Section 5.1 Problems

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1.

Just find the eigens:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} x = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 3)(\lambda - 2) = 0$$
$$\lambda_1 = 2 \quad \lambda_2 = 3$$

$$x_1 = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The second eigenvector is

$$\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

Sure enough, $3 + 2 = 1 + 4 = 5$ and $3(2) = 4(1) - (-1)(2) = 6$.

3.

The values are shifted down 7, so $\lambda_1 = -5$ and $\lambda_2 = -4$. But the vectors remain the same.

5.

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(3 - \lambda)(1 - \lambda)(0) = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = 1 \quad \lambda_3 = 0$$

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \implies x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \implies x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

And for the last eigenvalue 0, just the kernel of A itself,

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

For B, the determinant is -8, since it takes a row swap to diagonalize :

The eigenvalues must sum to 2 (the trace) and have a product of -8, which implies they are $\lambda = (2, 2, -2)$. So the vectors are

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \implies x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \implies x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

6.

If we consider $A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the characteristic polynomials $\lambda^2 - 7\lambda - 2$ and $\lambda^2 - 5\lambda - 2$ are clearly different. We have subtracted 2 from the trace, hence the difference. Zero eigenvalues are not changed by elimination because they represent a vector that *already* solves $Ax = 0x = 0$. Because row operations do not distrurb the kernel, since they respect row space, zero eigenvalues remain unchanged.

7.

a. If $B = A - 7I$, algebra shows that:

$$\begin{aligned} &Ax = \lambda x \\ &(A - 7I)x = \lambda x \\ &Ax - 7x = \lambda x \\ &Ax = \lambda x + 7x \\ &Ax = (\lambda + 7)x \end{aligned}$$

So the eigenvalue for A is 7 less than for B.

b. $\det(A^{-1}) = 1/\det(A)$

$$\begin{aligned} &Ax = \lambda x \\ &(A^{-1})^2 Ax = (A^{-1})^2 \lambda x \\ &A^{-1}x = A^{-1}x \end{aligned}$$

8.

If we set λ to 0...

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda)$$
$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

9.

$$\begin{aligned}
&(-1)^n \lambda^n + (-1)^{n-1}(trA)\lambda^{n-1} + \cdots + \det A = 0 \\
&-\lambda^n + (trA)(-\lambda)^{n-1} + \text{some convoluted polynomial} = 0 \\
&-\lambda^{n-1}(-\lambda + tr(A)) + \cdots = 0
\end{aligned}$$

10.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_A = 2, 3 \quad \lambda_b = 0, 2 \quad \lambda_{A+B} = \frac{7 \pm \sqrt{3}}{2}, \quad \lambda_{AB} = 0, 5$$

As we'd expect, $2 + 3 + 0 + 2 = 7$ and

$$2 \cdot 3 \cdot 0 \cdot 2 = 0 \cdot 5$$

, since

$$(A + B)x = \lambda x \implies Ax + Bx = \lambda x$$

11.

The eigen *values* of A and A^T are the same because $\det(A^T) = \det(A)$, and the trace remains constant as well. But the vectors are different. Consider $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$. The eigenvalues are both 1 and 0, but the vectors are very different: $(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix})$ for the first, $(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix})$ for the second.

12.

If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, then for A the eigens are ± 1, for B both are 2. But for $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, the eigenvalues are 3 and 1. But $3 + 1 = -1 + 1 + 2 + 2 = 4$, while

12.

a.

$$
\begin{aligned}
&A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \\
&(3-\lambda)(-3-\lambda) - 4(4) = 0 \\
&\lambda^2 - 15 = 0 \\
&\lambda_1 = 5, \lambda_2 = -5
\end{aligned}
$$

For the vectors:

$$
A - \lambda_1 I = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \implies x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

$$
A - \lambda_2 I = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \implies x_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}
$$

For the general $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ case, the polynomial works out:

$$
\begin{aligned}
&(a-\lambda)(a-\lambda) - b^2 = 0 \\
&\lambda^2 - 2a\lambda - b^2 = 0
\end{aligned}
$$

We resort to the quadratic formula and get a nice simplification:

$$
\begin{aligned}
\lambda &= \frac{2a \pm \sqrt{(2a)^2 - 4(a^2 - b^2)}}{2} \\
&= \frac{2a \pm \sqrt{4b^2}}{2} \\
&= \frac{2a \pm 2b}{2} \\
&= a \pm b
\end{aligned}
$$

13.

