure of Lévy processes; enabling the study of its path-wise and distributional properties in the sequel.

THEOREM 4.1 (**Lévy-Itô decomposition**). A Lévy process X with

- a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty; \tag{4.5}$$

- a jump measure J_X with intensity measure $\nu(dx)dt$;
- a drift $\gamma \in \mathbb{R}^d$ and Brownian motion B_t with covariance matrix A

has the *characteristic triplet* (A, ν, γ) and can be given in canonical form by

$$\begin{aligned} X_t &= \gamma t + B_t + X_t^l + \lim_{\varepsilon \downarrow 0} \widetilde{X}_t^\varepsilon, \\ \text{with} \qquad \gamma &= E\left[X_1 - \int_{\{s\in[0,1]\}\times\{|x|\geq 1\}} xJ_X(ds \times dx)\right] \\ &= E(X_1) - \int_{\{|x|\geq 1\}} x\nu(dx), \\ X_t^l &= \int_{\{s\in[0,t]\}\times\{|x|\geq 1\}} xJ_X(ds \times dx) \\ \text{and} \qquad \widetilde{X}_t^\varepsilon &= \int_{\{s\in[0,t]\}\times\{\varepsilon\leq|x|<1\}} x\widetilde{J}_X(ds \times dx) \\ &= \int_{\{s\in[0,t]\}\times\{\varepsilon\leq|x|<1\}} x[J_X(ds \times dx) - \nu(dx)ds]. \end{aligned} \tag{4.6}$$

PROOF. See Theorem 19.2 and 19.3 of [**36**]. Note the choice $|x| = 1$ to divide the large and small jumps is arbitrary. $\square$

By thinking of $\widetilde{X}_t^\varepsilon$ in a discrete sense, one can gain tractability of what occurs when $\varepsilon \to 0$. As the integrability condition (4.5) implies X has an infinite number of jumps as $\varepsilon \to 0$, one can think of $\widetilde{X}_t^\varepsilon$ as the sum of an infinite number of compensated jumps. $\widetilde{X}_t^\varepsilon$ almost surely converges uniformly, by the proof of Theorem 4.1, thus implying it is a martingale by Proposition 3.3. Hence, Theorem 4.1 provides a representation of X in terms of its martingale part $\widetilde{X}_t^\varepsilon + B_t$ and non-martingale part $\gamma t + X_t^l$.

Furthermore, by utilising the Lindeberg-Feller central limit Theorem [**23**, p. 262] (not considered here) and independence of terms in the Lévy-Itô decomposition, the continuous part $X_t - \widetilde{X}_t^\varepsilon - X_t^l$ is implied to be Gaussian. Thus, the Lévy-Itô decomposition also gives a unique canonical representation of a Lévy process X: it has a *continuous Gaussian* part $\gamma t + B_t$ and a *pure-jump* part $X_t^l + \widetilde{X}_t^\varepsilon$.

From (3.4), we observe that the pure-jump part is composed of Poisson-type processes, for which we have already found the characteristic function for (see Proposition 3.5). As noted at the end of chapter 3, we utilise this fact in the following Lévy-Khintchine representation Theorem 4.2.

THEOREM 4.2 (**Lévy-Khintchine representation**). *A random variable $\xi : \mathbb{R}^d \mapsto \mathbb{C}$ has infinitely divisible distribution μ, then*

$$\forall z \in \mathbb{R}^d : \widehat{\mu}(z) = -\frac{1}{2} z.Az + i\gamma.z + \int_{\mathbb{R}^d} (e^{iz.x} - 1 - iz.x\mathbf{1}_{|x|\leq 1})\nu(dx),$$

with $\gamma \in \mathbb{R}^d$, Gaussian covariance matrix A and Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ that satisfies (4.5).

PROOF. See [**36**, Theorem 8.1]. Note the choice of truncation function $\mathbf{1}_{|x|\leq 1}$ directly affects how γ is defined, as γt includes the compensation term $-E(X_t^l)$, see (4.6). □

For generality, the Lévy-Khintchine representation Theorem has been defined for all infinitely divisible distributions, now we handle the case for Lévy processes in particular.

PROPOSITION 4.2. *The characteristic function of a Lévy process X with characteristic triplet (A, ν, γ) is given by*

$$\forall z \in \mathbb{R}^d : E(e^{iz.X_t}) = e^{t\psi(z)},$$

$$\text{with} \quad \psi(z) = -\frac{1}{2} z.Az + i\gamma.z + \int_{\mathbb{R}^d} (e^{iz.x} - 1 - iz.x\mathbf{1}_{|x|\leq 1})\nu(dx).$$

TABLE 4.1. Examples of processes from each of the four categories of a Lévy process. For examples of infinte variation processes, see [**35**, **40**].

	Finite variation	Infinite variation
Finite Activity	Pure jump (see §3.4)	Jump-diffusion
Infinite Activity	Variance gamma (see §4.5)	Normal inverse gaussian

PROOF (*outline*). By Property 4.1 and Proposition 4.1, we have $E(e^{iz.X_1}) = \widehat{\mu}(z) = e^{\psi_1(z)}$, with the characteristic exponent ψ_t of X_t. The multiplicative Property 2.1 of a characteristic function implies that for $a \in \mathbb{Q}$: $a\psi_1 = \psi_a$. For $\{\psi_b : b \in \mathbb{R} \setminus \mathbb{Q}\}$, we can use *Lebesgue's Theorem of dominated convergence* [**42**, p. 109], in conjunction with the cádlág property of X, to show that a sequence $(\psi_a)_n$ converges to ψ_b such that $X_a \sim X_b$. Thus $\forall t \in \mathbb{R} : \psi_t = t\psi_1 = t\psi$ and we have ψ by Theorem 4.2.

Alternatively, one can use Theorem 4.1 and *Fatou's Lemma* (see [**42**, p. 110]) to directly prove the Lévy-Khintchine representation for Lévy processes and then extend it to all infinitely divisible distributions by Property 4.1. □

The Lévy-Khintchine representation Theorem provides an incredible result; the characteristic function of a Lévy process can be found in closed form. This will become crucial in the sequel, when we use Lévy processes and Fourier-based methods to price options.

