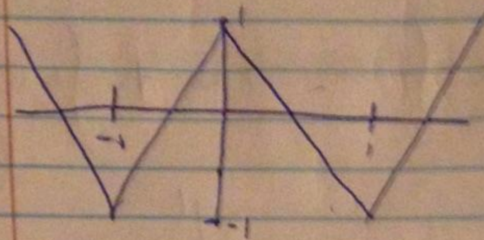
Fourier Serier Derivation – Ryan Brown



$$f(x) = a_0/2 + \sum_{K=1}^{\infty} a_K \cos\left(\frac{K\pi x}{L}\right) + \sum_{K=1}^{\infty} b_K \sin\left(\frac{K\pi x}{L}\right)$$

$L=1$ & $b_K = 0$ since f is even.

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 2x+1\,dx + \int_{0}^{1} -2x+1\,dx$$

$$= (x^2+x)\Big|_{-1}^{0} + (-x^2+x)\Big|_{0}^{1}$$

$$= 0-(1-1)=0 \quad + \quad (-1+1)-0 \quad =0$$

$a_0 = 0.$

$$a_K = \int_{-1}^{1} f(x)\cos(K\pi x)\,dx = \int_{-1}^{0} (2x+1)\cos(K\pi x)\,dx + \int_{0}^{1} (2x-1)\cos(K\pi x)\,dx$$

$$= \int_{-1}^{0} 2x\cos(K\pi x)\,dx + \int_{-1}^{0} \cos(K\pi x)\,dx$$

$$= \left(\frac{2x\sin(K\pi x)}{K\pi} + \frac{2\cos(K\pi x)}{K^2\pi^2}\right)\Big|_{-1}^{0} + \left(\frac{\sin(K\pi x)}{K\pi}\right)\Big|_{-1}^{0}$$

$$= \left(\frac{2}{K^2\pi^2} - \frac{2\cos(K\pi)}{K^2\pi^2}\right) + 0$$

$$= \frac{2-2(-1)^K}{K^2\pi^2} \;:\; \frac{4}{K^2\pi^2}, K \text{ odd}$$

$$\int_0^1 (2x+1)\cos(k\pi x)\,dx = -\int_0^1 2x\cos(k\pi x)\,dx + \cancel{\int_0^1 \cos(k\pi x)\,dx}\ 0$$

$$= -\left(\frac{2x\sin(k\pi x)}{k\pi} + \frac{2\cos(k\pi x)}{k^2\pi^2}\right)\Bigg|_0^1$$

$$-\left(\frac{2\cos(k\pi)}{k^2\pi^2} - \frac{2}{k^2\pi^2}\right) = -\left(\frac{2(-1)^k}{k^2\pi^2} - \frac{2}{k^2\pi^2}\right)$$

$$-\left(\frac{-4}{k^2\pi^2}\right) = \frac{4}{k^2\pi^2}, \quad k \text{ odd}$$

$$\Rightarrow a_k = \frac{8}{k^2\pi^2}, \quad k \text{ odd}$$

$$\Rightarrow \boxed{f(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(k\pi x)}{k^2}, \quad k \text{ odd}}$$