Notes

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1.

True. The eigenvalues of a diagonal matrix are the diagonal, and $A^T A = A^2$, and $\sqrt{\lambda^2} = \lambda$

2.

True . $2(3) > 5$, so the quadratic form is positive and therefore an ellipse.

3.

True. All symmetric matrices have orthogoanl egeinvectors, which guarantees distinctness and therefore diagonalizablity.

4.

True. It is not PD unless $ac > b^2$, but if a and c were both negative then the squared terms of the quadratic form would be negative.

5.

True. All orthogonal matrices are diagonalizable, so $A = S\Lambda S^{-1} \to \Lambda = S^{-1}AS$

6.

True.

$$A^T A = \begin{bmatrix} 25 \end{bmatrix}$$
$$\sqrt{25} = 5$$

7.

True, of the matrix

$$\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$$

8.

False, they are the square roots of those eigenvalues.

9.

True; the positive determinant condition implies this, while if all eigenvalues are negative then the quadratic form cannot be positive. ## 10. True, obviously.

11.

True. The eigenvalues of a triangualr matrix are the diagonal, and $A^T A$ is also triangualr, witht he squares of those eigenvalues.

12.

False. Conider:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^T A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \quad AA^T = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

13.

True, since σ_i is the length of v_i after transformation by u_i.

$$\begin{aligned} AV &= U\Sigma \\ U &= AV\Sigma^+ \\ u_i &= \frac{Av_i}{\sigma_i} \end{aligned}$$

14.

True. Negative-definite matrices of even dimension have positive determinants, as the first subdeterminant (in R) must be negative, the next positive.

15.

True. Symmetric matrices have orthogonal eigenvectors, so if v and W are eigenvectors $Av \cdot Aw = 0$.

16.

False, negative semidefinite. Subdeterminants are -2, 2, and

$$\begin{aligned} &-2[-2(-2) - 1(1)] - 1[1(-2) - 1(1)] + 1[-2(2) - 1(1)] \\ &-2(3) - 1(-3) + 1(-3) \\ &6 - 3 - 3 \\ &0 \end{aligned}$$

17.

False, they are diagonalizable over C. ## 18. True. Symmetric matrices always have distinct eigenvectors and are therefore diagonalizable.

19.

True,a s they must have a positive determinant, which guarantees it.

20.

True, as it is symmetric.

21.

An invertible symmetric has no nonzero eigenvalues, so the eigenvalues of A^2 are their squares, always positive.

22.

True. This constraint implies

$$A^T A = \begin{bmatrix} ||v||^2 & 0 \\ 0 & ||w||^2 \end{bmatrix}$$

So the singualr values are the square roots - the lengths.

23.

False. The relation is $B = S^{-1} A S$

24.

25.

True. Similar matrices have identical eigenvalues, and A is positive definite, so B must also have all positive eigenvalues and be psoitive definite.

26.

True.

$$A = Q \Lambda Q^T$$
$$\Lambda = Q^T A Q$$
$$S = Q^T$$

27.

True. Symmetric matrices always have orthogonal eigenvectors.

28.

True.

$$
\begin{aligned}
AV &= U\Sigma \\
V &= A^{+}U\Sigma
\end{aligned}
$$

29.

False. Squares of nonzero symmetric matrices never become zero because the nonzero terms are multiplied each iteration. $A^N = 0$ only for the symmetric zero matrix. But if there are no eiegnvalues above 1 the matrix approaches zero.

30.

False.

$$
\begin{aligned}
q(x) &= x_1^2 + x_2^2 \\
-q(x) &= -(x_1^2 + x_2^2)
\end{aligned}
$$

31.

True.

32.

True. Such a matrix must be symmetric and therefore diagonalizable:

$$
\begin{aligned}
Q + Q^{-1} &= Q + Q^T \\
(Q + Q^T)^T &= Q^T + Q = Q + Q^T
\end{aligned}
$$

33.

True.

$$
\begin{aligned}
C &= x^T A x x^T B x \\
&= x^T A D B
\end{aligned}
$$

The cetral term D is a symmetric matrix.

34.

False. This matrix is not symmetric.

35.

False. A matrix is negative definite only if its eigenvalues are all negative, but if they are then the even-numbered subdeterminants must be positive (as the products of eigenvalues). ## 36.

True. The quadratic form is $x^T Ax + x^T Bx$. Since both terms are separately positive, so is their sum.

$$\begin{aligned}
&x^T(A+B)x \\
&x^T(Ax+Bx) \\
&x^T Ax + x^T Bx
\end{aligned}$$

37.

True. No component of x may change sign for the quadratic form to be positive definite, so that means any shearing applied to x msut be less than $\pi/2$ radians.

Reflection matrices violate $ac > b^2$.

38.

True. Even if both matrices are sigaonal with those signualr values, the highest singular value of AB is 15. Theya re probably less, as implied by

$$AB = U_1 \Sigma_1 V_1^T U_2 \Sigma_2 V_2^T$$

39.

False, obviously.

40.

True. k is any number greater than $-a$ (if A is negative) or $\frac{b^2}{c} - a$ if not. (That's also the second pivot and the second part of the square completion).

41.

true. The overall determinant is

$$\begin{aligned}
&\det A a(df - e^2) - b(bf - ec) + c(be - cd) \\
&= adf - ae^2 - b^2 f + 2bce - c^2 d \\
&= d(af - c^2) + b(2ce - bf) - ae^2
\end{aligned}$$

If $af < c^2$, then none of the terms are positive definite, and neither is the overall matrix

42.

False.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

43.

False. Consider

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

44.

False. No souch vector is guaranteed to exist.

45.

True. Then the interaction term is negative semidefinite.

46.

False. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with singualr values 1 and 1 and determinant $1(-1) = -1$.

47.

True. They are the eigenvectors matrix S with the vectors in either order, and those two matrices' transposes.

48.

True. ## 49. False. They are equal to the absolute values of the eigenvalues. Consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

50.

True. they have the same eigenvalues, and the singular values given a set of eigenvalues are always the same.

51.

52.

True. Entry ij of A is

$$\sigma_i v_i^T u_j$$

The dot product of two normal vectors is at most 1, and no σ_i is equal to or greater than 5, so all entries of A are less than 5 ## 53.

True. $ac > b^2$ and $a > 0$ implies at least $a > |b|$ or $c > |b|$

54.

True. If $A^3 = B^3$, then from the eigenvalues of A^3 $A = B = Q\Lambda^{1/3}Q^T$. Since cube roots are unique, A and B have the same eigenvalues and eigenvectors and are therefore one and the same.