Section 7.6 Problems

Ryan Heslin

February 24, 2023

1.

$$z = 3(\cos(7\pi/4) + i\sin(7\pi/4))$$

2.

$\pm i$ and ± 1.

4.

$$\begin{aligned}
&w^2 = z \\
&(ac + di)^2 = a + bi \\
&c^2 + 2idc - d^2 = a + bi
\end{aligned}$$

implying $c^2 - d^2 = a$ and $2dc = b$ Depending on the number, the correct c^2 and d^2 will have either the same or opposite signs, giving two roots.

3.

$$\begin{aligned}
1^n(\cos(n\theta) + i\sin(n\theta)) &= 1 \\
\cos(n\theta) &= 1 \\
n\theta = \arccos(1) &= 0
\end{aligned}$$

meaning n rotations about itself brings the angle of the complex number to 0. So a root of unity is any angle for which $n\theta = 0 \bmod 2\pi$. This implies n distinct nth roots of unity (e.g., 0, $2\pi/3$, and $4\pi/3$ for $n = 3$).

5.

Let $z = r(cos(\alpha) + i\sin(\alpha))$ Then a root of z is $r(1/n)(cos(\theta) + i\sin(\theta))$ Because angles are unique up to a modulus of 2π, we can write

$$\alpha = n\theta \mod 2\pi$$

implying $\alpha = \theta \mod \frac{2\pi}{n}$. So there is a unique root for each of the n divisions of the complex circle.

6.

It has the inverse of z's radius and the opposite sign of its angle. The conjugate is the reflection of the original angle over the real axis, and has the same relationship to its inverse.

The inverse can be shown to be

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a - bi}{a^2 + b^2} \\ z^{-1}z &= \frac{(a - bi)(a + bi)}{a^2 + b^2} \\ &= \frac{a^2 + b^2}{a^2 + b^2} & = 1 \end{aligned}$$

7.

The imaginary part is added to the real, and the imaginary part is replaced with the negative sum of both parts.

8.

$\cos^3(\theta)$ and $\sin^3(\theta)$.

9.

Because $\arctan(0.7/0.8) \approx 0.71883$, and $r = \sqrt{(-0.7)^2 + 0.8^2} = 1.0630146$, the number can be written $1.06(\cos(.719) + i\sin(.718))$. Since $r > 1$, the number will spiral outward as it is raised to higher powers, each time rotating θ/n.

11.

```
polynomial <- function(lambda) lambda^3 - 3 * lambda^2 + 7 * lambda - 5
A <- cbind(c(0, 0), diag(nrow = 2)) |>
    rbind(c(5, -7, 3))
result <- polynomial(eigen(A)$values)
max(abs(result)) < 1e-8
```

```
## [1] TRUE
```

10.

If the polynomial has all real coefficients, then it can be mapped to a family of similar real matrices with it as their characteristic polynomial. The Frobenius companion matrix exists for any polynomial, since it is uniquely determined by their coefficients. So the polynomial must have n real or complex roots, counting multiplicities, because all $n \times n$ matrices have n eigenvalues that are the roots of their characteristic polynomials.

12.

Any complex eigenvalue satisfies $Ax = (a+bi)x$ for some eigenvector x, and some real $n \times n$ matrix A, where $\lambda = a+bi$. Since $A(\Re x) + A(\Im x) = ax + bix$, it follows that $A(\Re x) - A(\Im x) = ax - bix$ for the conjugate, so it is an eigenvalue as well. Since any polynomial with real coefficients can be mapped to a Frobenius matrix, all such polynomials must have complex roots in conjugate pairs.

17.

We solve for the eigenvectors, then create S by splitting into real and imaginary parts.

```r
A <- matrix(c(5, -5, 4, 1), nrow = 2)
S <- matrix(complex(
    real = c(-2, 5, -2, 5),
    imaginary = c(4, 0, -4, 0)
), nrow = 2)
S <- cbind(Im(S[, 2]), Re(S[, 2]))
print(S)
```

```
##      [,1] [,2]
## [1,]   -4   -2
## [2,]    0    5
```

```r
solve(S) %*% A %*% S
```

```
##      [,1] [,2]
## [1,]    3   -4
## [2,]    4    3
```

18.