3. Finite Variation Lévy Processes

The Lévy-Itô decomposition Theorem 4.1 implies that a Lévy process can be categorised into four types, as shown in Table 4.1; infinite activity models refer to when $\nu(\mathbb{R}^d) = \infty$ and infinite variation refers to a.s. *unbounded variation* of trajectories of X_t, i.e. $E\left[\mathcal{V}_{[0,t]}(X_t)\right] = \infty$.

DEFINITION 4.4 (cf. [**14**]). The **total variation** $\mathcal{V}$ of a $\mathbb{R}^d$-valued function f on $[a,b]$ is defined by

$$\mathcal{V}_{[a,b]}(f) = \sup_{\mathcal{P}} \sum_{i=0}^{p} |f(x_{i+1}) - f(x_i)|,$$

with $\mathcal{P}$ denoting the set of all finite partitions P of $[a,b]$ such that $P = \{x_0, \ldots, x_{p+1}\}$.

By bounding the Lévy measure ν, finite activity can be directly specified for a Lévy process and results in a jump-diffusion model. To study finite variation Lévy processes, we will impose the finite variation property on a Lévy process in Proposition 4.3, by doing so we find that Lévy processes must satisfy a stronger condition than (4.5).

Proposition 4.3. If a Lévy process X has finite variation, it must have characteristic triplet (A, ν, γ), with $A = 0$, and it satisfies

$$\int_{\mathbb{R}^d} (|x| \wedge 1)\nu(dx) < \infty.$$

Proof. Suppose we have a Lévy process X that has finite variation then, by Theorem 4.1, we must have $A = 0$, as Brownian motion has infinite total variation [**30**], and the integrability condition (4.5) ensures X_t^l has finite variation. Then, as the drift part is also of finite variation, we only have to consider X_t^ε to ensure finite variation.

By taking partitions that are dense in $[0,t]$, we have

$$E\left[\mathcal{V}_{[0,t]}(X_t^\varepsilon)\right] = E\left[\sup_{\mathcal{P}} \sum_{i=0}^{n} |X_{t_{i+1}}^\varepsilon - X_{t_i}^\varepsilon|\right],$$

with $0 < t_1 \leq \cdots < t_{n+1} \leq t$. Then, as $\sup|t_{i+1} - t_i| \to 0$, the term $|X_{t_{i+1}}^\varepsilon - X_{t_i}^\varepsilon| \to |dX_{t_i}^\varepsilon|$. Thus, we have

$$\lim_{n\to\infty} \sum_{i=0}^{n} |X_{t_{i+1}}^\varepsilon - X_{t_i}^\varepsilon| = \int_{\{s\in[0,t]\}} |dX_s^\varepsilon| = \int_{\{s\in[0,t]\}\times\{\varepsilon\leq|x|<1\}} |x| J_X(ds \times dx);$$

see [**9**, Lemma 3.1] for a proof of this that utilises stopping times. Then, the expected total variation of X_t^ε is given by

$$E\left[\mathcal{V}_{[0,t]}(X_t^\varepsilon)\right] = t \int_{\{\varepsilon\leq|x|<1\}} |x|\nu(dx).$$

Therefore, if Lévy process X has finite variation, $A = 0$ and X_t^ε must have a.s. bounded variation, i.e. it satisfies

$$\int_{\mathbb{R}^d} (|x| \wedge 1)\nu(dx) < \infty.$$

□

The proof of proposition 4.3 refers to the *drift* having finite variation, this does not refer to γ (which includes jumps of size $|x| < 1$, see (4.6)).

Instead, the drift refers to the continuous part of γ, given by

$$b = \gamma - \int_{\{|x|<1\}} x\nu(dx) = E(X_1) - \int_{\mathbb{R}^d} x\nu(dx).$$

Hence, this provides the reason for why we consider X_t^ε and not $\widetilde{X}_t^\varepsilon$ in the proof of Proposition 4.3, as the compensating part of the compensated Poisson process $\widetilde{X}_t^\varepsilon$ is included within b.

All this new information on finite variation Lévy processes warrants a new simplified formula for the Lévy-Itô decomposition of X with characteristic triplet $(0, \nu, b)$, given by

$$X_t = bt + \int\limits_{\{s\in[0,t]\}\times\mathbb{R}^d} xJ_X(ds \times dx). \tag{4.7}$$

Then, we immediately have the characteristic exponent of X_t by Theorem 4.2, given by

$$\forall z \in \mathbb{R}^d : \psi(z) = ib.z + \int_{\mathbb{R}^d} (e^{iz.x} - 1)\nu(dx).$$

4. Distributional Properties and Martingale Representation

In this section, we will investigate some of the distributional properties of Lévy processes, particularly the *skewness* (measure of symmetry) and *kurtosis* ('fatness' of tails) of the distribution. Then, we will find the necessary conditions for Lévy process $(X_t)_{t\in[0,T]}$ and exp-Lévy process $(e^{X_t})_{t\in[0,T]}$ to be $\mathcal{F}_t$-adapted martingales.

Prior to finding the cumulants of a Lévy process X, we must find the conditions for X to have bounded moments in the following proposition.

