Section 5.3 Problems

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1.

2.

3.

4.

No.

If A and B are orthogonal square matrices, which others also are?

5.

$3A$: No

6.

$-B$: Yes

7.

AB: Yes; orthogonal transformations preserve orthogonality.

8.

$A + B$: No.

9.

B^{-1}: Yes.

10.

$A^1 0$: Yes

$B^{-1}AB$: yes ,all are orthogonal, and orthogonal transformations preserve orthogonality.

11.

A^T: Yes, since $A^TA = I$, and orthogonal transformations preserve orthogonality.

We consider the symmetries of the same matrices, granted they are symmetric and B is invertible.

12.

$3A$: Yes

13.

$-b$: Yes

14.

Yes.

15.

AB: No. $AB_{ij} = [A_i \cdot B_j]$, not necessarily equal to $AB_{ji} = [A_j \cdot B_i]$.

16.

Yes.If $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$ where $i \neq j$, then $A_{ij} + B_{ij} = A_{ji} + B_{ji} \implies (A + B)_{ji} = (A + B)_{ji}$

Yes.

$$\begin{aligned} A + B &= A^T + B^T \\ &= (A + B)^T \end{aligned}$$

17.

B^{-1}: yes, since $[B_i^{-1}]\,[B_j] = [B_j^{-1}]\,[B_i] = 0$

18.

A^10: apparently so. The triangular element is consistent in the dot products

```
A <- square(2, 4, 4, -1)
A %^% 2
```

```
     [,1] [,2]
[1,]   20    4
[2,]    4   17
```

19.

$2I + 3A - 4A^2$: yes, we showed all these are symmetric.

20.

AB^2A: no, as AB is not necessarily symmetric.

For arbitrary square matrices,which are symmetric?

21.

A^TA: yes, since element ij and ji are equal.

22.

BB^T: yes, for the same reason

23.

$A - A^T$: no, if A is non-symmetric.

24.

A^TBA: No. Can be rearranged to BAA^T, not necessarily symmetric.

25

.... I think?

$$
\begin{aligned}
&A^TB^TBA \\
&(BA)^TBA \\
&((BA)^T)BA)^T = (BA)^TBA
\end{aligned}
$$

26.

$$
\begin{aligned}
&(B(A + A^T)B^T))^T = B(B(A + A^T))^T \\
&= B(A^T + A)B^T \\
&= B(A + A^T)B^T
\end{aligned}
$$

27.

$$
\begin{aligned}
&(AV) \cdot w = v \cdot (A^Tw) \\
&(Av)^Tw = v^t(A^Tw) \\
&(v^TA^T)w = v^T(A^Tw) \\
&v^TA^Tw = v^TA^Tw
\end{aligned}
$$

28.

$$
\begin{aligned}
&(Ax) \cdot (Ay) = x \cdot y \\
&(Ax)^T Ay = x^T y \\
&x^T A^T Ay = x^T y \\
&A^T Ay = y \\
&A^T A = I \\
&A^T = A^{-1}
\end{aligned}
$$

Of course, the transpose is the inverse only for orthogonal transformations

29.

$$
\begin{aligned}
\frac{v \cdot w}{||v||||w||} &= \frac{(Av) \cdot (Aw)}{||Av||||Aw||} \\
\frac{v \cdot w}{||v||||w||} &= \frac{v \cdot w}{||Av||||Aw||}
\end{aligned}
$$

Given that orthogonal transformations preserve lengths as well as dot products, we're done.

30.

If A is a transformation $R^m \to R^n$ that preserves length, then

$$
\begin{aligned}
&\sqrt{(Av) \cdot (Av)} = \sqrt{v \cdot v} \\
&(Av) \cdot (Av) = v \cdot v \\
&v^T A^T A = v^T v \\
&A^T A = I_m
\end{aligned}
$$

which means A^T is the left inverse of A. THis implies A has full column rank, which means it must have partial row rank if $n \neq m$. That means $A^T A$ is invertible but not AA^T (a $n \times n$ of a matrix with partial row rank, since the transpose preserves rank, and products cannot increase rank).

31.

The rows of an orthogonal A must also be orthonormal because A^T is also orthogonal.

32.

a. As shown above, $A^T A = I_m$ implies full column rank, which means AA^T cannot equal I_n.

b. BuT if $A^T A = I_n$ for an $n \times n$, then it is orthogonal, so $A^T = A^{-1} \implies AA^T = I_n$

33.

By multiplying out, we see that orthogonal matrices (for which the inverse is the transpose) satisfy the equations $a^2 + b^2 = c^2 + d^2 = 1$ and $ac = -bd$. That suggests the basis $\begin{bmatrix} a & 1 - a^2 \\ 1 - a^2 & -a \end{bmatrix}$. Opposite-signed diagonals also work.

