Calculus5 曲線・曲面 Curved line and surface

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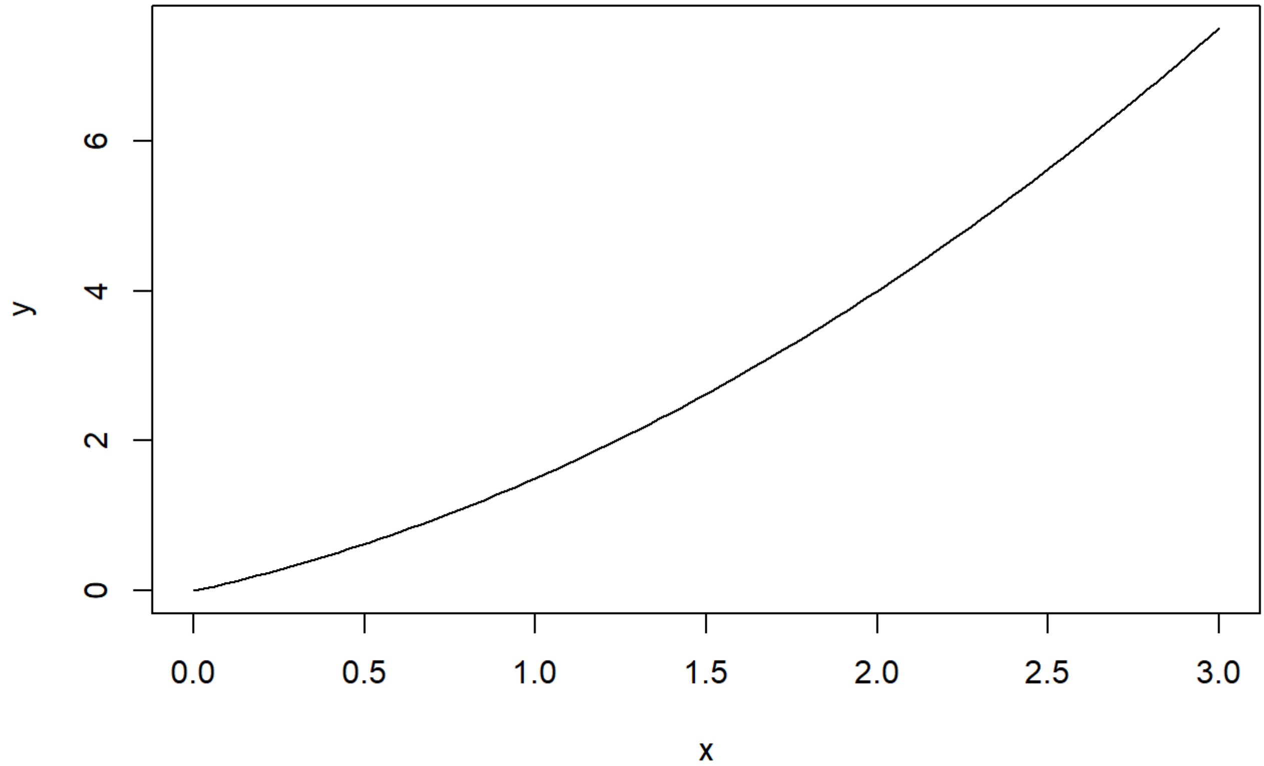
1 曲線の長さ Length of curve

$$y = f(x) = x + \frac{1}{2}x^2$$

```
x <- seq(from=0,to=3,length=100)
y <- x+ 1/2*x^2
plot(x,y,type="l")
```



$$\frac{d}{dx}f(x) = 1 + x$$

微分すると、単位x増分当たりのyの増加量がわかる。それらは、微小直角三角形。

Differentiation gives increase of y along with unit increase in x. They are two edges of a right triangle.