Proposition 4.4. The Lévy process X has a bounded nth moment $E(|X_t|^n)$ if, and only if, it satisfies

$$\forall t \geq 0 : \int_{\{|x|\geq 1\}} |x|^n \nu(dx) < \infty. \tag{4.8}$$

Proof (*outline*). By Theorem 4.1, it is clear that $E(|X_t^l|)$ is integrable only when (4.8) is satisfied for $n = 1$. By decomposing X_t into the sum of two Lévy processes, one that has jumps < 1 and one with jumps ≥ 1, one can prove (4.8) for $n \geq 2$. This is a special case of [**36**, Theorem 25.3], which proves this fact for all measurable functions on $\mathbb{R}^d$ that are submultiplicative and locally bounded. $\square$

Then, by Proposition 4.4 and the Lévy-Itô decomposition, a Lévy process X with bounded first moment and characteristic triplet (A, ν, γ) has expectation

$$E(X_t) = \gamma t + t \int_{\{|x|\geq 1\}} x\nu(dx). \tag{4.9}$$

We can also compute the nth cumulant $\mathcal{C}_n$ of X, given by

$$\mathcal{C}_n = i^{-n} t \cdot \left. \frac{\partial^n \psi(z)}{\partial z^n} \right|_{z=0},$$

with the characteristic exponent $\psi(z)$. Then $\mathcal{C}_1 = E(X_t)$ and $\mathcal{C}_2 = \mathrm{Var}(X_t)$, and we have skewness θ and (excess) kurtosis κ from [**15**, §2.2.5], given by

$$\theta(X_t) = \frac{\mathcal{C}_3(X_t)}{\mathcal{C}_2(X_t)^{\frac{3}{2}}} = t^{-\frac{1}{2}}\theta(X_1) \quad \text{and} \quad \kappa(X_t) = \frac{\mathcal{C}_4(X_t)}{\mathcal{C}_2(X_t)^2} = t^{-1}\kappa(X_1).$$

Note that $\mathcal{C}_4(X_t) > 0 \Rightarrow \kappa(X_t) > 0$, thus X_t is *leptokurtic*, and the skewness and kurtosis decays at $\frac{1}{\sqrt{t}}$ and $\frac{1}{t}$ over time t respectively. As the normal distribution has 0 skewness and kurtosis, this can be interpreted as a decay towards normality; this is shown by computing the Black-Scholes implied volatility surface of a call option price in the sequel (see Figure 5.1).

Now, we will show that $(X_t - E(X_t))_{t\in[0,T]}$ is a martingale in Proposition 4.5.

Proposition 4.5. If Lévy process $(X_t)_{t\in[0,T]}$ has a bounded first moment ($\forall t \in [0,T] : E(X_t) < \infty$), the $\mathcal{F}_t$-adapted process $(Y_t)_{t\in[0,T]}$, given by

$$Y_t = X_t - E(X_t),$$

is a martingale.

Proof. By the independent increments property of $(X_t)_{t\in[0,T]}$ and (4.9), we have

$$\begin{aligned}
E(Y_t|\mathcal{F}_s) &= E\left[X_t - E(X_t)|\mathcal{F}_s\right] \\
&= X_s + E\left\{ X_{t-s} - \left[\gamma t + t\int_{\{|x|\geq 1\}} x\nu(dx)\right] \middle| \mathcal{F}_s \right\} \\
&= X_s - \gamma t - t\int_{\{|x|\geq 1\}} x\nu(dx) + E(X_{t-s}|\mathcal{F}_s) \\
&= X_s - \gamma s - s\int_{\{|x|\geq 1\}} x\nu(dx) \\
&= X_s - E(X_s) = Y_s.
\end{aligned}$$

□

A special case of Proposition 4.5 is when $E(X_t) = 0$, then a Lévy process $(X_t)_{t\in[0,T]}$ with $\int_{\{|x|\geq 1\}} x\nu(dx) < \infty$ is a martingale if it satisfies

$$E(X_1) = \gamma + \int_{\{|x|\geq 1\}} x\nu(dx) = 0. \tag{4.10}$$

In order to find conditions for when $(e^{X_t})_{t\in[0,T]}$ is a martingale, we must first prove Proposition 4.6.

Proposition 4.6. The $\mathcal{F}_t$-adapted process

$$(Y_t)_{t\in[0,T]} = \left(\frac{e^{izX_t}}{E(e^{izX_t})}\right)_{t\in[0,T]},$$

with Lévy process $(X_t)_{t\in[0,T]}$, is a martingale for all $z \in \mathbb{R}$.

Proof. By the independent increments property of X_t and Proposition 4.2, we have $\forall z \in \mathbb{R}$ and $0 \leq s < t$:

$$\begin{aligned} E(Y_t|\mathcal{F}_s) = E\left[\frac{e^{izX_t}}{E(e^{izX_t})}\middle|\mathcal{F}_s\right] &= E\left[\frac{e^{izX_t}}{e^{t\psi(z)}}\middle|\mathcal{F}_s\right] \\ &= e^{-t\psi(z)}E\left[e^{iz(X_{t-s}+X_s)}|\mathcal{F}_s\right] \\ &= e^{-t\psi(z)}e^{(t-s)\psi(z)}E\left[e^{izX_s}|\mathcal{F}_s\right] \\ &= e^{-s\psi(z)}e^{izX_s} \\ &= \frac{e^{izX_s}}{E(e^{izX_s})} = Y_s. \end{aligned}$$

□

Suppose we extend Proposition 4.6 such that we allow for $z = -iu$, for all $u \in \mathbb{R}$, then we have

$$\forall t \in [0,T] : \frac{e^{izX_t}}{E(e^{izX_t})} = \frac{e^{uX_t}}{E(e^{uX_t})} = e^{uX_t - t\psi(-iu)}. \tag{4.11}$$

However, (4.11) is only possible if the exponential moment is bounded for large jumps [**36**, Theorem 25.17], i.e.

$$\int_{\{|x|\geq 1\}} e^{ux}\nu(dx) < \infty. \tag{4.12}$$

Thus, $(e^{X_t})_{t\in[0,T]}$ is a martingale when it satisfies (4.12) for $u = 1$ and $\psi(-i) = 0$, hence by the Lévy-Khintchine representation Theorem 4.2, we have

$$\gamma = -\frac{A}{2} - \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{|x|\leq 1})\nu(dx). \tag{4.13}$$

The result given by (4.11) is crucial, as we will use this result in Chapter 5 to correct for the drift in the Lévy component of a log-price process.

5. Subordinating Lévy Processes

In this section, we will introduce a special type of Lévy process, called *subordinators*, that will enable us to construct processes based on Brownian motion by using a stochastic time scale that is provided by a subordinator. This procedure of *Brownian subordination* results in an infinite activity Lévy process that does not require a Brownian component (due to small time behaviour of jumps [**11**]). Then we implement Brownian subordination by using a gamma subordinator, resulting in the *variance gamma process*; an infinite activity and finite variation Lévy process that we will utilise in the sequel for option pricing (see Chapter 5).

