Section 8.2 Problems

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1.

$$\begin{bmatrix} 6 & -7/2 \\ -7/2 & 8 \end{bmatrix}$$

3.

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & 7/2 \\ 3 & 7/2 & 7 \end{bmatrix}$$

4.

Positive definite.

5.

Negative definite.

7.

Indefinite.

8.

$\mathbf{A}^2$ is positive semidefinite because all its eigenvalues are 0 or positive. IF $\mathbf{A}$ is positive definite, so is $\mathbf{A}^2$ by that same token.

9.

a.

Symmetric. $(\mathbf{A}^{\mathbf{T}})^2 = (-\mathbf{A})^2 = \mathbf{A}^2$.

b.

It is negative semi-definite because the eigenvalues of $\mathbf{A}$ are either 0 or purely imaginary:

$$
\begin{aligned}
\mathbf{AX} &= \lambda x \\
x^{\mathbf{H}}\mathbf{A}^{\mathbf{H}} &= \bar{\lambda}x^{\mathbf{H}} \\
-x^{\mathbf{H}}\mathbf{A} &= \bar{\lambda}x^{\mathbf{H}} \\
-\bar{\lambda}x^{\mathbf{H}} &= \bar{\lambda}x^{\mathbf{H}} \\
-\bar{\lambda} &= \bar{\lambda}
\end{aligned}
$$

so the eigenvalues of $\mathbf{A}^2$ are either 0 or real positive.

c.

None, since there are imaginary eigenvalues.

10.

$$
\begin{aligned}
\mathbf{T}(\mathbf{X}) &= (x^{\mathbf{T}}\mathbf{A} + v^{\mathbf{T}}\mathbf{A})(x+v) - x^{\mathbf{T}}\mathbf{A}x - v^{\mathbf{T}}v \\
&= x^{t}\mathbf{A}x + v^{\mathbf{T}}ax + x^{\mathbf{T}}\mathbf{A}v + v^{\mathbf{T}}\mathbf{A}v - x^{\mathbf{T}}\mathbf{A}x - v^{\mathbf{T}}\mathbf{A}v \\
&= v^{\mathbf{T}}\mathbf{A}x + x^{\mathbf{T}}\mathbf{A}v \\
&= 2x^{\mathbf{T}}\mathbf{A}v
\end{aligned}
$$

11.

The same - the eigenvalues don't change sign.

12.

$\det \mathbf{A} < 0$ if and only if the eigenvalues have opposite signs. Thus $\lambda_1 c_1^2 + \lambda_2 c_2^2$ is indefinite because of the differing signs.

13.

Let d be a diagonal element less than or equal to zero. Then the term dx_i^2 is zero or negative definite, and so the matrix cannot be positive definite.

14.

Since $\det \mathbf{A} > 0$ and $a > 0$, then if $c \leq 0$ $ac \leq b^2$, which violates the constraint. And if $\det \mathbf{A} > 0$ then all eigenvalues are positive, so $\mathbf{A}$ is positive definite.

23.

No, since $\mathbf{M}$ is not symmetric.

24.

a.

25.

λ, since the unit eigenvectors eigenbasis coordinates are 1.

26.

True. Such a vector satisifies $\mathbf{A}x = \lambda x$ for $\lambda = 0$. Then$\mathbf{A}$ has zero determinant and is therefore not invertible.

27.

A hyperellipse whose axes are the unit eigenvectors.

30.

$\mathbf{A} = (\mathbf{Q}\sqrt{\Lambda}\mathbf{Q^T})^2$

If a matrix is positive definite, so is its square root, since the eigenvalues remain positive.

31.

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

32.

for the 2×2:

$$\begin{aligned} x &= \sqrt{a} \\ y &= b/\sqrt{a} \\ z &= \sqrt{c - b^2/a} \end{aligned}$$

33.

$$\mathbf{L} = \begin{bmatrix} \sqrt{8} & -2/\sqrt{8} & \\ & -2/\sqrt{8} & \sqrt{5} + 1/2 \end{bmatrix}$$

34.

