~~Cat~~ Lynx and ~~Mouse~~ Hare Games

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Abstract

The Canadian lynx and Snowshoe hare pelt data collected by the Hudson Bay Company (HBC) for over 100 years in the 19^{th} and 20^{th} centuries is a classic example used to study predator-prey systems in ecology and continuous modeling. It is assumed that the pelt data collection is an appropriate proxy and should be proportionate to the true population of the Canadian lynx and Snowshoe hare, therefore any conclusions made on the pelt data could be extended to the true lynx and hare population. We use widely accepted data from MacLulich (1937) and create lines of best fit using the Lotka-Volterra equations describing predator-prey systems. It is argued that the solutions to the Lotka-Volterra equations are periodic around an equilibrium, that a line of best fit to segments of the raw HBC data that are themselves similarly periodic to the Lotka-Volterra equations would be give reasonable estimates to the true parameters governing the lynx and hare relationship. The optimal parameters of the model are derived by minimizing the sum of square error between the HBC data and the Lotka-Volterra equations. The model is then tested by attempting to replicate trends seen in the data. In conclusion the model seems to reflect specific behaviors seen in the data set to a similar scale, but fails to explain overall fluctuations between maxima. Real world parameters are explored that could explain the deficiencies found in the best fit model developed, such as environmental capacity, disease, and climate oscillations.

1 Introduction

The Hudson Bay Company is one of the oldest companies in the English world. The company was founded in 1670 in response to the high demand of beaver pelts in England. French fur traders Pierre-Esprit Radisson and Médard Chouart des Groseillers convinced the English royalty to back a charter that would allow them to traverse the rivers of Hudson Bay, gaining access to the depths of the North American continent.

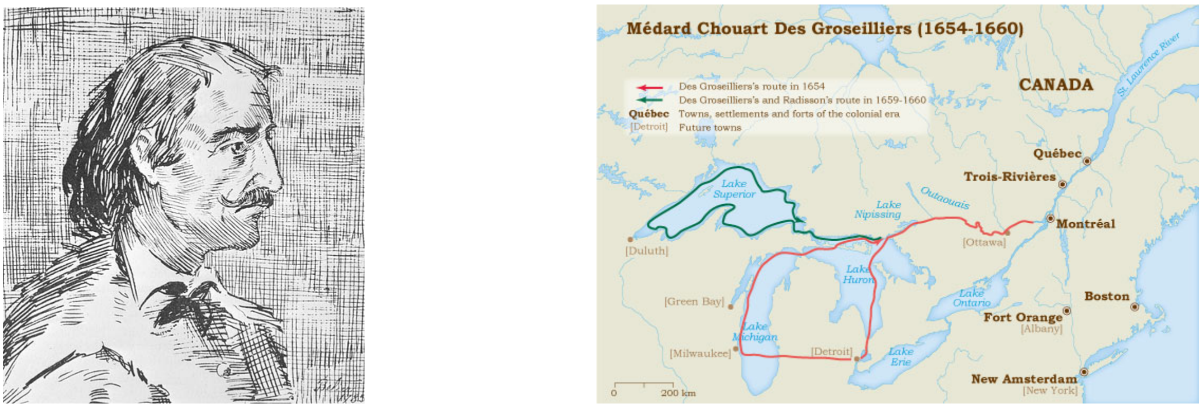

Figure 1: Pierre-Esprit Raddison and the Hudson Bay

Among other animal pelts, the Hudson Bay Company traded Canadian lynx and snowshoe hare. Most predators have a multitude of prey. This is true of the Canadian Lynx, which eat mice, squirrels, and birds. But above all, the Canadian Lynx eats the snowshoe hare (National Geographic, 2015). This fact makes the trappers data very useful for studying predator-prey systems.

The variation in the lynx-hare data is widely spread and consistent throughout the northern parts of North America. The variation is so consistent that researchers have suggested large scale meteorologic factors such as suns spots or El Niño climate cycles could be contributing to the population variations along with self-regulation and the classic predator-prey systems (Zhang et al. 2007).

It is important to recognize that this is data of pelt collection, not the total population of lynx and hare. The pelt collection data is meant to represent some proportion of the true population of lynx and hare, and therefore will vary proportionally to the true population. In this way if we can understand the rate of change of pelt collection, we may be able to understand the rate of change of the populations of Canadian lynx and Snowshoe hares.

2 General Predator and Prey Models

Figure 2: The Canadian Lynx and Snowshoe Hare

Vito Volterra was a famous mathematician who lived from 1840-1940. He first wrote the differential equations modeling predator-prey systems after his son-in-law approached him with a problem involving local fish and shark populations (Tung, K. 2007). Alfred Lotka (1880-1949) also independently developed a cyclic model to predator-prey systems during the same time period. The general equations are now known as the Lotka-Volterra equations:

$$\frac{d}{dt}x = rx - axy \tag{1}$$

$$\frac{d}{dt}y = -ky + bxy \tag{2}$$

$x(t)$ represents the population of the prey, which in this paper will be the snowshoe hare. The increase in population of hares (i.e. births) is proportional to the population of hares, with a constant of proportionality of r. A decrease in the population of the hares (i.e. death) is proportional to the product of the population of hares and lynx, representing a combination of natural death and predation. This constant of proportionality is a in equation 1.

$y(t)$ represents the population of the predator, the lynx. The population of lynx increases proportionally to the product of the lynx population and the hare population. This is because the hare population provides the lynx with the nutrition necessary to reproduce. This constant of proportionality is represented as b in equation 2. The death rate of the lynx population is assumed to be proportional to the lynx population (i.e. natural death) alone, and is represented by k in equation 2.

