Heaps & Priority Queues

Part 9

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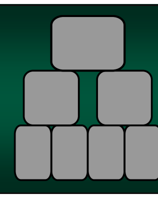


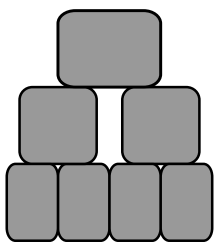
Heaps

Piles of data!

2

What is a heap?

- A *heap* is a binary tree, but a notable format to the nodes
- The value of a node is smaller (or larger) than both of its children
- Every subtree is a heap



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Terminology Warning

- The *heap data structure* is not the same as the operating system's heap
- They are often confused…
- The heap data structure is a tree that stores "heavier" objects at the bottom

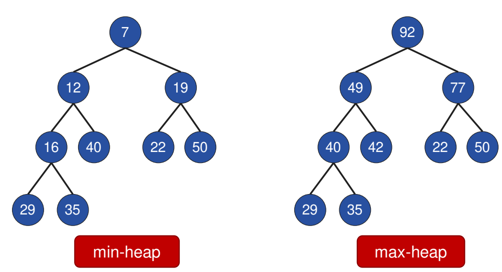
4

Min and max-heaps

- *Min-heap*
  - each of a node's descendants have a "heavier" value
  - stores smaller items (minimal items) at the top of the tree
- *Max-heap*
  - each node's parent has a "heavier" value
  - stores larger items (maximum items) at the top of the tree

5

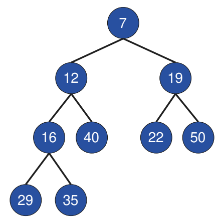
Min and max-heaps



6

Heaps

- Heaps are *complete* binary trees
- Nodes are added in breadth-first order
- The resulting tree is always optimal and *balanced*



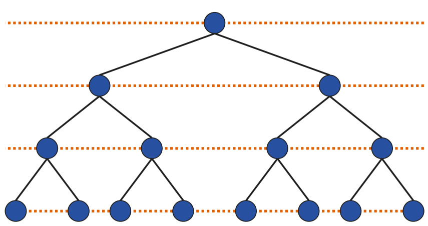
7

Height of a Heap

- Let i be the depth of a node
- Then, there are 2^i nodes of depth i
- Heap always has a height of **O(log n)**

8

Height of a Heap



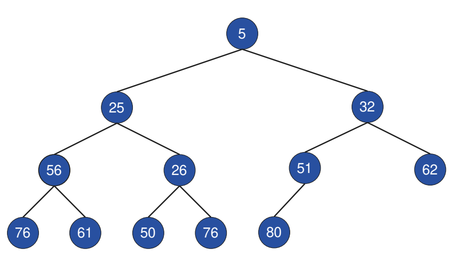
9

Adding a Node

1. Begin at next available position for a leaf
2. Now the item needs to be *up-heaped*
   - move the entry up depending on its value until a correct position is found
   - as this is done, nodes are swapped - parent to child change position
   - since a heap always has height O(log n), upheap runs in O(log n) time

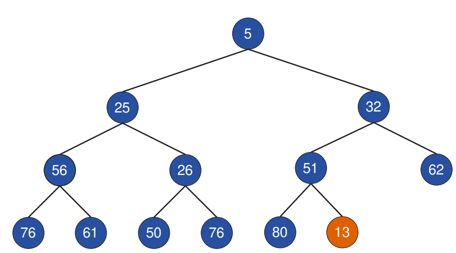
10

Min-heap: Adding a Node 13



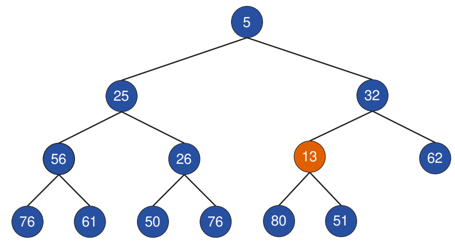
11

Min-heap: Adding a Node 13



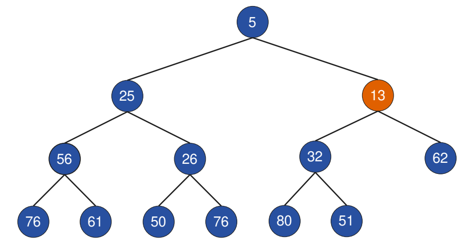
12

Min-heap: Adding a Node 13



13

Min-heap: Adding a Node 13



14

Total Up-heap-val

- Just to make matters confusing, up-heap is also known by various other terms *– which are* ***all*** *valid*
- These are some:
  - bubble-up
  - percolate-up
  - sift-up
  - heapify-up
  - cascade-up

