Section 3.4 Problems

Ryan Heslin

November 14, 2022

Coordinate transformations. I write a function to compute the pseudoinverse to save time.

6.

```
A_plus <- function(A) {
    SVD <- svd(A, nu = nrow(A), nv = ncol(A))
    S_plus <- diag(x = SVD$d, nrow = ncol(A), ncol = nrow(A))

    SVD$v %*% S_plus %*% t(SVD$u)
}

A_plus(matrix(c(1, 1, 0, 2, 0, 1), nrow = 3)) %*% 2:4
```

```
     [,1]
[1,]    5
[2,]    8
```

10.

```
mat2latex(A_plus(matrix(c(8, 4, -1, 5, 2, -1), nrow = 3)) %*%
    c(1, -2, -2))
```

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

17.

$$[x]_B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

18.

```r
A_plus(matrix(c(1, 1, 0, 0, 0, 1, 1, 0, 0, -1, 0, 1), nrow = 4)) %*%
    c(1, 1, 1, -1)
```

```
     [,1]
[1,]    2
[2,]    2
[3,]   -2
```

19.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$B = S^{-1}AS = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

23.

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$B = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$$

24.

```r
A <- square(13, 6, -20, -9)
S <- square(2, 1, 5, 3)

(solve(S) %*% A %*% S) |>
    mat2latex()
```

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

30.

```r
A <- matrix(c(-1, 0, 3, 1, -2, -9, 0, 2, 6), nrow = 3)
S <- matrix(c(rep(1, 3), 0, 1, 2, 1, 2, 4), nrow = 3)
mat2latex(solve(S) %*% A %*% S)
```

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

31.

$$
\begin{aligned}
&T(X)v_2 \times x \\
&= c_1(v_2 \times v_1) + c_2(v_2 \times v_2) + c_3(v_2 \times v_3) \\
&= -c_1 v_3 + c_3 v_1
\end{aligned}
$$

$$
B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}
$$

33.

$$
T(x) = (v2 \cdot x)v_2
$$

$$
B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

37.

$$
B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}
$$

38.

$$
\begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}
$$

39.

$$
\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & -2 \\ 3 & -1 & 1 \end{bmatrix}
$$

40.

$$
\begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 1 & -1/2 \\ 1 & 0 & 1 \end{bmatrix}
$$

41.

We have to find a basis for the plane, then a mutually orthogonal vector to create a basis for $\mathbb{R}$. That is

$$
\begin{bmatrix} 1/3 & 2/3 & -3/2 \\ 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}
$$

for which B is just the identity with e_3 zeroed.

43.

$$x = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

46.

I had to look up the solution: it's an ugly system of equations for each element of the two basis vectors.

```
A <- matrix(c(2, 0, 0, -1, 0, 0, 0, 2, 0, 0, -1, 0, 0, 0, 2,
    0, 0, -1, 1, 2, 1, 0, 0, 0, 0, 0, 0, 1, 2, 1), nrow = 5,
    byrow = TRUE)
answer <- A_plus(A) %*% c(1, -1, 1, 0, 0)
answer <- cbind(answer[1:3], answer[4:6])
answer <- answer/5
mat2latex(answer)
```

$$\begin{bmatrix} 0.4 & -0.2 \\ -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix}$$

47.

$$T = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

50.

a. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

b. It is at the center of the northeast hexagon from the one with the labeled vectors.

c. Center.

51.

Let S be the matrix of the basis of the coordinate system. As a basis, rank is m, so a left inverse is guaranteed to exist:

$$x = S|x|_B$$
$$[x]_b = S^+x$$

Because S^+ is a linear transformation, $S^+(x+y) = S^+(x) + S^+(y)$.

52.

See above. Being matrices, both S^+ and S^{-1} are linear transformations.

53.

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$B \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 40 \\ 58 \end{bmatrix}$$

54.

Yes. Let B be the matrix of basis B and F the basis of F.