For any real matrix it is the area of the parallelepiped traced out by the columns of the matrix.

19.

a.

The eigenvectors span only the subspace of projection, so the determinant is 0 because $m < v$, and the trace is m because the eigenvalues are only 0 and 1. (if diagonalized, some eigenvectors are zeroed out instantly, since $P^n = P$ for projection matrices, and diagonalization is possible because the formula yields a symmetric matrix).

b.

Since the matrix is involutory, $\lambda^2 = 1$ for all eigenvalues, leaving ± 1, since this is a real vector space. The determinant is thus -1.

27.

The implied equations $\lambda_1 + 2\lambda_2 = 1$ and $\lambda_1\lambda_2^2 = 3$ have real solution $\lambda_1 = 3$ and $\lambda_2 = -1$.

28.

The first equation $2a + 2ib - 2ib = 8$ implies $a = 2$. That means the second equation can be solved:

$$2(a^2 - b^2) = 504 - b^2 = 25b = \pm 6$$

so the complex eigenvalue is $2 \pm 6i$.

29.

Since $\det A = bcd$ and all these values are positive, the product of the eigenvalues is positive. That means zero or two negative eigenvalues. Since the eigenvalues also sum to 0, the sign of the greatest eigenvalue in either case must be positive, since in the latter case $\lambda_1 = -(|\lambda_2| + |\lambda_3|)$.

31.

a.

Each column approaches an even distribution

```
`%^%` <- function(X, n) {
    orig <- X
    for (i in seq(1, n)) {
        X <- X %*% orig
    }
    X
}
A <- 1 / 15 * matrix(c(4, 1, 3, 2, 5, 2, 3, 5, 1, 4, 5, 4, 1, 3, 2, 1, 5, 2, 4, 3, 3, 2, 4, 5, 1), ncol
A %^% 20
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]  0.2  0.2  0.2  0.2  0.2
## [2,]  0.2  0.2  0.2  0.2  0.2
## [3,]  0.2  0.2  0.2  0.2  0.2
## [4,]  0.2  0.2  0.2  0.2  0.2
## [5,]  0.2  0.2  0.2  0.2  0.2
```

b.

There is a spanning set of eigenvectors, so yes.

```
eigen(A)
```

```
## eigen() decomposition
## $values
## [1]  1.0000000+0.0000000i
## [2] -0.2004886+0.1355877i
## [3] -0.2004886-0.1355877i
## [4]  0.1338219+0.1322790i
## [5]  0.1338219-0.1322790i
```

```
##
## $vectors
##                [,1]                  [,2]
## [1,] 0.4472136+0i  0.2971576-0.1959434i
## [2,] 0.4472136+0i -0.0258714-0.3109976i
## [3,] 0.4472136+0i -0.0387812+0.5231144i
## [4,] 0.4472136+0i  0.3703023-0.0161734i
## [5,] 0.4472136+0i -0.6028073+0.0000000i
##                          [,3]
## [1,]  0.2971576+0.1959434i
## [2,] -0.0258714+0.3109976i
## [3,] -0.0387812-0.5231144i
## [4,]  0.3703023+0.0161734i
## [5,] -0.6028073+0.0000000i
##                          [,4]
## [1,] -0.5253915-0.2485400i
## [2,]  0.5963822+0.0000000i
## [3,]  0.1831591-0.2008921i
## [4,] -0.0969857+0.4455255i
## [5,] -0.1571642+0.0039066i
##                          [,5]
## [1,] -0.5253915+0.2485400i
## [2,]  0.5963822+0.0000000i
## [3,]  0.1831591+0.2008921i
## [4,] -0.0969857-0.4455255i
## [5,] -0.1571642-0.0039066i
```

c.

The first eigenvector, all with entry 1/5.

d.

All the eigenvalues other than 1 have a modulus less than 1. If we write in polar form, then $\lim_{n\to\infty} r^n(cos(\theta) + i\sin(\theta)) = 0$ if $r < 0$. So if we write $A = S\Lambda S^{-1}$, then only the first eigenvalue remains significant, leaving only the first eigenvector.

```r
Mod(eigen(A)$values)
```

```
## [1] 1.0000000 0.2420324 0.2420324 0.1881649
## [5] 0.1881649
```

33.

```r
set.seed(1)
A <- sample(1:9, size = 25, replace = TRUE) |>
    matrix(ncol = 5)
B <- A %^% 20

D <- diag(B[1, ])
C <- B %*% solve(D)
```

a.

