SACRAMENTO STATE

CSc 28
Discrete Structures

Chapter 9
Recurrence Relations

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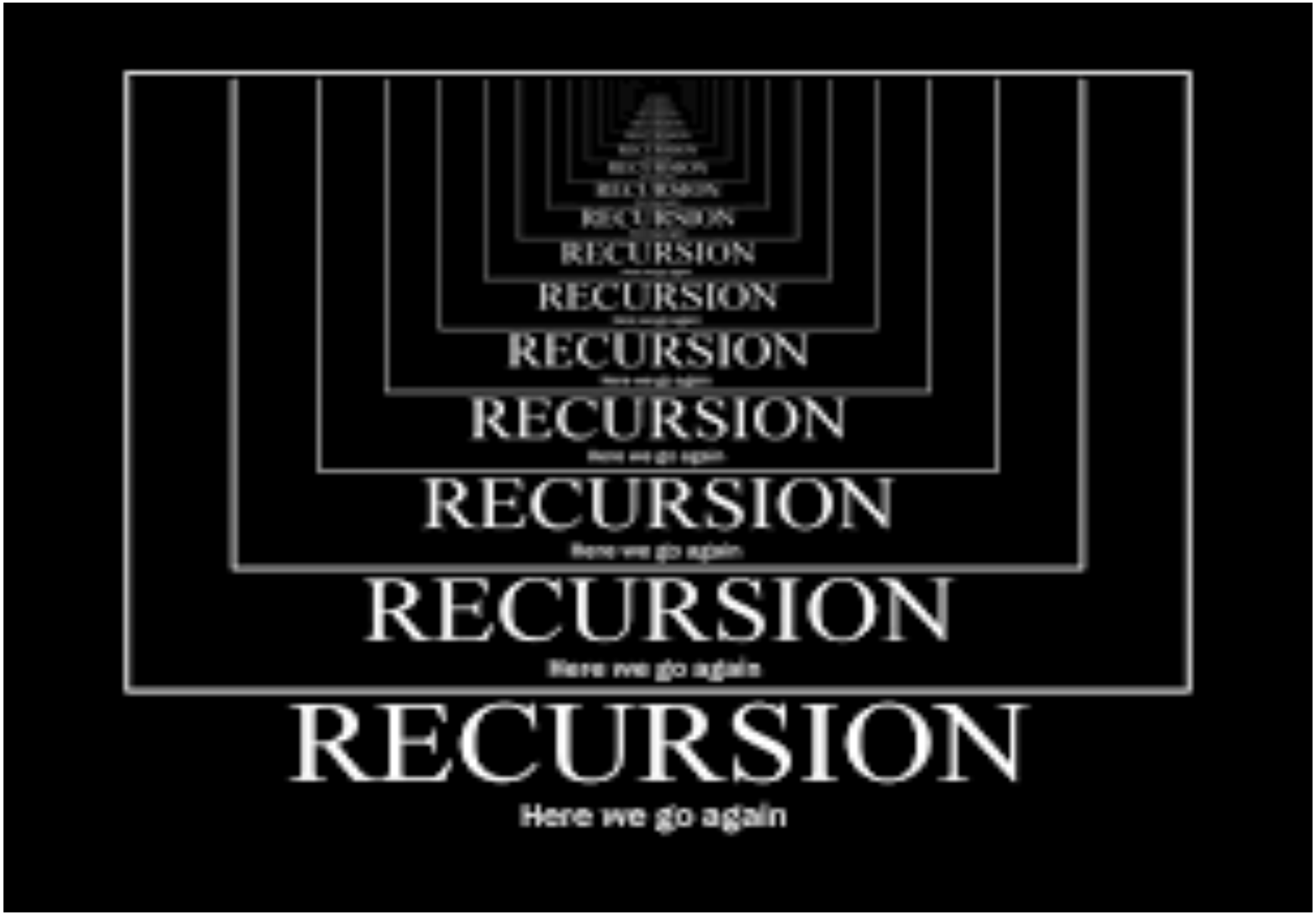
Syllabus

- Recursion
- Difference
- Recurrence Relations (RR)
- Modeling with RR
- Solving Recurrence Relations
- References

Recurrence Relations & Recursion

Recursion


RECURSION
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Here we go again
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Difference

- Recurrence relations look similar to recursive formulae
- Difference 1: A Recurrence Relation do not provide initial values; Recursive Definition must do so!
- Difference 2: Recurrence Relation defines mathematical formula, yet Recursive Definitions may define complex algorithms, way beyond complexity of math formulae
- Difference 3: Recurrence always directly expressed in some formula; recursion may be indirect. I.e. some recursive function f1() may not have any direct, recursive reference to f1() in its program body
- In that case we say the recursion is indirect, via some intermediate function f2(), or even further functions

