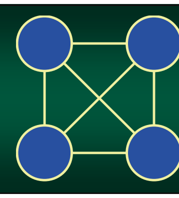
Graphs

Part 14

1

Introduction to Graphs



Nope, not the same as charts

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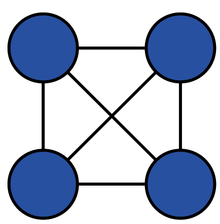
Graphs

- Lists and trees are just a special case of another structure - the *graph*
- Graphs are the basis for <u>all</u> of computer science



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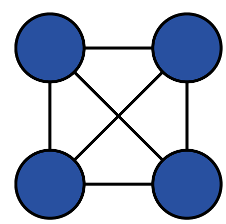
Graphs

- Computer science is not about chips, processors, etc...
- ... this is just implementation technology



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Where are Graphs Used?

- The easy answer is: *everywhere*
- In computer science
  - state machines
  - mazes and networks
- Other fields
  - chemistry
  - physics
  - Government?

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Motivation

- They are one of the pervasive data structures used in computer science
- Several real-life problems can be converted to problems on graphs
  - they are useful tool for modeling real-world problems
  - allows us to abstract details and focus on the problem

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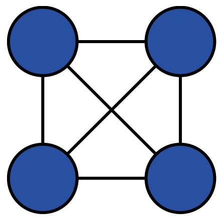
Terminology

- The terminology for graphs is a bit different from trees and linked lists
- Rather, it is more generalized
  - nodes are called *vertices*
  - branches are called *edges*



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Formal Definition

A graph $G = (V, E)$ is defined by a pair of two sets:

a finite set V of items called *vertices*

a finite set E of vertex ordered-pairs called *edges*

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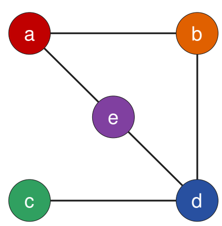
Example Graph

- Set of vertices V
  - a
  - b
  - c
  - d
  - e



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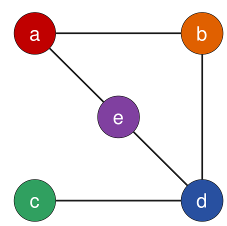
Example Graph

- Set of edges E
  - (a, b)
  - (a, e)
  - (b, d)
  - (c, d)
  - (d, e)



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Adjacent and Incident

- When two vertices share an edge (x, y)
  - they are said to be *adjacent*
  - in other words, they are connected
- The edge (x, y)
  - is called *incident* on vertices x and y
  - in other words, it is the connection

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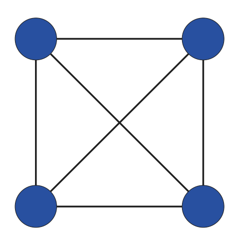
Complete Graphs

- A *complete graph* is one in which all pairs of vertices are adjacent
- In other words, every vertex is connected to every other vertex



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Complete Graphs

- The # of edges in a complete graph...
  - if *n* is the total number of vertices, then each vertex is incident to *n - 1* edges
  - we can compute $n \times (n - 1)$ edges, but this would count each edge twice!
  - so, the number of edges = $n \times (n - 1) / 2$
- For a noncomplete graph...
  - number of edges < $n \times (n - 1) / 2$

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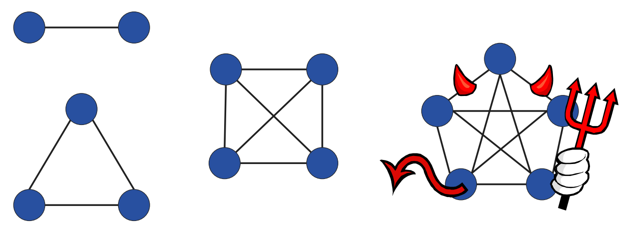
Example Complete Graphs



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Paths

Some edges are one-way streets!

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Paths

- A *path* is a sequence of vertices $v_1, v_2, . . . v_k$ such that consecutive vertices v_n and v_{n+1} are adjacent
- This can represent
  - a physical path
  - logical connection
  - etc...