Each column of C is that column of B divided by its first entry.

It is obtained

$$C = \sum_{i=1}^{n} B(0 + e_i)]$$

where 0 is the zero matrix.

b.

It shows that the ratios of the entries in each column to the first entry are identical.

c.

High powers of A are converging on the eigenvector of the dominant eigenvalue, which occurs for a diagonalizable matrix as discussed in the previous problem.

Normalizing C makes this clear

```
first <- eigen(B)$vectors[, 1]
C[, 1]
```

```
## [1] 1.0000000 0.9398333 0.9201034 0.6520367
## [5] 0.6685975
```

```
Re(first / first[[1]])
```

```
## [1] 1.0000000 0.9398333 0.9201034 0.6520367
## [5] 0.6685975
```

d.

The first row is the first eigenvalue of A. Normalizing each column by this value again yields the first eigenvector.

```
result <- A %*% C
result
```

```
##          [,1]     [,2]     [,3]     [,4]
## [1,] 24.77673 24.77673 24.77673 24.77673
## [2,] 23.28600 23.28600 23.28600 23.28600
## [3,] 22.79715 22.79715 22.79715 22.79715
## [4,] 16.15534 16.15534 16.15534 16.15534
## [5,] 16.56566 16.56566 16.56566 16.56566
##          [,5]
## [1,] 24.77673
## [2,] 23.28600
## [3,] 22.79715
## [4,] 16.15534
## [5,] 16.56566
```

```r
result / result[1, 1]
```

```
##            [,1]      [,2]      [,3]
## [1,] 1.0000000 1.0000000 1.0000000
## [2,] 0.9398333 0.9398333 0.9398333
## [3,] 0.9201034 0.9201034 0.9201034
## [4,] 0.6520367 0.6520367 0.6520367
## [5,] 0.6685975 0.6685975 0.6685975
##            [,4]      [,5]
## [1,] 1.0000000 1.0000000
## [2,] 0.9398333 0.9398333
## [3,] 0.9201034 0.9201034
## [4,] 0.6520367 0.6520367
## [5,] 0.6685975 0.6685975
```

34.

a.

Each is $\lambda_i - \lambda$, with the old first eigenvalue the smallest. $(A - \lambda I_n)^{-1}$ has their scalar inverses as eigenvalues. It has the same eigenvectors, since $(S\lambda S^{-1})^{-1} = (S\lambda^{-1}S^{-1})$. Now the first eigenvalue, the one that was originally estimated, has the smallest modulus. If this matrix is multiplied repeatedly, its columns will approach some multiple of λ_1's eigenvector.

b.

If we repeat the procedure, we get a good estimate of the first eigenvector.

```r
A <- matrix(c(1, 4, 7, 2, 5, 8, 3, 6, 10), nrow = 3)
estimate <- eigen(A, only.values = TRUE) |>
    unlist() |>
    min() |>
    round()
estimate
```

```
## [1] -1
```

```r
B <- A %^% 20

D <- diag(B[1, ])
C <- B %*% solve(D)
C[, 1] / C[1, 1]
```

```
## [1] 1.000000 2.254656 3.732727
```

35.

I looked up the actual proof, which exploits the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and the universality of the Jordan form to show that a matrix with the eigenvalues on the diagonal has $\sum_{i=1}^{n} \lambda_i$ for the trace.

36.

a.

The entries below the diagonals represent proportions of bracket n surviving to be counted in bracket $n+1$. The first row represents members of the younger brackets contributing to the very youngest by having children.

b.

```
top <- c(1.1, 1.6, 0.6, 0, 0, 0)
bottom <- cbind(diag(x = c(.82, .89, .81, .53, .29)), 0)
A <- rbind(top, bottom)
dimnames(A) <- NULL
eigenspace <- eigen(A)
greatest <- Mod(eigenspace$values) |>
    which.max()
eigenspace$vectors[, greatest]
```

```
## [1] 0.900673594+0i 0.387093899+0i
## [3] 0.180568245+0i 0.076658698+0i
## [5] 0.021294741+0i 0.003236722+0i
```

This eigenvector indicates the equilibrium distribution of the population. While the values will grow without bound, the proportions will stabilize, because only one eigenvalue has a modulus greater than 1.