Difference

- Example from [1]: Consider the recurrence relation

$$a_n = 5\ a_{n-1} - 6\ a_{n-2}$$

- It is a recurrence relation, not a recursive definition
- Why? Because no initial condition is provided! Hence: Is a recurrence relation!
- Once we provide initial conditions, e.g. $a_0 = 1$, $a_1 = 2$, then we get a directly recursive formula:

  $a_0 = 1$, $a_1 = 2$, $a_n = 5\ a_{n-1} - 6\ a_{n-2}$ for $n \geq 2$
- $a_0 = 1$, $a_1 = 2$, $a_2 = 5\ a_1 - 6\ a_0 = 5*2 - 6*1$ i.e. $a_2 = \ \ 4$

And a_3 and a_4?

Difference

- Example from [1]: Consider the recurrence relation
$$a_n = 5\,a_{n-1} - 6\,a_{n-2}$$
- It is a recurrence relation, not a recursive definition
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  $a_0 = 1$, $a_1 = 2$, $a_n = 5\,a_{n-1} - 6\,a_{n-2}$ for $n \geq 2$
- $a_0 = 1$, $a_1 = 2$, $a_2 = 5\,a_1 - 6\,a_0 = 5*2 - 6*1$ i.e. $a_2 = 4$
- $a_1 = 2$, $a_2 = 4$, $a_3 = 5\,a_2 - 6\,a_1 = 5*4 - 6*2$ i.e. $a_3 = 8$
- $a_2 = 1$, $a_3 = 2$, $a_4 = 5\,a_3 - 6\,a_2 = 5*8 - 6*4$ i.e. $a_4 = 16$
- Yields sequence: 1, 2, 4, 8, 16, . . . Powers of 2. You saw?

Difference

- Continue analyzing the recurrence relation

$$a_n = 5\, a_{n-1} - 6\, a_{n-2}$$

- If we provide a different initial condition, for example values $a_0 = 1,\ a_1 = 3$, we also get a recursive formula, yet a different one:

  $a_0 = 1,\ a_1 = 3,\ a_n = 5\, a_{n-1} - 6\, a_{n-2}$ for $n \geq 2$

- $a_0 = 1,\ a_1 = 3,\ a_2 = 5\, a_1 - 6\, a_0 = 5*3 - 6*1$ i.e. $a_2 = 9$

- And a_3 and a_4?

Difference

- Continue analyzing the recurrence relation

$$a_n = 5\, a_{n-1} - 6\, a_{n-2}$$

- If we provide a different initial condition, for example values $a_0 = 1, a_1 = 3$, we also get a recursive formula, yet a different one:

$a_0 = 1,\ a_1 = 3,\ a_n = 5\, a_{n-1} - 6\, a_{n-2}$ for $n \geq 2$

- $a_0 = 1,\ a_1 = 3,\ a_2 = 5\, a_1 - 6\, a_0 = 5*3 - 6*1$ i.e. $a_2 = 9$
- $a_1 = 3,\ a_2 = 9,\ a_3 = 5\, a_2 - 6\, a_1 = 5*9 - 6*3$ i.e. $a_3 = 27$
- $a_2 = 9,\ a_3 = 27,\ a_4 = 5\, a_3 - 6\, a_2 = 5*27 - 6*9$ i.e. $a_4 = 81$
- Yielding sequence: 1, 3, 9, 27, 81, . . . Powers of 3
- One recurrence relation used here generates 2 distinct recursive definitions; with many more feasible!

Difference

- When recursive formulae focus on mathematical formulae, the notions of *recurrence relation* and *recursion* appear very similar; almost identical!
- Key message: Initial conditions are not provided for recurrence relations!
- Once recursion is applied to say, the parsing of a complex context-free grammar, there is little (but some) resemblance left between recursion and recurrence relations
- For our simple purpose here, the 2 math concepts look quite similar

Recurrence

Recurrence Relations

Recursion? Recurrence? Infinite Regress?

