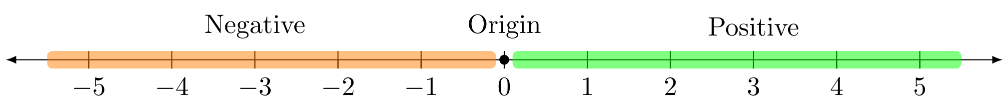
Chapter 1: Real Numbers and Functions

1 The Real Line and Inequalities

1.1 Real Numbers

The real numbers can be visualized using a coordinate system known as the **real line**. Every number on this line corresponds to some real number.



The point on the real line corresponding to zero is called the **origin**. All of the numbers to the right of the origin (in green) are **positive** and numbers to the left of the origin (in orange) are **negative**. We also use the terms **nonnegative** (resp. **nonpositive**) to refer to the positive numbers and 0 (resp. negative numbers and 0).

The following are a list of important sets of real numbers to know along with symbols commonly used to denote them.

- **Natural Numbers** ($\mathbb{N}$): $\{1, 2, \dots\}$ These are the counting numbers, the ones you grew up learning. Sometimes 0 is also considered to be a natural number, though I will not do so in this class (unless stated otherwise).

- **Integers** ($\mathbb{Z}$): $\{\dots, -2, -1, 0, 1, 2, \dots\}$ These are all of the counting numbers and their negatives. Everything in this set still has "whole-number values", so it does *not* include things like $\frac{1}{4}$ and $0.333\dots$

- **Rational Numbers** ($\mathbb{Q}$): $\{\frac{a}{b} \mid a \text{ and } b \text{ are integers and } b \neq 0\}$ The rational numbers include all fractions and repeating decimals, so things like $\frac{2}{3}$, $\frac{7}{5}$, and $4.111\dots$ are included in this set. It is important to note that the rational numbers also include the integers, since for example 3 can be written as $\frac{3}{1}$ or $\frac{6}{2}$.

- **Irrational Numbers**: The irrational numbers are all numbers which aren't rational. Some notable examples include $\sqrt{2}, \pi$, and e. One can tell these are all indeed irrational numbers since their decimal expansions do not repeat:
  - $\sqrt{2} = 1.414213562\dots$
  - $\pi = 3.14592654\dots$
  - $e = 2.718281828\dots$

From here, it looks like e repeats, but you can find more digits to show it does not. The irrational numbers don't have a special symbol.

1.2 Inequalities and Intervals

Since all of the real numbers can be arranged from left-to-right, we call them **ordered**. The following is some terminology related to the ordering on the real line.

Let x and y denote real numbers, then:

- We say x is **less than** y (denoted $x < y$) if x lies to the *left* of y on the real line.
- We say x is **greater than** y (denoted $x > y$) when x is to the *right* of y on the real line.
- We say x is **less than or equal to** y if $x < y$ or $x = y$.
- We say x is **greater than or equal to** y if $x > y$ or $x = y$.

Sometimes the inequalities $x < y$ and $x > y$ are referred to as *strict inequalities* since the two numbers x and y are not allowed to be equal. It will be important in this course to be able to manipulate/perform algebra on inequalities. For this reason, we now introduce several important properties of inequalities.

Properties of Inequalities

Let a, b, c, d and k be real numbers.

(Symmetric) $a < b$ if and only if $b > a$.

(Transitive) If $a < b$ and $b < c$, then $a < c$.

(Adding Constants) If $a < b$, then $a + k < b + k$.

(Adding Inequalities) If $a < b$ and $c < d$, then $a + c < b + d$.

(Positive Multiplication) If $a < b$ and k is positive, then $ak < bk$.

(Negative Multiplication) If $a < b$ and k is negative, then $ak > bk$

A few remarks about these properties: All of the above properties are true if $<$ or $>$ is replaced with $\leq$ or $\geq$ respectively. Similarly, all of the above properties are true for equations ($=$) as well, ignoring the need to "flip" signs when multipliying.

Inequalities are particularly useful in describing sets of numbers on the real line. For example, the positive numbers are all of the numbers x where $x > 0$. Similarly, the negative numbers are those numbers x which satisfy $x < 0$. At the moment, this is a very wordy way of describing sets of numbers. To fix this, we introduce **set notation**.