The models make some assumptions that may not be true in reality. Namely, we will assume that the only cause of death in both populations (predator-prey, lynx-hare) are caused by either natural death in the case of the lynx, or natural death and predation by lynx in the case of the hare. This, of course, is not the reality of the situation, but is assumed for the model.

In the spirit of using the Lotka-Volterra equations to model the predator-prey realtionship betweent the Canadian lynx and Snowshoe hare, we will call the population of hare at $time = t$, $h(t)$. The birth constant of proportionality is now a_1. The hare death rate constant of proportionality is now a_2.

The population of lynx at $time = t$ is now defined as $l(t)$. The constant of proportionality for the birth rate of lynx is now b_2. The death rate constant of proportionality for lynx is now referred to as b_1.

Our first Lotka-Volterra equations now becomes:

$$\frac{d}{dt}h = a_1 h(t) - a_2 h(t) l(t) \tag{3}$$

$$\frac{d}{dt}l = -b_1 l(t) + b_2 h(t) l(t) \tag{4}$$

We now have a system of ordinary differential equations. To interpret their behavior, first we find their equilibria.

2.1 Equilibria Points

To find the equilibria $(h^\star, l^\star)$ points for $\frac{d}{dt}h$ and $\frac{d}{dt}l$, we set both equations equal to zero and solve.

$$\tfrac{d}{dt}h(t) = 0 \quad \text{and} \quad \tfrac{d}{dt}l(t) = 0$$

Or written as;

$$a_1 h^\star - a_2 h^\star l^\star = 0 = h^\star(a_1 - a_2 l^\star)$$

$$-b_1 l(t) + b_2 h(t) l(t) = 0 = l^\star(-b_1 + b_2 h^\star)$$

Two possible equilibria are derived from these equations. The first one is;

$$(h_1^\star, l_1^\star) = (0, 0) \tag{5}$$

This is the equilibrium that signifies the extinction of both species.

The second equilibrium is;

$$(h_2^\star, l_2^\star) = (\frac{b_1}{b_2}, \frac{a_1}{a_2}) \tag{6}$$

2.2 Equilibria stability

To determine the stability of the two equilibria points we will first linearize the nonlinear equations about the equilibria solutions.

$$\frac{d}{dt}h = h(t)(a_1 - a_2 l(t)) = F_1(h, l)$$

$$\frac{d}{dt}l = l(t)(-b_1 + b_2 h(t)) = F_2(h, l)$$

changing variables gives;

$$H(t) = h(t) - h^\star$$

$$L(t) = l(t) - l^\star$$

We then take first order approximations of F_1 and F_2 we get the following system of linear differential equations for $H(t)$ and $L(t)$;

$$\frac{d}{dt}\begin{bmatrix} H(t) \\ L(t) \end{bmatrix} = J(h^\star, l^\star)\begin{bmatrix} H(t) \\ L(t) \end{bmatrix}$$

where the Jacobian matrix at $(h^\star, l^\star)$ is;

$$J(h^\star, l^\star) = \begin{bmatrix} \frac{\partial F_1(h^\star,l^\star)}{\partial H} & \frac{\partial F_1(h^\star,l^\star)}{\partial L} \\ \frac{\partial F_2(h^\star,l^\star)}{\partial H} & \frac{\partial F_2(h^\star,l^\star)}{\partial L} \end{bmatrix} = \begin{bmatrix} a_1 - a_2 l^\star & -a_2 h^\star \\ b_2 l^\star & -b_1 + b_2 h^\star \end{bmatrix}$$

At the first equilibrium $(h^\star, l^\star) = (0, 0)$

$$J(0, 0) = \begin{bmatrix} a_1 & 0 \\ 0 & -b_1 \end{bmatrix}$$

J(0,0) has eigenvalues and eigenvectors;

$$\lambda_1 = a_1,\ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad and \quad \lambda_2 = -b_1,\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because both eigenvalues are real and of opposite signs ($\lambda_1 > 0$ and $\lambda_2 < 0$) we know that this equilibrium point is an unstable saddle node. Small perturbations result in solutions that decay along the predator axis, or grow along the prey axis. This is easily conceptualized in the case of lynx and hare. If there are no lynx, the hare population will grow quickly. If there are no hare, the lynx population would die off quickly.

At the second equilibrium $(h^\star, l^\star) = (\frac{b1}{b2}, \frac{a1}{a2})$

$$J(\frac{b1}{b2}, \frac{a1}{a2}) = \begin{bmatrix} 0 & \frac{-a_2 b_1}{b_2} \\ \frac{b_2 a_1}{a_2} & 0 \end{bmatrix} \tag{A}$$

Where the trace of $J(h^\star, l^\star)$ is;

$$p = T_r(J(h^\star, l^\star)) = 0$$

and the $\det(J(h^\star, l^\star))$ is;

$$q = det(J(h^\star, l^\star)) = a_1 b_1$$

The eigenvalues $\lambda_{1,2}$ can be expressed as;

$$\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

therefore;

$$\lambda_{1,2} = \pm i\sqrt{a_1 b_1}$$

Because $q > \frac{p^2}{4}$ and $q = 0$ we have a center. This is indicative of a system that is periodic in nature and in fact is recognized as the harmonic oscillator with linear solution near the equilibrium;

$$h(t) = c_1 \cos(\sqrt{a_1 b_1} t) + c_2 \sin(\sqrt{a_1 b_1} t) + h^\star \tag{7}$$

and

$$l(t) = \frac{b_2}{a_2}\sqrt{\frac{a_1}{b_1}}[c_1 \sin(\sqrt{a_1 b_1} t) - c_2 \cos(\sqrt{a_1 b_1} t)] + l^\star \tag{8}$$

Because our solutions are periodic, we can define;

$$h_{average} = \frac{1}{T}\int_{t_0}^{t_0+T} h(t)dt,$$

$$l_{average} = \frac{1}{T}\int_{t_0}^{t_0+T} l(t)dt,$$

These integrals are solvable and yield the satisfying solutions;

$$h^\star = h_{average} = \frac{b_1}{b_2} \tag{9}$$

$$l^\star = l_{average} = \frac{a_1}{a_2} \tag{10}$$

The solutions to the integrals provide the coordinates to the equilibrium point around which the populations cycle.

