Section 7.3 Problems

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##1.

Just find some eigenvectors.

$$A = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$$
$$\lambda = (9, 4)$$
$$S = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

5.

None real, but $\lambda = 1 \pm i$

10.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

17.

$\lambda = (0, 0, 1, 1)$. Diagonalizable.

18.

Not diagonalizable; the only two eigenspace vectors are $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

21.

```
S <- matrix(c(1, 2, 2, 3), nrow = 2)
Lambda <- diag(x = c(1, 2), nrow = 2)
A <- S %*% Lambda %*% solve(S)
A
```

```
 [,1] [,2]
```

[1,] 5 -2 [2,] 6 -2

22.

Just $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

23.

For $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1$, but since that yields $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, the only eigenvector is e_1. That makes sense, since Ae_1 just selects the first column, which contains only single scalar.

25.

If c only is 0, then A has distinct eigenvectors. If all three are 0, then A has just one eigenvector for the repeat eigenvalue 0.

26.

Since $\det A = \prod_{i=1}^{n} \lambda_i$, if the determinant is negative but n is positive, there must be an odd number of negative eigenvalues.

27.

$\lambda = (1, 5)$.

28.

The eigenvalues are all just k, multiplicity n. They share an eigenspace of every standard vector but the last.

29.

Algebraic and geometric multiplicity are both $n - r$, since by rank-nullity $\ker(A)$ has dimension $n - r$.

30.

Algebraic multiplicity is $n - m$

31.

If an eigenbasis exist, both multiplicities sum to n, though geometric and algebraic multiplicity need not match for every distinct eigenvalue.

32.

The algebraic multiplicities are the same, since the eigenvalues are shared. The dimension of $\ker(A - I\lambda)$ is $n - rank(A^T - \lambda I)$. So the dimension of the transpose's eigenspace is the orthogonal complement of A's image, and vice versa. So if $n = 3$, then a two-dimensional eigenspace in A corresponds to a one-dimensional eigenspace in A^T and vice versa.

#33.

$$\begin{aligned}
&(B - \Lambda) = S^{-1}(A - \Lambda)S \\
&= S^{-1}(AS - \Lambda S) \\
&= S^{-1}AS - S^{-1}\Lambda S \\
&= B - \Lambda
\end{aligned}$$

34.

$$\begin{aligned}
&B = S^{-1}AS \\
&SB = AS \\
&S(Bx) = A(Sx)
\end{aligned}$$

So if $Bx = 0$, $Sx = 0$ as well.

b.

Invertible, so isomorphic.

$$\begin{aligned}
&T(X) = Sx \\
&T^{-1}x = S^{-1}Sx = x
\end{aligned}$$

c. Since S has full rank, Sx has the rank of $\ker B$, since x is some linear combination of the kernel, so the dimension remains the same. Since A and B both have n columns, if the kernels have dimension m since hey both have dimension $n - m$.

35.

No, the traces are different.

36.

No, for the same reasoning.

37.

a.

$$\begin{aligned}
&Av \cdot w = v \cdot Aw \\
&v^T A^T w = v^T Aw \\
&v^T Aw = v^T Aw
\end{aligned}$$

The proof of symmetric orthogonal eigenvectors b.

$$Av \cdot w = v \cdot Aw$$
$$\lambda_v v^T w = \lambda_w v^T w$$

Since $\lambda_w \neq \lambda_v$, this holds only if $v \cdot w = 0$.

38.

Since a rotation matrix rotates all vectors by θ, no real vector satisfies this criterion. But eigenvectors still exist because the characteristic polynomial must have roots, but they lie on C^3 and are complex. If the matrix is $\pm I_3$, the eigenvalues and eigenvectors are of course real.

39.

a. $n - m$ are 0, with equal geometric multiplicity because $\dim(\ker(A)) = n - m$ The remaining m are distinct and have orthogonal eigenvectors because all projection matrices are symmetric.

b. Reflection matrices only have eigenvalues ± 1 (this makes obvious geometric sense), so algebraic multiplicity is greater than 1. The eigenvectors are bases of the subspaces of reflection, since these retain their position.

40.

$a = 0$.

41.

All possible values.

42.

$b \neq 1$

43.

$a \neq 0$

44.

All values, since this is always symmetric.

45

Always diagonalizable.

46.

At least one of the constants is 0.

47.

$a = b = c = 0$.

49.

Actually diagonalizable, because I failed to take into account complex roots.

51.

Simple enough.

$$
\begin{aligned}
&\det A = -\lambda(-\lambda(c-\lambda) - b(1)) + a(1(1) - \lambda(0)) \\
&= -\lambda(-\lambda c + \lambda^2 - b) + a(1) \\
&= -\lambda^2 + \lambda^2 c - \lambda b + a
\end{aligned}
$$

52.

From that pattern:

$$\lambda^n - \lambda^{n-1} a_{n-1} + \lambda^{n-2} a_{n-2} + \ldots a_0$$

assuming n is even; otherwise the first term is negative, the second positive, and so on.

55.

```r
set.seed(1)
A <- matrix(c(0, 1, 0, 0, 0, 1, -7, 1, 7), nrow = 3)
S <- matrix(c(1, 0, 5, 2, 1, 6, 3, 4, 0), nrow = 3)
B <- S %*% A %*% solve(S)
eigen(B)
```

```
eigen() decomposition
$values
[1]  7 -1  1

$vectors
           [,1]      [,2]       [,3]
[1,]  0.2981424 0.4036037 0.21975617
[2,]  0.5962848 0.2690691 0.02746952
[3,] -0.7453560 0.8744746 0.97516801
```

56.

Just find the characteristic polynomial, write the Frobenius companion matrix, and diagonalize with a matrix copied from some website that has an integer inverse.

```
set.seed(1)
A <- matrix(c(0, 1, 0, 0, 0, 1, 6, -11, 6), nrow = 3)
S <- matrix(c(1, 0, 5, 2, 1, 6, 3, 4, 0), nrow = 3)
B <- S %*% A %*% solve(S)
eigen(B)
```

```
eigen() decomposition
$values
[1] 3 2 1

$vectors
           [,1]          [,2]          [,3]
[1,]  0.1230915  2.169305e-01 -7.071068e-01
[2,] -0.1230915 -6.223777e-15 -7.071068e-01
[3,]  0.9847319  9.761871e-01 -9.841075e-14
```