If $\mathbf{A}$ is positive definite, then its full quadratic form is positive definite. For a 3×3, the form of the first submatrix is ax_1^2. This must be positive definite, because if not then the form of the second leading submatrix $ax_1^2 + bx_1x_2 + cx_2^2$ would not be positive definite, and so forth.

And if all the leading submatrices have positive definite, then they have all positive eigenvalues, meaning they have positive determinant.

Then for the final decomposition, to find x we have to solve the system:

$$a_n1 = x_1 b_{11}$$
$$a_n2 = x_1 b_{12} + x_2 b_{22}$$
$$\vdots a_{nn-1} = x^{\mathbf{T}} bn - 1$$

which can clearly be solved in sequence, with the last element of x ignored because $\mathbf{B}$ is triangular with dimension $n-1$. The final equation is

$$a_{nn} = x^{\mathbf{T}} x + t$$

and t must be positive to split it into $\sqrt{t}$ across $\mathbf{L}$ and $\mathbf{L^T}$.

36.

```
library(matador)
```

```
Loading required package: magrittr


Attaching package: 'magrittr'

The following object is masked from 'package:purrr':

    set_names

The following object is masked from 'package:tidyr':

    extract
```

```
to_solve <- cholesky <- matrix(0, nrow = 3, ncol = 3)
A <- matrix(c(4, -4, 8, -4, 13, 1, 8, 1, 26), nrow = 3)
u <- sqrt(A[1, 1])
v <- A[1, 2] / u
w <- sqrt(A[2, 2] - v^2)
x <- A[3, 1] / u
y <- (A[2, 3] - x * v) / w
z <- sqrt(A[3, 3] - x^2 - y^2)
to_solve[lower.tri(to_solve, diag = TRUE)] <- letters[seq(length(letters) - 5, length(letters))]
cholesky[lower.tri(cholesky, diag = TRUE)] <- c(u, v, x, w, y, z)
cholesky %*% t(cholesky)
```

```
     [,1] [,2] [,3]
[1,]    4   -4    8
[2,]   -4   13    1
[3,]    8    1   26
```

```
A
```

```
     [,1] [,2] [,3]
[1,]    4   -4    8
[2,]   -4   13    1
[3,]    8    1   26
```

36.

$$\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathbf{T}} = \mathbf{Q}\mathbf{R} \implies \mathbf{L} = \mathbf{Q}\mathbf{R}\mathbf{L}^{\mathbf{T}}{-1}$$

37.

These two criteria are another way of saying that the derivative matrix is positive definite. In that case, the product of the separate derivatives is greater than the derivative of the products, which is a condition of the test for a global minimum

38.

$p - q > 0$.

40.

If λ_k has the greatest absolute value of the negative-signed eigenvalues, $k > \lambda_k$ does the job.

41.

3.

42.

$\frac{n(n+1)}{n}$

43.

Yes. Image is degree-0 polynomials, kernel all matrices with $a = 0$, rank 1, nullity 3.

44.

Yes, isomorphic since it preserves all three coordinates.

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

45.

$\mathbf{T}(x) = (a + 2c + +2e + f)x_1^2 + (b + d + e)x_2^2$ Image: all degree-1 polynomials Kernel: quadratic forms where only x_3^2 has a nonzero coefficient Rank 2, nullity 4.

46.

$\mathbf{T}(x) = 4ax_1^2 + x_2^2 + 4x_1x_2$. Image: all quadratic forms

52.

Consider a 3×3.

$$a = e_1^t a_1 e = 1(1)a = a$$

and

$$\begin{aligned}
b = 1/2((e_1 + e_2)^{\mathbf{T}}\mathbf{A}(e_1 + e_2)^{\mathbf{T}} - e_1^{\mathbf{T}}\mathbf{A}e_1 - e_2^{\mathbf{T}}\mathbf{A}e_2)& \\
&= 1/2(e_1^{\mathbf{T}}\mathbf{A}e_1 + e_2^{\mathbf{T}}\mathbf{A}e_2 + e_1^{\mathbf{T}}\mathbf{A}e_2 + e_2^{\mathbf{T}}\mathbf{A}e_1 - e_1^{\mathbf{T}}\mathbf{A}e_1 - e_2^{\mathbf{T}}\mathbf{A}e_2) \\
&= 1/2(e_1^{\mathbf{T}}\mathbf{A}e_2 + e_2^{\mathbf{T}}\mathbf{A}e_1) \\
&= 1/2(b + b) \\
&= b
\end{aligned}$$

51.

