Exercise 1.53 In the 1991 Gulf War, the Patriot missle defense system failed due to roundoff error. The troubles stemmed from a computer that performed the tracking calculations with an internal clock whose integer values in tenths of a second were converted to seconds by multiplying by a 24-bit binary approximation to $\frac{1}{10}$.

$$0.1_{10} \approx 0.00011001100110011001100_2$$

a. Convert the binary number above to a fraction by hand (common denominators would be helpful)

$$\begin{aligned}
&0.00011001100110011001100 \\
&= 2^{-4} + 2^{-5} + 2^{-8} + 2^{-9} + 2^{-12} + 2^{-13} + 2^{-16} + 2^{-17} + 2^{-20} + 2^{-21} \\
&= \frac{2^{17}}{2^{21}} + \frac{2^{16}}{2^{21}} + \frac{2^{13}}{2^{21}} + \frac{2^{12}}{2^{21}} + \frac{2^{9}}{2^{21}} + \frac{2^{8}}{2^{21}} + \frac{2^{5}}{2^{21}} + \frac{2^{4}}{2^{21}} + \frac{2^{1}}{2^{21}} + \frac{2^{0}}{2^{21}} \\
&= \frac{131072 + 65536 + 8192 + 4096 + 512 + 256 + 32 + 16 + 2 + 1}{2097152} \\
&= \frac{209715}{2097152}
\end{aligned}$$

b. The approximation of $\frac{1}{10}$ given above is clearly not equal to $\frac{1}{10}$. What is the absolute error in this value?

$$\left|\frac{209715}{2097152} - \frac{1}{10}\right| = \left|\frac{2097150}{20971520} - \frac{2097152}{20971520}\right| = \left|-\frac{2}{20971520}\right| = \frac{2}{20971520}$$

$$= \frac{1}{10485760}$$

c. What is the time error, in seconds, after 100 hours of operation?

$$100 \text{ hours} = 6,000 \text{ minutes} = 360,000 \text{ seconds} = 3,600,000 \text{ tenths of a second}$$

$$\frac{1}{10485760} \cdot 3600000 \text{ tenths of a second} = \frac{3600000}{10485760} \text{ tenths of a second} = \frac{360000}{1048576} \text{ seconds}$$

$$= \frac{1125}{32768} \text{ seconds}$$

d. During the 1991 war, a Scud missile traveled at approximately mach 5 (3750 mph). Find the distance that the Scud missile would travel during the time error computed in (c).

$$3750 \text{ mph} = \frac{3750}{3600} \text{ miles per second} = \frac{25}{24} \text{ miles per second}$$

$$\frac{25}{24} \text{ miles per second} \cdot \frac{1125}{32768} \text{ seconds} = \frac{28125}{786432} \text{ miles}$$

$$= \frac{9375}{262144} \text{ miles}$$

Exercise 1.54 Find the Taylor Series for $f(x) = \frac{1}{\ln(x)}$ centered at the point $x_0 = e$. Then use the Taylor Series to approximate the number $\frac{1}{\ln(3)}$ to 4 decimal places.

$$f^{(0)}(x) = \frac{1}{\ln(x)}, \qquad f_0 = \frac{f^{(0)}(e)}{0!}(x-e)^0 = 1$$

$$f^{(1)}(x) = -\frac{1}{x\ln(x)^2}, \qquad f_1 = \frac{f^{(1)}(e)}{1!}(x-e)^1 = \frac{e-x}{e}$$

$$f^{(2)}(x) = \frac{\ln(x)+2}{x^2\ln(x)^3}, \qquad f_2 = \frac{f^{(2)}(e)}{2!}(x-e)^2 = \frac{3}{2e^2}(x-e)^2$$

$$f^{(3)}(x) = \frac{-2\ln(x)^2+3\ln(x)+3}{x^3\ln(x)^4}, \qquad f_3 = \frac{f^{(3)}(e)}{3!}(x-e)^3 = -\frac{7}{3e^4}(x-e)^3$$

$$f^{(4)}(x) = \frac{6\ln(x)^3+22\ln(x)^2+36\ln(x)+24}{x^4\ln(x)^5}, \qquad f_4 = \frac{f^{(4)}(e)}{4!}(x-e)^4 = \frac{88}{24e^4}(x-e)^4$$

Thus,

$$f(x) \approx \frac{88}{24e^4}(x-e)^4 - \frac{7}{3e^4}(x-e)^3 + \frac{3}{2e^2}(x-e)^2 + \frac{e-x}{e} + 1$$

$$f(3) \approx 0.9103$$

Exercise 1.55 In this problem we will use Taylor Series to build approximations for the irrational number π.

a. Write the Taylor Series centered at $x_0 = 0$ for the function

$$f(x) = \frac{1}{1+x}$$

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{x^k}$$

b. Now we want the Taylor Series for the function $g(x) = \frac{1}{1+x^2}$. It would be quite time consuming to take all of the necessary derivatives to get this Taylor Series. Instead we will use our answer from part (a) of this problem to shortcut the whole process.

i. Substitute x^2 for every x in the Taylor Series for $f(x) = \frac{1}{1+x}$.

ii. Make a few plots to verify that we indeed now have a Taylor Series for the function $g(x) = \frac{1}{1+x^2}$

