Section 9.1 Problems

Ryan Heslin

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1.

$x = 73^{5t}$

3.

$P = 7e^{0.03t}$

7.

$x = \frac{-1}{t+1}$, so it approaches 0.

9.

$x^{-k}dx = dt$, so $\frac{x^{-k+1}}{-k+1} = t + 1$, implying $x = \frac{1}{((-k+1)(t+1))^{\frac{1}{1-k}}}$.

13.

$1990 - 1778 = 212$ years elapsed. Hence the sum is either $P = 450000e^{(1.06)212} = 1.5046598 \times 10^{11}$ or $450000(1.06)^{212} = 1.0424517 \times 10^{11}$.

15.

If k is a percentage, the growth rate is $k/100$. So we have

$$\begin{aligned}
2 &= e^t(k/100) \\
\ln 2 &= t(k/100) \\
t &= \frac{ln2}{k/100} \\
t &= \frac{100 \ln 2}{k} \qquad t \approx \frac{69}{k}
\end{aligned}$$

22.

By the sum rule of derivatives, $\frac{d(x_1+x_2)}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt}$. (where x_1 and x_2 are initial states). This means the expression equals $Ax_1 + Ax_2 = A(x_1 + x_2)$

23.

By linearity, $kx(t) = kSe^{\Lambda t}S^{-1}x_0$

24.

The second matrix is diagonal, with one distinct eigenvalue. Adding a diagonal matrix shifts eigenvalues elementwise but does not change eigenvectors. Therefore:

$$
\begin{aligned}
c(t) &= Se^{(\Lambda+k)t}S^{-1}c_0 \\
&= e^{kt}Se^{\Lambda t}S^{-1}c_0 \\
e^{kt}x(t) &= e^{kt}x(t)
\end{aligned}
$$

25.

Again by linearity, $x(kt) = kAx \implies kx(t) = A(kx) \implies x(t) = Ax$.

```r
# https://stackoverflow.com/questions/43223579/solve-homogenous-system-ax-0-for-any-m-n-matrix-a-in-r-f
kernel <- function(A) {
    m <- dim(A)[[1]]
    n <- dim(A)[[2]]

    ## QR factorization and rank detection
    QR <- base::qr.default(A)
    r <- QR$rank
    ## cases 2 to 4
    if (r == 0) {
        return(diag(x = 1, nrow = m))
    }
    if (r < min(m, n) || (m < n)) {
        R <- QR$qr[1:r, , drop = FALSE]
        P <- QR$pivot
        F <- R[, (r + 1):n, drop = FALSE]
        I <- diag(1, n - r)
        B <- -base::backsolve(R, F, r)
        Y <- rbind(B, I)
        X <- Y[order(P), , drop = FALSE]
        return(X)
    }
    ## case 1
    matrix(0, nrow = n, ncol = 1)
}
# Solve dx/dt = Ax for a matrix A, using
# standard factorization
solve_differential <- function(A) {
    lambda <- eigen(A, only.values = TRUE)[[1]]
    n <- nrow(A)
    S <- lapply(lambda, function(x) kernel(A -
        diag(x = x, n))) |>
        do.call(what = cbind)
```

```
    if (ncol(S) < n)
        stop("A cannot be diagonalized without using its Jordan form, everybody panic")
    Lambda <- diag(exp(lambda), nrow = n)
    S_inv <- solve(S)
    list(S = S, Lambda = Lambda, S_inv = S_inv)
}

compute_differential <- function(diff, x0, t) {
    diff$S %*% (diff$Lambda * exp(t)) %*% diff$S_inv %*%
        x0
}
```

42.

The relationship seems to be predator-prey. Only the eigenvectors stabilize in the long run, as is expected. The populations don't make much difference.

```
A <- matrix(c(1.4, 0.8, -1.2, -1.4), nrow = 2)
result <- solve_differential(A)
compute_differential(result, c(3, 1), 10)
```

```
          [,1]
[1,] 179622.43
[2,]  59874.14
```

45.

y kill 4 x for every member they lose, a 4:1 kill ratio.

```
A <- matrix(c(0, -1, -4, 0), nrow = 2)
result <- solve_differential(A)
combos <- expand.grid(x = 1:10, y = 1:10)
results <- combos |>
    as.matrix() |>
    asplit(MARGIN = 1) |>
    sapply(function(x) compute_differential(result,
        x, 30)) |>
    t()
results <- cbind(combos, results)
```

x seems to require about a 2:1 advantage to prevail.

46.

The x:y kill ratio is q/p. The general solution is:

$$\frac{1}{2\sqrt{p/q}}\begin{bmatrix}1 & 1\\ \sqrt{p/q} & -\sqrt{p/q}\end{bmatrix}\begin{bmatrix}e^{\sqrt{pq}t} & 0\\ 0 & e^{-\sqrt{pq}t}\end{bmatrix}\begin{bmatrix}-\sqrt{p/q} & -1\\ -\sqrt{p/q} & 1\end{bmatrix}x_0$$

51.

The derivative is a linear transformation, since $\frac{d}{dt}(x(t)+y(t)) = \frac{d}{dt}x(t) + \frac{d}{dt}y(t)$ and $\frac{d}{dt}kx(t) = k\frac{d}{dt}x(t)$.

This means S can be brought out of the derivative, since it is invariant with respect to the derivative, so $\frac{d}{dt}(Sx(t)) = S\frac{d}{dt}x(t)$.

52.

All are of the form $e^{\lambda t}x_0$.

53.

The solution works out to

$$x(t) = \frac{1}{2i}\begin{bmatrix} i(e^{(p+qi)t} + e^{(p-qi)t}) \\ e^{(p+qi)t} - e^{(p-qi)t} \end{bmatrix}$$

which is on the complex plane.