Definition 4.5. A Lévy process X is called a **subordinator** if it has trajectories that are a.s. monotonically increasing for all $0 \leq s \leq t$. By [**15**, Proposition 3.10], this is equivalent to a finite variation Lévy process that has no diffusion component and only positive jumps, i.e. it satisfies

$$A = 0, \quad \nu\{(-\infty, 0]\} = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (x \wedge 1)\nu(dx) < \infty.$$

As all subordinators are positive, it is more convenient use the Laplace exponent to describe the characteristics of the process (see [**20**] for the Laplace transform). Then, by (4.7), a subordinator $(S_t)_{t\geq 0}$ with triplet $(0, \rho, b)$ has Laplace exponent $\eta(z)$, given by [**6**]:

$$\eta(z) = bz + \int_{(0,\infty)} (e^{ux} - 1)\rho(dx).$$

Theorem 4.3 (see [**15**, Theorem 4.2]). Let X be a Lévy process with characteristic exponent $\psi(z)$ and triplet (A, ν, γ). Let $(S_t)_{t\geq 0}$ be a subordinator with Laplace exponent $\eta(z)$ and triplet $(0, \rho, b)$. Then, the process $Y(t, \omega) = X(S(t, \omega), \omega)$ is a Lévy process that is **subordinate** to X and it has characteristic function $E(e^{izY_t}) = e^{t\eta[\psi(z)]}$.

Suppose $(S_t)_{t\geq 0}$ is the gamma process $S_t \sim \Gamma(t; \alpha, \beta)$, with $\alpha, \beta > 0$. Then, $(S_t)_{t\geq 0}$ is a subordinator with characteristic triplet $(0, \rho, 0)$ and Lévy density ρ, given by

$$\rho(x) = \frac{\alpha e^{-\beta x}}{x} \mathbf{1}_{x>0}; \tag{4.14}$$

hence α and β control the intensity and size of jumps respectively.

However, it is more convenient to use an alternative unit time parametrisation of (4.14), where $\alpha = \frac{E(S_1)^2}{\mathrm{Var}(S_1)}$ and $\beta = \frac{E(S_1)}{\mathrm{Var}(S_1)}$ [**15**, §4.4.2]. As we only consider $E(S_t) = t$ in the sequel, we have

$$\rho(x) = \frac{1}{\kappa} \cdot \frac{e^{-\frac{x}{\kappa}}}{x} \mathbf{1}_{x>0},$$

with $\kappa = \mathrm{Var}(S_1)$.

Then the **variance gamma process** is a Lévy process $(X_t)_{t\geq 0}$, given by

$$X_t = \theta S_t + B(S_t),$$

with Brownian motion $(B_t)_{t\geq 0}$ that has drift θ and volatility σ.

By Theorem 4.3, the characteristic exponent ζ of X_t is acquired [**32**] by the composition of the Laplace exponent η of S_t and characteristic function ψ of B_t, it is given by

$$\zeta(z) = -\frac{1}{\kappa} \log\left(1 + \frac{z^2\sigma^2\kappa}{2} - i\theta\kappa z\right).$$

Then we have the Lévy density of $(X_t)_{t\geq 0}$, given by

$$\begin{aligned}
\nu(x) &= \int_{(0,\infty)} p_{B_s}\rho(ds) \\
&= \int_{(0,\infty)} \left(\frac{1}{\sigma\sqrt{2\pi s}} e^{\frac{(x-\theta s)^2}{2s\sigma^2}}\right)\left(\frac{1}{\kappa s} e^{-\frac{s}{\kappa}}\right) ds \\
&= \frac{1}{\kappa|x|} \exp\left(\frac{\theta}{\sigma^2} x + \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}|x|\right),
\end{aligned}$$

the derivation of the last equality has been omitted as it is quite lengthy and involves the the Bessel function of the second kind, see [**15**, §4.4.3] for a derivation for the general class of *normal tempered stable processes*; for which the variance gamma process is a special case.

Thus, the triple (σ, κ, θ) affects the Lévy density, we have drift θ and volatility σ, resulting from the Brownian subordination, and κ, the variance rate of the gamma subordinator. This provides much greater control on the distribution, with θ and κ characterising the skewness and kurtosis respectively [**32**].

CHAPTER 5

Option Pricing using exp-Lévy Models

This chapter is dedicated to the financial and computational application of Lévy processes in modelling asset prices and pricing financial instruments; namely the European call option.

It will begin by the introduction of the exp-Lévy model in Section 5.1, where we modify the a fundamental assumption of the Black-Scholes model, that asset returns following the normal distribution via Brownian motion, and replace it with a Lévy component instead. In Section 5.2, we address the fact that Lévy processes result in market incompleteness and note on methods for choosing the risk-neutral measure. Then, in Section 5.3, we use theory and methods introduced in Chapter 2, in order to present a Fourier-based method for call option pricing using the the exp-Lévy model.

We then implement this method in Section 5.4, using the variance gamma process (see Chapter 4) in order to: compare the variance gamma model to the Black-Scholes model, by computing the Black-Scholes implied volatility surface; analyse the effect of the 'damping' coefficient α (see Section 5.3) on the call option price; and examine the effect of changing skewness θ and kurtosis ν on the option price. Lastly, in Section 5.5, we assess the performance of the Fourier-based method, used in the prior section, by calculating the error of an at-the-money (ATM) call option price, relative to the value that is provided by using numerical integration.

1. exp-Lévy Models

In Black & Scholes ground-breaking paper [**8**], they were able to describe the risk-neutral dynamics of an asset price $(S_t)_{t\geq 0}$, based on geometric Brownian motion, within the Black-Scholes model:

$$S_t = S_0 e^{B_t}, \tag{5.15}$$

with Brownian motion $(B_t)_{t\geq 0}$ that has drift $r - \frac{1}{2}\sigma^2$ and volatility σ.

