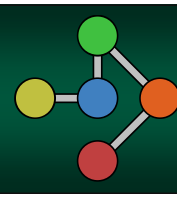
Graph Spanning Trees

Part 15

1

Spanning Trees



Connect the Dots (la la la la)

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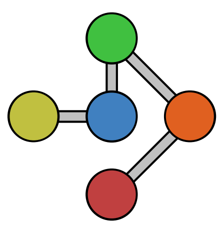
Spanning Trees

- In many cases, we want to make sure that every vertex is connected in a graph
- A *Spanning Tree* includes every vertex of the original graph (an no cycles, obviously)



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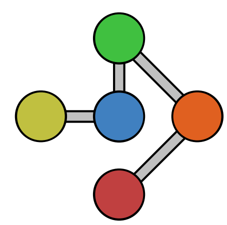
Why Spanning Trees?

- Often, we want to make sure all vertices are connected
- These can be anything from cities (connected by roads) to computers (connected by cables)
- ... just to name a few…



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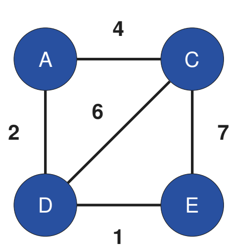
Example Graph

- The graph on the left is quite simple – containing just 4 vertices
- We need to create a tree that contains all 4 vertices



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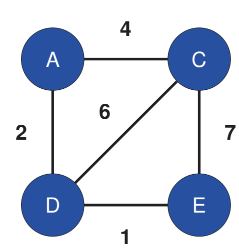
Example Graph

- As you can imagine, most graphs have multiple solutions
- Also note that this graph is weighted
- This can make some solutions better than others



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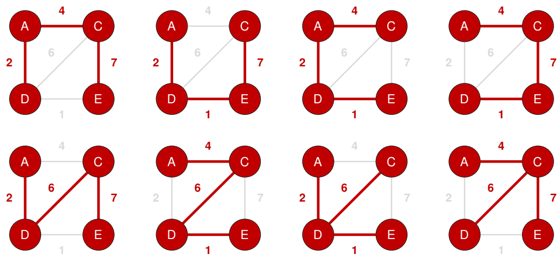
Example: Spanning Trees



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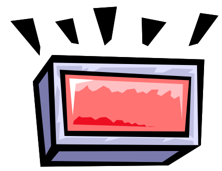
So How Many Spanning Trees?



- Let's look at the most complex graph possible – a complete one
- In this one, every vertex is connected to every other vertex
- *This results in a **large** number of possible solutions*

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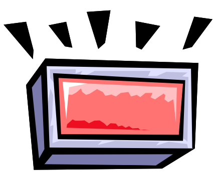
So How Many Spanning Trees?



- Given a complete graph of *N* vertices, there are N^{N-2} possible spanning trees
- Yes, that is N to the power of N – which is even *worst* than exponential growth!

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Complete Graph Spanning Trees

Vertices	Total Spanning Trees	Vertices	Total Spanning Trees
1	1	7	16,807
2	1	8	262,144
3	3	9	4,782,969
4	16	10	100,000,000
5	125	11	2,357,947,691
6	1,296	12	61,917,364,224

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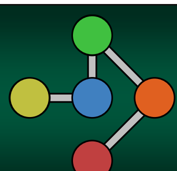

Minimum Spanning Tree

Getting the "Best" Tree

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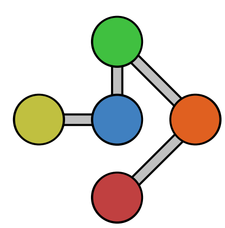
Minimum Spanning Tree

- *Minimum Spanning Tree (MST)* contains all the vertices of a graph with the minimum possible weight
- Naturally, depending on the weights, some solutions are more desirable than others



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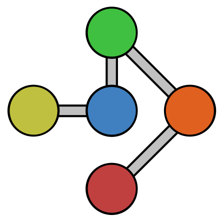
Minimum Spanning Tree

- Some uses (minimize cost)
  - building cable networks
  - building a road that joins cities
- So, creating an algorithm – that computes these trees - *is of vital importance*



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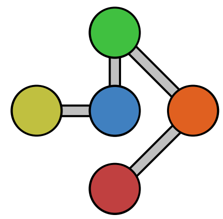
Uniformly Weighted Graph



