

## Section 5.6 Problems

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1.

$$\begin{aligned}B &= M^{-1}AM \\C &= N^{-1}BN \\C &= M^{-1}(N^{-1}AN)M\end{aligned}$$

$M$  and  $N$  are both invertible, so their product is as well. Only  $I$  itself is similar to  $I$ :

$$I = I^{-1}II$$

2.

Any matrices with the same pair of eigenvalues, such as  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  or  $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

3.

$A$  cannot be similar to  $A + I$  because  $A$  has eigenvalues  $\Lambda$ , while  $A + I$  has eigenvalues  $\Lambda + I$ .

4.

```
mat2latex(diag(x = rep(c(-1, 1), times = 2)))
```

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.

If  $S = B$ :

$$\begin{aligned}BA &= S^{-1}ABS \\BA &= BABB^{-1} \\BA &= BA\end{aligned}$$

6.

a.

$$CD = -DC$$

$$C = -DCD^{-1}$$

$$C = D(-C)D^{-1}$$

b. If  $C$  is similar to  $-C$ , then  $\Lambda = -\Lambda$ . So  $C$  must be of even dimension and have  $-\lambda_i$  for each  $\lambda_i$ .

c.  $D$  has the same property:  $D = C^{-1}(-D)C$ . So since  $Cx = \lambda x = -\lambda x$ ,  $C(Dx) = -\lambda Dx$

8.

```
A <- square(1, 1, 1, 4)
```

```
B <- square(2, 5, 1, 4)
```

```
M <- square(1, -1, 0, 1)
```

```
mat2latex(M)
```

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

9.

```
x <- c(3, 9)
```

```
c <- solve(A, x)
```

```
d <- solve(B, x)
```

```
mat2latex(M %*% solve(A, x))
```

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

10.

```
A %*% c
```

```
      [,1]
[1,]     3
[2,]     9
```

```
B %*% d
```

```
      [,1]
[1,]     3
[2,]     9
```

11.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

13.

a.

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

b. The zero matrix; the third derivative of degree-2 polynomial is 0.

c. Values all 0, vectors  $\pm e_1$ .

14.

Every number is an eigenvalue because  $T(\lambda e^{\lambda x}) = \lambda e^{\lambda x}$  for arbitrary  $\lambda$ .

15.

The “eigenmatrices” are the members of the basis that are diagonal -  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , or any symmetric matrix. Their eigenvalues are 1, since the transpose leaves the diagonal unchanged.

16.

Easy - just find the eigenbasis. A remarkable fact about zero eigenvalues - we can just pick any linear combination of the kernel and make up our own eigenvectors.

```
Q <- square(c(-1, 1, 0)/sqrt(2), c(-1, 0, 1)/sqrt(2), rep(1, 3)/sqrt(3))
```

```
Lambda <- diag(x = c(0, 0, 3))
```

```
mat2latex(Q %*% Lambda %*% t(Q))
```

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

```
Q2 <- square(c(normalize(c(-2, 1, 1))), normalize(c(0, -1, 1)), normalize(rep(1, 3)))
```

```
mat2latex(Q2 %*% Lambda %*% t(Q2))
```

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

## 18.

All permutation matrices are normal because they have orthogonal eigenvectors, since by definition they contain the standard vectors, which are orthogonal, in any order. Therefore their transpose is their inverse, satisfying  $A\bar{A} = \bar{A}A$ .

A possibility is a multiple of a permutation matrix:

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

19.

Proof  $T$  is normal:

$$TT^H = \begin{bmatrix} t_{11}^2 + |t_{12}|^2 + |t_{13}|^2 & t_{12}t_{22} + t_{13}\bar{t}_{23} & t_{13}t_{33} \\ t_{22}\bar{t}_{12} + t_{23}\bar{t}_{13} & t_{22}^2 + |t_{23}|^2 & t_{23}t_{33} \\ t_{33}\bar{t}_{13} & t_{33}\bar{t}_{23} & |t_{33}|^2 \end{bmatrix}$$

$$T^HT = \begin{bmatrix} t_{11}^2 & t_{11}t_{12} & t_{11}t_{33} \\ t_{12}t_{11} & |t_{12}|^2 + |t_{22}|^2 & \bar{t}_{12}t_{13} + t_{22}t_{33} \\ \bar{t}_{13}t_{11} & \bar{t}_{13}t_{12} + \bar{t}_{23}t_{22} & |t_{13}|^2 + |t_{23}|^2 + t_{33}^2 \end{bmatrix}$$

For these two to be equal, the cross-terms on the diagonal have to be 0. So  $T$  must be diagonal, so

$$T = T^H = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and all diagonal matrices are Hermitian and therefore normal.

20.

$$\begin{aligned} \|Nx\|^2 &= (NX)^H Nx \\ &= x^H N^H Nx \\ \|N^H x\|^2 &= (N^H x)^H N^H x \\ &= x^H N N^H x \\ N^H N &= N N^H \end{aligned}$$

Column  $i$  must have the same length as row  $i$ ; conjugation does not change length (length of a complex vector is the dot product *with* the conjugate transpose), but  $\|N_1 x\| = \|N_1^H x\|$ , and the rows of  $N^H$  are the columns of  $N$ .

21.

$$\begin{aligned}
N &= U\Lambda U^H \\
NN^H &= U\Lambda U^H U\Lambda^H U^H \\
&= U\Lambda^2 U^H \\
N^H N &= U\Lambda^H U^H U\Lambda U^H \\
&= U\Lambda^2 U^H
\end{aligned}$$

##23.

