Section 6.3 Problems

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1.

A simple SVD.

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \\
A^T A &= \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix} \\
AA^T &= \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix} \quad A = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} \sqrt{85} \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix}
\end{aligned}
$$

u_1 is a basis for the column space, u_2 a basis for the row space kernel, v_1 a basis for the row space, v_2 a basis for the kernel.

3.

Fibonacci matrix:

$$
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}
$$

I won't do this one by hand.

```
A <- square(1, 1, 1, 0)
AtA <- t(A) %*% A
V <- eigen(AtA)$vectors
Sigma <- diag(x = sqrt(eigen(AtA)$values))
U <- A %*% V %*% (diag(x = 1/diag(Sigma)))

mat2latex(U %*% Sigma %*% t(V))
```

$$
\begin{bmatrix} 1 & 1 \\ 1 & -0.000000000000000555111512312578 \end{bmatrix}
$$

5.

Alternate approach: find both matrices' vectors by hand

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$$

with relevant eigenvalues of $(3, 1)$.

Then

$$U = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The big decomposition:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

8.

$A^T A$ is a diagonal matrix of $\sigma_1^2, \ldots, \sigma_n^2$, so V^T is just I_m. U is a diagonal matrix of $Av_1/\sigma, \ldots, Av_n/\Sigma_n$. Summing up:

$$A = AV\Sigma^+ \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix} I_m$$

##9.

The formulation $A = \sigma_1 u_1 v^T + \cdots + \sigma_r u_r v_r$ breaks up$A$ into a series of matrices representing transformations of of the eigenspaces of A's row space into its column space. We need r terms because A's rank is the number of elements in its basis.

10.

From this information, the singular values are just the eigenvalues themselves (A is square). Since it is also symmetric, U is just the eigenvectors and $V^T = U^T$. Then:

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$

12.

a. If $A = 4A$ then $A^TA = 16A^TA$, so the singular values are increased by that factor. The unit eigenvectors are unaffected.

b. For A^T

$$\begin{aligned} A^T &= (U\Sigma V^T)^T & A^{-1} &= (U\Sigma V^T)^{-1} \\ &= V(U\Sigma)^T & &= (V^T)^{-1}(U\Sigma)^{-1} \\ &= V\Sigma U^T & &= V\Sigma^{-1}U^T \end{aligned}$$

Naturally, the inverse only exists if A^{-1} does, which requires a square matrix.

11

Add $-\sigma_1 I$ to A to get a singularity.

13.

If $A = A + I$, then $A^TA = (A+I)^T(A+I) = A^2 + A^T + A + I$, which implies $\Sigma = \Sigma^2 + 2\Sigma + I$

14.

The SVD of the zero matrix is just $I_n 0n \times mI_m$. Its pseudoinverse is the $m \times n$ zero matrix, not that that's very useful.

15.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

c.

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\
A^+ &= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}
\end{aligned}
$$

16.

If $m \times n$ Q has orthonormal columns, then $\Sigma = I_{n \times m}$, so the pseudoinverse is VU^T.

17.

$$
A = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}
$$

Diagonalize, find positive definite square root, then polar decomposition:

$$
\begin{aligned}
A^T A &= \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \\
V &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
\sqrt{\Sigma} &= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \\
V\sqrt{\Sigma}V^T &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}
\end{aligned}
$$

We now need to find $Q = UV^T$. Since we have A and $U = V\Sigma$:

$$
U = \frac{1}{\sqrt{20}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}
$$

18.

Use the generalized inverse for a least-squares solution.

```r
A <- square(c(rep(1, 3), 0, 0, 1, 0, 0, 1))
V <- eigen(t(A) %*% A)$vectors
Sigma <- sqrt(diag(x = eigen(t(A) %*% A)$values))
U <- A %*% V %*% diag(x = c(1/2, 1, 0))

A_plus <- V %*% diag(x = c(1/2, 1, 0)) %*% t(U)

b <- c(0, 2, 2)
mat2latex(A_plus %*% b)
```

$$\begin{bmatrix} 1 \\ 0.499999999999999 \\ 0.499999999999999 \end{bmatrix}$$

18+

If A has full rank in one dimension such that its generalized inverse is the left or right inverse, then A^+b is always in the row space, since both A^T and $(A^TA)^{-1}$are the leftmost matrices in either form, and the image of each is the row space.

Showing $A^TAx^+ = A^Tb$:

$$\begin{aligned}
&A^Tb = A^TA(A^TA)^{-1}A^Tb \\
&A^Tb = A^Tb \\
&A^Tb = A^TAA^T(AA^T)^{-1}b \\
&A^Tb = A^Tb
\end{aligned}$$

19.

$$\begin{aligned}
&A = U\Sigma V^T \\
&A = (U\Sigma U)(U^TV) \\
&= (V^TU)(U^T\Sigma^+U)
\end{aligned}$$

20.

$(AB)^+ \neq B^+A^+$ in general. If

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

21.

$$\begin{aligned}
&A^+ = U^T(UU^T)^{-1}(L^TL)^{-1}L^T \\
&A^TAb = U^TL^TLUU^T(UU^T)^{-1}(L^TL)^{-1}L^Tb \\
&= U^TL^TL(L^TL)^{-1}L^Tb \\
&= U^TL^Tb \\
&= A^Tb
\end{aligned}$$

s22.

AA^+ projects onto A's row space, A^+A onto its image. In either case only the first r vectors of U and V (well, first r rows of V^T) are selected; those r vectors provide bases for the image and row space, respectively.

$$\begin{aligned}
&AA^+ = U\Sigma V^TV\Sigma^+U^T \\
&= U\Sigma\Sigma^+U^T
\end{aligned}$$

$$
\begin{aligned}
A^+A &= V\Sigma^+U^TU\Sigma V^T \\
&= V\Sigma^+\Sigma V^T
\end{aligned}
$$

If A has full column rank, A^+A reduces it to I_n; if it has full row rank, A^+A reduces it to I_m. If not, the middle term selects the first r rows of U^T or V^T, yielding the projection. This matrix is idempotent and therefore a projection:

$$
\begin{aligned}
A^+A &= V\Sigma^+\Sigma V^T \\
(A^+A)^2 &= V\Sigma^+\Sigma V^TV\Sigma^+\Sigma V^T \\
&= V\Sigma^+\Sigma\Sigma^+\Sigma V^T \\
&= V\Sigma^+\Sigma V^T
\end{aligned}
$$

If A lacks full row or column rank, some singular values are 0, which is why we know it's a projection, not a rotation. If not, the product is the identity, also a projection.