- If each edge has the same weight, then the computation of the minimum spanning tree is trivial
- A breadth-first or depth-first search can be used
- …but many are weighted

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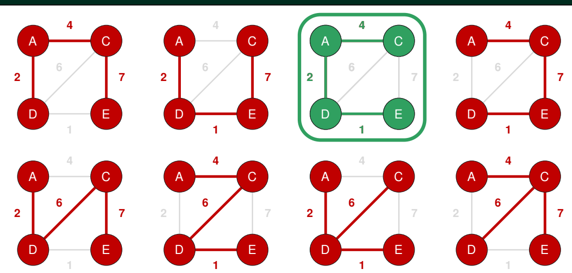
Example: Minimal Spanning Tree



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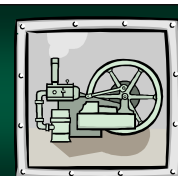

Brute Force

Doing in the hard way
(and impossible)

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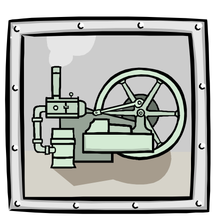
Brute Force

- One way to compute the minimum spanning tree is to simply ***try all of them!***
- Approach:
  - calculate every tree
  - keep track of the tree with the minimum total weight



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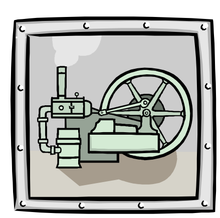
Brute Force

- Calculating just one tree will require O(n) time where *n* is the total number of vertices
- How many trees are there?
- For complete graphs, we know it's n^{n-2} – which is what we must use for Big-O



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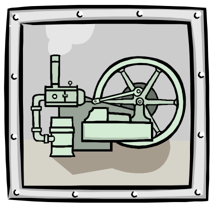
Brute Force

- So, there is a $n \times n^{n-2}$ computational requirement
- Which is $O(n^{n-1})$
- Naturally, this is a poor – and ***impracticable*** – solution



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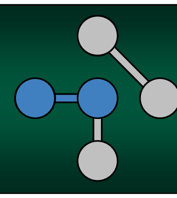

Kruskal's Algorithm

Computing a Minimum Spanning Tree

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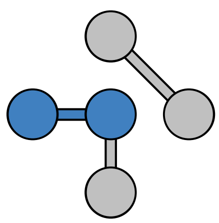
Kruskal's Algorithm

- *Kruskal's Algorithm* computes a minimum spanning tree
- It was invented in 1956 by *Joseph Kruskal* and published in *Proceedings of the American Mathematical Society*



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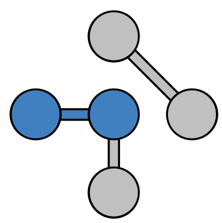
Kruskal's Algorithm



- It conceptually starts with a forest of single vertices
- It then adds edges – with the minimum weight – to connect the two vertices (and possible subtrees)

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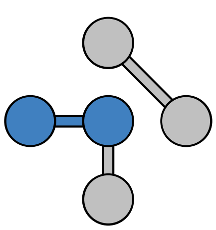
Kruskal's Algorithm



- So, it adds edges in sorted order of weight
- If an edge would cause a cycle, then it is rejected and not part of the solution
- Otherwise, add it.

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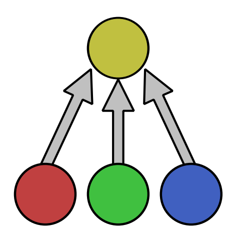
Kruskal's Algorithm

- Cycles are avoided by using a **union-find** to merge subtrees
- Remember: Union-Find allows objects to be unioned (grouped together)



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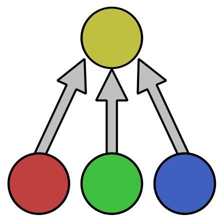
Kruskal's Algorithm

- If any two vertices have the same "representative", then they are already connected.
- So, don't add the edge – it would create a cycle



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Kruskal's Algorithm

```
Create a list of edges sorted by weight
Add each vertex to a Union-Find
loop through the list
   if the vertices have different representatives
      Add the edge
      Union the two vertices (in the Union-Find)
   end if
end loop
```