三角形 (x,y) ,(x+dx,y),(x+dx,y+dy)は直角三角形の3頂点。 曲線の長さはこの三角形の斜辺の積分

(x,y) ,(x+dx,y),(x+dx,y+dy) are three vertices of a right triangle. The length of curve is the integral of the obliques.

$$\int_0^3 \sqrt{dx^2 + dy^2} = \int_0^3 \sqrt{(\frac{dx}{dx})^2 + (\frac{dy}{dx})^2}dx = \sqrt{1 + (\frac{dy}{dx})^2}dx$$

積分できるのか？ Can we integrate this?

$$\int_0^3 \sqrt{2 + 2x + x^2}dx$$
$$= [\sqrt{2 + 2x + x^2}(x+1) + \sinh^{-1}(x+1)]_0^3$$

Mathematicaを使えば…。

Check Mathematica http://www.wolframalpha.com/examples/Math.html (http://www.wolframalpha.com/examples/Math.html) for "Calculus AND ANALYSIS" and http://d.hatena.ne.jp/ryamada22/20170128 (http://d.hatena.ne.jp/ryamada22/20170128)

積分できるかできないかわからなくても近似計算はできる。

Even when the integral is not easy shape, computational calculation is possible.

```
integrate(function(x){(2+2*x+x^2)^(1/2)},0,3)
```

```
## 8.145774 with absolute error < 9e-14
```

2 Exercise 1

2.1 Exercise 1-1

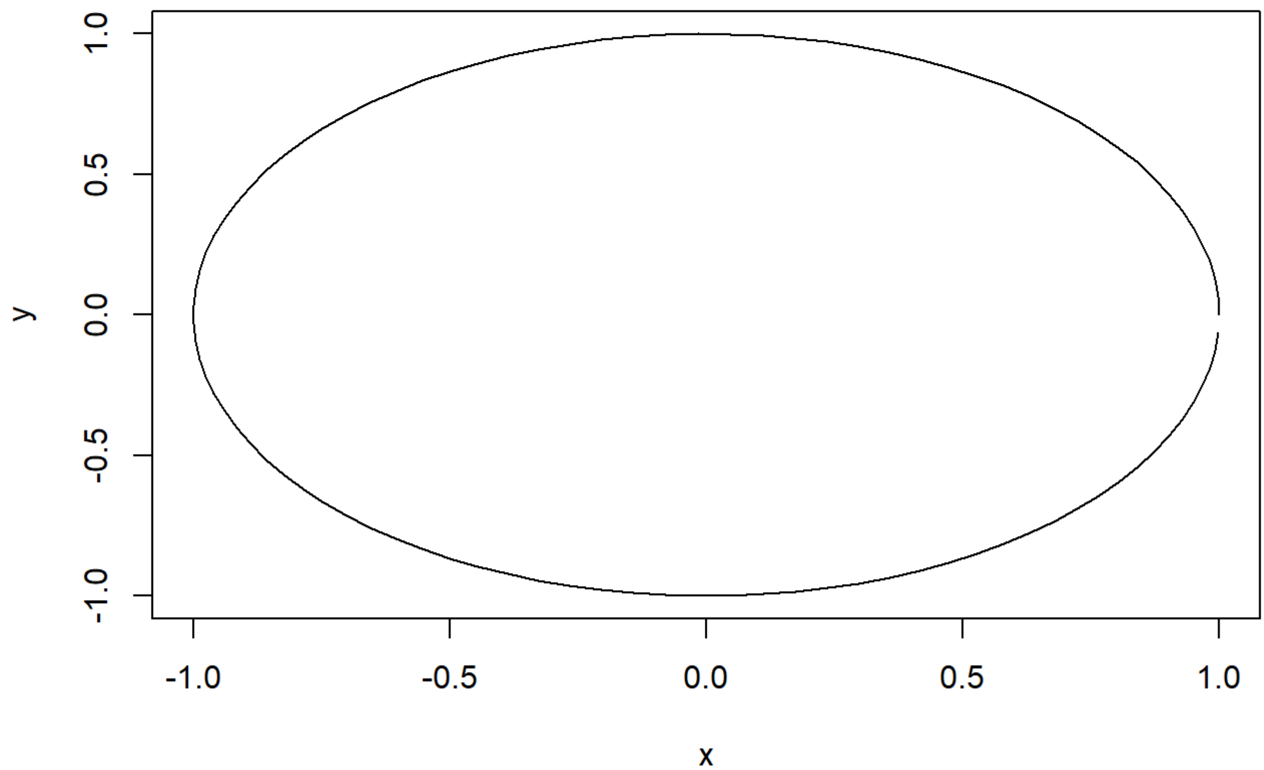
\$ y = f(x) = e^{x^2}\$ の $x \in [-1,1]$の長さを計算せよ。

Calculate the length of \$ y = f(x) = e^{x^2}\$ for $x \in [-1,1]$.