15

Deleting a Node

- Deleting a node is quite different from adding
- Heaps must maintain *completeness*
- So, the right-most leaf is needed to replace the deleted node
- Deletion:
  - remove the node
  - replace it with the right-most leaf
  - now, it needs to down-heaped (moved down) to the correct location
  - down-heap runs in O(log n) time

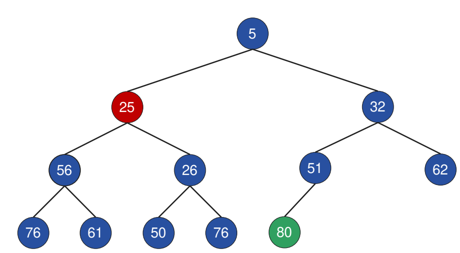
16

Downheap Algorithm

- With a heap, every node has two children
  - as you downheap, you swap nodes
  - so, which one do you select?
- Preserve the heap structure ← **vital**
  - on a min-heap, swap with the smallest child
  - on a max-heap, swap with the largest child

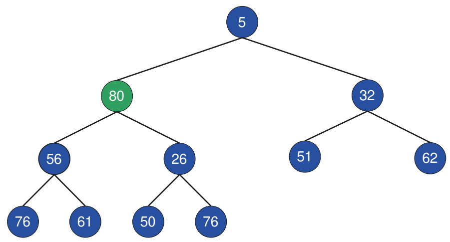
17

Min-heap: Deleting 25



18

Replace. Now must down-heap.



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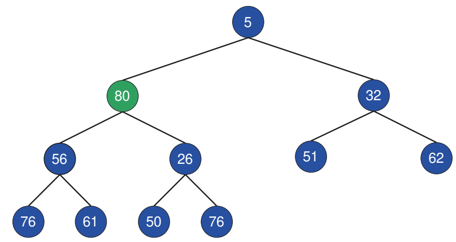

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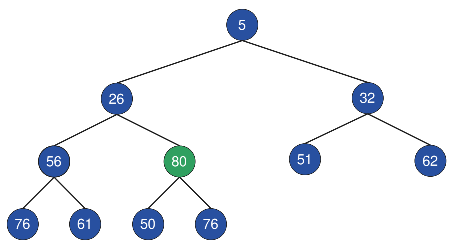

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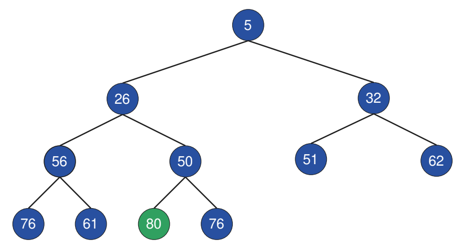
Replace. Now must down-heap.



22

As You Expected...

- Just like up-heap, down-heap has several other, completely valid, names
- These are some:
  - bubble-down
  - percolate-down
  - sift-down
  - heapify-down
  - cascade-down

23

Priority Queues

Queues can play favorites

24

Priority Queues

- A stack is first-in-last-out
- A queue is first-in-first-out
- A priority queue is modification of the queue ADT that follows the logic of *first-in-least-out*

25

First-in-Least-Out

- The *"least"* element is the first one that is removed
- If two items have the same "rank", items can be queued as normal
- The object's key can be used to determine if it is "least", but any field will do

26

What is the "Least" Item?

- Meaning of "least" is defined by the ADT
- It is abstract - does not mean "less than"
  - so, "least" can be any way of ranking items
  - ...if the items are mathematically transitive
  - "least" can be the largest value
- Examples of least:
  - the smallest / largest value
  - some ranking classification

27

Priority Queue API

public class PriorityQueue		
	`PriorityQueue()`	*Create an empty PQ*
`void`	`add(Item item)`	Same as enqueue()
`Item`	`removeLeast()`	Same as dequeue()
`Item`	`getLeast()`	Same as peek(), first()
`bool`	`isEmpty()`	
`int`	`size()`	

28

Implementation

- Before we select a data structure to implement a priority queue, we should look how data will be used
- The goal is to get the best time efficiency with as little overhead as possible