Then $[v_i]_F$ is column I of BF Because B and F have full rank, so does BF, so BF is a valid basis for R^n.

55.

The matrix that transforms standard coordinates into R-coordinates is the inverse of R's basis matrix, and the one that transforms B coordinates into standard coordinates is that of B's basis. Therefore:

$$P = \frac{1}{2} \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

56.

$$S \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$
$$S = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

57.

In R^3, a reflection matrix about a plane preserves the portion of a vector parallel to a plane and subtracts the perpendicular portion. So in R^3, $V\|$ has dimension 2 and $v^{\perp}$ has dimension 1. So the matrix must have eigenvalues $1, 1, -1$ to perform this operation, which matches

```
mat2latex(diag(x = c(1, 1, -2)))
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

58.

a.

$$
\begin{aligned}
&C_1 v + c_2 A v = 0 \\
&c_1 A^2 v + c_2 a^3 v = 0 \\
&c_1 A^2 v = 0 \\
&c_1 = 0
\end{aligned}
$$

$$
\begin{aligned}
&c_1 A^2 v + c_2 A v + c_3 v = 0 \\
&c_1 A^4 v + c_2 A^3 v + c_3 A^2 v = 0 \\
&c_3 A^2 v = 0 \\
&c_3 = 0
\end{aligned}
$$

b.

$$
\begin{aligned}
T(x) &= Ax \\
&= A^3 v x_1 + A^2 v x_2 + A v x_3
\end{aligned}
$$

implying the matrix

$$
Ax = \begin{bmatrix} 0 & | & | \\ 0 & A^2 v & Av \\ 0 & | & | \end{bmatrix}
$$

59.

Yes, diagonal matrices are similar to triangulars with the same diagonal because they have the same eigenvalue.

60.

No, the second matrix has complex eigenvalues, the first doesn't.

61.

```
A <- square(-5, 4, -9, 7)
B <- square(1, 0, 1, 1)
S <- B %*% solve(A)
```

62.

We just need to find the eigenvectors, since

$$
\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = S \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} A^{-1}
$$

Those eigenvectors are

$$
\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}
$$

63.

Yes; all rotation-scaling matrices in R^2 have eigenvalues

$$\pm\frac{i}{p^2+q^2}$$

64.

Yes, they both have the same characteristic polynomial.

65.

Proof of reflexivity and symmetry of similarity:

$$\begin{aligned}
&S = I \\
&A = S^{-1}AS = A \\
&A = S^{-1}BS \\
&SA = BS \\
&B = SAS^{-1}
\end{aligned}$$

67.

$$\begin{aligned}
B &= \frac{1}{c}\begin{bmatrix} c & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix} \\
&= \begin{bmatrix} 0 & bc-ad \\ 1 & a-d \end{bmatrix}
\end{aligned}$$

66.

Diagonalizing a reflection. Substituting gives the system

$$\begin{bmatrix} b & 1-a \\ 1-a & -b \end{bmatrix} = \begin{bmatrix} b & a-1 \\ 1-a & b \end{bmatrix} B$$

which obviously implies

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

69.

In the unlikely event anyone's reading this, I want to praise the author for smuggling in a diagonalization problem a few chapters before eigenvectors appear. Sneaky.

$$\begin{aligned}
S &= \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \\
S^{-1}AS &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}$$

70.

No. The first column of B must be $\begin{bmatrix} a \\ 0 \end{bmatrix}$, where a is any real number. Then column 1 of the output is a times column 1 of the input, which is incompatible with the permutation transformation.

71.

a. If $x \in \ker B$ and $B = S^{-1}AS$, then $S^{-1}ASx = 0 \implies Sx \in \ker A$. ($\ker S^{-1} = 0$).

b. Showing the nullities are identical. A just has its transformed by S first:

$$
\begin{aligned}
B &= S^{-1}AS \\
&= S^{-1}SBS^{-1}SSx \\
&= BSx
\end{aligned}
$$

72.

Since A and B must have the same eigenvalues, it follows that $\text{rank}A = \text{rank}B$, since rank is the number of nonzero eigenvalues.

73.

This isn't right; that isn't the correct rotation

