Systems of Linear Equations with Matrix 行列で連立方程式

ryamada

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1 変数の数と等式の数 Number of variables and number of equations

変数の数と等式の数が等しいとき(一次独立なら)解が求まる。

When the number of variables and the number of equations are the same, the mathematical solution of the system of equations is specifically obtained (if linearly independent).

それは、線形回帰において、Xが正方行列である、ということと、 Xに逆行列が存在するということに対応し、 解が求まるとは、$\sum_{i=1}^n (y_i - \hat{y_i})2 = 0$になるということ。

It is when X is a square matrix. it also means X^{-1} exists and it also means $\sum_{i=1}^n (y_i - \hat{y_i})2 = 0$.

$$y = X\mathbf{a}$$
$$\mathbf{a} = X^{-1}y$$

```
d <- 10
n <- 10
X <- matrix(rnorm(d*n),ncol=d)
a <- rnorm(d)
a
```

```
## [1] 0.9421766 1.4348187 0.5392254 -0.2558113 -0.1344681 1.3219515
## [7] -0.3662703 1.7935016 -0.7348274 -0.5735528
```

```
y <- X %*% a
a.est <- solve(X) %*% y
a.est
```

```
##           [,1]
##  [1,]  0.9421766
##  [2,]  1.4348187
##  [3,]  0.5392254
##  [4,] -0.2558113
##  [5,] -0.1344681
##  [6,]  1.3219515
##  [7,] -0.3662703
##  [8,]  1.7935016
##  [9,] -0.7348274
## [10,] -0.5735528
```

The following calculation was done in linear regression.

線形回帰では以下の計算をした。

$$a = (X^T X)^{-1} X^T y$$

Let's do it, too.

やってみる。

```
a.est2 <- solve(t(X)%*%X)%*%t(X)%*%y
a.est
```

```
##           [,1]
##  [1,]  0.9421766
##  [2,]  1.4348187
##  [3,]  0.5392254
##  [4,] -0.2558113
##  [5,] -0.1344681
##  [6,]  1.3219515
##  [7,] -0.3662703
##  [8,]  1.7935016
##  [9,] -0.7348274
## [10,] -0.5735528
```

```
a
```

```
##  [1]  0.9421766  1.4348187  0.5392254 -0.2558113 -0.1344681  1.3219515
##  [7] -0.3662703  1.7935016 -0.7348274 -0.5735528
```

確かに結果は同じ。

残差も0になっている。

The two outputs are the same.

No residuals.

```
y.hat <- X %*% a.est
y-y.hat
```

```
##              [,1]
##  [1,]  2.664535e-15
##  [2,] -2.442491e-15
##  [3,] -4.662937e-15
##  [4,]  3.108624e-15
##  [5,]  1.110223e-16
##  [6,]  5.329071e-15
##  [7,]  6.439294e-15
##  [8,]  8.881784e-16
##  [9,]  2.664535e-15
## [10,]  1.332268e-15
```

```
sum((y-y.hat)^2)
```

```
## [1] 1.240114e-28
```

つまり、

In other words,

$$X^{-1} = (X^T X)^{-1} X^T$$

なわけである。

Double check it with R.