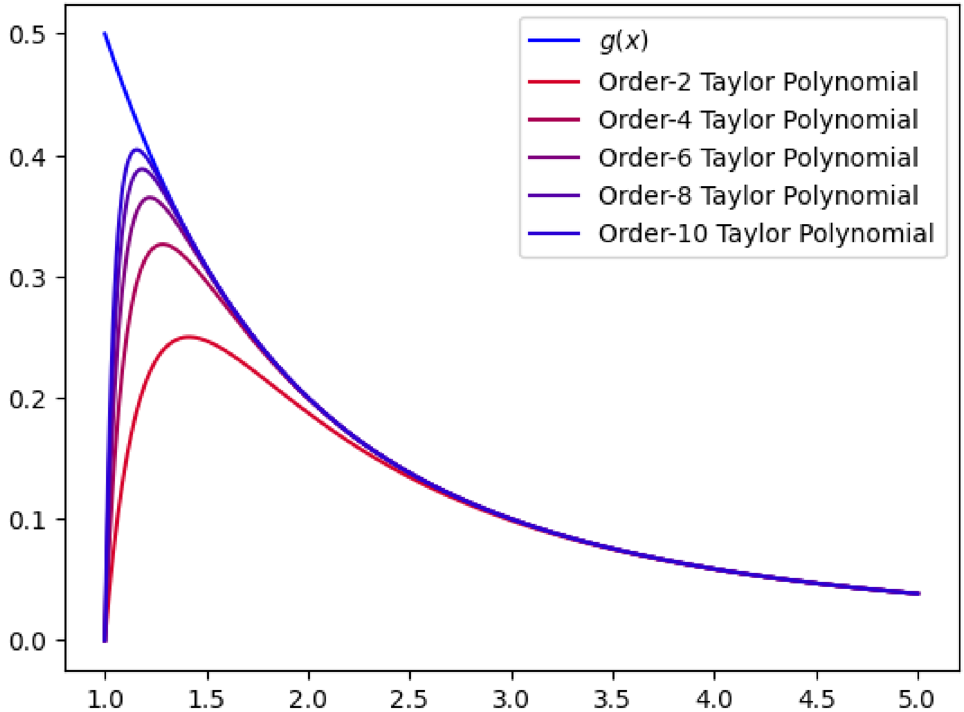
In [57]:
```python
import numpy as np
import matplotlib.pyplot as plt

f = lambda x: 1/(1+x**2)
def g(x, n):
    approx = 0
    for k in range(1, n+1):
        approx += (-1)**(k+1)*(1/x**(2*k))
    return approx
```

```
x = np.linspace(1, 5, 1000)
plt.plot(x, f(x), color='b', label='$g(x)$')

for n in range(2, 11, 2):
    plt.plot(x, g(x, n), color=(1-.083*n, 0, 0.083*n), label=
             'Order-{} Taylor Polynomial'.format(n))

plt.legend()
plt.show()
```



c. Recall from Calculus that

$$\int \frac{1}{1+x^2} dx = \arctan(x)$$

.

Hence, if we integrate each term of the Taylor Series that results from part (b) we should have a Taylor Series for $\arctan(x)$.

$$\arctan(x) = \int \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{x^{2k}} dx$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{x^{2k-1}}{2k-1} dx$$

d. Now recall the following from Calculus:

$$\tan(\pi/4) = 1$$

$$\text{so } \arctan(1) = \pi/4$$

$$\text{and thererefore } \pi = 4\arctan(1)$$

Let's use these facts along with the Taylor Series for $\arctan(x)$ to approximate π: we can just plug in $x = 1$ to the series, add up a bunch of terms, and then multiply by 4. Write a loop in Python that builds successively better and better approximations of π. Stop the loop when you have an approximation that is correct to 6 decimal places.

In [54]:
```
import numpy as np

estimated_pi = 0
n = 1
m = 4

while abs(np.pi - estimated_pi) > 1e-6:
    estimated_pi += (m/n)
    n += 2
    m *= -1
```

Exercise 1.57 In physics the *relativistic energy* of an object is defined as

$$E_{rel} = \gamma mc^2$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In these equations, m is the mass of the object, c is the speed of light ($c \approx 3 \times 10^8$m/s), and v is the velocity of the object. For an object of a fixed mass m we can expand the Taylor Series centered at $v = 0$ for E_{rel} to get

$$E_{rel} = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^2}{c^2} + \frac{5}{16}\frac{mv^6}{c^4} + \dots$$

a. What do we recover if we consider an object with zero velocity?

$$E_{rel} = mc^2$$

b. Why might it be completely reasonable to only use the quadratic approximation

$$E_{rel} = mc^2 + \frac{1}{2}mv^2$$

for the relativistic energy equation?

For most applications, $\frac{v}{c}$ is negligible because v is much less than c. Therefore, the first few terms of the Taylor Series captures almost all of the behavior since the higher-order terms are negligible.

c. What do you notice about the second term in the Taylor Series approximation of the relativistic energy function?

In the second term, the c^2 in the numerator cancels out the c^2 in the denominator, so velocity is non-negligible in this term.

d. Show all of the work to derive the Taylor Series centered at $v = 0$ given above.

$$E_{rel} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}mc^2$$

Recall the Taylor Series expansion for $\frac{1}{\sqrt{1-x}}$:

$$1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots$$

Then substituting $\frac{v}{c}$ for x and multiplying the result by the constant value mc^2

$$E_{rel}=mc^2+\frac{1}{2}mv^2+\frac{3}{8}\frac{mv^2}{c^2}+\frac{5}{16}\frac{mv^6}{c^4}+\ldots$$