CSc 28
Discrete Structures

Chapter 6
Counting

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Syllabus

- Basic Counting Principles
- Inclusion-Exclusion
- Permutation and Combination
- Binomial Coefficients
- Summary
- References

Counting

Basic Counting Principles

- Counting problems are of various kinds; subject sounds simple! ☺
- Here some counting examples:
  - How many different 8-letter passwords are there?
  - What is the probability of winning the lottery?
  - How many possible ways are there to pick 11 soccer players out of 20 members in a soccer club?
- Counting is the basis for computing probabilities of discrete events

Rule of Sum

Basic Counting Principles

The Rule of Sum in combinatorics, with a and b ways:

- ***Rule of sum*** **states: if there are *a* possible outcomes for an event –or *a* ways to do something**
- **. . . And there are *b* possible outcomes for another event –or *b* ways to do another thing**
- **. . . And the two events a and b cannot both occur (i.e. only one event shall happen)**
- **Then there will be a total of: *total = a + b* possible outcomes**
- ***a + b*** **possible choices of some event happening**
- **But only one shall happen**

Basic Counting Principles

Basic Counting Principles

Combinatorics per Wiki: https://en.wikipedia.org/wiki/Combinatorics

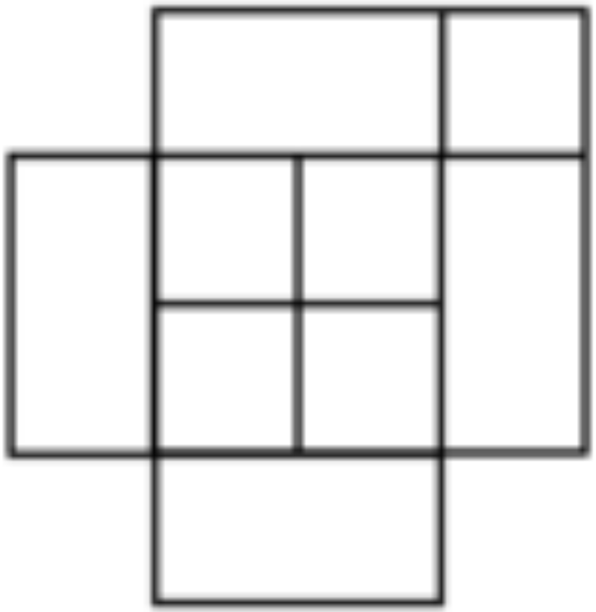
***Combinatorics* is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. It is closely related to many other areas of mathematics and has many applications ranging from logic to statistical physics, from evolutionary biology to computer science, etc.**

Basic Counting Principles

Example for the Rule of Sum:

- A Sac State student decides to shop at one store in Sacramento, either in the north or the south
- If she visits the north, she will shop at the mall, or at a furniture store, or at a jewelry store: 3 options
- Else, if she visits the south part, she will shop at a clothing store or a shoe store: 2 options
- Then there are $3 + 2 = 5$ possible ways she could end up shopping today
- But she shall go shopping just once! Not both South and North of Sacramento!

Basic Counting Principles



How many squares can you find in this figure?

Don't Count Rectangles, Only Squares!

Basic Counting Principles

Did you find all 11 squares? Yes 11!

Rule of Product

Basic Counting Principles

The Rule of Product in combinatorics:

- In contrast to the Rule of Sum: In combinatorics the Rule of Product (AKA multiplication principle) is another basic counting principle
- AKA a fundamental principle of counting
- Stated simply: It is the idea that if there are a ways of doing something
- . . . And b ways of doing another thing
- Then there are a * b ways of performing both actions if they both will be done

Basic Counting Principles

- **Example of:** ***How Many Permutations?***
  - **How many different license plates are there that contain exactly and only 3 of the 26 English letters?**
  - **Note that it is allowed to pick the same letter more than just once! The 'X' may be there 2 or even 3 times ☺**

- **Solution:**
  - **There are 26 possibilities to pick the first letter, then there are also 26 possibilities for the second one, and 26 for the last one**

- **So there are 26 * 26 * 26 = 17,576 different license plates, since repetition is allowed: 26^3 = 17,576**

Basic Counting Principles

Generalized Rule of Product:

If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_m$ that each can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, and they all shall be done, then there are $n_1 * n_2 * \ldots * n_m$ ways to carry out the whole procedure

Basic Counting Principles

How many permutations are there for all four of the *unique* capital letters A B C D? Unique = no repetition!

