Section 6.2 Problems

Ryan Heslin

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$QQ^T = I$ is analogous to $\pm 1^2 = 1$

1.

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

A: $a > 2$. B: no numbers

2.

A: pivots are $(2, 3/2, 0)$, so no

B: pivots are $(2, 3/2, 1)$, so yes

$$C^2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

By inspection, it's obvious some pivots are 0.

3.

$$\begin{aligned} &\det A = 1(1 - b^2) - b(b + b^2) + -b(b^2 + b) \\ &= 1 - b^2 - 2(b^3 + b^2) \\ &= -2b^3 - 3b^2 + 1 \end{aligned}$$

Can't find an obvious value to make it negative.

4.

We can diagonalize since A is symmetric.

$$\begin{aligned} &A = S\Lambda S^{-1} \\ &A^2 = S\Lambda^2 S^{-1} \\ &A^{-1} = S\Lambda^{-1} S^{-1} \end{aligned}$$

5.

$$x^T(A+B)x > 0$$
$$x^TAx + x^TBx > 0$$

Both terms are greater than zero, so their sum is too.

6.

Some factorizations

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\begin{aligned}
(L\sqrt{D})(\sqrt{D}L^T) &= \left(\begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3/\sqrt{5} \end{bmatrix} \right) \left(\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix} \right) \\
(Q\Lambda)(\Lambda Q^T) &= \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \right) \\
(Q\sqrt{\Lambda}Q^T)^2 &= \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \right)
\end{aligned}$$

7.

As a square root of symmetric PD A, $R = Q\sqrt{\Lambda}Q$ has the same signs of eigenvalues.

$$A = \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \right)$$

$\lambda = 10 \pm 6i$, so we find the square roots by DeMoivre's formula ($\sqrt{r}(\cos\theta/2 + i\sin\theta/2)$)

$$r = 10^2 + 6^2 = 136$$
$$\theta = \arctan(6/10) \approx .27$$

$$\begin{aligned}
A &= \begin{bmatrix} 10 & -6 \\ 6 & 10 \end{bmatrix} \\
A &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \sqrt{136}(\cos .27 + i\sin .27) & 0 \\ 0 & \sqrt{136}(\cos .27 - i\sin .27) \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}
\end{aligned}$$

8.

If A is positive definite, it may be expressed as $A = R^TR$. then

$$B = C^TAC$$
$$= C^TR^TRC$$

R^TR is positive definite,a s are all A^TA of full rank. given that C is invertible, C^TC is positive definite as well, so the product of the two matrices must be positive.

9.

The generalized Schwarz inequality. I think I have the steps backward, but the logic is sound.

$$
\begin{aligned}
&|x^T Ay|^2 \leq (x^T Ax)(y^T Ay) \\
&\leq (x^T R^T Rx)(y^T R^T Ry) \\
&\leq (Rx)^2(Ry)^2 \\
&|(RxRy)^2| \leq (Rx)^2(Ry)^2 \\
&|(x^T R^T Ry)^2| \leq (x^T R^T Rx)(y^T R^T Ry) \\
&|x^T Ay|^2 \leq x^T Axy^T Ay
\end{aligned}
$$

The right side is invariably positive, but the left only is for all vectors if A is positive definite or $x = y$

10.

Values obviously 1 and 4, vectors the standard vectors.

The major axis cuts through the x-axis, the minor axis the y-axis.

11.

From the polynomial $3u^2 - 2\sqrt{u}v + 2v^2$, we have:

$$S = \begin{bmatrix} \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} \end{bmatrix}$$

implying the eigenvector factorization:

$$4\left(\frac{\sqrt{2}}{\sqrt{3}}u + \frac{v}{\sqrt{3}}\right)^2 + \left(\frac{u}{\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}v\right)^2$$

This factorization just uses the eigenvalues as factors of squares to stand in for the effect of transforming $x^T x$ by A.

13.

Tests for a *negative* definite matrix:

I. $x^T kx < 0$ for all real x II. All $\lambda < 0$ III. All upper-left submatrices have determinants of alternating signs, starting with negative, and the overall determinant is negative. The overall determinant must be negative because the opposite-sign of the matrix is positive definite. IV. All pivots < 0

14.

