Notes

Ryan Heslin

July 3, 2022

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

(-1, 0, 1, 1).

13.

14.

Any eigenvalue of a block on the diagonal is also an eigenvalue of A, because subtracting the eigenvalue will make the whole matrix singular.

15.

$$\begin{aligned} &\lambda^2 - 2\lambda + 1 - k \\ &\frac{2 \pm sqrt4 - 4(1)(1-k)}{2} \quad 1 \pm \frac{4k}{2} 1 \pm k \end{aligned}$$

If $k = 0$ the eigenvalue 1 is repeated; if $k < 0$ they are both imaginary; otherwise, both are real.

16.

The quadratic formula ultimately leads to

$$\frac{a + c \pm 2\sqrt{1^{+}4b^2 + c^2 - 2ac}}{2}$$

If $a = -c$, then

$$\begin{aligned} &a^2 + 4b^2 + a^2 2a^2 = 0 \\ &4a^2 + 4b^2 = 0 \qquad\qquad a = b = 0 \end{aligned}$$

so repreat eigenvalues are possible only if both are zero. I'd rather not find out the imagniary and real cases.

17.

They work out to $\sqrt{(}a^2 + b^2)$. I think this is a rotation-scaling $\frac{2\pi}{2}$ left.

18.

By computing the determinant of

$$\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix}$$

we find

$$\begin{aligned}
&(a-\lambda^2) \\
&a^2 - 2a\lambda + \lambda^2 - b^2 \\
&\frac{2a \pm \sqrt{(-2a)^2 - 4(1)(a^2 - b^2)}}{2} \\
&\frac{2a \pm \sqrt{4b^2}}{2} \\
&a \pm b
\end{aligned}$$

19.

True. Since $\det A = \lambda_1\lambda_2$, it can only be negative if the eigenvalues are opposite-signed, which means they must be distinct.

20.

By the definition of an eigenvalue, $\dim(\ker(A - \lambda I)) \geq 1$ (because this matrix has zero determinant, so its kernel has nonzero dimension). So each eigenvector has geometric multiplicity of 1. The eigenvectors also must be distinct if the eigenvalues are. If they shared an eigenvector v, then $\lambda_1 v = \lambda_2 v$, because $\lambda_1 v = Av$ and $\lambda_2 v = Av$, which is impossible if $\lambda_1 \neq \lambda_2$. With two distinct eigenvectors, A can be diagonalized $A = S\Lambda S^{-1}$.

21.

22.

They are identical because $tr(A) = tr(A^T)$ and $\det A = \det A^T$. They also therefore have the same eigenvalues.

23.

If they are similar, then $B = S^{-1}AS$. Then if A can be diagonalized $A = T\Lambda T^{-1}$, $B = A$ if $S = TX$ for some other matrix X.(I think this holds for the Jordan diagonalization as well if A can't be diagonalized). So these two matrices have the same eigenvalues and thus the same characteristic polynomials, but different eigenvectors.

24.

The polynomial is $\frac{1}{4}(\lambda - 1)(\lambda - 4)$, so 1/4 and 1.

25.

By substituting into $Av = \lambda v$, we get

By solving

$$0 = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} v$$

for the two eigenvectors, we get>

$$a - \lambda - b = 0 \implies \lambda = 1 - b$$

for $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and

$$b(a - \lambda) = -bc \implies \lambda = a + c = 1$$

for $\begin{bmatrix} b \\ c \end{bmatrix}$

The first eigenvalue has an absolute value less than 1 because $a \leq 1$ and $b \geq 0$

26.

27.