```r
S <- matrix(c(3/5, 4/5, 0, -4/3, 1, 0, 0, 0, 1), nrow = 3)
B <- matrix(c(0, -1, 0, -1, 0, 0, 0, 0, 1), nrow = 3)
A <- S %*% B %*% solve(S)
mat2latex(A)
```

```
$$
\begin{bmatrix}
1.088 & 0.850666666666666 & 0\\
-0.216 & -1.088 & 0\\
0 & 0 & 1
\end{bmatrix}
$$
```

74.

a. 0.

T represents a rotation $2\pi/3$ radians about P_2; hence, it has no effect on that vector.

```r
solve(square(1, -1, -1, -1, 1, -1, -1, -1, 1)) %*% rep(1, 3)
```

```
 [,1]
```

[1,] -1 [2,] -1 [3,] -1

```
A <- square(-1, -1, 1, 1, -1, -1, rep(1, 3)) %*% solve(square(rep(1,
    3), -1, -1, 1, 1, -1, -1))

mat2latex(A %*% c(-1, 1, -1))
```

$$\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

```
S <- square(1, -1, -1, -1, 1, -1, -1, -1, 1)
B <- solve(S) %*% A %*% S
mat2latex(B)
```

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

75.

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The rotation does the same thing to any orthogonal basis, being an orthogonal transformation, so the matrix is the same.

76.

Here $S^{-1} = A$, so just the original rotation matrix.

77.

Column j of B is column $n - j + 1$ of A.

78.

```
S <- diag(x = c(2, 5, 10))
A <- square(0.3, 0.1, 0.2, 0.2, 0.3, 0.2, 0.1, 0.3, 0.1)
B <- solve(S) %*% A %*% S
```

Here, $B = S^{-1}AS$ first finds the total cost of the produced amounts of good j by industry i, then S^{-1} divides again by price to express the interindustry demands in terms of goods alone.

79.

Just

$$\begin{bmatrix} 11 & 30 \\ 4 & 11 \end{bmatrix}$$

Eigenvectors again.

$$\begin{bmatrix} 3 & 5/2 \\ 1 & 1 \end{bmatrix}$$

80.

No choice but to solve four equations. Solutions are multiples of

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

81.

Arrant cheating, but it works.

```
A <- square(0, 0, 0, 1, 0, 1, 0, 1, 1)
eigen(A)$vectors %>%
    apply(MARGIN = 2, function(x) x/x[1]) %>%
    mat2latex()
```

$$\begin{bmatrix} 1 & 1 & 1 \\ 1.61803398874989 & -0.618033988749895 & 0 \\ 2.6180339887499 & 0.381966011250105 & 0 \end{bmatrix}$$

82

I never would have figured this out if I tried this before learning about eigenvectors.

```
A <- matrix(c(0, 0, 0, 1, 0, -2, 0, 1, 3), nrow = 3)
S <- eigen(A)$vectors
S <- S[, c(1, 2:3)]
mat2latex(S)
```

```
$$
\begin{bmatrix}
-0.218217890235992 & -0.577350269189626 & 1\\
-0.436435780471985 & -0.577350269189626 & 0\\
-0.87287156094397 & -0.577350269189626 & 0
\end{bmatrix}
$$
```

Normalize the first elements to 1. The first standard vector is a trivial answer to the problem.

```r
S[, 1] <- S[, 1] * 1/S[1, 1]
S[, 2] <- S[, 2] * 1/S[1, 2]
mat2latex(S)
```

```
$$
\begin{bmatrix}
1 & 1 & 1\\
2 & 1 & 0\\
4 & 1 & 0
\end{bmatrix}
$$
```