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

- **4 for the first letter, any of the 4, no matter which!**
- **Times 3 for the second letter, but don't choose the first letter again! That is done already**
- **Times 2 . . .**

Basic Counting Principles

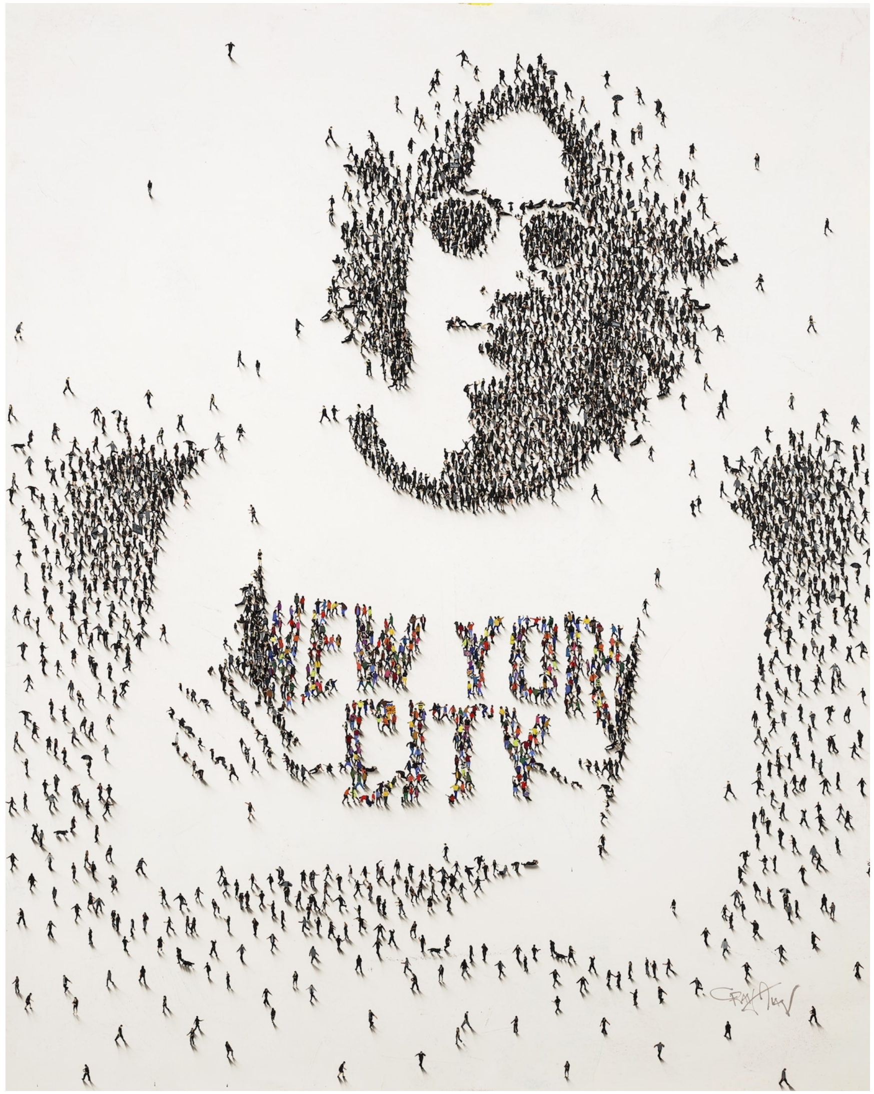
- **The rule of sum (Sum Rule) and rule of product (Product Rule) can be phrased in terms of set theory**
- **Sum rule: Let $A_1, A_2, \ldots, A_m$ be disjoint, finite sets**
- **Then the number of ways to choose one element *from one* of these sets is:**

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

- **Product rule: Let $A_1, A_2, \ldots, A_m$ be disjoint, finite sets**
- **Then the number of ways to choose one element *from each* of these sets is:**

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| * |A_2| * \ldots * |A_m|$$

Basic Counting



Count The Number of People in This Picture!

Probability

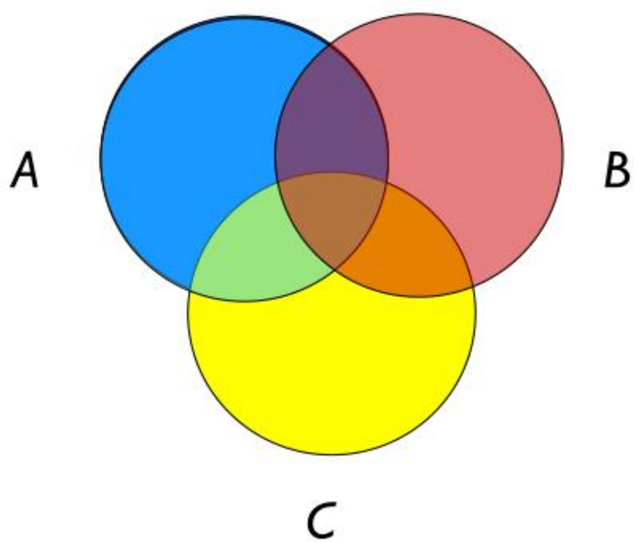
- Wiki: Probability is the extent to which something is probable; the likelihood of something happening or being the case
- Wiki: Probability is the extent to which an event is likely to occur, measured by the ratio of the favorable cases to the whole number of cases possible
- Probability is the likelihood of an event to occur. Thus it can vary between 0 and 1
- Probability is the number of ways of achieving success divided by the total number of possible outcomes