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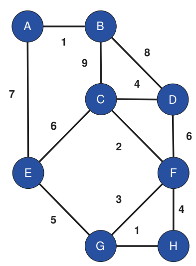
Kruskal's Algorithm Example

- The following is a graph with multiple weights
- Kruskal's Algorithm will create a list of the edges sorted by weight



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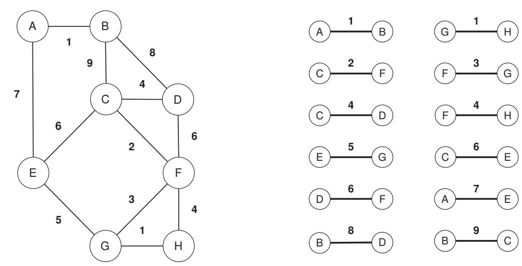
Kruskal's Algorithm Example



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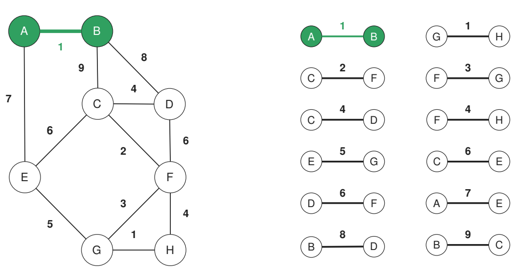
Add Edge



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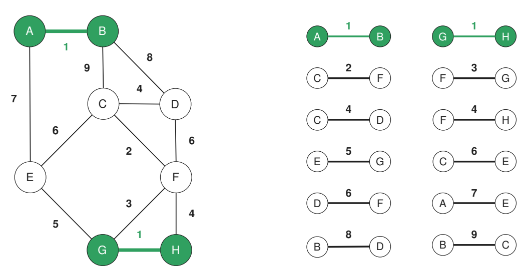
Add Edge



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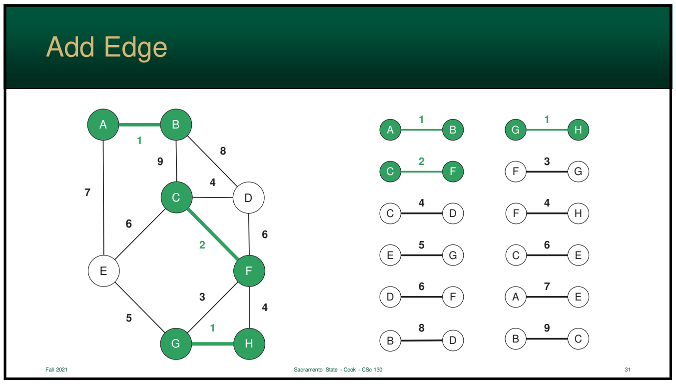

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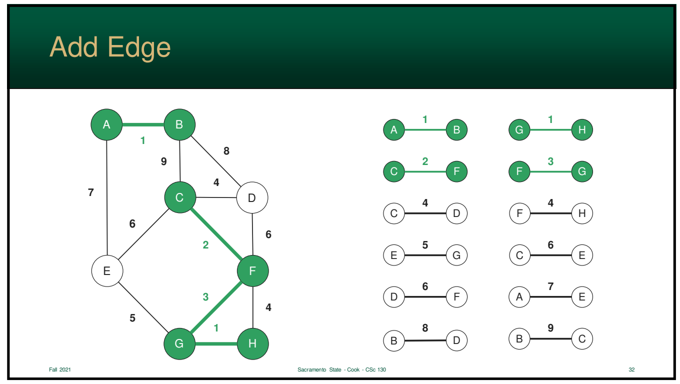

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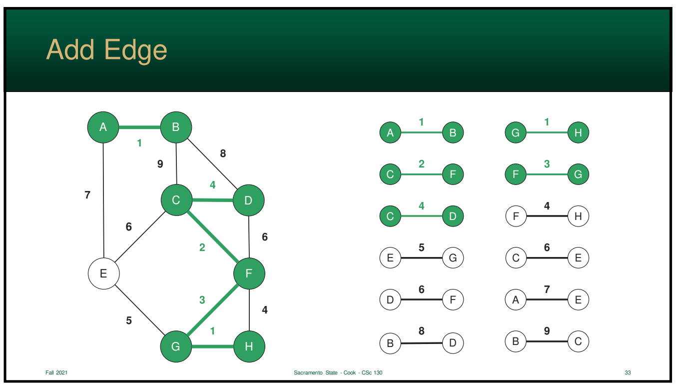