3 Lynx and Hare Data and Best Fit

Let's introduce the data from the Hudson Bay Company.

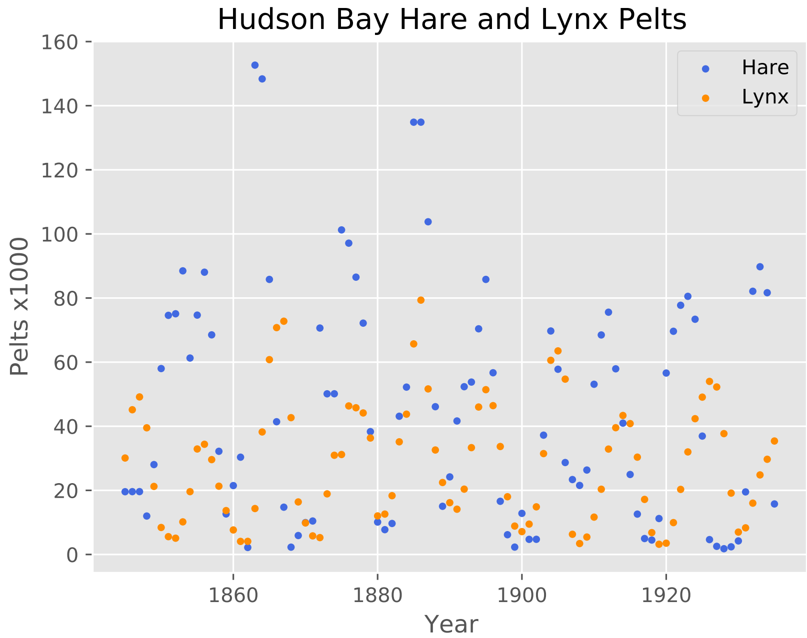

Figure 3: Hudson Bay Pelt Data

The above raw data from the Hudson Bay Company was taken from MacLulich(1937). This data is the number of pelts collected by the company from 1845 through 1935. Many trappers in the area would trap lynx and hare, and sell the pelts to the Hudson Bay Company (HBC). HBC would then take the pelts and sell them abroad. It is worth noting that the data was not collected uniformly from 1845-1935. Before 1903 the data was collected by counting the number of pelts traded by the HBC. After 1903, the data was collected by questionnaires and surveys (Zhang et al. 2007). Different researchers have approached the problem using different subsets of the data (Odum at al. 1971, Zhang et al. 2007). The difference in data collection leads to an interesting smoothing effect of the data and will be utilized in our best fit approach.

At first glance there may not seem to be much going on in this data. Let's make things a bit more clear by connecting the dots and drawing some lines.

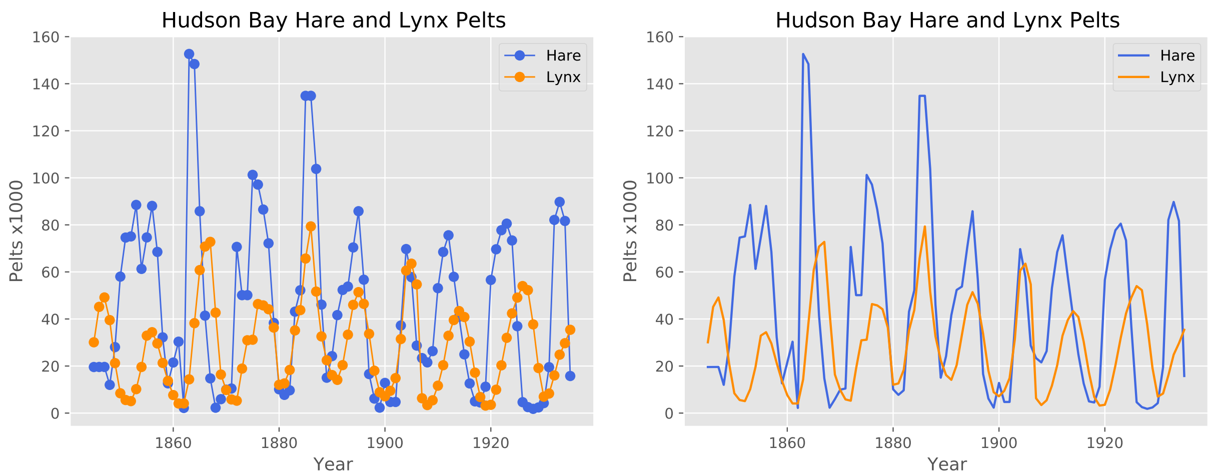

Figure 4: Hare and Lynx Pelt Data Presented Two Ways

Connecting the dots in the figure 3 we can see there is an overall periodic or cyclic nature to the data, although it is more clear in certain time periods than others. In general it seems that the hare pelts maximum precedes the lynx pelt maximum by a semi-constant phase shift, and is larger by a semi-constant scale.

