Section 4.1 Problems

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% Standard custom LaTeX commands

%

% 1: term 1 % 2: subscript 1 % 3: term 2 % 4: subscript 2 % 5. operation

Check some subspaces

1.

$$p(0) = 2$$

Not closed under addition:

$$ap(0) + bp(0) = 2a + 2b \neq 2$$

2.

$$p(2) = 0$$

Nice and closed.

$$ap(2) + bp(2) = 0a + 0a = 0$$

$$k(p2) = 0k = 0$$

3.

$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$1 + 1 = f(x) + g(x)$$

$$4a + 2b + c + 4d + 2e + f$$
$$4(a + d) + 2(b + e) + (f + c)$$

4.

Valid subspace

$$\int_0^1 (p(t)dt + \int_0^1 g(h)dh = 0 + 0 = 0$$
$$k\int_0^1 p(t)dt = k0 = 0$$

Basis: $(1, t, t^2)$

5.

$p(-t) = -p(t)$. This satisfies the scalar axiom by definition.

6.

3×3 invertibles are not a subspace because not closed under addition:

```
library(matador, quietly = TRUE)
try(solve(diag(nrow = 3) + square(0, 1, 1, 1, 0, 1, 1, 1, 0)))
```

```
Error in solve.default(diag(nrow = 3) + square(0, 1, 1, 1, 0, 1, 1, 1,  :
  Lapack routine dgesv: system is exactly singular: U[2,2] = 0
```

7.

Diagonals are obviously a subspace.

8.

Ditto upper triangular; the zero elements never become nonzero.

9.

3×3 with positive nonzero entries: yes.

10

Matrices whose kernel is $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$: yes. We can use the properties since all matrix transformations are linear.

$$\begin{aligned} &Av = 0 && Bv = 0 \\ &Av + Bv = (A+B)v = 0 \\ &kAv = 0 \\ &A(kv) = 0 \end{aligned}$$

11.

3×3 RREFS: not closed under scalar multiplication or addition, since scaling converts to non-RREF form.

The following concern the space of infinite sequences.

12.

$(a, a+k, a+2k, \dots)$ is a subspace.

$$
\begin{aligned}
&A = (a, a+k, a+2k, \dots) \\
&B = (b, b+c, b+2c, \dots) \\
&A + B = (a+b) + (a+b+k+c) + (a+b+2k+2c) \\
&cA = (ca + (ca+ck) + ca + 2kc) = (ca + (ca+ck) + ca + 2kc)
\end{aligned}
$$

13.

Geometric sequences $(a, ar, ar^2, ar^3, \dots)$ are not a subspace.

Not closed under addition:

$$
\begin{aligned}
&(a, ar, ar^2) + (b, bq, bq^2) = ((a+b), (a+b)(r+q), (a+b)(r^2+q^2) \\
&(a+b, ar+bq, ar^2+bq^2) \neq ((a+b), ar+br+aq+bq, ar^2+br^2+aq^2+bq^2
\end{aligned}
$$

Scalar multiplication

$$
\begin{aligned}
&k(a, ar, ar^2) = (ka, kar, kar^2) \\
&(ka, kar, kar^2) = (ka, kar, kar^2)
\end{aligned}
$$

14.

Sequences that converge on 0 are a subspace, because limits obey the adding and scalar multiplication axioms.

$$
\begin{aligned}
&A + B = 0 \\
&k(A) = 0 = A(k) = 0
\end{aligned}
$$

15.

Square-summable (converge on $\sum_{i=0}^{\infty} x_i^2$ are not a subspace. The squares of the summed sequence are not the same as those of the separate sequences.

$$
\begin{aligned}
&X + Y = (x_1 + y_1), (x_2 + y_2), \dots, (x_n + y_n) \\
&\sum_{i=0}^{\infty} = (x^2 + y^2 + 2xy, \dots, x_n^2 + y_n^2 + 2x_n y_n)
\end{aligned}
$$

Now we find bases.

16.

$R^{3\times 2}$: one-hot matrices with a 1 in each of the six elements, dimension 6.

17.

$R^{n\times m}$: mn one-hot matrices, dimension mn.

18.

All 2×2 with trace that sums to 0:

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ c & 0\end{bmatrix}$$

We don't need a basis for d because it's the opposite sign of a. So we've lost one degree of freedom.

19.

C^2: $(1, i)$

20.

All diagonal matrices: n one-hot matrices, one with 1 in each diagonal position

24.

Lower and upper triangular matrices: standard one-hots, with dimension $\sum_{i=1}^{n}$.

25.

All polynomials P_2 such that $f(1) = 0$: dimension is 2, since $a + 1b + 1c = 0$. A basis could be $1, t$.
Not at all right!

26.

27.

Such a matrix implies the system:

$$
\begin{aligned}
a^2 + bc &= 1 \\
ab + dc &= 0 \\
ac + bd &= 0 \\
bc + d^2 &= 2
\end{aligned}
$$

which requires the off-diagonal to be 0. So the components matrix itself are the basis

$$
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Note the basis doesn't have the ratio of components to each other of the final matrix: keep it one hot.

28.

29.

