Section 5.5 Problems

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An inner product in a linear space assigns a scalar result to the interaction of elements f and v, obeying the usual rules of the dot product. It is always a linear transformation and positive definite (i.e, $\langle f, f \rangle > 0$ for all nonzero f).

For example

$$\langle f, v \rangle = \int_a^b f(t)g(t)dt$$

This is a continuous generalization of the dot product, multiplying and summing the infinitely many values of the functions over their interval.

The *norm* generalizes the length:

$$||f|| = \sqrt{\langle f, f \rangle}$$

Two elements are orthogonal if $\langle f, v \rangle = 0$.

Distance is defined similarly:

$$\text{dist} = ||f - v||$$

As usual, the projection of a vector into a subspace is the least distant vector from that subsapce such that

$$proj_{\mathbf{W}} f = \langle g1, f \rangle g_1 + \cdots + \langle g_m, f \rangle g_m$$

where the gs are an orthonormal basis for $\mathbf{W}$. This is the continuous analog the least-squares criterion: the distance is assessed along *every point* of each function, not just several data points.

1.

$$\langle f, f \rangle = \int_a^b f(t)f(t) = \int_a^b (f(t))^2$$

and squares are always positive.

2.

$$\begin{aligned}
&\langle f, g + h \rangle \\
&\langle g + h, f \rangle = \langle g, f \rangle + \langle h, f \rangle \\
&= \langle f, g \rangle + \langle f, h \rangle
\end{aligned}$$

3.

a. Yes. $(\mathbf{S}x)^{\mathbf{T}}\mathbf{S}y = x^{\mathbf{T}}\mathbf{S}^{\mathbf{T}}\mathbf{S}y = (\mathbf{S}x)\cdot(\mathbf{S}y)$, and the dot product is an inner product.

b. Yes.

4.

a. $\mathbf{A}\cdot\mathbf{B}$.

b. $\mathbf{A}\mathbf{B}^{\mathbf{T}}$

5.

Yes. $tr((\mathbf{A}+\mathbf{C})\mathbf{B}^{\mathbf{T}}) = tr(\mathbf{A}\mathbf{B}^{\mathbf{T}}) + tr(\mathbf{A}\mathbf{C}^{\mathbf{T}})$, and we know from below $tr(\mathbf{A}\mathbf{B}^{\mathbf{T}}) = tr(\mathbf{B}^{\mathbf{T}}\mathbf{A})$.

6.

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{P}_{ij}\mathbf{Q}_{ji} = \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{Q}_{ij}\mathbf{P}_{ji}$$

They are identical.

7.

$k > 0$.

8.

Yes.

$$\begin{aligned}\mathbf{T}(v)+\mathbf{T}(u) &= \langle u,w\rangle + \langle v,w\rangle \\ \mathbf{T}(u+v) &= \langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle \\ k\mathbf{T}(v) = k\langle v,w\rangle = \langle kv,w\rangle = \mathbf{T}(kv)\end{aligned}$$

The kernel is all functions orthogonal to w.

9.

10.

11.

I hate integration.

The integral is [

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$$\int_{-} 1^1 f(t)g(t)$$

$$f(1)g(1) - f(-1)g(-1)$$

$$-f(1)g(-1) + f(1)g(-1) = 0$$

12.

13.

14.

Neither, because the inner product of a single function will be zero for nonzero f if those values lie in the kernel of the function.

15.

16.

17.

Those for which $\mathbf{T}(v) \neq 0$ for nonzero v

18.

They are the same.

19.

This is just the definition of positive definite matrices: $a > 0$, $b = c$, and $ad - b^2 > 0$. Otherwise the inner product fails positive definiteness.

20.

21.

Symmetry and positive definiteness are obvious. The scalar proprty follows from $||cv||^2 = \langle cv, cv \rangle = c^2 ||v||^2$. For addition:

$||u+v+w||^2 - ||u+v||^2 - ||w||^2 = ||v+w||^2 - ||v||^2 - ||w||^2 + ||u+w||^2 - ||v||^2 - ||w||^2$
$u \cdot u + v \cdot v + w \cdot w + 2(u \cdot v + v \cdot w + u \cdot w) - u \cdot u - 2u \cdot v - v \cdot v - w\ cdotw = v \cdot v + 2v \cdot w + w \cdot w - v \cdot v - w \cdot w + u \cdot u +$
$2(w \cdot v + w \cdot u) = 2(w \cdot v + w \cdot u)$

22.

Simple extension of Cauchy-Swarz.

$$\int_0^1 (f(t))^2 \leq \left(\int_0^1 f(t) \right)^2$$

23.

24.

a. $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle = 0 + 8 = 8$

b.

$$\begin{aligned}
||g+h|| &= \sqrt{\langle g+h, g+h \rangle} \\
&= \sqrt{\langle g, g \rangle + 2\langle g, h \rangle + \langle h, h \rangle} \\
&= \sqrt{1 + 2(3) + 50} \\
&= \sqrt{57}
\end{aligned}$$

c. $\text{proj}_{\mathbf{E}} h = \langle h, f/2 \rangle + \langle h, g \rangle h = 4h + 3h = 7h$

d. $f/2, g, \frac{h-7h}{|}|h-7h|| = \frac{-3h}{5\sqrt{3}}$

25.

Some research shows $1, 1/4, 1/9, ...1/n^2$ converges on $\pi^2/6$m hence $||x|| = pi/\sqrt{6}$

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.