A: positive, since pivots are 1, 1, 17.

B: Indefinite, since the pivots are 1, 2, 1, -7, and the subdeterminants don't go int he right pattern.

C: also indefinite.

D positive, as inverse of a positive.

15.

a. False.

b. True. This matrix is similar to A, and similar matrices have the same eigenvalues.

c. True, by similarity.

d. True; this is just the diagonal matrix of E raised to each eigenvalue, always positive.

17.

The product $a_{11}a_{22}\dots a_{nn}$ is the product of the trace of $R^T R$. When A is diagonal, then $R = R^T = \sqrt{A}$. By the volume interpretation, all of A's columns are orthogonal, so the parallelopiped is a perfect hyper-rectangle.

$$\begin{aligned}
&R = R^T \\
&\det A = \det(R^T R) \\
&= \det(R)^2
\end{aligned}$$

19.

Matrix 1 is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Matrix 2 fills in the remaining zeroes and is not positive definite because the third row now requires elimination to get the pivots..

20.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$

$$\begin{aligned}
&det(2) = 2 \\
&2(5) - 2(2) = 6 \\
&2(5(8) - 3(3)) - 2(2(8) - 3(0)) + 0(2(3) - 5(0)) = 2(40 - 9) - 2(16 - 0) = 30
\end{aligned}$$

Indeed, pivot 2 is $6/2 = 3$ and pivot 3 is $30/6 = 5$.

21.

The quadratic form $2(x_1^2 + x_1x_2 + x_3x_1 + 2x_2x_3 + 5x_2^2)$ is -71 if, for example, $(x_1, x_2, x_3) = 1, -10, 1$

22.

If entry A_{jj} of a PD matrix is smaller than any λ, then $A - a_{jj}I$ would be PD as well. Yet this matrix has a zero in the place of a_{jj}, so it cannot be PD.

23.

a. It cannot have zero eigenvalues, always present in invertible matrices.

b. All projection matrices have rank m, where m is the dimension of the subspace of projection. The only such matrix for which $m = n$ is the projection onto R^n - the identity.

c. The eigenvalues are the diagonal and are all positive.

d.Example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

24.

a.

$$\begin{aligned}
&\det A = s(s^2 + 16) + 4(-4s + 16) - 4(16 + 4s) \\
&= s^3 - 16s \\
&= s(s + 4)(s - 4)
\end{aligned}$$

$s \geq 4$

b.

$$\begin{aligned}
&\det A = t(t^2 - 9) - 3(3t - 0) \\
&= t^3 - 25t \\
&t^2 > 5 \\
&t \neq 0, -5, 5 \quad t > -5
\end{aligned}$$

25.

The coefficients of the completion of the square for the ellipse equation are $\frac{1}{\sqrt{\lambda}}$. For $9x^2 + 16y^2 = 1$ they are $1/3$ and $1/4$.

This works because the ellipse formula can be written as a a sum of $x_i^T \lambda_x x_i$ for each eigenvector, computing the elements of the final dot product separately.

26.

The matrix is just

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

So $\lambda = (3/2, 1/2)$, so the half axes are $\sqrt{2}/\sqrt{3}, \sqrt{2}$.

27.

```r
cholesky <- function(C) {
    L <- diag(nrow = nrow(C))
    L[lower.tri(L)] <- C[lower.tri(C)]
    D <- diag(C)
    L %*% sqrt(D) %*% sqrt(D) %*% t(L)
}
C <- square(3, 1, 0, 2)
mat2latex(cholesky(C))
```

$$\begin{bmatrix} 3 & 5.44948974278318 \\ 5.44948974278318 & 9.89897948556636 \end{bmatrix}$$

```r
A <- square(4, 8, 8, 25)
C <- square(1, 2, 0, 1) %*% sqrt(diag(x = c(4, 25)))
mat2latex(C)
```

$$\begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$$

28.

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$$
$$C = L\sqrt{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$$
$$L\sqrt{D} = C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{5} \end{bmatrix}$$

29.

