Intermediate Value Theorem

I am going to rigorously prove the intermediate value theorem (IVT) by using the existence of supremum. The claim of the theorem is as following.

<u>Theorem 1</u>

If f is continuous on $[a, b]$, and $f(a) < 0 < f(b)$, then $\exists x \in (a, b) \ni f(x) = 0$

<u>Proof</u>:
Let $A = [x \in [a, b] : f(y) < 0, \forall y \in [a, x]]$
Let C be the smallest element in $[a, b]$ which is larger than every element of A.

Claim: $f(C) = 0$
1) $C \neq a$, $f(a) < 0$
f is continuous at a, so

$$\lim_{x \to a^+} f(x) = f(a)$$

$\therefore \exists \delta > 0 \ni 0 \leq x - a < \delta$, then

$$|f(x) - f(a)| < -\frac{f(a)}{2}$$

Then, $f(x) - f(a) < -\frac{f(a)}{2}$, so

$$f(x) < \frac{f(a)}{2} < 0$$

$\therefore$ For all $x \in [a, a + \delta]$,

$$f(x) < 0$$

So all points in $[a, \delta]$ are in A.
$\therefore C$ is larger than all of these, so $C > a$.

2) $C \neq b$, $\exists \delta_2 > 0 \ni$ if $b - \delta_2 < x \leq b$, then

$$|f(x) - f(b)| < \frac{f(b)}{2}$$

Then, we get $-\frac{f(b)}{2} < f(x) - f(b) < \frac{f(b)}{2}$, so

$$f(x) > \frac{f(b)}{2} > 0$$

for all $x \in (b - \delta_2, b]$
So none of these xs are in A.
$\therefore$ all of the xs $\geq$ everything in A.
Since C was the smallest in $[a, b]$ which was larger than everything in A.
$\therefore C < x$ for all $x \in (b - \delta_2]$.
$\therefore C \neq b$.

Suppose $f(C) < 0$.

Since $f(x)$ is continuous at C,
$\exists \delta_3 > 0 \ni \forall x \in (C - \delta, C + \delta)$,

$$|f(x) - f(C)| < -\frac{f(C)}{2}$$

Then,

$$f(x) - f(C) < -\frac{f(C)}{2}$$

So we get

$$f(x) < \frac{f(C)}{2} < 0$$

Pick $s = C + \frac{\delta}{2}$, then

$$f(y) < 0, \forall y \in [a, s]$$

$$\therefore s \in A.$$

$$\therefore C \geq s = C + \frac{\delta}{2}.$$

This is contraction.

$$\therefore f(C) \geq 0.$$

Then, suppose $f(C) > 0$.

Since $f(x)$ is continuous at C,
$\exists \delta_4 > 0 \ni \forall x \in (C - \delta, C + \delta)$,

$$|f(x) - f(C)| < \frac{f(C)}{2}$$

Then,

$$\frac{f(C)}{2} < f(x) < \frac{3f(C)}{2}$$

Hence, we get

$$0 < \frac{f(C)}{2} < f(x)$$

Then, pick $s' = C - \frac{\delta}{2}$ and we have

$$f(y) > 0, \forall y \in [s', b]$$

$\therefore s'$ is bigger than every element in A.

$$\therefore C \leq s' = C - \frac{\delta}{2}$$

This is contradiction.

$$\therefore f(C) \leq 0$$

Hence,

$$f(C) = 0 \quad \blacksquare$$

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