Given this, $tr(A) = 1 + 2 + 3 + 7 + 8 + 9 = 30$

14.

Given this matrix:

```
mat2latex(matrix(rep(1, 16), nrow = 4))
```

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
$$

rank is clearly 1. Three eigenvalues are 0, and the remaining one is 4, because $\sum \lambda = \text{trace}(A)$

For the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

rank is 2, meaning two eigenvalues are 0. The other two are ± 2.

15.

For the first matrix in the previous problem, if it is $n \times n$, then $n-1$ eigenvalues are 0 and the reamining one is n. In the second matrix, all but 2 eigenvalues are 0, and the remaining 2 are $\pm n - 2$

16.

If the matrix is:

```
mat2latex(square(rep(1, 16)) - diag(nrow = 4))
```

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

All the eigenvalues are shifted down 1, so three are now -1 and the remaining one is 3 (in order to sum to a trace of 0).

18.

A has eigenvalues $(0, 3, 5)$ with eigenvectors u, v, w.

a. Then u is a basis for the kernel because $Au = 0u$ by definition. The other two eigenvectors provide a basis for the image.

b. $Ax = u$ has no solution because $Au = \lambda u = 0u = u$. All multiples of u are in the kernel.

19.

The eigenvalues of A are $(1, 1/2)$ and of A^2 $(1, 1/4)$. Since $A^n = S\Lambda^n S^{-1}$ and $(1/2)^2 = 1/2(1/2) = 1/4$, halving $A + A^\infty$ completes one-haldf of the exponential decay the eigenvalue 1/2 has to undergo before reaching 0 at the steady state.

20.

$A + I$ has the same eigenvectors as A, but its eigenvalues are increased by 1.

The eigenvalues are

$$
\begin{aligned}
&(1-\lambda)(3-\lambda) - 2(4) = 0 \\
&\lambda^2 - 4\lambda - 8 = 0 \\
&\frac{4 \pm \sqrt{16 - 4(1)(-8)}}{2(-4)} \\
&\frac{4 \pm 4\sqrt{3}}{2} \\
&2 \pm 2\sqrt{3}
\end{aligned}
$$

Not going to bother working out the vectors.

21.

The eigenvalues of A^{-1} are the inverse of those of A, and its eigenbasis is the inverse of A's eigenbasis.

22.

A^2 has the same eigenvectors as A, but its eigenvalues are squared ($\lambda^2 = 9, 4$).

$$
\begin{aligned}
&A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \\
&\lambda^2 + \lambda - 6 = 0 \\
&\lambda = -3, 2 \\
&S = \begin{bmatrix} -3/2 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

23.

If you know an eigenvector, find λ by multiplying it by A and finding the scaling factor.

If you know an eigenvalue, find the eigenvector by subtracting λI from A and finding a basis for the kernel.

24.

Some proofs.

a.

$$
\begin{aligned}
Ax &= \lambda x \\
A^2 x &= A\lambda x \\
A^2 x &= \lambda A x \\
A^2 x &= \lambda^2 x
\end{aligned}
$$

b.

$$
\begin{aligned}
&Ax = \lambda x \\
&(A^{-1})^2 A x = (A^{-1})^2 \lambda x \\
&A^{-1} x = (A^{-1})^2 \lambda x \\
&A^{-1} x = \lambda (\lambda^{-1})^2 x \\
&= \lambda^{-1} x
\end{aligned}
$$

c.

$$
\begin{aligned}
&Ax + x = \lambda x + x \\
&(A + I)x = \lambda_x + x \\
&= (\lambda + 1)x
\end{aligned}
$$

25.

```
u <- c(1, 1, 3, 5)/6

P <- tcrossprod(u)
mat2latex(round(P %*% u, digits = 3))
```

$$
\begin{bmatrix} 0.167 \\ 0.167 \\ 0.5 \\ 0.833 \end{bmatrix}
$$

Since P only has rank 1, and is symmetric, an orthogonal vector with $\lambda = 0$ would be $\begin{bmatrix} -11 \\ 0 \\ 0 \end{bmatrix}$.

