Section 3.3 Problems

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1.

The least squares solution to

$$
\begin{aligned}
3x &= 10 \\
4x &= 5
\end{aligned}
$$

is

```r
fit <- function(A) {
    (solve(t(A) %*% A) %*% t(A))
}

fitted <- c(3, 4) %*% fit(c(3, 4)) %*% c(10, 5)
t((c(10, 5) - fitted)) %*% c(3, 4)
```

```
     [,1]
[1,]    0
```

I check the solution is orthogonal, just to be safe.

2.

$D = 3$.

3.

The equation is:

$$E^2 = (u-1)^2 + (v-3)^2 + (u+v--4)^2$$

So:

$$
\begin{aligned}
&\frac{\partial f}{\partial u} = 2(v-3) + 2(u+v-4) \\
&= 2v + u - 7 \\
&\frac{\partial f}{\partial u} = 2(u-1) + 2(u+v-4) \\
&= 2u + v - 5
\end{aligned}
$$

Then just solve those two equations to get $(1, 3)$.

```
fit(rbind(diag(nrow = 2), c(1, 1))) %*% c(1, 3, 4)
```

```
     [,1]
[1,]    1
[2,]    3
```

b happens to lie within the span of A here.

6.

```
A <- matrix(c(1, 1, -2, 1, -1, 4), nrow = 3)
parallel <- A %*% fit(A) %*% c(1, 2, 7)
c(1, 2, 7) - parallel
```

```
          [,1]
[1,] -1.090909
[2,]  3.272727
[3,]  1.090909
```

7.

Find a projection matrix:

```
A <- cbind(c(1, 1, -2), c(1, -1, 4))
A %*% solve(t(A) %*% A) %*% t(A)
```

```
           [,1]       [,2]        [,3]
[1,] 0.90909091  0.2727273  0.09090909
[2,] 0.27272727  0.1818182 -0.27272727
[3,] 0.09090909 -0.2727273  0.90909091
```

8.

If P projects onto the k-dimensional subspace, then P's image is k, and its rank is the dimension of k.

9.

IF $P = P^T P$, then P is a projection matrix. Projection matrices are both symmetric and respect unit length, so $P^T P = P^2 = P$.

b. $P = 0$ projects into the kernel of the transpose, since 0_n (from the right-hand matrix) resides there.

10.

Say v, w, and b are orthogonal, then $A^T A = I_m$ and $A^T b$ maps b onto A's column space.

11.

Say P projects onto S and Q onto $S^{\perp}$. Then $P+Q=I$ because every vector consists of $Px+Qx$, so $(P+Q)x=x$ $PQ=0$ because $Px \cdot Qx = 0$. Then

$$\begin{aligned}
&(P-Q)^2 = I \\
&P^2 - QP - PQ + Q^2 = I \\
&P + Q = I \\
&I = I
\end{aligned}$$

12.

The kernel of the transpose is

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

```r
A <- cbind(c(-1, 1, 0, 0), c(-1, 0, 0, 1))
A %*% fit(A)
```

```
           [,1]       [,2] [,3]       [,4]
[1,]  0.6666667 -0.3333333    0 -0.3333333
[2,] -0.3333333  0.6666667    0 -0.3333333
[3,]  0.0000000  0.0000000    0  0.0000000
[4,] -0.3333333 -0.3333333    0  0.6666667
```

Since all vectors in V and $V^{\perp}$ are orthogonal, the projection of a vector in one onto the other is 0.

14.

One such vector is

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

15.

Show the reflection matrix R is involutory:

$$\begin{aligned}
R^2 &= (I - 2P)^2 \\
&= I^2 + 4P^2 - 2PI - 2IP + I^2 \\
&= I^2 \\
&= I
\end{aligned}$$

16.

Show $P = uu^T$.

Symmetry is obvious. For idempotence, consider the first element of P^2:

$$
\begin{aligned}
&= (u_1^2)^2 + (u_1 + u_2)^2 \\
&= u_1^2(u_1^2 + u_2^2) \\
&= u_1^2
\end{aligned}
$$

17.