Inclusion - Exclusion

Inclusion-Exclusion 3 Sets

Inclusion-Exclusion (3 sets)

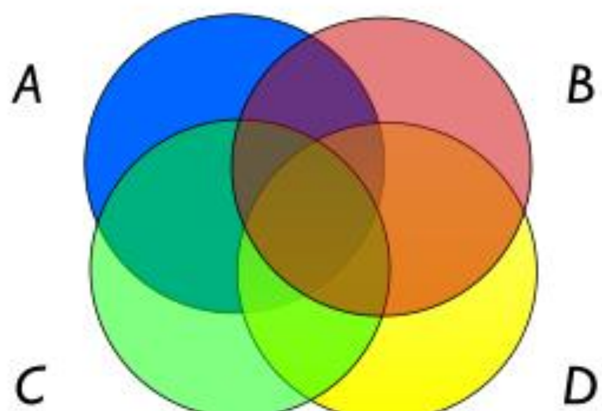
$$\begin{aligned}|A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| \\ + |A \cap B \cap C|\end{aligned}$$



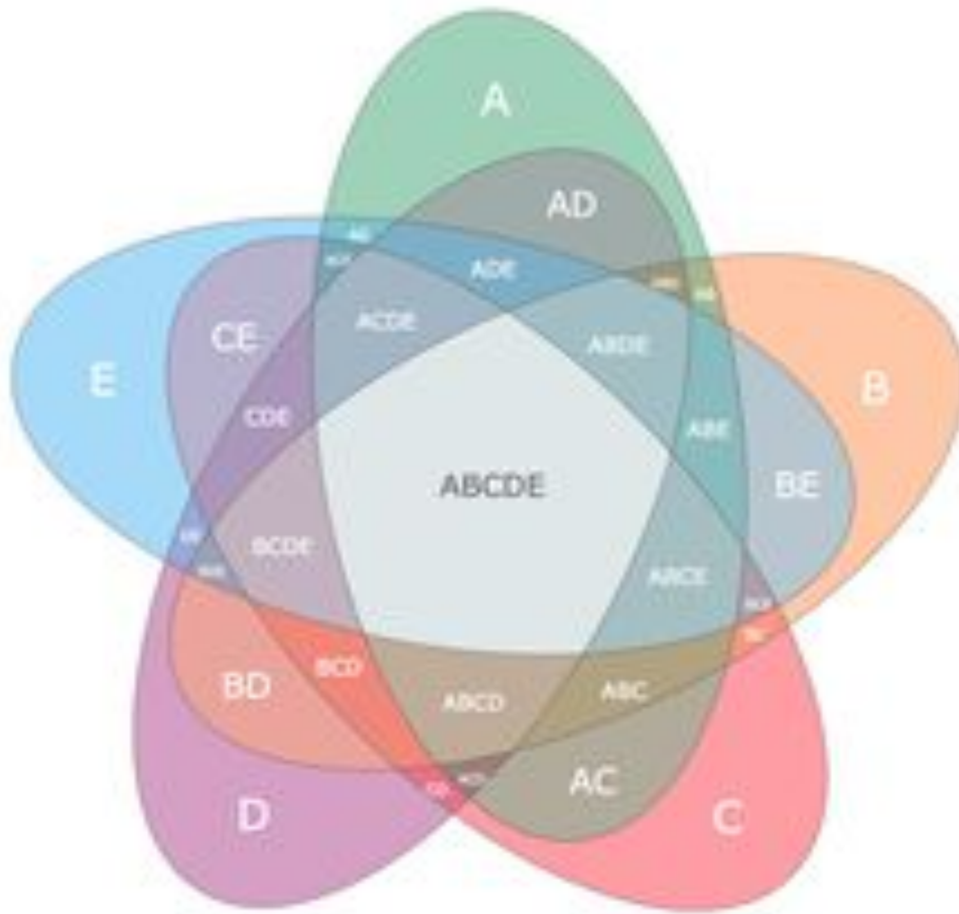
Inclusion-Exclusion 4 Sets

Inclusion-Exclusion (4 sets)

$$
\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\
& - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
& + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
& - |A \cap B \cap C \cap D|
\end{aligned}
$$