The three eigenvectors with $\lambda = 0$ are the bases of $A^{\perp}$, which is also the kernel since A is symmetric. A^T reduces to one row $\begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix}$, which implies a kernel basis of

$$
\begin{bmatrix} -1 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

26.

Given the rotation matrix

$$
Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix}
$$

The determinant works out to

$$
\begin{aligned}
&\det(Q - \lambda I) = (\cos - \lambda)^2 - (\sin)(-\sin) \\
&= \cos^2 + \lambda^2 - 2\lambda\cos - \sin^2
\end{aligned}
$$

27.

Every permutation matrix has an eigenvalue of 1 (for a vector fo ones). We can find others.

For

$$
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
$$

determinant is 1 (two row swaps from the identity, and trace 0, so $1 + \lambda_2 + \lambda_3 = 0$ and $1 \times \lambda_2\lambda_3 = 1$). Since trace is 0, the characterictic polynomial reduces to $-\lambda^3 + 0 + 1/2(0) + 1 = 0$. SO $\lambda^3 = 1$. FO the others:

$$\lambda(\lambda^2 + \lambda + 1) = 0)$$

By the quadratic formula:

$$\lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

So there are two complex roots. For

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

determinant is -1 (1 row swap) and trace is 1. This implies the remaining eigenvalues are ± 1.

28.

Technically correct, though lazy.

$$\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 5 & -1000000 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix}$$

29.

We know the rank of B (2), the determinant of B^TB ($\det(B)^2 = 0^2 = 0$), and the eigenvalues of $B + I)^{-1}$ (1, 1/2, and 1/3). We don't know the eigenvalues of B^TB.

30.

Given

$$\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$$

$c = 4 + 7 = 11$ and $0(1) - d = 28$ so $d = -28$.

31.

$c = 0$ so the trace sums to 0. $a = 0$ and $b = 9$ so the determinant remains 0 no matter which eigenvalue is subtracted; those choices make row 3 a multiple of 2 if a nonzero. eigenvalue is subtracted and of row 1 and 2 if the eigenvalue is 0.

32.

$\lambda = (0, 1, -1/2)$ because all Markovs have 1 as an eigenvalue, singularity means 0 is an eigenvalue, and the eigenvalues must sum to the trace. ## 33.

Three such matrices are:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

34.

Given rank 1, two eigenvalues are 0, leaving one as six.

The eigenvectors:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -2 \\ 1 & 1/2 & 1 \end{bmatrix}$$

35.

$Ax = Bx = \lambda x$ ## 36.

a. $\lambda = (1, 4, 6)$

b. $\lambda = (2, \pm\sqrt{6})$

c. $\lambda = (0, 0, 6)$

37.

Given $a + b = c + d$, $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + b \\ c + d \end{bmatrix}$, which means λx has equal values and is therefore a scalar multiple of x. Then the eigenvalues are $a + b$ and $c + d$, which must be equal.

39.

Yes. This is easy to show geometrically, the matrix of a 120-degree rotation cubed gives the identity.

```
theta <- (4 * pi)/3

mat2latex(round(matador::square(cos(theta), sin(theta),
    -sin(theta), cos(theta)) %^% 3))
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

40.

For permutation matrices in general, determinants are ± 1 (depending on the number of swaps), the trace varies from 0 to 3, and the possible eigenvalues are ± 1 and nasty complex roots of the characteristic polynomial. Pivots are always 1 or 0.