As noted in [**11**], Brownian motion does not provide a realistic description of asset returns, in the risk-neutral setting asset returns consistently show higher kurtosis and negative skew. Thus, we replace the Brownian motion in (5.15) with a Lévy process $(X_t)_{t\geq 0}$, resulting in the **exp-Lévy**

model, given by

$$S_t = S_0 e^{\mathcal{X}_t}, \tag{5.16}$$

with $\mathcal{X}_t = rt + X_t - \psi(-i)t$ and characteristic exponent ψ of $(X_t)_{t\geq 0}$. Note the restrictions (4.12) and (4.13) apply to $e^{\mathcal{X}}$. Also, note the inclusion of the drift correcting term $-\psi(-i)t$ (often characterised as the 'martingale correction' term) ensures that the discounted asset price process $(\widetilde{S}_t)_{t\in[0,T]} = (e^{-r(T-t)}S_T)_{t\in[0,T]}$ is a martingale under risk-neutral measure Q, i.e.

$$E^Q(\widetilde{S}_T|\mathcal{F}_t) = \widetilde{S}_t.$$

2. Market Incompleteness

Now, I must emphasise that the risk-neutral measure Q is *not unique* and X does not have to be a Lévy process under Q. However, it does remain a semimartingale (the sum of a local martingale and $\mathcal{F}_t$-adapted càdlàg process with bounded variation, see [**27**, Ch. I, §4c]).

By *Girsanov's Theorem* for semimartingales (see [**27**, Ch. III, §3d]), a Lévy process X, with bounded first moment and characteristic triplets (A, ν, γ) and (A^Q, ν^Q, γ^Q) under equivalent measures P and Q, has characteristics relative to Q given by

$$\begin{aligned} \gamma^Q &= \gamma + A\chi + x\mathbf{1}_{|x|\leq 1}(Y-1)\cdot\nu, \\ A^Q &= A \qquad \text{and} \qquad \nu^Q = Y\nu, \end{aligned} \tag{5.17}$$

with non-negative function Y and predictable process χ; a process measurable with respect to a σ-algebra generated by left continuous $\mathcal{F}_t$-adapted processes.

By (4.13), we have the the martingale condition under Q, given by

$$\gamma^Q = -\frac{A^Q}{2} - \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{|x|\leq 1})\nu^Q(dx). \tag{5.18}$$

Thus, we have a system of equations given by (5.17) and (5.18), with the parameters Y and χ that provide infinitely many solutions for the equivalent martingale measure Q.

Note the measure change to Q is related to changing the probabilities associated with each trajectory of X, it does not change the trajectories itself. This is intuitive as the pathwise properties, namely A and random measure J_X, are invariant under the measure change to Q. As mentioned before, X may not remain a Lévy process under measure Q, this is due to the fact X is contingent on Y and χ being deterministic and time-independent.

Now, we introduce the *second fundamental theorem of asset pricing*, given by the following theorem.

THEOREM 5.1 (see [**7**, Theorem 11.17]). An arbitrage-free market is **complete** if, and only if, there exists unique martingale measure Q that is equivalent to $\mathbb{P}$.

Hence, exp-Lévy models give rise to incomplete markets and you cannot replicate the option payoff perfectly, as one would by delta-hedging in the Black-Scholes model for instance. This calls attention to the problem of hedging and model calibration, which we discuss in Section 6.2.

3. Option Pricing with Fourier-based Methods

In this section, we will utilise the Fourier transform (see Chapter 2) to price a European call option using the exp-Lévy model we introduced in Section 5.1.

We have value of a call option at maturity T, with initial asset price S_0 and strike price K, given by

$$C_T(k) = e^{-rT}E^Q\left[(S_T - K)^+\right] = e^{-rT}E^Q[(S_0e^{\mathcal{X}_T} - e^k)^+], \tag{5.19}$$

with $k = \log K$, note from this point onwards we will let $S_0 = 1$ without loss of generality.

As noted at the end of Chapter 2, the risk-neutral probability density function ϱ_T associated with with the log-asset price $\mathcal{X}_T$ is not analytically tractable, thus $E[(e^{\mathcal{X}_T} - e^k)^+]$ cannot be computed. However, the characteristic function, given by

$$\phi_T(z) = \int_{\mathbb{R}} e^{izx}\varrho_T(x)dx,$$

is known in closed form and for values of k, we have

$$C_T(k) = e^{-rT}E^Q[(e^{\mathcal{X}_T} - e^k)^+] = \int_{\mathbb{R}} e^{-rT}(e^x - e^k)^+\varrho_T(x)dx.$$

Now, recall the *Parseval relation* in Theorem 2.1, suppose $e^{-rT}(e^x - e^k)^+ \in L^2(\mathbb{R})$ and $\varrho_T(x) \in L^2(\mathbb{R})$, then we have

$$C_T(k) = \frac{1}{2\pi}\int_{\mathbb{R}} 2\pi \cdot \mathcal{F}^{-1}\left[e^{-rT}(e^x - e^k)^+\right](z) \cdot \phi_T(z)dz,$$

see Section 2.3 for the note on the 2π normalistion factor. However, the supposition $e^{-rT}(e^x - e^k)^+ \in L^2(\mathbb{R})$ is not true, because it must be *square* *integrable*, i.e.

$$\mathcal{I} = \int_{\mathbb{R}} \left[e^{-rT}(e^x - e^k)^+\right]^2 dx < \infty,$$

which it clearly is not, as $e^{-rT}(e^x - e^k)^+ \to e^{-rT}$ when $k \to -\infty$, hence $\mathcal{I} \not< \infty \Rightarrow e^{-rT}(e^x - e^k)^+ \notin L^2(\mathbb{R})$.

In order to make make $C_T(k)$ square integrable, we must introduce the *modified* call price C_T^α, given by

$$\forall \alpha > 0 : C_T^\alpha(k) = e^{\alpha k} C_T(k),$$

then we have $e^{\alpha k - rT}(e^x - e^k)^+ \to 0$ as $k \to -\infty$, thus $e^{\alpha k - rT}(e^x - e^k)^+ \in L^2(\mathbb{R})$.

