Notes

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1.

The least squares solution to

$$\begin{aligned} 3x &= 10 \\ 4x &= 5 \end{aligned}$$

is

```r
fit <- function(A) {
    (solve(t(A) %*% A) %*% t(A))
}

fitted <- c(3, 4) %*% fit(c(3, 4)) %*% c(10, 5)

t((c(10, 5) - fitted)) %*% c(3, 4)
```

```
     [,1]
[1,]    0
```

I check the solution is orthogonal, jsut to be safe.

7.

Find a projection matrix:

```r
A <- cbind(c(1, 1, -2), c(1, -1, 4))

matador::mat2latex(A %*% fit(A))
```

$$\begin{bmatrix} 0.909090909090909 & 0.272727272727273 & 0.0909090909090906 \\ 0.272727272727273 & 0.181818181818182 & -0.272727272727273 \\ 0.0909090909090911 & -0.272727272727273 & 0.909090909090909 \end{bmatrix}$$

8.

If P projects onto the k-dimensional subspace, then P's image is k, and its rank is the dimension of k.

9.

IF $P = P^T P$, then P is a projection matrix. Projection matrices are both symmetric and respect unit length, so $P^T P = P^2 = P$.

b. $P = 0$ projects into the kernel of the transpose, since 0_n (from the right-hand matrix) resides there.

10.

Say v, w, and b are orthogonal, then $A^T A = I_m$ and $A^T b$ maps b onto A's column space.

11.

Say P projects onto S and Q onto $S^\perp$. Then $P + Q = I$ because every vector consists of $Px + Qx$, so $(P + Q)x = x$ $PQ = 0$ because $Px \cdot Qx = 0$. Then

$$
\begin{aligned}
&(P - Q)^2 = I \\
&P^2 - QP - PQ + Q^2 = I \quad P + Q = I \\
&I = I
\end{aligned}
$$

12.

The kernel of the transpose is

$$
\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

```r
A <- cbind(c(-1, 1, 0, 0), c(-1, 0, 0, 1))

A %*% fit(A)
```

```
           [,1]       [,2] [,3]       [,4]
[1,]  0.6666667 -0.3333333    0 -0.3333333
[2,] -0.3333333  0.6666667    0 -0.3333333
[3,]  0.0000000  0.0000000    0  0.0000000
[4,] -0.3333333 -0.3333333    0  0.6666667
```

Since all vectors in V and $V^\perp$ are orthogonal, the projection of a vector in one onto the other is 0.

15.

Show the reflection matrix R is involutory:

$$
\begin{aligned}
R^2 &= (I - 2P)^2 \\
&= I^2 + 4P^2 - 2PI - 2IP + I^2 \\
&= I^2 \\
&= I
\end{aligned}
$$

16.

Show $P = uu^T$.

Symmetri is obvious. For idempotence, consider the first element of P^2:

$$
\begin{aligned}
&= (u_1^2)^2 + (u_1 + u_2)^2 \\
&= u_1^2(u_1^2 + u_2^2) \\
&= u_1^2
\end{aligned}
$$

17.

That matrix is 0.5, -0.5, -0.5, 0.5

19.

The row space projection is of course $A^T(AA^T)^{-1}A$

23.

The best fit to a constant function is the average. Then $||\hat{x} - x|| = ||\bar{x} - x|| = \sqrt{\bar{x} - x}$, which is the exact definition of the residual sum of squares.

24.

Quadratic fit:

```
A <- matrix(c(1, -1, 1, 1, 0, 0, 1, 1, 1, 1, 2, 4), nrow = 4)
fit(A) %*% c(2, 0, -3, -5)
```

```
                          [,1]
[1,]  1.0000000000000006661338
[2,] -5.0000000000000017763568
[3,]  0.0000000000000004440892
```

The equation is $1 - 5t$; the coefficeint on t^2 is barely significant.