Section 5.3 Exercises

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1.

$F_k = F_{k-1} + F_{k-2}$. So F_k is even if both the preceding two numbers are both even and odd, odd otherwise. Given the starting sequence $0, 1$, F_3 is odd, F_4 is even, F_5 is odd, as is F_6, and F_7 is again even. The pattern repeats indefinitely.

2.

```
m <- square(0, 0.5, 0, 0, 0, 1/3, 6, 0, 0)
lambdas <- eigen(m, only.values = TRUE)

mat2latex(m %^% 3)
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b.Each iteration represents 1 year, so after two three-year cycles there will be 3000 again.

3.

```
mat_pows(square(1, 1, 1, 0), 2:4)
```

```
$`2`
     [,1] [,2]
[1,]    2    1
[2,]    1    1

$`3`
     [,1] [,2]
[1,]    3    2
[2,]    2    1

$`4`
     [,1] [,2]
[1,]    5    3
[2,]    3    2
```

##4.

The "Gibonacci" matrix is:
$$\begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

$$\lambda^2 + 1/2\lambda - 1/2 = 0 \quad \lambda = (1, -1/2)$$
$$S = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

b. The limit is

$$\begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

c.

```
S <- square(1, 1, -1/2, 1)
C <- solve(S) %*% c(1, 0)

mat2latex(S %*% diag(x = c(1, -1/2)^100) %*% C)
```

$$\begin{bmatrix} 0.666666666666667 \\ 0.666666666666667 \end{bmatrix}$$

5.

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{l\lambda_1 - \lambda_2} \\ -\frac{1}{l\lambda_1 - \lambda_2} \end{bmatrix}$$
$$F_k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} \frac{1}{l\lambda_1 - \lambda_2} \\ -\frac{1}{l\lambda_1 - \lambda_2} \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 \lambda_1^k & \lambda_2 \lambda_1^k \\ \lambda_1^k & \lambda_2^k \end{bmatrix} \begin{bmatrix} \frac{1}{l\lambda_1 - \lambda_2} \\ -\frac{1}{l\lambda_1 - \lambda_2} \end{bmatrix}$$
$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$

7.

The Lucas matrix is a Fibonacci variant that starts with $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The sequence goes $2, 1, 3, 4, 7, 11, 18, 39, 57, 76, 133$. And $\frac{1}{2}(1 - \sqrt{5})^1 0 = 133$.

8.

$\lambda = 1, 3/4, 1/2$, so the steady state is $\begin{bmatrix}1\\0\\0\end{bmatrix}$ - all dead.

##9.

$$\begin{bmatrix}5/6 & 1/6 & 0\\1/6 & 5/6 & 0\\1/4 & 1/3 & 1\end{bmatrix}$$

10.

The provided matrix *is* A, so we diagonalize it and multiply by u_0 to complete $u_k = A_k u_0$ The eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix}3/2\\1\end{bmatrix}$, so that is the equilibrium.$

```
A <- square(0.8, 0.2, 0.3, 0.7)
eigen(A)
```

```
eigen() decomposition
$values
[1] 1.0 0.5

$vectors
          [,1]       [,2]
[1,] 0.8320503 -0.7071068
[2,] 0.5547002  0.7071068
```

$$\begin{aligned}
u_k &= \frac{2}{5}\begin{bmatrix}3/2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1^k & 0\\0 & .5^k\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 3/2\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix}\\
&= \frac{2}{5}\begin{bmatrix}3/2 & -(.5^k)\\1 & .5^k\end{bmatrix}\begin{bmatrix}5\\15/2\end{bmatrix}\\
&= \frac{2}{5}\begin{bmatrix}15/2 - \frac{15(-.5k)}{2}\\5 + \frac{15(.5^k)}{2}\end{bmatrix}\\
&= \begin{bmatrix}5 - 5(.5)^k\\2 + 5(.5)^k\end{bmatrix}
\end{aligned}$$

11.

a. $v_1 + v_2 = 2v_3$, so for $\lambda = 0$ the eigenvector is $x = \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$. So the

I cheat a little by borrowing a function from StackOverflow to compute the standard eigenvectors. I use it to find the formula.

```r
# From
# https://stackoverflow.com/questions/43223579/solve-homogenous-system-ax-0-for-any-m-n-matrix-a-in-r-f
NullSpace <- function(A) {
    m <- dim(A)[1]
    n <- dim(A)[2]
    ## QR factorization and rank detection
    QR <- base::qr.default(A)
    r <- QR$rank
    ## cases 2 to 4
    if ((r < min(m, n)) || (m < n)) {
        R <- QR$qr[1:r, , drop = FALSE]
        P <- QR$pivot
        F <- R[, (r + 1):n, drop = FALSE]
        I <- base::diag(1, n - r)
        B <- -1 * base::backsolve(R, F, r)
        Y <- base::rbind(B, I)
        X <- Y[base::order(P), , drop = FALSE]
        return(X)
    }
    ## case 1
    return(base::matrix(0, n, 1))
}
A <- square(0.2, 0.4, 0.3, 0.4, 0.2, 0.3, rep(0.4,
    3), byrow = TRUE)

