Section 5.2 Problems

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7.

Got it in one!

```
v_1 <- c(2, 2, 1)
v_2 <- c(-2, 1, 2)
v_3 <- c(18, 0, 0)

u_1 <- normalize(v_1)
v_2p <- v_2 - (t(v_2) %*% u_1) * u_1
u_2 <- normalize(v_2p)
v_3p <- v_3 - (t(v_3) %*% u_2) * u_2 - (t(v_3) %*%
    u_1) * u_1
u_3 <- normalize(v_3p)

R <- diag(x = apply(cbind(v_1, v_2p, v_3p), 2, l2))
R[1, 2] <- t(v_2) %*% u_1
R[1, 3] <- t(v_3) %*% u_1
R[2, 3] <- t(v_3) %*% u_2
print_eqn(c("R =", mat2latex(R, sink = TRUE)))
```

$$R = \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}$$

```
Q <- cbind(u_1, u_2, u_3)
print_eqn(c("Q =", mat2latex(round(Q, 3), sink = TRUE)))
```

$$Q = \begin{bmatrix} 0.667 & -0.667 & 0.333 \\ 0.667 & 0.333 & -0.667 \\ 0.333 & 0.667 & 0.667 \end{bmatrix}$$

```
mat2latex(Q %*% R)
```

$$\begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

11.

$$A = \begin{bmatrix} 4/5 & -1/5 \\ 0 & 2/15 \\ 0 & 14/15 \\ 3/5 & 4/15 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 0 & 15 \end{bmatrix}$$

31.

The QR on the generic matrix

$$\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$$

is just

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$$

A convenient property of upper-triangular matrices, since only one dimension of vector j is left after subtracting u_{j-1}.

32.

A basis of $x_1 + x_2 + x_3 = 0$ is $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

36.

M is already in QR form.

37.

Just A with the columns normalized and R with lengths in the diagonal, since the vectors are already orthogonal.

39.

We can construct an orthonormal basis given some vectors by finding a normal vector from the kernel of the transpose.

```r
u_1 <- normalize(1:3)
u_2 <- normalize(c(1, 1, -1))
u_3 <- normalize(c(5, -4, 1))

mat2latex(pracma::rref(t(cbind(u_1, u_2))))
```

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix}$$

```
mat2latex(normalize(c(5, -4, 1)))
```

```
Argument is not a matrix. Attempting to coerce
```

$$\begin{bmatrix} 0.771516749810459 \\ -0.617213399848368 \\ 0.154303349962092 \end{bmatrix}$$

40.

If a matrix has orthogonal but not orthonormal columns, then Q is each vector scaled to unit length and R is a diagonal matrix with diagonal $||v_j||$, since none of the vectors has any component parallel to another.

41.

As I showed above, the QR of an upper triangular matrix consists of the standard basis and the original matrix

44.

If A lacks full column rank, then $v^\perp$ for redundant columns is 0, because they are linear combinations of the others. So column j of Q is 0 and element jj of R is 0 as well, since the zero vector has no length. That means the matrix is not invertible, meaning there is no unique factorization.

45.

In this case, $v^\perp$ is always nonzero because the vectors are all linearly independent, so none point in the same direction. By induction, $v_2^\perp$ is nonzero because v_2 is not a multiple of v_1, and $v_3^\perp$ is nonzero because no constants satisfy $c_1 v_1 + c_2 v_2 = v_3$. If R is instead lower triangular L, then the diagonal remains the lengths of each $v^\perp$, and th lower triangle reflects the upper triangle of R. In other words, $L = R^T$.

$$\begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$