Section 4.2 Problems

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2023-02-24

We check the linearity of some transformations.

1.

$T(M) = M + I_2$: nonlinear

3.

$$T(M) = \text{trace}(M)$$

Obviously not isomorphic; many matrices have the same trace.

##5.

$T(m) = M^2$: not isomorphic

7.

$$T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$$

Obviously linear, given matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

Also an isomorphism, since this is clearly invertible.

9.

$$T(M) = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} M \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Not an isomorphism, since M itself may be noninvertible.

Actually yes:

$$T(M) = S^{-1}MS$$
$$S^{-1}M = S^{-1}$$
$$M = SAS^{-1}$$

12.

$$T(c) = cM \quad c \to R^2$$

where M is a particular matrix. Linear - $c(M_1 + M_2) = cM_1 + cM_2$ and $c(kM_1) = kcM_1$.

Also isomorphic: $M = 1/cT(c)$

16.

$$T(M) = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M$$

The matrix is:

$$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \end{bmatrix}$$

Linear, not invertible.

17.

$$T(x + iy) = x$$

Linear, not invertible, as the complex part is just zeroed out. ## 18.

$T(x + iy) = x^2 + y^2$

Non-linear ($cT(x) = cx^2 + cy^2$), but $T(cx) = c^2x^2 + c^2y^2$); x can be positive or negative.

21.

$$T(x + iy) = y + ix$$

Linear: $T(x_1 + iy_1 + x_2 + iy_2) = y_1 + y_2 + i(x_1 + x_2) = T(x_1 + iy_1) + T(x_2 + iy_2)$ and $T(kx_1 + kiy_1) = ky_1 + kx_1 i_y = kT(x + iy)$

Isomorphism, and involutory too: $T^{-1}y = y + ix$.

22.

$$T(f(t)) = \int_{-} 2^3 f(t)dt$$

$$\begin{bmatrix} 3 & 3/2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2/3 \end{bmatrix}$$

Clearly noninvertible; infinitely many functions with the same definite integral. ## 23.

$$T(f(t)) = f(7)$$
$$T(f(t)) = a + 7b + 49c$$

Linear, nonisomorphic, with matrix

$$\begin{bmatrix} 1 & 7 & 49 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

24.

$$T(f(t)) = f''(t)f(t)$$

$$T(1) = 0(1) = 0 \qquad T(t) = b(0)(t) = 0$$
$$T(t^2) = c(2)(t^2) = 2ct^2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Not isomorphic.

27.

$$T(f(2t))$$

$$T(1) = 1a = 1$$
$$T(t) = b(2t) = 2bt$$
$$T(t^2) = c(4t^2) = 4ct^2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Isomorphic.

28.

$$T(f(t)) = f(2t) - f(t)$$

Nonisomorhpic; the constant term gets zeroed out, with amtrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 3 \end{bmatrix}$$

29.

$$T = f'(t)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Linear, non-isomorphic.

31.

$$T = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix}$$

from P_2 to $R^{2\times 2}$ Substituting, we find

$$T = \begin{bmatrix} a & a+b+c \\ a+2b+4c & a+3b+9c \end{bmatrix}$$

The matrix should be:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

```r
cbind(matrix(rep(rep(c(1, 0, 0), times = 2),
    times = 3), ncol = 3), sapply(list(c(2,
    0, 0), c(5, 0, 0), c(13, 0, 0)), rep, 2)) %>%
    mat2latex()
```

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 5 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 5 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

32.

$$T(f(t)) = f'(t) + t^2 = b + 2ct + t^2$$

Nonlinear; that t^2 is affine and has no matrix representation.

33.

Drop every other term.

$$T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_0, x_2, x_4, \dots)$$

Linear, since the sums and scalar multiples of parallel terms are the same before and after the transformation, but not isomorphic because there is no way to recover the terms.

34.

$$T(x_0, x_1, x_2, \dots) = (0, x_1, x_2, \dots$$

First rule:

$$T(x+y) = (0, x_0 + y_0, x_1 + y_1)$$
$$T(x) + T(y) = (0, x_0, x_1, \dots) + (0, y_0, y_1, \dots)$$

Second rule:

$$
\begin{aligned}
kT(x) &= (0, kx_1, kx_2, \dots) \\
T(kx) &= (0, kx_1, kx_2, \dots) = kT(x)
\end{aligned}
$$

Linear, isomorphic.

35.

From P to V:

$$T(f(t)) = (f(0), f'(0), f''(0), \dots$$

The first property is satisfied by the sum rule of derivatives.

Second property satisfied:

$$
\begin{aligned}
kT(f(t)) &= (kf(0), kf'(0), kf''(0), \dots) \\
T(kf(t)) &= ((kf(0), kf'(0), kf''(0), \dots))
\end{aligned}
$$

Not an isomorphism; many functions are undefined at 0.

39.

Let's find the matrix:

$$
\begin{aligned}
T(f) &= f'' + 2f' + f \\
&= 2c + 2(b + 2ct) + a + bt + ct^2 \\
&= a + 2b + 2c + bt + 4ct + ct^2
\end{aligned}
$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

This is clearly invertible, so it's an isomorphism.