3.1 Equation Parameters and Best Fit

The Lotka-Volterra coupled differential equations produce solutions that are periodic about an equilibrium point. Because of this we reason the portions of the HBC data that most resemble periodic motion are best described by the Lotka-Volterra couple equations, and best fit curves could give insights into the true equilibrium point of the lynx-hare predator-prey relationship.

Looking at the data in figure 4, we can see that the time period shortly after the turn of the 20^{th} century (1908-1935) seems to best resemble a periodic motion. Therefore we will attempt to fit a line to the data during the time period of 1908-1935.

We will attempt to create the best fit line to the HBC data using the Lotka-Volterra equations for predator-prey systems. To do this, we must first determine reasonable estimates to the parameters ($a_{1,est}$, $a_{2,est}$, $b_{1,est}$, $b_{2,est}$) of equations 3 and 4. Using these reasonable estimates as a starting point, we minimize the sum of square error between the observed data from HBC and the numerically derived curves created by the Lotka-Volterra coupled equations by using a downhill simplex algorithm (scipy.optimize.fmin).

To start, we remember that $a_{1,est}$ is a measure of the reproductive rate of hares, without influence of predation. This is assumed to be simple exponential growth. It is then reasonable to assume that $a_{1,est}$ is most clearly seen when the the lynx population $l(t)$ is low. From figure 4 we can see that the Lynx population is low around the year 1910. Therefore we choose the hare growth between 1910-1911 as a reasonable estimate of some proportion of the true intrinsic exponential growth rate of hare.

To find this estimate of the growth rate for the hare from year 1910 to 1911 we take the natural log of the pelts in 1910 divided by the pelts in 1911:

$$a_{1,est} = \ln\left(\frac{53.1}{26.34}\right) = .7011$$

We now use equations 9 and 10 to derive further parameter estimates. It reasons a periodic motion will have average values equal to the point around which it cycles. To this end we choose maxima for each population near the time period we wish to model (1908-1935) and determine the average lynx and hare pelt value between the chosen maxima. We chose 1904-1912 for the hares, and 1905-1914 for the lynx. Note that the maxima do not take place on the same years. This is consistent with predator-prey systems where the predator maxima is typically phase shifted in respect to the prey. From this we have;

$$h_{mean}(1904 - 1912) = 43.62625 = \frac{b_1}{b_2}$$

and

$$l_{mean}(1905 - 1914) = 26.42111 = \frac{a_1}{a_2}$$

From equations 7 and 8 we know that the frequency of the period of oscillation for the predator-prey system is;

$$\pm i\omega = \pm i\sqrt{a_1 b_1}$$

Therefore if we approximate the period from measuring and averaging the time between one set of maxima for the hare(1904-1912) and lynx(1905-1914) and cycles, we will have our final equation needed to estimate the parameters of the Lotka-Volterra equations.

$$T \approx \frac{(1912 - 1904) + (1914 - 1905)}{2} = 8.5\ years$$

and

$$a_1 b_1 = \omega^2 \approx .5464$$

Now we can solve for our estimates of the Lotka-Volterra equation parameters.

$$\begin{aligned} a_{1,est} &= .7011 \ \ (from\ above) \\ a_{2,est} &= \frac{a_{1,est}}{l_{mean}} = .0265 \\ b_{1,est} &= \frac{\omega^2}{a_{1,est}} = .7794 \\ b_{2,est} &= \frac{b_{1,est}}{h_{mean}} = .0179 \end{aligned}$$

Using parameters $a_{1,est}$, $a_{2,est}$, $b_{1,est}$, $b_{2,est}$ and initial population sizes given from the data for the year 1908 $(21.54, 3.41)$ $(in\ thousands)$, we can use the python library scipy.optimize.fmin to minimize the sum of square error between the coupled differential equations and the data. The code used to produce the minimized sum of square error is provided at the end of this paper.

The optimized parameters are found to be;

$$\begin{aligned} a_{1\,opt} &= 0.4837153 \\ a_{2\,opt} &= 0.02959469 \\ b_{1\,opt} &= 0.90424203 \\ b_{2\,opt} &= 0.02473212 \end{aligned} \tag{11}$$

Plugging these parameters into equations 3 and 4 we finally have the optimized Lotka-Volterra equations that will be used to model the relationship between the Canadian lynx and Snowshoe hare.

$$\frac{d}{dt}h = 0.4837 \times h(t) - 0.0296 \times h(t)l(t) \tag{12}$$

$$\frac{d}{dt}l = -0.9042l \times l(t) + 0.0247 \times h(t)l(t) \tag{13}$$

The optimized parameters found in this study are within reasonable estimates of other researchers who developed best fit lines (Mahaffy, J. 2018) using different programming methods and subsections of the data. Discrepancies can be easily understand if the time period to which a model is being fitted differs between researchers.

The optimized parameters result in the estimated equilibria;

$$h^{\star}_{est} = 36.56$$

and

$$l^{\star}_{est} = 16.34$$

4 Interpretation of the Prediction Model

We can now map the predicted Lotka-Volterra equations against the HBC data during the time period in question (1908-1935).

