Mean Value Theorem

Riku Yamada

I am going to rigorously prove the mean value theorem (MVT). Then, I will prove the Cauchy's mean value theorem. The claim of the theorem is as following.

Theorem 1

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists \varepsilon \in (a, b)$ s.t.

$$f'(\varepsilon) = \frac{f(b) - f(a)}{b - a}$$

I will first prove the Rolle's theorem before proving the mean value theorem.

Rolle's theorem

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then $\exists x \in (a, b)$ s.t.

$$f'(x) = 0$$

Proof (Rolle's theorem):

Since f is continuous on $[a, b]$, f has a maximum and minimum value on $[a, b]$.
Suppose first that the maximum value occurs at point x in (a, b).
Then,

$$f'(x) = 0.$$

Suppose next that the minimum value occurs at point x in (a, b).
Then,

$$f'(x) = 0.$$

Finally, suppose minimum and maximum values occur at the end points.
Then, since $f(a) = f(b)$, f is constant on $[a, b]$.

$$\therefore f'(x) = 0 \quad \forall x \in (a, b) \quad \blacksquare$$

Next, I will prove the mean value theorem.

Proof (Mean Value Theorem):

Let $g(x) = f(x) - (\frac{f(b)-f(a)}{b-a}(x - a) + f(a))$
Then,

$$g(a) = 0, \quad g(b) = 0$$

Here, g is continuous on $[a, b]$ and differentiable in (a, b), and $g(a) = g(b)$.
So by Rolle's theorem, $\exists x \in (a, b)$ s.t. $g'(x) = 0$

$$\therefore f'(x) = \frac{f(b) - f(a)}{b - a} \quad \blacksquare$$

I will then introduce and prove the Cauchy's mean value theorem. This theorem is a more generalized version of the mean value theorem. The claim of this theorem is as following.

Cauchy's Mean Value Theorem

If f, g are continuous on $[a, b]$ and differentiable on (a, b), then $\exists x \in (a, b)$ s.t.

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$$

proof:

Let $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$.
Then,

$$h(a) = f(b)g(a) - g(b)f(a)$$
$$h(b) = -f(a)g(b) + g(a)f(b).$$

Since f, g are continuous on $[a, b]$ and differentiable in (a, b), h is also continuous on $[a, b]$ and differentiable in (a, b).
$\therefore \exists x \in (a, b)$ s.t. $h(x) = 0$

$$\therefore h'(x) = (f(b) - f(a))g'(x) - (g(b) - f(a))f'(x) = 0$$
$$\therefore (f(b) - f(a))g'(x) = (g(b) - g(a))f'(x) \quad \blacksquare$$

Note: You get MVT if you set $g(x) = x$.