

## Section 4.2 Problems

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We check the linearity of some transformations.

**1.**

$T(M) = M + I_2$ : nonlinear

**3.**

$$T(M) = \text{trace}(M)$$

Obviously not isomorphic; many matrices have the same trace.

##5.

$T(m) = M^2$ : not isomorphic

**7.**

$$T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$$

Obviously linear, given matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

Also an isomorphism, since this is clearly invertible.

**9.**

$$T(M) = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} M \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Not an isomorphism, since  $M$  itself may be noninvertible.

Actually yes:

$$T(M) = S^{-1}MS$$

$$S^{-1}M = S^{-1}$$

$$M = SAS^{-1}$$

**12.**

$$T(c) = cM \quad c \rightarrow R^2$$

where  $M$  is a particular matrix. Linear -  $c(M_1 + M_2) = cM_1 + cM_2$  and  $c(kM_1) = kcM_1$ .

Also isomorphic:  $M = 1/cT(c)$

**16.**

$$T(M) = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M$$

The matrix is:

$$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \end{bmatrix}$$

Linear, not invertible.

**17.**

$$T(x + iy) = x$$

Linear, not invertible, as the complex part is just zeroed out. ## 18.

$$T(x + iy) = x^2 + y^2$$

Non-linear ( $cT(x) = cx^2 + cy^2$ ), but  $T(cx) = c^2x^2 + c^2y^2$ ;  $x$  can be positive or negative.

**21.**

$$T(x + iy) = y + ix$$

Linear:  $T(x_1 + iy_1 + x_2 + iy_2) = y_1 + y_2 + i(x_1 + x_2) = T(x_1 + iy_1) + T(x_2 + iy_2)$  and  $T(kx_1 + kiy_1) = ky_1 + kx_1i_y = kT(x + iy)$

Isomorphism, and involutory too:  $T^{-1}y = y + ix$ .

**22.**

$$T(f(t)) = \int_{-} 2^3 f(t) dt$$

$$\begin{bmatrix} 3 & 3/2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2/3 \end{bmatrix}$$

Clearly noninvertible; infinitely many functions with the same definite integral. ## 23.

$$T(f(t)) = f(7)$$

$$T(f(t)) = a + 7b + 49c$$

Linear, nonisomorphic, with matrix

$$\begin{bmatrix} 1 & 7 & 49 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

24.

$$T(f(t)) = f''(t)f(t)$$

$$T(1) = 0(1) = 0 \qquad T(t) = b(0)(t) = 0$$

$$T(t^2) = c(2)(t^2) = 2ct^2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Not isomorphic.

27.

$$T(f(2t))$$

$$T(1) = 1a = 1$$

$$T(t) = b(2t) = 2bt$$

$$T(t^2) = c(4t^2) = 4ct^2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Isomorphic.

28.

$$T(f(t)) = f(2t) - f(t)$$

Nonisomorphic; the constant term gets zeroed out, with matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 3 \end{bmatrix}$$

29.

$$T = f'(t)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Linear, non-isomorphic.

