Approximation by Polynomial Functions

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I am going to introduce Taylor series which can be used to approximate various types of functions.

Taylor polynomials

Let

$$a_k = \frac{f^k(a)}{k!}, \quad 0 \leq k \leq n,$$

and define

$$P_{n,a,f}(x) = a_0 + a_1(x-1) + \cdots + a_n(x-a)^n.$$

The polynomial $P_{n,a,f}$ is called the Taylor polynomial of degree n for f at a.

Theorem 1

Suppose that f is a function for which

$$f'(a), ..., f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \leq k \leq n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n.$$

Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0.$$

proof:

Writing out $P_{n,a}(x)$ explicitly, we obtain

$$\frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \frac{f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n} - \frac{f^{(n)}(a)}{n!}.$$

Now, it will help to introduce the new functions

$$Q(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i \quad \textit{and} \quad g(x) = (x-a)^n;$$

now we need to show that

$$\lim_{x \to a} \frac{f(x) - Q(x)}{g(x)} = \frac{f^{(n)}(a)}{n!}.$$