Section 7.6 Problems

Ryan Heslin

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Is the zero state a stable equilibrium?

##1.

Stable, since the diagonal eigenvalues are less than 0.

##2.

Not stable.

3.

$$\lambda^2 - 1.6\lambda + 1.13$$
$$\lambda = .8 \pm .98i$$

Since $1.6^2 + .98^2 = 1.6004$, not stable.

11.

$-1.1 \leq k \leq .1$.

12.

$|k| \leq .8$

13.

All possible values.

14.

$|k| < 1/2$

15.

No possible values, since $\lambda_1 + \lambda_2 = 2$ means one or both has absolute value greater than 1.

16.

17.

$\lambda = .6 \pm .8i$

I pick $.6 + .8i$.

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$
$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$c = S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The big dumb formula, with $\theta = \arctan .8/.6 \approx .92$:

$$u_{k+1} = (1^k) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos 1^k\theta & -\sin 1^k\theta \\ \sin 1^k\theta & \cos 1^k\theta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Interesting - is r always the square root of the determinant for skew-symmetric?

19.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$\lambda = 2 \pm 3i$

$$S_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$c = S^{-1}x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$\theta = \arctan\frac{3}{2} = 0.9827937$$
$$r = \sqrt{13}$$

$$u_{k+1} = \sqrt{13}^k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta k & -\sin\theta k \\ \sin\theta k & \cos\theta k \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$= \sqrt{13}^k \begin{bmatrix} -\cos\theta k \\ \sin\theta k \end{bmatrix}$$

21.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 7 \end{bmatrix}$$

$$\begin{aligned}
S_1 &= \begin{bmatrix} 1 \\ 3/5 + 5/i \end{bmatrix} \\
S &= \begin{bmatrix} 0 & 1 \\ 1/5 & 3/5 \end{bmatrix} \\
r &= \sqrt{17} \\
\theta &= \arctan \frac{1}{4} \\
c &= -5 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
u_{k+1} &= \sqrt{17} \begin{bmatrix} 0 & 1 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} \cos\theta k & -\sin\theta k \\ \sin\theta k & \cos\theta k \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\
&= \sqrt{17} \begin{bmatrix} 0 & 1 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 5\cos\theta k \\ 5\sin\theta \end{bmatrix} \\
&= \sqrt{17} \begin{bmatrix} 5\sin\theta k \\ \cos\theta k + 3\sin\theta k \end{bmatrix}
\end{aligned}$$

25.

Unstable. If A is stable,e ach λ must be less than 1 in absolute value, so $1/\lambda$ of the inverse must be greater.

26.

Stable, since A^T has the same eigenvalues as A.

27.

Stable, since the eigenvalues just have to be less than 1 in absolute value.

28.

Unstable.

$$\begin{aligned}
|\lambda| &< 1 \\
|\lambda - 2| &< -1
\end{aligned}$$

29.

Stable if $\lambda < 0$, the square is smaller.

30.

31.

By the rule that $|\lambda| < 1$ for stability:

$$
\begin{aligned}
&|trA| - 1 < \det A < 1 \\
&|\lambda_1 + \lambda_2| - 1 < \lambda_1\lambda_2 < 1 \\
&1 < 1 < 1
\end{aligned}
$$

This holds only if both have absolute value less than 1.

33.

If x_0 is real, then \$ c_1iw = -c_2iw\$ - the complex components cancel. Since $\bar{c}_1 iw = -c_1 iw$, that means $c_1 = \bar{c}_2$.

34.

a. It cannot be stable, because at least one iegenvalue is too large.

b. If $|\det A| < 1$, it may be stable. Otherwise, at least one eigenvalue is too large.

35.

a. $u_{k+1} = Au_k$ may be rewritten $A^k u_0$. Since all eigenvalues are less than 1 in absolute value, A is stable at zero, so A^k approaches 0 as k increases. Thus the length of the vector dieminishes with each iteration

b. No. All eigenvalues are 1, so the matrix is unstable and approaches no limit.

38.

a.

$$
\begin{aligned}
x &= Ax + b \\
Ax &= x - b \\
x &= A^{-1}(x - b) \\
x &= \frac{1}{\lambda}(x - b)
\end{aligned}
$$

b. If x is an eigenvector with a nonzero eigenvalue.

39.

The eigenvector with highest λ is $/m12$, so

$$
\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}
$$

because $\begin{bmatrix}1\\2\end{bmatrix}$ gets added after *each iteration*.