Writing out $x^T Ax = x^T LDL^T x$ reveals the square completion - $(c - b^2/a)^2$ is the second term. The formula squares the sum of x and the ratio of base b to a, then adds the square of the difference of the third base and b^2 in units of a.

$$2(x + 2y)^2 + (10 - 2)y^2)$$
$$2x^2 + 8y^2 + 8xy + 8y^2$$
$$2x^2 + 8xy + 16y^2$$

30.

a. $\det A = 2 \times 5 = 10$

b. $\lambda = 2, 5$

c. $\begin{bmatrix} 1 \pm i \end{bmatrix}$

d. It admits an LDL^T factorization, which can be expressed CC^T

31.

$B = (x_1 + x_x + x_3)^2$

32.

a. No; second subdeterminant is $1^2 - 1^2 = 0$.

b. All eigenvalues are 0 or positive, so positive semidefinite.

35.

It does. After eliminating one row we have:

$$\begin{bmatrix} 2.5 & 3 & 0 \\ 3 & 5.9 & 7 \\ 0 & 7 & 7.5 \end{bmatrix}$$

Clearly $7(7/5.9) > 7.5$.

36.

a. The squared eigenvectors magically disappear because they're orthornormal

$$\begin{aligned} &z = a_1 x_1 + \cdots + a_p x_p = v_1 C y_1 + \cdots + b_q C y_q \\ &z = S\alpha \\ &z^T A z = z^T A S \alpha \\ &= z^T (\sum_{i=1}^{p} \lambda_i a_i x_i) \\ &= \lambda_1 a_1^2 + \cdots + \lambda_p a_p^2 \geq 0 \end{aligned}$$

Since

$$\begin{aligned} C^T A C y &= \mu y \\ A C y &= (C^T)^{-1} \mu y \end{aligned}$$

$$\begin{aligned} &z = b_1 C y_1 + \cdots + b_1 C y_1 \\ &z^T A z = z^T A (b_1 C y_1 + \cdots + b_1 C y_1) \\ &= (b_1 y_1^T C^T + \cdots + b_q y^T C^T)(\mu_1 b_1 (C^T)^{-1} y + \cdots + \mu_q b_1 (C^T)^{-1} y) \\ &= \mu_1 b_1^2 + \cdots + \mu_1 b_q^2 \end{aligned}$$

b. Because these expressions of z are equal, they fail to hold if any term of either side is nonzero, since $\lambda_p a_p^2$ is always positive and $\mu_q b_q^2$ always negative. So all a, b are zero. And if the eigenvectors are independent, then $p + q \leq n$.

c.

$$\begin{aligned}
&n - p + n - q \leq n \\
&2n \leq n + p + q \\
&n \leq p + q
\end{aligned}$$

This is compatible with $p + q \leq n$ only if $p + q = n$.

37.

If C is nonsingular, then only the zero vector solves $C^T x = 0$. Then the kernel of $C^T AC$ is that of AC. Let $Cx = y$. Then $C^T Ay$ has the kernel of A and therefore its rank.

39.

C has to be square.

Orthogonal eigenvectors for a similarity transformation:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad x_i = \begin{bmatrix} \sqrt{3} - 1 \\ 1 \end{bmatrix} \quad x_j = \begin{bmatrix} \sqrt{3} + 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned}
&\begin{bmatrix} \sqrt{3} - 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3} + 1 \\ -1 \end{bmatrix} \\
&\begin{bmatrix} \sqrt{3} - 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3} + 1 \\ -1 \end{bmatrix} \\
&(3 + -1) + 2 \\
&0
\end{aligned}$$

41.

$$\begin{aligned}
&\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} x = \frac{\lambda}{18} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} x \\
&R = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1/2 \\ 0 & 2 \end{bmatrix} \\
&R^{-1} = C = \begin{bmatrix} 2 & -1/2 \\ 0 & 2 \end{bmatrix} \\
&C^T AC = \begin{bmatrix} 12 & 0 \\ -6 & 57/2 \end{bmatrix}
\end{aligned}$$

The values are 216 and 513, the vectors the standard vectors.