計算してみる。

```
X1 <- solve(X)
X2 <- solve(t(X)%*%X) %*% t(X)
round(X1,8)
```

```
##             [,1]        [,2]       [,3]       [,4]        [,5]        [,6]
##  [1,]  0.21142499  1.50985712 -2.1598397  1.0527930 -0.46643720  0.11543315
##  [2,]  0.76116385 -1.48873213  2.7336181 -0.5334814  0.38768663 -0.49645710
##  [3,]  0.70851629 -1.39513665  3.4543647 -0.4929913  0.13993402 -0.49979332
##  [4,] -0.88762765  2.13623233 -4.1645534  0.7171552 -0.45373187  0.81185410
##  [5,]  0.20472275 -0.39121670  1.0377338 -0.2883462  0.31296973 -0.15263636
##  [6,] -0.23002019  0.44223011 -0.8518778  0.0742593  0.05103471  0.39200194
##  [7,]  0.00937429 -0.05125614  0.1022624  0.2213403 -0.04202622 -0.03558288
##  [8,] -0.53318960 -1.41362265  1.2567550 -1.4718635  0.05899155 -0.12198128
##  [9,] -0.15344496  0.12903434 -0.3682408 -0.2618192  0.08333357  0.07725073
## [10,] -1.06772578  2.17434021 -4.2622222  0.7320476 -0.27695195  0.67866583
##             [,7]        [,8]        [,9]       [,10]
##  [1,] -1.87693634 -0.52238378 -0.10487325  0.63042971
##  [2,]  1.37590334  0.98377961  0.50029863 -0.10560209
##  [3,]  2.05724231  1.20378257  0.48935452  0.16988925
##  [4,] -2.85353918 -1.37734338 -0.61724163  0.07520244
##  [5,]  0.72421862  0.10019740  0.27008146 -0.14937520
##  [6,] -0.37638705 -0.43471530 -0.26925697 -0.08900303
##  [7,]  0.01976739 -0.15353237  0.01847788 -0.13230975
##  [8,]  1.58322703 -0.00785856 -0.24362761 -0.51163353
##  [9,]  0.07636913 -0.15423770 -0.11063189 -0.51526612
## [10,] -2.68286003 -1.45211946 -0.84223872  0.13139762
```

```
round(X2,8)
```

```
##              [,1]        [,2]       [,3]       [,4]        [,5]        [,6]
##  [1,]  0.21142499  1.50985712 -2.1598397  1.0527930 -0.46643720  0.11543315
##  [2,]  0.76116385 -1.48873213  2.7336181 -0.5334814  0.38768663 -0.49645710
##  [3,]  0.70851629 -1.39513665  3.4543647 -0.4929913  0.13993402 -0.49979332
##  [4,] -0.88762765  2.13623233 -4.1645534  0.7171552 -0.45373187  0.81185410
##  [5,]  0.20472275 -0.39121670  1.0377338 -0.2883462  0.31296973 -0.15263636
##  [6,] -0.23002019  0.44223011 -0.8518778  0.0742593  0.05103471  0.39200194
##  [7,]  0.00937429 -0.05125614  0.1022624  0.2213403 -0.04202622 -0.03558288
##  [8,] -0.53318960 -1.41362265  1.2567550 -1.4718635  0.05899155 -0.12198128
##  [9,] -0.15344496  0.12903434 -0.3682408 -0.2618192  0.08333357  0.07725073
## [10,] -1.06772578  2.17434021 -4.2622222  0.7320476 -0.27695195  0.67866583
##              [,7]        [,8]        [,9]       [,10]
##  [1,] -1.87693634 -0.52238378 -0.10487325  0.63042971
##  [2,]  1.37590334  0.98377961  0.50029863 -0.10560209
##  [3,]  2.05724231  1.20378257  0.48935452  0.16988925
##  [4,] -2.85353918 -1.37734338 -0.61724163  0.07520244
##  [5,]  0.72421862  0.10019740  0.27008146 -0.14937520
##  [6,] -0.37638705 -0.43471530 -0.26925697 -0.08900303
##  [7,]  0.01976739 -0.15353237  0.01847788 -0.13230975
##  [8,]  1.58322703 -0.00785856 -0.24362761 -0.51163353
##  [9,]  0.07636913 -0.15423770 -0.11063189 -0.51526612
## [10,] -2.68286003 -1.45211946 -0.84223872  0.13139762
```

```
round(X1-X2,8)
```

```
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
##  [1,]    0    0    0    0    0    0    0    0    0     0
##  [2,]    0    0    0    0    0    0    0    0    0     0
##  [3,]    0    0    0    0    0    0    0    0    0     0
##  [4,]    0    0    0    0    0    0    0    0    0     0
##  [5,]    0    0    0    0    0    0    0    0    0     0
##  [6,]    0    0    0    0    0    0    0    0    0     0
##  [7,]    0    0    0    0    0    0    0    0    0     0
##  [8,]    0    0    0    0    0    0    0    0    0     0
##  [9,]    0    0    0    0    0    0    0    0    0     0
## [10,]    0    0    0    0    0    0    0    0    0     0
```

確かにそうなっている。

もう一度式を見てみる。

Take a second look at the formula.

$$X^{-1} = (X^T X)^{-1} X^T$$

両辺に右から$(X^T)^{-1}$を掛けて

multiplying $(X^T)^{-1}$ from right on both sides,

$$X^{-1}(X^T)^{-1} = (X^T X)^{-1}$$

となる。

これは逆行列の一般的な性質

This is a basic rule of matrix multiplication and inverse.

$$(AB)^{-1} = B^{-1}A^{-1}$$

について、$A = X^T$,$B = X$と置いたものである。

You can show $(AB)^{-1}$ is $B^{-1}A^{-1}$ as below.

ちなみに、ABの逆行列$(AB)^{-1}$が$B^{-1}A^{-1}$であることは

$$ABB^{-1}A^{-1}$$

を二通りでかっこで区切ることで解る。

$$ABB^{-1}A^{-1} = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

右辺は、

$$A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

これから中辺が

$$(AB)(B^{-1}A^{-1}) = I$$

となるから、この式の意味することはABの逆行列が$B^{-1}A^{-1}$であることである。

2 Exercise 1

2.1 Excercise 1-1

連立方程式

A system of equation is solvable as if it is linear regression as below.

$$\begin{gathered}3a_1 + 2a_2 - 4a_3 = 4 \\ 2a_1 - 6a_2 + 3a_3 = -1 \\ 5a_1 + a_2 + 4a_3 = 3\end{gathered}$$

は

$$\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3, 2, -4 \\ 2, -6, 3 \\ 5, 1, 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$y = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

$$X = \begin{pmatrix} 3, 2, -4 \\ 2, -6, 3 \\ 5, 1, 4 \end{pmatrix}$$

と置けば

$$y = X\mathbf{a}$$

となり、線形回帰の形である。

これを利用して、上の連立方程式を解け。

Solve the system using linear regression function.

2.2 Exercise 1-2

変数の数と等式の数が等しい連立方程式を作って、連立方程式として書き、それを行列を使って解け。

Make a system of equations where the number of variables and the number of equations and write the system and solve it with linear algebra.