Inclusion-Exclusion 5 Sets



Inclusion-Exclusion

- **How many bit strings of length 8 either start with one digit 1 or end with the two digits 00?** (Inclusive or)
- **Compute both Task 1 and Task 2:**

- **Task 1: Construct bit strings of length 8 starting with a single digit 1**
  - **There is one way to pick the first bit (1)**
  - **Two ways to pick the second bit (0 or 1),**
  - **Two ways to pick the third bit (0 or 1),**
  - **. . .**
  - **And two ways to pick the eighth bit (0 or 1)**

- **Task 1 can be done in $1 * 2^7 = 128$ ways**

Inclusion-Exclusion

- Task 2: Construct bit string, length 8 that ends with 00
  - There are two ways to pick the first bit (0 or 1),
  - Two ways to pick the second bit (0 or 1),
  - . . . for third to fifth
  - Two ways to pick the sixth bit (0 or 1),
  - One way to pick the seventh bit (0), and
  - One way to pick the eighth bit (0)

- Task 2 can be done in $1 * 1 * 2^6 = 64$ ways

Inclusion-Exclusion

- There are 128 ways to do Task 1 and 64 ways to do Task 2
- Does this mean that there are 128 + 64 = 192 bit strings either starting with 1 or ending with 00?
- No, because here Task 1 and Task 2 can be done at the same time
- When we carry out Task 1 and create strings starting with 1, some of these strings end with 00
- We cannot double-count them!
- Therefore, we sometimes do Tasks 1 and Task 2 at the same time, so the sum rule does not apply

Inclusion-Exclusion

- **To count correctly: subtract cases when Tasks 1 and Task 2 are done simultaneously!**
- **I.e. eliminate duplicates!**
- **How many duplicate cases are there, that is, how many strings start with 1 *and* end with 00?**
  - **There is one way to pick the first bit (1),**
  - **Two ways for the second,**
  - **Two ways for the third bit (0 or 1)**
  - **. . . fourth to sixth**
  - **One way for the seventh bit (0)**
  - **One way for the eighth bit (0)**
- **Thus 5 bits can be varied! = 2^5**
- **In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time**

Inclusion-Exclusion

- Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are: 128 + 64 - 32 = 160 ways to do either task

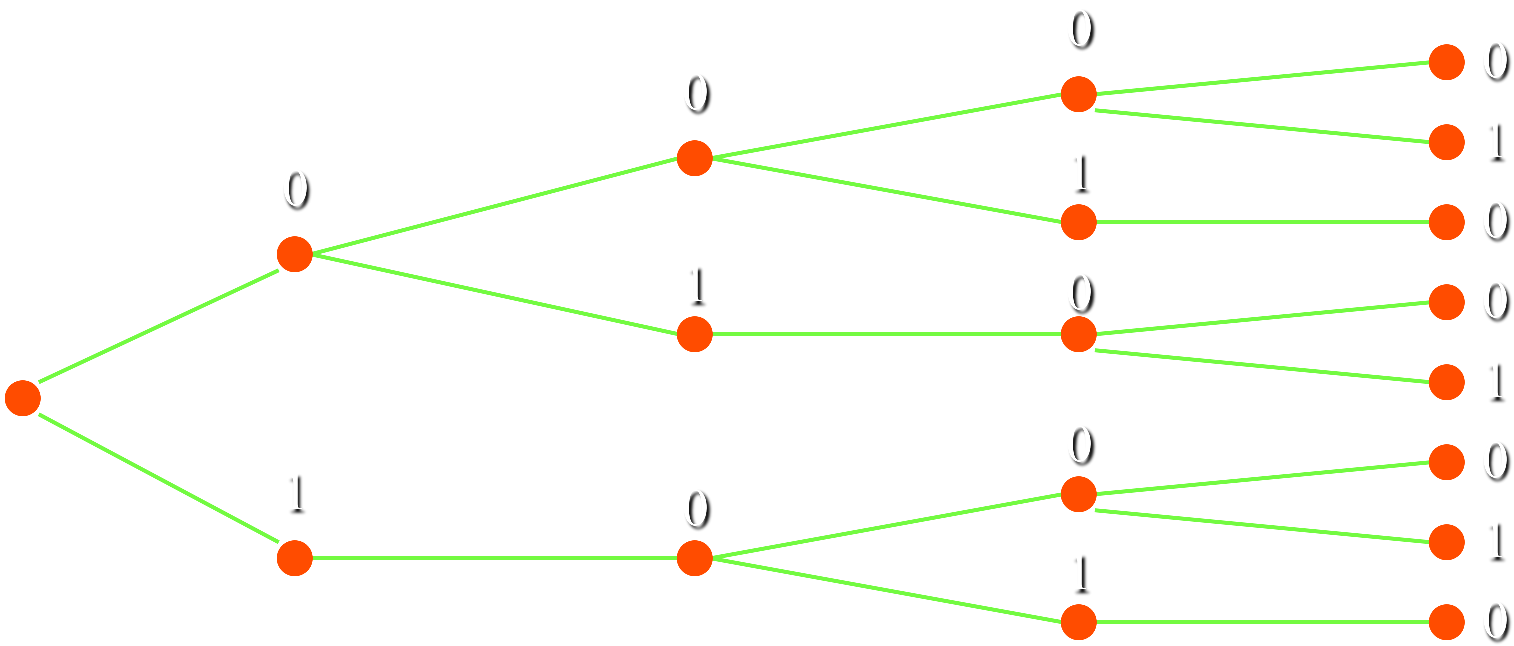
- In set theory, this corresponds to sets A_1 and A_2 that are not disjoint. Then we have:
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