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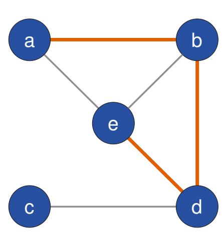
Paths

- In a *simple path*, all edges of a path are distinct
- The *length* of a path is measured by either the total number of edges or vertices



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Connected and Unconnected

- A *connected* graph
  - has a path from every vertex to all other vertex
  - so, everything is connected somehow
- An *unconnected* graph
  - at least one vertex exists in which no path exists to another vertex
  - so, there are 2+ sub-graphs that are unlinked

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Connected and Unconnected

- The *connected component* is the maximum connected subgraph of a given graph
- If the graph is connected, then the whole graph is one single connected component

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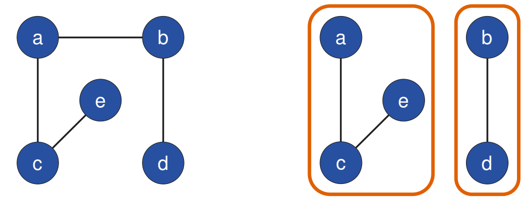
Connected and Unconnected



Connected **Unconnected**

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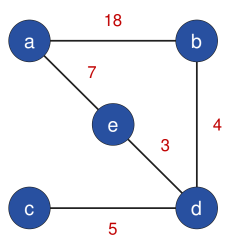
Weighted Graphs

- *Weighted graph* has values on each edge
- These values can be anything – and are defined by the ADT using the graph



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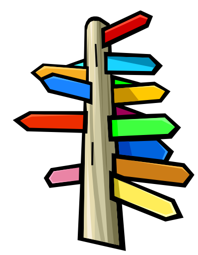
Weighted Graphs

- These weights are abstract and can represent *anything*
- Examples
  - distances – driving, flight paths
  - costs
  - server latency times
  - etc...



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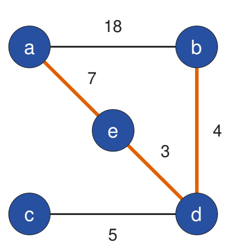
Minimum Path

- Often it is useful to find the minimum path – e.g. the smallest sum of edges
- Example: minimum path from a to b is: a → e → d → b
- We'll cover that soon!



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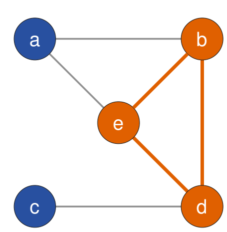
Cycles

- A *cycle* is sequence of vertices v_1, v_2, …v_k where…
  - v_n and v_{n+1} are adjacent
  - $v_1 = v_k$
- So, it's a loop of vertex



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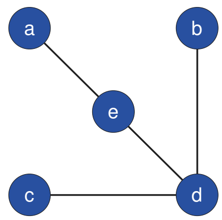
Cycles

- An *acyclic graph* does not contain any possible cycles
- This definition will come up later as an essential attribute for some solutions



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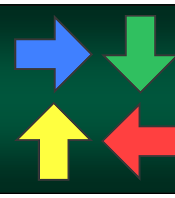

Directed & Undirected

Some edges are one-way streets!

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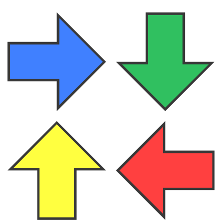
Directed & Undirected

- Sometimes, an edge can be travelled both directions
- However, sometimes, they have a distinct source and destination
- Think of it as streets – some are two-way, and some are one-way



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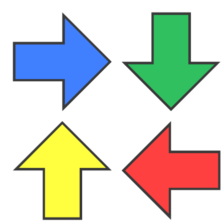
Directed Graph

- A *directed graph (digraph)* is a graph where each edge has a source and target vertex
- This is the basis most of the data structures used today:
  - trees
  - linked lists



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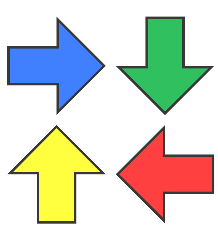
Undirected Graphs

- Undirected graphs have edges that have the same meaning for both directions
- So, if there is an edge (a, b) then there is also an edge (b, a)



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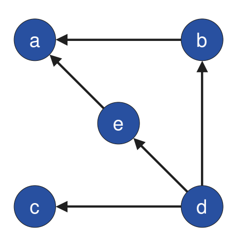
Directed Acyclic Graphs

- When a directed graph lacks cycles, it is called a *Directed Acyclic Graph* (DAG)
- Since edges only travel one way, this leads to some interesting graphs



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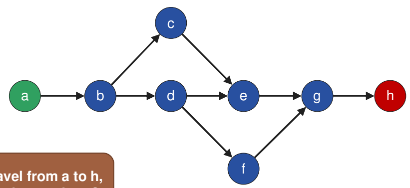
Examine this DAG: Path from a to h



If we want travel from a to h, how many paths are there?

