Section 8.2 Problems

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1.

$$\begin{bmatrix} 6 & -7/2 \\ -7/2 & 8 \end{bmatrix}$$

3.

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & 7/2 \\ 3 & 7/2 & 7 \end{bmatrix}$$

4.

Positive definite.

5.

Negative definite.

7.

Indefinite.

8.

$\mathbf{A}^2$ is positive semidefinite because all its eigenvalues are 0 or positive. IF $\mathbf{A}$ is positive definite, so is $\mathbf{A}^2$ by that same token.

9.

a.

Symmetric. $(\mathbf{A^T})^2 = (-\mathbf{A})^2 = \mathbf{A}^2$.

b.

It is negative semi-definite because the eigenvalues of $\mathbf{A}$ are either 0 or purely imaginary:

$$\mathbf{A}\mathbf{X} = \lambda x$$
$$x^{\mathbf{H}}\mathbf{A}^{\mathbf{H}} = \bar{\lambda}x^{\mathbf{H}}$$
$$-x^{\mathbf{H}}\mathbf{A} = \bar{\lambda}x^{\mathbf{H}}$$
$$-\bar{\lambda}x^{\mathbf{H}} = \bar{\lambda}x^{\mathbf{H}}$$
$$-\bar{\lambda} = \bar{\lambda}$$

so the eigenvalues of $\mathbf{A}^2$ are either 0 or real positive.

c.

None, since there are imaginary eigenvalues.

10.

$$\mathbf{T}(\mathbf{X}) = (x^{\mathbf{T}}\mathbf{A} + v^{\mathbf{T}}\mathbf{A})(x + v) - x^{\mathbf{T}}\mathbf{A}x - v^{\mathbf{T}}v$$
$$= x^{t}\mathbf{A}x + v^{\mathbf{T}}ax + x^{\mathbf{T}}\mathbf{A}v + v^{\mathbf{T}}\mathbf{A}v - x^{\mathbf{T}}\mathbf{A}x - v^{\mathbf{T}}\mathbf{A}v$$
$$= v^{\mathbf{T}}\mathbf{A}x + x^{\mathbf{T}}\mathbf{A}v$$
$$= 2x^{\mathbf{T}}\mathbf{A}v$$

11.

The same - the eigenvalues don't change sign.

12.

$\det \mathbf{A} < 0$ if and only if the eigenvalues have opposite signs. Thus $\lambda_1 c_1^2 + \lambda_2 c_2^2$ is indefinite because of the differing signs.

13.

Let d be a a diagonal emenet less than or equal to zero. Then the term dx_i^2 is zero or negative definite, and so the matrix cannot be positive definite.

14.

Since $\det \mathbf{A} > 0$ and $a > 0$, then if $c \leq 0$ $ac \leq b^2$, which violates the constraint. And if $\det \mathbf{A} > 0$ then all eigenvalues are positive, so $\mathbf{A}$ is positive definite.

23.

No, since $\mathbf{M}$ is not symmetric.

24.

a.

25.

λ, since the unit eigenvectors eigenbasis coordiantes are 1.

26.

True. Such a vector satisifeis $\mathbf{A}x = \lambda x$ for $\lambda = 0$. Then $\mathbf{A}$ has zero determinant and is therefore not invertible.

27.

A hyperellipse whose axes are the unit eigenvectors.

30.

$\mathbf{A} = (\mathbf{Q}\sqrt{\Lambda}\mathbf{Q^T})^2$

If amatrix is positive definite, so is its square root, since the eigenvalues remain positive.

32.

for the 2×2

Solving the system gets su

$$x = \sqrt{a}$$
$$y = b/\sqrt{a}$$
$$z = \sqrt{c - b^2/a}$$

33.

$$\mathbf{L} = \begin{bmatrix} \sqrt{8} & -2/\sqrt{8} \\ -2/\sqrt{8} & \sqrt{5} + 1/2 \end{bmatrix}$$

34.