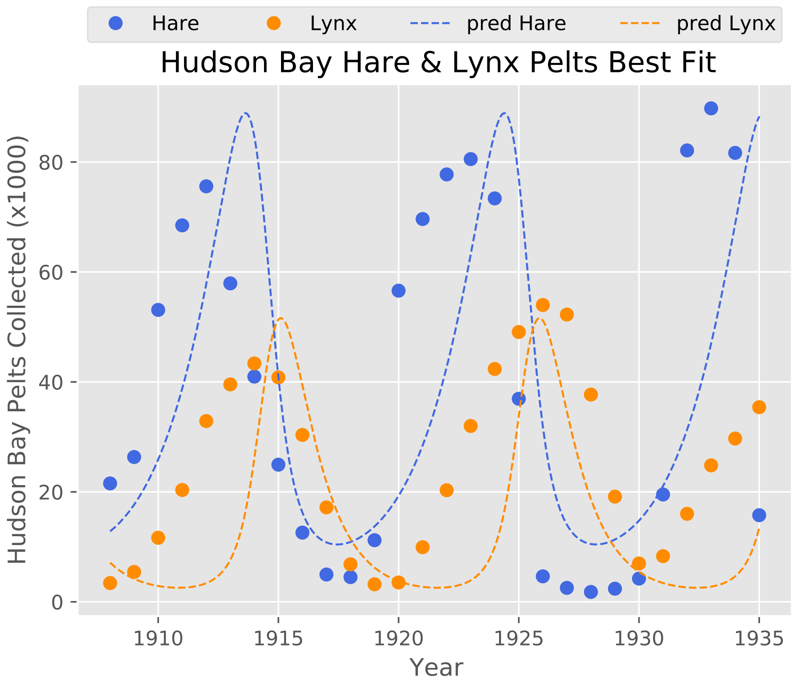

Figure 5: Best fit model overlapped with HBC data 1908-1935

Figure 5 shows obvious deviations from the data, but the overall fit during this time period seems to fit the data well with a sum of squared error = 11070. The minimized sum of square error is larger than the results from Mahaffy (2018) who minimized the error to under 600. This difference can be explained by the difference in time periods used to create the models. Mahaffy (2018) uses the time period of 1900-1920 while this research uses 1908-1935. Other considerations such as environment carrying capacity, climate, and animal harvesting by humans could lead to a more accurate model.

The period of our optimized equations is;

$$T = \frac{2\pi}{\sqrt{a_{1\,opt} b_{1\,opt}}} = 9.5\ years$$

The period obtained using our model is well within the estimated period of 8-10 years of the lynx and hare abundance cycles, noted by other published works (Stenseth et al. (1997), Leigh(1968), and Zhang et al.(2007)).

4.1 Harsh Winter Simulation

Taking another look at figure 4 we can see that there is a lot of variation in the data. This is to be expected, because in reality there are many other factors affecting the Canadian lynx and Snowshoe hare populations such as weather, disease, and even the fur trappers of the HBC.

In the years from 1861 to 1862 the hare pelts collected dropped from over 30,000 pelts to nearly 2,000 pelts, while the lynx pelts varied only slightly. This should indicate a severe and sudden drop in the hare population. Let's now simulate this using our model, and see if the model reacts in a way that matches the data.

A sudden drop in the hare population could be due to any of the aforementioned reasons, but let's suppose that the winter of 1861 was unusually warm, and resulted in a fast melting of the snow cover. The Snowshoe hare change the color of their fur in the winter months, from brown to snow white, so they can have adequate camouflage during the winter months. With the unusually warm weather and snow melt of the winter 1861, many hares were left with the wrong color camouflage, and as a result the lynx were wildly successful in their hunting, decimating the hare population throughout the winter. The hare population downturn is then proceeded by a fall in the lynx population, which is followed by a massive increase in the hare population in the year 1863. This increase marks the highest measure of hare population. This is classic behavior of a predator-prey system. The model created using the optimized parameters (11) should yield similar results.

To simulate such a situation, we will use equations 3 and 4 with the optimized parameters (11). We start the simulation using the same initial population levels given from the HBC data, but in year 1861 we will drastically reduce the hare population by the same ratio as seen in the HBC data, and observe the behavior of our model.

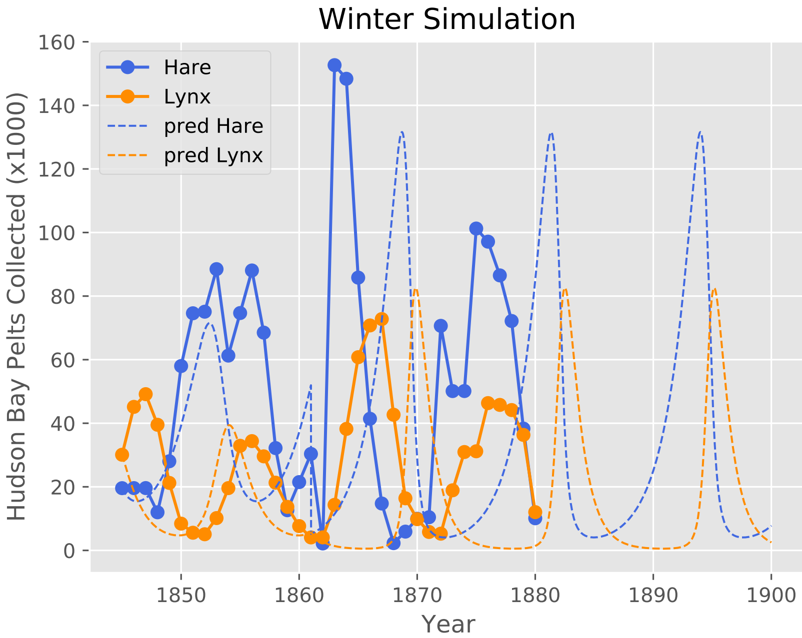

Figure 6: Simulation of a harsh winter for the hare population

As can be seen in figure 6 our model shows a fall in the lynx population after the winter of 1861, followed by a massive spike in the hare population. In this way, our model matches the data to a reasonable degree. Although, our model settled into a new periodic cycle with a higher peak, the real data showed a decline in both the hare and lynx populations after the year 1863. This is indicative of a population capacity (Tung, K. 2007), perhaps limited by other resources in the environment such as the vegetation available to the hares. This is not considered in the Lotka-Volterra equations used for our model.