We know the eigenvalues, so we just have to find the eigenvectors, diagonalize, and multiply out. That comes to:

$$\begin{bmatrix} 1 & 0 \\ 0 & -(1/4)^k \end{bmatrix}$$

The matrix approaches its steady state. The eigenvector associated with the dominant eigenvalue stays in place ($A^k v = \lambda^k v = v$ for $\lambda = 1$), and the one associated with the eigenvalue less than 1 in absolute value dies.

```
`%^%` <- function(lhs, rhs) {
  out <- lhs
  for (i in seq_len(rhs)) {
    out <- out %*% lhs
  }
  out
}
A <- matrix(c(.5, .5, .25, .75), nrow = 2)
A %^% 100
```

```
          [,1]      [,2]
[1,] 0.3333333 0.3333333
[2,] 0.6666667 0.6666667
```

```
A %^% 100 %*% c(1, -1)
```

```
     [,1]
[1,]    0
[2,]    0
```

```
A %^% 100 %*% c(1, 2)
```

```
     [,1]
[1,]    1
[2,]    2
```

28.

```
A <- matrix(c(.8, .2, .1, .9), nrow = 2)
A %^% 100
```

```
          [,1]      [,2]
[1,] 0.3333333 0.3333333
[2,] 0.6666667 0.6666667
```

b.

The full formula is

$$w(0)\begin{bmatrix}1+2(7/10)^t\\2-2(7/10)^t\end{bmatrix}+m(0)\begin{bmatrix}1-(7/10)^t\\2+(7/10)^t\end{bmatrix}$$

c.

Never, because the steady state has 1/3 of the townspeople at the Wipfs', which is 400.

29.

It always has eigenvalue 1 because $A1$ (if 1 designates a vector of 1s) is 1 (since the columns sum to 1). Therefore $A1 = \lambda 1 = 1$, so $\lambda = 1$.

30.

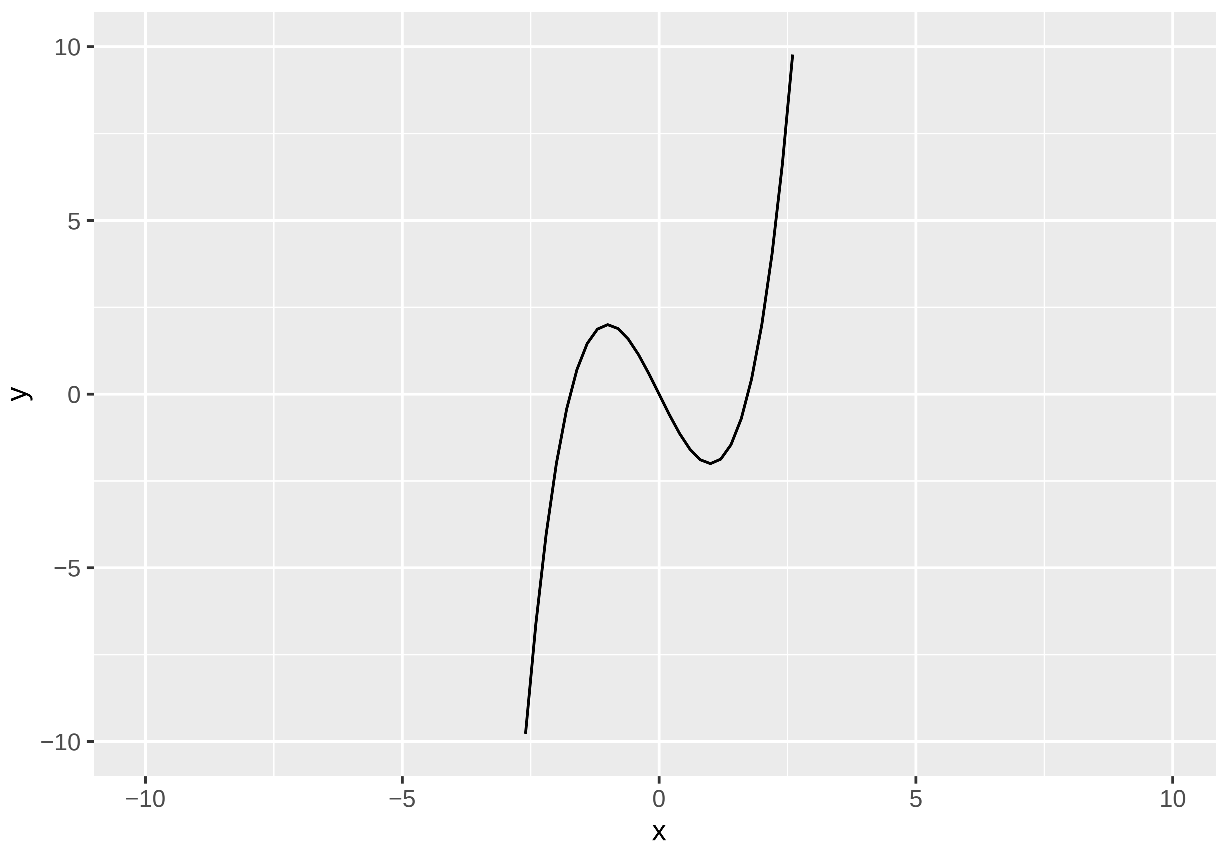
31.