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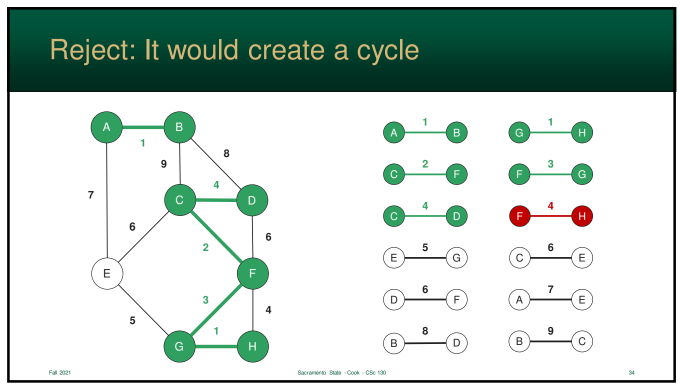

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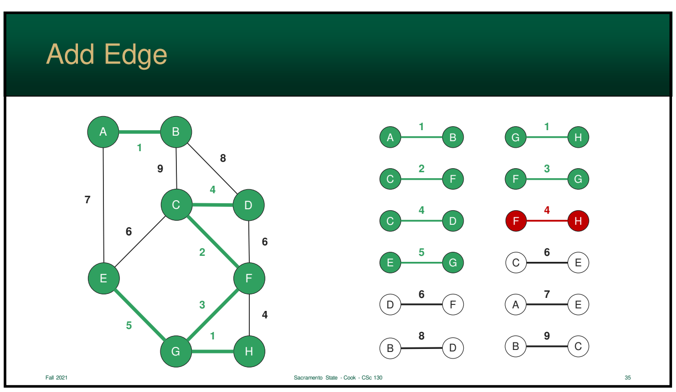

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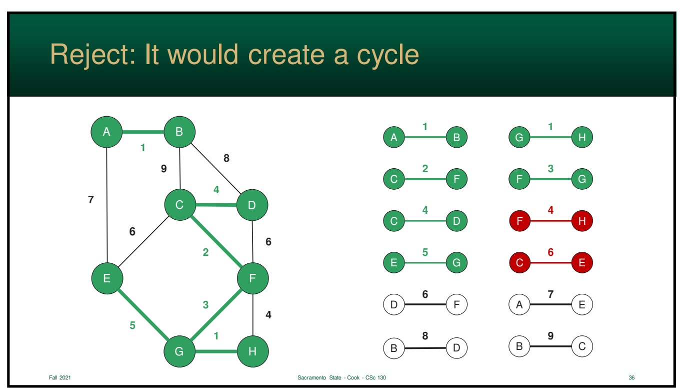

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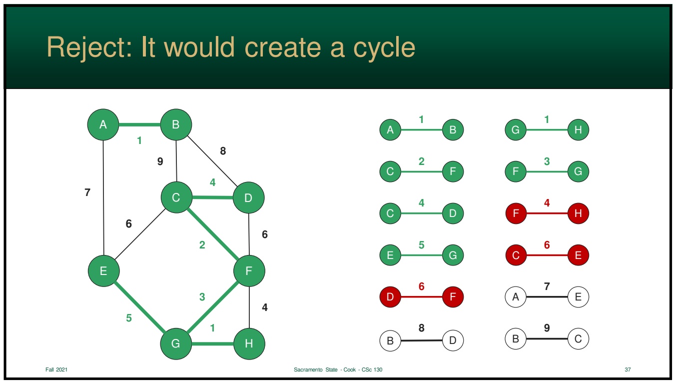

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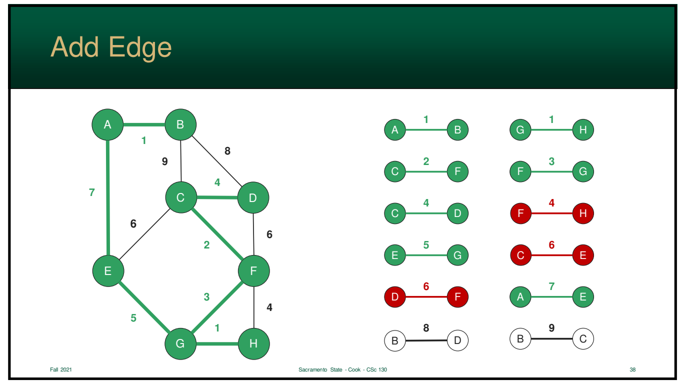