43.

$$T(f(t) = \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 11 & 121 \end{bmatrix}$$

Non-isomporhic; the constant term gums up the workss.

52.

The kernel of $T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is all matrices which have $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in their kernel, of the form $v_1 = -2v_2$.

25.

$$T(f(t)) = f''(t) + 4f'(t)$$

$$\begin{aligned} &= 2c + 4(b + 2ct) \\ &= 4b + 2c + 8ct \end{aligned}$$

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The kernel is constant functions you idiot.

57.

$$T(f(t)) = f'' - 5f' + 6f$$

$$\begin{aligned} &= 2c - 5(b + 2ct) + 6(a + bt + ct^2) \\ &= 2c - 5b - 10ct + 6a + 6bt + 6ct^2 \\ &= 6a - 5b + 2c + 6bt - 10ct + 6ct^2 \end{aligned}$$

The image is P_2, the kernel $f(t) = 0$.

59.

From P_2 to R^n

$$T(f(T)) = f(7)$$

The matrix:

$$\begin{bmatrix} 1 & 7 & 49 \end{bmatrix}$$

The matrix kernel makes this clearer, I think.

$$\begin{bmatrix} -7 & -49 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

66.

For the infinite-dimensional $T(f) = f - f'$, the kernel is e^x, the image all other functions.

Working with the subspace of polynomials of degree $\leq n$ such that $f(0) = 0$; no constant term.

72.

Is it a subspace? Yes, because $f(0 + g(0)) = 0 + 0 = 0$, and $kf(0) = 0(0) = 0$.The dimension of Z_n is n, because a polynomial with no constant term has n terms.

75.

$$
\begin{aligned}
&T(0_V) = 0_W \\
&0_V(T) = 0_W \\
&0_V = 0_W
\end{aligned}
$$

$$
\begin{aligned}
&T(0_V) = 0_W \\
&T^{-1}(T(0_V)) = T^{-1}(0_W) \\
&0_V = T^{-1}(0_W)
\end{aligned}
$$

76.

We know T is linear, so we need to prove T^{-1} obeys the properties as well.

$$
\begin{aligned}
&T^{-1}(f+g) = T^{-1}(T(T^{-1}(f+g))) \\
&= T^{-1}(T(T^{-1}(f)) + T(T^{-1}(g))) \\
&= T^{-1}(f) + T^{-1}(g)
\end{aligned}
$$

and

$$
\begin{aligned}
&kT^{-1}(f) = T^{-1}(kT(T^{-1}(f)) \\
&= T^{-1}(T(k)T^{-1}f)) \\
&= kT^{-1}f
\end{aligned}
$$

81.

a.

$\ker(T)$ must be finite dimensional because it is a subspace of a finite-dimensional space. $\mathrm{im}(T)$ must be finite-dimensional because r elements form a basis for it, and infinite-dimensional spaces cannot be spanned by finitely many elements.

b.

$$
\begin{aligned}
&c_1u_1 + \cdots + c_ru_r + d_1v_1 + \cdots + d_nv_n = 0 \\
&T(c_1u_1 + \cdots + c_ru_r + d_1v_1 + \cdots + d_nv_n) = T(0) \\
&c_1w_1 + \cdots + c_rw_r = 0
\end{aligned}
$$

This holds for any linear combination of $\ker(T)$ and any linear combination of the domain equal to 0, which means it is linearly independent of T's domain.

c.

Since $T(v - d_1u_1 - \cdots - d_ru_r) = T(v) - d_1u_1 - \cdots - d_ru_r$, the left term must be 0, because we have subtracted a linear combination of the domain of T. That means the result is in $\ker(T)$, so the v_i that form a basis for the kernel span it. Suppose v is not linearly independent of $\ker(T)$. That means both the u_i and v_i are required to form a basis for this expression V, and likewise for V itself, since $v \in V$. So $\dim(V) = \mathrm{rank}(T) + \mathrm{nullity}(T)$.

82.

Let A be the matrix of the transformation. The rank of A is the dimension of its image, and its nullity is the dimension of the space in V it maps to 0. If A has m columns, then m is the dimension of V. Then $\text{nullity}(A) = m - \text{rank}(A)$, because the rank is the dimension of the space not mapped to 0. So $m = \dim(V) = \text{rank}(A) + \text{nullity}(A)$.

Alternately, use the above exercise to demonstrate a finite basis for V exists and consists of the domain and kernel.

83.

By the rank-nullity theorem $\dim(W) = \text{rank}(L) + \text{nullity}(L)$. The combined kernel consists of any elements of $\text{im}(T) \in \ker(L)$, which get mapped to 0, as well as all of $\ker(T)$, since $L(0) = 0$. Form a basis for $\ker(L)$ in W consisting of $w_1, \ldots, w_n$. If $T(v) = w_i$ for some $v \in V$, then a mapping exists from T's domain to L's kernel. (Any v is linearly independent of T's kernel by definition). Since a basis for L's kernel requires $\text{nullity}(L)$ elements, the size of the combined kernel is at most $\text{nullity}(T) + \text{nullity}(L)$

84.

Since $\text{im}(T) = W$, all elements of $\ker L$ are possible outputs of T. So the composition maps all elements of $\ker L$, plus all elements of $\ker T$, to 0, proving the claim.