Set Notation

To define a set of real numbers, the *set notation* $\{x \,|\, \text{condition on } x\}$ denotes all of the real numbers x which satisfy the given condition. These are a few common sets of numbers written in set notation:

- Positive Numbers: $\{x \,|\, x > 0\}$
- Negative Numbers: $\{x \,|\, x < 0\}$
- Even Numbers: $\{x \,|\, x = 2k \text{ where } k \text{ is an integer}\}$
- Odd Numbers: $\{x \,|\, x = 2k + 1 \text{ where } k \text{ is an integer}\}$
- Powers of 2: $\{x \,|\, x = 2^k \text{ where } k \text{ is an natural number}\}$

While real numbers are commonly denoted with lowercase letters like x, y, z, sets are denoted with capital letters like A and B. Commonly-used sets like the integers and rational numbers are given bolded letters like $\mathbb{Z}$ and $\mathbb{Q}$ to denote their importance. The following is some important terminology involving sets.

Let x be a real number and A, B be sets of real numbers, then:

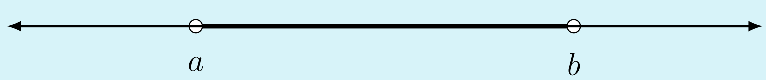
- When x is **contained in** the set A, we write $x \in A$. Similarly if x is **not contained in** A, we write $x \notin A$.
- $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$ is the **union** of A and B, and it consists of all of the numbers in either A, B, or both.
- $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$ is the **intersection** of A and B, and it consists of all of the numbers in both A and B.
- We say the sets A and B are **disjoint** if they have no elements in common (i.e. $A \cap B$ is empty).

One important class of sets used frequently in calculus are **intervals** - visually, these are the sets which look "connected". Below is a table of different types intervals.

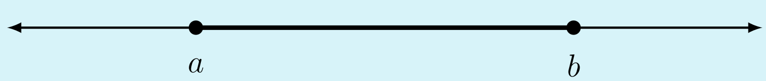
Types of Intervals

Let a and b be real numbers where $a < b$, then the following are all different types of intervals on the real line.

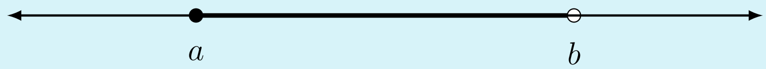
Open Interval: $(a, b) = \{x \,|\, a < x < b\}$



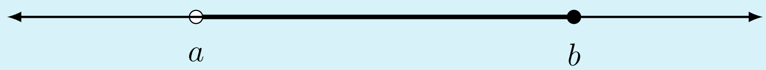
Closed Interval: $[a, b] = \{x \,|\, a \leq x \leq b\}$



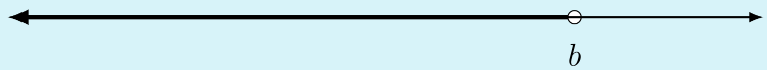
Half-Open Intervals: $[a, b) = \{x \mid a \leq x < b\}$



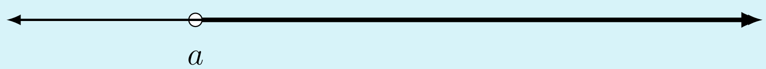
$(a, b] = \{x \mid a < x \leq b\}$



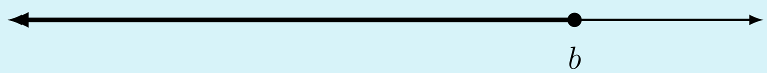
Open Rays: $(-\infty, b) = \{x \mid x < b\}$



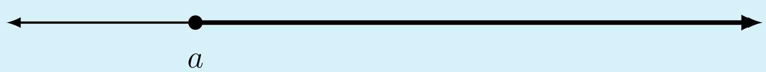
$(a, \infty) = \{x \mid a < x\}$



Closed Rays: $(-\infty, b] = \{x \mid x \leq b\}$



$[a, \infty) = \{x \mid a \leq x\}$



All Real Numbers: $(-\infty, \infty)$

Interval notation is central to many of the problems at the heart of calculus. Solutions to inequalities, increasing/decreasing intervals for functions and domain restrictions are just a few applications which will come up this semester, so being comfortable with this sort of notation is important.

2 The Cartesian Coordinate System

3 Graphing Equations

4 Lines in the Plane

5 Functions