Notes

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```
<!-- % 5. operation -->
\newcommand{\dotsn}[5]{#1_{1}#3_{1}#5{#1}_{2}#3_{2}{#5}\dots{#5}#1_{#2}#3_{#4}}
```

8.

Acute.

$$\theta = r`angle(rep(c(1, -1), times = 2), c(3, 4, 5, 3))`$$

11.

Using Cauchy-Swarz to prove the triangle inequality

$$\begin{aligned}
||v + w|| &\leq ||v|| + ||w|| \\
||v + w||^2 &= (v \cdot w) \cdot (v\ cdotw) \\
||v + w||^2 &= v \cdot v + 2(v \cdot w) + w \cdot \\
w||v + w||^2 &= ||v||^2 + ||w^2|| + 2(w \cdot v) \\
||v + w|| &\leq ||v|| + ||w|| + \sqrt{2(w \cdot v)}
\end{aligned}$$

Of course, the dot product is zero only for orthogonal vectors.

16.

Solve for the kernel of the transpose.

```
m <- cbind(rep(1/2, 4), c(1/2, 1/2, -1/2, -1/2), c(1/2, -1/2, 1/2, -1/2))
pracma::rref(t(m))
```

```
 [,1] [,2] [,3] [,4]
```

[1,] 1 0 0 -1 [2,] 0 1 0 1 [3,] 0 0 1 1

```
mat2latex(c("1/2", "-1/2", "-1/2", "1/2"))
```

```
Argument is not a matrix. Attempting to coerce
```

$$\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

22.

A vector is orthogonal to a subspace only if it is orthogonal to every vector of its basis.

Say this was not true. Then $\frac{(x \cdot w)}{w \cdot w} x > 0$. Then $x^{\parallel} > 0$, which means $x^{\perp} < x$, so the vector cannot be orthogonal.

Remember that dividing by the sum of squares produces the unit vector, proving a basis by which to scale x $(A^T A)^{-1}$ merely generalizes this to higher dimension, accounting for directions among the column vectors. Multiplying this by A^T maps R^M onto the matrix.

23.

Proof that $V^{\perp})^{\perp} = V$.

If two vectors are orthogonal, $v \cdot w = w \cdot v = 0$. V consists of all vectors orthogonal to $V^{\perp}$ by this definition. This is the same as $V^{\perp})^{\perp} = V$. Also, $\dim(V) = \dim(V^{\perp})^{\perp})$. If $\dim(V) = p$, $\dim(V^{\perp}) = n - p$. Since the two spaces are complementary, $\dim(V^{\perp})^{\perp}) = n - (n - p) = p$,

##24.

Proving the linearity of $T(x) = x^{\parallel}$. Since the dot product is a linear transformation:

$$
\begin{aligned}
&T(x) + T(y) = (x \cdot u)u + (y \cdot)u \\
&T(x + y) = ((x + y) \cdot u)u \\
&= (x \cdot u + y \cdot u)u \\
&= (x \cdot u)u + (y \cdot u)u
\end{aligned}
$$

25.

Given the absolute value, we can ignore nonreal roots.

$$
\begin{aligned}
&||kv|| = |k|||v|| \\
&= |k|\sqrt{v \cdot v} \\
&= \sqrt{kv \cdot kv} \\
&= ||kv||
\end{aligned}
$$

30.

Consider a subspace where y is the projection of x. Then

$$||y||^2 \geq y \cdot x$$

because $||x|| \leq ||y||$ (projections are always the same length or shorter), $so||y||^2 = y \cdot x$ only if $x = y$, in which case x is in the subspace.

31.

Along similar lines, $(u_1 \cdot x)^2 + (u_2 \cdot x)^2 + \cdots + (u_m \cdot x)^2 \leq ||x^2||$ because the sum of squares of the projections can only equal the sum of squares of x when x lies entirely in the subspace of projection.

32.

The matrix

$$\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{bmatrix}$$

Is invertible if and only if v_1 and v_2 are orthogonal, meaning the off-diagonal is zeroes. This is an example of an orthogonal (but not orthonormal) amtrix.

33.

All elements $\pm\frac{1}{n}$. In that case, the vector is the hypotenuse of a right isoceles triangle, minimizing its length.

34.

$$\sqrt{x_1^2 + \ldots x_n^2} = 1$$
$$x_1^2 + \ldots x_n^2 = 1$$

This is maximal for a basis vector ($\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, etc.), because for any $|x| < 1$ $x^2 \leq x$.

35.

There is a more involved computation, but obviously this is minimized by e_1.

36.

```r
fit <- function(A) {
    solve(t(A) %*% A) %*% t(A)
}

A <- c(0.2, 0.3, 0.5)
fit(A)
```

```
          [,1]      [,2]     [,3]
[1,] 0.5263158 0.7894737 1.315789
```

37.

Let A be the basis of the plane. Then the reflection is $2A(A^TA)^{-1}A^T - I_3$. Or subtract from x twice its projection onto $u_1 \times u_2$.

38.

$v_1 \cdot v_2 = v_1 \cdot v_3 = 1/2$. Since they are unit vectors, $\theta = \arccos 1/2 \approx 1.047$ - about 60 degrees. So v_1 and v_2 are on opposite sides of v_3. So $\cos\frac{2\pi}{3} = \cos(v_2 \cdot v_3) \implies v_2 \cdot v_3 \approx 2\pi/3)$

39.

$x \cdot proj_l x$ is invariably positive. We have:

$$
\begin{aligned}
&x = x^{\|} + x^{\perp} \\
&x \cdot proj_l x = (x^{\|} + x^{\perp})x^{\perp} \\
&= ||x^{\|}||^2 + x^{\|}x^{\perp} \\
&= ||x^{\|}||^2
\end{aligned}
$$

which is never negative.

40.

Given

$$
\begin{bmatrix} 3 & 5 & 11 \\ 2 & 9 & 20 \\ 11 & 20 & 49 \end{bmatrix}
$$

$||v2|| = 3$

The angle of v_2 and v_3 is `r acos((20)/(3 * 7))`

42.

$||v_1 + v_2|| = \sqrt{(v + w) \cdot (v + w)}$, so:

$$
\begin{aligned}
&||v_1 + v_2|| = \sqrt{(v + w) \cdot (v + w)} \\
&= \sqrt{v \cdot v + w \cdot w + 2(v \cdot w)} \\
&= \sqrt{3 + 9 + 10} \\
&= \sqrt{22}
\end{aligned}
$$

43.

$proj_{v_2} v_1 = 5/9 v_2$

44.

A vector in $\text{Span}(v_2, v_3)$ orthogonal to v_3 is $v_2 - \text{proj}_{v_3}(v_2)$, which is $v_2 - 20/49 v_3$.

45.

$proj_v(v_3) = 5/9 v_2 + 11/49 v_3$

46.

$proj_V(v_3) = 11/3 v_1 + 20/9 v_2$