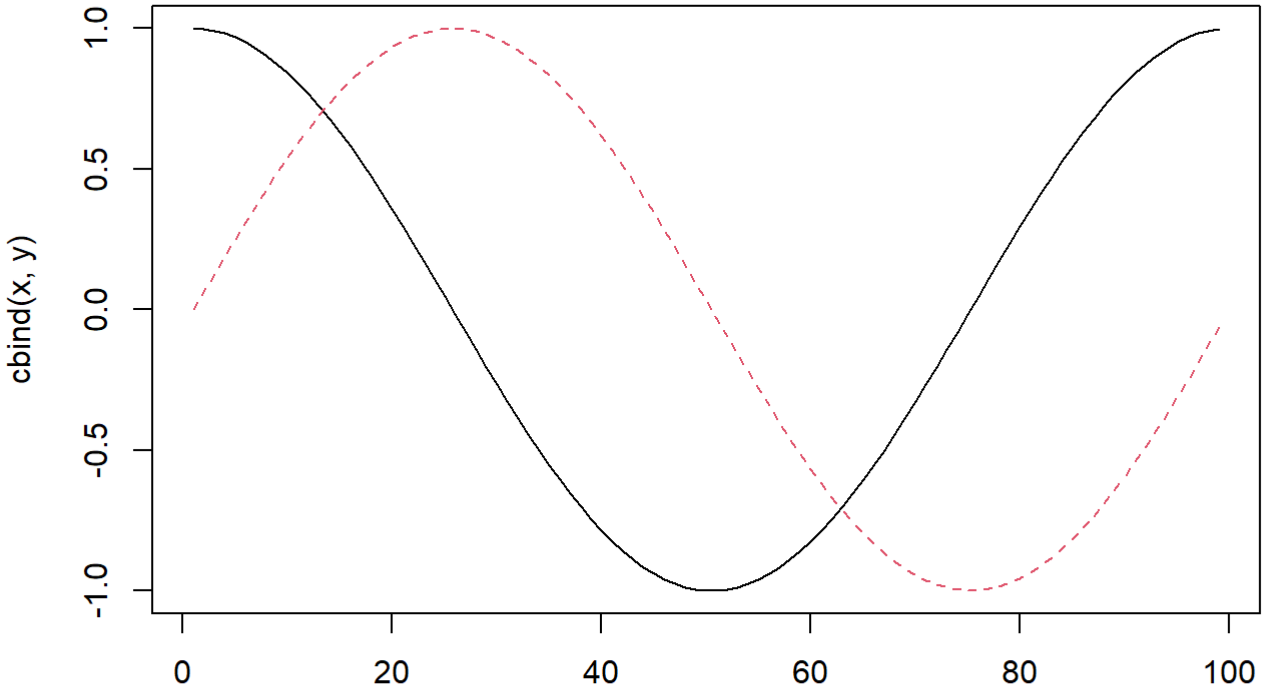
3 曲線のパラメタ表示 Parameteric expression of curve.

$$x = \cos t$$
$$y = \sin t$$
$$t \in [0, 2\pi)$$

```
t <- seq(from=0,to=1,length=100)*2*pi
t <- t[-length(t)]
x <- cos(t)
y <- sin(t)
plot(x,y,type="l")
```



```
matplot(cbind(x,y),type="l")
```



$$L(\tau) = \int_0^\tau \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$L(\tau) = \int_0^\tau \sqrt{(\sin t)^2 + (\cos t)^2} dt$$