4.2 Phase Diagram

Figure 7 is a phase diagram meant to show the cyclic relationship between the lynx and hare. The blue line shows the periodic cycle of the lynx and hare using our best fit Lotka-Volterra equations (equations 12 and 13). As we can see from Figure 7 the oscillation is about the equilibrium point $(h^{\star}, l^{\star}) = (36.56, 16.34)$ with a peak of around 90,000 hare pelts and 50,000 lynx pelts.

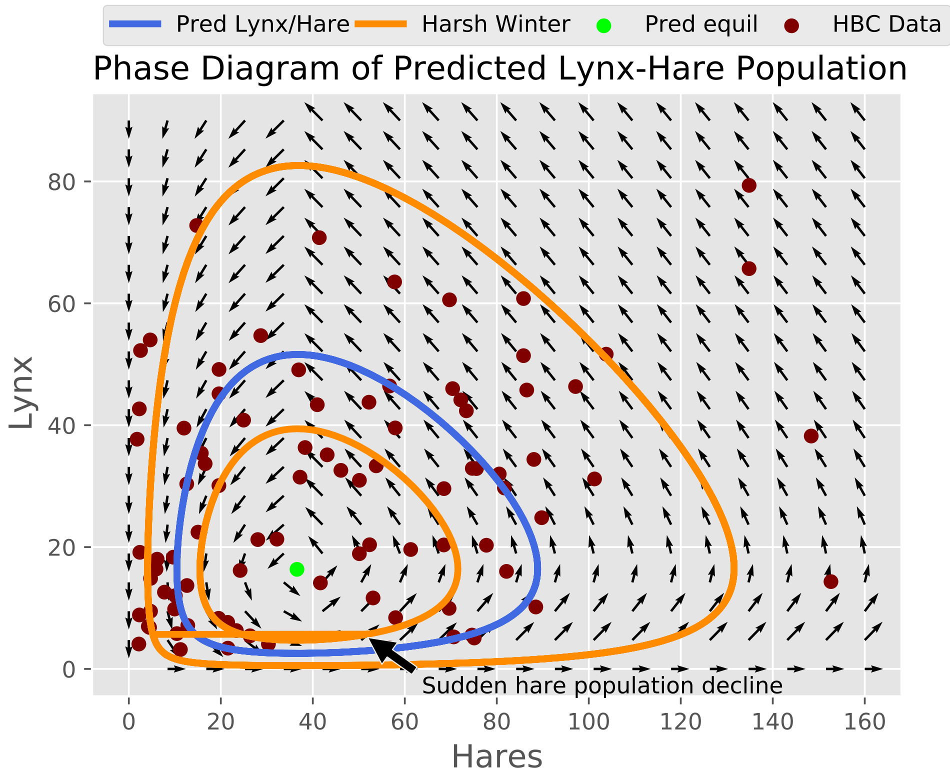

Figure 7: Phase Diagram

The orange line in Figure 7 is the Harsh winter simulation from the previous section. In the harsh winter simulation the lynx and hare populations have different starting points from the best fit equations, resulting in the smaller orange oscillation that lies inside the blue oscillation. After a few cycles, the hare population experiences the simulated winter of 1861 that ends up decimating their population while not affecting the lynx population. This decimation results in the nearly horizontal line along the bottom of the phase diagram (marked by the black arrow in Figure 7). In response to the sudden drop in hare population, the lynx population declines, resulting in a massive jump in the hare population. Our simulation predicts the hare and lynx populations to increase to a point where around 130,000 hare pelts and over 80,000 lynx pelts will be collected yearly. This is reasonably similar to the HBC data, where the pelt collection spike to around 150,000 for the hare and 70,000 for the lynx. From the HBC data we know that the hare and lynx populations settle back down into a more familiar range. This is not reflected in the model. Again, this may be the result of an environmental carrying capacity that is not considered in the Lotka-Volterra equations used to create our model.

5 Conclusion

The optimized parameters seemed to create a model that fits periodic motion of the data. The model also accurately simulated population response to dramatic population fluctuations. The final question is whether or not the equilibrium obtained from the best fit equations is an accurate estimate of the true equilibrium around which the lynx and hare populations cycle. Further research could be done to investigate whether this equilibrium is an accurate estimate by measuring the lynx and hare populations after sudden drops and spikes. If the equilibrium found in this paper is an accurate estimate of the true equilibrium, the population response after a large loss (as in the Harsh Winter simulation) should be close to those seen in the simulations.

The model created using best fit practices seems to be able to recreate the effects that a sudden change in hare population would have on the lynx-hare system to a reasonable degree of accuracy. The model fails to show the effect that an environment carrying capacity would have on the lynx-hare system. Namely, after a large spike in population, the model predicts a steady state with much higher maxima. This does not reflect reality or the HBC data. Future work involving this best fit practice could incorporate an environment carrying capacity and simulations such as the Harsh Winter simulation carried out in section 4.1 should be much more accurate.

