Section 5.2 Problems

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1.

Factor into $S\Lambda S$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$
$$S = \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$A = \begin{bmatrix} -5 & -1 \\ 0 & 1 \end{bmatrix}$$

3.

$\lambda_1 = 3, \lambda_2 = \lambda_3 = 0$

The eigenbasis is

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Any scalar of this matrix will also diagonalize.

4.

If a triangular matrix has n distinct values on its trace, those values are its eigenvalues, so it has ll distinct eigenvalues and can therefore be diagonalized.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

5.

$A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ is not diagonalizable because the repeated eigenvalues have the same eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, meaning S is not invertible.

6.

a. If $A^2 = I$, then $A = A^{-1}$, meaning the eigenvalues must be ± 1.

b. $tr(A) = \pm n$; $\det(A) = -1^n$

7.

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

8.

a.

$$\begin{aligned} A &= uv^T \\ Au &= uv^t u \\ &= u\lambda = \lambda u \end{aligned}$$

b. If the first eigenvalue is $u \times v$, the second must be 0 for the trace to sum to that value.

c. This is a cross-products matrix, so $tr(A) = u_1 v_1 + u2 v_2 = u \cdot v$

9.

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mat2latex(check_commute(m = square("q", "s", "r", "t")))
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$$\begin{bmatrix} qa + sb = qa + rc & ra + tb = qb + rd \\ qc + sd = sa + tc & rc + td = sb + td \end{bmatrix}$$

Evidently $qa + sb + rc + td = qa + rc + sb + td$.

So $tr(AB - BA) = 0$

10.

The eigenvalues of A^2 are $1, 4, 8$, so $tr(A^2) = 1 + 4 + 8 = 13$. For the determinant:

$$\begin{aligned} \det(A) &= 1 \times 2 \times 4 = 8 \\ \det(A^{-1})^T &= \det A^{-1} = 1/8 \end{aligned}$$

11.

a. True. Only noninvertible matrices have zero eigenvalues.

b. False. Non-distinct eigenvalues do not guarantee diagonalizability. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

for which the eigenspace for $\lambda = 1$ has 1 dimension but the algebraic multiplicity is 2.

c. False. If A is already diagonal:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then $S = S^{-1} = I_3$.

12.

a. True. If S has rank 1, it is noninvertible and therefore diagonalization is impossible.

b. True. A must have rank 1, since S has rank 1 as well. Since $n = 3$, two eigenvalues must be 0.

c. False. A itself may be diagonal.

13.

14.

If S is orthogonal, then:

$$\begin{aligned} S\Lambda S^{-1} &= S\Lambda S^T \\ &= (\Lambda S^T)^T S^T \\ &= S\Lambda^T S^T \\ &= S\Lambda S^T \end{aligned}$$

15.

16.

$A^3 = S\Lambda^3 S^{-1}$ and $A^{-1} = S\Lambda^{-1}S^{-1}$.

17.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

18.

The eigenvalues are just $\lambda + 2$ and the eigenvectors are the same. So $A + 2I = S(\Lambda + 2I)S^{-1}$

$$\begin{aligned} &= (S\Lambda + 2S)S^{-1} \\ &= S\Lambda S^{-1} + 2SS^{-1} \\ &= S\Lambda S^{-1} + 2I \\ &= A + 2I \end{aligned}$$

19.

a. False. Consider a diagonal matrix with zeroes on the diagonal.

b. True.

c. True.

d. False; invertible matrices do not necessarily have distinct eigenvalues.

20.

S is the identity if A is already diagonal.

21.

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$
$$S = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

To get A^{-1} we just take the inverse of Λ, $\begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}$

22.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

23.

24.

$$\lambda_A = (1, 1)$$
$$\lambda_B = (1, 1)$$
$$\lambda_(A + B) = (3, 1)$$

25.

a. True.

b. False.

c. False.

26.

A is noninvertible, and so is A^2 The eigenvalues and eigenvectors of A^2 are also the same ($1^2 = 1$ and $0^2 = 0$). Both have determinant zero and trace 1. The transposes of each have the same eigenvalues. IF A is diagonalizable, then $A^2 = A$ because $\Lambda^2 = \Lambda$

27.

Any values will work. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a possible eigenvector.

28.

Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, diagonalization is impossible because $A - 3I$ has rank 1, but that eigenvalue has algebraic multiplicity 2. Changing either entry of the diagonal or $(2, 1)$ to a nonzero value would make it diagonalizable.

29.

A^k approaches 0 only if if every λ is fractional.