$$= \int_0^\tau 1 dt$$

$$= [t]_0^\tau = \tau$$

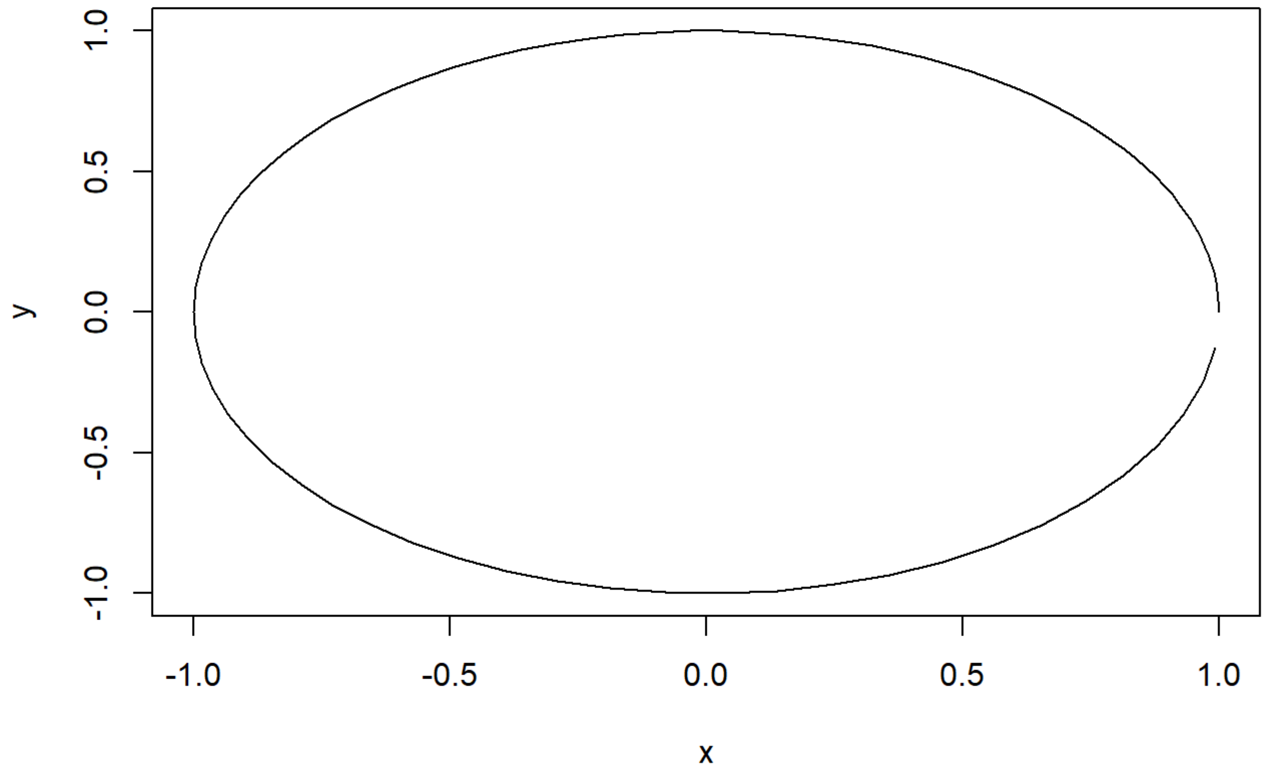
3.1 同じ軌道、スピードが違う Same trajectory but different speed