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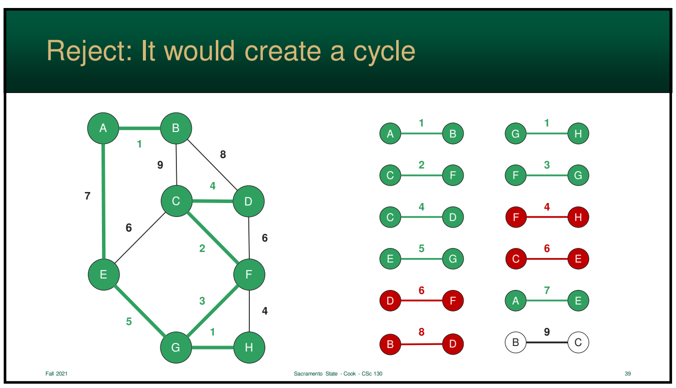

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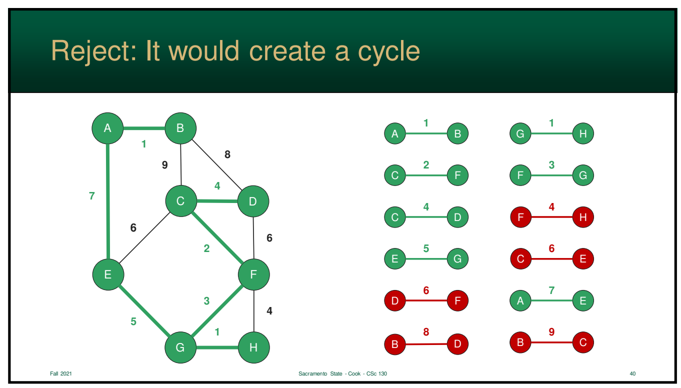

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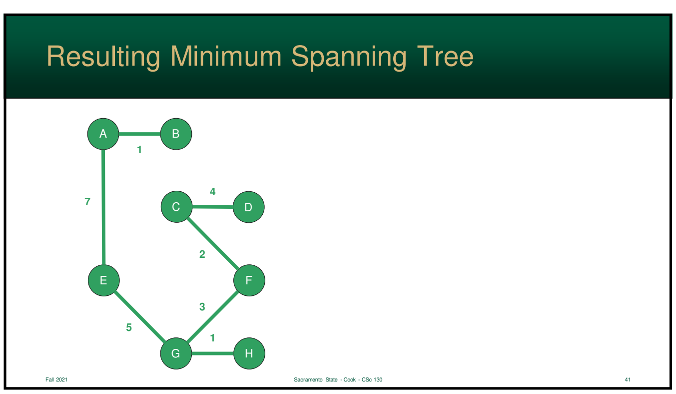

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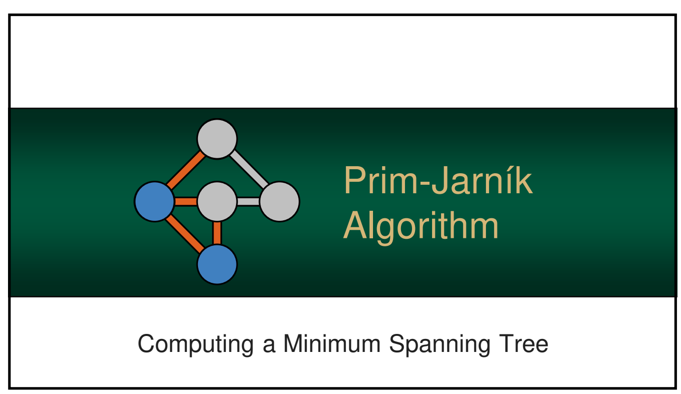

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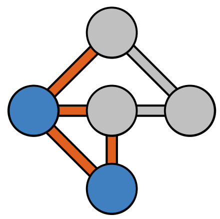
Prim-Jarník Algorithm

- *Prim's Algorithm* computes a minimum spanning tree
- It was invented in 1956 by computer scientist *Robert C. Prim*
- … same year as Kruskal