If $\mathbf{A}$ is positive definite, then it sfull quadratic form is positive definie. For a 3×3, the form of the first submatrix is ax_1^2. This must be positive definite, because if not then the form of the second leading submatrix $ax_1^2 + bx_1x_2 + cx_2^2$ would not be positive definite, and so forth.

And if all the elading submatrices have psoitive definite, then they have all positive eigenvalues, meaning they have positive determinant.

Then for the final decomposition, to find x we ahve to solve the system:

$$a_n 1 = x_1 b_{11}$$
$$a_n 2 = x_1 b_{12} + x_2 b_{22}$$
$$\vdots a_{nn-1} = x^{\mathbf{T}} bn - 1$$

which can clearly be solved in sequence, with the last element of x ignored because $\mathbf{B}$ is triangular with dimension $n - 1$. The final equation is

$$a_{nn} = x^{\mathbf{T}} x + t$$

and t must be positive to split it into $\sqrt{t}$ across $\mathbf{L}$ and $\mathbf{L}^{\mathbf{T}}$.

36.

```r
library(matador)
```

```
Loading required package: magrittr
```

```
Attaching package: 'magrittr'
```

```
The following object is masked from 'package:purrr':

    set_names
```

```
The following object is masked from 'package:tidyr':

    extract
```

```r
to_solve <- cholesky <- matrix(0, nrow = 3, ncol = 3)
A <- matrix(c(4, -4, 8, -4, 13, 1, 8, 1, 26), nrow = 3)
u <- sqrt(A[1, 1])
v <- A[1, 2] / u
w <- sqrt(A[2, 2] - v^2)
x <- A[3, 1] / u
y <- (A[2, 3] - x * v) / w
z <- sqrt(A[3, 3] - x^2 - y^2)
to_solve[lower.tri(to_solve, diag = TRUE)] <- letters[seq(length(letters) - 5, length(letters))]
cholesky[lower.tri(cholesky, diag = TRUE)] <- c(u, v, x, w, y, z)
cholesky %*% t(cholesky)
```

```
     [,1] [,2] [,3]
[1,]    4   -4    8
[2,]   -4   13    1
[3,]    8    1   26
```

```r
A
```

```
     [,1] [,2] [,3]
[1,]    4   -4    8
[2,]   -4   13    1
[3,]    8    1   26
```

36.

$$\mathbf{A} = \mathbf{LL^T} = \mathbf{QR} \implies \mathbf{L} = \mathbf{QRL^T} {-1}$$

37.

These two criteria are another way of saying that the derivative matrix is positive definite. In that case, the product of the separate derivatives is greater than the derivative of the products, which is a condition of the test for a global minimum

38.

$p - q > 0$.

40.

If λ_k has the greatest absolute value of the negative-signed eigenvalues, $k > \lambda_k$ does the job.

41.

   3.

42.

$\frac{n(n+1)}{n}$

43.

Yes. Image is degree-0 polynomials, kernel all matrices with $a = 0$, rank 1, nullity 3.

44.

Yes, isomorphic since it preserves all three coordinates.

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

45.

$\mathbf{T}(x) = (a + 2c + +2e + f)x_1^2 + (b + d + e)x_2^2$ Image: all degree-1 polynomials Kernel: quadratic forms where only x_3^2 has a nonzero coefficient Rank 2, nullity 4.

46.

$\mathbf{T}(x) = 4ax_1^2 + x_2^2 + 4x_1x_2$. Image: al qaudratic forms

52.