$$x = \cos t^2$$

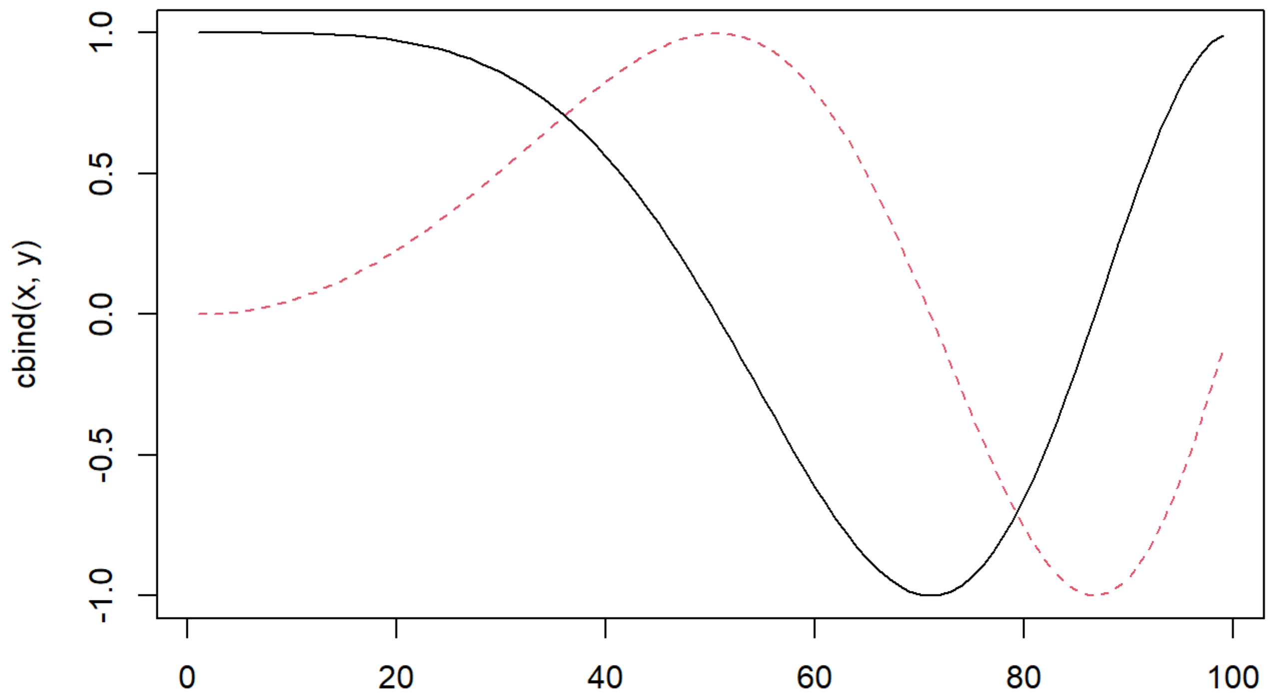
$$y = \sin t^2$$

$$t \in [0, \sqrt{2\pi})$$

```
t <- seq(from=0,to=1,length=100)*sqrt(2*pi)
t <- t[-length(t)]
x <- cos(t^2)
y <- sin(t^2)
plot(x,y,type="l")
```




```
matplot(cbind(x,y),type="l")
```



$$L(\tau) = \int_0^\tau \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$
$$= \int_0^\tau \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} dt$$
$$= \int_0^\tau 2t\sqrt{\sin^2(t^2) + \cos^2(t^2)} dt$$
$$= \int_0^\tau 2t dt$$
$$= [t^2]_0^\tau$$
$$= \tau^2$$

4 Exercise 2

4.1 Exercise 2-1

曲線

$$x = \cos t$$
$$y = \sin t$$
$$t \in [0, 2\pi)$$

を描き、その曲線の上に適当に多数の点を取り、その点での進行方向ベクトル$(\frac{dx}{dt}, \frac{dy}{dt})$を描け。

Draw the curve and put its speed vectors $(\frac{dx}{dt}, \frac{dy}{dt})$ on many points on the curve.

4.2 Exercise 2-2

$$x = \cos t^2$$
$$y = \sin t^2$$
$$t \in [0, \sqrt{2\pi})$$

について同様のことをせよ。

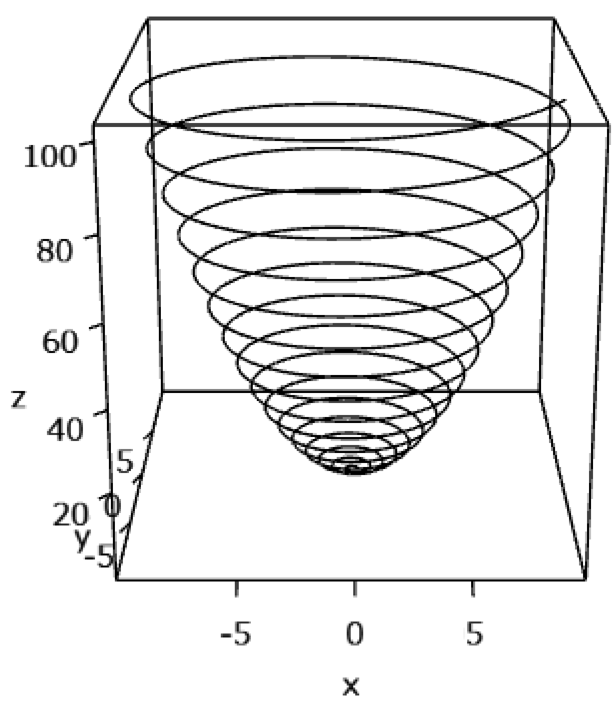
Do the same for this parameterization.