1 must be an eigenvalue because if A is a transition matrix, A^T must have rows that sum to 1. As shown above, 1 is an eigenvalue, so 1 must be an eigenvalue of A as well. But the eigenvector is not necessarily the same, since the eigenvectors of A are not generally those of A^T.

32.

They are ± 2.

```
library(ggplot2)
polynomial <- function(lambda) lambda^3 - 3 * lambda
X <- data.frame(x = seq(-10, 10))
ggplot(X, aes(x = x)) +
  geom_function(fun = polynomial) +
  lims(x = c(-10, 10), y = c(-10, 10))
```




```
optimize(polynomial, interval = c(-5, 5))
```

```
$minimum
[1] 1.000007

$objective
[1] -2
```

```r
optimize(polynomial, maximum = TRUE, interval = c(-5, 5))
```

```
$maximum
[1] -1.000007

$objective
[1] 2
```

33.

34.

35.

36.

37.

38.

By algebra, 7 and -2.

39.

For $n \times n$ matrices, element ii of the trace of AB is $a_i^T b_i$ (where the subscript denotest the column). For $BA, it is $b_i^T a_i$, which is equivalent.

40.

As shown above.

41.

If they are similar, they have the same eigenvalues, so the sums of eigenvalues, and therefore the traces, are equal.

42.

If $BA = 0$, then $tr(BA) = 0$, and we just showed that $tr(AB) = tr(BA)$. Therefore:

$$
\begin{aligned}
&tr((A + B)^2) = tr(A^2) + tr(B^2) \\
&= tr(A^2) + tr(B^2) + AB + BA \\
&= tr(A^2) + tr(B^2
\end{aligned}
$$

43.

No, because since $tr(AB) = tr(BA)$, tr(AB - BA) = tr(BA - AB) = 0$.

44.

45.

The eigenvalue satisfies (we can ignore the negaitve branch, since it makes no impact):

$$
\begin{aligned}
&1 \pm 2\sqrt{1 + k} = 5 \\
&2\sqrt{1 + k} = 4 \\
&\sqrt{1 + k} = 2 \\
&1 + k = 4 \\
&k = 3
\end{aligned}
$$

46.

a.

$$
\begin{aligned}
&(\lambda_1+\lambda_2)^2=(a+d)^2\\
&\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2=a^2+2ad+d^2\\
&\lambda_1^2+\lambda_2^2+2(ad-bc)=a^2+2ad+d^2\\
&\lambda_1^2+\lambda_2^2=a^2+d^2+2bc
\end{aligned}
$$

###.b

Subtract a^2+d^2 from both sides to get $2bc \leq b^2+c^2$, an old algebraic identity that provides the basis for Cauchy-Swarz. (I had to look it up, but of course the sum of squares has to be greater than the cross term to be positive definite in cases where one term is subtracted from the other).

c.

Diagonal matrices, for which $b=c=0$

47.

48.

49.

50.