- This is called the principle of inclusion-exclusion

Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?

Task 1 (1st bit) **Task 2 (2nd bit)** **Task 3 (3rd bit)** **Task 4 (4th bit)**



There are 8 strings

Pigeonhole Principle

Pigeonhole Principle

At Most One Occupant Per Slot; Not Two!

Pigeonhole Principle

- Pigeonhole principle: If $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects

- Example 1: If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored two goals or more!

- Example 2: If you have 6 classes at Sac State Monday through Friday, there must be at least one day on which you have two or more classes

Pigeonhole Principle

- The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects

- Example 3: In some 60-student class, at least 12 students got the same letter grade A, B, C, D, or F

Pigeonhole Principle

- Example 4: Assume you have a drawer containing a random distribution of one dozen brown socks and one dozen black socks. It is dark, so you just grab some. How many socks must you grab minimally to be sure that among them be a matching pair?

- There are 2 types of socks, so if you pick at least 3 socks, there must be either at least 2 brown socks or at least 2 black socks

Permutation & Combination

Permutation and Combination

Source Wiki: https://medium.com/i-math/combinations-permutations-fa7ac680f0ac

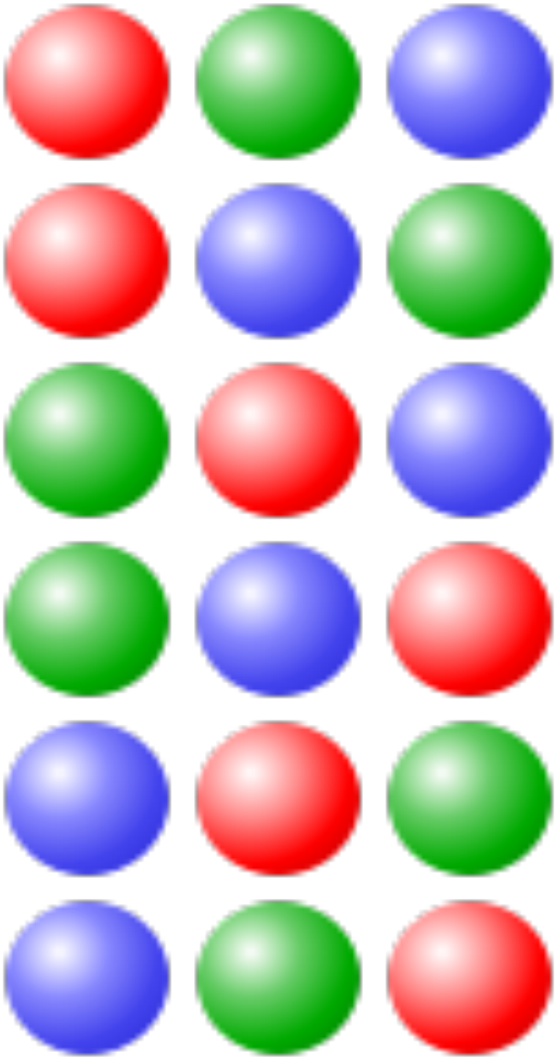
The difference between combinations and **permutations** is ordering. With **permutations** we ***care about the order*** of the elements, whereas with **combinations** we ***don't***. For example, say your locker "combo" is **5432**. If you enter 4325 into your locker it won't open because that is a different ordering of the 4 digits (i.e. a different **permutation**) yet has all the same digits! Just in a different order!