4.3 Exercise 2-3

3次元空間の曲線

$$x = r\cos t$$
$$y = r\sin t$$
$$z = r^2$$
$$r = 0.1t + 0.1$$
$$t \in [0, 100]$$

```
t <- seq(from=0,to=1,length=1000)*100
r <- 0.1*t+0.1
x <- r*cos(t)
y <- r*sin(t)
z <- r^2
plot3d(x,y,z,type="l")
```



$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$を計算し、曲線の長さとしてその積分をRのintegrate()関数を使って計算せよ。

This is a curve in 3D space. Calculate $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ and calculate the length of curve with R's integrate() function.

また、曲線の長さを折れ線の長さの集まりとして

```
t <- seq(from=0,to=1,length=n)*100
```

$n = 10, 100, 1000, 10000, 100000$について計算せよ。

For multiple n values, calculate the length of broken lines that approximate the curve.

5 単位円の面積 Area of a unit circle.

単位円の面積は、The following is the 1/4 of unit circle's area.

$$y = \sqrt{(1-x^2)}$$
$$y = 0$$
$$x \in [0,1]$$

の4倍として求めることができる。

$$4\int_0^1 \sqrt{(1-x^2)}dx$$

この積分を解くのは難しい。 It's no easy to solve this.

```
integrate(function(x){4*sqrt(1-x^2)},0,1)
```

```
## 3.141593 with absolute error < 0.00016
```

微小底辺dlを円周上に持ち、高さが半径1の三角形の和として計算することもできる。

中心からの半径方向を$\theta \in [0, 2\pi)$とすれば

The area of circle is the sum of small triangles whose base lines are small arcs dl of the circle and whose hight are the radius. The directions of the triangles are parameterized as $\theta \in [0, 2\pi)$.

$$\int_0^{2\pi} \frac{1}{2} \times dl$$
$$dl = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta = d\theta$$

これは、円周の長さ2πを底辺の長さとする三角形の面積。

It corresponds to the area of triangle whose base is a whole circle.

別の方法は円を半径が0から1に変化する円のあつまりとみなすものである。

Another way is to consider the area of circle as the sum of cicles with radius ranging from 0 to 1.

半径rでの微小面積は$2\pi r \times dr$.

The small area attached to the circle with radius r is $2\pi r \times dr$.

$$\int_0^1 2\pi r dr = 2\pi \int_0^1 r dr = 2\pi[\frac{1}{2}r^2]_0^1 = \pi$$

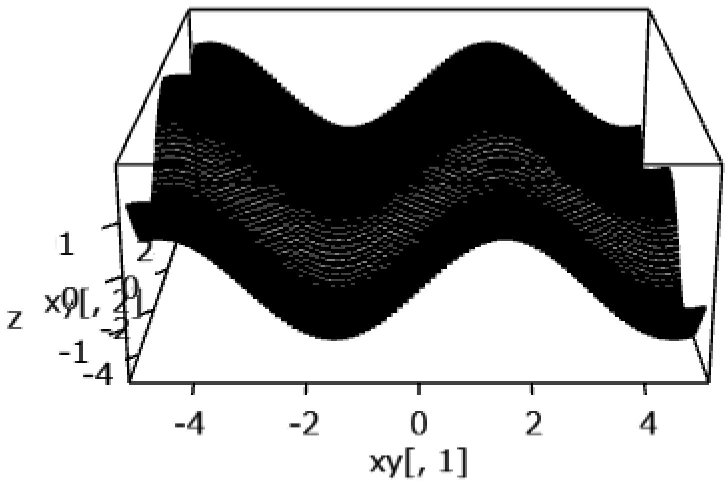
曲面 Curved surface

以下のような曲面の面積はどうなるか？

How do we approach to the area of the curved surface as below?

$$z = \sin x + \cos y$$
$$x, y \in [-5, 5]$$

```
x <- y <- seq(from=-1,to=1,length=100) * 5
xy <- as.matrix(expand.grid(x,y))
z <- sin(xy[,1])+cos(xy[,2])
xlim <- max(xy[,1])-min(xy[,1])
ylim <- max(xy[,2])-min(xy[,2])
zlim <- max(z)-min(z)
plot3d(xy[,1],xy[,2],z,aspect=c(xlim,ylim,zlim))
```



(x,y),(x+dx,y),(x,y+dy),(x+dx,x+dy)が作る「斜面」の面積dSを足し合わせる。

Small areas that are slopes over (x,y),(x+dx,y),(x,y+dy),(x+dx,x+dy), dS, should be summed up.