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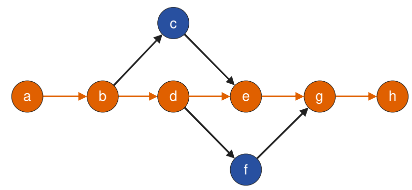
Examine this DAG: Path 1



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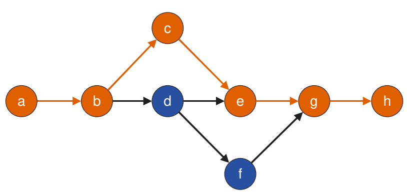
Examine this DAG: Path 2



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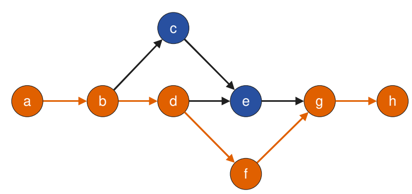
Examine this DAG: Path 3



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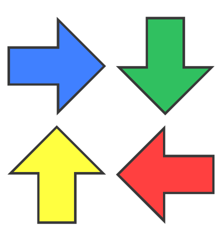
Directed Acyclic Graphs

- In the last example, there are 3 paths from a to h
- Yet, there are **no** cycles
- So, while a DAG contains no cycles, there may be multiple paths from one node to another



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Trees & Arborescence

A graph is a tree… if…

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Arboresence

- An *arborescence* is a directed graph where, if there is a path from one node to another, it is **unique**
- So, there is at most one path from a vertex A to B

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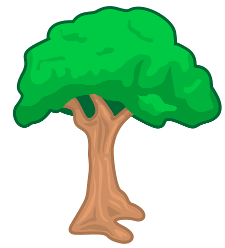
Alternative terms for Arborescence

- There are alternative terms (all of which are valid)
- Such as….
  - directed rooted tree
  - out-arborescence,
  - out-tree
  - branching



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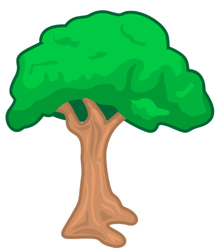
Trees

- All trees are acyclic graphs (which may be directed)
- But, not all DAGs are trees
  - DAGS allow multiple paths to the same node
  - since they directed, this won't create cycles



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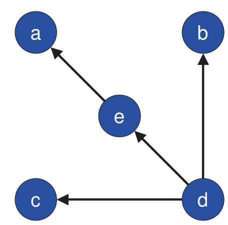
Trees

- *Rooted Tree* selects an arbitrary vertex as the root
- *Forest* is a collection of trees



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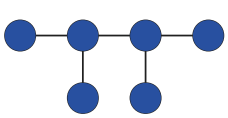
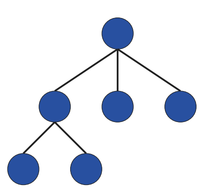
Trees



Tree



Same Tree

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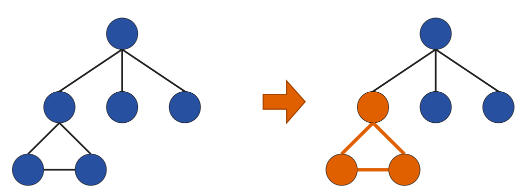
Trees



NOT a Tree

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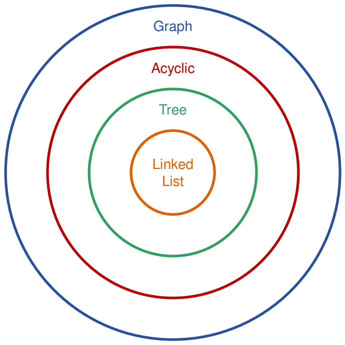

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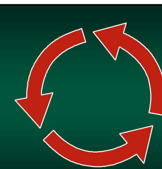
Graph Traversal



Not as easy as trees

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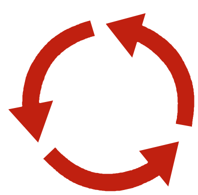
Graph Traversal

- Typically when you search a tree, you can use a simple depth-first search
- However, graphs can contain cycles
- Your program can get stuck in an graph loop and never escape
- This has to be taken into account