**31.**

$$T = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix}$$

from  $P_2$  to  $R^{2 \times 2}$  Substituting, we find

$$T = \begin{bmatrix} a & a+b+c \\ a+2b+4c & a+3b+9c \end{bmatrix}$$

The matrix should be:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

```
cbind(matrix(rep(rep(c(1, 0, 0), times = 2), times = 3), ncol = 3), sapply(list(c(2,
0, 0), c(5, 0, 0), c(13, 0, 0)), rep, 2)) %>%
  mat2latex()
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$$\begin{bmatrix} 1 & 1 & 1 & 2 & 5 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 5 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**32.**

$$T(f(t)) = f'(t) + t^2 = b + 2ct + t^2$$

Nonlinear; that  $t^2$  is affine and has no matrix representation.

**33.**

Drop every other term.

$$T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_0, x_2, x_4, \dots)$$

Linear, since the sums and scalar multiples of parallel terms are the same before and after the transformation, but not isomorphic because there is no way to recover the terms.

**34.**

$$T(x_0, x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

First rule:

$$\begin{aligned} T(x+y) &= (0, x_0+y_0, x_1+y_1) \\ T(x)+T(y) &= (0, x_0, x_1, \dots) + (0, y_0, y_1, \dots) \end{aligned}$$

Second rule:

$$kT(x) = (0, kx_1, kx_2, \dots)$$

$$T(kx) = (0, kx_1, kx_2, \dots) = kT(x)$$

Linear, isomorphic.

**35.**

From  $P$  to  $V$ :

$$T(f(t)) = (f(0), f'(0), f''(0), \dots)$$

The first property is satisfied by the sum rule of derivatives.

Second property satisfied:

$$kT(f(t)) = (kf(0), kf'(0), kf''(0), \dots)$$

$$T(kf(t)) = ((kf(0), kf'(0), kf''(0), \dots))$$

Not an isomorphism; many functions are undefined at 0.

**39.**

Let's find the matrix:

$$T(f) = f'' + 2f' + f$$

$$= 2c + 2(b + 2ct) + a + bt + ct^2$$

$$= a + 2b + 2c + bt + 4ct + ct^2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

This is clearly invertible, so it's an isomorphism.

**43.**

$$T(f(t)) = \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 11 & 121 \end{bmatrix}$$

Non-isomorphic; the constant term gums up the works.

**52.**

The kernel of  $T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$  is all matrices which have  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in their kernel, of the form  $v_1 = -2v_2$ .

**25.**

$$T(f(t)) = f''(t) + 4f'(t)$$

$$= 2c + 4(b + 2ct)$$

$$= 4b + 2c + 8ct$$

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The kernel is constant functions you idiot.

**57.**

$$T(f(t)) = f'' - 5f' + 6f$$

$$= 2c - 5(b + 2ct) + 6(a + bt + ct^2)$$

$$= 2c - 5b - 10ct + 6a + 6bt + 6ct^2$$

$$= 6a - 5b + 2c + 6bt - 10ct + 6ct^2$$

The image is  $P_2$ , the kernel  $f(t) = 0$ .

**59.**

From  $P_2$  to  $R^n$

$$T(f(T)) = f(7)$$

The matrix:

$$\begin{bmatrix} 1 & 7 & 49 \end{bmatrix}$$

The matrix kernel makes this clearer, I think.

$$\begin{bmatrix} -7 & -49 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**66.**

For the infinite-dimensional  $T(f) = f - f'$ , the kernel is  $e^x$ , the image all other functions.

Working with the subspace of polynomials of degree  $\leq n$  such that  $f(0) = 0$ ; no constant term.

**72.**

Is it a subspace? Yes, because  $f(0 + g(0)) = 0 + 0 = 0$ , and  $kf(0) = 0(0) = 0$ . The dimension of  $Z_n$  is  $n$ , because a polynomial with no constant term has  $n$  terms.

**75.**

$$T(0_V) = 0_W$$

$$0_V(T) = 0_W$$

$$0_V = 0_W$$

$$T(0_V) = 0_W$$

$$T^{-1}(T(0_V)) = T^{-1}(0_W)$$

$$0_V = T^{-1}(0_W)$$

**76.**

We know  $T$  is linear, so we need to prove  $T^{-1}$  obeys the properties as well.

$$T^{-1}(f + g) = T^{-1}(T(T^{-1}(f + g)))$$

$$= T^{-1}(T(T^{-1}(f)) + T(T^{-1}(g)))$$

$$= T^{-1}(f) + T^{-1}(g)$$

and

$$kT^{-1}(f) = T^{-1}(kT(T^{-1}(f)))$$

$$= T^{-1}(T(k)T^{-1}(f))$$

$$= kT^{-1}f$$