Clearly a "combination lock" should more correctly be named a "**permutation lock**"! ☺

Permutation and Combination

Rule of thumb: **The number of permutations is generally greater than the number of combinations**

Permutations



3 Balls, 3 Colors, No Repeat Color, Yields 6 Permutations

Permutation and Combination

In general, a permutation P(n,k) has more choices than a combination C(n,k)

Permutation Formula: **P(n,k) = n! / (n-k)!**
Combination Formula: **C(n,k) = P(n,k) / k!**
Combination Formula: **C(n,k) = n! / ((n-k!) * k!)**
Combination Formula: **C(n,k) = n choose k**

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!\,(n-k)!} & \text{for } 0 \le k < n \\ 0 & \text{otherwise,} \end{cases}$$

Case k = 0: C(n,0) = n! / ((n-0)! * 0!) = n! / (n! * 1) = 1

Permutation and Combination

All permutations of 2, 3, 4, and 5, are shown below

Note: digits in this example must be unique, thus we cannot, for example, pick the 3 twice! But as we discuss permutations, the 3 may be at any of the four positions!

There are 4 choices to chose the first digit; 3 choices for the second; 2 for the third position; only 1 left for the fourth position, yielding 4 * 3 * 2 * 1 = 24 permutations!

5432, 5423, 5324, 5342, 5234, 5243, 4532, 4523, 4325, 4352, 4253, 4235, 3542, 3524, 3425, 3452, 3254, 3245, 2543, 2534, 2435, 2453, 2354, 2345

Permutation & Combination Detail

Permutation and Combination

- How many choices are there to pick 3 distinct people from a group of 6 persons? Persons = A B C D E F

- Initial quick ☺ claim: There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are: 6 * 5 * 4 = 120 ways to do this

Be very careful! Is this correct?

- Consider: Picking person C first; then person A; then E
- Leads to the same group as first picking person E, then C, and then A; selection order does not matter here!

- These cases were counted separately in the initial above computation of 120, and that was wrong! See 46

Permutation and Combination

- Back to the general challenge: compute how many different subsets of people can be picked?
  - That is, we ignore the selection order – in combinations!
  - We know subgroup (person1, person2) is the same as subgroup (person2, person1)
  - Order of listing them for this type of problem doesn't matter

- To identify the combinations, start by first computing all permutations:

- A permutation of a set of distinct objects is an ordered arrangement of these objects

- An ordered arrangement of r elements of a set is called an r-permutation

Permutation and Combination

- Example: Let S = { 1, 2, 3 }
- The arrangement 3, 1, 2 is a 3-permutation of S
- The arrangement 3, 2, 1 is also a 3-permutation of S
- The arrangement 3, 2 is a 2-permutation of S
- The arrangement 1, 3 is also a 2-permutation of S
- The number of r-permutations of a set with n distinct elements is denoted by $P(n,r)$

- We can calculate $P(n,r)$ with the product rule:

- $P(n,r) = n * (n - 1) * (n - 2) * \ldots * (n - r + 1)$

- n choices for the first element, (n – 1) for the second one, (n – 2) for the third one . . .

Permutation and Combination

- **Example:**
- **$P(8,3) = (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) / (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 8 \cdot 7 \cdot 6 = 336$**
- **General formula for permutation P(n,r):**

$$P(n,r) = n! / (n - r)!$$

- **Knowing this, we can return to our initial question:**
- **How many ways (i.e. how many combinations) are there to pick 3 people from a group of 6**
- **That will mean: regardless of order of selecting?**

Permutation and Combination

- An r-combination of elements of a set S is an unordered selection of r elements from S
- Example: Let S = { 1, 2, 3, 4 }
- Then { 1, 3, 4 } is a 3-combination from S
- Note that { 3, 1, 4 } is the same 3-combination from S
- The number of r-combinations of a set with n distinct elements is denoted by C(n,r)
- Goal, to compute C(n,r)
- Example: C(4,2) = 6, since the 2-combinations of sample set S are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}

Permutation and Combination

- How can we calculate C(n,r)?
- Consider that we can obtain the r-permutation P(n,r) of a set with n elements in the following way:
- First, we form all r-combinations of set; there are C(n,r) such r-combinations
- Then, we generate all possible orderings in each of these r-combinations
- There are P(r,r) such orderings in each case

Therefore: $P(n,r) = C(n,r) * P(r,r)$

Permutation and Combination

$C(n,r) = P(n,r) / P(r,r)$
$C(n,r) = (n! / (n - r)!) / (r! / (r - r)!)$
$C(n,r) = n! / (r! (n - r)!)$

- Remember: $0! = 1$
- Now we can answer: How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

- $C(6,3) = 6! / (3! * 3!) = (4*5*6) / 3! = 4*5 = 20$

- There are 20 different ways, that is there are 20 different groups that can be picked

Permutation and Combination

Example: A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations (i.e. combinations) are there?

