Chapter 5 Review Problems

Ryan Heslin

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1.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B = \frac{2}{3} \begin{bmatrix} 1 & 1 \\ 3/2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -3/2 & 1 \end{bmatrix}$$

2.

$\det A = \lambda_1 \lambda_2$; $\det A^{-1} = \frac{1}{\lambda_1 \lambda_2}$

3.

The eigenvectors are orthogonal and the values real, a pair of properties unique to symemtric matrices. The trace is 1, the sum of the eigenvalues, the determinant 0, the product.

I multiply out the diagonalziation to get:

$$\frac{-1}{5} \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

4.

The eigenvectors remain the same, as will the eigenvalues, because $0^2 = 0$ and $1^2 = 1$.

5.

For real r, $A + ri$ is invertible if $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$ because skew-symmetirc matrices have complex eigenvalues, so there will never be a zero eigenvalue. Likewise, adding cI to an invertible Hermitian does not make it noninvertible because Hermitians always have real eigenvalues.

6.

$$A^t = \frac{1}{2} \begin{bmatrix} e^{4t} + e^{2t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{4t} + e^{2t} \end{bmatrix}$$

From this the solutions for $u_0 = e_1$ and $u_0 = e_2$ just select the columns.

7.

If t is 1 year:

$$P = (1 + \frac{.4}{4})^{4t} = 1.1^{4t}$$
$$P = (1 + \frac{1}{2})^{t} = 1.5^{t}$$

The annual interest is somewhat better.

8.

a. False. Row exchanges change the eigenvalues.

b. False. $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

c. True. If a matrix is unitary and Hermitian, its transpose is real and its inverse, so $A^2 = I$. If $A^2 = I$ and the matrix is Hermitian, it must be unitary because $A^{-1} = A$. If unitary and $A^2 = I$, the inverse is the Hermitian and also A itself, so A is Hermitian.

d. False. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

9.

The bizarro Fibonacci sequence starts with $F_{-1} = 1$, so $F_{-2} + 1 = 0 \implies F_{-2} = 1$. from there it's the regular sequence: $1, -1, 2, -3, 5, -8$. Matrix I think is $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

10.

11.

For any vector s in S, $Ps = s$ because $s^{\perp} = 0$, so $Ax = \lambda s \implies \lambda = 1$. For vectors v orthogonal to the subspace, $Pv = 0$, again satisfying the criterion. The eigenvalues are $n - \text{rank}(P)$ zeroes and the rest ones.

12.

Probably not the intended reasoning, but this could be done by splitting the column vectors over two matrices, filling with zeroes, and summing.

13.

14.

$$A = \begin{bmatrix} -7 & 20 \\ -4 & 11 \end{bmatrix}$$

$$Au = \begin{bmatrix} 17 \\ 8 \end{bmatrix}$$

$$
\begin{aligned}
u_{k+1} &= 1(1^k)\begin{bmatrix}5\\2\end{bmatrix} + 2(3^k)\begin{bmatrix}2\\1\end{bmatrix}\\
&= \begin{bmatrix}5\\2\end{bmatrix} + 3^k\begin{bmatrix}4\\2\end{bmatrix}
\end{aligned}
$$

15.

The eigenvectors, as expected, are complex but orthogonal.

```
A <- matrix(complex(9, imaginary = c(0, 1, 0, -1, 1,
    -1, 0, 1, 0)), nrow = 3)
mat2latex(round(eigen(A)$vectors, digits = 3))
```

$$
\begin{bmatrix}
-0.177-0.468i & -0.177+0.468i & 0.707+0i\\
0.707+0i & 0.707+0i & 0+0i\\
-0.177-0.468i & -0.177+0.468i & -0.707+0i
\end{bmatrix}
$$

16.

From the multiplication

$$
\begin{bmatrix}
a^2+bc & b(a+d)\\
c(a+d) & bc+d^2
\end{bmatrix}
$$

Neither a nor d can be zero, so $bc = -\sqrt{a} = -\sqrt{d}$. But then c must be zero for the bottom left to be 0 yet nonzero for the diagonal to be 0. If $A = \begin{bmatrix}4 & 1\\0 & 4\end{bmatrix}$, then a possible square root is $\begin{bmatrix}2 & 1/4\\0 & 2\end{bmatrix}$

17.

a.

$$
du/dt = \frac{1}{8}\left(500e^t\begin{bmatrix}4\\1\end{bmatrix} - 300e^{-t}\begin{bmatrix}4\\-1\end{bmatrix}\right)
$$

18.

a. True.

b. False. Consider $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

c. False. Consider

$$
\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}
$$

d. False. Consider

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

e. False. This is true only of Hermitian matrices.

19.

a. The column vectors are linearly independent, and 1 cannot be an eigenvalue, so $K - I$ is invertible.

b. The orthogonal eigenvectors can be converted to a unitary matrix and used to diagonalize K.

c. Since $e^{\lambda i} e^{-\lambda i} = e^{\lambda i - \lambda i} = e^0 = 1$, $e^{\Lambda t}$ is unitary.

d. From the above,

$$\begin{aligned} KK^H &= Ue^{\Lambda t}U^H U(e^{\Lambda}t)^H U \\ &= Ue^{\Lambda t}(e^{\Lambda t})^H U^H \\ &= UU^H \\ &= I \end{aligned}$$

21.

a.

$$M = \begin{bmatrix} d & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^3 \end{bmatrix}$$

b.

$$M^{-1}AM = \begin{bmatrix} 1 & d & d^2 \\ 1/d & 1 & d \\ 1/d^2 & 1/d & 1 \end{bmatrix}$$

22.

The eigenvalues must be all -1, since the standard vectors are all flipped. But such a rotation is possible only if $\det A = 1$ (orientation does not change), and this is impossible if n is odd, since then $\det A = -1$. So A must be a 180-degree rotation matrix, e.g., $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

23.

$$\begin{aligned} &x^t A^T = \lambda_x x^T \\ &x^T A^T A^T y = \lambda_y x^T A^T y \\ &\lambda_x x^T A^T y = \lambda_y x^T A^T y \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, this is only true if the eigenvectors are orthogonal.

24.

```
A <- sin(1/sqrt(2) * outer(1:3, 1:3) * (pi/4))

S <- eigen(A)$vectors
mat2latex(round(S %*% t(S)))
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

25.

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

a. $\lambda^3 x = N^3 x$, and $N^3 = 0$

b. Let element $ji = ij$ be nonzero. Then $N^2_{ji} = N^2_{ij}$ would be nonzero as well, so the matrix could never reach 0. This also holds if any part of the trace is nonzero.

26.

a.

$$A = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

b. $\lambda = 1$, since the dimension of the subspace is 1.

c. Since $P^n = P$, we just have to solve Pu_0.

$$Pu_k = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

27.

From the trace and determinant rules:

$$A = \begin{bmatrix} 7 & 6 \\ -25/3 & -7 \end{bmatrix}$$

28.

a. Any real d and $c = 5$.

b. $c = 5$ and $d \neq 37/3$

29.

In each case the eigenvectors are the columns of S and the eigenvalues are $(2, 1)$ because the matrices are similar.

30.

The dominant eigenvector is $ $,so

$$\frac{-a}{3}\begin{bmatrix}1\\2\end{bmatrix}$$