Section 2.5 Problems

Ryan Heslin

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An incidence matrix has a row for every edge. For each row, there is -1 in the column indexed by the "from" node, and 1 in the column corresponding to the "to" node.

the dimensions are edges × nodes.

The vector 1 is always in the kernel because an arbitrary constant can always be added to the solution. The graph only expresses relative relationships, not absolute ones. So rank is at most $m - 1$.

A solution to $Ax = 0$ represents a distribution for which it is impossible to find the actual potentials. A solution to $A^T y = 0$ represents a distribution that circulates endlessly (a cycle). A basis for the row space thus contains those edges for which cycles are impossible.

For an incidence, the dimensions are:

Subspace	Dimension
Kernel	$1; 1$
Row	$m - 1$
Image	$m - 1$
Transpose kernel	$n - m + 1$

Euler's formula states that in a directed graph, the numbers of edges, nodes, and loops sum to 1, apparent from the dimensions above.

A network is a graph where each edge is assigned a weight by the diagonal $n \times n$ matrix C.

Kirchoff's law states that the inflows to any node (the potentials) sum to 0. From Ohm's law, there are implicit equations for equilibrium:

$$y = C(b - Ax) \text{or} C^{-1}y + Ax = b$$
$$A^T y = f$$

The first says any edge distribution is obtained by scaling the kernel component of a potentials vector by C. The second says any potentials vector can be recovered. By some rearrangement:

$$A^T CAx = A^T Cb - f$$

This holds only if the last element of the potentials vector $x_m = 0$, ignoring the last column of A. The reduced graph has rank m, so it is a tree - no loops are possible.

The least squares problem $Ax - b$ is a special case of this.

1.

The graph is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

and its kernel is 1. The kernel of A^T is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

2.

From the rows and columns, it is obvious summing the first two gives the third.

3.

Again, immediate from attempting to sum the rows. It means the last difference has to cancel the previous ones.

4.

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

A few elimination steps gives a zero row. The submatrix has inverse

$$\frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

5.

Now it's

$$\frac{1}{5(c_1 + c_2)}\begin{bmatrix} 2c_1 & 1c_2 \\ 1c_1 & 2c_2 \end{bmatrix}$$

6.

```r
A <- matrix(c(
  -1, 1, 0, 0,
  -1, 0, 1, 0,
  0, -1, 1, 0,
  0, -1, 0, 1,
  -1, 0, 0, 1,
  0, 0, -1, 1
), nrow = 6, byrow = TRUE)
```

7.

Probably screwed up, but the kernel is

$$\begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.

Row and column are both $m - 1 = 3$. Kernel is 1, making transpose kernel $6 - (4 - 1) = 3$.

9.

```
C <- diag(x = complex(6, 0, 1:6))
t(A) %*% C %*% A
```

```
     [,1] [,2]  [,3]  [,4]
[1,] 0+8i 0-1i 0- 2i 0- 5i
[2,] 0-1i 0+8i 0- 3i 0- 4i
[3,] 0-2i 0-3i 0+11i 0- 6i
[4,] 0-5i 0-4i 0- 6i 0+15i
```

I believe it's the sum of the constat associated with all nodes that eventually reach that node.

10.

Remove one row and the matrix has full rank, so the remaining rows are a basis for the potentials.

11.

The big ugly system:

$$\begin{aligned} &y_1 = -x_1 + x_2 \\ &1/2y_2 = -x_1 + x_3 \\ &1/2y_3 = x_2 \\ &y_4 = -x_3 \\ &-y_1 - y_2 = f_1 \\ &y_1 + y_3 = f_2 \\ &y_2 - y_4 = f_4 \end{aligned}$$

I tried by hand but decided it wasn't worth it.

```
A <- matrix(c(-1, 1, 0, -1, 0, 1, 0, 1, 0, 0, 0, -1), nrow = 4, byrow = TRUE)
C <- diag(x = c(1, 2, 2, 1))
f <- c(1, 1, 6)
solution <- solve(t(A) %*% C %*% A, -f)
solution
```

```
[1] -4.000000 -1.666667 -4.666667
```

```
A %*% solution
```

```
           [,1]
[1,]  2.3333333
[2,] -0.6666667
[3,] -1.6666667
[4,]  4.6666667
```

Converting back to f:

```
y <- C %*% A %*% solution
t(A) %*% y
```

```
     [,1]
[1,]   -1
[2,]   -1
[3,]   -6
```

12.

Rank is 6. $\ker(A)$ has dimension 1, $\ker(A^T)$ dimension 6. We'd need to lop off $12 - 6 = 6$ edges.

13.

14.

Princeton beats Harvard by 1, MIT beats Yale by 35, MIT beats Princeton by 34

15.

It's accounted for by the potential differences.

16.

$\binom{n}{2}$

17.

18.

Other nodes, connected.

19.

A constant can be added to any initial potential without distrubing the solution. The kernel has dimension $n - 1$.

20.

That is $\binom{6}{2}$.

21.

I won't write the matrix, but $A_i \cdot A_j$ adds 1 for every overlapping pair of nodes, and subsequent multiplications compound.