C(8, 6) * C(7, 5)   = (8! / (6! * 2!)) * (7! / (5! * 2!))
= 28 * 21
= 588

There are 588 different combinations of forming soccer teams under the above configuration conditions

Permutation and Combination

Corollary:
Let n and r be nonnegative integers with $r \leq n$

$$\text{Then } C(n, r) = C(n, n - r)$$

Note that "picking a subgroup of r people from a group of n" is the same as "splitting a group of n people into one group of r people, and into another group of $n-r$ people"

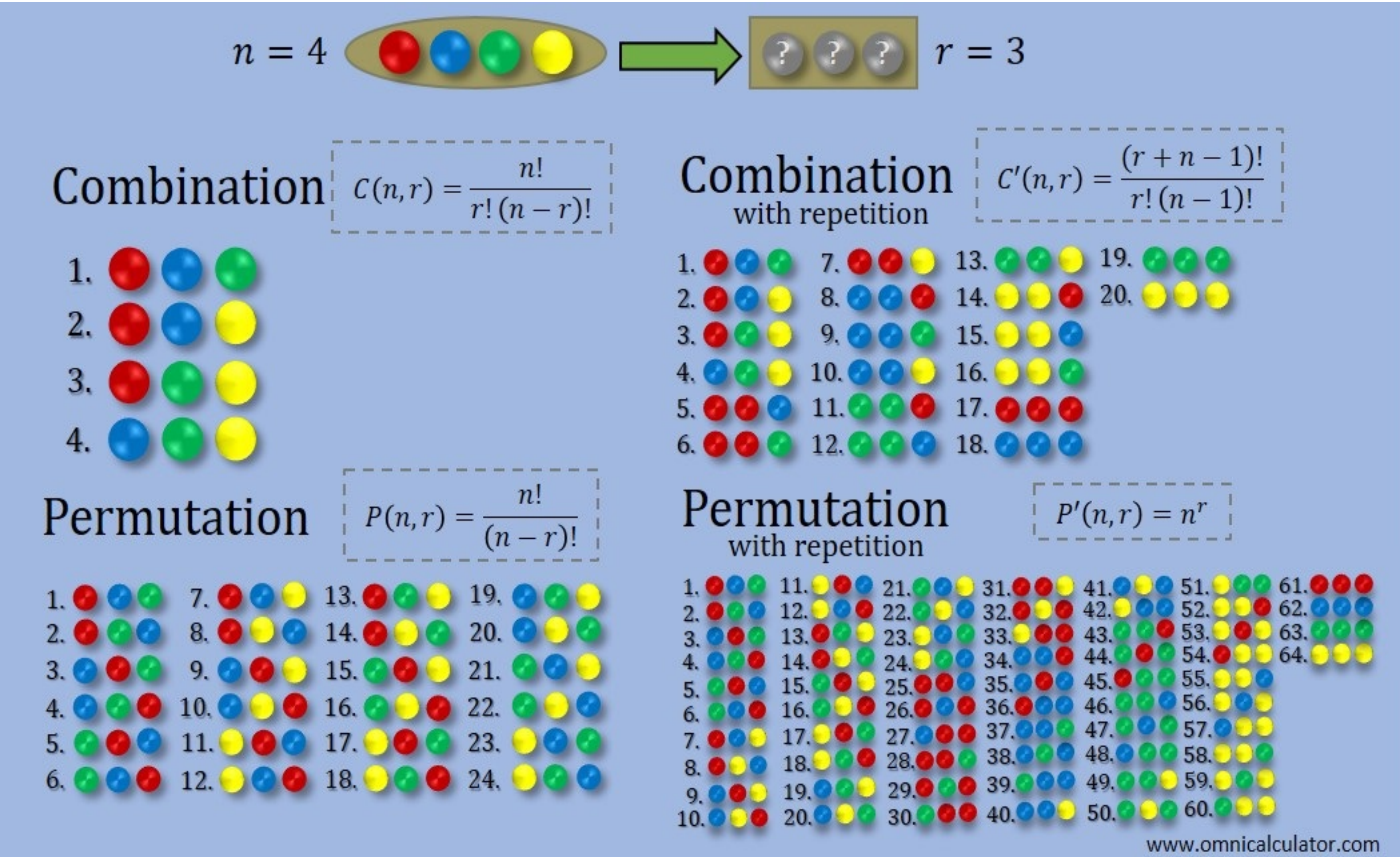
Permutation and Combination

$C(n, r) = n! / ((n-r)! * r!)$

$$\begin{aligned} C(n, n-r) &= n! / ((n-(n-r))! * (n-r)!) \\ &= n! / (r! * (n-r)!) \\ &= n! / ((n-r)! * r!) \end{aligned}$$

q.e.d.