Recurrence Relations

- **From [2] Math Insight:**
- **Definition of Recurrence Relation: An equation that defines a sequence that can compute the next term as a function of a number of previous terms; previous implying at least one term**
- **In its simplest form, a recurrence relation depends only on the immediately previous term, i.e. one single term**
- **If we denote the n^{th} term in the sequence by x_n, then such a simple recurrence relation is of form:**

$$x_{n+1} = f(x_n)$$

for some function f()

Recurrence Relations

- Here another simple Recurrence Relation: $x_{n+1} = 2\, x_n$
- However, recurrence relations can be of higher order, in which case term x_{n+1} not only depends on previous term x_n but also on x_{n-1}, x_{n-2} etc., so could be of form:

$$x_{n+1} = f(\, x_{n,}\; x_{n-1}\,)$$

- or some function f(n) with multiple inputs!
- Here a practical example: the recurrence relation

$$x_{n+1} = x_n + x_{n-1}$$

- Simply generates the well known Fibonacci sequence; assuming correct initial conditions for x_0 and x_1

Recurrence Relations

- **Similar definition:**
- **A recurrence relation for sequence $\{a_n\}$ is an equation that expresses a_n in terms of *one or more* previous terms of the sequence, for example a_0, a_1, …, a_{n-1}, for all n, with index n being a nonnegative integer**
- **A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation**

Recurrence Relations

- A recurrence relation is reminiscent of a recursively defined sequence, but does not specify initial values; AKA: does not specify initial conditions
- Therefore, the same recurrence relation can have (and usually has) multiple incarnations (i.e. solutions)
- If both the initial conditions and the recurrence relation are specified, then the sequence is uniquely determined to evolve into a recursive formula

Recurrence Relations

Example:

- Consider **recurrence relation**: $a_n = 2a_{n-1} - a_{n-2}$ for values $n = 2, 3, 4, \ldots$
- Is the sequence $\{a_n\}$ with $a_n = 3n$ some **solution** of this recurrence relation?

For $n \geq 2$: $a_n = 2a_{n-1} - a_{n-2} = 2(\,3(\,n - 1\,)\,) - 3(\,n - 2\,)$

$$a_n = 2a_{n-1} - a_{n-2} = 6n - 6 - 3n + 6 = 3n$$

- Therefore, $\{a_n\}$ with $a_n = 3n$ is a solution of the above recurrence relation

Modeling with Recurrence Relations

For example: Compute Compound Interest

Modeling with Recurrence Relations

Example:

- **Someone deposits $100,000 in a savings account at a bank yielding 5% per year interest, compounded annually**
- **How much money will be in the account after 30 years?**

Solution:

- **Let P_n denote the amount in the account after n years**
- **How can we determine P_n as a function of P_{n-1} i.e. compute P_n as a function of the previous year?**

Modeling with Recurrence Relations

Solution Cont'd: Derive the **interest recurrence relation:**

- $P_n = P_{n-1} + 0.05\ P_{n-1} = 1.05\ P_{n-1}$
- The initial condition is $P_0 = 100{,}000$

Then we have:

- $P_1 = 1.05\ P_0$
- $P_2 = 1.05\ P_1 = (1.05)^2\ P_0$
- $P_3 = 1.05\ P_2 = (1.05)^3\ P_0$
- …
- $P_n = 1.05\ P_{n-1} = (1.05)^n\ P_0$
- We now have a formula to calculate P_n for any natural number n and can avoid iteration altogether!