Many other researchers have added parameters that better fit the data, while some others question the data altogether. Weinstein (1997) calls into question the speed in which the lynx population seems to react to the hare population, citing a the lynx reproduction and maturity rates would require the population increases to lag farther behind the hare population than the data shows. Weinstein suggests that the relationship has more to do with the trappers and their ability to hunt lynx more readily after years in which the rabbit hunts were successful.

References

[1] Canadian Encyclopedia, The. (2009, October 8) *Hudson's Bay Company* thecanadianencyclopedia.ca. https://www.thecanadianencyclopedia.ca/en/article/hudsons-bay-company

[2] Leigh ER (1968) *The ecological role of Volterra's equations.* American Mathematical Society, Providence, RI

[3] MacLulich DA (1937) *Fluctuation in the number of the varying hare.* Univ. of Toronto Press, Toronto

[4] Mahaffy, J. (2018) Mathematical Modeling-Continuous Models, Lotka-Volterra[Powerpoint slides] Retrieved from https://jmahaffy.sdsu.edu/courses/f17/math636/beamer/lotvol.pdf

[5] National Geographic *Lynx* nationalgeographic.com.
https://www.nationalgeographic.com/animals/mammals/facts/lynx

[6] Odum EP, Barrett GW (1971) *Fundamentals of ecology, vol 3.* Saunders, Philadelphia

[7] Stenseth NC, Falck W, Bjørnstad ON, Krebs CJ (1997) *Population regulation in snowshoe hare and Canadian lynx: asymmetric food web configurations between hare and lynx.* Proc Natl Acad Sci

[8] TUNG, K. (2007). *Interactions: Predator–Prey, Spraying of Pests, Carnivores in Australia. In Topics in Mathematical Modeling (pp. 161-190).* Princeton; Oxford: Princeton University Press

[9] Weinstein, Martin S. (1977) *Hares, Lynx, and Trappers* Source: The American Naturalist, Vol. 111, No. 980 (Jul. - Aug., 1977), pp. 806-808 Published by: The University of Chicago Press for The American Society of Naturalists

[10] Zhang, Z., Tao, Y., & Li, Z. (2007). *Factors affecting hare–lynx dynamics in the classic time series of the Hudson Bay Company, Canada. Climate Research, 34(2), 83-89.* `http://www.jstor.org/stable/24869389`

