Notes

Ryan Heslin

‘r format(Sys.Date(), ‘%B %d, %Y’)

1.

A 90 degree rotation followed by an x-axis projection is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

4.

$$\begin{aligned} z &= \frac{y - z}{2} \\ Az &= A\left(\frac{y - z}{2}\right) \\ &= \frac{1}{2}Ay - \frac{1}{2}Az \end{aligned}$$

6.

3 by 3’s for:

a. x-axis projection:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. xy reflection:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

c. xy rotation:

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d. Composed 90-degree rotations:

```r
sin_90 <- 1
cos_90 <- 0
compose_trans(list(square(sin_90, -cos_90, 0, cos_90,
    sin_90, 0, 0, 0, 1), square(sin_90, 0, -cos_90,
    0, 1, 0, sin_90, 0, cos_90), square(1, 0, 0, 0,
    sin_90, -cos_90, 0, cos_90, sin_90))) %>%
    mat2latex()
```

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

e. Composed 180-degree:

‘

```r
sin_90 <- sin(pi)
cos_90 <- cos(pi)

compose_trans(list(square(sin_90, -cos_90, 0, cos_90,
    sin_90, 0, 0, 0, 1), square(sin_90, 0, -cos_90,
    0, 1, 0, sin_90, 0, cos_90), square(1, 0, 0, 0,
    sin_90, -cos_90, 0, cos_90, sin_90))) %>%
    round() %>%
    mat2latex()
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

7.

Find the P_3 to P_4 matrix of $3t+2$

$$\begin{aligned} a &= 3t+2 \\ b &= b(t(3t+2)) = b(3t^2+2t) \\ c &= c(t^2(2+3t)) = c(3t^3+2t^2) \\ d &= d(t^3(3t+2)) = d(3t^4+2t^3) \end{aligned}$$

Now the matrix:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

12.

$A^{-1}(x+y) = A^{-1}x + A^{-1}y$ by the linearity of the inverse transformation.

14.

If T is linear, it has matrix A. The $T^2 = A(A) = A^2$, which is a composition of linear transformations

15.

What's the transpose matrix for $R^{2\times 2}$?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

17.

a. Matrix of the "right shift" $T(x) = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The opposite "left shift":

$$\begin{bmatrix} ,0 & 1 & 0 & 0 \\ ,0 & 0 & 1 & 0 \\ ,0 & 0 & 0 & 1, \end{bmatrix}$$

Naturally, they are left and right inverses of each other.

19.

Which nonlinears are invertible?

a. x^3: yes

b.e^x: no, limited to positive numbers.

c $x+11$: yes

c.$\cos x$: yes

21.

Why linear transformations can't move the zero vector:

$$T(0+w) = T(0) + T(w) = T(w)$$

This holds only if $T(0) = 0$. Similarly, $T(0v) = 0T(v)$.

23.

If S and T are linear and equal v, then $S(T(v)) = S(v) = v$

24.

Say $T(0, v_2) = (0, 0)$. Then $kT(0, v_2) = (0, kv_2) = 0 = (0, kv_2)$, but:

$$\begin{aligned} &T((0, v_2) + w) = (w_1, v_2) \\ &T((0, v_2)) + T(w) = (0, 0) + w = w \end{aligned}$$

25.

Which transformations satisfy which linearity property?

a. $T(v) = v/||v||$: neither

$$T(v+w) = \frac{v+w}{||v+w||} \neq v/||v|| + w/||wT(cv) = cv/||cv|| \neq cv/c||v||$$

b. $T(v) = v_1 + v_2 + v_3$: both

$$\begin{aligned} &T(v+w) = v_1 + v_2 + v_3 + w_1 + w_2 + w_3 = T(v) + T(w) \\ &kT(v) = k(v_1 + v_2 v_3) = kv_1 + kv_2 + kv_3 \end{aligned}$$

c. $T(v) = (v1, 2v_2, 3v_3)$: obviously a legit diagonal matrix.

d. $T(v) = \max(v)$

Say $\max(w) = k\max(v)$ and the index of $\max(v)$ is not that of $\max(w)$. It satisfies the scalar property only.

$$T(v+w) = \max(v+w) \quad T(v)+T(w) = \max(v) + k\max(v)$$

$$kT(v) = k\max(v) = \max(kv)$$

28.

Find some ranges and kernels:

a. $T(v_1, v_2) = (v_2, v_1)$: R^2 and 0.

b. $T(v_1, v_2, v_3) = (v_1, v_2)$ R^2 and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

c. \$T(v_1, v_2) = (0, 0): $0andR^2$.

d. $(v_1, v_2) = v_1, v_1$: $(1,0)and(0,1)$.

31.

A matrix transformation is linear because the dot product, by which multiplication is done, is itself linear.

33.

A "universal" transposing matrix cannot exist because we would need to swap the $2,1$ and $1,2$ elements. But this can only be achieved by a matrix with a a 1 in one of those positions and zeroes elsewhere. Yet the diagonal of the transpose must be $(1,1)$ in order to select columns in the same order. No single matrix can have both these properties.

Wait a minute, it satisfies the rules of linearity properties but has no matrix!

34.

The transpose is involutory ($(A^T)^T = A$), has the zero kernel, and has a range of all matrices, but $T(M) = -M$ only for diagonal or skew-symmetric matrices.

40.

$$M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

The solution is $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ - the first column of M^{-1}, which reverses the effect of the transformation.

43.

The matrix of T is Λ, since the transformation is just scaling eigenvectors.

43.

Because the matrix of a linear transformation is defined as the the transformation applied to the columns of the identity. That means each basis vector has a unique representation in the transformation, which may be inverted.

44.

A reflection across the x axis followed by the y axis is also a reflection over $y = -x$

45.

$S(T(v)) = \begin{bmatrix} y \\ -x \end{bmatrix}$, also a reflection over the line $y = -x$.

49.

If we know $T(V)$ for n distinct vectors, we don't know the transformation for all v unless they're linearly independent.

50.

a. A parallelogram

b. Scaling matrices with one value in the trace.

c. Matrices of rank 1.

d. Rotation and reflection matrices.