Modeling with Recurrence Relations

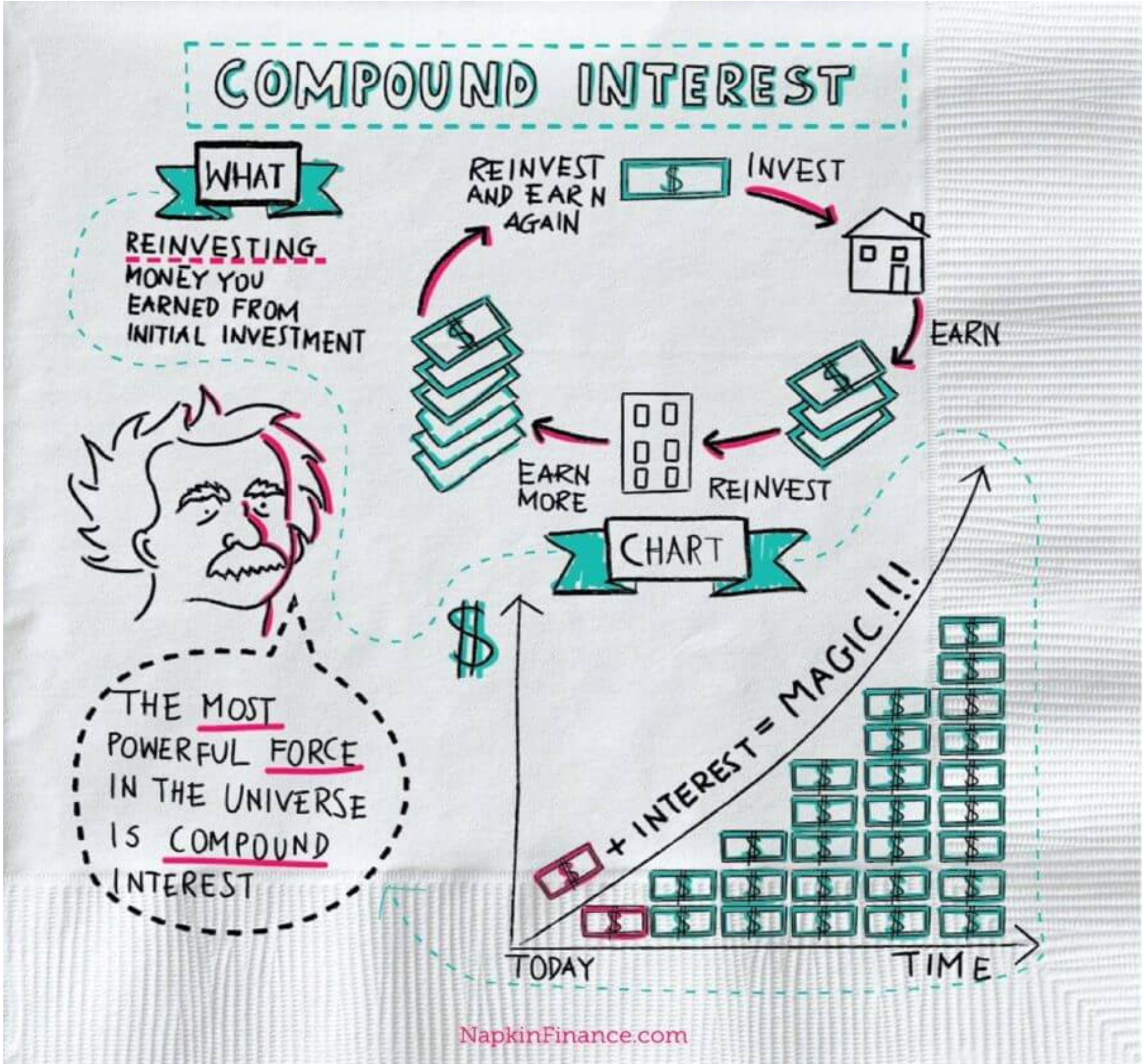
- **Use the formula just found to identify P_{30} under initial condition:**

$$P_0 = 100{,}000$$

- **$P_{30} = (1.05)^{30} * 100{,}000 = 432{,}194.20$**
- **After 30 years, the account contains $432,194.20**

But who wants to wait that long? ☺

Compound Interest



Modeling with Recurrence Relations

Another example:

- **Let a_n denote the number of bit strings of length n that do not have two consecutive 0s ("valid strings")**
- **String 10001 also excluded: has ≥ 2 consecutive 0s!**
- **Find a recurrence relation and give initial conditions for the sequence $\{a_n\}$**

Solution Idea:

- **The number of valid strings is the number of strings ending with a 0 (i.e. the bit before is not also a 0) . . .**
- **Plus the number of strings ending with a 1 (o.k. if the bit before that ending 1 is a 0)**

Modeling with Recurrence Relations

- We don't know the number a_n
- Let us assume that we do know the solution a_{n-1} of valid strings of length n-1
- Then how large is the subset of valid strings of length n, if these strings end with digit 1?
- There are a_{n-1} such strings, namely the set of all valid strings of length n-1 with 1 appended to them
- Whenever we append a 1 to a valid string, that string remains valid, even if there are two 1s in a row or a 01 at the end, which is ok

Modeling with Recurrence Relations

- We still need to know: How many valid strings of length n are there that end with digit 0?
- Valid strings of length n ending with 0 must have a 1 as their $(n-1)^{st}$ bit, otherwise they would end with 00, and that would be invalid
- So what is the total number of valid strings of length n-1 that end with digit 1?
- We already know that there are a_{n-1} strings of length n that end with digit 1
- Therefore, there are a_{n-2} strings of length n-1 that end with digit 1

Modeling with Recurrence Relations

- So there are a_{n-2} valid strings of length n that end with a 0: i.e. all valid strings of length n-2 with 10 appended to them
- Reiterate: The total number of valid strings is the number of valid strings ending with a 0, plus the number of strings ending with a 1
- The solution is the recurrence relation:

$$a_n = a_{n-1} + a_{n-2}$$

Modeling with Recurrence Relations

- What are the initial conditions?