34.

$$\begin{bmatrix} a & 1-a^2 & 0 \\ 0 & 0 & 1 \\ e & 1-e^2 & 0 \end{bmatrix}$$

35.

36.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \sqrt{2} \begin{bmatrix} -3/8 \\ -3/8 \\ 3/2 \end{bmatrix}$$

38.

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

A^2 must be symmetric, since $A^2_{ij} = A^2_{ji}$, both negative.

39.

If we have a line spanned by a unit vector, entry ij of the projection matrix is $u_i u_j/||u||$, with the squares on the diagonal.

40.

41.

The projection onto the unit line in R^n is given by a matrix consisting entirely of $1/n$, since each element of the unit vector is $1/\sqrt{n}$.

$$\begin{aligned} &P = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T = \quad A(A^TA)^{-1} \\ &= A(A^TA)^{-1} \end{aligned}$$

42.

We know by now projection matrices are idempotent: they rebalance the elements of a vector so it becomes part of the subspace, but a vector already in the subspace has exactly the correct ratios of elements already.

43.

Given a unit vector in R^3, the matrix $A = 2uu^T - I_3$ describes the reflection $(2P - I)$,w while the opposite sign $B = I_3 - 2uu^T$ is

44.

Given an $n \times m$ matrix, the dimension of the image and of the kernel of the transpose sum to n, because all vectors in R^n belong to one of those two spaces.

45.

$\dim(\ker(A)) = \dim(\ker(A^T))$ for matrices for which rank is exactly $n/2$ - only then is R^n partitioned equally.

46.

Trivial.

$$
\begin{aligned}
M &= QR \\
R &= Q^{-1}M \\
R &= Q^T M
\end{aligned}
$$

47.

For $A = QR$, then:

$$
\begin{aligned}
A^T A &= (QR)^T QR \\
&= R^T Q^T QR \\
R^T R
\end{aligned}
$$

so $A^T A = R^T R$. This makes sense - each column of R decomposes vectors of A into projections along unit vectors and the residual $v^\perp$, so $R^T R$ collects the intersection of column vectors in the same way as $A^T A$.

48.

We can also write:

$$
\begin{aligned}
A &= QR \\
A^T &= R^T Q^T
\end{aligned}
$$

since Q^T is also orthogonal.

49.

50.

a. Element $1,1$ of the matrix can only be 1. The nonzero entries a, b of the second column must satisfy:

$$
\begin{aligned}
a + 0b &= 0 \\
a^2 + b^2 &= 1
\end{aligned}
$$

which only $a = 0, b = \pm 1$ satisfy. Proceeding column by column and restricting b to be positive, that leaves only the identity.

b.

$$
\begin{aligned}
&A = Q_1 R_1 = Q_2 R_2 \\
&Q_2^{-1} Q_1 R_1 = R_2 \\
&Q_2^{-1} Q_1 = R2 R_1^{-1}
\end{aligned}
$$

$Q_2^{-1}Q_1 = I$ because the product of the triangular matrices must be orthogonal, but the only orthogonal triangular matrix possible here is the identity, so $Q_2^{-1} = Q_1^{-1}$, so $Q_2 = Q_1$,

and the same for the Rs.

51.

a.

$$
\begin{aligned}
&Q_1 = Q_2 S \\
&Q^T Q1 = (Q_2 S)^T Q_2 S \\
&I = S^T Q_2^T Q_2 S \\
&I = S^T S
\end{aligned}
$$

so S must be orthogonal.

b.

$$
\begin{aligned}
&M = Q_1 R_1 = Q_2 R_2 \\
&Q_1 = Q_2 R_2 R_1^{-1}
\end{aligned}
$$

$R_2 R_1^{-1}$ must be orthogonal for the reasons given above, and again the only possible orthogonal triangular matrix is the identity. so $Q_1 = Q_2$ and $R_1 = R_2$. ## 52.

53.

54.

55.

$$
\frac{n^2 - n}{2}
$$

56.

57.

Yes. $L^{-1}(A^T) = A$ is the transpose from $R^{m \times n}$ to $R^{n \times m}$

58.

The kernel is 0. For the image, the diagonal remains the same and the off-diagonal element ij is $\frac{A_{ij}+A_{ji}}{2}$, so the resulting matrix is symmetric

59.

The kernel is all symmetric matrices. The image is a skew-symmetric matrix with a zero diagonal and element ij is $\frac{A_{ij}-A_{ji}}{2}$.

60.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

61.

Applying the transformation zeroes out all but the third element of the basis, which gets doubled:

$$T\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

So the coordinates vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ and the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

62.

63.

64.

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71.

72.

The projection onto

$$\begin{bmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{bmatrix}$$

is a Hankel matrix (positive sloping diagonals) of the same element) because the first column multiplies each element of the vector by 1, the second by a, and so on. This ensures that the diagonal elements are the same as the echelon diagonals.

73.