All matrices such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \\ & 0 & 0 \end{bmatrix}$: dimension 2, since the kernel is a single vector, requiring only 2 unique elements,w with one having the opposite sign of the other.

This is wrong.

$$
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

30.

$$
\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

Dimension 2. Rows have to be multiples of $(1, -3)$

$$
\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}
$$

31.

$$
\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}
$$

dimension 2.

32.

The basis is $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, from the implied system $a = c$ and $b = -d$.

```r
S <- matrix(c(1, 1, -1, 1), nrow = 2)

matrix(rep(1, 4), nrow = 2) %*% S
```

```
     [,1] [,2]
[1,]    2    0
[2,]    2    0
```

```r
S %*% matrix(c(2, rep(0, 3)), nrow = 2)
```

```
     [,1] [,2]
[1,]    2    0
[2,]    2    0
```

34.

```r
solve(matrix(c(3, 4, 2, 5), nrow = 2))
```

```
           [,1]       [,2]
[1,]  0.7142857 -0.2857143
[2,] -0.5714286  0.4285714
```

Find S for $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} S = S$.

The implied system:

$$\begin{aligned} a &= 3a + 2c \\ b &= 3b + 2d \\ c &= 4a + 5c \\ d &= 4b + 5d \end{aligned}$$

Thus:

$$\begin{aligned} a &= -c \\ b &= -d \end{aligned}$$

So a basis is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

35.

We want to find the basis for matrices commuting with $\begin{bmatrix} ,2 & 0 & 0 \\ ,0 & 3 & 0 \\ ,0 & 0 & 4, \end{bmatrix}$. Obviously we need three for the diagonal elements, since all diagonal matrices commute. But we need an additional three for the upper and lower triangles, since A would still commute if it were symmetric. So adding the bases together,one possibility is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

36.

Our matrix is $$ The space of matrices that commute with this matrix has dimension 5. We need three values to account for each element of the diagonal, but since 3 is repeated we gain two degrees of freedom, for the $(2,3),(3,2)$ interactions, since those values in the commuting matrix ar arbitrary. because the row-column interaction is the same for either side.

37.

The possibilities for an 3×3 matrix are 3 (all unique values), 5 (two distinct values), 9, (the same value). In general, each repeated value adds $n-1$ to the basis, except for the last.

38.

For a 4×4 diagonal, the possibilities for the basis dimension are 4, 6, 10, and 16. It looks like the pattern is $\dim(A) + 2(\dim(A) - N)$, where n is the distinct values on the diagonal.

```
LHS <- paste0(letters[1:16], rep(letters[23:26], 4)) %>%
    matrix(nrow = 4, byrow = TRUE)

RHS <- paste0(letters[1:16], rep(letters[23:26], each = 4)) %>%
    matrix(nrow = 4, byrow = TRUE)
```

39.

The dimension of the space of all upper triangular is $\sum_{i=1}^{n}$. For a 3×3 it is 6. # 40.

$n^2 - n, n^2 - 2n, \ldots, 0$. Each increment of rank adds n elements to the basis for the matrix. If c is the zero vector, the dimension could b3 n^2, since full-rank matrices have only the zero kernel.

41.

If B is the zero matrix, any dimension. If B has full rank, 0. Otherwise $dim(\ker(B)) * 3$

42.

$$n(n - rank(B))$$

43.

A is a reflection matrix about L. We are given:

$$AS = S\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

A is involutory, so $A = A^{-1}$ Therefore:

$$S = \begin{bmatrix}v\\w\end{bmatrix}$$
$$A\begin{bmatrix}v\\w\end{bmatrix} = \begin{bmatrix}v\\-w\end{bmatrix}$$

So A has to reverse the signs of S's second column while being a reflection. This is guaranteed only if v is parallel to L and w orthogonal. But in that case the vectors must be orthogonal to each other as well, so a dimension of 2 is sufficient

$$v = x^{\parallel}$$
$$w = x^{\perp}$$
$$v \cdot w = 0$$

44.

45.

46

Simply (a, k), so dimension 2.

47.

Even functions satisfy the scalar property:

$$f(-t) = f(t)$$
$$kf(-t) = kf(t)$$

$$f(-t) + g(-h) = f(t) + g(h)$$

They are. The scalar multiples remain part of the subspace because the evenness condition does not apply to them. The same is true of odd functions

48.

49.

50.

51.

52.

53.

Say a space C of dimension n has a basis with $n+1$ elements. By definition, a unique linear combination of this basis describes every member of the space. These coordinates may be mapped to vectors in R^n using the coordinate transformation. (The standard coordinate transformation could not be used for R^{n+1} because C has dimension n.). Let V designate the subspace containing the coordinate vector for every member of C's basis. If the basis is valid, then the members of V are linearly independent, such that $c_1 v_1 + \cdots + c_{n+1} v_{n+1} = 0$ has only the solution $c_1, \ldots, c_{n+1} = 0$. But a set of vectors in R^n can contain at most n linearly independent vectors, so v_{n+1} must be redundant. Because it is not linearly independent, V cannot form coordinates for a a basis of C. But the basis would be valid if its coordinates were n linearly independent vectors. So a linear space of n dimensions admits at most n linearly independent elements.

54.

55.

56.

57.

58.

59.

60.