29

Implementation

- The type data to be stored will influence how the priority queue is implemented
- We have quite a few options:
  - array
  - linked-list
  - tree / heap

30

Implementation with an Array

- Unsorted array
  - enqueue requires **O(n)** – resize array
  - dequeue requires **O(n)** – search and moving
- Sorted array
  - enqueue requires **O(n)** – find a position to insert and then move the rest
  - dequeue requires **O(n)**

31

Implementation with a Linked List

- Unsorted linked list
  - enqueue takes **O(1)**
  - dequeue requires **O(n)** – find & remove node
- Sorted linked list
  - enqueue requires **O(n)** – must find a position and insert
  - dequeue requires **O(1)** – just remove the head/tail

32

We Need Another Data Structure!

- Arrays have a time complexity of **O(n)** for Enqueue and Dequeue
- Linked Lists did have a single **O(1)** operation, but the other was **O(n)**
- Given priority queues are updated often (just like normal queues), arrays and linked lists are **poor** solutions

33

Hybrid Implementations

- In some cases, the key value can have a minor range of values – possibly just a few
- Examples:
  - hospital triage – immediate, delayed, minor
  - computer processes – OS, application, GUI
- We can make clever hybrid structures that maximize efficiency

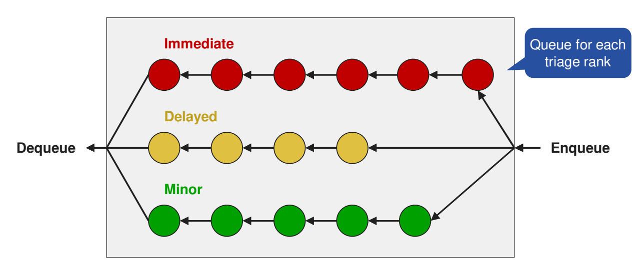
34

Hybrid Implementations

- If the key contains a small number of values, you can use multiple queues – one for each key value
- Basically, the priority queue, internally, will have an array of queues
- Adding/removing items will always be O(1)
  - **O(1)** for the queue head
  - **O(1)** for enqueue/dequeue (using linked list)

35

Medical Triage – Array of Queues



36

... But Heaps are Universal

- However, in most cases, the key values have large ranges
- For example, if the key is a 32-bit integer, do you want to create 4 million queues?
- Didn't think so….
- So, this only works in limited situations

37

Implementation with a Heap

- However, a priority queue can be implemented as a heap
- Remember...
  - in a heap, all the items below a node have a greater value
  - so, *the root is the least item!*
  - heaps *naturally* implement a priority queue

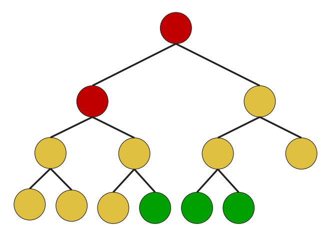
38

Implementation with a Heap

- To enqueue an item...
  - just add to it the heap
  - it will up-heap to the correct position
  - requires **O(log n)**
- To dequeue an item…
  - just remove the root
  - requires **O(log n)** rebalance

39

Medical Triage - Heap



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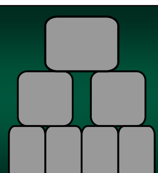


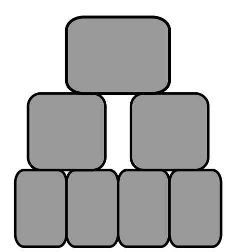
Heaps and Arrays

Not a Heap O' Trouble

41

Heaps and Arrays

- Heaps are *complete*, balanced, binary trees
- This rigid, predictable, structure...
  - lends itself to being stored in an array
  - each node has a pre-ordained location



42

Heaps and Arrays

- Using an array, links between items are not explicitly stored
- Finding the location of an array item can found using simple mathematics
- Heaps are **no** different - due to their predictable structure

43

Heaps and Arrays

- Any node's parent and children can be computed *mathematically*
- Heap ADTs only need to…
  - track the index of the end of the heap
  - all new items are added here – before upheap
  - and this is where the last item will be swapped for a deleted item (before it is downheaped)