6 Code

The figures, optimizing of the Lotka-Volterra equations, and simulations were all run using the python code below:

```
    ## Predator Prey Models Lotka-Volterra Lynx-Hare

import numpy as np
import matplotlib.pyplot as plt
from scipy import integrate
from scipy import optimize
from scipy import linalg
import pandas as pd
import math
%matplotlib notebook

plt.style.use('ggplot')

## Lynx-Hare

Pred_prey_data = np.genfromtxt('hare_lynx_data.csv', delimiter=',', skip_header=1)

year = Pred_prey_data[:,0]
hare = Pred_prey_data[:,1]
lynx = Pred_prey_data[:,2]

fig0, ax0 = plt.subplots()

ax0.plot(year,hare, color="royalblue",label="Hare")
ax0.plot(year,lynx, color = "darkorange",label="Lynx")
ax0.set_xlabel("Year")
ax0.set_ylabel("Pelts x1000")
ax0.set_title("Hudson Bay Hare and Lynx Pelts")
ax0.legend()

fig00, ax00= plt.subplots()

ax00.scatter(year,hare,color = "royalblue", s=10, label="Hare")
ax00.scatter(year,lynx, color = "darkorange",s=10, label="Lynx")
ax00.set_xlabel("Year")
ax00.set_ylabel("Pelts x1000")
ax00.set_title("Hudson Bay Hare and Lynx Pelts")
ax00.legend()

fig000, ax000 = plt.subplots()

ax000.plot(year, hare,'-o',color = "royalblue", linewidth=1, label="Hare")
ax000.plot(year, lynx,'-o', color = "darkorange",linewidth=1, label="Lynx")
ax000.set_xlabel("Year")
```

```
ax000.set_ylabel("Pelts x1000")
ax000.set_title("Hudson Bay Hare and Lynx Pelts")
ax000.legend()

fig1, ax1 = plt.subplots()

ax1.plot(year[63:], hare[63:], '-o',color = "royalblue",label = "Hare")
ax1.plot(year[63:],lynx[63:],'-o',color = "darkorange", label="Lynx")

ax1.set_xlabel("Year")
ax1.set_ylabel("Pelts x1000")
ax1.set_title("Hudson Bay Hare and Lynx Pelts (1908-1935)")
ax1.legend()

fig10, ax10 = plt.subplots()

ax10.plot(Pred_prey_data[:36,0], Pred_prey_data[:36,1], '-o',color
= "royalblue",label = "Hare")
ax10.plot(Pred_prey_data[:36,0],Pred_prey_data[:36,2],'-o',color
= "darkorange", label="Lynx")

ax10.set_xlabel("Year")
ax10.set_ylabel("Pelts x1000")
ax10.set_title("Hudson Bay Hare and Lynx Pelts (1845-1880)")
ax10.legend()

## General Predator Prey

### Equilibrium positions

## fmin

year=Pred_prey_data[63:,0]
hare=Pred_prey_data[63:,1]
lynx=Pred_prey_data[63:,2]

year

def sumSquaredError(hare, lynx, a1, a2, b1,b2) :
    dt = .01; max_time = 1935

    # initial time and populations size
    t = 1908; x = 21.54; y = 3.41

    # empty lists
    time_list = []; prey_list = []; pred_list = []

    # initialize lists
```

```
    time_list.append(t); prey_list.append(x); pred_list.append(y)

    while t < max_time:
        # calc new values for t, x, y
        t = t + dt
        x = x + (a1*x - a2*x*y)*dt
        y = y + (-b1*y + b2*x*y)*dt

        # store new values in lists
        time_list.append(t)
        prey_list.append(x)
        pred_list.append(y)


    error = sum( (hare-prey_list[0:-1:100])**2 + (lynx-pred_list[0:-1:100])**2 )


    return error

sumSquare_adapter = lambda p: sumSquaredError(hare, lynx, p[0], p[1], p[2], p[3])

p = np.array([0.45372498, 0.02959179, 0.90423199, 0.0247327])#a1,a2,b1,b2

A5 = optimize.fmin(sumSquare_adapter, p)


print(A5)
print(A5[0]/A5[1],A5[2]/A5[3])

# model parameters
a1 = A5[0]; a2 = A5[1]; b1 = A5[2];  b2 = A5[3]
dt = .01; max_time = 1935

# initial time and populations
t = 1908; x = 12.82; y = 7.13

# empty lists
time_list = []; prey_list = []; pred_list = []

# place first value in lists
time_list.append(t); prey_list.append(x); pred_list.append(y)

while t < max_time:
    # calc new values for t, x, y
    t = t + dt
    x = x + (a1*x - a2*x*y)*dt
    y = y + (-b1*y + b2*x*y)*dt
```

```
    # store new values in lists
    time_list.append(t)
    prey_list.append(x)
    pred_list.append(y)

ODE_time=time_list
ODE_prey=prey_list
ODE_pred=pred_list

fig, ax=plt.subplots()

# Plot the results
ax.plot(year, hare,'o', label = "Hare", color="royalblue")
ax.plot(year,lynx,'o', label="Lynx", color="darkorange")
ax.set_xlabel('Year')
ax.set_ylabel('Hudson Bay Pelts Collected (x1000)')
ax.plot(time_list, prey_list, "--",label="pred Hare", linewidth=1, color='royalblue')
ax.plot(time_list, pred_list,"--",label="pred Lynx", linewidth=1, color= 'darkorange')
ax.legend(bbox_to_anchor=(0,1.06,1,.2), loc="lower left", mode="expand", ncol=4)
ax.set_title("Hudson Bay Hare & Lynx Pelts Best Fit")

fig3, ax3 = plt.subplots()

# model parameters
a1 = A5[0]; a2 = A5[1]; b1 = A5[2];  b2 = A5[3]
dt = 0.01; max_time = 1900

# initial time and populations
t = 1845; x = 19.58; y = 30.09

# empty lists in which to store time and populations
time_list = []; prey_list = []; pred_list = []

# initialize lists
time_list.append(t); prey_list.append(x); pred_list.append(y)

Harsh_winter = True

while t < max_time:
    # calc new values for t, x, y
    t = t + dt

    if Harsh_winter and t>1861:
        x = 0.1 * x
        Harsh_winter = False

    x = x + (a1*x - a2*x*y)*dt
    y = y + (-b1*y + b2*x*y)*dt
```

```
    # store new values in lists
    time_list.append(t)
    prey_list.append(x)
    pred_list.append(y)



ax10.plot(time_list, prey_list, "--",label="pred Hare", linewidth=1, color='royalblue')
ax10.plot(time_list, pred_list,"--",label="pred Lynx", linewidth=1, color= 'darkorange')
ax10.set_xlabel('Year')
ax10.set_ylabel('Hudson Bay Pelts Collected (x1000)')
ax10.set_title("Winter Simulation")
ax10.legend()

def dZ_dt(x,y):
    x, y = x,y
    dxdt, dydt = x*(A5[0] - A5[1]*y), -y*(A5[2] - A5[3]*x)
    return [dxdt, dydt]

fig6, ax6 = plt.subplots()

x=np.linspace(0,160,20)
y=np.linspace(0,90,20)
X,Y = np.meshgrid(x,y)
U,V = dZ_dt(X,Y)
U = U / np.sqrt(U**2 + V**2);
V = V / np.sqrt(U**2 + V**2);
ax6.quiver(X,Y,U,V)

ax6.plot(ODE_prey,ODE_pred, linewidth=3,color = "royalblue",label="Pred Lynx/Hare")
ax6.plot(prey_list,pred_list, linewidth=3,color="darkorange", label="Harsh Winter")
ax6.scatter(A5[2]/A5[3],A5[0]/A5[1], color="lime", label = "Pred equil")
ax6.set_xlabel("Hares", fontsize=14)
ax6.set_ylabel("Lynx", fontsize=14)
ax6.scatter(Pred_prey_data[:,1],Pred_prey_data[:,2], color = "maroon", label="HBC Data")
ax6.set_title("Phase Diagram of Predicted Lynx-Hare Population")
ax6.legend(bbox_to_anchor=(0,1.06,1.075,.2), loc="lower left", mode="expand", ncol=4)

ax6.annotate('Sudden hare population decline', xy=(51.3, 5.5), xytext=(63.7, -4),
            arrowprops=dict(facecolor='black', shrink=0.05))

# fig.savefig('Hare_lynx_bestfit.pdf')
# fig0.savefig('Hare_Lynx_raw.pdf')
# fig00.savefig("Hare_Lynx_scatter.pdf")
# fig000.savefig("Hare_Lynx_dotdash.pdf")
# fig1.savefig('Hare_lynx_1908_1935.pdf')
# fig3.savefig('Harsh_winter.pdf')
```

```
#fig6.savefig("phase_diagram_lynx_hare.pdf")
```