Section 4.2 Problems

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We check the linearity of some transformations.

1.

$T(M) = M + I_2$: nonlinear

3.

$$T(M) = \text{trace}(M)$$

Obviously not isomorphic; many matrices have the same trace.

##5.

$T(m) = M^2$: not isomorphic

7.

$$T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$$

Obviously linear, given matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

Also an isomorphism, since this is clearly invertible.

9.

$$T(M) = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} M \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Not an isomorphism, since M itself may be noninvertible.

Actually yes:

$$T(M) = S^{-1}MS$$
$$S^{-1}M = S^{-1}$$
$$M = SAS^{-1}$$

12.

$$T(c) = cM \quad c \to R^2$$

where M is a particular matrix. Linear - $c(M_1 + M_2) = cM_1 + cM_2$ and $c(kM_1) = kcM_1$.

Also isomorphic: $M = 1/cT(c)$

16.

$$T(M) = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M$$

The matrix is:

$$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \end{bmatrix}$$

Linear, not invertible.

17.

$$T(x + iy) = x$$

Linear, not invertible, as the complex part is just zeroed out. ## 18.

$T(x + iy) = x^2 + y^2$

Non-linear ($cT(x) = cx^2 + cy^2$), but $T(cx) = c^2x^2 + c^2y^2$); x can be positive or negative.

21.

$$T(x + iy) = y + ix$$

Linear: $T(x_1 + iy_1 + x_2 + iy_2) = y_1 + y_2 + i(x_1 + x_2) = T(x_1 + iy_1) + T(x_2 + iy_2)$ and $T(kx_1 + kiy_1) = ky_1 + kx_1 i_y = kT(x + iy)$

Isomorphism, and involutory too: $T^{-1}y = y + ix$.

22.

$$T(f(t)) = \int_{-} 2^3 f(t) dt$$

$$\begin{bmatrix} 3 & 3/2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2/3 \end{bmatrix}$$

Clearly noninvertible; infinitely many functions with the same definite integral. ## 23.

$$T(f(t)) = f(7)$$
$$T(f(t)) = a + 7b + 49c$$

Linear, nonisomorphic, with matrix

$$\begin{bmatrix} 1 & 7 & 49 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

24.

$$T(f(t)) = f''(t)f(t)$$

$$T(1) = 0(1) = 0 \qquad T(t) = b(0)(t) = 0$$
$$T(t^2) = c(2)(t^2) = 2ct^2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Not isomorphic.

27.

$$T(f(2t))$$

$$T(1) = 1a = 1$$
$$T(t) = b(2t) = 2bt$$
$$T(t^2) = c(4t^2) = 4ct^2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Isomorphic.

28.

$$T(f(t)) = f(2t) - f(t)$$

Nonisomorhpic; the constant term gets zeroed out, with amtrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 3 \end{bmatrix}$$

29.

$$T = f'(t)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Linear, non-isomorphic.

31.

$$T = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix}$$

from P_2 to $R^{2\times 2}$ Substituting, we find

$$T = \begin{bmatrix} a & a+b+c \\ a+2b+4c & a+3b+9c \end{bmatrix}$$

The matrix should be:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

```r
cbind(matrix(rep(rep(c(1, 0, 0), times = 2), times = 3),
    ncol = 3), sapply(list(c(2, 0, 0), c(5, 0, 0),
    c(13, 0, 0)), rep, 2)) %>%
    mat2latex()
```

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 5 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 5 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

32.

$$T(f(t)) = f'(t) + t^2 = b + 2ct + t^2$$

Nonlinear; that t^2 is affine and has no matrix representation.

33.

Drop every other term.

$$T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_0, x_2, x_4, \dots)$$

Linear, since the sums and scalar multiples of parallel terms are the same before and after the transformation, but not isomorphic because there is no way to recover the terms.

34.

$$T(x_0, x_1, x_2, \dots) = (0, x_1, x_2, \dots$$

First rule:

$$T(x+y) = (0, x_0 + y_0, x_1 + y_1)$$
$$T(x) + T(y) = (0, x_0, x_1, \dots) + (0, y_0, y_1, \dots)$$

Second rule:

$$kT(x) = (0, kx_1, kx_2, \dots)$$
$$T(kx) = (0, kx_1, kx_2, \dots) = kT(x)$$

Linear, isomorphic.

35.

From P to V:

$$T(f(t)) = (f(0), f'(0), f''(0), \dots$$

The first property is satisfied by the sum rule of derivatives.

Second property satisfied:

$$kT(f(t)) = (kf(0), kf'(0), kf''(0), \dots)$$
$$T(kf(t)) = ((kf(0), kf'(0), kf''(0), \dots))$$

Not an isomorphism; many functions are undefined at 0.

39.

Let's find the matrix:

$$\begin{aligned} T(f) &= f'' + 2f' + f \\ &= 2c + 2(b + 2ct) + a + bt + ct^2 \\ &= a + 2b + 2c + bt + 4ct + ct^2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

This is clearly invertible, so it's an isomorphism.

43.

$$T(f(t) = \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 11 & 121 \end{bmatrix}$$

Non-isomporhic; the constant term gums up the workss.

52.

The kernel of $T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is all matrices which have $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in their kernel, of the form $v_1 = -2v_2$.

25.

$$T(f(t)) = f''(t) + 4f'(t)$$

$$\begin{aligned} &= 2c + 4(b + 2ct) \\ &= 4b + 2c + 8ct \end{aligned}$$

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The kernel is constant functions you idiot.

57.

$$T(f(t)) = f'' - 5f' + 6f$$

$$\begin{aligned} &= 2c - 5(b + 2ct) + 6(a + bt + ct^2) \\ &= 2c - 5b - 10ct + 6a + 6bt + 6ct^2 \\ &= 6a - 5b + 2c + 6bt - 10ct + 6ct^2 \end{aligned}$$

The image is P_2, the kernel $f(t) = 0$.

59.

From P_2 to R^n

$$T(f(T)) = f(7)$$

The matrix:

$$\begin{bmatrix} 1 & 7 & 49 \end{bmatrix}$$

The matrix kernel makes this clearer, I think.

$$\begin{bmatrix} -7 & -49 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

66.

For the infinite-dimensional $T(f) = f - f'$, the kernel is e^x, the image all other functions.

Working with the subspace of polynomials of degree $\leq n$ such that $f(0) = 0$; no constant term.

72.

Is it a subspace? Yes, because $f(0 + g(0)) = 0 + 0 = 0$, and $kf(0) = 0(0) = 0$.The dimension of Z_n is n, because a polynomial with no constant term has n terms.

75.

$$T(0_V) = 0_W$$
$$0_V(T) = 0_W$$
$$0_V = 0_W$$

$$T(0_V) = 0_W$$
$$T^{-1}(T(0_V)) = T^{-1}(0_W)$$
$$0_V = T^{-1}(0_W)$$

76.

We know T is linear, so we need to prove T^{-1} obeys the properties as well.

$$T^{-1}(f+g) = T^{-1}(T(T^{-1}(f+g)))$$
$$= T^{-1}(T(T^{-1}(f)) + T(T^{-1}(g)))$$
$$= T^{-1}(f) + T^{-1}(g)$$

and

$$kT^{-1}(f) = T^{-1}(kT(T^{-1}(f))$$
$$= T^{-1}(T(k)T^{-1}f))$$
$$= kT^{-1}f$$