これを斜めになった平行四辺形の面積とみなす。 They are considered small parallelograms.

平行四辺形の第1の辺は(x,y,z)と$(x+dx,y,z+\frac{\partial x}{\partial z}dx)$とが作るベクトル。 第2の辺は$(x,y,z)$と$(x,y+dy+z+\frac{\partial y}{\partial z}dy)$。

The parallelogram's two edges are; one connecting (x, y, z)と$(x + dx, y, z + \frac{\partial x}{\partial z} dx)$ and and the other connecting (x, y, z) and $(x, y + dy + z + \frac{\partial y}{\partial z} dy)$.

2つのベクトルは They are:

$$v_1 = (dx, 0, \frac{\partial z}{\partial x} dx)$$
$$v_2 = (0, dy, \frac{\partial z}{\partial y} dy)$$

2つのベクトルv_1, v_2が作る平行四辺形の面積は The area of parallelogram is;

$$dS = ||v_1||v_2| \sin \theta|$$

$$\cos \theta = \frac{(v_1, v_2)}{|v_1||v_2|}$$

ただし、(v_1, v_2)は内積。 where (v_1, v_2) is inner product.

したがって Therefore,

$$dS = ||v_1||v_2| \sqrt{1 - (\frac{(v_1, v_2)}{|v_1||v_2|})^2}|$$
$$= \sqrt{|v_1|^2 |v_2|^2 - (v_1, v_2)^2}$$

ここで Using

$$|v_1|^2 = (1 + (\frac{\partial z}{\partial x})^2 dx^2$$
$$|v_2|^2 = (1 + (\frac{\partial z}{\partial y})^2 dy^2$$
$$(v_1, v_2) = (\frac{\partial z}{\partial x} \frac{\partial z}{\partial y})^2 dxdy$$

以上より、Then,

$$dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dxdy$$

5.1 Exercise 3

5.1.1 Exercise 3-1

2つの式を比較せよ。

Compare two formulas.

$$\int_0^3 \sqrt{dx^2 + dy^2} = \int_0^3 \sqrt{(\frac{dx}{dx})^2 + (\frac{dy}{dx})^2} dx = \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dxdy$$

5.2 曲面の前に、平面 Before using this to curved surface, apply it to the flat one.

$$z = ax + by + c$$

$$\frac{\partial z}{\partial x} = a$$
$$\frac{\partial z}{\partial y} = b$$
$$dS = \sqrt{1+a^2+b^2}dxdy$$
$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} dSdxdy = \sqrt{1+a^2+b^2} \times (x_1 - x_0)(y_1 - y_0)$$

単なる、平行四辺形の面積。

Simple parallelogram's area.

5.3 単位球面積 Surface area of unit sphere.

$$x^2 + y^2 + z^2 = 1$$
$$z = \pm\sqrt{1-(x^2+y^2)}$$

$ z= $, $0 \le x^2 + y^2 \le 1; x, y \ge 0$を扱うことにする。

Calculate a part of the sphere of $ z= $, $0 \le x^2 + y^2 \le 1; x, y \ge 0$.

$$dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dxdy$$
$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{-2x}{\sqrt{1-(x^2+y^2)}}$$
$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{-2y}{\sqrt{1-(x^2+y^2)}}$$

$$dS = \sqrt{1 + \frac{x^2}{1-(x^2+y^2)} + \frac{y^2}{1-(x^2+y^2)}} dxdy$$
$$= \sqrt{\frac{1}{1-(x^2+y^2)}} dxdy$$

$$S = \int_{x^2+y^2 \le 1} dS = \int_{x^2+y^2 \le 1} \sqrt{\frac{1}{1-(x^2+y^2)}} dxdy$$

これは解けないが、うまくパラメタ表示を変えることで解けるようにできる。

This is not easy but converting to different parameterization, then it can be.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$

$(r,r),((r+dr),(r+dr)),(r,r),((r+dr),(r+dr))が作る微小面積は

Small area should be changed accordingly.

$$dr \times (r \times d\theta)$$

したがって$dxdy$に相当するのは$rdrd\theta$

Means $dxdy$ is converted to $rdrd\theta$.