If

$$A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}$$

it does not approach 0, because $\lambda_1 = 1$.

$$\lambda^2 - 1.2\lambda + .2 = 0$$
$$(\lambda - 1)(\lambda - .2) = 0$$

But if

$$B = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}$$

it does, because

$$\lambda^2 - 1.2\lambda + .27 = 0$$
$$(\lambda - .9)(\lambda - .3) = 0$$

because the lambdas are fractional.

30.

For the previous A, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$. So

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So the factorization is

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Λ's second column gradually zeroes out as $k \to \infty$.

31.

32.

Prove a formula for A^k.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$
$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

This clearly represents the formula $\frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}$

33.

Doing the same for B, we get

$$\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$$
$$S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix} \begin{bmatrix} 1/6 & 1/2 \\ -1/2 & 1/6 \end{bmatrix}$$

34.

If A is diagonalizable, we can prove:

$$\begin{aligned}
&A = S\Lambda S^{-1} \\
&\det A = \det S \det \Lambda \frac{1}{\det S} \\
&\det A = \det \Lambda \\
&\det A = \prod_{i=1}^{n} \lambda_n
\end{aligned}$$

35.

$tr(\Lambda) = tr(A)$ because eigenvalues always sum to the trace.

36.

37.

Proof matrices with the same eigenbasis form a subspace:

$$\begin{aligned}
&S\Lambda_1 S^{-1} + S\Lambda_2 S^{-1} \\
&= S(\Lambda_1 S^{-1} + \Lambda_2 S^{-1}) \\
&= S(\Lambda_1 + \Lambda_2) S^{-1}) \\
&= S\Lambda_1 S^{-1} + S\Lambda_2 S^{-1}
\end{aligned}$$

Scalar multiples:

$$kS\Lambda S^{-1} = S(k\Lambda)S^{-1}$$

If $S = I$, then the subspace is R^4.

38.

Given $A^2 = A$, $\lambda = 1$ belong in the image, with eigenvectors of A's nonzero column vectors, $\lambda = 0$ in the kernel ($Ax = 0x \implies Ax = 0$) with dimension $n - r$. That eigenbasis consists of e_n where row (or column) n of A is all zeroes.

$$\begin{aligned}
&A^2 = A \\
&A^2 x = Ax \\
&\lambda^2 x = \lambda x \\
&x = \lambda x
\end{aligned}$$

39.

The eigenvectors are *in* the spaces but do necessarily span them. Eigenvectors may fail to exist, or a multiple eigenvalues may correspond to the same eigenvector.

40.

$$
\begin{aligned}
&(S\Lambda S^{-1} - \lambda_1 I)(S\Lambda S^{-1} - \lambda_2 I)\dots(S\Lambda S^{-1} - \lambda_n I) = 0\\
&S\Lambda^2 S^{-1} - \lambda_1 S\Lambda S^{-1} - \lambda_2 S\Lambda S^{-1} - \lambda_1\lambda_2 I \dots = 0\\
&S\Lambda^n S^{-1} - \prod_{i=0}^{n} \lambda_i S\Lambda S^{-1} - \prod_{i=0}^{n} \lambda_i I = 0\\
&= S\Lambda^n S^{-1} - \det S\Lambda S^{-1}(S\Lambda S^{-1} + I) = 0\\
&A^n - (\det A)A - (\det A)I = 0
\end{aligned}
$$

This is just a generalization of the characteristic polynomial, with $S\Lambda^n S^{-1}$ replacing $-\lambda^n$

41.

Demonstrate Cayley-Hamilton on the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $A^2 - A - I = 0$ for this matrix, since its determinant is 1

$$
\begin{aligned}
&\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0\\
&\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0\\
&0 = 0
\end{aligned}
$$

42.

43.

44.

If $AB - BA = 0$, the matrix representing the equation is singular because it always has the nonzero solution $B = I$.

45.

Using the $m \pm \sqrt{m^2 - p}$ trick:

$$
A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}
$$

$$
\begin{aligned}
\Lambda &= \frac{.6 + .8}{2} \pm \sqrt{\left(\frac{.6 + .8}{2}\right)^2 - ((.6)(.8) - (.2)((.4)))}\\
&= .7 \pm \sqrt{.49 - .4}\\
&= .7 \pm .3 S = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

For A^∞:

$$
\begin{aligned}
\lambda &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - 0} \\
&= 1, 0 \\
S &= \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

$A^{100} \approx A^{\infty}$ because $\Lambda^{1}00 \approx \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the eigenvalues of A^{∞} The eigenvalue .4 all but disappears, and A becomes computationally indistinguishable from a one-dimensional matrix.