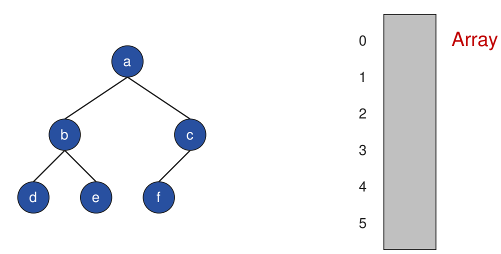
44

Heap Array Math

Find	0 Indexed Array	1 Indexed Array
Parent of node i	`(i – 1) / 2`	`i / 2`
Left child of node i	`(2 * i) + 1`	`2 * i`
Right child of node i	`(2 * i) + 2`	`(2 * i) + 1`

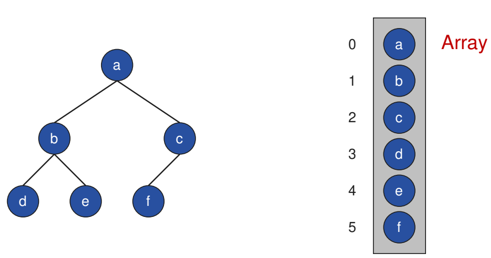
45

Heap in an Array

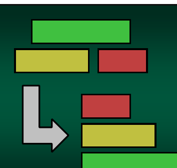


46

Heap in an Array



47



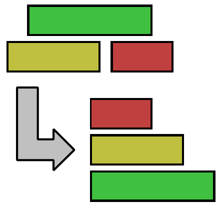
Heap Sort

Heap-a-rific algorithm!

48

Heap Sort

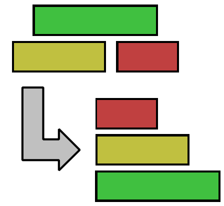
- *Heap Sort* is an ingenious algorithm that uses a max-heap to sort an array
- *John W. J. Williams* invented both heaps and the Heap Sort in 1964
- The same year, the sort was improved by *Robert Floyd*



49

Heap Sort

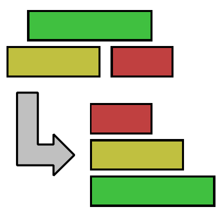
- Heap Sort takes advantage of the fact that a heap is a natural priority queue
- … and that a heap will always add / remove from the right-most leaf



50

Heap Sort Works in Two Phases

- Phase 1: Heapify
  - converts the existing array into a max-heap
- Phase 2: Empty the heap
  - removes all the nodes (treating it as priority queue)
  - sorted data is added to the end of the array



51

Implementation

- Both the "heap" and the remaining array can be used in memory at the same time
- The sorted array is stored at the empty space after the end of the heap
- This concept works for both Phase 1 and Phase 2

52

Phase 1: Array → Heap

- In Phase 1, we convert the array into a max-heap. This step is called *heapify*.
- Remember....
  - a heap can be stored in an array
  - so, we can just *look at the array as a heap*
  - ...but, its not quite a heap yet
  - data needs to be rearranged to turn the array into a heap

53

How do we convert it?

- First approach: *top-down*
  - start building the heap at the top of the array
  - iterate *i* starting at 0 and build a heap above *i*
  - item are upheaped
- Second approach: *bottom-up*
  - fastest approach is to downheap all the leaves
  - run the downheap, at the root, all the leaves

54

Phase 1: Heapify (top down)

```
procedure heapify(array, count)
   last = count – 1
   n = 0                //First item

   while (n <= last)
      upHeap(array, 0, n – 1)
      n++
   end while
end procedure
```


55

Phase 1: Heapify (top down)

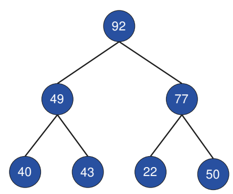
```
procedure heapify(array, count)
   last = count – 1
   n = last                 //last item

   while (n >= 0)
      downHeap(array, n + 1, last)
      n--
   end while
end procedure
```


56

Phase 2: Root Deletion

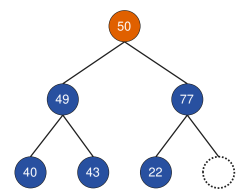
- Now that the array is a **max-heap**, the root contains the maximum item
- If we remove the root, we have the *last item of the sorted array!*



57

Phase 2: Root Deletion

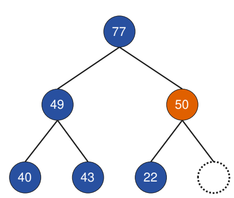
- When we remove the root... right-most leaf is moved to the root
- …and then downheaped into the correct position



58

Phase 2: Root Deletion

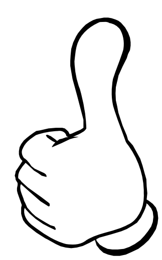
- Now the root contains the second-largest item in the array
- *This leaf position is now empty*



59

Phase 2: Root Deletion

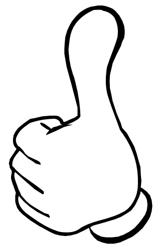
- We can put the root, that was just removed, in this new empty space
- *What a sec!* We just put the largest item in the last position in the array!



