Section 7.3 Problems

Ryan Heslin

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##1.

Just find some eigenvectors.

$$
\begin{aligned}
A &= \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix} \\
\lambda &= (9, 4) \\
S &= \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

5.

None real, but $\lambda = 1 \pm i$

10.

$$
A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}
$$

17.

$\lambda = (0, 0, 1, 1)$. Diagonalizable.

21.

```
S <- matador::square(1, 2, 2, 3)
Lambda <- diag(x = c(1, 2), nrow = 2)
A <- S %*% Lambda %*% solve(S)
A
```

```
 [,1] [,2]
```

[1,] 5 -2 [2,] 6 -2 ## 22.

Just $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ ## 23.

For $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1$, but since that yields $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, the only eigenvector is e_1. That makes sense, since Ae_1 just selects the first column, which contains only single scalar.

25.

If c only is 0, then A has distinct eigenvectors. If all three are 0, then A has just one eigenvector for the repeat eigenvalue 0. ## 26.

Since $\det A = \prod_{i=1}^{n} \lambda_i$, if the determinant is negative but n is positive, there must be an odd number of negative eigenvalues.

27.

$\lambda = (1, 5)$.

28.

The eigenvalues are all just k, multiplicity n. They all have e_1 as the eigenvector.

29.

Algebraic and geometric multiplicity are both $n - r$, since by rank-nullity $\ker(A)$ has dimension $n - r$.

30.

Algebraic multiplicity is $n - m$

31.

If an eigenbasis exist, both multiplicities sum to n, though geometric and algebraic multiplicity need not match for every distinct eigenvalue.

32.

The algebraic nultiplicities are the same, since the eigenvalues are shared. The dimension of $\ker(A - I\lambda)$ is $n - rank(A^T - \lambda I)$. So the dimension of the transposes eigenspace is the orthogonal complement of A's image, and vice versa. So if $n = 3$, then a two-dimensional eigenspace in A corresponds to a one-dimensional eigenspace in A^T and vice versa.

#33.

$$
\begin{aligned}
(B - \Lambda) &= S^{-1}(A - \Lambda)S \\
&= S^{-1}(AS - \Lambda S) \\
&= S^{-1}AS - S^{-1}\Lambda S \\
&= B - \Lambda
\end{aligned}
$$

34.

$$
\begin{aligned}
B &= S^{-1}AS \\
SB &= AS \\
S(Bx) &= A(Sx)
\end{aligned}
$$

So if $Bx = 0$, $Sx = 0$ as well.

b.

Invertible, so isomorphic.

$$
\begin{aligned}
T(X) &= Sx \\
T^{-1}x &= S^{-1}Sx = x
\end{aligned}
$$

c. Since S has full rank, Sx has the rank of $\ker B$, since x is some linear combination of the kernel, so the dimension remains the same. Since A and B both have n columns, if the kernels have dimension m since hey both have dimension $n - m$.

35.

No, the traces are different.

37.

a.

$$
\begin{aligned}
Av \cdot w &= v \cdot Aw \\
v^T A^T w &= v^T Aw \\
v^T Aw &= v^T Aw
\end{aligned}
$$

The proof of symmetric orthogonal eigenvectors

b.

$$
\begin{aligned}
Av \cdot w &= v \cdot Aw \\
\lambda_v v^T w &= \lambda_w v^T w
\end{aligned}
$$

Since $\lambda_w \neq \lambda_v$, this holds only if $v \cdot w = 0$.

38.

Since $\det A = 1$ and A is orthogonal, 1 or 3 eigenvalues are 1. Thus $Av = 1v = v$ for those eigenvectors.

```
x <- c(1, 0, 0, 1, 0)
y <- c(0, 0, 1, 0, 1, 0, 0)
z <- c(0, 0, 0, 0, 0)
sum(cumsum(x == 0) == seq_along(x))
```

```
[1] 0
```

```r
sum(cumsum(y == 0) == seq_along(y))
```

```
[1] 2
```

```r
sum(cumsum(z == 0) == seq_along(z))
```

```
[1] 5
```

39.

a. $n-m$ are 0, with equal geometric multiplicity because $\dim(\ker(A)) = n-m$ The remaining m are distinct and have orthogonal eigenvectors because all projection matrices are symmetric.

b. Reflection matrices only have eigenvalues ± 1 (this makes obvious geometric sense), so algebraic multiplicity is greater than 1. The eigenvectors are bases of the subspaces of reflection, since these retain their position.

40.

$a = 0$.

41.

All possible values.

42.

$b \neq 1$

43.

$a \neq 0$

44.

All values, since this is always symmetric.

45

Always diagonalizable.

46.

At least one of the constants is 0.

47.

$a = b = c = 0$.