Section 6.1 Problems

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% 1: term 1 % 2: subscript 1 % 3: term 2 % 4: subscript 2 % 5. operation

1.

$$x^2 + 4xy + 2y^2$$

This can be written $(x + 2y)^2 - 2y^2$

2.

a. Negative. $x^2 + 6xy + 5y^2$.
b. Semidefinite. $x^2 - 2xy + y^2$
c. PD. $2x^2 + 6xy + 5y^2$.
d. Not PD. $-x^2 + 4xy - 8y^2$

For b, the valley runs along $x = y$.

3.

The equation is

$$(a - \lambda)(c - \lambda) = b^2$$

Both terms must ave the same sign. If $a > 0$ and $ac > b^2$, then both a and c are positive. Because $ac > b^2$ for this determinant equation to be true both a and c must be reduced, so the eigenvalues subtracted from them are positive.

4.

a. Maximum; all four derivatives are increasing.

$$F = 1 + 4e^x - 4x - 5x \sin y + 6y^2$$

$$
\begin{aligned}
&F_{xx} = 4e^x \\
&F_{YY} = 5x \sin y + 12 \\
&F_{xy} = 5cosy \\
&4e^0(5(0)\sin 0 + 12) < 5\cos 0 \\
&0 < 5
\end{aligned}
$$

b. Minimum

$$
\begin{aligned}
&F = (x^2 - 2x)\cos y \\
&F_{xx} = 2\cos y \\
&F_{yy} = x^2 \cos y - 2x \sin y \\
&F_{xy} = -2x \sin y + 2\sin y \\
&(2\cos \pi)(\cos \pi - 2\sin \pi) > (-2\sin \pi + 2\sin \pi)^2 \\
&(-2)(-1) > 0 \\
&2 > 0
\end{aligned}
$$

c. $|b| < 9$.

d. The LDL^T is:

$$
\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}
$$

d. Some googling tells me the minimizing solution is given by the eigenvector of the minimum eigenvalue. If $b = 3 \implies \lambda_{min} = 0$, that's a unit vector of the kernel basis $\begin{bmatrix} 1 \\ -1/3 \end{bmatrix}$

6.

$$
A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}
$$

7.

$$
A_1 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}
$$

The function of this matrix is 0 where $x_1 = 1$ and $x_2 = 1$ and $x_3 = 0$ or vice versa, from its kernel.

$$
\begin{aligned}
&A_2 = LL^T \\
&L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}
\end{aligned}
$$

8.

from the inverse formula, the quadratic form of the inverse is

$$\frac{cx^2 - 2bxy + a^2}{ac - b^2}$$

In terms of the usual rules, this means The test is

$$\frac{c}{ac^x} > \frac{b^2}{a^2c^2 - 2b^2ac + b^4}$$

9.

The factorization for $3(x_1 + 2x_2)^2 + 4x_2^2$ is:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The elements of d are the coefficients of the diagonal terms, and the off-diagonals of l are the interaction coefficients

10.

$$a = \begin{bmatrix} p & q \\ q & r \end{bmatrix} \quad a^2 = \begin{bmatrix} p^2 + q^2 & pq + qr \\ pq + qr & q^2 + r^2 \end{bmatrix}$$

If the matrix is singular, than for a^2 $b^2 = p^2q^2 + 2q^4 + 2q^2r^2$, always positive and potentially equal to $p^2 + q^2$.

11.

a. $\det a = ac - |b|^2$, with pivots a and $c - \frac{|b|}{a}$

b. $a|x_1|^2 + 2Reb|x_1|x_2 + c|x_2| = a|x + \frac{bx_2}{a}|^2 + |c - \frac{|b|}{a}||x_2|$

c. Given this, the second pivot must be positive,e ensuring the matrix is positive semi-definite.

d. The first is not, since $2 >= 1^2 + 1^2$, but the second its, since $18 > 4^2 + 1^2$

12.

$$f = x^2y^2 - 2x - 2y$$
$$f_{xx} = 2y^2$$
$$f_{yy} = 2x^2$$
$$f_{xy} = 4xy$$

at $(1, 1)$:

$$\begin{aligned}
&2(1^2)2(1^2) > 4(1)(1) \\
&4 = 4
\end{aligned}$$

Not quite a minimum or maximum.

13.

$$\begin{aligned}
f_x &= 2ax + 2by \\
f_{xx} &= 2a \\
f_y &= 2(bx + cy) \\
f_{yy} &= 2c
\end{aligned}$$

Since the original function's partial derivatives are $2x$ and $2y$ and 2, a, b, and c must be greater than 1.

14.

Do they have two positive eigenvalues a. No. $x = \begin{bmatrix} 1/4 \\ -1/4 \end{bmatrix}$

b. No. $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c. No. No x exists because it's positive semidefinite.

d. Yes. None exists; it's pd.

15.

a. $f = a^2 + 4xy + 9^2$ and $ f = (x +2y)^2 + (9-4)y^2$

b. $f = x^2 + 6xy + 9y^2 $ and $f = (x + 3y)^2$.

16.

This is not a minimum:

$$\begin{aligned}
f_{xx} &= 2 \\
f_{yy} &= 6 \\
f_{xy} &= 4 \\
2(6) &< 4^2
\end{aligned}$$

As a difference of squares, $f = x^2 + 4xy + 3y^2 = (x + 2y)^2 - y^2$. so $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, for example, gives $z = -1$.

17.

$$\begin{aligned}
&x^T a^T ax \\
&= (ax)^T (ax)
\end{aligned}$$

Clearly this is always positive for nonzero x.

18.

All are clearly PD.

$$a^t a = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$$
$$a^t a = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$
$$a^t a = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

19.

$4(x_1 - x_2 + 2x_3)^2$ expands to

$$4x_1^2 - 4x_1x_2 + 8x_1x_3 - 4x_1x_2 + 4x_2^2 + 8x_2x_3 + 8x_1x_3 - 8x_2x_3 + 32x_3^2$$

implying a matrix of

$$\begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 32 \end{bmatrix}$$

determinant is 0, rank is 2, not bothering with the eigenvalues.

20.

$$a_1 = \begin{bmatrix} 9/4x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix}$$
$$a_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

21.

The first derivatives have opposite signs: one function is increasing, the other decreasing.

22.

For $f(x) = 4x^2 + 12xy + cy^2$ the derivatives matrix is

$$\begin{bmatrix} 8 & 12 \\ 12 & 2c \end{bmatrix}$$

which is positive definite where

$$16c > 144$$
$$c > 9$$