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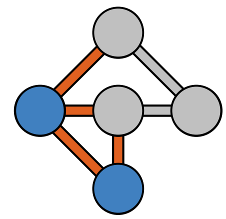
Prim-Jarník Algorithm

- However, it was discovered that computer scientist *Vojtěch Jarník* had also invented it in 1930
- So, it is also called the *Prim-Jarník Algorithm*



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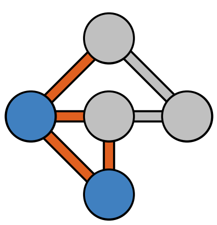
Prim-Jarník Algorithm



- This algorithm starts with an *arbitrarily chosen*, vertex
- The algorithm then adds the minimal edge – that the entire current tree can currently "see"

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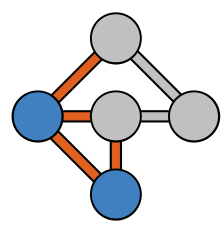
Prim's Algorithm



- So, it checks all the edges connect to unvisited vertices
- The algorithm is *greedy* – always following the local optimal path – to grow the tree

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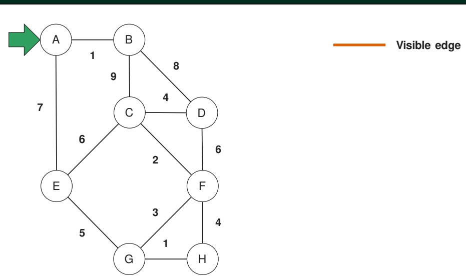
Prim's Algorithm Example



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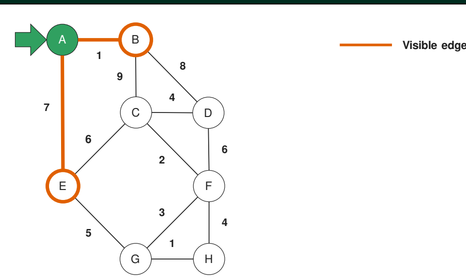
Prim's Algorithm Example



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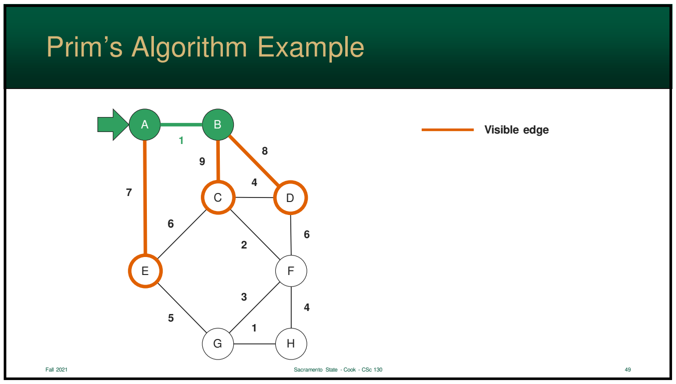

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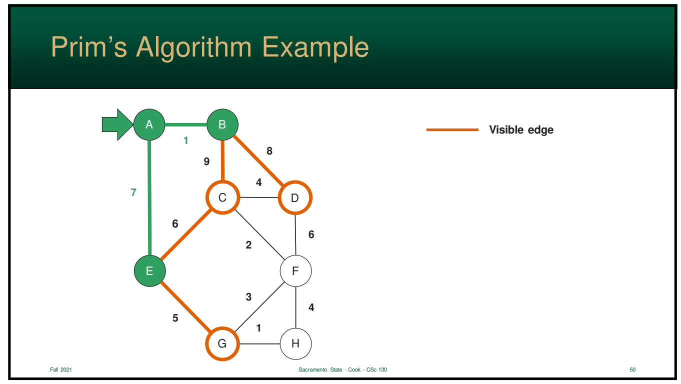