Now, we take the approach formulated by Carr & Madan [**12**]. First, we have the expression of the call price by the inverse Fourier transform (see Definition 2.2), given by

$$C_T(k) = \mathcal{F}^{-1}\mathcal{F}\left[C_T^\alpha\right](k) = \frac{e^{-\alpha k}}{2\pi} \int_{[0,\infty)} e^{izk} \mathcal{F}\left[C_T^\alpha(k)\right](z) dz. \tag{5.20}$$

Note the integral is evaluated on $[0, \infty)$ as $C_T(k)$ is real, hence implying the integrand is an even function.

Then, the expression for the Fourier transform of the call price $\mathcal{F}\left[C_T^\alpha(k)\right](z)$ is given in terms of ϕ_T:

$$\mathcal{F}\left[C_T^\alpha(k)\right](z) = \frac{e^{-rT}\phi_T((\alpha+1)i - z)}{\alpha^2 + \alpha + z^2 - i(2\alpha - 1)z}.$$

The derivation [**12**] employs *Fubini's theorem* [**42**, p. 90] and is not considered here.

To numerically solve (5.20) the discrete Fourier transform is used (see (2.1)), for which we use the fast Fourier transform (FFT) in Section 5.4. I have decided to omit this part as the logic and procedure behind the truncation and discretisation is similar to the Example 2.1; it can be found in the original paper [**12**].

4. Analysis of Option Pricing Implementation

In this section, we use the variance gamma process (see Section 4.5) in the exp-Lévy model (5.16), to price a call option using the Fourier-based method introduced in Section 5.3. All call option prices that are used in the sequel were computed using the `price_vg` function created in MATLAB (see Appendix A.1).

By using the Black-Scholes implied volatility `blsimpv` function, available in the Financial Toolbox in MATLAB, one can find the implied volatility of

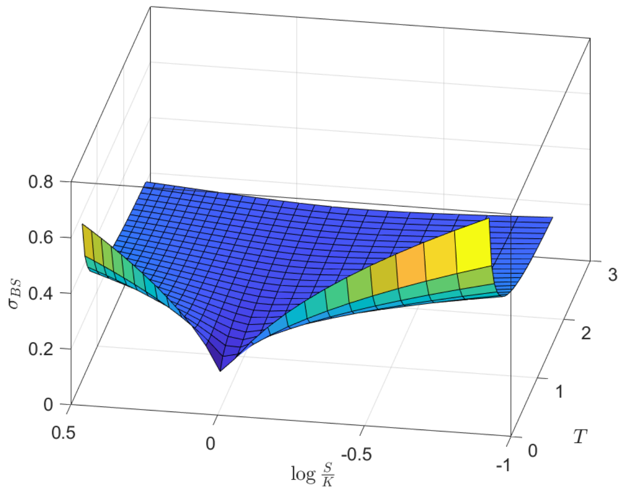


FIGURE 5.1. Implied volatility as function of maturity and log-moneyness for the variance gamma model, note flattening of smile with maturity. $r = 0$, $\sigma = 0.25$, $\kappa = 1$ and $\theta = -0.1$.

the option prices given by the variance gamma model. This is done in Figure 5.1, which shows the Black-Scholes implied volatility surface as a function of option maturity and log-moneyness; it illustrates the 'flattening' of the volatility smile and skew as maturity increases.

As pointed out by [**15**, p. 359] and seen in Section 4.4, this is due to the decaying skewness and kurtosis over time in finite variation Lévy processes; one of the shortcomings of the variance gamma model. Also note the negative skew present in Figure 5.1 for the implied volatilities relative to log-moneyness, this is due to the fact we have a negatively skewed jump distribution as $\theta = -0.1$. Madan et al. [**32**] found, by fitting the variance gamma model, that the risk-neutral density implied option data is negatively skewed, in turn this implies risk aversion.

Figure 5.2 shows the effect of 'damping' coefficient α on the ATM call price; it remains stable for values (approximately) in the range $[0.5, 6.2]$, with the call price diverging outside the range, thus increasing error in Fourier-based method.

Now, we analyse the parameters κ and θ that affect the jump distribution of the variance gamma model, Figure 5.3 shows the price curve for a call option relative to log-moneyness. It shows how increasing the kurtosis κ causes the convexity of the call price curve to increase. This is expected, as

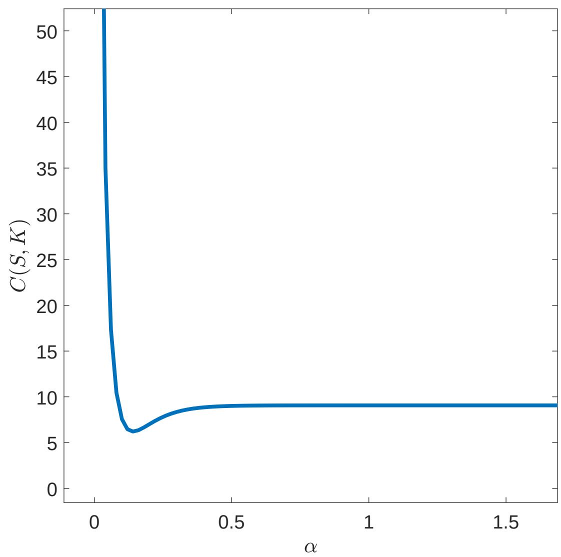


FIGURE 5.2. ATM Call price as $\alpha \to 0$. For $\sigma = 0.25$, $\theta = -0.1$, $\kappa = 1$: values of $\alpha \in [0.5, 6.2]$ remain stable

increasing kurtosis results in 'fatter' tails in the distribution of asset returns in the risk-neutral setting [**11**], thus the value of options decrease near the at-the-money level and increases symmetrically for in-the-money (ITM) and out-of-the-money (OTM) options.

Similarly, Figure 5.4 for variable θ also shows the price curve for a call option relative to log-moneyness. It shows how increasing θ skews the curve negatively, this is due to the risk-neutral asset returns having negative skew, thus the ITM call options increase in value and OTM options decrease in value.