60

Phase 2: Root Deletion

- So, to sort the array....
  - so, we just keep removing the root and placing it position where the leaf was located
  - the "heap" section of the array shrinks as the sorted array grows from the bottom
- *OMG! Sooooo, awesome!*



61

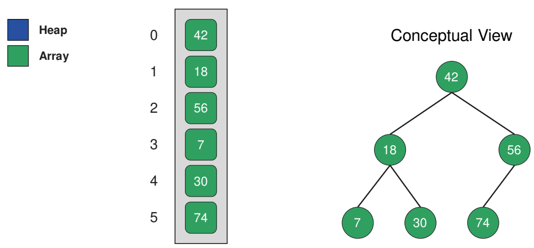
Heap Sort Algorithm

```
last = count - 1
heapify(array, 0, last)

while (last > 0)
   swap array[0] and array[last]
   downHeap(0, last - 1)
   last--
end while
```

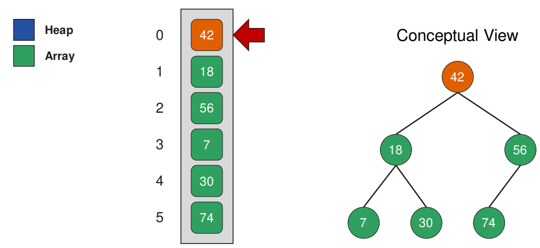

62

Phase 1 – Top-down Upheap



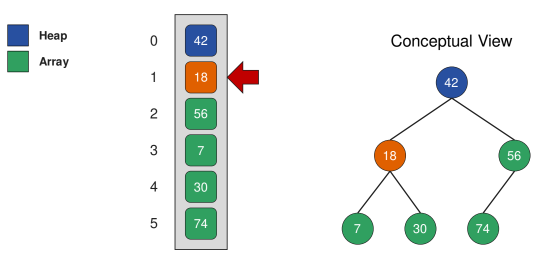
63

Phase 1 – Top-down Upheap



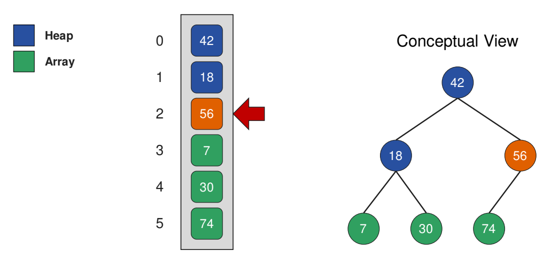
64

Phase 1 – Top-down Upheap



65

Phase 1 – Top-down Upheap



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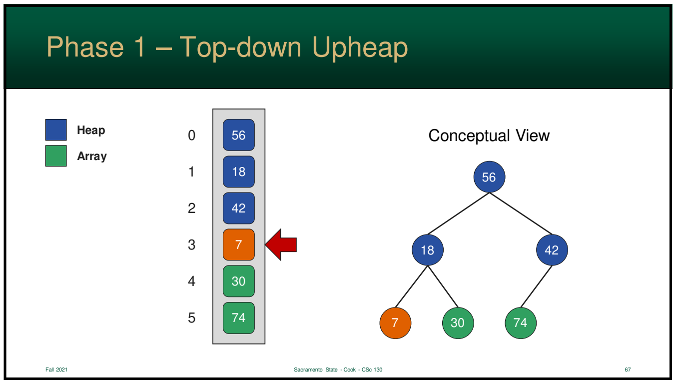
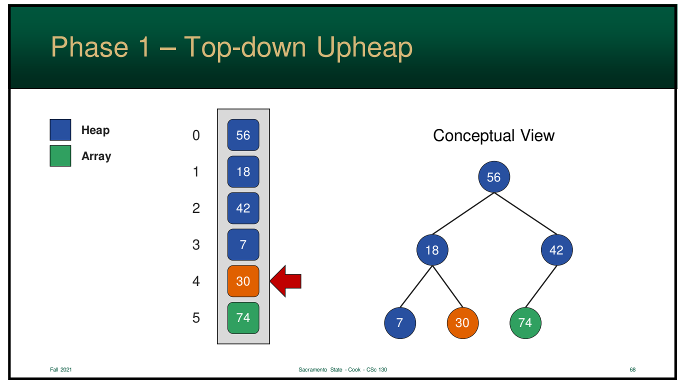