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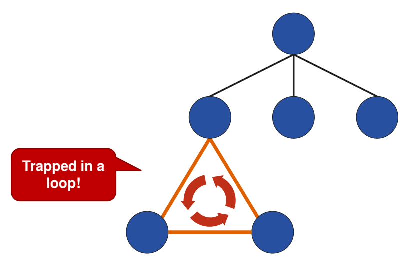
Graph Traversal



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Graph Traversal

- So, to before you search…
  - the vertices need to visited only ONCE
  - otherwise, we have a loop
- *Solution:* vertex has a "visited" property
  - before the search, each vertex is set to false
  - when the search visits them, it is set true
  - search never follows an edge to a visited vertex

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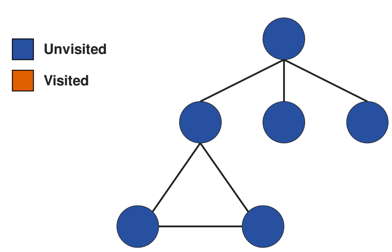
Graph Traversal



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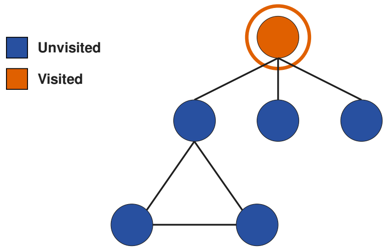
Graph Traversal



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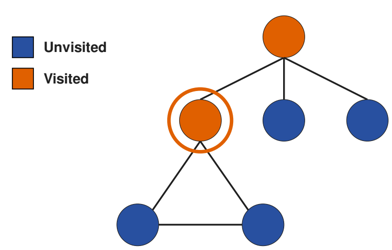
Graph Traversal



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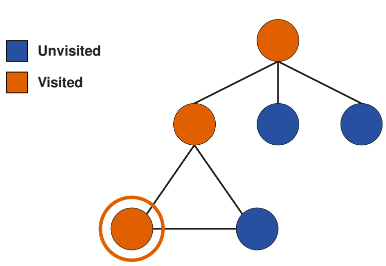
Graph Traversal



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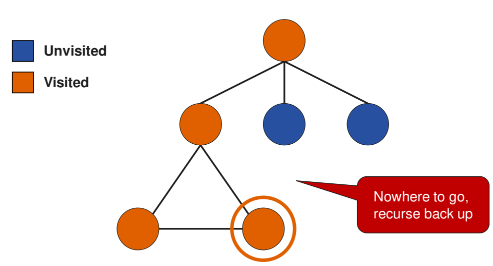
Graph Traversal



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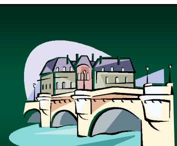

Birth of Graph Theory

... and, later, computer science!

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Birth of Graph Theory

- Graph theory was created due to a perplexing question troubling mathematician *Leonhard Euler*
- Lived in Konigsberg
  - now "Kaliningrad" in Russia
  - city occupied 2 islands plus areas on both banks
  - 7 bridges over the Pregel River

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Birth of Graph Theory

- People wondered they could:
  - leave home…
  - cross every bridge <u>once</u>
  - and return to their starting point
- This is known as the *Konigsberg Bridge Problem* & was unsolved in the 1700's

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Konigsberg Bridge Problem

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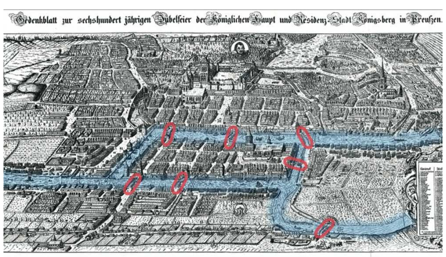
Konigsberg Bridge Problem



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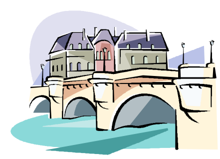
Konigsberg Bridge Problem



- The problem reduces to 4 points and several links to between the points
- From this, Euler created the first graph and began the study of their properties

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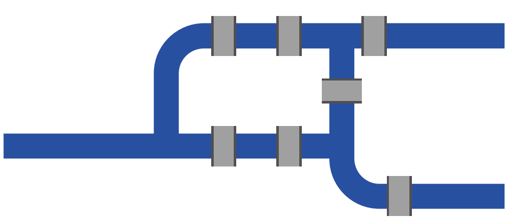
Konigsberg Bridge Problem