Alternating odd, even.

53.

a.

$p(x) = x^2 a_{ii} + 2xya_{ij} + y^2 a_{jj}$

b.

If so, then a_{ii} and a_{jj} are positive and $a_{ii}a_{jj} > a_{ij}^2$, which mean the matrix of p has positive eigenvalues and is positive definite.

c.

Then $2_{aji} = a_{jj}a_{ii}$, which translates to a zero eigenvalue and an indeterminate matrix.

54.

They must be 0; otherwise the final determinant is negative, so some eigenvalues are negative and the matrix is not positive semidefinite.

55.

If this were not true the determinant would be negative, preventing he matrix from being positive definite.

56.

A symmetric $\mathbf{A}$ may be factorized $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q^T}$, where $\mathbf{Q}$ is the orthogonal eigenvectors matrix. Then $a_1 1 = \lambda_1 x_1^2 + \cdots + \lambda_n z_1$. $x_i^2 \leq 1.$, (where $x, \ldots z$ are the eigenvectors). Since these are unit vectors., this quantity is maximized if the first element of $\mathbf{Q^T}$'s first column is 1. If so, then $a_1 1 = 1\lambda_i = \lambda_i$. This is a quadratic form with the eigenvalues as the constants. The greatest eigenvalue maximizes the quantity $x^{\mathbf{T}}\mathbf{A}x$ for a unit vector x for a symmetric matrix, because if not then the vector that did maximize $x^{\mathbf{T}}\mathbf{A}x$ would be an eigenvector with a greater eigenvalue, contradicting the premise, and the quadratic form for an eigenvector, which lies entirely on one vector of the eigenbasis, is just λ. This vector has eigenbasis coordinates $1\lambda_{max} = \lambda_{max}$, By Cauchy-Swarz, the inner product of the vector of lambdas with any unit vector (here the first column of $\mathbf{Q^T}$) is at most the norm of the lambdas vector. The inner product of any unit vector with the lambdas vector is then its eigenbasis coordinates, and we have shown that is maximally λ_{max}, proving the claim.

$$
\begin{aligned}
&|\Lambda q_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}| \leq ||\Lambda 1_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}|| \cdot ||q_1^{\mathbf{T}}|| \\
&\leq ||\Lambda 1_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}|| \\
&\leq \lambda_{max}
\end{aligned}
$$

so a cannot be greater than the greatest λ_i.

58.

A spheroid around the origin.

58.

An ellipse around the origin

59.

A line.

60.

Concave surfaces oriented positively.

61.

Concave surfaces oriented negatively.

63.

The expression is eigenbasis coordinates divided by the appropriate lambdas, which cancel.

$$q(c_1 w_1 + \cdots + c_n w_n) = \lambda_1 c_1^2/\lambda_1 + \cdots + \lambda_n c_n^2/\lambda_n = c_1^2 + \cdots + c_n^2$$

64.

```r
eigens <- eigen(matrix(c(8, -2, -2, 5), nrow = 2))
eigens$vectors / sqrt(eigens$values)
```

```
           [,1]       [,2]
[1,] -0.2981424 -0.1490712
[2,]  0.2236068 -0.4472136
```

65.

For the negative-signed λ

$$c_i^2 \frac{v_i}{\sqrt{|\lambda_i}}^2 = \lambda_i c_i^2 / |lambda_i| = -c_i^2$$

66.

```r
eigens <- eigen(matrix(c(3, -5, -5, 3), nrow = 2))
eigens$vectors[, order(eigens$values, decreasing = TRUE)] * 1 / sqrt(abs(sort(eigens$values, decreasing
```

```
      [,1]  [,2]
[1,] -0.25 -0.25
[2,]  0.50 -0.50
```

67.

By the above logic, if we scale the eigenvectors with negative-signed eigenvalues by $1/\sqrt{|\lambda_i}$, c_i^2 is opposite-signed.

68.

$\mathbf{R^T A R}$.

69.

$\mathbf{R}$ has rank m.