If we put the eigenvalues in a matrix like this:

$$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

we can see the products all come to 0.

24.

Demonstrating Cayley-Hamilton

$$\begin{aligned}
(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) &= 0 \\
(T^2 - \lambda_1 T - \lambda_2 T + \lambda_1 \lambda_2 I)(T - \lambda_3 I) &= 0 \\
T^3 - \lambda_1 T^2 - \lambda_2 T^2 + \lambda_1 \lambda_2 T - \lambda_3 T^2 + \lambda_1 \lambda_3 T + \lambda_2 \lambda_3 T + \lambda_1 \lambda_2 \lambda_3 I &= 0 \\
T^3 - (\lambda_1 + \lambda_2 + \lambda_3)T^2 + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)T - \lambda_1 \lambda_2 \lambda_3 I &= 0
\end{aligned}$$

In this matrix, the columns gives the eigenvalues of  $(A - \lambda_k I)$ . It is clear the final product has all zero eigenvalues.

$$\begin{bmatrix} 0 & \lambda_1 - \lambda_2 & \dots & \lambda_1 - \lambda_n \\ \lambda_2 - \lambda_1 & 0 & \dots & \lambda_2 - \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n - \lambda_1 & \lambda_n - \lambda_2 & \dots & 0 \end{bmatrix}$$

For the first eigenvalue of the product:

$$\begin{aligned}
\lambda_1^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda_1^2 + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)\lambda_1 - \lambda_1 \lambda_2 \lambda_3 &= 0 \\
\lambda_1^3 - \lambda_1^3 - \lambda_1^2 \lambda_2 - \lambda_1^2 \lambda_3 + \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 + \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_2 \lambda_3 &= 0 \\
0 &= 0
\end{aligned}$$

and so on.

25.

Proving Cayley-Hamilton:

$$\begin{aligned}
 A^2(a+d)A + (ad-bc)I_2 &= 0 \\
 \begin{bmatrix} a^2+bc & ab+ad \\ ac+dc & bc-ad \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\
 &= \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

26.

```

Jordan <- function(A) {
  lambda <- unique(eigen(A)$values)
  lapply(lambda, function(x) {
    J <- diag(x = x, nrow = nrow(A))
    J[col(J) - row(J) == 1] <- 1
    J
  })
}
A <- matrix(rep(0, 16), nrow = 4)
A[upper.tri(A)] <- 1
mat2latex(Jordan(A))

```

Argument is not a matrix. Attempting to coerce

$$[c(0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0)]$$

## 29.

$$\begin{aligned}
 A &= \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \\
 A^{10} &= M^{-1} \begin{bmatrix} 1024 & 10 \\ 3 & 1024 \end{bmatrix} M \\
 e^A &= M^{-1} \begin{bmatrix} e^2 & 1e^2 \\ 0 & e^2 \end{bmatrix}
 \end{aligned}$$

30.

$$\begin{aligned}
 \text{a. } M &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 \text{b. } M &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

31.

Only  $A_1$  has eigenvalues 1 and 1, so it is similar to none. The same for  $A_2$  and  $(-1, -1)$ . The latter two pairs are similar to each other.

32.

Family	Description	Number	$\lambda$
1	Zero matrix or one 1	5	(0, 0)
2	Row or column of ones	4	(1, 0)
3	All 1s	1	(2, 0)
4	Upper triangular 1	1	(1, 1)
5	Lower triangular 1	1	(1, 1)
6	Identity transpose	1	(-1, -1)
7	Identity	1	(1, 1)

35.

$$\begin{aligned}
 J &= \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} \\
 J^2 &= \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix} \\
 J^3 &= \begin{bmatrix} c^3 & 3c^2 \\ 0 & c^3 \end{bmatrix} \\
 J^k &= \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix} \\
 J^0 &= I_2 \\
 J^{-1} &= \begin{bmatrix} c^{-1} & -1c^{-2} \\ 0 & c^{-1} \end{bmatrix}
 \end{aligned}$$

33.

Similar matrices have kernels with the same basis.

34.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

36.

Squaring an  $A$  like this just moves the off-diagonal 1s one diagonal upward. So there are two eigenvectors:

```

A <- square(rep(0, 25))
A[col(A) - row(A) == 1] <- 1
A <- A %^% 2

J <- square(rep(0, 25))
J[2, 1] <- 1

```

```
J[3, 2] <- 1
mat2latex(J)
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**38.**

$J$  has non-distinct eigenvectors  $e_1, e_3$ , while  $K$  has non-distinct eigenvectors  $e_1, e_4$ . So they cannot be similar.  
# 41.

- a. True. A singular matrix will have a 0 eigenvalue somewhere, an invertible one will not.
- b. False.  $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$  is similar to  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
- c. True.  $\Lambda_A = -\Lambda_{-A}$ .
- d. True; they obviously have different eigenvalues.

**42.**

The respective Jordan forms are

$$\begin{aligned} AB &= M^{-1} J M N^{-1} K N \\ BA &= N^{-1} K N M^{-1} J M \end{aligned}$$

**44.**

- a.  $A^2$  and  $B^2$  both have eigenvalues  $\Lambda^2$ .
- b. The square is not a one-to-one transformation, since squares have two roots
- c.  $B$  is a Jordan block with the same eigenvalues ( $\Lambda$ )
- d.  $B$  is a non-Jordan form that cannot be diagonalized.
- e. Swapping two rows and then two columns is a similarity relation defined by a permutation matrix with the initial block  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  (the identity with  $e_1$  and  $e_2$  swapped). So the output of the transformation is similar to the original matrix and therefore has the same eigenvalues.