ゆえに

$$S = \int_{x^2+y^2\le 1}\sqrt{\frac{1}{1-(x^2+y^2)}}dxdy$$
$$= \int_{0\le r\le 1, 0\le\theta\le 2\pi}\sqrt{\frac{1}{1-r^2(\cos^2\theta+\sin^2\theta)}}rdrd\theta$$
$$= \int_0^{2\pi} 1 d\theta \int_0^1 r\sqrt{\frac{1}{1-r^2}}dr$$
$$\frac{d}{dr}\sqrt{1-r^2} = \frac{1}{2}(-2r)\sqrt{\frac{1}{1-r^2}}$$
$$= -r\sqrt{\frac{1}{1-r^2}}$$

したがって

$$S = 2\pi$$

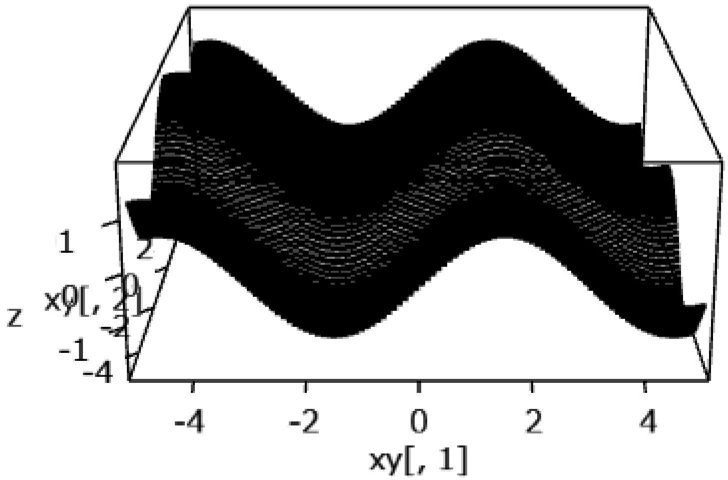
これは球の上半分であるから、球面全体の表面積は、4π.

This is the upper half's area, therefore the are of unit sphere is 4π.

5.4 曲面 Curverd surface

$$z = \sin x + \cos y$$
$$x, y \in [-5, 5]$$

```
x <- y <- seq(from=-1,to=1,length=100) * 5
xy <- as.matrix(expand.grid(x,y))
z <- sin(xy[,1])+cos(xy[,2])
xlim <- max(xy[,1])-min(xy[,1])
ylim <- max(xy[,2])-min(xy[,2])
zlim <- max(z)-min(z)
plot3d(xy[,1],xy[,2],z,aspect=c(xlim,ylim,zlim))
```



$$\frac{\partial z}{\partial x} = -\sin x$$
$$\frac{\partial z}{\partial y} = \cos y$$

2つのベクトルは　The vecors are;

$$(dx, 0, -\sin x dx)$$
$$(0, dy, \cos y dy)$$

したがって Then

$$dS = ||v_1||v_2|\sqrt{1 - (\frac{(v_1, v_2)}{|v_1||v_2|})^2}|$$
$$= \sqrt{|v_1|^2 |v_2|^2 - (v_1, v_2)^2}$$

ここで Using

$$|v_1|^2 = (1 + \sin^2 x) dx^2$$
$$|v_2|^2 = (1 + \cos^2 y) dy^2$$
$$(v_1, v_2) = -\sin x \cos y dx dy$$

以上より、Therefere

$$dS = \sqrt{(1 + \sin^2 x)(1 + \cos^2 y) - \sin^2 x \cos^2 y} dx dy = \sqrt{1 + (\sin^2 x + \cos^2 y)} dx dy$$

解けないけれど、計算機的には計算できる。 Computationally calculable.

5.5 Exercise 4

5.6 Exercise 4-1

Mathematicaのサイトを使ってこの面積を計算せよ。

Calculate the area using Mathematica's site.