Permutation and Combination



Combination

We saw:

$$C(n,n-r)=\frac{n!}{(n-r)![n-(n-r)]!}=\frac{n!}{(n-r)!r!}=C(n,r)$$

- **This symmetry is intuitively plausible. For example, let us consider a set containing six elements: n = 6**
- **Picking two elements and leaving four is essentially the same as picking four elements and leaving two**
- **We can say: we are "not picking" four out of six; our action being: *"not picking"***
- **In either case, the number of choices is the number of possibilities to divide the set into one set containing two elements and another set containing four elements**

Combination

- **Pascal's Identity:**

- **Let n and k be positive integers with $n \geq k$. Then**

$$C(n + 1, k) = C(n, k - 1) + C(n, k)$$

- **How can this be explained? What is it good for?**

Combination

- Imagine a set S containing n elements and a set T containing (n + 1) elements, namely all elements in S plus a new element a
- Calculating C(n + 1, k) is equivalent to answering the question: How many subsets of T that contain k items are there?
- Case I: The subset contains (k – 1) elements of S plus the element a: C(n, k – 1) choices
- Case II: The subset contains k elements of S and does not contain a: C(n, k) choices
- Sum Rule: C(n + 1, k) = C(n, k – 1) + C(n, k)

Pascal’s Triangle

In Pascal’s Triangle, initiate both outsides with 1. Each number is the sum of numbers to its upper left + right:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

Pascal's Triangle

Pascal's Triangle incorporates the Choose function

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\iddots \qquad \vdots \qquad \ddots$$

Detail of Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

Pascal’s Triangle

Since we have C(n + 1, j) = C(n, j – 1) + C(n, j) and C(0, 0) = 1, we use Pascal’s triangle to simplify C(n, j):

j

n

$C(0, 0) = \mathbf{1}$

$C(1, 0) = \mathbf{1}$ $C(1, 1) = \mathbf{1}$

$C(2, 0) = \mathbf{1}$ $C(2, 1) = \mathbf{2}$ $C(2, 2) = \mathbf{1}$

$C(3, 0) = \mathbf{1}$ $C(3, 1) = \mathbf{3}$ $C(3, 2) = \mathbf{3}$ $C(3, 3) = \mathbf{1}$

$C(4, 0) = \mathbf{1}$ $C(4, 1) = \mathbf{4}$ $C(4, 2) = \mathbf{6}$ $C(4, 3) = \mathbf{4}$ $C(4, 4) = \mathbf{1}$

Binomial Coefficients

- Expressions of the form C(n, k) are called binomial coefficients. Reason?
- Binomial expression sums two terms at various powers: $(a + b)^n$
- Now consider $(a + b)^2 = (a + b) * (a + b)$
- When expanding such expressions, this forms all possible products of a term with 2 summands a and b:

$$(a + b)^2 = a * a + a * b + b * a + b * b$$

- Sum up identical terms:

$$(a + b)^2 = a^2 + 2 * a * b + b^2$$

coefficients being: 1 2 1

Binomial Coefficients

- For $(a + b)^3 = (a + b)(a + b)(a + b)$ we have:
- $(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- There is only one term a^3, as there is exactly one way to form it: Choose a from all three factors: C(3,0) = 1
- Ditto with single term b^3 and C(3,3) = 1
- Three times a^2b, as there are three possibilities to choose a from two of three factors: C(3, 2) = 3
- Similarly, term ab^2 occurs three times
- C(3,1) = 3 plus the factor for term b^3 equals C(3, 0) = 1

Binomial Coefficients

Leading to formula:

$$(a+b)^n = \sum_{j=0}^{n} C(n,j) \cdot a^{n-j} b^j$$

Coefficient C(n,j): See p. 57

- **With the help of Pascal's triangle, this formula considerably simplifies process of expanding powers of binomial expressions**
- **For example, the fifth row of Pascal's triangle on p. 54 (1 4 6 4 1) helps us to compute $(a + b)^4$:**

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Summary

- Number of permutations of n distinct objects = n!
- Combination is a unordered selection of k items from a set of n, with n >= k. The number of combinations is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

AKA: ”n choose k”

- Pascal’s Triangle computes coefficients for all summands of $(a + b)^n$

References

1. Wiki rule of sum: https://en.wikipedia.org/wiki/Rule_of_sum
2. Wiki rule of product: https://en.wikipedia.org/wiki/Rule_of_product
3. Probability: https://en.wikipedia.org/wiki/Probability
4. Combinatorics: https://en.wikipedia.org/wiki/Combinatorics