5. Error Analysis

In this section, we assess the performance of the Fourier-based method used in the prior section. To do this, we use the final equation (5.20), from Section 5.3, that provides the solution of the call price (for values of k) in terms of the integral of the Fourier transform of the modified call price. Now, recall that we have the closed-form solution for the Fourier transform, thus we can use numerical integration to approximate the value within a given tolerance. Hence, we have an accurate benchmark to assess the error in Fourier-based methods.

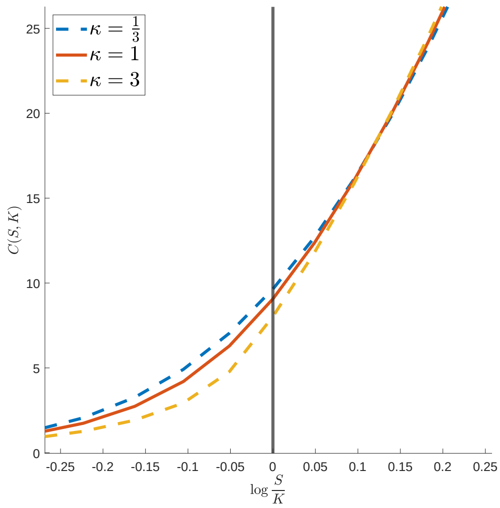


FIGURE 5.3. Call price as a function of log-moneyness and κ; with $\sigma = 0.25$, $\theta = -0.1$, $r = 0$ and $T = 1$.

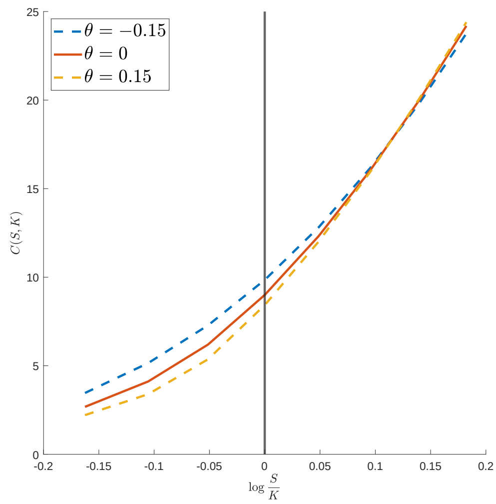


FIGURE 5.4. Call price as a function of log-moneyness and θ; with $\sigma = 0.25$, $\kappa = 2$, $r = 0$ and $T = 1$.

In order to obtain the value of an ATM call price, with $\sigma = 0.25$, $r = 0$ and $T = 1$, the function `num_int` has been created in MATLAB (see Appendix A.2). This function uses the function `integral` in MATLAB with

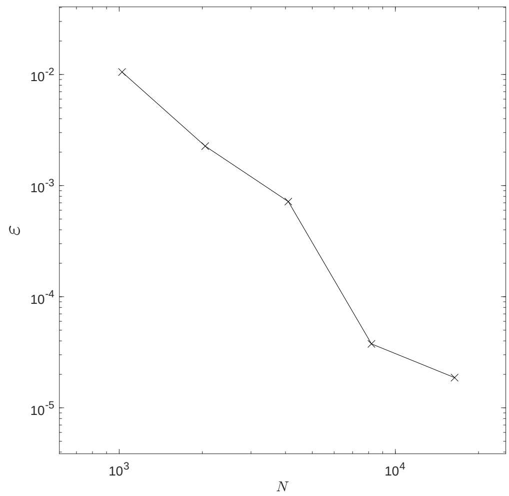


FIGURE 5.5. Error ε of an ATM call option price under the variance gamma model relative to N. Computed using `price_vg`, relative to a benchmark computed using `num_int`, with $r = 0$, $T = 1$, $\theta = -0.1$, $\sigma = 0.25$, $\kappa = 1$ and $\Delta = 0.25$.

an absolute tolerance of 10^{-16}, thus this is an appropriate benchmark for assessing error.

As shown in Figure 5.5, we make the grid of strikes finer by increasing the magnitude of points N, in doing so the error ε decreases with a linear relationship (in magnitudes). This illustrates the speed of convergence, $\mathcal{O}(N \log N)$, that the FFT algorithm displays [**17**]; this is noted on in Section 2.1. Note that there is the initial grid step Δ (see (2.1)) that is associated with the Fourier transform of the modified call price (5.20). One must be careful in the specification of Δ, as shown in Figure 5.6, as decreasing the size of Δ leads to dispersed strikes and an increased need for interpolation [**19**].

Furthermore, the Fourier-based method has some limitations: the number of points N must a power of 2, a limitation from using the FFT; possible divergence or singularity in the integrand has to be treated with caution [**19**]; and we cannot increase the resolution in selective regions, i.e. irregularity in the integrand in the at-the-money region [**15**].

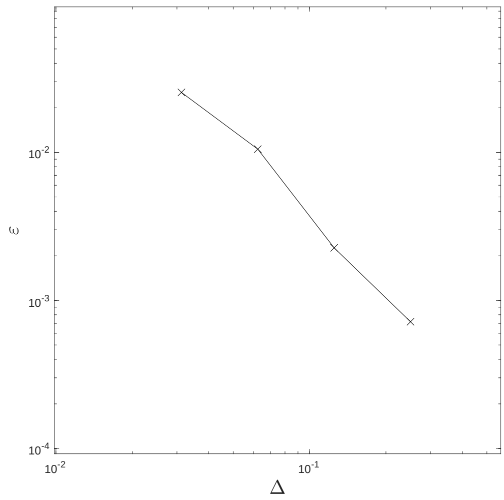


FIGURE 5.6. Error ε of an ATM call option price under the variance gamma model relative to Δ. Computed using `price_vg`, relative to a benchmark computed using `num_int`, with $r = 0$, $T = 1$, $\theta = -0.1$, $\sigma = 0.25$, $\kappa = 1$ and $N = 2^{12}$.

CHAPTER 6

Conclusion and Further Work

1. Conclusion

The body of work regarding Lévy processes and its applications in finance have a relatively high barrier to entry; making it inaccessible to new practitioners in the field of quantitative finance. Thus, this paper has attempted to develop the theory of Lévy processes from its foundational aspects in measure theory and Fourier analysis. By using Poisson-type processes and infinitely divisible distributions, we were able to establish landmark theorems in the study of stochastic processes, which we then utilised in the analysis of Lévy processes.