37.

For addition:

$$\begin{bmatrix} z+x & -\bar{z}-\bar{y} \\ z+y & \bar{w}+\bar{x} \end{bmatrix} = \begin{bmatrix} w\prime & -\bar{z}\prime \\ z\prime & \bar{w}\prime \end{bmatrix}$$

For multiplication, the same holds for the product of the matrices:

$$\begin{bmatrix} wx-y\bar{z} & -w\bar{y}-\bar{x}\bar{z} \\ xz+w\bar{y} & -\bar{y}z+\bar{x}\bar{w} \end{bmatrix} = \begin{bmatrix} w\prime & -\bar{z}\prime \\ z\prime & \bar{w}\prime \end{bmatrix}$$

```
w <- complex(real = 2, imaginary = 5)
z <- complex(real = -1, imaginary = 2)
A <- matrix(c(w, z, -Conj(z), Conj(w)), nrow = 2)
```

The determinant works out to $a^2 - b^2 + c^2 + 2icd + d^2$, which is invertible unless $c = d = 0$ and $a = b$. If invertible, we have the matrix:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a-bi & c+di \\ -c-di & a+bi \end{bmatrix}$$

which satisfies the requirements of the field if we set $w = a - bi$ and $z = -c - di$.

Generating such a matrix:

```r
find_dissimilar <- function() {
    space <- seq(-10, 10)
    repeat({
        w <- complex(real = sample(space, 1), imaginary = sample(space, 1))
        z <- complex(real = sample(space, 1), imaginary = sample(space, 1))
        x <- complex(real = sample(space, 1), imaginary = sample(space, 1))
        y <- complex(real = sample(space, 1), imaginary = sample(space, 1))
        A <- matrix(c(w, z, -Conj(z), Conj(w)), nrow = 2)
        B <- matrix(c(x, y, -Conj(y), Conj(x)), nrow = 2)
        if (!all(A %*% B == B %*% A)) {
            return(list(A, B))
        }
    })
}
set.seed(1)
result <- find_dissimilar()
result
```

```
## [[1]]
##        [,1]  [,2]
## [1,]  -7-4i 10-9i
## [2,] -10-9i -7+4i
##
## [[2]]
##      [,1]  [,2]
## [1,] 0+3i -7+8i
## [2,] 7+8i  0-3i
```

42.

For $\overline{AB}$, an element can be written

$$\overline{AB}_{ij} = \sum_{i=1}^{n} \overline{(a+bi)(c+di)}$$
$$= \sum_{i=1}^{n} \overline{ac - bd + i(ad+bc)}$$
$$= \sum_{i=1}^{n} ac - bd - i(ad+bc)$$

For $\overline{AB}$, the expression is $\sum_{i=1}^{n}(a-bi)(c-di) = \sum_{i=1}^{n} ac - bd - i(ad+bc)$, which is equivalent.

b.

We already showed that complex eigenvalues of real matrices occur in conjugate pairs. From the given information, $A(v+iw) = (p+iq)(v+iw) = pv - qw + i(qv+pw)$. If we try the same for the conjugate, $A(v-iw) = (p-iq)(v-iw) = pv-qw-i(qv+pw)$ Both equations imply $Av = pv-qw$ and $Aiw = i(qv+pw)$, so one implies the other.

52.

No, because, as described, it lacks commutative multiplication.

72.

Using the formula outlined yields

$$\begin{bmatrix} 0 & 1 & b \\ 0 & 0 & ac \\ 0 & 0 & 0 \end{bmatrix}$$

so the whole upper triangle must be 0.

73. (from section 4)

If we substitute λ for A in the expression, we get the factored form of the characteristic polynomial by the definition of eigenvalues. Since the expression must equal 0 for any diagonalizable matrix, and since A is used in place of λ in the polynomial, the theorem's claim is satisfied.