S <- sapply(eigen(A)$values, function(x) NullSpace(A -
    diag(x = x, nrow = 3)))
S
```

```
     [,1]          [,2] [,3]
[1,] 0.75 -1.000000e+00 -0.5
[2,] 0.75  1.000000e+00 -0.5
[3,] 1.00 -2.775558e-16  1.0
```

b., c.

$$= \begin{bmatrix} .75 & -1 & -.5 \\ .75 & 1 & -.5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 & 0 \\ 0 & -.2^k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} .4 & .4 & .4 \\ -.5 & .5 & 0 \\ -.4 & -.4 & .6 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} .75(1^k) & -1(-.2^k) & 0 \\ .75(1^k) & -.2^k & 0 \\ 1^k & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = 5(-.2^k)$$

```r
NullSpace(A)
```

```
     [,1]
[1,] -0.5
[2,] -0.5
[3,]  1.0
```

12.

The matrix is

$$A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

For $\lambda = 1$, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Scale to unit length for

$$U_\infty = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

13.

a. $0 \le a, b \le 1$.

b. The eigenvalues are 1 and $a - b$ (since the trace is $1 + a - b$)

The eigenvectors:

$$\begin{aligned} S &= \begin{bmatrix} \frac{1-a}{b} & -1 \\ 1 & 1 \end{bmatrix} \\ S^{-1} &= \left(\frac{b}{1-a} + 1\right) \begin{bmatrix} 1 & 1 \\ -1 & \frac{1-a}{b} \end{bmatrix} \\ \Lambda &= \begin{bmatrix} (1-a)^k & 0 \\ 0 & 1^k \end{bmatrix} \end{aligned}$$

$$\begin{aligned} c = S^{-1} &= \left(\frac{b}{1-a} + 1\right) \begin{bmatrix} 1 & 1 \\ -1 & \frac{1-a}{b} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2(1 + \frac{b}{1-a} \\ 0 \end{bmatrix} \\ \lambda c_1 x_1 &= 2(1 + \frac{b}{1-a}) \begin{bmatrix} \frac{1-a}{b} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2(1 + \frac{b}{1-a}) \end{bmatrix} \end{aligned}$$

I think it's $\begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$

14.

a. For the stable state:

```r
A <- square(rep(0.5, 3), 0.25, 0.5, 0, 0.25, 0, 0.5,
    byrow = TRUE)

S <- eigen(A)$vectors
Lambda <- diag(eigen(A)$values)

stable <- S[, 1]
MASS::fractions(stable/sum(stable))
```

```
[1] 1/2 1/4 1/4
```

The diagonalization is:

b.

```r
S <- map(eigen(A)$values, ~NullSpace(A - diag(nrow = 3,
    x = .x))) %>%
    reduce(cbind)
invisible(sapply(list(S, Lambda, solve(S)), mat2latex))
```

$$\begin{bmatrix} 2 & -6.9726111936842e-32 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -1.66533453693773e-16 \end{bmatrix}$$

$$\begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 5.55111512312578e-17 & -0.5 & 0.5 \\ -0.25 & 0.25 & 0.25 \end{bmatrix}$$

15.

Say $m = 3$ and $S = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

$$\begin{aligned} SA &= S \\ Sx &= \sum_{i=1}^{m} x_i \\ SAx &= Sx = \sum_{i=1}^{m} x_i \end{aligned}$$

18.

19.

$$\begin{aligned} I &= (I - A)(I + A + A^2 + \cdots + A^n) \\ &= (I^2 - A) + (A - A^2) + (A^2 - A^3) + (A^n - A^{n+1}) \\ &= I \end{aligned}$$

This works when no eigenvalue is greater than 1; the sequence has a finite sum, so $(A - I)$ has an exact inverse.

```
A <- diag(x = 0, nrow = 3)
A[upper.tri(A)] <- 1

mat2latex((diag(x = 3) - A) %*% (mat_pows(A, 1:100) %>%
    reduce(`+`) + diag(x = 3)), sink = FALSE)
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

20.

```
A <- square(0, 0, 0.2, 0.5)
A %^% 100
```

```
     [,1]         [,2]
[1,]    0 3.155444e-31
[2,]    0 7.888609e-31
```

$$A^k = \frac{1}{6} \begin{bmatrix} 1 & .4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & .5^k \end{bmatrix} \begin{bmatrix} 1 & -.4 \\ 0 & 1 \end{bmatrix}$$

This does sum to

24.

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

26.

Identical.

28.

a. The eigenvectors for $\lambda = 0$ always span the kernel, since they are solutions for $Ax = 0$.

b. When they are real.

29.

If $B^4 = I$ and $C^3 = -I$, they are rotations about $\pi/2$ and $\pi/3$ radians respectively.

Since an angle on the complex plane is is $\cos\theta i \sin\theta +$

$$\lambda = e^{\pm i(\pi/2)} = \pm i \quad \lambda = e^{i(\pi/3)} = \frac{1 \pm \sqrt{-3}}{2}$$