Then the theories were applied in financial modelling to construct the exp-Lévy model, and we were able to attain reasoning for the market incompleteness that models based on Lévy processes exhibit. This was then used to present a method for pricing options using the exp-Lévy model and Fourier-based methods.

We implemented these methods to determine call option prices based on the variance gamma model. By analysing the model computationally, we were able to deduce the effect of varying the parameters in a statistical and financial perspective, namely the control over the skewness and kurtosis of the distribution of risk-neutral asset returns, thus allowing for the rectification of pricing biases present in the Black-Scholes model. The error analysis of the Fourier-based method revealed that one has to be cautious with parameter specification and aware of the limitations of the method. However, the fast convergence of the FFT algorithm makes it optimal for pricing a large quantity of options that share the same maturity [**15**].

The three main aims outlined at the beginning of the paper have been addressed, and hopefully the work that this paper comprises of will be of use to readers looking to gain a grasp on the theory of Lévy processes and Fourier transforms, with its applications in mathematical finance, probability theory, and numerous other fields; such as quantum groups [**1**].

2. Further Work

There are numerous topics that this paper is unable to address, due to the extensive impact of Lévy processes in mathematical finance and other fields. Here, I will discuss some topics for further research and considerations for practitioners within the field of mathematical finance.

Hedging and model calibration: as seen in Section 5.2, exp-Lévy models give rise to incomplete markets, thus one needs to pay special consideration to the choice of martingale measure Q. One can opt for a method such as the *Esscher transform* to choose measure Q, for which there is a wealth of literature regarding this topic [**7**, **15**, **27**], however, this is usually inconsistent with the prices of traded options on the market [**15**]. This leads to the *inverse problem* of model calibration, which is concerned with obtaining the the parameters associated with the measure Q when you have option price data available. However, there is a lower bound in the estimation error, as noted by [**5**], which implies that this inverse problem is ill-defined and requires further treatment.

Partial integro-differential equation methods: As discussed in Section 5.5, there are many limitations to the Fourier-based method I used for pricing European options. Thus, when you have more complex derivatives, such as Barrier or Asian options, or want to price large amounts of options with differing maturity, partial integro-differential (PIDE) methods are superior and there are many methods for numerical approximation of a PIDE, such as finite difference methods [**16**].

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APPENDIX A

Option Pricing Implementations

1. Fourier-based Method

Listing A.1 shows the MATLAB implementation of the function `price_vg`, which is used in Section 5.4 to determine option prices with the variance gamma model and Fourier-based method introduced in Section 5.3.

LISTING A.1. MATLAB function `price_vg`.

```
function price = price_vg(spot, strike, maturity, r, ...
    sigma, nu, theta, alpha, p, delta)
% Price call option with Variance Gamma exp-Levy model

  % Default values for the arguments p, delta and alpha
  if nargin < 8
    alpha = 1.5;
  end
  if nargin < 9
    delta = 0.25;
  end
  if nargin < 10
    p = 12;
  end

  N = 2^p;
  k_step = (2*pi)/(N*delta);

  % u_m = m*delta, m = 0, ..., N-1
  u = 0 : delta : ((N-1)*delta);
  % k_n = - theta/2 + n*k_step, n = 0, ..., N-1
  k = ((-k_step*N)/2) : k_step : ((k_step*(N-1))/2);

  % (F) is the argument that we will apply FFT to
  F = VarianceGamma(u - (alpha+1)*1i, log(spot), ...
        maturity, r, sigma, nu, theta);
  F = F ./ (alpha^2 + alpha - u.^2 + 1i * (2*alpha+1)*u);
```

```
  F = exp(-r*maturity)* delta* F.* exp(1i*(k_step*N/2)*u);
  % apply simpson's rule to x
  F = (F/3) .* (3+(-1).^(1:N) - ((0:(N-1)) == 0));
  % apply FFT and get call prices
  C_k = real((exp(-alpha*k)/pi) .* fft(F));

  % range of indices [min_i, max_i] containing log(strike)
  % +2 in range if cubic/spline interpolation is used
  logK = log(strike);
  min_i = floor((min(logK) + (k_step*N)/2)/k_step+1) - 2;
  max_i = ceil((max(logK) + (k_step*N)/2)/k_step+1) + 2;
  x = k(min_i : 1 : max_i);
  y = C_k(min_i : 1 : max_i);

  % interpolate values of log(strike)
  price = interp1(x, y, logK);
end

function cf= VarianceGamma(u, lnS, T, r, sigma, nu, theta)
% c.f. of V.G. process with martingale correction
  omega = (1/nu) * log(1 - theta*nu - 0.5*(sigma^2)*nu);
  exponent = 1i * u * (lnS + (r+omega)*T) - (T/nu) * ...
    log(1 - 1i*theta*nu*u + 0.5*(sigma^2)*nu*(u.^2));
  cf = exp(exponent);
end
```

2. Numerical Integration Method

Listing A.2 shows the MATLAB implementation of the function `num_int`, which is used in Section 5.5 to assess the performance of the Fourier-based method from Section 5.3, which is implemented in `price_vg` from Appendix A.1.

LISTING A.2. MATLAB function `num_int`.

```
function sol = num_int(alpha, S0, K, r, T, theta, ...
    sigma, nu)
%Numerical integration with Variance Gamma exp-Levy model
  k = log(K);
  lnS = log(S0);
  f = @(u) real(func(u, k, alpha, lnS, r, T, theta, ...
      sigma, nu));
  sol = (exp(-alpha*k - r * T ))/(pi).* ...
  integral(f, 0, Inf, 'AbsTol', 1e-16);
end

function f = func(u, k, alpha, lnS, r, T, theta, ...
    sigma, nu)
  cf = exp(- 1i*u*k).*VarianceGamma(u-(alpha+1)*1i, ...
      lnS, T, r, sigma, nu, theta);
  f = cf ./(alpha .^ 2 + alpha - u .^ 2 + 1i .* ...
      (2 * alpha + 1) .* u);
end
```