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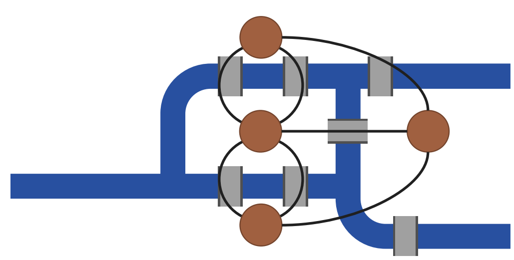
Konigsberg Bridge Problem



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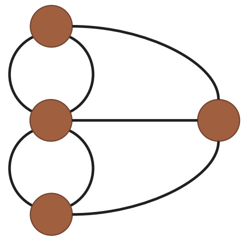
Konigsberg Bridge Problem



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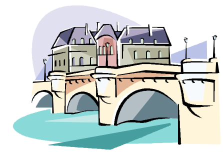
The Solution to Konigsberg

- In 1736, Euler proved that no such traversal exists
- *Eulerian circuit,* in a graph...
  - is cycle containing all the edges in the graph
  - and only traversing each edge once



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The Solution to Konigsberg

- Euler proved:
  - a graph may have an Eulerian circuit if and only if there is no vertex with an odd number of edges
- Konigsberg Bridge Problem
  - 4 vertices, all with an odd number of edges
  - Sorry people of Konigsberg, there is no solution!

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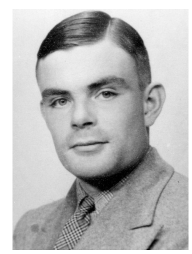
Alan Turing

- Mathematician, logician & cryptographer
- *Father of Computer Science*
  - Highest award in Computer Science is the *Turing Award*
  - Developed Turing Machines



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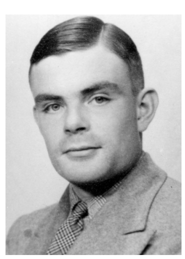
Major Work: Turing Machines

- Invented in 1937
- Logical model – not an actual computer or machine
- Based on 2 graphs (and sets on each of the edge)



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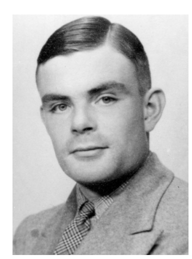
Major Work: Turing Machines

- One graph is simple array, but the other could be anything
- From this, he proved programming



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Major Work: Turing Test

- Used in artificial intelligence
- Consists of a human operator *texting* a human or computer
- If the operator can't ascertain if it is a computer or human, the computer is "intelligent"
- No computer has passed it

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Real World Examples

The origin and the usage

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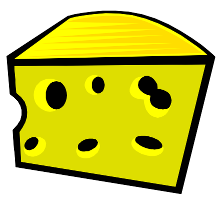
Real World Examples

- How can we lay cable at minimum cost to make every network reachable from every other?
- What is the fastest route from the national capital to each state capital?
- How can *n* jobs be filled by *n* people with maximum total utility?



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The London Underground Subway

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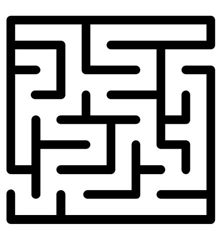
Maze Traversal

- One example of where a graph is useful is a maze traversal
- Basically, any maze can be represented with a graph
- ... and this is not so much different to how networks actually work
- ... a source must find a destination through various vertices



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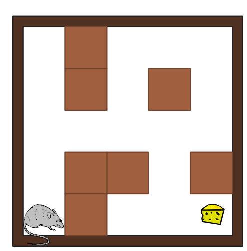
Maze Traversal

- This is a simple maze – though not to the mouse!
- We can help him find the cheese if we convert this to a graph



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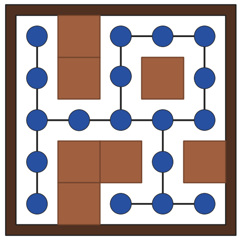
Maze Traversal

- The empty spaces are vertices
- The bordering ones are connected with an edge
- Is this a directed graph?



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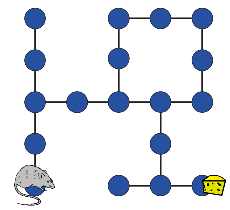
Maze Traversal

- So, to help the mouse, we can get to depth-first search on the maze
- If we find it, we can print off the vertices post-order



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