67



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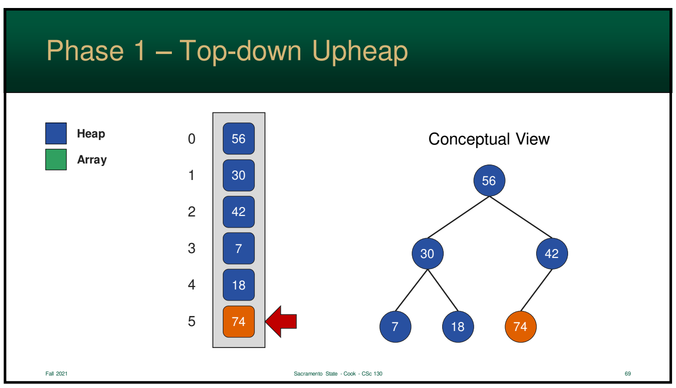

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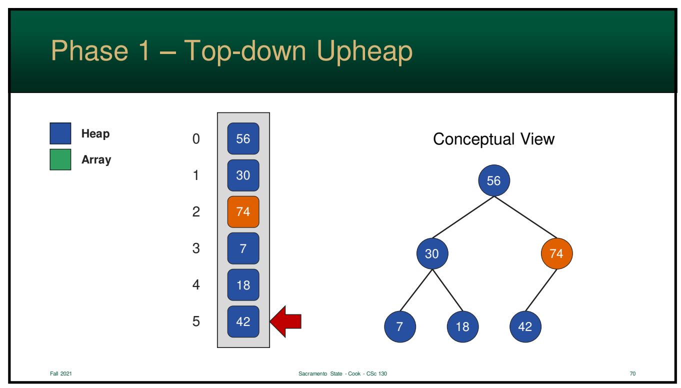

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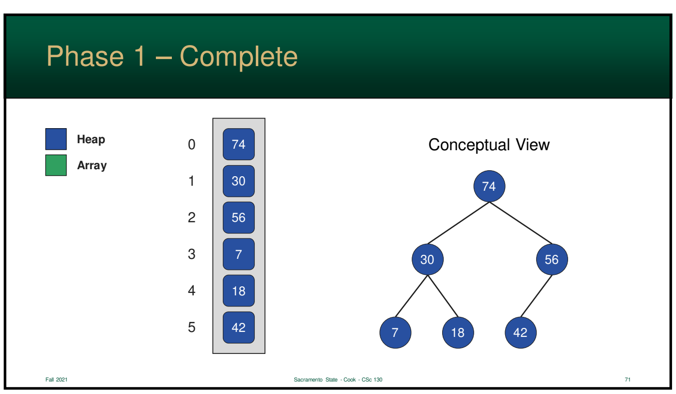

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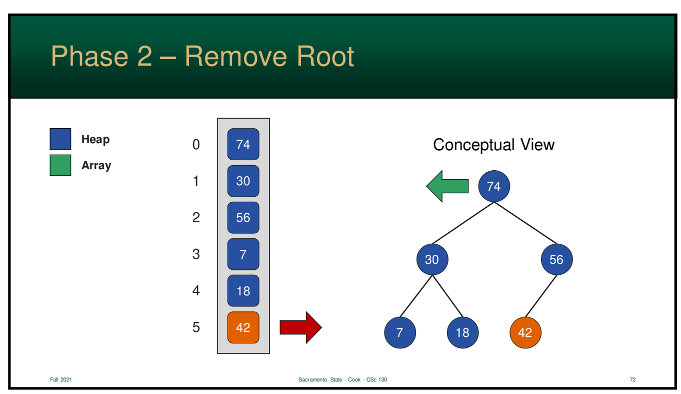
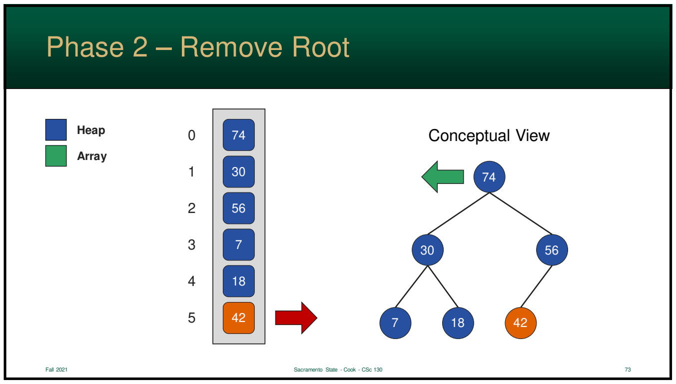