  $a_1 = 2$ 0 and 1, neither has 2 consecutive 0s ☺

  $a_2 = 3$ 01, 10, and 11

  $a_3 = a_2 + a_1 = 3 + 2 = 5$

  $a_4 = a_3 + a_2 = 5 + 3 = 8$

  $a_5 = a_4 + a_3 = 8 + 5 = 13$ . . .

- This sequence satisfies the same recurrence relation as the Fibonacci sequence
- Since $a_1 = f_3$ and $a_2 = f_4$, we have $a_n = f_{n+2}$
- Recall f_i sequence: $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, . . .

Solving Recurrence Relations

- In general, we would prefer an explicit formula to compute the value of a_n rather than conducting n iteration steps
- For some class of recurrence relations, we can obtain such formulae in a systematic way
- Those are the recurrence relations that express the terms of a sequence as a linear combination of previous terms

Solving Recurrence Relations

- **Definition**: A **linear homogeneous recurrence relation** of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

- Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$
- A sequence satisfying such a recurrence relation is uniquely determined by that relation and **k initial conditions**

Solving Recurrence Relations

Examples:

- The recurrence relation $P_n = (1.05)P_{n-1}$ is a linear homogeneous recurrence relation of degree one
- Recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two
  - But due to index n-2, not due to the two summands ☺
  - Requiring 2 initial conditions
- The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five
  - As it goes back 5 generations of prior solutions
  - Requiring 5 initial conditions

Solving Recurrence Relations

- When solving such recurrence relations, we aim to find solutions of the form $a_n = r^n$, where r is constant
- $a_n = r^n$ is a recurrence relation solution $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if:

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$$

- Divide this equation by r^{n-k} and subtract the right-hand side from the left:
- $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_{k-1}r - c_k = 0$
- This is called the **characteristic equation** of the recurrence relation

Solving Recurrence Relations

- Solutions of this equation are called the characteristic roots of the recurrence relation
- Let us consider linear homogeneous recurrence relations of degree two
- Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2
- Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants

Solving Recurrence Relations

- **Example**: What is the solution of recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
- **Solution**: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$
- Its roots are $r = 2$ and $r = -1$
- Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:
- $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2

Solving Recurrence Relations

- Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2$$

$$a_1 = 7 = \alpha_1 * 2 + \alpha_2 * (-1)$$

- Solving these two equations gives $\alpha_1 = 3$ and $\alpha_2 = -1$
- Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 * 2^n - (-1)^n$$

Solving Recurrence Relations

- $a_n = r^n$ is a solution of the linear homogeneous recurrence relation
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$
- if and only if
- $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.
- Divide this equation by r^{n-k} and subtract the right-hand side from the left:
- $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$
- This is called the characteristic equation of the recurrence relation

Solving Recurrence Relations

- **Solutions of this equation are called the characteristic roots of the recurrence relation**
- **Consider linear homogeneous recurrence relations of degree two**
- **Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2**
- **Then the sequence $\{ a_n \}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants**

Solving Recurrence Relations

- **Example: Give an explicit formula for Fibonacci Numbers**
- **Solution: Fibonacci Numbers satisfy recurrence relation**

  $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$
- **The characteristic equation is: $r^2 - r - 1 = 0$**
- **Its roots are** $r_1 = \dfrac{1+\sqrt{5}}{2}, \quad r_2 = \dfrac{1-\sqrt{5}}{2}$

Solving Recurrence Relations

Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2. We can determine values for these constants so that the sequence meets

$f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0 \qquad f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

Solving Recurrence Relations

The unique solution to this system of two equations and two variables is:

$$\alpha_1 = \frac{1}{\sqrt{5}}\ ,\ \ \alpha_2 = -\frac{1}{\sqrt{5}}$$

Explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations

- What happens if the characteristic equation has just one root?
- How can we match our equation with initial conditions a_0 and a_1?
- Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0
- A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if
- $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants

Solving Recurrence Relations

- **Example**: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?
- Solution: The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$. Hence, the recurrence relation solution is
- $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2
- To match the initial condition, we need

$$a_0 = 1 = \alpha_1$$
$$a_1 = 6 = \alpha_1 * 3 + \alpha_2 * 3$$

- Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$
- Consequently, the overall solution is given by

$$a_n = 3^n + n * 3^n$$

Summary

- Recursion and recurrence relations share strong similarities
- Yet recurrence relations don’t provide initial values
- Thus are open to “customization” for any particular formula
- Recursion in contrast must define initial conditions
- Moreover, recursion is not constrained to mathematical formulae
- Recursion can be applied to complex algorithms; not constraint to formal, mathematical expressions
- Typical applications of recursive definitions in addition to mathematical formulations: Language definitions, spec. programming languages!

References

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