Consider a 3×3.

$$a = e_1^t a_1 e = 1(1)a = a$$

and

$$b = 1/2((e_1 + e_2)^{\mathbf{T}}\mathbf{A}(e_1 + e_2)^{\mathbf{T}} - e_1^{\mathbf{T}}\mathbf{A}e_1 - e_2^{\mathbf{T}}\mathbf{A}e_2)$$
$$= 1/2(e_1^{\mathbf{T}}\mathbf{A}e_1 + e_2^{\mathbf{T}}\mathbf{A}e_2 + e_1^{\mathbf{T}}\mathbf{A}e_2 + e_2^{\mathbf{T}}\mathbf{A}e_1 - e_1^{\mathbf{T}}\mathbf{A}e_1 - e_2^{\mathbf{T}}\mathbf{A}e_2)$$
$$= 1/2(e_1^{\mathbf{T}}\mathbf{A}e_2 + e_2^{\mathbf{T}}\mathbf{A}e_1)$$
$$= 1/2(b + b)$$
$$= b$$

51.

Alternating odd, even.

53.

a.

$$p(x) = x^2 a_{ii} + 2xy a_{ij} + y^2 a_{jj}$$

b.

If so, then a_{ii} and a_{jj} are positive and $a_{ii}a_{jj} > a_{ij}^2$, which mean the matrix of p ahs positive eigenvalues and is positive definite.

c.

Then $2_{aji} = a_{jj}a_{ii}$, which translates to a zeroeigenvalue and an indeterminate matrix.

54.

They must be 0; otherwise the final determinant is negative, so some eigenvalues are negative and the matrix is not positive semidefinite.

55.

If this were not true the determinant would be negative, preventing he matrix from being positive definite.

56.

A symmetric $\mathbf{A}$ may be factorized $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{\mathbf{T}}$, where $\mathbf{Q}$ is the orthogonal eigenvectors matrix. Then $a_1 1 = \lambda_1 x_1^2 + \cdots + \lambda_n z_1$. $x_i^2 \leq 1$., (where $x, \ldots z$ are the eigenvectors). Since these are unit vectors., this quantity is maximized if the first column of $\mathbf{Q}^{\mathbf{T}}$ is a unit vector. If so, then $a_1 1 = 1\lambda_i = \lambda_i$. This is a quadratic form with the eigenvalues as the contants. So a is maximally λ_{max}. Since $a = \Lambda q_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}$, by Cauchy-Swarz

$$|\Lambda q_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}| \leq ||\Lambda 1_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}|| \cdot ||q_1^{\mathbf{T}}||$$
$$\leq ||\Lambda 1_1^{\mathbf{T}} \cdot q_1^{\mathbf{T}}||$$
$$\leq \lambda_{max}$$

so a cannot be greater than the greatest λ_i.

58.

A spheroid around the origin.

58.

An ellipse around the origin

59.

A line.

60.

Concave surfaces oriented positively.

61.

Concave surfaces oriented negatively.

63.

The expression is eigenbasis coordinates divided by the appropriate lambdas, which cancel.

$$q(c_1 w_1 + \cdots + c_n w_n) = \lambda_1 c_1^2 / \lambda_1 + \cdots + \lambda_n c_n^2 / \lambda_n = c_1^2 + \cdots + c_n^2$$

64.

```
eigens <- eigen(matrix(c(8, -2, -2, 5), nrow = 2))
eigens$vectors / sqrt(eigens$values)


          [,1]       [,2]
[1,] -0.2981424 -0.1490712
[2,]  0.2236068 -0.4472136
```

65.

For the negative-signed λ

$$c_i^2 \frac{v_i}{\sqrt{|\lambda_i|}}^2 = \lambda_i c_i^2/|lambda_i| = -c_i^2$$

66.

```
eigens <- eigen(matrix(c(3, -5, -5, 3), nrow = 2))
eigens$vectors[, order(eigens$values, decreasing = TRUE)] * 1 / sqrt(abs(sort(eigens$values, decreasing
```

```
       [,1]  [,2]
[1,] -0.25 -0.25
[2,]  0.50 -0.50
```

67.

By the above logic, if we scale the eigenvectors with negative-signed eigenvalues by $1/\sqrt{|\lambda_i|}$, c_i^2 is opposite-signed.

68.

$\mathbf{R^T A R}$.

69.

$\mathbf{R}$ has rank m.