72



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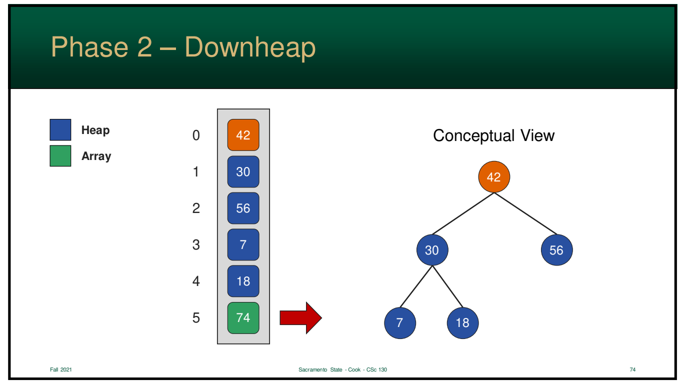
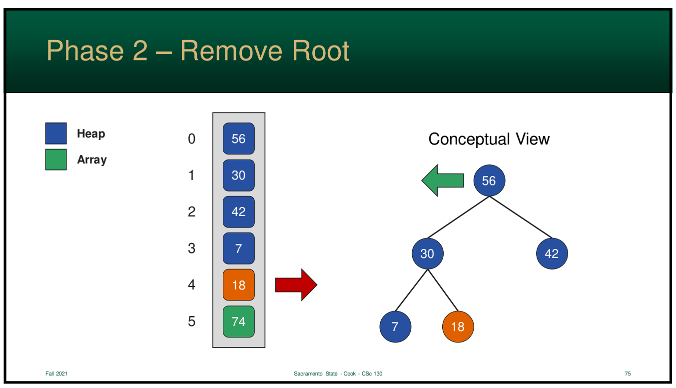
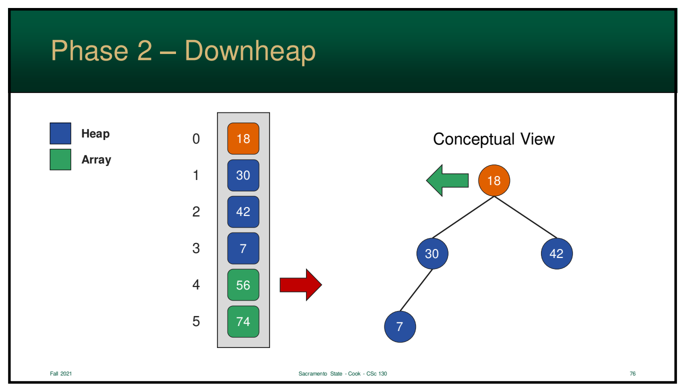


74



75



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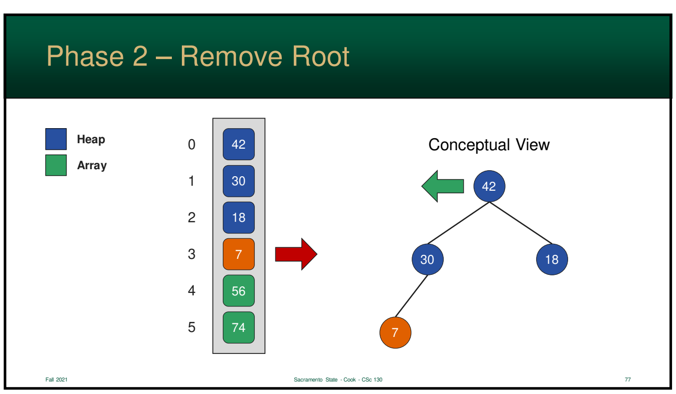
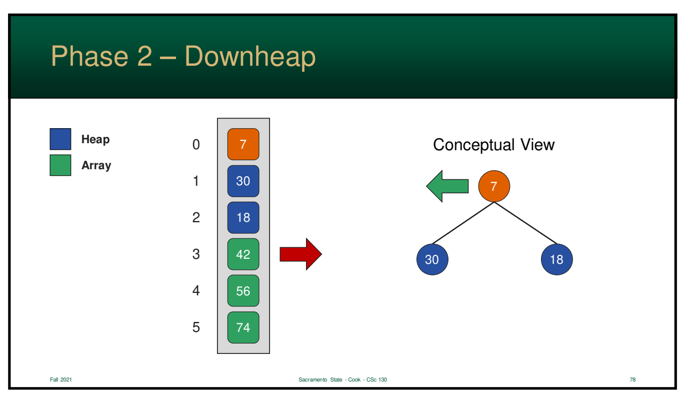
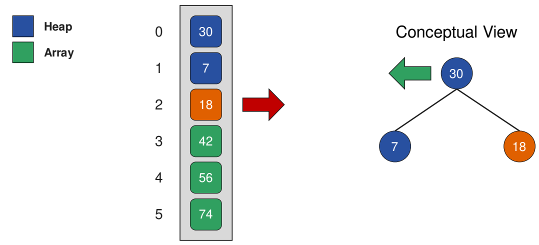


