

Notes

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`'r format(Sys.Date(), '%B %d, %Y')`

1.

A 90 degree rotation followed by an x-axis projection is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

4.

$$\begin{aligned} z &= \frac{y - z}{2} \\ Az &= A\left(\frac{y - z}{2}\right) \\ &= \frac{1}{2}Ay - \frac{1}{2}Az \end{aligned}$$

6.

3 by 3's for:

a. x-axis projection:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. xy reflection:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

c. xy rotation:

$$\begin{bmatrix} \sin \theta & -\cos \theta & 0 \\ \cos \theta & \sin \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d. Composed 90-degree rotations:

```
sin_90 <- 1
cos_90 <- 0
compose_trans(list(square(sin_90, -cos_90, 0, cos_90, sin_90, 0, 0, 0, 1), square(sin_90,
0, -cos_90, 0, 1, 0, sin_90, 0, cos_90), square(1, 0, 0, 0, sin_90, -cos_90,
0, cos_90, sin_90))) %>%
mat2latex()
```

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

e. Composed 180-degree:

```
sin_90 <- sin(pi)
cos_90 <- cos(pi)

compose_trans(list(square(sin_90, -cos_90, 0, cos_90, sin_90, 0, 0, 0, 1), square(sin_90,
0, -cos_90, 0, 1, 0, sin_90, 0, cos_90), square(1, 0, 0, 0, sin_90, -cos_90,
0, cos_90, sin_90))) %>%
round() %>%
mat2latex()
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

7.

Find the P_3 to P_4 matrix of $3t + 2$

$$a = 3t + 2$$

$$b = b(t(3t + 2)) = b(3t^2 + 2t)$$

$$c = c(t^2(2 + 3t)) = c(3t^3 + 2t^2)$$

$$d = d(t^3(3t + 2)) = d(3t^4 + 2t^3)$$

Now the matrix:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

12.

$A^{-1}(x + y) = A^{-1}x + A^{-1}y$ by the linearity of the inverse transformation.

14.

If T is linear, it has matrix A . The $T^2 = A(A) = A^2$, which is a composition of linear transformations

15.

What's the transpose matrix for $R^{2 \times 2}$?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

17.

a. Matrix of the “right shift” $T(x) = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The opposite “left shift”:

$$\begin{bmatrix} ,0 & 1 & 0 & 0 \\ ,0 & 0 & 1 & 0 \\ ,0 & 0 & 0 & 1, \end{bmatrix}$$

Naturally, they are left and right inverses of each other.

19.

Which nonlinear are invertible?

a. x^3 : yes

b. e^x : no, limited to positive numbers.

c. $x + 11$: yes

c. $\cos x$: yes

21.

Why linear transformations can't move the zero vector:

$$T(0 + w) = T(0) + T(w) = T(w)$$

This holds only if $T(0) = 0$. Similarly, $T(0v) = 0T(v)$.

23.

If S and T are linear and equal v , then $S(T(v)) = S(v) = v$

24.

Say $T(0, v_2) = (0, 0)$. Then $kT(0, v_2) = (0, kv_2) = 0 = (0, kv_2)$, but:

$$\begin{aligned} T((0, v_2) + w) &= (w_1, v_2) \\ T((0, v_2)) + T(w) &= (0, 0) + w = w \end{aligned}$$

25.

Which transformations satisfy which linearity property?

a. $T(v) = v/||v||$: neither

$$T(v + w) = \frac{v + w}{||v + w||} \neq v/||v|| + w/||w|| \quad T(cv) = cv/||cv|| \neq cv/c||v||$$

b. $T(v) = v_1 + v_2 + v_3$: both

$$\begin{aligned} T(v + w) &= v_1 + v_2 + v_3 + w_1 + w_2 + w_3 = T(v) + T(w) \\ kT(v) &= k(v_1 + v_2 + v_3) = kv_1 + kv_2 + kv_3 \end{aligned}$$

c. $T(v) = (v_1, 2v_2, 3v_3)$: obviously a legit diagonal matrix.

d. $T(v) = \max(v)$

Say $\max(w) = k \max(v)$ and the index of $\max(v)$ is not that of $\max(w)$. It satisfies the scalar property only.

$$T(v + w) = \max(v + w) \quad T(v) + T(w) = \max(v) + k \max(v)$$

$$kT(v) = k \max(v) = \max(kv)$$

28.

Find some ranges and kernels:

a. $T(v_1, v_2) = (v_2, v_1): R^2 \text{ and } 0.$

b. $T(v_1, v_2, v_3) = (v_1, v_2) R^2 \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

c. $T(v_1, v_2) = (0, 0): 0 \text{ and } R^2.$

d. $(v_1, v_2) = v_1, v_1: (1, 0) \text{ and } (0, 1).$

31.

A matrix transformation is linear because the dot product, by which multiplication is done, is itself linear.

33.

A “universal” transposing matrix cannot exist because we would need to swap the 2, 1 and 1, 2 elements. But this can only be achieved by a matrix with a 1 in one of those positions and zeroes elsewhere. Yet the diagonal of the transpose must be (1, 1) in order to select columns in the same order. No single matrix can have both these properties.

Wait a minute, it satisfies the rules of linearity properties but has no matrix!

34.

The transpose is involutory ($(A^T)^T = A$), has the zero kernel, and has a range of all matrices, but $T(M) = -M$ only for diagonal or skew-symmetric matrices.

40.

$$M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

The solution is $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ - the first column of M^{-1} , which reverses the effect of the transformation.

43.

The matrix of T is Λ , since the transformation is just scaling eigenvectors.

43.

Because the matrix of a linear transformation is defined as the the transformation applied to the columns of the identity. That means each basis vector has a unique representation in the transformation, which may be inverted.

44.

A reflection across the x axis followed by the y axis is also a reflection over $y = -x$

45.

$S(T(v)) = \begin{bmatrix} y \\ -x \end{bmatrix}$, also a reflection over the line $y = -x$.

49.

If we know $T(V)$ for n distinct vectors, we don't know the transformation for all v unless they're linearly independent.

50.

- a. A parallelogram
- b. Scaling matrices with one value in the trace.
- c. Matrices of rank 1.
- d. Rotation and reflection matrices.