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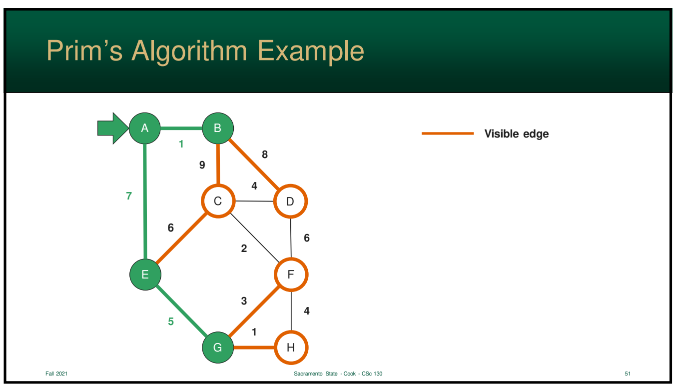

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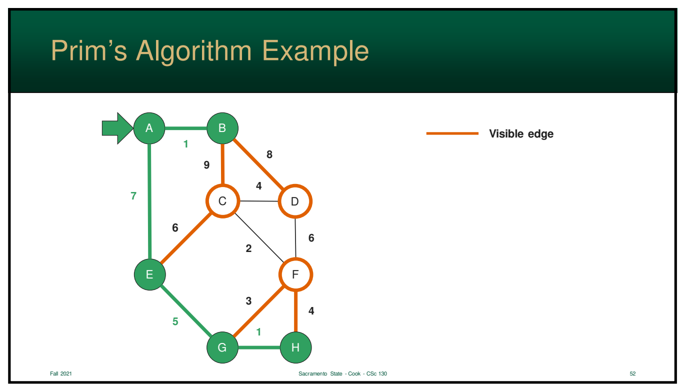

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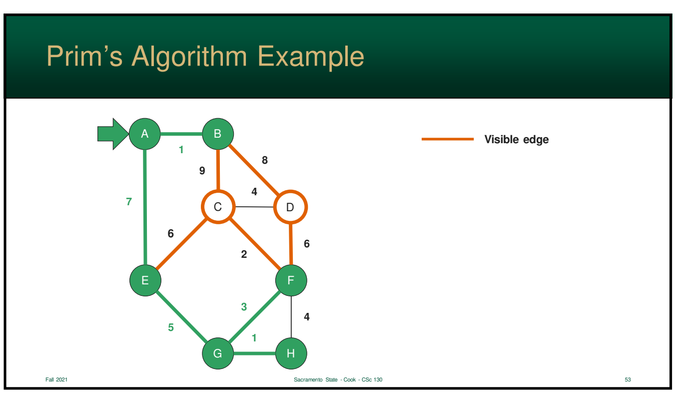

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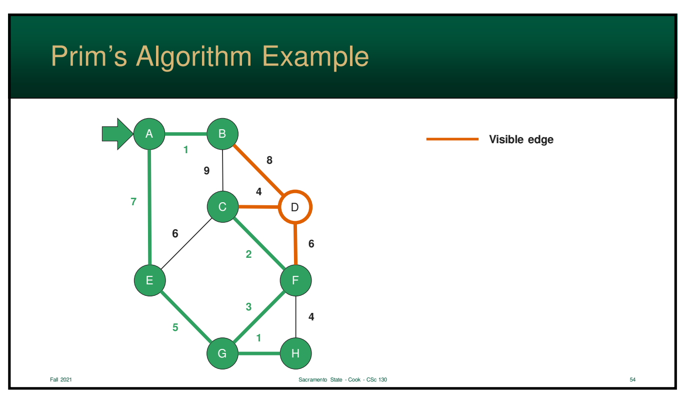

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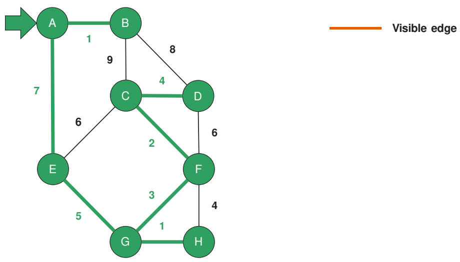
Prim's Algorithm Example



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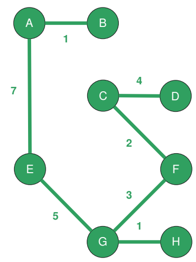
Resulting Minimum Spanning Tree



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Kruskal vs. Prim-Jarník

- Both algorithms find the minimum spanning tree
  - *though not necessarily identical*
  - *… if multiple, equal, solutions exist (they might)*
- Both are O(|E| log |V|) where |E| is the number of edges and |V| is the number of vertices
- Kruskal is far easier to conceptualize, but Prim is far easier to implement

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