77



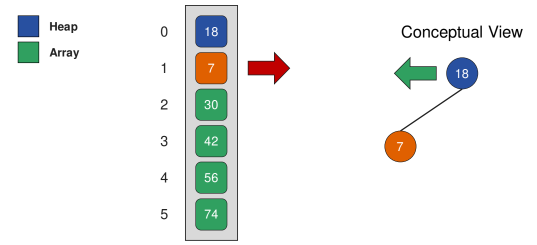
78

Phase 2 – Remove Root



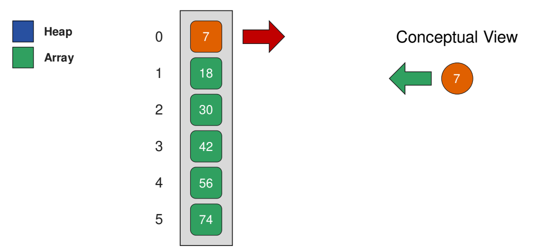
79

Phase 2 – Remove Root



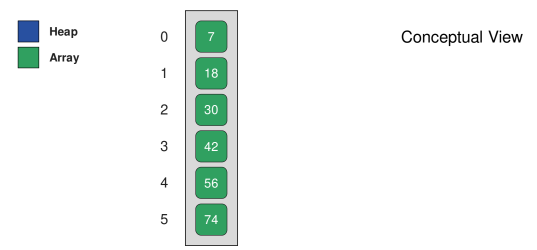
80

Phase 2 – Remove Root



81

Complete



82

Merge Sort vs. Heap Sort

- Heap-Sort allows us to sort any array in O(n log n) just like Merge-Sort & Quicksort
- However, there is no overhead
  - Heap-Sort can be sorted in-place, meaning auxiliary storage is O(1)
  - Merge-Sort, however, requires O(n)
  - Quick-Sort can become O(n^2)

83

Merge Sort vs. Heap Sort

- However, in some cases, the recursive nature of Merge Sort is better
  - easy to distribute to multiple computers
  - Heap-Sort uses the entire array – *not online*
- But...in the Real World, it gets complex
  - you can cut an array into sub-lists, send them to different machines which Heap-Sort them
  - ... and then you Merge

84

Heap Sort Summary

Heap Sort	
Time Average	O(n log n)
Time Best	O(n log n)
Time Worst	O(n log n)
Auxiliary space	O(1)
Stable	No – Equal element order not preserved
Online?	No

85

Summary of Sorting Algorithms

Sort Algorithm	Best	Average	Worst	Aux. Storage
Bubble	$O(n^2)$	$O(n^2)$	$O(n^2)$	O(1)
Selection	$O(n^2)$	$O(n^2)$	$O(n^2)$	O(1)
Insertion	O(n)	$O(n^2)$	$O(n^2)$	O(1)
Shell	O(n log n)	$O(n^{5/4})$	$O(n^{3/2})$	O(1)
Merge	O(n log n)	O(n log n)	O(n log n)	O(n)
Quick	O(n log n)	O(n log n)	$O(n^2)$	O(1)
Heap	O(n log n)	O(n log n)	O(n log n)	O(1)
Radix	$O(k \times n)$	$O(k \times n)$	$O(k \times n)$	O(b + n)

86