That matrix is $0.5, -0.5, -0.5, 0.5$

18.

I wind up with the system:

$$
\begin{aligned}
&4c - +3t + 5z = 14 \\
&2c + 3t + 2z = 8 \\
&3c + 3t + 3z = 14
\end{aligned}
$$

19.

The row space projection is of course $A^T(AA^T)^{-1}A$

20.

Since they are orthogonal complements, $I - A(A^TA)^{-1}A^T$

21.

22.

Algebra gives a solution of $x_1 = 2, x_2 = -1$, yielding points $(2, 2, 0)$ and $2, 0, 4$.

23.

The best fit to a constant function is the average. Then $||\hat{x} - x|| = ||\bar{x} - x|| = \sqrt{\bar{x} - x}$, which is the exact definition of the residual sum of squares.

24.

Quadratic fit:

```r
A <- matrix(c(1, -1, 1, 1, 0, 0, 1, 1, 1, 1, 2, 4),
    nrow = 4)
fit(A) %*% c(2, 0, -3, -5)
```

```
              [,1]
[1,]  1.000000e+00
[2,] -5.000000e+00
[3,]  4.440892e-16
```

The equation is $1 - 5t$; the coefficient on t^2 is barely significant.

27

a.

$$
\begin{aligned}
&a^T a\hat{x} = a^T b \\
&n\hat{x} = \sum b \\
&x = \frac{\sum b}{n} \\
&x = \bar{b}
\end{aligned}
$$

b.

The error is the centered vector, the variance $(b - a\hat{x})(b - a\hat{x})^T$, the standard deviation the square root of this.

c.

All works as it should.

```r
a <- c(1, 2, 6)
fitted <- a %*% fit(a) %*% c(1, 2, 6)
crossprod(a - fitted, c(1, 2, 6))
```

```
              [,1]
[1,] -5.329071e-15
```

28.

$$
\begin{aligned}
&A^T A^{-1} A^T (b - Ax) \\
&A^T A^{-1} A^T b - A^T A^{-1} A^T Ax \\
&x - x \\
&0
\end{aligned}
$$

30.

$(b - \bar{b})(b - \bar{b})^T/4$

31.

9/10.

37.

A proof:

$$
\begin{aligned}
&A^T A x = A^T b \\
&n x_1 + x_2 \sum_{i=1}^{n} t = \sum_{i=1}^{n} b \\
&x_1 + x_2 \hat{t} = \hat{b} \\
&x_1 = \hat{b} - x_2 \hat{t}
\end{aligned}
$$

38.

$$
\begin{aligned}
\hat{x}_w &= \frac{w_1^2 b_1 + w_2^2 b_2}{w_1^2 + w_2^2} \\
&= \frac{w_2^2 b_2}{w_2^2} \\
&= b_2
\end{aligned}
$$

39.

$$
\frac{\sum_{i=1}^{m} w_i^2 b_i}{\sum_{i=1}^{m} w_i^2}
$$

40.

Respectively 11 and 5. The perpendicular line is given by $(1, -4)$.

41.

Weighted least squares!

```r
W <- diag(x = c(2, 1, 0))
A <- cbind(1, c(0, 1, 2))
b <- c(0, 1, 1)
x_w <- solve(t(A) %*% W^2 %*% A) %*% t(A) %*% W^2 %*%
    b
fitted <- A %*% x_w
t(b - fitted) %*% W^2 %*% b
```

```
     [,1]
[1,]    0
```

42.

Since the expectation is 0, just square to get the variance.

```
e <- c(-2, -1, 5)
probs <- c(0.5, 0.25, 0.25)
crossprod(e, probs)
```

```
     [,1]
[1,]    0
```

```
crossprod(e, e)/3
```

```
     [,1]
[1,]   10
```

```
e_2 <- c(-1, 0, 1)
crossprod(e_2, e_2)/3
```

```
          [,1]
[1,] 0.6666667
```

The inverses of the variances, $(1/10, 3/2)$